BIGRADED PATH HOMOLOGY AND THE MAGNITUDE-PATH SPECTRAL SEQUENCE

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Abstract. Two important invariants of directed graphs, namely magnitude homology and path homology, have recently been shown to be intimately connected: there is a *magnitude-path spectral sequence* or *MPSS* in which magnitude homology appears as the first page, and in which path homology appears as an axis of the second page. In this paper we study the homological and computational properties of the spectral sequence, and in particular of the full second page, which we now call *bigraded path homology*. We demonstrate that every page of the MPSS deserves to be regarded as a homology theory in its own right, satisfying excision and Künneth theorems (along with a homotopy invariance property already established by Asao), and that magnitude homology and bigraded path homology also satisfy Mayer–Vietoris theorems. We construct a homotopy theory of graphs (in the form of a cofibration category structure) in which weak equivalences are the maps inducing isomorphisms on bigraded path homology, strictly refining an existing structure based on ordinary path homology. And we provide complete computations of the MPSS for two important families of graphs—the directed and bi-directed cycles—which demonstrate the power of both the MPSS, and bigraded path homology in particular, to distinguish graphs that ordinary path homology cannot.

Contents

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1. Introduction

There is, by now, an abundance of homological invariants of directed graphs, including path homology [24, 22], magnitude homology [28] and reachability homology [27]. In the context of undirected graphs, even more invariants are applicable, including clique homology [14, 40] and discrete cubical homology [5]. These theories have some notable successes, such as work of Asao relating magnitude homology to curvature of metric spaces [1]; work of Asao, Hiraoka and Kanazawa relating magnitude homology to girth of graphs [1, 4]; work of Tajima and Yoshinaga developing a corresponding magnitude homotopy type [45]; a version of discrete Morse theory relevant to path homology developed by Lin, Wang and Yau [38]; and path homology analogues of classical geometric results developed by Kempton, Münch and Yau [33]. Many of these homology theories possess formal properties analogous to the Eilenberg–Steenrod axioms, yet they tend to disagree when evaluated on even simple classes of graphs. It is desirable, then, to understand the relationships among them, where such relationships exist.

Asao has recently shown that two of these homology theories—namely magnitude homology and path homology—are indeed closely related, appearing on consecutive pages in a certain spectral sequence [3]. We follow Di *et al* [15] in referring to that spectral sequence as the *magnitude-path spectral sequence* or *MPSS*. Page 1 of the MPSS is exactly magnitude homology, while page 2 contains path homology on its horizontal axis, and the target of the spectral sequence is reachability homology [27]. The MPSS thus encompasses three existing invariants of directed graphs and clarifies their relationships, while adding infinitely many new ones. In particular, it extends path homology (the horizontal axis of the second page) to a bigraded theory (the entire second page) that we now call *bigraded path homology*.

The objective of this paper is to demonstrate the properties, strength and usefulness of the MPSS, and of bigraded path homology in particular. Asao has shown that each page of the MPSS has a homotopy-invariance property, whose strength increases as one turns through the pages of the sequence [2]. Building on that observation, we demonstrate that the spectral sequence as a whole possesses formal properties that justify calling *every* page a homology theory for directed graphs—each one distinct, but systematically related. Concerning the MPSS, our main results are as follows.

- (A) Every page of the MPSS satisfies a Künneth theorem with respect to the box product (Theorem 5.6).
- (B) Every page of the MPSS satisfies an excision theorem with respect to a class of subgraph inclusions first studied in [9] (Theorem 6.5).

We also provide a new and detailed proof of the fact (which appeared first in [15]) that the spectral sequence preserves filtered colimits:

- (C) Every page of the MPSS is a finitary functor on the category of directed graphs (Proposition 7.4, proved in Appendix A).

In particular these three properties all hold for magnitude homology and bigraded path homology. Additionally, magnitude homology and bigraded path homology each satisfy a Mayer–Vietoris theorem (Theorems 6.6 and 6.8) that we are able to deduce from the excision property using straightforward homological algebra.

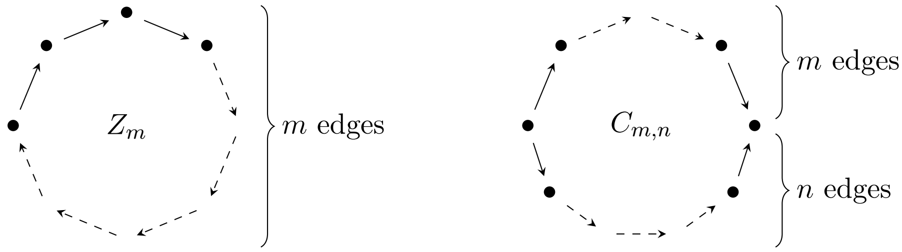

FIGURE 1. The directed cycle Z_m and the bi-directed cycle $C_{m,n}$

The diversity of homological viewpoints on directed graphs has motivated a recent drive towards consolidation using formal homotopy theory, and Section 7 of this paper contributes to that development. In [9], Carranza *et al* prove that the category of directed graphs carries a cofibration category structure for which the weak equivalences are maps inducing isomorphisms on path homology. By specializing properties (A)–(C) to bigraded path homology, we can prove that their structure admits a natural refinement:

(D) The category of directed graphs carries a cofibration category structure in which the cofibrations are those of [9] and the weak equivalences are maps inducing isomorphisms on bigraded path homology (Theorem 7.2).

That this structure is indeed strictly finer than the one in [9] is demonstrated by complete computations of bigraded path homology and the MPSS for two important families of directed graphs. These are the *directed cycles* Z_m for $m \geq 1$, and the *bi-directed cycles* $C_{m,n}$ for $m, n \geq 1$, depicted in Figure 1.

Path homology can only distinguish Z_1 and Z_2 (which it sees as 'contractible') from the Z_m for $m \geq 3$ (which it sees as 'circle-like'). Similarly, path homology cannot distinguish any of the $C_{m,n}$ from one another so long as $\max(m, n) \geq 3$. We find, however, that the MPSS, and even bigraded path homology, can do much better:

(E) Bigraded path homology can distinguish all of the Z_m for $m \geq 2$ by inspecting the bidegrees in which generators lie. Further, the MPSS of Z_m characterises $m \geq 1$ as the first value of r for which the E^r-page is trivial, i.e. concentrated in bidegree $(0, 0)$. (Theorem 8.2 and the subsequent paragraph.)
(F) The bigraded path homology of $C_{m,n}$ depends only on the value of $\mathfrak{m} = \max(m, n)$, and for $\mathfrak{m} \geq 2$ one can determine the value of $\mathfrak{m}$ by inspecting the bidegrees in which generators lie. Further, the MPSS of $C_{m,n}$ characterises $\mathfrak{m} \geq 2$ as the first value of r for which the E^r-page is trivial, i.e. concentrated in bidegree $(0, 0)$. (Proposition 9.2 and Theorem 9.3.)

We believe that the results (A)–(F) demonstrate clearly the theoretical and computational strength of the MPSS, and of bigraded path homology in particular. Indeed, it appears that bigraded path homology shares many of the strengths and advantages of path homology, whilst being a more sensitive and informative invariant. On a technical front, our methods illustrate a useful principle: that properties of path homology are frequently (though not always) 'inherited' from corresponding properties of magnitude homology—and that what holds true for either of these will often hold true throughout the MPSS. Indeed, proofs of results about path homology that use the standard construction of [24, 22] are often rather involved, and we believe that passage to the bigraded theory and the MPSS can serve to clarify such proofs, rather than complicating them.

Moreover, it is our belief that the MPSS as a whole will eventually cast more light on the homotopy theory of directed graphs. Elaborating on speculations made by Asao in the introduction to [2], it is tempting to conjecture that the cofibration category we describe may belong to a nested family of structures, one for each page of the sequence, with r*-homotopy* (Definition 2.2) functioning as the relevant notion of homotopy for the theory in which weak equivalences induce isomorphisms on the $(r+1)$-page. Zooming in a little, consider the excision result mentioned above, which shows that certain maps of pairs obtained from cofibrations induce isomorphisms on the MPSS from the E^1 page onwards. This is in contrast to the situation for homotopies, where r-homotopic maps induce equal maps on the MPSS from only the $(r+1)$-page onwards. We wonder whether there may be a notion of r*-cofibration*, more general than the existing notion of cofibration, which induces excision isomorphisms from the E^{r+1} page onwards; such cofibrations might feature in a homotopy theory in which weak equivalences induce isomorphisms on the $(r+1)$-page. The development of this idea is left to future work.

Open questions. Magnitude homology, path homology and the MPSS are all relatively new invariants, and as such there are many open questions and opportunities for further research. Here, we highlight just a few that we anticipate will be fruitful:

- To what extent can entries of the MPSS be nonzero, and to what extent can this be controlled? For example, given an arbitrary location $E^r_{i,j}$ in the MPSS, can we find a graph G for which $E^r_{i,j}(G) \neq 0$? And can the diameter of such G be chosen to be r? (The answer to such questions will often be 'no', if only because the MPSS is concentrated in a specific octant, but we anticipate further restrictions still. Our computations for Z_m, for instance, provide many examples, but still only succeed in occupying bidegrees (a,b) for which $\frac{a}{2}|b$ if a is even, or $\frac{a-1}{2}|(b-1)$ if a is odd.)
- To what extent can torsion appear in the MPSS, and to what extent can it be controlled? For example, given a specific finitely-generated torsion abelian group, and a specific location $E^r_{i,j}$ in the MPSS, can we find a graph whose MPSS features that torsion group in that location? (We do not know whether this question has been investigated in full even for magnitude homology. Again, the answer to this question will often be 'no', but the nature and extent of any restrictions will themselves be of interest.)
- To what extent do the results and methods of this paper extend to the category $\mathbb{N}$**Met** of generalised metric spaces and short maps with distances in $\mathbb{N} \cup \{\infty\}$? (The usefulness of this category is made apparent in our Appendix A. We anticipate that extending to this category will elucidate and facilitate any attacks on 'realisation problems' like the two above.)
- Kaneta and Yoshinaga gave, under certain conditions, a decomposition of magnitude homology as a direct sum indexed by certain *frames* [32, Theorem 3.12].

To what extent does this decomposition extend to the MPSS? In particular, how does it interact with the differentials d^1 of the MPSS?

- Is there a Mayer–Vietoris theorem applying to pages E^r for $r \geq 3$? (We prove Mayer–Vietoris theorems for E^1 and E^2, which take the form of a short and long exact sequence respectively, the latter obtained from the former by passing to homology with respect to the d^1 differential. However, applying homology in a long exact sequence does not produce a long exact sequence, but rather a spectral sequence converging to 0. A significant sub-question, then, is to identify the relevant algebraic structures, generalising short and long exact sequences, that will relate the E^r-pages.)
- Are there versions of magnitude corresponding to the later pages of the MPSS of a graph G that can be described directly from the graph, without reference to homological algebra? (The notion of magnitude of a finite graph preceded the introduction of magnitude homology, and has a simple and direct definition [36].)

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2. The magnitude-path spectral sequence

We begin by fixing terminology and notation concerning the categories of directed graphs and metric spaces, before proceeding to describe the magnitude-path spectral sequence. We will assume familiarity with spectral sequences in general, but for the reader who is new to the subject, we recommend Sections 5.1 and 5.2 of [47], and Section 2.1 of [39].

2.1. **The category of directed graphs.** This paper is concerned with directed graphs, which are permitted to contain loops but not parallel edges. Thus, a directed graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G) \times V(G)$ of (directed) edges. We depict an edge (x, y) by an arrow $x \to y$. A *(directed) path* from a vertex x to a vertex y in G is a sequence of consistently-oriented edges

$$x = x_0 \to x_1 \to \cdots \to x_k = y.$$

(Note that we do not require the vertices in a path to be distinct.) A *map of directed graphs* $G \to H$ is a function $f \colon V(G) \to V(H)$ such that for every edge $x \to y$ in G, either $f(x) = f(y)$ or there is an edge $f(x) \to f(y)$ in H (or both). The category of directed graphs and maps of directed graphs is denoted by **DiGraph**.

We will also be interested in the category **Met** of *(generalized) metric spaces* and *short maps*. A generalized metric space is a set X equipped with a function $d_X \colon X \times X \to [0, +\infty]$ such that $d_X(x, x) = 0$ for every $x \in X$ and the triangle inequality is satisfied. We will always call the function d_X a *metric*, though it may not be separated or symmetric. A short map is a function $f \colon X \to Y$ such that $d_Y(f(x), f(x')) \leq d_X(x, x')$ for every $x, x' \in X$.

Every directed graph G carries a metric on its vertex set, in which $d(x, y)$ is the minimal number of edges in a directed path from x to y, or $+\infty$ if no such path exists.

This is called the *shortest path metric* on G. The operation of equipping a directed graph with the shortest path metric extends to a functor

$$M\colon \mathbf{DiGraph} \hookrightarrow \mathbf{Met}$$

which is full and faithful, making **DiGraph** into a full subcategory of **Met**.

2.2. **Reachability chains and the length filtration.** Throughout the paper we work over a fixed commutative ground ring R. The *reachability chain complex* of a directed graph G is the chain complex $\mathrm{RC}_*(G)$ of R-modules defined as follows. In degree k the R-module $\mathrm{RC}_k(G)$ has basis given by all tuples $(x_0, \ldots, x_k)$ of vertices of G in which consecutive entries are distinct, and in which there is a directed path in G from each entry to the next. The differential of $\mathrm{RC}_*(G)$ is given by

$$\partial(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_k) \tag{1}$$

where any term with repeated consecutive entries is omitted. Note that in the summand $(x_0, \ldots, \widehat{x_i}, \ldots, x_k)$ a path between the adjacent terms x_{i-1} and x_{i+1} can be obtained by concatenating a path from x_{i-1} to x_i with one from x_i to x_{i+1}, these existing due to the original assumption on $(x_0, \ldots, x_k)$. The *reachability homology* $\mathrm{RH}_*(G)$ of G is simply the homology of $\mathrm{RC}_*(G)$, and was studied in detail in [27].

The *length* of a tuple $(x_0, \ldots, x_k)$ of vertices in G is defined, using the shortest path metric, by

$$\ell(x_0, \ldots, x_k) = d(x_0, x_1) + \cdots + d(x_{k-1}, x_k).$$

The triangle inequality guarantees that $\ell(x_0, \ldots, \widehat{x_i}, \ldots, x_k)$ is at most $\ell(x_0, \ldots, x_k)$. This allows us to define a filtration

$$F_0\mathrm{RC}_*(G) \subseteq F_1\mathrm{RC}_*(G) \subseteq F_2\mathrm{RC}_*(G) \subseteq \cdots$$

of $\mathrm{RC}_*(G)$ by defining $F_\ell\mathrm{RC}_*(G)$ to be the subcomplex spanned by the generators of length at most ℓ.

Remark 2.1. The reachability chains of a directed graph G can also be described as the normalized simplicial chains of a certain simplicial set—a perspective that will be useful in Section 4 and Appendix A. We follow [15, Section 1.2] in calling this simplicial set the '(filtered) nerve'; it is defined as follows.

The *nerve* of G is the simplicial set $\mathcal{N}(G)$ whose set of k-simplices is

$$\mathcal{N}_k(G) = \left\{ (x_0, \ldots, x_k) \;\middle|\; \begin{array}{l} x_0 \ldots x_k \in V(G) \text{ and for } 0 \leq i < k \text{ there} \\ \text{exists a directed path from } x_i \text{ to } x_{i+1} \text{ in } G \end{array} \right\}.$$

(Observe that here we do not insist that adjacent terms be distinct.) For $0 \leq i \leq k$ the face operator $\delta_i\colon \mathcal{N}_k(G) \to \mathcal{N}_{k-1}(G)$ discards the i^{th} vertex, and the degeneracy operator $\sigma_i\colon \mathcal{N}_k(G) \to \mathcal{N}_{k+1}(G)$ duplicates the i^{th} vertex. For each $\ell \in \mathbb{N}$, there is a sub-simplicial set $F_\ell\mathcal{N}(G)$ of $\mathcal{N}(G)$ whose set of k-simplices comprises all tuples in $\mathcal{N}_k(G)$ of length at most ℓ. These make $\mathcal{N}(G)$ into a filtered simplicial set called the *filtered nerve* of G.

Given a simplicial set A, we write $N(A)$ or simply NA for the complex of normalized simplicial chains on A; if A is filtered, we equip NA with the filtration defined by

$F_\ell NA = N(F_\ell A)$. The reachability complex of G is precisely the normalized complex of simplicial chains in the nerve of G [27, Section 4]—

$$\mathrm{RC}_*(G) = N(\mathcal{N}(G))$$

—and with the filtration by length, this becomes an equality of filtered chain complexes.

To any filtered chain complex we can associate a spectral sequence that, roughly speaking, allows us to understand the homology of the original complex in terms of the homology of its filtration quotients. (For details, see Section 5.4 of [47], and Section 2.2 of [39].) In the case of the reachability chain complex with its length filtration, this spectral sequence is called the *magnitude-path spectral sequence*, or MPSS for short, and denoted by $\{E^r_{*,*}(G), d^r\}_{r\geq 0}$. Since $\mathrm{RC}_*(-)$ and its filtration are functorial with respect to maps of graphs, each group $E^r_{i,j}(-)$ determines a functor from directed graphs to R-modules. The MPSS was first mentioned, for undirected graphs, in the original paper on magnitude homology [28, Remark 8.7], but it was Asao who first demonstrated its importance in [3]. For more details on aspects of what follows, see [3] and Section 1 of [15].

2.3. **The E^0-page.** The E^0-page of the spectral sequence associated to a filtered chain complex is given by the filtration quotients, and the differential d^0 is induced from the differential on the original complex. In the case of the MPSS, this recovers the *magnitude chains* $\mathrm{MC}_{*,*}(G)$ of a directed graph G [28]. This is the graded chain complex spanned in bidegree k, ℓ by the generators of $\mathrm{RC}_*(G)$ that have degree k and length precisely ℓ, and whose differential is given by (1), but with any terms of length less than ℓ omitted. (Note in particular that the first and last terms of the sum are always omitted.) It is easy to see that $\mathrm{MC}_{*,\ell}(G) = F_\ell\mathrm{RC}_*(G)/F_{\ell-1}\mathrm{RC}_*(G)$, so that taking degree shifts into account, we find that

$$E^0_{i,j}(G) = F_i\mathrm{RC}_{i+j}(G)/F_{i-1}\mathrm{RC}_{i+j}(G) = \mathrm{MC}_{i+j,i}(G)$$

with the magnitude chains differential.

Certain bounds on k and ℓ must be satisfied in order for magnitude chain groups to be nonzero. It follows that the E^0-page of the MPSS is concentrated in bidegrees i, j for which $i \geq 0$ and $-i \leq j \leq 0$; see Figure 2. If G has an upper bound on its finite lengths, i.e. if there is K such that $d(x,y) \leq K$ or $d(x,y) = \infty$ for all vertices x, y, then j is constrained to lie in the smaller range $-\frac{K-1}{K}\cdot i \leq j \leq 0$. And the terms $E^0_{i,-i} = \mathrm{MC}_{0,i}(G)$ on the negative diagonal all vanish for $i > 0$. (See [28, Proposition 2.10] and [37, Theorem 4.1].)

Observe that we are now working with two different kinds of bigrading, sometimes on the same object. First, the bigrading of $\mathrm{MC}_{k,\ell}(G)$, which is well established in the literature on magnitude homology, and which exactly picks out the homological degree and length. Second, the bigrading of $E^r_{i,j}(G)$, which has been established in the literature on spectral sequences of this type since their classical times. Although it is awkward to use two different bigradings, it seems that we are stuck with both. We will always use notation to distinguish which bigrading system is at play, writing $E^1_{i,j}(G)$ on the one hand and $\mathrm{MC}_{k,\ell}(G)$ on the other.

$$
\boxed{E^0_{*,*}(G)} \qquad
\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 \\
0 & \mathrm{MC}_{0,0}(G) & \mathrm{MC}_{1,1}(G) \arrow[d, "d"'] & \mathrm{MC}_{2,2}(G) \arrow[d, "d"'] & \mathrm{MC}_{3,3}(G) \arrow[d, "d"'] & \cdots \\
-1 & & \mathrm{MC}_{0,1}(G) & \mathrm{MC}_{1,2}(G) \arrow[d, "d"'] & \mathrm{MC}_{2,3}(G) \arrow[d, "d"'] & \cdots \\
-2 & & & \mathrm{MC}_{0,2}(G) & \mathrm{MC}_{1,3}(G) \arrow[d, "d"'] & \cdots \\
-3 & & & & \mathrm{MC}_{0,3}(G) & \cdots \\
-4 & & & & & \ddots
\end{tikzcd}
$$

FIGURE 2. Page E^0 of the MPSS is the magnitude chain complex.

2.4. **The E^1-page.** The E^1-page of the spectral sequence associated to a filtration is precisely the homology of the associated filtration quotients. In the case of the MPSS, this recovers the *magnitude homology* $\mathrm{MH}_{*,*}(G)$ of the graph G, defined to be precisely the homology of the magnitude chains, $\mathrm{MH}_{k,\ell}(G) = H_k(\mathrm{MC}_{*,\ell}(G))$. Thus, taking the degree shifts into account, we have

$$E^1_{i,j}(G) = \mathrm{MH}_{i+j,i}(G),$$

which we depict as in Figure 3.

$$
\boxed{E^1_{*,*}(G)} \qquad
\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 \\
0 & \mathrm{MH}_{0,0}(G) & \mathrm{MH}_{1,1}(G) \arrow[l, "d^1"'] & \mathrm{MH}_{2,2}(G) \arrow[l, "d^1"'] & \mathrm{MH}_{3,3}(G) \arrow[l, "d^1"'] & \cdots \\
-1 & & \mathrm{MH}_{0,1}(G) & \mathrm{MH}_{1,2}(G) \arrow[l, "d^1"'] & \mathrm{MH}_{2,3}(G) \arrow[l, "d^1"'] & \cdots \\
-2 & & & \mathrm{MH}_{0,2}(G) & \mathrm{MH}_{1,3}(G) \arrow[l, "d^1"'] & \cdots \\
-3 & & & & \mathrm{MH}_{0,3}(G) & \cdots \\
-4 & & & & & \ddots
\end{tikzcd}
$$

FIGURE 3. Page E^1 of the MPSS is magnitude homology.

The differential d^1 in the E^1 term of the spectral sequence associated to a filtered chain complex C_* is given by applying the differential of C_* to appropriate representatives [47, 5.4.6]. In the case of the MPSS, this amounts to the following.

- Take an element $x \in \mathrm{MH}_{i+j,i}(G)$.
- Represent x by a cycle in $\mathrm{MC}_{i+j,i}(G)$.
- Regard the cycle as an element of $\mathrm{RC}_{i+j}(G)$. It is a combination of generators of length i.
- Apply the differential of $\mathrm{RC}_{i+j}(G)$ to obtain an element of $\mathrm{RC}_{i+j-1}(G)$. This is a combination of generators of length at most $i-1$.
- Discard all generators of length $i-2$ or less and regard the result as an element of $\mathrm{MC}_{i+j-1,i-1}(G)$. This element is a cycle.
- Then $d^1(x)$ is the associated homology class in $\mathrm{MH}_{i+j-1,i-1}(G)$.

2.5. **The E^2-page and beyond.** The E^2-page of the MPSS does not admit so direct a description as the preceding pages. Nevertheless, Asao showed that it contains an important invariant, namely the path homology, as its horizontal axis:

$$E^2_{i,0}(G) = \mathrm{PH}_i(G)$$

See [3, Theorem 1.2]. Though it is standard in some parts of the literature to denote path homology simply by H_*, we write PH_* to distinguish it more clearly from the other homology theories at play.

Path homology has an important homotopy-invariance property [22, Theorem 3.3]. Asao showed that this extends to the rest of the E^2-page and, with increasing strength, to the subsequent pages, as we now explain.

