# Bredon motivic cohomology of the real numbers 

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#### Abstract

Over the real numbers with $\mathbb{Z} / 2$-coefficients, we compute the $C_{2}$-equivariant Borel motivic cohomology ring, the Bredon motivic cohomology groups and prove that Bredon motivic cohomology of reals is a proper subring in $R O\left(C_{2} \times C_{2}\right)$-graded Bredon cohomology of a point.

This generalizes Voevodsky's computation of the motivic cohomology ring of the real numbers to the $C_{2}$-equivariant setting. These computations are extended afterwards to any real closed field.


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## 1 Introduction

A fundamental principle of modern homotopy theory is that group actions on homotopical objects reveal interesting and otherwise hard to find information about the underlying homotopical objects. Given the importance of motivic stable homotopy theory and its relations with equivariant stable homotopy theory (see for example [3], [2]), it is therefore important to study equivariant motivic stable homotopy theory (see for example [15], [20], [5] [17]). Bredon motivic cohomology, given by the equivariant study of Voevodsky's motivic cohomology spectrum, was introduced in [19] and [20], and belongs to a larger group of $C_{2}$-motivic invariants, such as Hermitian K-theory or motivic real cobordism. Bredon motivic cohomology also appears as the zero slice of the equivariant motivic sphere [12].

Concrete computations in Bredon motivic cohomology are essential for applications of the theory to other motivic and topological invariants. In many cases, these computations shed new light on the well-known computations of classical motivic cohomology, and, as in the case of Bredon cohomology, they are more difficult and contain more information about the underlying object, even in the case of a trivial $C_{2}$-action.

In [21], the second author together with J.Heller and P.A.Østvær computed completely the Bredon motivic cohomology rings of the complex numbers and of $\mathbf{E} C_{2}$ (over the complex numbers). In this paper, we compute the Bredon motivic cohomology of the real numbers and of $\mathbf{E} C_{2}$ (over the real numbers). In particular, we generalize the classical motivic cohomology computation of Voevodsky of the motivic cohomology of the real numbers by computing the Bredon motivic cohomology groups of real numbers and by showing that the Bredon motivic cohomology ring of the real numbers is a subring of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point. Moreover, we show that the Borel motivic cohomology ring of the real numbers is a Laurent/polynomial series over the $R O\left(C_{2}\right)$-graded Bredon cohomology of a point, generalizing and shedding new light on Voevodsky's computation of the motivic cohomology of $\mathbf{B} C_{2}$ (over the reals). Relating these motivic invariants to their equivariant topological counterparts through realization we also obtain previously unknown results in equivariant topology, especially about the periodicity of $R O\left(C_{2} \times \Sigma_{2}\right)$ graded Bredon cohomology of $E_{\Sigma_{2}} C_{2}$.

One of the advantages of the computation in the complex case when compared with our paper is that the one dimensional $R O\left(C_{2}\right)$-graded cohomology of a point (computed originally by Stong and used in [21]) is much simpler than the higher dimensional $R O\left(C_{2} \times \Sigma_{2}\right)$-graded cohomology of point (computed by Holler and Kriz in [13] and used in this paper). This makes most of arguments in [21] not to extend to our case. Another difficulty in the real case as opposed to the complex case is that the $C_{2} \times \Sigma_{2}$ topological isotropy sequence that is needed here is more complicated than its complex counterpart and to our knowledge, not previously studied. Both computations in the cases of real and complex numbers (and therefore, by the usual rigidity theory, of real closed fields and algebraically closed fields of characteristic zero) are part of the understanding of the hard to compute and largely unknown Bredon motivic cohomology ring of an arbitrary field as well as of the $C_{2}$-equivariant motivic Steenrod algebra of cohomology operations.

Our computations are organized via modules over the $R O\left(C_{2}\right)$-graded Bredon cohomology ring of a point. Before presenting our computations, we recall this ring and introduce some notation used to explain our results. We also recall some basics of equivariant motivic homotopy theory and Bredon motivic cohomology.

### 1.1 Equivariant motivic homotopy theory

The stable equivariant motivic homotopy category $\mathrm{SH}^{C_{2}}(k)$ is the stabilization of Voevodsky's category of equivariant motivic spaces [5], with respect to Thom spaces of representations. We recall a few key facts and the notation that we use in the case where $G=C_{2}$. See [14] or [20] for details.

Let $V=a+p \sigma$ be a $C_{2}$-representation, where $a$ denotes the $a$-dimensional trivial representation and $p \sigma$ is the $p$-dimensional sign representation. We write $\mathbb{A}(V)$ and $\mathbb{P}(V)$ for the $C_{2}$-schemes $\mathbb{A}^{\operatorname{dim}(V)}$ and $\mathbb{P}^{\operatorname{dim}(V)-1}$ equipped with the corresponding action coming from $V$. The associated motivic representation sphere is

$$
T^{V}:=\mathbb{P}(V \oplus 1) / \mathbb{P}(V)
$$

Indexing is based on the following four spheres. There are two topological spheres $S^{1}, S^{\sigma}$ and two algebro-geometric spheres $S_{t}=\left(\mathbb{A}^{1} \backslash\{0\}, 1\right)$ equipped with trivial action, and $S_{t}^{\sigma}=\left(\mathbb{A}^{1} \backslash\{0\}, 1\right)$ equipped the $C_{2}$-action $x \rightarrow x^{-1}$. We write

$$
S^{a+p \sigma, b+q \sigma}:=S^{a-b} \wedge S^{(p-q) \sigma} \wedge S_{t}^{b} \wedge S_{t}^{q \sigma}
$$

In this indexing, we have $T \simeq S^{2,1}$ and $T^{\sigma} \simeq S^{2 \sigma, \sigma}$. The stable equivariant motivic homotopy category $\mathrm{SH}^{C_{2}}(k)$ is the stabilization of (based) $C_{2}$-motivic spaces with respect to the motivic sphere $T^{\rho}$ corresponding to the regular representation $\rho=1+\sigma$.

We make use of two fundamental cofiber sequences in $\mathrm{SH}^{C_{2}}(k)$. The first is

$$
\begin{equation*}
C_{2+} \rightarrow S^{0} \rightarrow S^{\sigma} \tag{1.1}
\end{equation*}
$$

The second is

$$
\begin{equation*}
\mathbf{E} C_{2+} \rightarrow S^{0} \rightarrow \widetilde{\mathbf{E}} C_{2} \tag{1.2}
\end{equation*}
$$

Here, $\mathbf{E} C_{2}$ is the universal free motivic $C_{2}$-space. It has a geometric model, $\mathbf{E} C_{2} \simeq \operatorname{colim}_{n} \mathbb{A}(n \sigma) \backslash\{0\}$, see [9, Section 3]. The quotient $\mathbf{E} C_{2} / C_{2} \simeq \operatorname{colim}_{n}(\mathbb{A}(n \sigma) \backslash\{0\}) / C_{2}$ is the geometric classifying space $\mathbf{B} C_{2}$ constructed by Morel-Voevodsky [26] and Totaro [28]. Note that $\widetilde{\mathbf{E}} C_{2}=\operatorname{colim}_{n} S^{2 n \sigma, n \sigma}$. In particular, the maps $S^{0} \rightarrow T^{\sigma}$ and $S^{0} \rightarrow S^{\sigma}$ induce equivalences

$$
\widetilde{\mathbf{E}} C_{2} \xrightarrow{\simeq} T^{\sigma} \wedge \widetilde{\mathbf{E}} C_{2} \text { and } \widetilde{\mathbf{E}} C_{2} \xrightarrow{\simeq} S^{\sigma} \wedge \widetilde{\mathbf{E}} C_{2},
$$

see [20, Proposition 2.9].
Equipping a variety with the trivial action yields an embedding $\mathrm{Sm}_{k} \rightarrow \mathrm{Sm}_{k}^{C_{2}}$ which induces a functor $\mathrm{SH}(k) \rightarrow \mathrm{SH}^{C_{2}}(k)$.