Definition 2.2. Let $f, g\colon G \to H$ be maps of directed graphs, and let $r \geq 0$. We say *there is an r-homotopy from f to g*, and write $f \rightsquigarrow_r g$, if every vertex x of G satisfies $d(f(x), g(x)) \leq r$. We say *there is a long homotopy from f to g*, and write $f \rightsquigarrow_\infty g$, if every vertex x of G satisfies $d(f(x), g(x)) < \infty$.

Thus, for example, there is a 1-homotopy from f to g if, for each x, either $f(x)$ and $g(x)$ are equal or there is a directed edge from the former to the latter. In general, r-homotopy (or long homotopy) is a condition on the pair of maps f and g which requires the existence of certain paths in H, but does not demand us to make a particular choice of such paths. The relation $\rightsquigarrow_r$ is not symmetric; nor, when $r \neq 0, \infty$, is it transitive. However, Asao proved the following, which we state explicitly here for future reference.

Proposition 2.3 (Asao [2, Theorem 1.3])**.** *If there is an r-homotopy from f to g, then the induced maps $E^s(f), E^s(g)\colon E^s_{*,*}(G) \to E^s_{*,*}(H)$ are equal for $s \geqslant r+1$.*

In particular, this says that the entirety of the E^2-term of the MPSS is invariant under 1-homotopy. (See also [3, Proposition 5.7] for this statement.) Meanwhile, reachability homology is invariant under long homotopy:

Proposition 2.4 ([27, Theorem 4.6])**.** *If there is a long homotopy from f to g, then the induced maps $\mathrm{RH}_*(f), \mathrm{RH}_*(g)\colon \mathrm{RH}_*(G) \to \mathrm{RH}_*(H)$ are equal.*

The notion of r-homotopy and long homotopy give rise immediately to corresponding notions of r-homotopy and long homotopy *equivalence* for pairs of directed graphs. In particular, one can define r-contractibility for any $r \geq 0$. For this, note first that the terminal object in the category **DiGraph** is the directed graph with a unique vertex; we denote it by $\bullet$. Its MPSS is trivial: for every r, the page $E^r(\bullet)$ is concentrated in bidegree $(0,0)$, where it is given by a single copy of the ground ring R.

Definition 2.5. Directed graphs G and H are said to be *r-homotopy equivalent* (respectively, *long homotopy equivalent*) if there exist maps $f\colon G \rightleftarrows H : g$ such that $g \circ f$ is related to the identity on G by a zig-zag of r-homotopies (resp. long homotopies) and $f \circ g$ is related to the identity on H by a zig-zag of r-homotopies (resp. long homotopies). A directed graph G is said to be *r-contractible* if the terminal map $G \to \bullet$ is part of an r-homotopy equivalence.

Corollary 2.6. *If G and H are r-homotopy equivalent, then $E^s(G) \cong E^s(H)$ for all $s \geq r+1$.* □

For instance, if G has diameter r (meaning that $r = \sup_{g,g' \in V(G)} d(g, g')$) then G is r-contractible; it follows that its magnitude-path spectral sequence is trivial from page E^{r+1} onwards. For recent progress in the study of r-homotopy equivalence for directed graphs and metric spaces, we refer the reader to Ivanov [29].

2.6. **The E^∞-page.** Let us now consider the E^∞-page of the MPSS. The target of the MPSS is the homology of the reachability chains from which it was constructed, i.e. the reachability homology $\mathrm{RH}_*(G)$. In order to guarantee convergence, let us assume that the graph G has an upper bound on its finite distances, i.e. that there is K such that for each pair of vertices x, y either $d(x, y) \leq K$ or $d(x, y) = \infty$. Then the filtration of $\mathrm{RC}_*(G)$ is bounded: in each degree n, $F_p\mathrm{RC}_n(G)$ vanishes for $p \leq n-1$, and coincides with $\mathrm{RC}_n(G)$ for $p \geq n \cdot K$. (See page 123 of [47].) It follows that in each bidegree i, j the terms $E^r_{i,j}(G)$ are eventually independent of r, and that their common value $E^\infty_{i,j}(G)$ is isomorphic to the relevant filtration quotient of $\mathrm{RH}_*(G)$ in the filtration it inherits from the length filtration:

$$E^\infty_{i,j}(G) = F_i\mathrm{RH}_{i+j}(G)/F_{i-1}\mathrm{RH}_{i+j}(G).$$

Note that the above condition on G always holds for finite graphs. However, it seems possible that other conditions on G may guarantee that the filtration is bounded, and certainly other conditions besides boundedness can guarantee convergence of a spectral sequence.

3. Bigraded path homology

We saw in the last section that the E^2-page of the MPSS of a directed graph G contains the path homology $\mathrm{PH}_*(G)$ as its horizontal axis, and that the entire page has the same homotopy invariance property that $\mathrm{PH}_*(G)$ does, namely invariance under 1-homotopy. This motivates the following definition.

Definition 3.1. Let G be a directed graph. The *bigraded path homology* of G, denoted $\mathrm{PH}_{*,*}(G)$, is defined by

$$\mathrm{PH}_{k,\ell}(G) = E^2_{\ell,k-\ell}(G) \tag{2}$$

for all k, ℓ, so that we have

$$E^2_{i,j}(G) = \mathrm{PH}_{i+j,i}(G)$$

in precise analogy with the relationship between E^1 and MH, and also

$$\mathrm{PH}_{k,k}(G) = \mathrm{PH}_k(G).$$

We may depict the former as in Figure 4. Just as the MPSS is functorial with respect to maps of graphs, so the same holds for the bigraded path homology.

$$\boxed{E^2_{*,*}(G)} \qquad \begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & \mathrm{PH}_{0,0}(G) & \mathrm{PH}_{1,1}(G) & \mathrm{PH}_{2,2}(G) & \mathrm{PH}_{3,3}(G) & \cdots \\
-1 & & \mathrm{PH}_{0,1}(G) & \mathrm{PH}_{1,2}(G) & \mathrm{PH}_{2,3}(G) & \cdots \\
-2 & & & \mathrm{PH}_{0,2}(G) & \mathrm{PH}_{1,3}(G) & \cdots \\
-3 & & & & \mathrm{PH}_{0,3}(G) & \cdots \\
-4 & & & & & \ddots
\end{array}$$

FIGURE 4. Page E^2 of the MPSS is bigraded path homology.

We will see later in the paper that the bigraded path homology groups satisfy many of the same formal properties as path homology, but that they contain strictly more information.

Definition 3.2. Let X be a directed graph and let A be a subgraph of X. Then $\mathrm{RC}(A)$ is a subcomplex of $\mathrm{RC}(X)$, and we define the *relative reachability chains* of the pair (X, A) to be the quotient chain complex

$$\mathrm{RC}(X, A) = \mathrm{RC}(X)/\mathrm{RC}(A).$$

We equip this with the filtration inherited from the filtration on $\mathrm{RC}(X)$, so that $F_\ell\mathrm{RC}(X, A)$ is the image of $F_\ell\mathrm{RC}(X)$ in $\mathrm{RC}(X, A)$. This results in the *relative magnitude path-spectral sequence* $\{E^r(X, A), d^r\}_{r\geq 0}$ and associated *magnitude chains*, *magnitude homology* and *bigraded path homology* groups of the pair, defined by

$$\begin{aligned}
\mathrm{MC}_{k,\ell}(X, A) &= E^0_{\ell,k-\ell}(X, A) \\
\mathrm{MH}_{k,\ell}(X, A) &= E^1_{\ell,k-\ell}(X, A) \\
\mathrm{PH}_{k,\ell}(X, A) &= E^2_{\ell,k-\ell}(X, A)
\end{aligned}$$

for all k, ℓ.

Recall that a subgraph A of X is said to be *convex* if for every pair of vertices a, a' in A we have $d_A(a, a') = d_X(a, a')$ [36, Definition 4.2].

Theorem 3.3 (Exact sequences of a pair)**.** *Let X be a graph and let A be a subgraph of X. If A is convex in X, then there is a short exact sequence of magnitude chains:*

$$0 \to \mathrm{MC}(A) \to \mathrm{MC}(X) \to \mathrm{MC}(X, A) \to 0 \tag{3}$$

Consequently there is a long exact sequence of magnitude homology groups:

$$\cdots \to \mathrm{MH}_{*,*}(A) \to \mathrm{MH}_{*,*}(X) \to \mathrm{MH}_{*,*}(X, A) \to \mathrm{MH}_{*-1,*}(X, A) \to \cdots \tag{4}$$

If, in addition, there are no edges from $X \setminus A$ into A, then (3) is split by a chain map, (4) splits into short exact sequences, and we obtain a long exact sequence of bigraded path homology groups:

$$\cdots \to \mathrm{PH}_{*,*}(A) \to \mathrm{PH}_{*,*}(X) \to \mathrm{PH}_{*,*}(X, A) \to \mathrm{PH}_{*-1,*-1}(X, A) \to \cdots \tag{5}$$

Proof. We begin with the proof that if $A \subseteq X$ is convex, then we obtain the sequence (3). Convexity means that the length of a generator of $\mathrm{RC}(A)$ does not depend on whether we regard it as a generator of $\mathrm{RC}(A)$ or $\mathrm{RC}(X)$. The result can now be proved directly from the explicit description of magnitude chains given in Section 2.3. Alternatively, recall that a map of filtered chain complexes $f\colon C \to D$ is called *strict* if for each ℓ we have $f(F_\ell C) = f(C) \cap F_\ell D$. Note that the latter condition is equivalent to $f^{-1}(F_\ell D) = F_\ell C + \ker(f)$. If $C \hookrightarrow D$ is a strict inclusion of chain complexes, then one obtains a short exact sequence of filtration quotients:

$$0 \to \frac{F_\ell C}{F_{\ell-1} C} \to \frac{F_\ell D}{F_{\ell-1} D} \to \frac{F_\ell\,(D/C)}{F_{\ell-1}\,(D/C)} \to 0$$

See [44, Section 0120] or [13, Lemme 1.1.9]. Our assumption ensures that the inclusion map $\mathrm{RC}(A) \hookrightarrow \mathrm{RC}(X)$ is strict, so the result follows from the last paragraph.

Applying homology to the short exact sequence (3) now gives the long exact sequence (4).

Let us assume for the rest of the proof that there are no edges into A from $X \setminus A$. This 'no-entry' condition on A means that the only vertices of X that admit paths into A are those that already lie in A. (In particular, the only paths in X between vertices of A are those that lie wholly in A, so this condition alone ensures that $A \subseteq X$ is convex.)

We now show that (3) is split. Observe that, by the no-entry condition, a generator of $\mathrm{MC}(X)$ lies in $\mathrm{MC}(A)$ if and only if its final entry lies in A. Thus we may define a map $p\colon \mathrm{MC}(X) \to \mathrm{MC}(A)$ by the following rule.

$$p(x_0, \ldots, x_k) = \begin{cases} (x_0, \ldots, x_k) & \text{if } x_k \in A \\ 0 & \text{if } x_k \notin A \end{cases}$$

Then p is a chain map because the differential of magnitude chains sends a generator to a linear combination of generators with the same start and end points. And it is a splitting because it sends the generators of $\mathrm{MC}(A)$ to themselves.

So we have shown that our stronger assumption gives us a splitting of (3). It follows that the connecting maps of (4) vanish and (4) splits into short exact sequences. The maps in these short exact sequences are obtained from maps of filtered complexes, and so they commute with the differential d^1 of the magnitude-path spectral sequence. In terms of magnitude chains, the differential d^1 has the form $d^1\colon \mathrm{MH}_{*,*}(-) \to \mathrm{MH}_{*-1,*-1}(-)$, and consequently we obtain the long exact sequence (5) with the specified degree shifts. □

4. Eilenberg–Zilber theorems

This section and the next concern the behaviour of the pages of the magnitude-path spectral sequence with respect to the box product of directed graphs.

Definition 4.1. The *box product* of directed graphs G and H is the directed graph $G\square H$ with vertex set $V(G\square H) = V(G)\times V(H)$, and $((g_1,h_1),(g_2,h_2)) \in E(G\square H)$ if either $g_1 = g_2$ and $(h_1,h_2)\in E(H)$, or $h_1 = h_2$ and $(g_1,g_2)\in E(G)$.

In this section we will prove two Eilenberg–Zilber-style theorems. The first of these, Theorem 4.7, says that for any directed graphs G and H the filtered chain complex $\mathrm{RC}(G\square H)$ is naturally chain homotopy equivalent to the tensor product $\mathrm{RC}(G)\otimes\mathrm{RC}(H)$ via filtration-preserving chain maps and chain homotopies. The second, Theorem 4.9, relates the magnitude-path spectral sequence $E(G\square H)$ to the spectral sequences $E(G)$ and $E(H)$ of the factors.

Before we can state those results formally and prove them (which we do in Section 4.2), we must collect some terminology and facts concerning *pairings* and *tensor products* of spectral sequences.

4.1. Pairings and tensor products of spectral sequences.

First, let us briefly recall the relevant facts about the homology of tensor products of ordinary chain complexes. Given any chain complexes C and D over a ring R, there is a map

$$\alpha\colon H_*(C)\otimes H_*(D)\to H_*(C\otimes D) \tag{6}$$

determined by $[c]\otimes[d]\mapsto[c\otimes d]$. We shall refer to this map as the *homology product*. If the ring R is a principal ideal domain (P.I.D.) and C happens to consist of flat R-modules, then the classical Künneth theorem for chain complexes says that the homology product fits into a short exact sequence, as follows.

Theorem 4.2 (Algebraic Künneth theorem)**.** *Let R be a P.I.D. and let C be a chain complex of flat R-modules. Then, given any chain complex D of R-modules, we have for each $n\in\mathbb{N}$ a short exact sequence*

$$0\to\bigoplus_k H_k(C)\otimes H_{n-k}(D)\xrightarrow{\alpha} H_n(C\otimes D)\to\bigoplus_k \mathrm{Tor}(H_k(C),H_{n-k-1}(D))\to 0$$

natural in D and with respect to chain maps $C\to C'$ where C' is also flat. The sequence splits, but not naturally.

A proof of Theorem 4.2 can be found in [11, VI.3.3]. (The statement there is for hereditary rings, which includes the case of P.I.D.s [11, p.13].)

The algebraic Künneth theorem has the following corollary.

Corollary 4.3. *If C and D are chain complexes over a field, then the homology product $\alpha\colon H_*(C)\otimes H_*(D)\to H_*(C\otimes D)$ is an isomorphism.* □

Now, fix a ground ring R (not necessarily a P.I.D.) and let E and $'E$ be any two spectral sequences of R-modules. For each $r\geq 0$, we can form the tensor product $E^r\otimes{}'E^r$ of bigraded R-modules—

$$(E^r\otimes{}'E^r)_{pq} = \bigoplus_{\substack{s+u=p\\ t+v=q}} E^r_{st}\otimes{}'E^r_{uv}$$

—and endow it with the differential $d^r_\otimes(x \otimes y) = d^r_E(x) \otimes y + (-1)^{s+t} x \otimes d^r_{'E}(y)$. In general, the family $(E^r \otimes {'E^r}, d^r_\otimes)_{r\geq 0}$ need not be a spectral sequence, for, although the homology product always provides a natural map

$$E^{r+1} \otimes {'E^{r+1}} \cong H_*(E^r) \otimes H_*('E^r) \xrightarrow{\alpha} H_*(E^r \otimes {'E^r}),$$

this need not be an isomorphism. If the ring R is a field, however, then Corollary 4.3 implies we have a *tensor product spectral sequence* $\{E^r \otimes {'E^r}, d^r_\otimes\}$.

Definition 4.4. Let $E, 'E$ and $''E$ be spectral sequences of R-modules. A *pairing* $\phi^*\colon (E, 'E) \to {''E}$ is a sequence of maps of bigraded R-modules $\phi^r\colon E^r \otimes {'E^r} \to {''E^r}$ with the following properties:

(1) Each ϕ^r is a chain map with respect to the differentials on page r.
(2) For every r this diagram commutes:

$$\begin{tikzcd}
E^{r+1} \otimes {'E^{r+1}} \arrow[r, "\phi^{r+1}"] \arrow[d, "\cong"'] & {''E^{r+1}} \\
H_*(E^r) \otimes H_*('E^r) \arrow[d, "\alpha"'] & \\
H_*(E^r \otimes {'E^r}) \arrow[r, "H_*(\phi^r)"'] & H_*(''E^r). \arrow[uu, "\cong"']
\end{tikzcd}$$

If the ground ring R is a field, a pairing is precisely a map of spectral sequences.

Definition 4.5. Let A and B be filtered chain complexes. Their *(filtered) tensor product* is the chain complex $A \otimes B$ equipped with the filtration in which

$$F_\ell(A \otimes B) = \sum_{s+t=\ell} F_sA \otimes F_tB.$$

A chain map $f\colon A \otimes B \to C$ is filtered if and only if for every s and t we have $f(F_sA \otimes F_tB) \subseteq F_{s+t}C$. Any such map induces a map

$$\bigoplus_{p+q=n} \frac{F_pA}{F_{p-1}A} \otimes \frac{F_qB}{F_{q-1}B} \to \frac{F_nC}{F_{n-1}C}$$

since $f(F_{p-1}A \otimes F_qB) \subseteq F_{p+q-1}C$ and $f(F_pA \otimes F_{q-1}B) \subseteq F_{p+q-1}C$. It follows that there is an induced map $\overline{f}\colon E^0(A) \otimes E^0(B) \to E^0(C)$. Indeed, f induces a map $E^r(A) \otimes E^r(B) \to E^r(C)$ for every r, and these maps comprise a pairing of spectral sequences. (For a detailed proof, see [26, Lemma 3.5.2].)

Lemma 4.6. *Let A, B and C be filtered chain complexes over R. Any filtered chain map $f\colon A \otimes B \to C$ induces a pairing $\phi\colon (E(A), E(B)) \to E(C)$ with $\phi^0 = \overline{f}$. If R is a field, then f induces a map of spectral sequences $\phi\colon E(A) \otimes E(B) \to E(C)$.* □

We are now equipped to state and prove the Eilenberg–Zilber theorems.

4.2. **The Eilenberg–Zilber theorems.** In Theorem 5.1 of [27], the classical Eilenberg–Zilber theorem for simplicial sets is applied to prove that there is a chain homotopy equivalence

$$\nabla\colon \mathrm{RC}(G)\otimes\mathrm{RC}(H)\rightleftarrows\mathrm{RC}(G\Box H):\Delta.$$

Here we extend this to an equivalence of *filtered* chain complexes, from which we will obtain, via Lemma 4.6, an induced pairing of spectral sequences

$$\nabla^*\colon E(G)\otimes E(H)\to E(G\Box H).$$

Theorem 4.7 (Filtered Eilenberg–Zilber for the reachability chain complex)**.** *Let G and H be directed graphs. Then $\mathrm{RC}(G)\otimes\mathrm{RC}(H)$ and $\mathrm{RC}(G\Box H)$ are chain homotopy equivalent, via maps and chain homotopies that are natural and that respect the filtrations of the two sides. The chain homotopy equivalence*

$$\nabla\colon \mathrm{RC}(G)\otimes\mathrm{RC}(H)\to\mathrm{RC}(G\Box H)$$

can be described explicitly on generators as follows:

$$\nabla\left((g_0,\ldots,g_p)\otimes(h_0,\ldots,h_q)\right)=\sum_\sigma \mathrm{sign}(\sigma)((g_{i_0},h_{j_0}),\ldots,(g_{i_r},h_{j_r})). \tag{7}$$

Here $r=k+k'$, and σ runs over all sequences $((i_0,j_0),\ldots,(i_r,j_r))$ in which $0\leq i_s\leq k$, $0\leq j_s\leq k'$, and in which each term (i_{s+1},j_{s+1}) is obtained from (i_s,j_s) by increasing exactly one of the components by 1. The coefficient $\mathrm{sign}(\sigma)$ is defined to be $(-1)^n$ where n is the number of pairs (i,j) for which $i=i_k\implies j<j_k$.

In the statement of Theorem 4.7, the sequences σ can be regarded as the paths in $\mathbb{Z}^2$ from $(0,0)$ to (k,k') that may only go upwards or to the right. With this in mind, $\mathrm{sign}(\sigma)$ is $(-1)^n$ where n is the number of lattice points that are on or above the x-axis, and strictly below the path itself. And if we regard each (g_i,h_j) as being a label on lattice point (i,j), then the summand $((g_0,h_0),\ldots,(g_k,h_{k'}))$ associated to σ is precisely the list of vertices visited by the path.

To prove Theorem 4.7, we will make use of a filtered variant of the classical Eilenberg–Zilber theorem. The classical theorem says that we have, for any simplicial sets A and B, the Eilenberg–Zilber map $\nabla\colon NA\otimes NB\to N(A\times B)$ and the Alexander–Whitney map $\Delta\colon N(A\times B)\to NA\otimes NB$, which satisfy $\Delta\circ\nabla=\mathrm{Id}_{NA\otimes NB}$, and a chain homotopy SHI between $\nabla\circ\Delta$ and $\mathrm{Id}_{N(A\times B)}$. (For details, see Section 5 of [16] or Section 2 of [18]; in particular the latter reference contains explicit descriptions of Δ, ∇ and SHI.) For present purposes, it is important to know that if A and B are filtered, then all these maps are automatically filtration-preserving.

Lemma 4.8. *Let A and B be filtered simplicial sets. We equip their product $A\times B$ with the filtration given by $F_\ell(A\times B)=\bigcup_{i+j=\ell}F_iA\times F_jB$. Then the Alexander–Whitney map $\Delta\colon N(A\times B)\to NA\otimes NB$, the Eilenberg–Zilber map $\nabla\colon NA\otimes NB\to N(A\times B)$, and the chain homotopy* SHI *between $\nabla\circ\Delta$ and $\mathrm{Id}_{N(A\times B)}$ are all filtration-preserving. Thus, $NA\otimes NB$ and $N(A\times B)$ are chain homotopy equivalent as filtered chain complexes.*

Proof. Observe that $F_p[NA\otimes NB]$ is the span of the images of the maps $N(F_iA)\otimes N(F_jB)\to N(A)\otimes N(B)$ for $i+j=p$, and $F_p[N(A\times B)]$ is the span of the images of the maps $N(F_iA \times F_jB) \to N(A \otimes B)$ for $i + j = p$. Thus we can prove that Δ, say, preserves filtrations by showing that the following diagram commutes:

$$\begin{tikzcd} N(F_iA \times F_jB) \arrow[r, "\Delta"] \arrow[d] & N(F_iA) \otimes N(F_jB) \arrow[d] \\ N(A \times B) \arrow[r, "\Delta"] & N(A) \otimes N(B). \end{tikzcd}$$

But this is an immediate consequence of the naturality of Δ. The proof that ∇ and SHI are filtration-preserving is similar. $\square$

We now prove Theorem 4.7. The proof makes use of the fact that the shortest path metric on the box product $G\square H$ coincides with the ℓ_1-metric on the product of the vertex sets:

$$d_{G\square H}((g_0, h_0), (g_1, h_1)) = d_G(g_0, g_1) + d_H(h_0, h_1) \tag{8}$$

for each $(g_1, h_1), (g_2, h_2) \in V(G) \times V(H)$. (See, for example, [15, Corollary 1.3].)

Proof of Theorem 4.7. By Remark 2.1 and Lemma 4.8, there is a chain homotopy equivalence of filtered chain complexes

$$\mathrm{RC}(G) \otimes \mathrm{RC}(H) = N(\mathcal{N}(G)) \otimes N(\mathcal{N}(H)) \rightleftarrows N(\mathcal{N}(G) \times \mathcal{N}(H)). \tag{9}$$

It remains to identify $N(\mathcal{N}(G) \times \mathcal{N}(H))$ with $N(\mathcal{N}(G\square H)) = \mathrm{RC}(G\square H)$.