### 1.2 Bredon motivic cohomology

Bredon motivic cohomology is represented in $\mathrm{SH}^{C_{2}}(k)$ by the spectrum $M \underline{A}$ associated to an abelian group $A$, where $M \underline{A}_{n}=A_{t r, C_{2}}\left(T^{n \rho}\right)$ is the free presheaf with equivariant transfers, see [20] for details. Here $k$ is an arbitrary field and $\rho$ denotes the $C_{2}$ regular representation $k\left[C_{2}\right]$.
Definition 1.3 ([20]). The Bredon motivic cohomology of a motivic $C_{2}$-spectrum $E$ with coefficients in an abelian group $A$ is defined by

$$
\widetilde{H}_{C_{2}}^{a+p \sigma, b+q \sigma}(E, \underline{A})=\left[E, S^{a+p \sigma, b+q \sigma} \wedge M \underline{A}\right]_{\mathrm{SH}^{C_{2}}(k)}
$$

If $X \in \mathrm{Sm}_{k}^{C_{2}}$ we typically write

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \underline{A}):=\widetilde{H}_{C_{2}}^{a+p \sigma, b+q \sigma}\left(X_{+}, \underline{A}\right) .
$$

When $A$ is a ring, then $H_{C_{2}}^{\star, \star}(X, \underline{A})$ is a graded commutative ring by [20, Proposition 3.24]. Specifically this means that if $x \in H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \underline{A})$ and $y \in H_{C_{2}}^{c+s \sigma, d+t \sigma}(X, \underline{A})$, then

$$
x \cup y=(-1)^{a c+p s} y \cup x .
$$

Notice that when $A=\mathbb{Z} / 2$, the corresponding Bredon motivic cohomology is a commutative ring.
A few features of this theory, which we use are the following (see [20], [21]).

- If $E$ is in the image of $\mathrm{SH}(k) \rightarrow \mathrm{SH}^{C_{2}}(k)$, i.e. it has "trivial action", then there is an isomorphism in integral bidegrees with ordinary motivic cohomology,

$$
\widetilde{H}_{C_{2}}^{a, b}(E, \underline{A}) \cong \widetilde{H}^{a, b}(E, A)
$$

- If $X$ has free action, then there is an isomorphism in integral bidegrees with ordinary motivic cohomology,

$$
H_{C_{2}}^{a, b}(X, \underline{A}) \cong H^{a, b}\left(X / C_{2}, A\right)
$$

- $H_{C_{2}^{*}}^{\star, \star}\left(\mathbf{E} C_{2}, \underline{A}\right)$ is $(-2+2 \sigma,-1+\sigma)$-periodic. The periodicity is given by multiplication with an invertible element $\kappa_{2} \in H_{C_{2}}^{2 \sigma-2, \sigma-1}\left(\mathbf{E} C_{2}, \underline{A}\right)$ over real numbers. Over complex numbers, we denote the invertible element with $u \in H_{C_{2}}^{2 \sigma-2, \sigma-1}\left(\mathbf{E} C_{2}, \underline{A}\right)$.
- Over complex numbers, Borel motivic cohomology ring is

$$
H_{C_{2}}^{\star \star \star}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{M}_{2}^{C_{2}}\left[\tau_{\sigma}, u^{ \pm 1}\right]
$$

with $\tau_{\sigma} \in H_{C_{2}}^{0, \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)$ being the only nontrivial element. We write $\mathbb{M}_{n}^{C_{2}}:=H_{B r}^{*+* \sigma}(p t ., \underline{Z} / n)$.

- The additive groups of Bredon motivic cohomology of a complex numbers are given by the below diagram:


Figure 1: Regions of $H_{C_{2}}^{\star, b+q \sigma}(\mathbb{C}, \mathbb{Z} / 2)$ determined by $\mathbf{E} C_{2}$, Betti realization, and $\widetilde{\mathbf{E}} C_{2}$. The degrees of the displayed elements are $|\xi|=(-2+2 \sigma,-1+\sigma),|\mu|=$ $(0,1-\sigma),\left|\tau_{\sigma}\right|=(0, \sigma)$.

A point $(b, q)$ in the Figure 1 is given by the graded $\mathbb{Z} / 2$-vector space $\oplus_{a, p \in \mathbb{Z}^{2}} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{C}, \mathbb{Z} / 2)$. When we say that the realization is an isomorphism for a point $(b, q)$ we mean that the realization

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{C}, \mathbb{Z} / 2) \rightarrow H_{B r}^{a+p \sigma}(p t ., \mathbb{Z} / 2)
$$

is an isomorphism for any choices of $(a, p) \in \mathbb{Z}^{2}$.
The map of sites $\operatorname{Sm}_{\mathbb{R}}^{C_{2}} \rightarrow \operatorname{Top}^{C_{2} \times \Sigma_{2}}$, given by $X \rightarrow X(\mathbb{C})$, where the set of complex points is equipped with the analytic topology, extends to a functor Re: $\mathrm{SH}^{C_{2}}(\mathbb{R}) \rightarrow \mathrm{SH}^{C_{2} \times C_{2}}$ between the stable equivariant motivic homotopy category over $\mathbb{R}$ and the classical stable equivariant homotopy category. We refer to this functor as "Betti realization," or simply "realization".