In fact, $\mathcal{N}(G) \times \mathcal{N}(H)$ and $\mathcal{N}(G\square H)$ are isomorphic as filtered simplicial sets. This is Proposition 1.4 in [15], but for clarity we give some details here. The isomorphism of simplicial sets $\mathcal{N}(G) \times \mathcal{N}(H) \cong \mathcal{N}(G\square H)$, given by

$$((g_0, \ldots, g_k), (h_0, \ldots, h_k)) \leftrightarrow ((g_0, h_0), \ldots, (g_k, h_k)),$$

was established in the proof of Theorem 5.1 of [27]; we just need to show that the tuple $((g_0, h_0), \ldots, (g_k, h_k))$ lies in filtration p if and only if the same is true of the pair $((g_0, \ldots, g_k), (h_0, \ldots, h_k))$. But this follows from the fact that

$$\ell((g_0, h_0), \ldots, (g_k, h_k)) = \ell(g_0, \ldots, g_k) + \ell(h_0, \ldots, h_k),$$

which holds since, by (8), we have

$$\begin{aligned} \ell\left((g_0, h_0), \ldots, (g_k, h_k)\right) &= \sum_{m=0}^{k-1} d_{G\square H}((g_m, h_m), (g_{m+1}, h_{m+1})) \\ &= \sum_{m=0}^{k-1} (d_G(g_m, g_{m+1}) + d_H(h_m, h_{m+1})) \\ &= \sum_{m=0}^{k-1} d_G(g_m, g_{m+1}) + \sum_{m=0}^{k-1} d_H(h_m, h_{m+1}) \\ &= \ell(g_0, \ldots, g_k) + \ell(h_0, \ldots, h_k). \end{aligned}$$

Thus we have an isomorphism of filtered chain complexes

$$N(\mathcal{N}(G) \times \mathcal{N}(H)) \cong N(\mathcal{N}(G\square H)) = \mathrm{RC}(G\square H),$$

and this, combined with (9), proves the theorem. $\square$

From Theorem 4.7 we can derive an Eilenberg–Zilber-type theorem for the magnitude-path spectral sequence of the box product, as follows. In Section 5 we will use this result to prove Künneth theorems for each page of the MPSS.

Theorem 4.9 (Eilenberg–Zilber for the MPSS)**.** *For any directed graphs G and H there is a pairing of spectral sequences*

$$\nabla^*\colon (E(G), E(H)) \longrightarrow E(G\Box H)$$

which is natural in G and H, and for which ∇^0 is a chain homotopy equivalence.

Proof. Theorem 4.7 gives us a map of filtered chain complexes

$$\nabla\colon \mathrm{RC}(G)\otimes \mathrm{RC}(H) \to \mathrm{RC}(G\Box H)$$

natural in G and H. From this, via Lemma 4.6, we obtain the pairing of spectral sequences

$$\nabla^*\colon (E(G), E(H)) \longrightarrow E(G\Box H)$$

in which the map ∇^0 of the E^0-terms is the map of filtration quotients induced by ∇. Since ∇'s homotopy inverse Δ and the chain homotopy SHI are also filtration-preserving, they both descend to the filtration quotients, making ∇^0 a chain homotopy equivalence. □

Remark 4.10. Though the box product is sometimes referred to as the *cartesian product* of directed graphs, it is not the categorical product in **DiGraph**. The categorical product of G and H is their *strong product*: the directed graph $G\boxtimes H$ whose vertices are elements of $V(G)\times V(H)$, with $((g_1,h_1),(g_2,h_2))\in E(G\boxtimes H)$ if $g_1=g_2$ and $(h_1,h_2)\in E(H)$; or $h_1=h_2$ and $(g_1,g_2)\in E(G)$; or $(g_1,g_2)\in E(G)$ and $(h_1,h_2)\in E(H)$. The shortest path metric on $G\boxtimes H$ coincides with the ℓ_∞-metric on the product of the vertex sets:

$$d_{G\boxtimes H}((g_0,h_0),(g_1,h_1)) = \max\{d_G(g_0,g_1), d_H(h_0,h_1)\}$$

for each $(g_0,h_0),(g_1,h_1)\in V(G)\times V(H)$.

Just as in the case of the box product, there is an isomorphism of simplicial sets $\mathcal{N}(G)\times\mathcal{N}(H)\cong\mathcal{N}(G\boxtimes H)$, given by

$$((g_0,\dots,g_k),(h_0,\dots,h_k)) \leftrightarrow ((g_0,h_0),\dots,(g_k,h_k)).$$

(This is part of Theorem 5.1 in [27].) However, this is not an isomorphism of *filtered* simplicial sets: while the function $\mathcal{N}(G)\times\mathcal{N}(H)\to\mathcal{N}(G\boxtimes H)$ is always filtration-preserving, its inverse usually is not.

5. Künneth theorems

In general, one obtains a Künneth theorem for some homology theory by combining an appropriate Eilenberg–Zilber theorem with the classic algebraic Künneth theorem for chain complexes. The Eilenberg–Zilber theorem usually establishes a chain homotopy equivalence, so is in some sense as good as can be hoped. However, relating the homology of a tensor product of chain complexes with the tensor product of the homologies entails loss of information, as quantified by the relevant Tor term in the algebraic Künneth theorem.

Our Eilenberg–Zilber theorem for the magnitude-path spectral sequence (Theorem 4.9) is, in this setting, as good as can be hoped: a pairing of spectral sequences that is a chain homotopy equivalence on the initial term. However, in order to access, say, the E^2-term $E^2(G\square H)$, we need to take homology twice, and therefore potentially twice encounter the discrepancies expressed by the Tor terms. It may be the case that there is a general framework for encapsulating and understanding these cascading errors in the setting of a spectral sequence, but in the present paper our approach is to make assumptions in order to ensure that no such cascades arise.

Our strongest Künneth theorem holds under the assumption that R is a field. In this case, the pairing ∇^* in Theorem 4.9 is a map of spectral sequences. Since ∇^0 is a quasi-isomorphism, it follows that ∇^r is an isomorphism

$$E^r(G)\otimes E^r(H)\xrightarrow{\cong}E^r(G\square H)$$

for every $r\geq 1$. This gives the following result.

Theorem 5.1 (Künneth theorem for the MPSS over a field)**.** *Fix a ground ring R which is a field. Then for every pair of directed graphs G and H there is a map of spectral sequences*

$$E(G)\otimes E(H)\to E(G\square H)$$

natural in G and H and consisting of isomorphisms from E^1 onwards. □

In particular, Theorem 5.1 gives us Künneth isomorphisms for magnitude homology and bigraded path homology with coefficients in a field. In the absence of the assumption that R is a field, we can still obtain Künneth theorems of the usual form in magnitude homology and in the original path homology, as we see in the next two results.

Specialized to undirected graphs, the following theorem recovers Theorem 5.3 of [28]. It is in turn a special case of the Künneth formula for the magnitude homology of generalized metric spaces [42, Theorem 4.6], which extends that for classical metric spaces proved as Proposition 4.3 in [7].

Theorem 5.2 (Künneth theorem for magnitude homology)**.** *Fix a ground ring R which is a P.I.D. For any directed graphs G and H there is short exact sequence*

$$\begin{aligned} 0\to \bigoplus_{\substack{i+j=k\\ a+b=\ell}}\mathrm{MH}_{i,a}(G)\otimes\mathrm{MH}_{j,b}(H)\to\mathrm{MH}_{k,\ell}(G\square H)\\ \to\bigoplus_{\substack{i+j=k-1\\ a+b=\ell}}\mathrm{Tor}(\mathrm{MH}_{i,a}(G),\mathrm{MH}_{j,b}(H))\to 0,\end{aligned}\tag{10}$$

natural in G and H.

Proof. Apply the algebraic Künneth theorem (Theorem 4.2) to the tensor product chain complex $E^0(G)\otimes E^0(H)$. Then use the quasi-isomorphism $\nabla^0\colon E^0(G)\otimes E^0(H)\to E^0(G\square H)$ to replace the middle term—the result is a short exact sequence relating $E^1(G)$, $E^1(H)$ and $E^1(G\square H)$. Now use the identification $E^1_{p,q}(-)=\mathrm{MH}_{p+q,p}(-)$ to replace these with $\mathrm{MH}(G)$, $\mathrm{MH}(H)$ and $\mathrm{MH}(G\square H)$, and re-index appropriately. □

Using Theorem 5.2 we can recover two known Künneth formulae for ordinary path homology with respect to the box product: [23, Theorem 4.7] and [30, Theorem 9.5]. (The latter applies to more general objects than graphs.)

First, we record a fact that will be useful here and later.

Lemma 5.3. *Fix a ground ring R which is a P.I.D. Let G be any directed graph. For every $k \in \mathbb{N}$, the R-module $\mathrm{MH}_{kk}(G)$ is freely generated.*

Proof. Since $\mathrm{MC}_{k+1,k}(G) = 0$ for every k, we have $\mathrm{MH}_{kk}(G) = \ker(\partial_{kk})$, which is a submodule of the free R-module $\mathrm{MC}_{kk}(G)$. ☐

Theorem 5.4 (Künneth theorem for ordinary path homology)**.** *Fix a ground ring R which is a P.I.D. For any directed graphs G and H there is a short exact sequence*

$$\begin{aligned}0 \to \bigoplus_{i+j=k} \mathrm{PH}_i(G) \otimes \mathrm{PH}_j(H) \to \mathrm{PH}_k(G \square H) \\ \to \bigoplus_{i+j=k} \mathrm{Tor}(\mathrm{PH}_i(G), \mathrm{PH}_{j-1}(H)) \to 0,\end{aligned} \tag{11}$$

natural in G and H.

Proof. Recall from Section 2.5 that the ordinary path homology of a directed graph is the diagonal of its bigraded path homology or, equivalently, the horizontal boundary row on page E^2 of the magnitude-path spectral sequence:

$$\mathrm{PH}_i(G) = \mathrm{PH}_{i,i}(G) = E^2_{i,0}(G).$$

Applying the algebraic Künneth theorem in the horizontal boundary row of page E^1 gives the short exact sequence

$$\begin{aligned}0 \to \bigoplus_{i+j=k} \mathrm{PH}_{ii}(G) \otimes \mathrm{PH}_{jj}(H) \to H_k\left(E^1_{*0}(G) \otimes E^1_{*0}(H)\right) \\ \to \bigoplus_{i+j=k} \mathrm{Tor}(\mathrm{PH}_{ii}(G), \mathrm{PH}_{j-1,j-1}(H)) \to 0;\end{aligned}$$

the claim is that the middle term is isomorphic to $\mathrm{PH}_{kk}(G \square H)$. To see this, we can use the Künneth formula for magnitude homology.

Consider the Künneth sequence for magnitude homology (10) in the case $k = \ell$. Since $\mathrm{MH}_{pq}(-)$ vanishes for $p > q$, the first term $\bigoplus \mathrm{MH}_{ia}(G) \otimes \mathrm{MH}_{jb}(H)$ reduces in this case to just the part involving diagonal terms, i.e. those terms where $a = i$ and $b = j$. For the same reason, the third term $\bigoplus \mathrm{Tor}(\mathrm{MH}_{ia}(G), \mathrm{MH}_{jb}(H))$ reduces to just those terms in which $a = i$ and $b = j - 1$, or $a = i - 1$ and $b = j$. Thus, in all cases the Tor term features a diagonal group $\mathrm{MH}_{ii}(G)$ or $\mathrm{MH}_{jj}(H)$ as one of its arguments. Since the diagonal magnitude homology modules are always free (Lemma 5.3), it follows that the third term vanishes in this case, yielding for every k an isomorphism

$$\bigoplus_{i+j=k} \mathrm{MH}_{ii}(G) \otimes \mathrm{MH}_{jj}(H) \xrightarrow[\cong]{\nabla^1} \mathrm{MH}_{kk}(G \square H).$$

Since $\mathrm{MH}_{ii}(G) \otimes \mathrm{MH}_{jj}(H) = E^1_{i0}(G) \otimes E^1_{j0}(H)$, taking homology gives

$$H_k\left(E^1_{*0}(G) \otimes E^1_{*0}(H)\right) \cong H_k\left(\mathrm{MH}_{**}(G \square H)\right) = \mathrm{PH}_{kk}(G \square H),$$

as claimed. ☐

Looking at the entire second page of the magnitude-path spectral sequence yields a Künneth formula for bigraded path homology. However, to access this we need to make a flatness assumption.

Theorem 5.5 (Künneth theorem for bigraded path homology)**.** *Fix a ground ring R which is a P.I.D., and let G be a directed graph with flat magnitude homology. Then for any directed graph H there is a short exact sequence*

$$\begin{aligned} 0 \to \bigoplus_{\substack{i+j=k \\ a+b=\ell}} \mathrm{PH}_{i,a}(G) \otimes \mathrm{PH}_{j,b}(H) \to \mathrm{PH}_{k,\ell}(G \square H) \\ \to \bigoplus_{\substack{i+j=k \\ a+b=\ell}} \mathrm{Tor}(\mathrm{PH}_{i,a}(G), \mathrm{PH}_{j-1,b-1}(H)) \to 0, \end{aligned} \tag{12}$$

natural in H and with respect to maps $G \to G'$ where G' also has flat magnitude homology. If R is a field then for every *pair of directed graphs G and H there is an isomorphism*

$$\mathrm{PH}_{k,\ell}(G \square H) \cong \bigoplus_{\substack{i+j=k \\ a+b=\ell}} \mathrm{PH}_{i,a}(G) \otimes \mathrm{PH}_{j,b}(H), \tag{13}$$

natural in G and H.

Proof. Since G has flat magnitude homology—meaning that for every p, q the R-module $E^1_{pq}(G) = \mathrm{MH}_{p+q,p}(G)$ is flat—we can apply the algebraic Künneth theorem in each row of $E^1(G) \otimes E^1(H)$ to obtain, for each p and q, a short exact sequence

$$\begin{aligned} 0 \to \left(E^2(G) \otimes E^2(H)\right)_{pq} \to H_p\left((E^1(G) \otimes E^1(H))_{*q}\right) \\ \to \bigoplus_{\substack{m+u=p \\ n+v=q}} \mathrm{Tor}(E^2_{mn}(G), E^2_{u-1,v}(H)) \to 0, \end{aligned} \tag{14}$$

which has the claimed naturality in G and H. The assumption that G has flat magnitude homology ensures that in the statement of Theorem 5.2 the Tor term vanishes, so that using the identification $E^1_{pq}(-) = \mathrm{MH}_{p+q,p}(-)$ we obtain a natural isomorphism

$$(E^1(G) \otimes E^1(H))_{p,q} = (\mathrm{MH}(G) \otimes \mathrm{MH}(H))_{p+q,p} \cong \mathrm{MH}(G \square H)_{p+q,p} = E^1(G \square H)_{p,q}.$$

Using this isomorphism we may replace the middle term of (14) with $H_p(E^1(G \square H)_{*,q}) = E^2_{p,q}(G \square H)$. Using the identification $E^2_{pq}(-) = \mathrm{PH}_{p+q,p}(-)$ and reindexing appropriately, this yields the required short exact sequence (12).

Over a field, the flatness assumption is guaranteed to be satisfied and the torsion term in (12) vanishes, yielding the isomorphism (13). $\square$

Our final Künneth formula holds throughout the magnitude-path spectral sequence, though only under more restrictive flatness assumptions.

Theorem 5.6 (Künneth theorem for the MPSS over a P.I.D.)**.** *Fix a ground ring R which is a P.I.D. Let G and H be directed graphs, and suppose that there is $s \geq 1$ such that, for each $0 \leq r < s$, each term of $E^r(G)$ is flat. Then for $1 \leq r < s$ the map ∇^r is an isomorphism*

$$\bigoplus_{\substack{m+u=p \\ n+v=q}} E^r_{mn}(G) \otimes E^r_{uv}(H) \xrightarrow{\cong} E^r_{pq}(G \square H),$$

while ∇^s fits into a short exact sequence

$$0 \to \bigoplus_{\substack{m+u=p\\ n+v=q}} E^s_{mn}(G) \otimes E^s_{uv}(H) \xrightarrow{\nabla^s} E^s_{pq}(G \square H)$$
$$\to \bigoplus_{\substack{m+u=p\\ n+v=q}} \mathrm{Tor}(E^s_{mn}(G), E^s_{u-s+1,v+s-2}(H)) \to 0$$

which is natural in H and with respect to maps $G \to G'$ for G' satisfying the same flatness property as G.

Proof. We prove this by induction on $s \geq 1$. The case $s = 1$ is just the Künneth theorem for magnitude homology (Theorem 5.2); the case $s = 2$ follows from that theorem and the Künneth theorem for bigraded path homology (Theorem 5.5). Now we take $s > 2$ and assume the statement holds for all $r < s$.

For $1 \leq r < s$, the map on page r is the composite

$$\begin{tikzcd}
E^r(G) \otimes E^r(H) \arrow[rr, "\nabla^r"] \arrow[d, "\cong"'] & & E^r(G \square H) \\
H(E^{r-1}(G)) \otimes H(E^{r-1}(H)) \arrow[d, "\alpha"'] & & \\
H(E^{r-1}(G) \otimes E^{r-1}(H)) \arrow[rr, "H(\nabla^{r-1})"'] & & H(E^{r-1}(G \square H)) \arrow[uu, "\cong"']
\end{tikzcd}$$

where α is the homology product. By the assumptions of the theorem, $E^{r-1}(G)$ and $E^r(G)$ both consist of flat R-modules, so the algebraic Künneth theorem tells us that α has the claimed naturality and is an isomorphism. By the inductive assumption, the map $H(\nabla^{r-1})$ is an isomorphism and has the claimed naturality too. It follows that the same holds for ∇^r.

Applying the algebraic Künneth theorem on page E^{s-1}, to the chain complex lying along each line of slope $-(s-1)/(s-2)$, yields for each p, q a short exact sequence

$$0 \to (E^s(G) \otimes E^s(H))_{pq} \xrightarrow{\alpha} H\left((E^{s-1}(G) \otimes E^{s-1}(H))_{pq}\right)$$
$$\to \bigoplus_{\substack{m+u=p\\ n+v=q}} \mathrm{Tor}(E^s_{mn}(G), E^s_{u-s+1,v+s-2}(H)) \to 0,$$

which has the claimed naturality. We may use the isomorphism $H(\nabla^{s-1})$ to replace the middle term by $E^s_{p,q}(G \square H)$, obtaining the short exact sequence

$$0 \to (E^s(G) \otimes E^s(H))_{pq} \xrightarrow{H(\nabla^{s-1}) \circ \alpha} E^s_{pq}(G \square H)$$
$$\to \bigoplus_{\substack{m+u=p\\ n+v=q}} \mathrm{Tor}(E^s_{mn}(G), E^s_{u-s+1,v+s-2}(H)) \to 0.$$

Since $\nabla^s = H(\nabla^{s-1}) \circ \alpha$, this completes the proof. □

6. Excision and Mayer–Vietoris theorems

This section contains three key results. We will prove an excision theorem which holds for every page of the magnitude-path spectral sequence from E^1 onwards (Theorem 6.5). From the excision theorem we are able to derive a Mayer–Vietoris theorem for magnitude homology (Theorem 6.6) and for bigraded path homology (Theorem 6.8).

It is well known from the literature on both magnitude homology and path homology that Mayer–Vietoris theorems do not hold for arbitrary unions of graphs. Rather, one needs to assume that the union in question is 'nice' in some appropriate sense. Indeed, here we follow [9] in considering pushouts (rather than unions) along a class of subgraph inclusions termed *cofibrations*. (This terminology will be justified in Section 7.) Thus, we begin in Section 6.1 with a recollection on cofibrations. Then in Section 6.2 we state our main results, leaving the lengthy proof of Theorem 6.5 to Section 6.3.

6.1. **Cofibrations.** The maps we call 'cofibrations' are essentially the same as those defined in [9, Definition 2.8], except that we have reversed the directions of edges. This superficial modification and the reasons for it are explained in Remark 6.4. The definition runs as follows.

Definition 6.1. Let X be a directed graph, and $A \subseteq X$ a subgraph. The *reach* of A, denoted rA, is the induced subgraph of X on the set of all vertices that admit a path from some vertex in A.

Definition 6.2 ([9, Definition 2.8])**.** A *cofibration* of directed graphs is an induced subgraph inclusion $A \hookrightarrow X$ for which:

(1) There are no edges from vertices not in A to vertices in A.
(2) For each $x \in rA$ there is a vertex $\pi(x) \in A$ with the property that

$$d(a, x) = d(a, \pi(x)) + d(\pi(x), x) \qquad \text{for every } a \in A.$$

Before stating the theorems, we make a few remarks on the definition. First, note that condition (1) is equivalent to saying that there are no paths in X from vertices outside A to vertices inside A. This guarantees, in particular, that A is a convex subgraph of X. Condition (2) says that X *projects* to A in the sense of Leinster [36, Definition 4.6] and its precursor (also Leinster) [35, Definition 2.3.1]. As noted in [36], the vertex $\pi(x)$ is the vertex of A closest to x, and this determines $\pi(x)$ uniquely. In particular, taking $x = a$ in condition (2) we see that $\pi(a) = a$ for every $a \in V(A)$. Thus, we have a *projection function* $\pi\colon V(rA) \to V(A)$. In general, though, π does not determine a map of graphs.

We are concerned in this section with the behaviour of cofibrations, and the homology of pushouts along cofibrations. The category **DiGraph** is cocomplete, so in particular it has all pushouts; this is explained in [9] before Lemma 1.10 (and in this paper after Lemma A.1). Moreover, it is shown in [9, Proposition 2.13] that the class of cofibrations is closed under pushout. That is, given a cofibration $i\colon A \to X$ and an arbitrary map of directed graphs $f\colon A \to Y$, the map j in the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"'] & X \cup_A Y \end{tikzcd} \tag{15}$$

is also a cofibration.

Example 6.3 (A cone)**.** Let I denote the following directed graph.

$$\begin{array}{ccc} -1 & 0 & +1 \\ \bullet \longrightarrow & \bullet & \longleftarrow \bullet \end{array}$$

Given an arbitrary directed graph X, we define CX to be the directed graph obtained from $X\Box I$ by identifying the induced subgraph $X\Box\{+1\}$ to a single vertex that we denote simply $+1$. We think of CX as a form of 'cone' on X. We now identify X with the induced subgraph of CX on the vertices of form $(x,-1)$ for x a vertex of X, so that we obtain the induced subgraph inclusion

$$X \hookrightarrow CX.$$

This is a cofibration. To see this, we verify the two properties of Definition 6.2:

(1) Since there is no edge of I from 0 to -1, there are no edges of CX from $CX\setminus X$ to X.
(2) The only paths in I that begin at -1 are the trivial path and the edge $-1\to 0$, so that the reach rX of X is the induced subgraph on the vertices of the form $(x,0)$ and $(x,-1)$ for x a vertex of X. We then define π on such vertices by

$$\pi(x,-1)=\pi(x,0)=(x,-1).$$

The required property of π then states that, for vertices x,y of X, and for $j=-1,0$,

$$d((x,-1),(y,j)) = d((x,-1),\pi(y,j)) + d(\pi(y,j),(y,j))$$

or in other words

$$d((x,-1),(y,j)) = d((x,-1),(y,-1)) + d((y,-1),(y,j)),$$

and this is immediately verified; see Equation (8).

Note also that CX is 1-contractible: the identity map $CX\to CX$ is 1-homotopic to the constant map with value $+1$. Indeed, if we denote by $d\colon CX\to CX$ the map defined by $d(+1)=+1$ and $d(x,j)=(x,\max\{j,0\})$ for $x\in X$ and $j=0,-1$, then one can check that there are 1-homotopies from both the identity map, and the constant map, to d. This example will be used later in Corollary 6.10.

Remark 6.4. As the observations after Definition 6.2 suggest, the definition of cofibrations in [9] is an adaptation, to the directed setting, of the notion of *projecting decomposition* appearing in the literature on the magnitude and magnitude homology of undirected graphs and metric spaces [35, 36, 28]. Our Definition 6.2 is taken directly from Definition 2.8 of [9], except that we have reversed the directionality: $A\hookrightarrow X$ is a cofibration in the sense of this paper if and only if the corresponding map between the transpose graphs—in which the direction of every edge has been reversed—is a cofibration in the sense of [9].

This superficial change means that Definition 6.2 can also be seen as a strengthening, suited to this paper's filtered techniques, of the notion of *long cofibration* of directed graphs given in [27, Definition 6.2]: every cofibration in the sense of this paper is also a long cofibration. As is shown in [27, Proposition 6.7], a long cofibration $A\hookrightarrow X$ is precisely a map of directed graphs that induces a *Dwyer morphism* between the preorders generated by A and X. The class of Dwyer morphisms contains all cofibrations in the Thomason model structure on the category of small categories, which is Quillen equivalent to the classical model structure on the category of simplicial sets [46, 12]. Thus, the theory being developed here and in Section 7 bears a close relationship to more classical homotopy theoretic constructions. The details of that relationship remain to be explored.

6.2. **The excision and Mayer–Vietoris theorems.** We can now state our excision and Mayer–Vietoris theorems, which apply in the context of a pushout along a cofibration. Our Mayer–Vietoris theorems, which hold for magnitude homology and bigraded path homology, are fairly direct consequences of the excision theorem and so we prove them within this subsection. The excision theorem, on the other hand, applies to every page of the MPSS, but its proof is intricate and so is deferred to the next subsection.

Recall from Definition 3.2 that, given any subgraph $A \subseteq X$, one can consider the relative magnitude-path spectral sequence $\{E^r_{*,*}(X,A), d^r\}_{r\geq 0}$. Given a pushout of the form in (15), our excision theorem says that the relative magnitude-path spectral sequence of the pair (X,A) coincides with that of the pair $(X \cup_A Y, Y)$.

Theorem 6.5 (Excision in the magnitude-path spectral sequence)**.** *Let $i\colon A \to X$ be a cofibration and let $f\colon A \to Y$ be an arbitrary map of directed graphs, so that we obtain the pushout diagram of the form in* (15)*. Then for all $r \geq 1$ the induced map*

$$E^r_{*,*}(X,A) \xrightarrow{\cong} E^r_{*,*}(X \cup_A Y, Y)$$

is an isomorphism. In particular, on magnitude homology and bigraded path homology we have

$$\mathrm{MH}_{*,*}(X,A) \xrightarrow{\cong} \mathrm{MH}_{*,*}(X \cup_A Y, Y)$$

and

$$\mathrm{PH}_{*,*}(X,A) \xrightarrow{\cong} \mathrm{PH}_{*,*}(X \cup_A Y, Y).$$

In the classical topological setting, the excision theorem for spaces gives rise to the Mayer–Vietoris sequence by 'stitching together', using a standard argument of homological algebra, the long exact sequences of pairs of the form (X,A) and $(X \cup_A Y, Y)$. The same reasoning can be applied here to either the short exact sequences of magnitude homology groups, or the long exact sequences of bigraded path homology groups, that are associated to a pair by Theorem 3.3. That is how we obtain the next two theorems, which give a short exact Mayer–Vietoris sequence for magnitude homology, and a long exact Mayer–Vietoris sequence for path homology.

However, we are not at this time able to give a Mayer–Vietoris result for the E^r-pages for any $r \geq 3$. Indeed, the snake lemma allows us to passed from short exact sequences of chain complexes to long exact sequences of homology groups, and this is how we pass from E^1 to E^2 in the proof of Theorem 3.3. But we do not know of any account of the structure obtained from *long* exact sequences of chain complexes by taking homology, and this prevents us from obtaining results of Mayer–Vietoris type on the later pages of the MPSS.

We now state the theorems formally, before proceeding to the proofs.

Theorem 6.6 (Mayer–Vietoris for magnitude homology of directed graphs)**.** *Suppose given a cofibration $i\colon A \to X$ and any map $f\colon A \to Y$, so that we have a pushout of the form in (15). Then we have the* Mayer–Vietoris sequence *in magnitude homology of directed graphs, in the form of a short exact sequence:*

$$0 \to \mathrm{MH}_{*,*}(A) \xrightarrow{(i_*,-f_*)} \mathrm{MH}_{*,*}(X) \oplus \mathrm{MH}_{*,*}(Y) \xrightarrow{g_* \oplus j_*} \mathrm{MH}_{*,*}(X \cup_A Y) \to 0 \tag{16}$$

This sequence is split.

Remark 6.7. This theorem can be compared with the Mayer–Vietoris theorem for magnitude homology of *undirected* graphs that appeared as Theorem 6.6 in Hepworth and Willerton [28]. The result for undirected graphs applies in the presence of a projecting decomposition; we refer the reader to [28, Definition 6.3] for the definition.

The category of undirected graphs embeds canonically into **DiGraph** by taking a graph G and replacing each of its edges $\{x, y\}$ by a pair of directed edges $x \rightleftarrows y$. However, given a projecting decomposition $G = X \cup_A Y$, it is almost never the case that the inclusion of A into X (or into Y) induces, under this operation, a cofibration in the sense of Definition 6.2. Indeed, that occurs only when X (respectively, Y) is the disjoint union of A and its complement, in which case the split exact sequence (16) holds trivially for $\mathrm{MH}_{*,*}(X \cup_A Y)$. On the other hand, there are many non-trivial examples of projecting decompositions of undirected graphs; see, for example, Corollaries 4.13 and 4.14 of Leinster [36]. Thus, our Mayer–Vietoris theorem and that in [28] are quite independent.

Theorem 6.8 (Mayer–Vietoris for bigraded path homology)**.** *Suppose given a cofibration $i\colon A \to X$ and any map $f\colon A \to Y$, so that we have a pushout of the form in (15). Then we have the* Mayer–Vietoris sequence *in bigraded path homology, meaning that there is a long exact sequence:*

$$\cdots \to \mathrm{PH}_{*,*}(A) \xrightarrow{(i_*,-f_*)} \mathrm{PH}_{*,*}(X) \oplus \mathrm{PH}_{*}(Y) \xrightarrow{g_* \oplus j_*} \mathrm{PH}_{*,*}(X \cup_A Y) \xrightarrow{\partial_*} \mathrm{PH}_{*-1,*}(A) \to \cdots$$

Theorem 6.5 is the hardest to prove of the three theorems in this section, and we therefore leave its proof to the end of the section. The proofs of the other two theorems are straightforward once we have excision, so we tackle them first.

Proof of Theorem 6.6, assuming Theorem 6.5. Extracting long exact sequences of Mayer–Vietoris type from excision theorems is common in the algebraic topology literature. Indeed, if we consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{MH}_{*,*}(A) \arrow[r, "i"] \arrow[d, "f"] & \mathrm{MH}_{*,*}(X) \arrow[r] \arrow[d, "g"] & \mathrm{MH}_{*,*}(X, A) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \mathrm{MH}_{*,*}(Y) \arrow[r, "j"'] & \mathrm{MH}_{*,*}(X \cup_A Y) \arrow[r] & \mathrm{MH}_{*,*}(X \cup_A Y, Y) \arrow[r] & \cdots
\end{tikzcd}$$

whose rows are obtained from Theorem 3.3, and identify every pair of third terms using the excision isomorphism, then we may apply the result of exercise 38 of [25, p.159] to obtain the following long exact sequence:

$$\cdots \longrightarrow \mathrm{MH}_{*,*}(A) \xrightarrow{(i_*,-f_*)} \mathrm{MH}_{*,*}(X) \oplus \mathrm{MH}_{*,*}(Y) \xrightarrow{g_* \oplus j_*} \mathrm{MH}_{*,*}(X \cup_A Y) \longrightarrow \cdots$$

Since the inclusion $A \subseteq X$ satisfies the condition that there are no edges from $X \setminus A$ to A, Theorem 3.3 shows that i_* is split, and it follows that $(i_*, -f_*)$ is also split, so that we obtain the short exact sequence of the claim. □

Proof of Theorem 6.8, assuming Theorem 6.6. This is similar to the proof of Theorem 6.6, except that one uses the long exact sequences of bigraded path homology groups obtained from Theorem 3.3. We leave the details to the reader. □

Let us give here an immediate application of our excision theorem. A further application will be seen in the more substantial computation in Section 9.

Definition 6.9. Let X be a directed graph. We define SX to be the graph obtained from X by adding two new vertices, $+1$ and -1, together with an edge from each of the new vertices to each vertex of X. Thus SX is a form of 'unreduced suspension' of X; see Example 6.11 for an illustrative example.

The following result can be compared with Proposition 5.10 of Grigor'yan *et al* [21].

Corollary 6.10 (A suspension theorem)**.** *Let X be a nonempty directed graph, and $k, \ell \geq 0$. Then there is a natural isomorphism*

$$\ker\bigl(\mathrm{PH}_{k,\ell}(SX) \to \mathrm{PH}_{k,\ell}(\bullet)\bigr) \cong \ker\bigl(\mathrm{PH}_{k-1,\ell-1}(X) \to \mathrm{PH}_{k-1,\ell-1}(\bullet)\bigr).$$

In particular, $\mathrm{PH}_{k,\ell}(SX)$ *and* $\mathrm{PH}_{k-1,\ell}(X)$ *are naturally isomorphic unless $k = \ell = 0$ or $k = \ell = 1$, in which case they differ only by a single summand of R.*

Note that the kernels appearing in the statement could be called *reduced* bigraded path homology groups by analogy with algebraic topology. Since reduced groups do not appear elsewhere in the paper, we restrain ourselves from defining these in general.

Proof. Recall from Example 6.3 the graph I given by $-1 \to 0 \leftarrow +1$, and the graph CX obtained from $X \Box I$ by identifying $X \Box \{+1\}$ to a single vertex $+1$. We showed that the inclusion $X \hookrightarrow CX$ that identifies X with the subgraph $X \Box \{-1\}$ is a cofibration. Now define a map $CX \to SX$ sending $X \Box \{0\}$ to X, $X \Box \{+1\}$ to $+1$, and $X \Box \{-1\}$ to -1. Then we have a pushout square

$$\begin{array}{ccc} X & \longrightarrow & \bullet \\ \big\downarrow & & \big\downarrow \\ CX & \longrightarrow & SX \end{array}$$

where the right-hand map includes the one-vertex graph as the subgraph with single vertex -1. Excision gives an isomorphism

$$\mathrm{PH}_{*,*}(SX, \bullet) \cong \mathrm{PH}_{*,*}(CX, X).$$

Next, the map $\mathrm{PH}_{k,\ell}(SX) \to \mathrm{PH}_{k,\ell}(SX, \bullet)$ induces an isomorphism

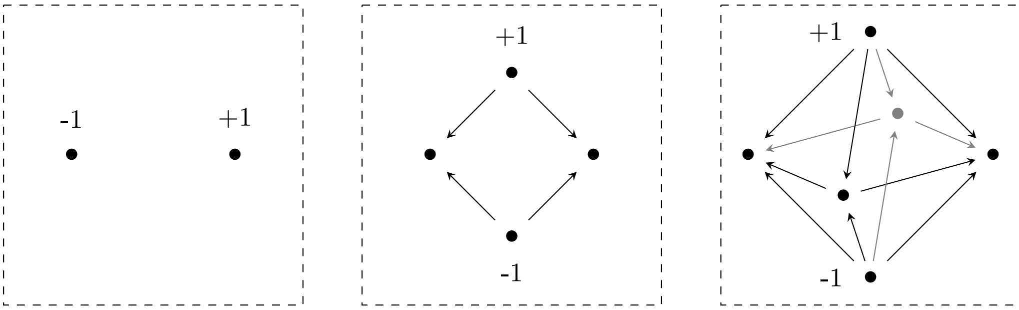
$$\ker\bigl(\mathrm{PH}_{k,\ell}(SX) \to \mathrm{PH}_{k,\ell}(\bullet)\bigr) \cong \mathrm{PH}_{k,\ell}(SX, \bullet).$$

This is seen by using the map of pairs $(SX, \bullet) \to (\bullet, \bullet)$ to compare the associated long exact sequences. Finally, the connecting morphism $\mathrm{PH}_{k,\ell}(CX, X) \to \mathrm{PH}_{k-1,\ell-1}(X)$ induces an isomorphism

$$\mathrm{PH}_{k,\ell}(CX, X) \cong \ker\bigl(\mathrm{PH}_{k-1,\ell-1}(X) \to \mathrm{PH}_{k-1,\ell}(\bullet)\bigr),$$

as one sees using the fact that $CX \to \bullet$ induces an isomorphism in path homology thanks to 1-contractibility of CX. The three isomorphisms obtained in this paragraph are natural in X and, combined, they complete the proof. $\square$

Example 6.11 (A family of spheres)**.** Denote the empty graph by $\emptyset$, and for each $n \geq 0$ let $\mathbb{S}^n$ denote the $(n+1)$-fold suspension $S^{n+1}\emptyset$. The directed graph $\mathbb{S}^n$ is then analogous to an 'n-sphere'. (Indeed, it is the face poset of the regular CW decomposition of the topological n-sphere by hemispheres.) Here we depict, from left to right, $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^2$, with the new vertices ± 1 labelled in each case:



The bigraded path homology of $\mathbb{S}^0$ is concentrated in bidegree $(0,0)$, where it is given by $R \oplus R$, so that $\ker\big(\mathrm{PH}_{*,*}(\mathbb{S}^0) \to \mathrm{PH}_{*,*}(\bullet)\big)$ is a single copy of R concentrated in bidegree $(0,0)$. Applying Corollary 6.10 and inducting on n, one finds that, for $n \geq 0$, $\ker\big(\mathrm{PH}_{*,*}(\mathbb{S}^n) \to \mathrm{PH}_{*,*}(\bullet)\big)$ is a single copy of R concentrated in bidegree (n,n). It then follows, for $n > 0$, that $\mathrm{PH}_{k,\ell}(\mathbb{S}^n) = 0$ when $k \neq \ell$, while

$$\mathrm{PH}_{k,k}(\mathbb{S}^n) \cong \begin{cases} R & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

That is, for every $n \geq 0$, the bigraded path homology of $\mathbb{S}^n$ is concentrated on the diagonal, where it consists of a copy of the singular homology of the topological n-sphere.

6.3. **Proof of the excision theorem.** We now embark on the proof of the excision theorem, Theorem 6.5. Our proof owes a significant debt to the proof in Section 9 of [28], and indeed Lemma 6.14 is closely related to Lemma 9.2 of [28]. However, the overall structure of the proof here is essentially different from that one, in order to cope with the fact that the map $f\colon A \to Y$ appearing in our pushout square (15) is not necessarily the inclusion of an induced subgraph.

The proof of Theorem 6.5 is rather intricate, so we offer a sketch here. The objective is to prove that

$$E^r_{*,*}(X, A) \to E^r_{*,*}(X \cup_A Y, Y)$$

is an isomorphism for all $r \geq 1$. Since an isomorphism of chain complexes induces an isomorphism on homology, it is sufficient to prove that this is an isomorphism when $r = 1$, and this, by the definition of magnitude homology of a pair, is equivalent to showing that the map

$$\frac{\mathrm{MC}_{*,*}(X)}{\mathrm{MC}_{*,*}(A)} \longrightarrow \frac{\mathrm{MC}_{*,*}(X \cup_A Y)}{\mathrm{MC}_{*,*}(Y)}$$

is a quasi-isomorphism. The map $X \to X \cup_A Y$ induces a map $X \setminus A \to (X \cup_A Y) \setminus Y$ that is in fact an isomorphism of directed graphs. By the five-lemma it is therefore sufficient to show that the map

$$\frac{\mathrm{MC}_{*,*}(X)}{\mathrm{MC}_{*,*}(A) + \mathrm{MC}_{*,*}(X \setminus A)} \longrightarrow \frac{\mathrm{MC}_{*,*}(X \cup_A Y)}{\mathrm{MC}_{*,*}(Y) + \mathrm{MC}_{*,*}((X \cup_A Y) \setminus Y)}$$

is a quasi-isomorphism. In order to prove this, we equip the domain and codomain with compatible filtrations

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\ell-1} = \frac{\mathrm{MC}_{*,*}(X)}{\mathrm{MC}_{*,*}(A) + \mathrm{MC}_{*,*}(X \setminus A)}$$

and

$$0 = F'_0 \subseteq F'_1 \subseteq \cdots \subseteq F'_{\ell-1} = \frac{\mathrm{MC}_{*,*}(X \cup_A Y)}{\mathrm{MC}_{*,*}(Y) + \mathrm{MC}_{*,*}((X \cup_A Y) \setminus Y)},$$

so that we are reduced to proving that the induced maps of filtration quotients

$$F_i/F_{i-1} \longrightarrow F'_i/F'_{i-1} \tag{17}$$

are all quasi-isomorphisms. It turns out that each of these filtration quotients has a decomposition as a direct sum of suspensions of more elementary complexes that we denote

$$A_{*,\ell}(-,-) \qquad \text{and} \qquad A'_{*,\ell}(-,-)$$

respectively. These complexes all have very simple homology that can be computed directly. With that computation in hand, it is then possible to verify directly that (17) is an isomorphism on homology, which completes the proof.

We now embark on the proof in detail.

Definition 6.12. Let $A \hookrightarrow X$ be a cofibration, let $\ell \geq 0$, let $x \in X \setminus A$ and let $a \in A$. Define $A_{*,\ell}(a,x)$ to be the subcomplex of $\mathrm{MC}_{*,\ell}(X)$ spanned by those tuples $(x_0, \ldots, x_k)$ for which $x_0 = a$, $x_k = x$ and $x_1, \ldots, x_{k-1} \in A$.

Lemma 6.13. *Suppose we are in the situation of Definition 6.12, and that $a = \pi(x)$ and $\ell = d(a,x)$. Then the homology of $A_{*,\ell}(a,x)$ is a single copy of R in degree 1, generated by the homology class of the tuple $(\pi(x), x)$.*

Proof. Any generator of $A_{k,\ell}(\pi(x),x)$ has form $(\pi(x), x_1, \ldots, x_{k-1}, x)$. The length of such a tuple satisfies

$$\begin{aligned} d(\pi(x),x) &= \ell(\pi(x), x_1, \ldots, x_{k-1}, x) \\ &\geq d(\pi(x), x_i) + d(x_i, x) \\ &= d(\pi(x), x_i) + d(x_i, \pi(x)) + d(\pi(x), x) \end{aligned}$$

where the second line follows from the triangle inequality, and the third follows from the second property of a cofibration, using the fact that $x_i \in A$. It follows that $d(\pi(x), x_i) = d(x_i, \pi(x)) = 0$, so that $x_i = \pi(x)$. This can only happen if $k = 1$, and the result follows. $\square$

Lemma 6.14. *Suppose we are in the situation of Definition 6.12, and that at least one of $a = \pi(x)$ and $\ell = d(a,x)$ fails. Then $A_{*,\ell}(a,x)$ is acyclic.*

Proof. Define a map

$$s\colon A_{*,\ell}(a,x) \longrightarrow A_{*+1,l}(a,x)$$

by

$$s(x_0,\ldots,x_k) = \begin{cases} (-1)^k(x_0,\ldots,x_{k-1},\pi(x_k),x_k) & \text{if } x_{k-1}\neq\pi(x_k),\\ 0 & \text{if } x_{k-1}=\pi(x_k).\end{cases}$$

We claim that s is a chain homotopy from the identity of $A_{*,\ell}(a,x)$ to the zero map, or in other words that

$$\partial\circ s + s\circ\partial = \mathrm{Id}. \tag{18}$$

This immediately shows that the homology of $A_{*,\ell}(a,x)$ is trivial, and the result follows.

Equation (18) is equivalent to the claim that

$$\sum_{i=1}^{k}(-1)^i\partial_i s(x_0,\ldots,x_k) + \sum_{i=1}^{k-1}(-1)^i s\partial_i(x_0,\ldots,x_k) = (x_0,\ldots,x_k) \tag{19}$$

for each generator $(x_0,\ldots,x_k)$ of $A_{*,\ell}(a,x)$.

To begin we consider the case $k=1$. In this case the only possible generator is (a,x), but this is only present when $\ell = d(a,x)$. So by our assumption that we do not have both $a=\pi(x)$ and $\ell=d(a,x)$, we must have $a\neq\pi(x)$. Then (19) becomes $-\partial_1 s(a,x)=(a,x)$, which follows from the definition of s.

For the remainder of the proof we consider the case $k\geq 2$. We have $\partial_i s = -s\partial_i$ for $1\leq i\leq k-2$, since then ∂_i affects only the first $k-1$ terms of a tuple, while s affects, and depends upon, only the remaining terms. So (19) reduces to the claim that:

$$\begin{aligned} &(-1)^{k-1}\ \partial_{k-1}s(x_0,\ldots,x_k)\\ +&(-1)^{k}\quad\ \partial_k\ \ s(x_0,\ldots,x_k)\\ +&(-1)^{k-1}\ s\partial_{k-1}(x_0,\ldots,x_k) = (x_0,\ldots,x_k)\end{aligned} \tag{20}$$

We now divide into the following cases:

- Assume that $x_{k-1}=\pi(x_k)$.

  Here we see immediately that the first two terms of (20) vanish by the definition of s. For the third term, we have

  $$\begin{aligned}(-1)^{k-1}s\partial_{k-1}(x_0,\ldots,x_k) &= (-1)^{k-1}s(x_0,\ldots,x_{k-2},x_k)\\ &= (x_0,\ldots,x_{k-2},\pi(x_k),x_k)\\ &= (x_0,\ldots,x_k)\end{aligned}$$

  as required. The first of these equations holds because $d(x_{k-2},x_{k-1})+d(x_{k-1},x_k) = d(x_{k-2},x_k)$ by the defining property of $\pi(x_k)$, and the second holds since $x_{k-2}\neq x_{k-1}=\pi(x_k)$.
- Assume that $x_{k-1}\neq\pi(x_k)$ and $d(x_{k-2},x_{k-1})+d(x_{k-1},x_k) > d(x_{k-2},x_k)$.

  Applying the defining property of $\pi(x_k)$ to the assumed inequality, we find that we also have $d(x_{k-2},x_{k-1})+d(x_{k-1},\pi(x_k)) > d(x_{k-2},\pi(x_k))$ so that we have $\partial_{k-1}(x_0,\ldots,x_{k-1},\pi(x_k),x_k)=0$ and the first term of (20) vanishes. The assumed inequality also shows that $\partial_{k-1}(x_0,\ldots,x_k)=0$ so that the third term of (20) vanishes. It remains to show that the second term of (20) is $(x_0,\dots,x_k)$. Thanks to the assumption $x_{k-1}\neq\pi(x_k)$ we have

$$(-1)^k\partial_k s(x_0,\dots,x_k)=\partial_k(x_0,\dots,x_{k-1},\pi(x_k),x_k)=(x_0,\dots,x_{k-1},x_k)$$

as required.

- Assume that $x_{k-1}\neq\pi(x_k)$ and $d(x_{k-2},x_{k-1})+d(x_{k-1},x_k)=d(x_{k-2},x_k)$.

  By applying the defining property of $\pi(x_k)$ to the assumed equation, we get $d(x_{k-2},x_{k-1})+d(x_{k-1},\pi(x_k))=d(x_{k-2},\pi(x_k))$, and in particular $x_{k-2}\neq\pi(x_k)$. The first term of (20) is given by:

$$\begin{aligned}(-1)^{k-1}\partial_{k-1}s(x_0,\dots,x_k)&=-\partial_{k-1}(x_0,\dots,x_{k-1},\pi(x_k),x_k)\\&=-(x_0,\dots,x_{k-2},\pi(x_k),x_k),\end{aligned}$$

  where the first equation used the assumption $x_{k-1}\neq\pi(x_k)$, and the second equation used the fact that $d(x_{k-2},x_{k-1})+d(x_{k-1},\pi(x_k))=d(x_{k-2},\pi(x_k))$ and $x_{k-2}\neq\pi(x_k)$. The second term of (20) is

$$\begin{aligned}(-1)^k\partial_k s(x_0,\dots,x_k)&=\partial_k(x_0,\dots,x_{k-1},\pi(x_k),x_k)\\&=(x_0,\dots,x_{k-1},x_k),\end{aligned}$$

  where again in the first equation we used the assumption $x_{k-1}\neq\pi(x_k)$, and in the second equation we used the defining property of $\pi(x_k)$ to see that $d(x_{k-1},\pi(x_k))+d(\pi(x_k),x_k)=d(x_{k-1},x_k)$. The third term of (20) is

$$\begin{aligned}(-1)^{k-1}s\partial_{k-1}(x_0,\dots,x_k)&=(-1)^{k-1}s(x_0,\dots,x_{k-2},x_k)\\&=(x_0,\dots,x_{k-2},\pi(x_k),x_k)\end{aligned}$$

  where in the first equation we used the assumption that $d(x_{k-2},x_{k-1})+d(x_{k-1},x_k)=d(x_{k-2},x_k)$, and in the second we used the fact $x_{k-2}\neq\pi(x_k)$. So altogether, the left hand side of (20) is

$$-(x_0,\dots,x_{k-2},\pi(x_k),x_k)+(x_0,\dots,x_{k-1},x_k),+(x_0,\dots,x_{k-2},\pi(x_k),x_k)$$

  which is precisely $(x_0,\dots,x_k)$ as required.

In all three cases above, the equation (20) holds. This completes the proof. □

Now we take a cofibration $A\hookrightarrow X$ and study the chain complex

$$\frac{\mathrm{MC}_{*,*}(X)}{\mathrm{MC}_{*,*}(A)+\mathrm{MC}_{*,*}(X\setminus A)}$$

This quotient has a basis consisting of the tuples $(x_0,\dots,x_k)$ of vertices of X that have finite length and that do not lie entirely in A or entirely in $X\setminus A$. Since $A\hookrightarrow X$ is a cofibration, there are no paths from vertices of $X\setminus A$ into A, and so any such tuple satisfies $x_0,\dots,x_i\in A$ and $x_{i+1},\dots,x_k\in X\setminus A$ for some i in the range $0\le i<k$.

Definition 6.15. Let $A\hookrightarrow X$ be a cofibration and let $\ell\geq 0$. Define a filtration

$$0=F_0\subseteq F_1\subseteq\cdots\subseteq F_{\ell-1}=\frac{\mathrm{MC}_{*,\ell}(X)}{\mathrm{MC}_{*,\ell}(A)+\mathrm{MC}_{*,\ell}(X\setminus A)}$$

by setting F_i to be the span of all those tuples $(x_0,\ldots,x_k)$ for which $x_0,\ldots,x_{k-i}\in A$. In other words, the final i entries of a tuple in F_i are allowed to be outside A, but no more.

To see that the F_i are closed under the boundary map, observe that deleting an entry of a tuple either preserves or reduces the number of its entries that lie in $X\setminus A$. And to see that $F_{\ell-1}$ does indeed exhaust our chain complex, observe that an arbitrary tuple $(x_0,\ldots,x_k)$ has at least its final term in $X\setminus A$ and therefore lies in F_{k-1}, but that we always have $k\leq\ell$.

Lemma 6.16. *There is an isomorphism*

$$F_i/F_{i-1}\xrightarrow{\cong}\bigoplus_{a,(z_1,\ldots,z_i)}\Sigma^i A_{*,\ell-\ell'}(a,z_1)$$

where the direct sum is over all $a\in A$ and all tuples $(z_1,\ldots,z_i)$ of elements of $X\setminus A$ with $\ell(z_1,\ldots,z_i)\leq\ell$, and ℓ' is shorthand for $\ell(z_1,\ldots,z_i)$.

Proof. Since F_i and F_{i-1} are spans of generators, with those for the latter being a subset of those for the former, the quotient F_i/F_{i-1} has basis consisting of all those generators that lie in F_i but not F_{i-1}. These are precisely the tuples for which $x_0,\ldots,x_{k-i}$ lie in A (so that the tuple is in F_i), while x_{k-i+1} and all subsequent entries of the tuple do not (so that the tuple does not lie in F_{i-1}). So we may write an arbitrary generator of F_i/F_{i-1} in the form $(x_0,\ldots,x_{k-i},z_1,\ldots,z_i)$ where $x_0,\ldots,x_{k-i}\in A$ and $z_1,\ldots,z_i\in X\setminus A$.

The boundary map on F_i/F_{i-1} is, as usual, given by the alternating sum of all ways of removing an element from a tuple, ignoring any terms for which the length is decreased. If, in a tuple $(x_0,\ldots,x_{k-i},z_1,\ldots,z_i)$, it is one of the x_j that is removed, then we find that we still have the first $k-i-1=(k-1)-i$ terms in A. On the other hand, if one of the z_j is removed, then in this new tuple the first $k-i=(k-1)-(i-1)$ terms lie in A, so that the tuple lies in F_{i-1} and therefore vanishes in F_i/F_{i-1}. From this, we see that the differential of F_i/F_{i-1} is given by the alternating sum of all ways to delete one of the x_j from a tuple $(x_0,\ldots,x_{k-i},z_1,\ldots,z_i)$, omitting any terms where the length is decreased.

The last two paragraphs now enable us to form the required isomorphism

$$F_i/F_{i-1}\xrightarrow{\cong}\bigoplus_{a,(y_1,\ldots,y_i)}\Sigma^i A_{*,\ell-\ell'}(a,y_1)$$

by sending $(x_0,\ldots,x_{k-i},z_1,\ldots,z_i)\in F_i/F_{i-1}$ to the element $(x_0,\ldots,x_{k-i},z_1)$ in the summand indexed by x_0 and $(z_1,\ldots,z_i)$. The first paragraph shows that this is a bijection on generators, and therefore an isomorphism of graded R-modules, while the second paragraph shows that it respects the differentials on both sides. $\square$

Suppose now that we are given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"'] & X\cup_A Y \end{tikzcd}$$

in which i, and consequently j, are cofibrations. Since the square commutes, g sends $A \subseteq X$ into $Y \subseteq X \cup_A Y$. And since (by construction—see Lemma 1.10 of [9]) g identifies $X \setminus A$ with $(X \cup_A Y) \setminus Y \subseteq X \cup_A Y$, we have an induced map

$$\frac{\mathrm{MC}_{*,*}(X)}{\mathrm{MC}_{*,*}(A) + \mathrm{MC}_{*,*}(X \setminus A)} \longrightarrow \frac{\mathrm{MC}_{*,*}(X \cup_A Y)}{\mathrm{MC}_{*,*}(Y) + \mathrm{MC}_{*,*}(X \cup_A Y \setminus Y)}. \tag{21}$$

Lemma 6.17. *The map* (21) *is a quasi-isomorphism.*

Proof. Let us fix some $\ell \geq 0$ and restrict to length ℓ—that is, fix the second grading to be ℓ.

The domain and codomain of (21) both admit the filtration of Definition 6.15. For clarity, we denote the filtration of the domain by $\{F_i\}$ and that of the codomain by $\{F'_i\}$. Similarly, both $A \hookrightarrow X$ and $Y \hookrightarrow X \cup_A Y$ admit the complexes $A_{*,\ell}(-,-)$ of Definition 6.12. Again for clarity, we will write the the complexes associated with $Y \hookrightarrow X \cup_A Y$ as $A'_{*,\ell}(-,-)$.

By construction, the induced map (21) preserves the filtration, and thus induces maps on filtration quotients

$$F_i/F_{i-1} \longrightarrow F'_i/F'_{i-1}. \tag{22}$$

Since both filtrations terminate after $\ell - 1$ steps, to prove the lemma it will be sufficient to prove that each map (22) is a quasi-isomorphism. The domain and codomain of (22) both admit the isomorphism of Lemma 6.16, so that (22) becomes a map

$$\bigoplus_{a,(z_1,\ldots,z_i)} \Sigma^i A_{*,\ell-\ell'}(a, z_1) \longrightarrow \bigoplus_{y,(z_1,\ldots,z_i)} \Sigma^i A'_{*,\ell-\ell'}(y, z_1). \tag{23}$$

It is straightforward to see—since the map $g\colon X \to X \cup_A Y$ identifies $X \setminus A$ with $X \cup_A Y \setminus Y$—that this map now sends the summand corresponding to a and $(z_1, \ldots, z_i)$ to precisely the summand corresponding to $f(a)$ and $(z_1, \ldots, z_i)$, and that on this summand it is given by the evident induced map $A_{*,\ell}(a, z_1) \to A'_{*,\ell}(f(a), z_1)$. Lemma 6.14 shows that the only summands that do not have vanishing homology are those of the form $A_{*,d(\pi(x),x)}(\pi(x), x)$ and $A'_{*,d(f(\pi(x),x)}(f(\pi(x)), x)$, and Lemma 6.13 shows that in these cases the induced map $A_{*,d(\pi(x),x)}(\pi(x), x) \to A'_{*,d(f(\pi(x)),x}(\pi(x), x)$ is a quasi-isomorphism. This is sufficient to complete the proof. □

Proof of Theorem 6.5. We wish to prove that the maps $E^r_{*,*}(X, A) \to E^r_{*,*}(X \cup_A Y, Y)$ are isomorphisms for $r \geq 1$. These maps are induced by a map of filtered complexes, and so it is enough to show that there is a quasi-isomorphism between the E^0-pages, or in other words that the map $\mathrm{MC}_{*,*}(X, A) \to \mathrm{MC}_{*,*}(X \cup_A Y, Y)$ is a quasi-isomorphism. The domain and codomain contain within them a copy of $\mathrm{MC}_{*,*}(X \setminus A)$ and $\mathrm{MC}_{*,*}(X \cup_A Y \setminus Y)$ respectively, and these subcomplexes are identified by the map in question. It is therefore sufficient to prove that the map of quotients

$$\frac{\mathrm{MC}_{*,*}(X, A)}{\mathrm{MC}_{*,*}(X \setminus A)} \longrightarrow \frac{\mathrm{MC}_{*,*}(X \cup_A Y, Y)}{\mathrm{MC}_{*,*}(X \cup_A Y \setminus Y)}$$

is a quasi-isomorphism. But this map is precisely (21), which is a quasi-isomorphism by Lemma 6.17. This completes the proof. □

7. A cofibration category of directed graphs

In the drive to develop the formal homotopy theory of directed graphs, results so far have mainly been negative: for various natural notions of weak equivalence, it is known that no model structure can exist [20, 19, 10]. There has, however, been one positive result. Carranza *et al* [9] exhibit a *cofibration category* structure on **DiGraph** for which the weak equivalences are maps inducing isomorphisms on path homology. A cofibration category structure is, roughly speaking, 'one half' of a model category structure: it comprises a class of weak equivalences and a class of cofibrations, satisfying axioms that enable the construction of homotopy colimits (but not homotopy limits). This type of structure was first introduced in its dual form (then known as a *category of fibrant objects*) by Brown in 1973 [8]; for a classical account, see Baues' book [6]. Various definitions of cofibration categories can now be found in the literature—we adopt the one used in [9].

Definition 7.1 (Definition 1.33 in [9])**.** A *cofibration category* is a category $\mathcal{C}$ together with two distinguished classes of morphisms, the class of *weak equivalences* and the class of *cofibrations*, which satisfy axioms (C1)–(C6) below. An *acyclic cofibration* is a morphism that is both a cofibration and a weak equivalence.

- (C1) The class of cofibrations and the class of weak equivalences are each closed under composition, and for every object X in $\mathcal{C}$, the identity morphism Id_X is an acyclic cofibration.
- (C2) The class of weak equivalences satisfies the *2-out-of-6 property*: given a triple of composable morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, if $g \circ f$ and $h \circ g$ are weak equivalences, then so are f, g, h and $h \circ g \circ f$.
- (C3) The category $\mathcal{C}$ admits an initial object $\emptyset$, and every object X in $\mathcal{C}$ is *cofibrant*, meaning that the unique morphism from the initial object to X is a cofibration.
- (C4) The category $\mathcal{C}$ admits pushouts along cofibrations, and the pushout of an (acyclic) cofibration is an (acyclic) cofibration.
- (C5) For every object X in $\mathcal{C}$, the codiagonal map $X \sqcup X \to X$ can be factored as a cofibration followed by a weak equivalence.
- (C6) The category $\mathcal{C}$ admits all small coproducts.
- (C7) The transfinite composite of (acyclic) cofibrations is again an (acyclic) cofibration.

Axioms (C1)–(C6) imply that the cofibrations and weak equivalences in a cofibration category $\mathcal{C}$ satisfy various properties one would expect to hold in a model category. In particular, every morphism in $\mathcal{C}$ can be factored as a cofibration followed by a weak equivalence [8, p. 421], and the pushout of a weak equivalence along a cofibration is a weak equivalence [8, Lemma I.4.2]. For further discussion of the definition, we refer the reader to Section 1 of [9].

Theorem 4.1 in [9] says that **DiGraph** carries a cofibration category structure in which the cofibrations are those of Definition 6.2 and the weak equivalences are maps inducing isomorphisms on path homology. Equipped with the results of the previous sections, we will prove that that structure has a refinement, in which the weak equivalences are maps inducing isomorphisms on *bigraded* path homology.

Theorem 7.2 (A cofibration category for bigraded path homology)**.** *Fix a ground ring R which is a P.I.D. The category* **DiGraph** *admits a cofibration category structure in which the cofibrations are those in Definition 6.2 and the weak equivalences are morphisms inducing isomorphisms on bigraded path homology.*

Before proving the theorem, we record a remark, an example, and two auxiliary statements that will be required for the proof.

Remark 7.3. The maps that induce isomorphisms on bigraded path homology are strictly finer than those that induce isomorphisms on ordinary path homology, as the examples of Section 8 demonstrate. Consider the directed cycles Z_m, for $m \geq 3$ (see Figure 1). Theorem 8.2 shows that for each $i \geq 0$ there is exactly one $j \geq 0$ for which $\mathrm{PH}_{i,j}(Z_m)$ is nonzero, and the first three cases are $\mathrm{PH}_{0,0}(Z_m)$, $\mathrm{PH}_{1,1}(Z_m)$ and $\mathrm{PH}_{2,m}(Z_m)$. The first two of these are precisely the (ordinary) path homology groups $\mathrm{PH}_0(Z_m)$ and $\mathrm{PH}_1(Z_m)$, and are the only nonzero path homology groups. On the other hand, the third, $\mathrm{PH}_{2,m}(Z_m)$, is not an ordinary path homology group at all. Now take $n > m \geq 3$ and consider any map

$$Z_n \longrightarrow Z_m$$

that contracts precisely $n - m$ edges. This does *not* induce an isomorphism of bigraded path homology groups, because $\mathrm{PH}_{2,m}(Z_n)$ vanishes while $\mathrm{PH}_{2,m}(Z_m)$ does not. On the other hand, it *does* induce an isomorphism on ordinary path homology groups. (The latter can be seen by direct computation. Tracking through the proof of Theorem 8.12 shows that $\mathrm{PH}_{0,0}(Z_m)$ is a single copy of R represented by the reachability chain (v) for any vertex $v \in V(Z_m)$, while $\mathrm{PH}_{1,1}(Z_m)$ is again a single copy of R represented by the reachability chain $\sum_{(a,b)\in E(Z_m)}(a,b)$, and similarly for Z_n. These representatives are preserved by the map $Z_n \to Z_m$.)

The proof of Theorem 7.2 follows the structure of the proof of Theorem 4.1 in [9] closely, and makes use of several facts established in that paper. It also depends on the Künneth theorem and the excision theorem for bigraded path homology proved in Sections 5 and 6 of this paper, and on the fact that bigraded path homology is a *finitary* functor on the category of directed graphs: it preserves filtered colimits.

Proposition 7.4. *Fix a commutative ground ring R. For each $r \geq 1$ and every $p, q \in \mathbb{Z}$, the functor $E^r_{pq}(-)$:* **DiGraph** $\to$ $\mathbf{Mod}_R$ *preserves filtered colimits. In particular this holds for magnitude homology and bigraded path homology.*

Proposition 7.4 is proved as Proposition 1.14 in [15]; for the interested reader, we give an alternative and more detailed proof in Appendix A. Before proceeding to the proof of Theorem 7.2, let us consider an application of the Proposition.

Example 7.5 (An infinite sphere)**.** Recall from Example 6.11 the 'sphere'-like directed graphs $\mathbb{S}^n$. By construction, each $\mathbb{S}^n$ is the suspension of $\mathbb{S}^{n-1}$; in particular, $\mathbb{S}^{n-1}$ includes into $\mathbb{S}^n$, for each $n > 0$. This sequence of inclusions gives a filtered diagram

$$\mathbb{S}^0 \to \mathbb{S}^1 \to \mathbb{S}^2 \to \cdots \tag{24}$$

in **DiGraph**, whose colimit we denote by

$$\mathbb{S}^\infty = \mathrm{colim}_{\mathbb{N}} \mathbb{S}^n.$$

We can apply Proposition 7.4 to see that the bigraded path homology of $\mathbb{S}^\infty$ is

$$\mathrm{PH}_{k,\ell}(\mathbb{S}^\infty) = \begin{cases} R & \text{if } k = \ell = 0 \\ 0 & \text{otherwise,} \end{cases}$$

analogous to the vanishing in positive dimensions of the singular homology of the infinite topological sphere.

Indeed, we saw in Example 6.11 that the bigraded path homology of $\mathbb{S}^n$ consists of two copies of R, one in bidegree $(0,0)$ and a second in bidegree (n,n). Thus $\mathrm{PH}_{0,0}(\mathbb{S}^n)$ is a copy of R for all $n \geq 1$, and the maps $\mathbb{S}^{n-1} \to \mathbb{S}^n$ induce the identity on this, so that we obtain $\mathrm{PH}_{0,0}(\mathrm{colim}_{\mathbb{N}}\mathbb{S}^n) = \mathrm{colim}_{\mathbb{N}}\mathrm{PH}_{0,0}(\mathbb{S}^n) = R$. And in any other bidegree (i,j) there is at most one value of n for which $\mathrm{PH}_{i,j}(\mathbb{S}^n)$ is nonzero, so that the induced maps $\mathrm{PH}_{i,j}(\mathbb{S}^{n-1}) \to \mathrm{PH}_{i,j}(\mathbb{S}^n)$ all vanish, and $\mathrm{PH}_{i,j}(\mathrm{colim}_{\mathbb{N}}\mathbb{S}^n) = \mathrm{colim}_{\mathbb{N}}\mathrm{PH}_{i,j}(\mathrm{colim}_n\mathbb{S}^n) = 0$.

We require one more lemma. To state it, we recall that a directed graph G is called *diagonal* if $\mathrm{MH}_{k\ell}(G) = 0$ whenever $k \neq \ell$. (This definition was first made for undirected graphs in [28], and is used in the context of directed graphs in [3], for instance.) Observe that the terminal directed graph $\bullet$ is also the unit object for the box product: for every directed graph H, we have $\bullet\Box H \cong H \cong H\Box\bullet$.

Lemma 7.6. *Fix a ground ring R which is a P.I.D. Suppose G is 1-contractible and diagonal. Let $t\colon G \to \bullet$ denote the terminal map. Then, for any directed graph H, the map of graphs $t\Box\mathrm{Id}\colon G\Box H \to H$ induces an isomorphism on bigraded path homology.*

Proof. By Lemma 5.3, the assumption that G is diagonal implies that its magnitude homology is free. We can therefore apply the Künneth theorem for bigraded path homology (Theorem 5.5) to get, for every H, a short exact sequence involving $\mathrm{PH}(G) \otimes \mathrm{PH}(H)$ and $\mathrm{PH}(G\Box H)$ and a Tor term. The assumption that G is 1-contractible implies that its bigraded path homology is concentrated in bidegree $(0,0)$, where it is a single copy of the ground ring R. Thus, the Tor term vanishes, leaving the isomorphism $\alpha\colon \mathrm{PH}(G) \otimes \mathrm{PH}(H) \xrightarrow{\cong} \mathrm{PH}(G\Box H)$. The naturality of that isomorphism tells us that this square commutes:

$$\begin{tikzcd} \mathrm{PH}(G) \otimes \mathrm{PH}(H) \arrow[r, "\cong"] \arrow[d, "\mathrm{PH}(t)\otimes\mathrm{PH}(\mathrm{Id})"'] & \mathrm{PH}(G\Box H) \arrow[d, "\mathrm{PH}(t\Box\mathrm{Id})"] \\ I \otimes \mathrm{PH}(H) \arrow[r, "\cong"] & \mathrm{PH}(H) \end{tikzcd}$$

where $I = \mathrm{PH}(\bullet)$ is is the bigraded module given by a single copy of R in bidegree $(0,0)$. Since the left leg is an isomorphism, the right leg is an isomorphism too. $\square$

Proof of Theorem 7.2. First, recall that **DiGraph** is cocomplete. So axiom (C6) certainly holds, and in particular **DiGraph** admits an initial object (the empty graph) and pushouts along cofibrations.

That the class of cofibrations is closed under composition and contains all identities is proved as Proposition 2.10 in [9]; the corresponding fact for weak equivalences follows immediately from the functoriality of bigraded path homology. This establishes axiom (C1). Axiom (C2) is immediate from the corresponding fact for isomorphisms and the functoriality of bigraded path homology. Axiom (C3)—that all directed graphs are cofibrant—follows immediately from the definitions.

For axiom (C4), consider a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"'] & X \cup_A Y \end{tikzcd}$$

in which i is a cofibration. Then j is also a cofibration, by Proposition 2.13 in [9]. In particular, our Theorem 3.3 applies to both (X, A) and $(X \cup_A Y, Y)$, giving a long exact sequence on bigraded path homology in both cases. If i is an *acyclic* cofibration, so $\mathrm{PH}_{*,*}(A) \cong \mathrm{PH}_{*,*}(X)$, then the long exact sequence of the pair (X, A) tells us the relative homology $\mathrm{PH}_{*,*}(X, A)$ vanishes. The excision theorem (Theorem 6.5) says that $\mathrm{PH}_{*,*}(X \cup_A Y, X) \cong \mathrm{PH}_{*,*}(X, A) = 0$ and it follows, by considering the long exact sequence of the pair $(X \cup_A Y, Y)$, that $\mathrm{PH}_{*,*}(X \cup_A Y) \cong \mathrm{PH}_{*,*}(Y)$. This establishes (C4).

To prove that axiom (C5) holds, let J denote the directed graph

$$\begin{array}{ccccccccc} -2 & & -1 & & 0 & & 1 & & 2 \\ \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \end{array}$$

and let ∂J denote the subgraph consisting of just the vertices labelled -2 and 2. Then the inclusion $\iota\colon \partial J \to J$ is a cofibration, and, for every X, the codiagonal map $X \sqcup X \to X$ factors as

$$X \sqcup X = \partial J \square X \xrightarrow{\iota \square \mathrm{Id}_X} J \square X \xrightarrow{t \square \mathrm{Id}_X} \bullet \square X = X \tag{25}$$

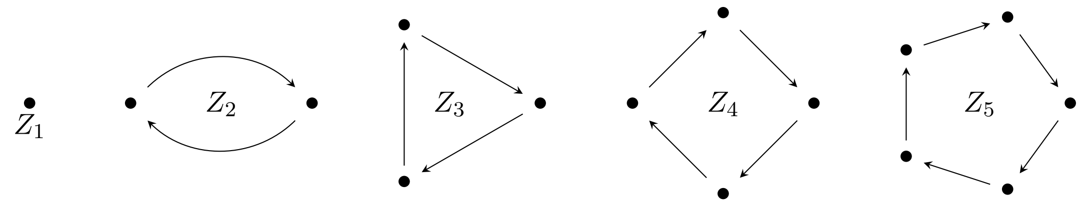
where t is the terminal map. The first morphism here is a cofibration by Proposition 2.12 in [9], which says that the box product of cofibrations is a cofibration. To see that the second morphism is a weak equivalence, observe that J is both 1-contractible and diagonal. (Since it contains no paths of length greater than 1, its magnitude chains are concentrated in bidegrees $(0, 0)$ and $(1, 1)$.) It follows from Lemma 7.6, then, that $t \square \mathrm{Id}_X$ is a weak equivalence. This establishes (C5).

That the transfinite composite of cofibrations is again a cofibration is proved as Proposition 2.16 in [9]; that the transfinite composite of weak equivalences is a weak equivalence follows from Proposition 7.4 together with the fact that a transfinite composite of isomorphisms is an isomorphism. This establishes (C7). □

8. Directed cycles

In this section we will explore in detail the magnitude-path spectral sequence of the directed cycles. These examples, together with those in Section 9, will clearly demonstrate the strength of the bigraded theory.

Definition 8.1. For $m \geq 1$, let Z_m denote the *directed cycle of length m*, i.e. a graph with m cyclically ordered vertices, and with a single directed edge between adjacent vertices, consistently oriented:



Observe that each Z_m has diameter $m-1$, so that all Z_m are long homotopy equivalent to one another and to the singleton, and in particular they all have trivial reachability homology. In contrast, when we consider 1-homotopy, it is not difficult to see that for $m \geq 2$ no two of the Z_m are 1-homotopy equivalent (when $m < n$ the only maps $Z_m \to Z_n$ are constant, and for $n > 2$ the constant maps $Z_n \to Z_n$ are only 1-homotopic to other constant maps). Nevertheless, for $m \geq 3$ the Z_m all have the same path homology, namely a single copy of R in degrees 0 and 1. So path homology cannot distinguish the different oriented cycles. In contrast, we will see that *bigraded* path homology can distinguish all except for Z_1 and Z_2.

Theorem 8.2 (Magnitude homology, bigraded path homology, and MPSS of directed cycles)**.** *Let $m \geq 3$. Then the magnitude homology $\mathrm{MH}_{*,*}(Z_m)$ and bigraded path homology $\mathrm{PH}_{*,*}(Z_m)$ are both concentrated in bidegrees of the form $(2i, mi)$ and $(2i+1, mi+1)$, in each of which they are free of rank m and rank 1 respectively. Moreover, the MPSS of Z_m satisfies $E^2(Z_m) = \cdots = E^{m-1}(Z_m)$ while $E^m(Z_m)$ is trivial, consisting of a single copy of R in bidegree $(0,0)$.*

In the case $m = 1$ the magnitude and bigraded path homology of Z_1 are both trivial, consisting of a single copy of R in bidegree $(0,0)$. In the case $m = 2$ the description of the magnitude homology given in the theorem remains true, while the bigraded path homology becomes trivial.

Corollary 8.3. *The E^r-page of the MPSS distinguishes all of the directed cycles Z_m for $m \geq r$, and is trivial for $m \leq r$. In particular, bigraded path homology distinguishes all the directed cycles Z_m for $m \geq 2$.*

Let us describe the MPSS of Z_m and preview the work of the rest of the section; we assume $m \geq 3$ for simplicity. In Theorem 8.11 we will see that the magnitude homology of Z_m vanishes except for copies of R^m that occur in pairs, one in each total degree of the spectral sequence, each pair being connected by the differential d^1, as depicted in Figure 5. This result is based on Theorem 8.6, which computes the homology of an auxiliary chain complex, which we call the *complex of partitions with upper bound.* The magnitude homology of Z_m is then a direct sum of copies of the homology groups of these complexes of partitions.

Returning to the E^1 term, we will see in Theorem 8.12 that by computing the differentials d^1, it follows that the E^2-term—or in other words, the bigraded path homology—has the form depicted in Figure 6. The nonzero groups here lie in bidegrees of the form $(im, -i(m-2))$ and $(im+1, -i(m-2))$, and in particular these bidegrees mean that the bigraded path homology of Z_m determines m for $m \geq 3$. Finally, the position of the nonzero terms means that the only subsequent differential can be in the $(m-1)$-page of

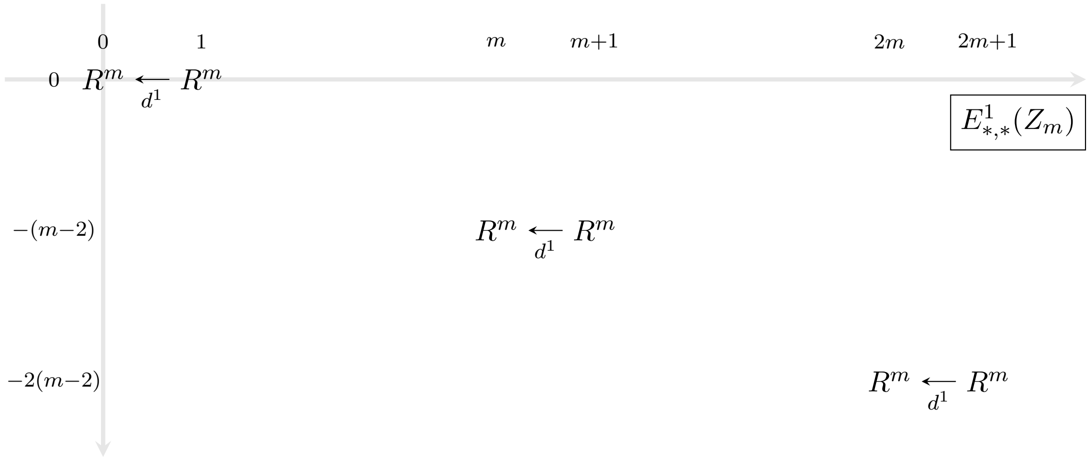


FIGURE 5. The magnitude homology of the directed m-cycle.

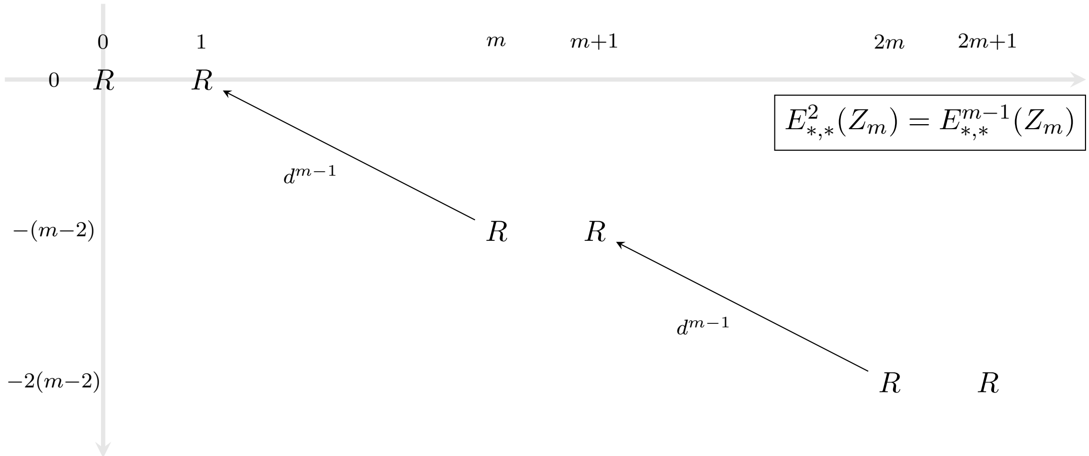


FIGURE 6. The bigraded path homology of the directed m-cycle.

the spectral sequence. Since we know that the homology of the reachability complex is trivial, the same must be true of the term $E^m = E^\infty$, so that each d^{m-1} depicted must be an isomorphism.

8.1. **The complex of ordered partitions with upper bound.**

Definition 8.4. Let $\ell \in \mathbb{Z}$. An *ordered partition* of ℓ is an ordered tuple $(a_1, \dots, a_k)$ of $k \geq 0$ positive integers a_i for which $a_1 + \cdots + a_k = \ell$.

In the last definition, observe that there is a unique ordered partition of $\ell = 0$, namely the empty tuple $()$, but that any ordered partition of $\ell > 0$ must have $k > 0$ entries. Note also that the definition admits the cases $\ell < 0$, but that in these cases there are no ordered partitions. (This apparently pointless decision will be useful in formulating a later lemma.)

Definition 8.5. Given $\ell, m \in \mathbb{Z}$ with $m \geq 2$, we let $\mathcal{OP}_*(\ell, m)$ be the chain complex of R-modules defined as follows. In degree k, $\mathcal{OP}_k(\ell, m)$ has basis given by the ordered partitions $(a_1, \ldots, a_k)$ of ℓ such that $a_i < m$ for all i. The differential d vanishes for $k \leq 1$, and is given in degrees $k \geq 2$ by summing adjacent entries,

$$d(a_1, \ldots, a_k) = \sum_{i=1}^{k-1} (-1)^i (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k),$$

and omitting any terms in which the summed entry fails the requirement $a_i + a_{i+1} < m$.

In the last definition, note that $\mathcal{OP}_*(\ell, m)$ is defined—and vanishes—for $\ell \leq -1$, while $\mathcal{OP}_*(0, m)$ has a single basis element $() \in \mathcal{OP}_0(0, m)$, and $\mathcal{OP}_*(1, m)$ has the single basis element $(1) \in \mathcal{OP}_1(1, m)$.

We will see later that the magnitude homology of a directed cycle of size m has a description in terms of the homology of the complexes $\mathcal{OP}_*(\ell, m)$. Our aim now is to compute that homology.

Theorem 8.6. *Let $\ell, m, k \in \mathbb{Z}$ with $m \geq 2$ and $k \geq 0$. Then $H_k(\mathcal{OP}(\ell, m)) = 0$, except in the following cases.*

- *Let $i \geq 0$. Then $H_{2i}(\mathcal{OP}(mi, m))$ is a copy of R generated by the element*
$$[(1, m-1, \ldots, 1, m-1)] = [(m-1, 1, \ldots, m-1, 1)],$$
where each tuple has $2i$ entries.
- *Let $i \geq 0$. Then $H_{2i+1}(\mathcal{OP}(mi+1, m))$ is a copy of R generated by the element*
$$[(1, m-1, \ldots, 1, m-1, 1)],$$
where the tuple has $2i+1$ entries.

This theorem will be used as a black box in our computation of the magnitude homology of the directed m-cycle. The remainder of this subsection is dedicated to the proof of the theorem, which the reader might therefore wish to skip.

We will see that, modulo a degree shift of 2, $H_*(\mathcal{OP}(\ell, m))$ is in fact periodic in ℓ with period m. This will allow us to prove the theorem by induction. We begin with the base cases.

Lemma 8.7. *For $\ell \leq 1$, the homology of $\mathcal{OP}_*(\ell, m)$ is as follows:*

- *For $\ell < 0$, $H_*(\mathcal{OP}(\ell, m))$ vanishes.*
- *For $\ell = 0$, $H_*(\mathcal{OP}(\ell, m))$ is a single copy of R concentrated in degree 0, generated by the class of the empty word $()$.*
- *For $\ell = 1$, $H_*(\mathcal{OP}(\ell, m))$ is a single copy of R concentrated in degree 1, generated by the class of the word (1).*

Proof. These claims are immediate: $\mathcal{OP}_*(\ell, m)$ vanishes for $\ell < 0$, while each of $\mathcal{OP}_*(0, m)$ and $\mathcal{OP}_*(1, m)$ has a single basis element, namely $()$ and (1) respectively. □

Definition 8.8. Let $\ell \geq 2$ and $m \geq 2$, and define a chain map $\phi\colon \mathcal{OP}_{*-2}(\ell - m, m) \to \mathcal{OP}_*(\ell, m)$ by

$$\phi(a_1, \ldots, a_k) = (1, m-1, a_1, \ldots, a_k).$$

It is straightforward to check that this is a chain map, thanks to the fact that $1 + (m-1) \geq m$ and $(m-1) + a_1 \geq m$.

Lemma 8.9. *Let $\ell, m \geq 2$. Then $\phi\colon \mathcal{OP}_{*-2}(\ell-m, m) \to \mathcal{OP}_*(\ell, m)$ is a quasi-isomorphism.*

Note that we include the cases $2 \leq \ell \leq m$, where the lemma tells us that $\mathcal{OP}_*(\ell, m)$ is acyclic.

Proof. Since ϕ is injective, it will suffice to show that its cokernel, which we denote by C_*, is acyclic. Observe that C_k has basis given by the basis elements $(a_1, \ldots, a_k)$ of $\mathcal{OP}_k(\ell, m)$ for which either $a_1 > 1$, or $a_1 = 1$ and $a_2 < m - 1$. To prove that C_* is acyclic, we use the map $s\colon C_* \to C_{*+1}$ defined on basis elements as follows.

$$s(a_1, \ldots, a_k) = \begin{cases} 0 & \text{if } a_1 = 1 \\ -(1, a_1 - 1, a_2, \ldots, a_k) & \text{if } a_1 > 1 \end{cases}$$

We will show that s is a chain contraction, i.e. that $sd + ds$ is the identity map, and this will complete the proof. Let us therefore fix a basis element $(a_1, \ldots, a_k)$ and show that $(sd + ds)(a_1, \ldots, a_k) = (a_1, \ldots, a_k)$. We will do this in three separate cases, as follows.

Case 1. We begin with the case $k = 1$, so that the only possible basis element of C_1 is (ℓ). (Even this element will not be present when $\ell \geq m$.) Then $sd(\ell) = s(0) = 0$ and $ds(\ell) = -d(1, \ell - 1) = (\ell)$, so that $(sd + ds)(\ell) = (\ell)$ as required. (Note that the restriction to $\ell \geq 2$ was necessary since case 1 fails when $\ell = 1$.)

Case 2. Let us assume now that $k > 1$, that $a_1 = 1$ and that $a_2 < m - 1$. Then $ds(a_1, \ldots, a_k) = 0$, while

$$\begin{aligned} sd(a_1, \ldots, a_k) &= \sum_{i=1}^{k-1} (-1)^i s(a_1, \ldots, a_i + a_{i+1}, \ldots, a_k) \\ &= -s(a_1 + a_2, a_3, \ldots, a_k) \\ &= (a_1, \ldots, a_k). \end{aligned}$$

In the first line, the i-th term of the sum is omitted if $a_i + a_{i+1} \geq m$. This omission only happens for $i > 1$, and in these cases the resulting partition has initial entry $a_1 = 1$ and therefore vanishes under s, giving us the second line, and the third then follows immediately since $a_1 + a_2 > 1$. Thus in this case $sd + ds$ sends $(a_1, \ldots, a_k)$ to itself as required.

Case 3. Let $(a_1, \ldots, a_k)$ be a basis element of C_k, and assume that $1 < a_1 \leq m - 1$. Then

$$\begin{aligned} sd(a_1, \ldots, a_k) &= \sum_{i=1}^{k-1} (-1)^i s(a_1, \ldots, a_i + a_{i+1}, \ldots, a_k) \\ &= -s(a_1 + a_2, a_3, \ldots, a_k) + \sum_{i=2}^{k-1} (-1)^i s(a_1, \ldots, a_i + a_{i+1}, \ldots, a_k) \\ &= (1, a_1 + a_2 - 1, a_3, \ldots, a_k) + \sum_{i=2}^{k-1} (-1)^{i+1} (1, a_1 - 1, \ldots, a_i + a_{i+1}, \ldots, a_k) \end{aligned}$$

where the first term is omitted if $a_1 + a_2 \geq m$, and the i-th term of each sum is omitted if $a_i + a_{i+1} \geq m$. And

$$\begin{aligned} ds(a_1, \ldots, a_k) &= -d(1, a_1 - 1, a_2, \ldots, a_k) \\ &= (a_1, \ldots, a_k) - (1, a_1 + a_2 - 1, a_3, \ldots, a_k) \\ &\quad + \sum_{i=2}^{k-1} (-1)^i (1, a_1 - 1, \ldots, a_i + a_{i+1}, \ldots, a_k) \end{aligned}$$

where the second term is omitted if $a_1 + a_2 - 1 \geq m$, and the i-th term of the sum is omitted if $a_i + a_{i+1} \geq m$. Thus when we compute $(sd + ds)(a_1, \ldots, a_k)$ we find that the sums cancel out, the question of which of their terms are omitted being the same in each case, and that $(a_1, \ldots, a_k)$ always remains. Thus $(sd + ds)(a_1, \ldots, a_k) = (a_1, \ldots, a_k)$ will hold so long as we can show that the two potential terms $(1, a_1 + a_2 - 1, a_3, \ldots, a_k)$ contribute 0 overall, the question being exactly when each one is omitted. If $a_1 + a_2 \geq m + 1$ then both of these terms are omitted; if $a_1 + a_2 \leq m - 1$ then both are present and cancel out; and if $a_1 + a_2 = m$ then one is present and the other is not. But in this last case we in fact have $(1, a_1 + a_2 - 1, a_3, \ldots, a_k) = (1, m - 1, a_3, \ldots, a_k) = 0$ because we work in the cokernel of ϕ. This completes the proof that $(ds + sd)(a_1, \ldots, a_k) = (a_1, \ldots, a_k)$ in this case. □

Proof of Theorem 8.6. The two cycles listed in the first bullet point are homologous. Indeed, let us define $c \in \mathcal{OP}_{2i+1}(mi, m)$ to be the sum

$$c = \sum_{j=1}^{i} (1, m - 1, \ldots, 1, m - 2, 1, \ldots, m - 1, 1),$$

where each tuple has $2i + 1$ entries, and $m - 2$ appears in position $2j$ in the j-th term. Then

$$dc = (1, m - 1, \ldots, 1, m - 1) - (m - 1, 1, \ldots, m - 1, 1)$$

where now each tuple has $2i$ entries.

For $\ell \leq 1$ the theorem follows from Lemma 8.7, while for $\ell \geq 2$ we may repeatedly apply Lemma 8.9 to one of the cases from Lemma 8.7 to obtain the result, including the description of the generators. □

8.2. **The magnitude homology of Z_m.** If x is a vertex of Z_m, then we write x^+ for the next vertex in the cyclic order, and x^- for the previous vertex. In other words, $x^\pm$ are characterised by the existence of edges $x^- \to x \to x^+$.

Definition 8.10. We define two families of classes in the magnitude homology of Z_m as follows.

- Given $x \in V(Z_m)$ and $i \geq 0$, define $\kappa_x^i \in \mathrm{MH}_{2i,mi}(Z_m)$ by
$$\kappa_x^i = [(x, x^+, \ldots, x, x^+, x)] = [(x, x^-, \ldots, x, x^-, x)],$$
where the tuple has a total of $2i + 1$ entries.
- Given $e \in E(Z_m)$ and $i \geq 0$, define $\lambda_e^i \in \mathrm{MH}_{2i+1,mi+1}(Z_m)$ by
$$\lambda_e^i = [(x, x^+, \ldots, x, x^+)],$$
where $e = xx^+$ and the tuple has a total of $2i + 2$ entries.

In the spectral sequence grading, these classes lie in positions

$$\kappa_x^i \in E^1_{mi,(2-m)i}(Z_m) \quad \text{and} \quad \lambda_e^i \in E^1_{mi+1,(2-m)i}(Z_m).$$

Theorem 8.11. *Let $m \geq 2$. The magnitude homology $\mathrm{MH}_{*,*}(Z_m)$ is the free R-module with basis given by the elements κ_x^i and λ_e^i for $i \geq 0$, $x \in V(Z_m)$ and $e \in E(Z_m)$. In particular, it is free of rank m in bidegrees of the form $(2i, mi)$ and $(2i+1, mi+1)$, and it is zero in all other bidegrees.*

Proof. Observe that if we fix a vertex $x \in V(Z_m)$ and a length $\ell \geq 0$, then the tuples of the form $(x_0, \dots, x_k)$, where $x_0 = x$, $k \geq 0$ and $\ell(x_0, \dots, x_k) = \ell$, span a subcomplex $\mathrm{MC}_*(x, \ell)$ of $\mathrm{MC}_{*,\ell}(Z_m)$. Moreover, observe that $\mathrm{MC}_{*,\ell}(Z_m)$ is the direct sum of the $\mathrm{MC}_*(x, \ell)$ for $x \in V(Z_m)$.

There is an isomorphism of chain complexes

$$\mathrm{MC}_*(x, \ell) \xrightarrow{\cong} \mathcal{OP}_*(\ell, m)$$

defined by

$$(x, x_1, \dots, x_k) \longmapsto (d(x, x_1), d(x_1, x_2), \dots, d(x_{k-1}, x_k)).$$

That this is an isomorphism of R-modules follows immediately from the fact that, given $x \in V(Z_m)$ and $a \in \{1, \dots, m-1\}$, there is a unique vertex $y \neq x$ with $d(x, y) = a$. That it is a chain map follows from the fact that, given $x, y, z \in V(Z_m)$, we have $d(x, y) + d(y, z) = d(x, z)$ if and only if $d(x, y) + d(y, z) < m$.

The isomorphism above identifies the elements

$$(x, x^+, \dots, x, x^+, x), (x, x^-, \dots, x, x^-, x) \in \mathrm{MC}_{2i}(x, mi)$$

with the elements

$$(1, m-1, \dots, 1, m-1), (m-1, 1, \dots, m-1, 1) \in \mathcal{OP}_{2i}(mi),$$

and identifies the element

$$(x, x^+, \dots, x, x^+) \in \mathrm{MC}_{2i+1}(x, mi+1)$$

with

$$(1, m-1, \dots, m-1, 1) \in \mathcal{OP}_{2i+1}(mi+1).$$

The theorem now follows directly from Theorem 8.6. ∎

8.3. **Bigraded path homology.**

Theorem 8.12. *Let $m \geq 3$. The bigraded path homology of Z_m is the free R-module with basis given by classes $\alpha^{2i}, \beta^{2i+1}$ for all $i \geq 0$, where*

$$\alpha^{2i} \in E^2_{mi,-(m-2)i}(Z_m), \qquad \beta^{2i+1} \in E^2_{mi+1,-(m-2)i}(Z_m)$$

or equivalently

$$\alpha^{2i} \in \mathrm{PH}_{2i,mi}(Z_m), \qquad \beta^{2i+1} \in \mathrm{PH}_{2i+1,mi+1}(Z_m).$$

In particular, $\mathrm{PH}_{*,*}(Z_m)$ is free of rank 1 in bidegrees of form $(2i, mi)$ and $(2i+1, mi+1)$ for $i \geq 0$, and vanishes in all other bidegrees.

Lemma 8.13. *Let $m \geq 3$. If $e = xy \in E(Z_m)$, then $d^1(\lambda_e^i) = \kappa_y^i - \kappa_x^i$. And if $x \in V(Z_m)$ then $d^1(\kappa_x^i) = 0$.*

Proof. We have $y = x^+$, so that

$$\lambda_e^i = [(x, x^+, \ldots, x, x^+)] \in \mathrm{MH}_{2i+1,mi+1}(Z_m)$$

and consequently

$$d^1(\lambda_e^i) = [d(x, x^+, \ldots, x, x^+)] \in \mathrm{MH}_{2i,mi}(Z_m).$$

Here $d(x, x^+, \ldots, x, x^+) \in \mathrm{MC}_{2i,mi}(Z_m)$ is the alternating sum of the tuples obtained by omitting terms from $(x, x^+, \ldots, x, x^+)$. When we delete anything other than the first or last term, we obtain a tuple with repeated consecutive entries, which therefore vanishes in the magnitude chain group. Thus

$$\begin{aligned} d(x, x^+, \ldots, x, x^+) &= (x^+, x, \ldots, x, x^+) - (x, x^+, \ldots, x^+, x) \\ &= (y, y^-, \ldots, y, y^-) - (x, x^+, \ldots, x^+, x) \end{aligned}$$

so that, taking classes in $\mathrm{MH}_{2i,mi}(Z_m)$, we find $d^1(\lambda_e^i) = \kappa_y^i - \kappa_x^i$ as required. Finally, $d^1(\kappa_x^i) = 0$ because it lies in a bidegree where magnitude homology vanishes. □

Proof of Theorem 8.12. Lemma 8.13 shows that the homology of $E^1_{*,*}(Z_m)$ with respect to d^1 is freely spanned over R by the following homology classes, for each $i \geq 0$:

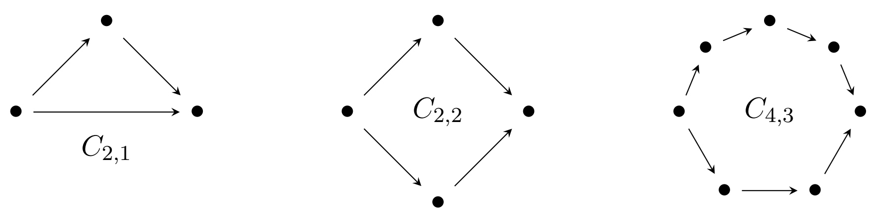
(1) $[\kappa_x^{2i}]$ for any chosen vertex x of Z_m. A different choice of vertex defines the same homology class.
(2) $\left[\sum_{e \in E(Z_m)} \lambda_e^{2i+1}\right]$.

We may now define α^{2i} and β^{2i+1} to be the elements of $E^2_{*,*}(Z_m)$ corresponding to these classes under the isomorphism $H(E^1_{*,*}(Z_m)) \cong E^2_{*,*}(Z_m)$. □

9. Bi-directed cycles

In this section we determine the magnitude-path spectral sequence for the bi-directed cycles. Again, we find that the bigraded path homology contains strictly more information than the path homology itself, but that in contrast with the case of oriented cycles, it is not sufficient to determine graphs of this class up to isomorphism.

Definition 9.1. Let m and n be integers with $m, n \geq 1$. The *bi-directed cycle* or (m, n)*-cycle* $C_{m,n}$ is obtained by taking directed intervals of length m and n, and then identifying their initial points to a single point, and their final points to a single point. Put differently, it is obtained from an unoriented $m + n$ cycle by orienting a set of m contiguous edges in one direction, and the remaining n edges in the opposite direction. We often think of $C_{m,n}$ as having its two intervals oriented from left to right, with the interval of length m across the top, and the one of length n across the bottom, as follows:



Observe that $C_{m,n}$ has both an initial and a terminal vertex; in our diagrams these are the ones at the left and right respectively. Moreover, $C_{m,n}$ has diameter $\mathfrak{m}-1$ where $\mathfrak{m}=\max(m,n)$.

Throughout the section we will use the letter $\mathfrak{m}$ to denote $\max(m,n)$. The following result tells us that the homological behaviour of $C_{m,n}$ is largely determined by $\mathfrak{m}$ alone.

Proposition 9.2. *The bigraded path homology of $C_{m,n}$ depends only on $\mathfrak{m}=\max(m,n)$. More precisely, choose the shorter of the two directed intervals of $C_{m,n}$ (or either one if $m=n$), and contract all of its edges except the last, to obtain a map $C_{m,n}\to C_{\mathfrak{m},1}$. This map is an isomorphism on bigraded path homology.*

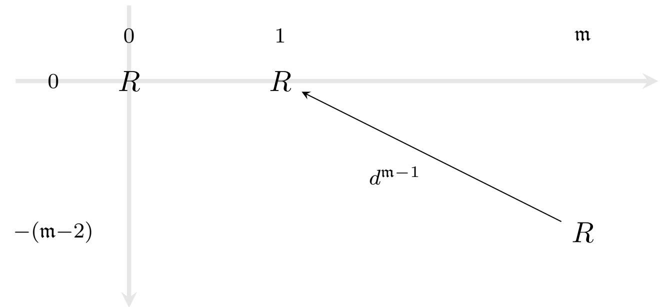
This proposition will be used to reduce our task to computing the MPSS for the graphs $C_{m,1}$. Although it is possible to compute the MPSS for all $C_{m,n}$ directly, that is significantly more onerous, and in particular requires the tedious separation of the cases $m=n$ and $m\neq n$.

Theorem 9.3 (Bigraded path homology and MPSS of bi-directed cycles)**.** *Let $m,n\geq 1$ and assume further that $\mathfrak{m}=\max(m,n)\geq 3$. Then the bigraded path homology $\mathrm{PH}_{*,*}(C_{m,n})$ is concentrated in bidegrees $(0,0)$, $(1,0)$ and $(\mathfrak{m},-(\mathfrak{m}-2))$, in each of which it is free of rank 1. Moreover, the MPSS of $C_{m,n}$ satisfies $E^2(C_{m,n})=\cdots=E^{\mathfrak{m}-1}(C_{m,n})$ while $E^{\mathfrak{m}}(C_{m,n})$ is trivial, consisting of a single copy of R in bidegree $(0,0)$. If $\mathfrak{m}=1,2$ then the bigraded path homology is concentrated in bidegree $(0,0)$, where it is free of rank 1.*

Corollary 9.4. *The MPSS of $C_{m,n}$ depends only on the value of $\mathfrak{m}=\max(m,n)$. The E^r-page of the MPSS of $C_{m,n}$ determines $\mathfrak{m}$ for $\mathfrak{m}\geq r$, and is trivial for $\mathfrak{m}\leq r$. In particular, bigraded path homology determines the value of $\mathfrak{m}$ for $\mathfrak{m}\geq 2$.*

In this section we will describe the MPSS for $C_{m,n}$ from the E^2-term onwards. In the case $\mathfrak{m}=2$, the bigraded path homology of $C_{m,n}$ is trivial, consisting of a single copy of R in degree $(0,0)$. For $\mathfrak{m}\geq 3$, the bigraded path homology for $C_{m,n}$ is given as follows:

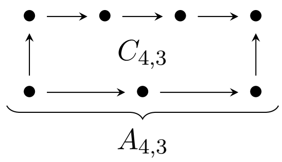
$\boxed{E^2_{*,*}(C_{m,n})}$



The MPSS is then determined by the value of the lone remaining differential, which is $d^{\mathfrak{m}-1}$ as shown. Since $C_{m,n}$ has both initial and terminal vertices, its reachability homology is trivial, and therefore E^∞ vanishes in positive total degrees. Thus in $E^{\mathfrak{m}-1}$, the terms in degrees $(1,0)$ and $(\mathfrak{m},-(\mathfrak{m}-2))$ cannot survive, and $d^{\mathfrak{m}-1}$ must therefore be an isomorphism. (The apparent separation between the cases $\mathfrak{m}=2$ and $\mathfrak{m}\geq 3$ arises because, when $\mathfrak{m}=2$, the $d^{\mathfrak{m}-1}$ differential takes place on the E^1 page.) Thus, we see that $E^2_{*,*}(C_{m,n})$ determines the value of $\mathfrak{m}$ via the placement of its nonzero groups. The

MPSS of $C_{m,n}$ also determines $\mathfrak{m}$ as the first page that is trivial, i.e. concentrated in degree $(0,0)$.

We now move to the proof of Proposition 9.2. We now assume without loss that $m \geq n$ so that $\mathfrak{m} = m$. Let us write $A_{m,n}$ for the subgraph of $C_{m,n}$ consisting of the first $n-1$ edges of the directed interval of length n, which in our diagrams we think of as going across the bottom. To make it easier to visualise the arguments to come, it can help to redraw $C_{m,n}$ and $A_{m,n}$ in the following manner.

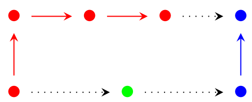


Lemma 9.5. *Let $m, n \geq 1$ with $m \geq n$ and $m \geq 2$. Then the inclusion $A_{m,n} \hookrightarrow C_{m,n}$ is a cofibration.*

Proof. There are certainly no edges into $A_{m,n}$ from vertices not in $A_{m,n}$, and so the first condition of a cofibration holds. Observe that the reach of $A_{m,n}$ is then the whole of $C_{m,n}$, since the initial vertex of $A_{m,n}$ can reach every other. We may now define π as follows:

- Each vertex of $A_{m,n}$ is sent to itself.
- The terminal vertex of $C_{m,n}$ is sent to the terminal vertex of $A_{m,n}$.
- The remaining vertices of $C_{m,n}$ are sent to the initial vertex of $A_{m,n}$.

Observe that in all cases there is a path from $\pi(x)$ to x. For example, in the case of $C_{4,3}$ the map π identifies all vertices of the same colour in this diagram:



With respect to the given map π, we may now verify the second condition of a cofibration, namely that
$$d(a, x) = d(a, \pi(x)) + d(\pi(x), x)$$
for $x \in C_{m,n}$ and $a \in A_{m,n}$. If a and x are, respectively, the initial and final vertices of $C_{m,n}$, then there are two distinct paths from a to x, but the shortest distance is always given by the path that travels through $\pi(x)$ (the lower path), and consequently the condition holds in this case. In all other cases there is at most one path from a to x, and that path passes through $\pi(x)$, so that the condition holds in these cases too. $\square$

Proof of Proposition 9.2. We may assume that $m \geq n$ and $m \geq 2$. Then the map from the statement takes the form $C_{m,n} \to C_{m,1}$, and fits into a pushout diagram

$$\begin{tikzcd}
A_{m,n} \arrow[r] \arrow[d] & A_{m,1} \arrow[d] \\
C_{m,n} \arrow[r] & C_{m,1}
\end{tikzcd}$$

whose upper map collapses $A_{m,n}$ to $A_{m,1}$, the latter being a single vertex. The maps $\mathrm{PH}_{*,*}(A_{m,n}) \to \mathrm{PH}_{*,*}(A_{m,1})$ are isomorphisms since $A_{m,n} \to A_{m,1}$ is a 1-homotopy equivalence. And the maps $\mathrm{PH}_{*,*}(C_{m,n}, A_{m,n}) \to \mathrm{PH}_{*,*}(C_{m,1}, A_{m,1})$ are isomorphisms by the excision theorem, which applies since the vertical maps of our pushout are cofibrations. Then the long exact sequences of the pairs $(C_{m,n}, A_{m,n})$ and $(C_{m,1}, A_{m,1})$ are related by a commutative ladder, and the last sentence allows us to apply the five lemma to this ladder to obtain the result. $\square$

Let us now turn to the MPSS of $C_{m,1}$ for $m \geq 2$. To do so we need to establish some notation. We write the vertices of $C_{m,1}$ as $a_0, \ldots, a_m$, with edges $a_{i-1} \to a_i$ for $i = 1, \ldots, m$, and $a_0 \to a_m$. So for $C_{4,1}$ we have the following:

$$\begin{tikzcd}
 & & a_2 \arrow[dr] & & \\
 & a_1 \arrow[ur] & C_{4,1} & a_3 \arrow[dr] & \\
a_0 \arrow[ur] \arrow[rrrr] & & & & a_m
\end{tikzcd}$$

Next, we define some magnitude homology classes.

Definition 9.6. We define magnitude homology classes

$$\kappa_x \in \mathrm{MH}_{0,0}(C_{m,1}), \qquad \lambda_e \in \mathrm{MH}_{1,1}(C_{m,1}), \qquad \mu \in \mathrm{MH}_{2,m}(C_{m,1}),$$

for $x \in V(C_{m,1})$ and $e \in E(C_{m,1})$, by the rules

$$\kappa_x = [(x)], \qquad \lambda_e = [(a,b)], \qquad \text{and } \mu = [(a_0, a_i, a_m)]$$

where $e = ab$. In the definition of μ one can make any choice of $1 < i < m$; the value of μ does not change. Note that in the spectral sequence grading we have

$$\kappa_x \in E^1_{0,0}(C_{m,1}), \qquad \lambda_e \in E^1_{1,0}(C_{m,1}), \qquad \mu \in E^1_{m,-m+2}(C_{m,1}).$$

We can now state our main result.

Theorem 9.7. *Let $m \geq 2$. The magnitude homology $\mathrm{MH}_{*,*}(C_{m,1})$ is the free bigraded R-module with basis given by the κ_x, λ_e and μ for $x \in V(C_{m,1})$ and $e \in E(C_{m,1})$. The differential d^1 acts on these classes by the rules $d^1(\kappa_x) = 0$, $d^1(\lambda_e) = \kappa_b - \kappa_a$ when $e = ab$, and, if $m = 2$, $d^1(\mu) = \lambda_{a_0,a_1} + \lambda_{a_1,a_2} - \lambda_{a_0,a_2}$.*

We immediately obtain the following description of the bigraded path homology.

Corollary 9.8. *If $m > 2$ then the bigraded path homology of $C_{m,1}$ consists of a single copy of R in bidegrees $(0,0)$, $(1,0)$ and $(m, -m+2)$. If $m = 2$ then the bigraded path homology of $C_{m,1}$ vanishes in positive degrees.*

Let us turn to the proof of Theorem 9.7. For the final sentence of the statement we have the following.

Lemma 9.9. *The description of d^1 in Theorem 9.7 holds.*

Proof. The map d^1 is obtained by applying the differential of the reachability complex to any representative. Thus, for example, when $m = 2$ we have $d^1(\mu) = d^1[(a_0, a_1, a_2)] = [d(a_0, a_1, a_2)] = [(a_1, a_2) - (a_0, a_2) + (a_0, a_1)] = [(a_1, a_2)] - [(a_0, a_2)] + [(a_0, a_1)] = \lambda_{a_1 a_2} - \lambda_{a_0 a_2} + \lambda_{a_0 a_1}$ as claimed. The other cases are similar and left to the reader. $\square$

In order to give our proof of Theorem 9.7, we use the decomposition of magnitude chains and magnitude homology by initial and final points of tuples, which we now recall. (The decomposition is well known, and to the best of our knowledge its first explicit use is in Section 5.4 of [32].) Let G be a directed graph. Let a, b be vertices of G, possibly equal, and let $\ell \geq 0$. Then we write $\mathrm{MC}_{*,\ell}(a,b)$ for the subcomplex of $\mathrm{MC}_{*,\ell}(G)$ spanned by all tuples of the form $(x_0,\ldots,x_k)$ with $x_0 = a$ and $x_k = b$, and we write $\mathrm{MH}_{*,\ell}(a,b)$ for its homology. Note that $\mathrm{MC}_{*,\ell}(a,b)$ is indeed a subcomplex of $\mathrm{MC}_{*,\ell}(G)$, thanks to the fact that omitting the first or last entries of a nondegenerate tuple always strictly decreases its length. Then the magnitude chains and magnitude homology of G both decompose as direct sums:

$$\mathrm{MC}_{*,*}(G) = \bigoplus_{a,b\in V(G)} \mathrm{MC}_{*,*}(a,b), \qquad \mathrm{MH}_{*,*}(G) = \bigoplus_{a,b\in V(G)} \mathrm{MH}_{*,*}(a,b)$$

Note that $\mathrm{MC}_{*,*}(a,b)$ is nonzero if and only if $d(a,b) < \infty$.

The description of $\mathrm{MH}_{*,\ell}(C_{m,n})$ for $\ell = 0,1$ is straightforward and left to the reader. Compare, for example, with Proposition 2.9 of [28] or Theorems 4.1 and 4.3 of [37]. Therefore it remains to prove Theorem 9.7 in the case $\ell \geq 2$, which is given by the following.

Lemma 9.10. *Let $\ell \geq 2$. Then the groups $\mathrm{MH}_{*,\ell}(a_i,a_j)$ for $0 \leq i \leq j \leq m$ all vanish, with the single exception of $\mathrm{MH}_{2,m}(a_0,a_m)$, which is a copy of the ring R spanned by the class μ.*

Proof. We first address the case where $(i,j) \neq (0,m)$. The generators of $\mathrm{MC}_{k,*}(a_i,a_j)$ are the tuples

$$(a_i, a_{i_1}, \ldots, a_{i_k}, a_j)$$

for $k \geq 0$ and $i < i_1 < \cdots < i_k < j$, and in particular these all have length $j-i$. So we may assume $\ell = j - i$. Define $s\colon \mathrm{MC}_{*,j-i}(a_i,a_j) \to \mathrm{MC}_{*+1,j-i}(a_i,a_j)$ by the rule

$$s(a_i, a_{i_1}, \ldots, a_{i_k}, a_j) = \begin{cases} -(a_i, a_{i+1}, a_{i_1}, \ldots, a_{i_k}, a_j) & \text{if } i_1 \neq i+1 \\ 0 & \text{if } i_1 = i+1. \end{cases}$$

Then $(sd+ds)(a_i, a_{i_1}, \ldots, a_{i_k}, a_j) = (a_0, a_{i_1}, \ldots, a_{i_k}, a_m)$, and it follows that the homology of $\mathrm{MC}_{*,j-i}(a_i,a_j)$ vanishes in all degrees.

Next we address the case of $\mathrm{MC}_{*,*}(a_0,a_m)$, whose generators are the tuples of the form $(a_0, a_{i_1}, \ldots, a_{i_k}, a_m)$ for $k \geq 1$. (We no longer allow the case $k = 0$ because $\ell(a_0,a_m) = 1$ and we have assumed $\ell \geq 2$.) Any one of these has length m, and so we may assume that $\ell = m$. Define $s\colon \mathrm{MC}_{*,m}(a_0,a_m) \to \mathrm{MC}_{*+1,m}(a_0,a_m)$ by the rule

$$s(a_0, a_{i_1}, \ldots, a_{i_k}, a_m) = \begin{cases} -(a_0, a_1, a_{i_1}, \ldots, a_{i_k}, a_m) & i_1 \neq 1 \\ 0 & i_1 = 1. \end{cases}$$

Then $(sd+ds)(a_0, a_{i_1}, \ldots, a_{i_k}, a_m)$ is equal to $(a_0, a_{i_1}, \ldots, a_{i_k}, a_m)$ except in the case $k = 1$, $i_1 = 1$, when it vanishes. It follows that the homology of $\mathrm{MC}_{*,m}(a_0,a_m)$ vanishes except for $\mathrm{MC}_{2,m}(a_0,a_m)$, which is generated by the class of (a_0,a_1,a_m), this class being precisely μ. $\square$

Appendix A. Preservation of filtered colimits

It is proved in Proposition 1.14 of [15] that each page of the magnitude-path spectral sequence is a finitary functor on the category of directed graphs. We have chosen to include here an alternative (and much more detailed) proof of this fact, because we believe it illuminates the construction of the MPSS, and because the lemmas in this Appendix may be useful to others working with the pages of the sequence. In particular, our proof makes clear that each page of the MPSS extends to a finitary functor on the category $\mathbb{N}\mathbf{Met}$ whose objects are metric spaces with integer distances—but *not* to a finitary functor on the larger category $\mathbf{Met}$ (see Remark A.10).

Proposition 7.4. *For each $r \geq 1$ and every $p, q \in \mathbb{Z}$, the functor*

$$E^r_{pq}(-)\colon \mathbf{DiGraph} \to \mathbf{Mod}_R$$

preserves filtered colimits. In particular this holds for magnitude homology and bigraded path homology.

As our proof takes place in several stages, let us first outline the strategy. Recall that an object X in a category $\mathcal{C}$ is called *finitely presentable* (or *compact*) if the covariant functor it represents (the functor $\mathrm{Hom}(X, -)\colon \mathcal{C} \to \mathbf{Set}$) preserves filtered colimits. Every directed graph with finitely many vertices is finitely presentable in **DiGraph**, so in particular this is true of the 'k-path graph'

$$p_0 \to p_1 \to \cdots \to p_k.$$

It is readily seen that, for each $k \in \mathbb{N}$, the k^{th} diagonal piece of the filtered nerve—the functor $F_k\mathcal{N}_k(-)\colon \mathbf{DiGraph} \to \mathbf{Set}$ (see Remark 2.1)—is represented by the k-path graph, and thus is finitary. This fact is exploited in Propositions 4.4 and 4.5 of [9] to prove that ordinary path homology is a finitary functor.

To deal with the off-diagonal modules in bigraded path homology and the rest of the spectral sequence, it is enough to prove that for each ℓ and k the functor $F_\ell\mathcal{N}_k(-)$ from **DiGraph** to **Set** is finitary. However, when $k \neq \ell$, the functor $F_\ell\mathcal{N}_k(-)$ is no longer representable. (This can be seen, for example, by observing that it does not preserve the categorical product; recall Remark 4.10.) Instead, to construct the set

$$F_\ell\mathcal{N}_k(X) = \{(x_0, \ldots, x_k) \mid x_i \in X \text{ and } \sum d(x_i, x_{i+1}) \leq \ell\}$$

for a given directed graph X, one needs to consider short maps into X from a big family of generalized metric spaces: those of the form

$$p_0 \underset{\infty}{\overset{d_1}{\rightleftarrows}} p_1 \underset{\infty}{\overset{d_2}{\rightleftarrows}} \cdots \underset{\infty}{\overset{d_k}{\rightleftarrows}} p_k \tag{26}$$

where the distances $d_i = d(p_{i-1}, p_i)$ are integers satisfying $\sum d_i = \ell$; we have $d(p_i, p_j) = \sum_{i+1}^{j} d_i$ for $i \leq j$; and $d(p_i, p_j) < \infty$ for $j < i$. We call these spaces *straight paths*.

Of course, there are in general more such maps than there are elements of $F_\ell\mathcal{N}_k(X)$, but that can be remedied by taking a quotient to identify those maps whose images coincide inside X. We prove in Proposition A.6 that

$$F_\ell\mathcal{N}_k(-) \cong \mathrm{colim}_D \mathrm{Hom}(\mathbf{d}, -) \tag{27}$$

where D is a diagram involving all straight paths $\mathbf{d}$ with k points and length at most ℓ.

Since colimits commute with colimits, to deduce from (27) that $F_\ell \mathcal{N}_k(-)$ is finitary it is enough to see that every straight path is finitely presentable. And that is the case—but not in **DiGraph** (after all, most straight paths are not directed graphs). Nor is it true in the much larger category **Met** of generalized metric spaces (see Remark A.8). Rather, we show in Proposition A.7 that straight paths are finitely presentable in the full subcategory of **Met** whose objects are spaces with integer distances.

One virtue of this rather long and involved proof is that it offers an alternative description of the filtered nerve as a particular colimit of especially 'nice' representable functors; we anticipate that this description will prove technically useful in future work. Another virtue is that by explaining *why* the MPSS is finitary as a functor on the category of directed graphs, it also makes clear that (and why) this property does not extend to the category of metric spaces, whose 'spectral homology'—a natural extension of the magnitude-path spectral sequence—is studied by Ivanov in [29].

A.1. **Colimits in the category $\mathbb{N}$Met.** Recall from Section 2.1 that **Met** is the category of generalized metric spaces and short maps. We denote by $\mathbb{N}$**Met** the full subcategory on objects whose metric takes values in $\mathbb{N} \cup \{+\infty\}$. There is a chain of full subcategory inclusions

$$\mathbf{DiGraph} \hookrightarrow \mathbb{N}\mathbf{Met} \hookrightarrow \mathbf{Met}.$$

In fact, each of these is the inclusion of a *coreflective* subcategory, as the next lemma shows.

Lemma A.1. *The inclusions* $M\colon \mathbf{DiGraph} \hookrightarrow \mathbb{N}\mathbf{Met}$ *and* $\iota\colon \mathbb{N}\mathbf{Met} \hookrightarrow \mathbf{Met}$ *each have a right adjoint.*

Proof. We consider first the functor M. Given an object X of $\mathbb{N}$**Met**, let $\Gamma(X)$ be the directed graph whose set of vertices is the set of points in X, with an edge $x \to y$ whenever $d_X(x,y) = 1$. This construction extends to a functor $\Gamma\colon \mathbb{N}\mathbf{Met} \to \mathbf{DiGraph}$, which we claim is right-adjoint to M.

Observe that for any G in **DiGraph** we have $G = (\Gamma \circ M)(G)$. Meanwhile, for any X in $\mathbb{N}$**Met**, the identity on points determines a short map $\epsilon_X\colon (M\circ\Gamma)(X) \to X$. To see this, take any pair of points x, x' in $(M\circ\Gamma)(X)$; suppose $d_{(M\circ\Gamma)(X)}(x,x') = n$. By definition of the shortest path metric, there exists a directed path $x = x_0 \to \cdots \to x_n = x'$ in ΓX, and by definition of Γ, we must have $d_X(x_i, x_{i+1}) = 1$ for $i = 0, \dots, n-1$. Thus, $d_X(x,x') \leq \sum_{i=0}^{n-1} d_X(x_i,x_{i+1}) = n = d_{(M\circ\Gamma)(X)}(x,x')$. The maps $\eta_G = \mathrm{Id}_G\colon G \to (\Gamma\circ M)(G)$ and $\epsilon_X\colon (M\circ\Gamma)(X) \to X$ are automatically natural and satisfy the unit-counit identities, so the claim follows.

Now define $\lceil - \rceil\colon \mathbf{Met} \to \mathbb{N}\mathbf{Met}$ as follows. For each X in **Met** the space $\lceil X \rceil$ has the same points as X, with $d_{\lceil X\rceil}(x,y) = \lceil d_X(x,y)\rceil$ for each $x, y \in X$. To see that this indeed defines a metric, note that $d_{\lceil X\rceil}(x,x) = \lceil 0 \rceil = 0$ for every $x \in X$, and for every triple $x, y, z \in X$ we have

$$\lceil d_X(x,y)\rceil + \lceil d_X(y,z)\rceil \geq \lceil d_X(x,y) + d_X(y,z)\rceil \geq \lceil d_X(x,z)\rceil.$$

Meanwhile, given a short map $f\colon X \to Y$ we have

$$d_{\lceil Y\rceil}(f(x), f(y)) = \lceil d_Y(f(x), f(y))\rceil \leq \lceil d_X(x,y)\rceil = d_{\lceil X\rceil}(x,y),$$

which shows that $\lceil - \rceil$ is functorial.

To see that $\lceil - \rceil$ is right-adjoint to ι, observe that for each X in $\mathbb{N}$**Met**, the identity on points is an isometry $\eta_X \colon X \to \lceil \iota(X) \rceil$, while for each X in **Met**, the identity on points is a short map $\epsilon_X \colon \iota(\lceil X \rceil) \to X$. These maps are automatically natural and satisfy the unit-counit identities, so the statement follows. □

The category **Met** is, equivalently, the category of categories enriched in the poset $[0, +\infty]$ with monoidal operation given by addition [34]. As $[0, +\infty]$ is cocomplete, **Met** is cocomplete too [48, Corollary 2.14]. Lemma A.1 then implies that **DiGraph** and $\mathbb{N}$**Met** are both cocomplete, since the inclusion of a coreflective subcategory creates all colimits that exist in the ambient category. In particular, $\mathbb{N}$**Met** is closed under taking colimits in **Met**.

We shall want an explicit description of colimits in **Met**. The following formula can be extracted from the discussion in Appendix A of [17] or Lemma 3 in [31], for example. (Both those references deal with symmetric metrics, but the proof in our setting is the same.)

Given a diagram $F \colon J \to \mathbf{Met}$, we will write $F(j) = X_j$ for each object j in J.

Lemma A.2. *The colimit of a small diagram $F \colon J \to \mathbf{Met}$ is constructed by first taking the colimit in* **Set** *of the underlying sets: $\coprod_{j \in J} X_j / \sim$ where $\sim$ is the equivalence relation generated by setting $x \sim x'$ if $x' = F(\phi)(x)$ for some morphism ϕ in J. The colimit metric is then specified by*

$$d([x],[y]) = \inf \left\{ \sum_{i=0}^{k-1} d_{X_{j_i}}(x'_i, x_{i+1}) \right\} \tag{28}$$

where the infimum is taken over all tuples

$$(x'_0, x_1, x'_1, \ldots, x_{k-1}, x'_{k-1}, x_k)$$

such that $0 \leq k < \infty$; we have $x \sim x'_0$, $x_k \sim y$, and $x_i \sim x'_i$ for each i; and each pair (x'_i, x_{i+1}) belongs to one of the spaces X_{j_i} for some $j_i \in J$. □

The important point for what follows is that, when it comes to filtered diagrams in $\mathbb{N}$**Met**, this formula for the colimit metric can be simplified.

Proposition A.3. *Let $F \colon J \to \mathbb{N}\mathbf{Met}$ be a filtered diagram. Then for each $[x], [y] \in \mathrm{colim}_J X_j$ we have*

$$d([x],[y]) = \min \left\{ n \;\middle|\; \begin{array}{l} \text{there exist } j \in J \text{ and } x', y' \in X_j \text{ such that} \\ {[x] = [x'], [y] = [y'] \text{ and } d_{X_j}(x', y') = n} \end{array} \right\}.$$

Proof. Given *any* small diagram $J \to \mathbb{N}\mathbf{Met}$, the colimit metric is defined as in Lemma A.2, but since the metric in each space X_j is valued in $\mathbb{N} \cup \{+\infty\}$, the defining infimum is actually a minimum. That is, if $d([x],[y]) = n$ in the colimit, there must exist a tuple

$$(x = x_0 \sim x'_0, x_1 \sim x'_1, \ldots, x_{k-1} \sim x'_{k-1}, x_k \sim x'_k = y) \tag{29}$$

such that each pair (x'_i, x_{i+1}) belongs to some X_{j_i}, and $\sum d_{X_{j_i}}(x'_{i-1}, x_i) = n$. We claim that if J is filtered, then there is a tuple with $k = 1$ that will do the job.

By the formula for filtered colimits in **Set** [44, Section 04AX], the equivalences in (29) are witnessed by morphisms

$$\begin{tikzcd}
j_0 \arrow[dr, "\phi_{01}"'] & & j_1 \arrow[dl, "\psi_{01}"] \arrow[dr, "\phi_{01}"'] & & \cdots & j_{k-1} \arrow[dr, "\phi_{k-1,k}"'] & & j_k \arrow[dl, "\psi_{k-1,k}"] \\
& j_{01} & & j_{12} & \cdots & & j_{k-1,k} &
\end{tikzcd}$$

in J such that $F(\phi_{i,i+1})(x_i) = F(\psi_{i,i+1})(x'_i)$ for each i. As J is filtered, this diagram has a cocone: we can find an object $j \in J$ and morphisms $\phi_i \colon j_i \to j$ for $i = 0, \dots, k$ such that, for each i,

$$F(\phi_i)(x_i) = F(\phi_{i+1})(x'_i). \tag{30}$$

Then $[x] = [F(\phi_0)(x)]$ and $[y] = [F(\phi_k)(y)]$; I claim that $d(F(\phi_0)(x), F(\phi_k)(y)) = n$ in the space X_j. Indeed, we have

$$\begin{aligned}
n = d([x],[y]) &\le d_{X_j}(F(\phi_0)(x), F(\phi_k)(y)) \\
&\le \sum_{i=0}^{k-1} d_{X_j}(F(\phi_i)(x_i), F(\phi_{i+1})(x_{i+1})) \\
&= \sum_{i=1}^{k} d_{X_j}(F(\phi_i)(x'_{i-1}), F(\phi_i)(x_i)) \\
&\le \sum_{i=1}^{k} d_{X_{j_i}}(x'_{i-1}, x_i) = n,
\end{aligned}$$

where the first inequality holds by definition of the colimit metric and the second by the triangle inequality in X_j; the equality in line three follows from equation (30); and the final inequality holds because each $F(\phi_i)$ is a short map. □

A.2. **The filtered nerve as a colimit of representables.** The nerve functor extends naturally to a functor $\mathcal{N}_*(-) \colon \mathbb{N}\mathbf{Met} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$, with

$$\mathcal{N}_k(X) = \{(x_0, \dots, x_k) \mid x_i \in X \text{ and } d(x_i, x_{i+1}) < \infty \text{ for each } i\}.$$

This simplicial set carries a filtration by length, as described in Remark 2.1: for each ℓ and k, we define

$$F_\ell \mathcal{N}_k(X) = \left\{ (x_0, \dots, x_k) \;\middle|\; \sum_{i=0}^{k-1} d(x_i, x_{i+1}) \le \ell \right\}.$$

We are going to exhibit the functor $F_\ell \mathcal{N}_k(-) \colon \mathbb{N}\mathbf{Met} \to \mathbf{Set}$ as a certain colimit of representables. Later we will see that each of the representables involved is finitary, and it will follow that each piece of the filtered nerve is too.

The strategy of this section, and in particular the proof of Proposition A.6, is guided by the proof of Proposition D.3.5 in [41].

Definition A.4. For each $k, \ell \in \mathbb{N}$, let

$$\mathbf{Path}_{k,\le\ell} = \left\{ \mathbf{d} = (d_1, \dots, d_k) \in \mathbb{N}^k \;\middle|\; \sum_{i=1}^{k} d_i \le \ell \right\}.$$

We equip this set with the partial order inherited from the product order on $\mathbb{N}^k$; that is, $\mathbf{c} \leq \mathbf{d}$ if and only if $c_i \leq d_i$ for $i = 1, \ldots, k$.

Regarded as a category in the standard way, $\mathbf{Path}_{k,\leq\ell}$ has an arrow $\mathbf{c} \to \mathbf{d}$ if and only if $\mathbf{c} \leq \mathbf{d}$. For each $k, \ell \in \mathbb{N}$ there is a functor $P\colon \mathbf{Path}_{k,\leq\ell}^{\mathrm{op}} \to \mathbb{N}\mathbf{Met}$ taking the tuple $\mathbf{d} = (d_1, \ldots, d_k)$ to the generalized metric space with $k+1$ points $p_0, \ldots, p_k$ such that

- for $0 \leq i \leq j \leq k$ we have $d(p_i, p_j) = \sum_{k=i+1}^{j} d_k$, and
- for $0 \leq j < i \leq k$ we have $d(p_i, p_j) = \infty$.

(Such a space was depicted in (26).) Given $\mathbf{c} \leq \mathbf{d}$ in $\mathbf{Path}_{k,\leq\ell}$, the induced short map $P(\mathbf{d}) \to P(\mathbf{c})$ is determined by $p_i \mapsto p_i$. This functor is faithful and injective on objects, but not full.

Definition A.5. The spaces in the image of the functor $P\colon \mathbf{Path}_{k,\leq\ell}^{\mathrm{op}} \to \mathbb{N}\mathbf{Met}$ will be called *straight k-paths of length ℓ*, or just *straight paths*. Hereafter we will denote the metric space $P(\mathbf{d})$ by $\mathbf{d}$.

The functor $P\colon \mathbf{Path}_{k,\leq\ell}^{\mathrm{op}} \to \mathbb{N}\mathbf{Met}$ induces a functor

$$\begin{aligned} \text{よ}_P\colon \mathbf{Path}_{k,\leq\ell} &\to [\mathbb{N}\mathbf{Met}, \mathbf{Set}] \\ \mathbf{d} &\mapsto \mathrm{Hom}(\mathbf{d}, -). \end{aligned}$$

Note that this diagram is cofiltered—as $\mathbf{Path}_{k,\leq\ell}$ has an initial object, namely the tuple $(0, \ldots, 0)$—but not filtered (except, trivially, when $k = 0$).

Proposition A.6. *For each $k, \ell \in \mathbb{N}$ there is a natural bijection*

$$\mathrm{colim}_{\text{よ}_P} \mathrm{Hom}(\mathbf{d}, X) \xrightarrow{\cong} F_\ell \mathcal{N}_k(X)$$

specified by $[\mathbf{d} \xrightarrow{f} X] \mapsto (f(p_0), \ldots, f(p_k))$.

Proof. As $\mathbf{Set}$ is cocomplete, colimits in $[\mathbb{N}\mathbf{Met}, \mathbf{Set}]$ can be computed pointwise. So, fix an object $X \neq \emptyset$ in $\mathbb{N}\mathbf{Met}$. (For the empty space the proposition is trivial.)

First, let $k = 0$. For every ℓ, the poset $\mathbf{Path}_{0,\leq\ell}$ has a unique element—the unique element of the empty product $\mathbb{N}^0$—and the functor $P\colon \mathbf{Path}_{0,\leq\ell}^{\mathrm{op}} \hookrightarrow \mathbb{N}\mathbf{Met}$ picks out the singleton space, $\bullet$. Thus,

$$F_\ell \mathcal{N}_0(X) = \{(x_0) \mid x_0 \in X\} \cong \mathrm{Hom}(\bullet, X) = \mathrm{colim}_{\text{よ}_P} \mathrm{Hom}(\mathbf{d}, X).$$

For the remainder of the proof we assume that $k > 0$.

By the general formula for colimits in $\mathbf{Set}$, we have

$$\mathrm{colim}_{\text{よ}_P} \mathrm{Hom}(\mathbf{d}, X) = \coprod_{\mathbf{d} \in \mathbf{Path}_{k,\leq\ell}} \mathrm{Hom}(\mathbf{d}, X) / \sim$$

where $\sim$ is the equivalence relation generated by setting $(f\colon \mathbf{c} \to X) \sim (g\colon \mathbf{d} \to X)$ if $\mathbf{c} \leq \mathbf{d}$ and $g = f \circ h$ for the induced short map $h\colon \mathbf{d} \to \mathbf{c}$. That is, $f \sim g$ in the generating relation if and only if $g(p_i) = f(h(p_i)) = f(p_i)$ for each $i = 1, \ldots, k$. It follows that there is a well defined function from the colimit to $F_\ell \mathcal{N}_k(X)$ taking an equivalence class $[f]$ to the tuple

$$\mathrm{Im}(f) = (f(p_0), \ldots, f(p_k))$$

of points in X.

In the other direction, given a tuple $\mathbf{x} = (x_0, \ldots, x_k)$ of points in X such that $\sum d(x_i, x_{i+1}) \leq \ell$, we can consider the straight k-path

$$s(\mathbf{x}) = (d(x_0, x_1), \ldots, d(x_{k-1}, x_k)) \in \mathbf{Path}_{k,\leq \ell}$$

and the short map $g\colon s(\mathbf{x}) \to X$ specified by $g(p_i) = x_i$. By design, $\mathrm{Im}(s(\mathbf{x})) = \mathbf{x}$. Moreover, any short map $f\colon \mathbf{d} \to X$ factors through the straight k-path

$$s(\mathrm{Im}(f)) = (d(f(p_0), f(p_1)), \ldots, d(f(p_{k-1}), f(p_k))),$$

since $d(f(p_{i-1}), f(p_i)) \leq d(p_{i-1}, p_i) = d_i$ for each i. Thus, $[s(\mathrm{Im}(f))] = [f]$. □

A.3. **Every page of the MPSS is a finitary functor.** Having exhibited each functor $F_\ell \mathbb{N}_k(-)$ as a particular colimit of representables, the goal is to show that each of the representing objects is finitely presentable.

Proposition A.7. *Every straight path is finitely presentable in* $\mathbb{N}\mathbf{Met}$.

Proof. For each natural number $n \geq 1$, let $\vec{n}$ denote the space with two points a, b such that $d(a, b) = n$ and $d(b, a) = \infty$. Each straight path $\mathbf{d} = (d_1, \ldots, d_k)$ can be built by a finite sequence of pushouts in $\mathbb{N}\mathbf{Met}$ of the objects $\vec{d_1}, \ldots, \vec{d_k}$ over the singleton space $\bullet$. As the class of finitely presentable objects is closed under finite colimits, it will suffice to show that $\bullet$ and each space $\vec{n}$ is finitely presentable.

Let J be a filtered category and $F\colon J \to \mathbb{N}\mathbf{Met}$ a functor. Write $X_j = F(j)$ for each $j \in J$. The bijection

$$\mathrm{colim}_J \mathrm{Hom}(\bullet, X_j) \cong \mathrm{Hom}(\bullet, \mathrm{colim}_J X_j)$$

is immediate from the construction of colimits in $\mathbb{N}\mathbf{Met}$ (Lemma A.2). Meanwhile, for each $n \geq 1$, the universal property of the colimit provides a function

$$\begin{aligned} u\colon \mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j) &\to \mathrm{Hom}\,(\vec{n}, \mathrm{colim}_J(X_j)) \\ [(x, y)_j] &\mapsto ([x_j], [y_j]). \end{aligned}$$

The goal is to prove that this is a bijection.

First, observe that if $x, x' \in X_i$ and $y, y' \in X_j$ are such that $d(x, x') \leq n$ and $d(y, y') \leq n$, and we have $[x] = [y]$ and $[x'] = [y']$ in $\mathrm{colim}_J(X_j)$, then $[(x, x')] = [(y, y')]$ in $\mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j)$. Indeed, the relations $x \sim y$ and $x' \sim y'$ are witnessed by maps

$$\begin{tikzcd} i \arrow[dr, "\phi_{ik}"'] & & j \arrow[dl, "\phi_{jk}"] \arrow[dr, "\phi_{jk'}"'] & & i \arrow[dl, "\phi_{ik'}"] \\ & k & & k' & \end{tikzcd}$$

in J such that $F(\phi_{ik})(x) = F(\phi_{jk})(y)$ and $F(\phi_{ik'})(x') = F(\phi_{jk'})(y')$. As J is filtered, this diagram has a cocone; let p be its apex and $i \xrightarrow{\phi} p \xleftarrow{\psi} j$ two of its legs. A brief diagram chase shows that $F(\phi)(x) = F(\psi)(y)$ and $F(\phi)(x') = F(\psi)(y')$ in X_p, which says that (x, x') and (y, y') coincide under the induced maps

$$\mathrm{Hom}(\vec{n}, \phi)\colon \mathrm{Hom}(\vec{n}, X_i) \to \mathrm{Hom}(\vec{n}, X_p) \leftarrow \mathrm{Hom}(\vec{n}, X_j) : \mathrm{Hom}(\vec{n}, \psi).$$

Hence, $[(x, x')] = [(y, y')]$.

Now, consider the map $u\colon \mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j) \to \mathrm{Hom}\left(\vec{n}, \mathrm{colim}_J(X_j)\right)$. To define a function in the other direction, take $([x_j],[x_{j'}]) \in \mathrm{Hom}\left(\vec{n}, \mathrm{colim}_J(X_j)\right)$. By Proposition A.3, the distance $d([x_j],[x_{j'}])$ is attained by some pair x, x' in one of the spaces X_i such that $[x_j] = [x]$ and $[x_{j'}] = [x']$. If $x, x' \in X_i$ and $y, y' \in X_k$ both attain the distance, then $[x] = [x_j] = [y]$ and $[x'] = [x_{j'}] = [y']$ in $\mathrm{colim}_J(X_j)$, so by the previous paragraph we have $[(x,x')] = [(y,y')]$ in $\mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j)$. It follows that there is a well defined function

$$
\begin{aligned}
r\colon \mathrm{Hom}\left(\vec{n}, \mathrm{colim}_J(X_j)\right) &\to \mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j)\\
([x_j],[x_{j'}]) &\mapsto [(x,x')_i]
\end{aligned}
$$

where the pair $x, x' \in X_i$ has been chosen to attain the distance $d([x_j],[x_{j'}])$.

To see this is a bijection, take any $([x_j],[x_{j'}]) \in \mathrm{Hom}(\vec{n}, \mathrm{colim}_J(X_j))$. Then

$$
(u \circ r)([x_j],[x_{j'}]) = u([(x,x')]) = ([x],[x'])
$$

where, by definition of the map r, we have $[x] = [x_j]$ and $[x'] = [x_{j'}]$. Thus, $u \circ r = \mathrm{Id}$. Meanwhile, given any $[(x,x')] \in \mathrm{colim}_J \mathrm{Hom}(\vec{n}, X_j)$ we have

$$
(r \circ u)([(x,x')]) = r([x],[x']) = [(y,y')]
$$

where, by definition of the map r, we have $[y] = [x]$ and $[y'] = [x']$. It follows that $[(x,x')] = [(y,y')]$, and hence that $r \circ u = \mathrm{Id}$. $\square$

Remark A.8. The natural analogue of Proposition A.7 does not hold in **Met**, where the infimum defining the metric on a colimit need not be attained. This can be illustrated by the following example, based on Remark 2.5 of [43]. Fix $n \in \mathbb{N}_{>0}$, and for each $m \in \mathbb{N}_{>0}$ let X_m denote the space with two points a, b such that $d(a,b) = n + \frac{1}{m}$ and $d(b,a) = +\infty$. There is a filtered diagram $X_1 \to X_2 \to X_3 \to \cdots$ in **Met**, in which every map is the identity on points. This diagram has as its colimit the space $\vec{n}$, but the identity on $\vec{n}$ does not factor through any of the colimit maps $X_m \to \vec{n}$. Thus, $\vec{n}$ is not finitely presentable.

Proposition A.9. *For each $k, \ell \in \mathbb{N}$, the functor $F_\ell \mathcal{N}_k(-)\colon \mathbb{N}\mathbf{Met} \to \mathbf{Set}$ is finitary.*

Proof. For any filtered diagram $I \to \mathbb{N}\mathbf{Met}$ we have

$$
\begin{aligned}
F_\ell \mathcal{N}_k(\mathrm{colim}_I(X_i)) &\cong \mathrm{colim}_{\text{よ}P} \mathrm{Hom}(\mathbf{d}, \mathrm{colim}_I(X_i))\\
&\cong \mathrm{colim}_{\text{よ}P} \left(\mathrm{colim}_I \mathrm{Hom}(\mathbf{d}, X_i)\right)\\
&\cong \mathrm{colim}_I \left(\mathrm{colim}_{\text{よ}P} \mathrm{Hom}(\mathbf{d}, X_i)\right)\\
&\cong \mathrm{colim}_I (F_\ell \mathcal{N}_k(X_i)),
\end{aligned}
$$

where the first and last isomorphisms are given by Proposition A.6, the second by Proposition A.7, and the third by the commuting of colimits with colimits. $\square$

With this, we can complete the proof.

Proof of Proposition 7.4. As homology commutes with filtered colimits in $\mathbf{Mod}_R$, it is enough to prove the statement for $r = 1$, i.e. that the magnitude homology functor $\mathrm{MH}_{k\ell}(-) = E^1_{\ell,k-\ell}(-)\colon \mathbf{DiGraph} \to \mathbf{Mod}_R$ preserves filtered colimits.

The free R-module functor, being a left adjoint, preserves all small colimits; together with Proposition A.9 and Lemma A.1, this implies that

$$R \cdot F_\ell \mathcal{N}_k(-)\colon \mathbf{DiGraph} \to \mathbf{Mod}_R$$

preserves filtered colimits. For a given directed graph X, the R-module of *unnormalized* magnitude chains [37, Definition 5.7] in bidegree (k, ℓ) is the quotient

$$\widetilde{\mathrm{MC}}_{k\ell}(X) = \frac{R \cdot F_\ell \mathcal{N}_k(X)}{R \cdot F_{\ell-1}\mathcal{N}_k(X)}.$$

Since colimits commute with colimits, it follows that $\widetilde{\mathrm{MC}}_{k\ell}(-)$ preserves filtered colimits. Since $\mathrm{MH}_{k\ell}(-) \cong H_k(\widetilde{\mathrm{MC}}_{k\ell}(-))$ [37, Remark 5.11] and homology commutes with filtered colimits, this completes the proof. □

Remark A.10. Since the filtered nerve extends naturally from **DiGraph** to $\mathbb{N}\mathbf{Met}$, the same is true for the magnitude-path spectral sequence. Our proof of Proposition 7.4 shows, in fact, that for each $r \geq 1$ and $p, q \in \mathbb{Z}$, the functor

$$E^r_{pq}(-) : \mathbb{N}\mathbf{Met} \to \mathbf{Mod}_R$$

preserves filtered colimits.

Magnitude homology can be extended further still: it is defined for arbitrary generalized metric spaces (and indeed for a much broader class of enriched categories; see [37]). However, magnitude homology is *not* finitary as a functor on **Met**. To see this, consider the filtered diagram described in Remark A.8. For each $m \in \mathbb{N}_{>0}$ we have $\mathrm{MH}_{1,\ell}(X_m) = R$ when $\ell = n + \frac{1}{m}$ and 0 otherwise, so that $\mathrm{colim}_{\mathbb{N}_{>0}}\mathrm{MH}_{1,\ell}(X_m)$ vanishes for every ℓ. On the other hand $\mathrm{colim}_{\mathbb{N}_{>0}} X_m = \vec{n}$, and $\mathrm{MH}_{1,n}(\vec{n}) = R \neq 0$.

Thus, magnitude homology is finitary on the 'combinatorial' category **DiGraph**, but is no longer finitary after extending to the 'continuous' category **Met**. This is analogous to the situation for topological spaces: singular homology is finitary on the 'combinatorial' category of simplicial complexes and simplicial maps (because there it can be computed using simplicial homology, for which the finitary property can be proved directly), but fails to be finitary on the 'continuous' category of topological spaces and continuous maps. To see that singular homology is not finitary on spaces, one can consider for example the system of circles $\{\mathbb{R}/\frac{1}{n}\mathbb{Z}\}_{n\in\mathbb{N}}$, indexed by the natural numbers under divisibility, and equipped with the evident quotient maps $\mathbb{R}/\frac{1}{m}\mathbb{Z} \to \mathbb{R}/\frac{1}{n}\mathbb{Z}$ whenever $m|n$. In this case the colimit of the circles is the indiscrete space $\mathbb{R}/\mathbb{Q}$, which has vanishing first singular homology, while the colimit of the first singular homology groups of the $\mathbb{R}/\frac{1}{n}\mathbb{Z}$ is $\mathbb{Q}$.

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