The way the motivic spheres interact with the topological spheres through Betti realization is

$$
\operatorname{Re}\left(S^{a+p \sigma, b+q \sigma}\right) \simeq S^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}
$$

Here we have the following nontrivial one dimensional $C_{2} \times \Sigma_{2}$-representations: $\sigma$ ( $C_{2}$ nontrivial action for the first component), $\epsilon$ ( $C_{2}$ nontrivial action for the second component) and $\sigma \otimes \epsilon$. We have the following four maps giving four $C_{2} \times \Sigma_{2}$-one dimensional irreducible representations

$$
\mathbb{Z} / 2 \times \mathbb{Z} / 2 \rightarrow \mathbb{Z} / 2 \hookrightarrow G L_{1}(\mathbb{R})
$$

given by $\left(k_{1}, k_{2}\right) \rightarrow i k_{1}+j k_{2}$ where $i, j \in\{0,1\}$ are fixed choices. Now $\sigma$ is the choice $(i, j)=(1,0), \epsilon$ is the choice $(i, j)=(0,1)$ and $\sigma \otimes \epsilon$ is the choice $(i, j)=(1,1)$. The identity representation is given by the choice $(i, j)=(0,0)$. The way $\mathbb{Z} / 2=\{0,1\}$ embeds in $G L_{1}(\mathbb{R})$ is by sending 0 to multiplication by 1 and sending 1 to multiplication by -1 .

By [20, Theorem A.29], $\operatorname{Re}(M \underline{A}) \simeq H \underline{A}$, where $H \underline{A}$ is the equivariant Eilenberg-MacLane spectrum associated to the constant Mackey functor $\underline{A}$.

In particular, for any smooth $C_{2}$-scheme over $\mathbb{R}$ there is a realization map

$$
\operatorname{Re}: H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \underline{A}) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(X(\mathbb{C}), \underline{A})
$$

Betti realization takes the cofiber sequences (1.1) and (1.2) to the corresponding ones in $\mathrm{SH}^{C_{2} \times \Sigma_{2}}$. These are

$$
\begin{equation*}
C_{2+} \rightarrow S^{0} \rightarrow S^{\sigma} \tag{1.4}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{\Sigma_{2}} C_{2+} \rightarrow S^{0} \rightarrow \tilde{E}_{\Sigma_{2}} C_{2} \tag{1.5}
\end{equation*}
$$

The cofiber sequence 1.4 is associated to the $C_{2} \times \Sigma_{2}$ representation $\sigma$; consequently we have by symmetry two more similar topological cofiber sequences associated to the $C_{2} \times \Sigma_{2}$ representations $\epsilon$ and $\sigma \otimes \epsilon$. The same happens for the cofiber sequence 1.5; there are two other symmetric topological cofiber sequences depending on which two one dimensional $C_{2} \times \Sigma_{2}$ representations are chosen. According to [20], over the reals we have

$$
\operatorname{Re}\left(\mathbf{E} C_{2}\right)=E_{\Sigma_{2}} C_{2}
$$

where by $\Sigma_{2}$ we denote the second copy of $C_{2}$ in $C_{2} \times C_{2}$.
Here $E_{\Sigma_{2}} C_{2}$ is the $\Sigma_{2}$-equivariant universal free $C_{2}$-space. By construction we have that

$$
E_{\Sigma_{2}} C_{2}=\operatorname{colim}_{n} S(n \sigma+n \sigma \otimes \epsilon)
$$

We write $S(V)$ for the unit sphere included in the disk $D(V)$ given by any actual $C_{2} \times \Sigma_{2}$-representation $V$. For a discussion of the cofiber sequence 1.5 in more detail see Section 2. Throughout this paper, we refer to this cofiber sequence as the " $C_{2} \times \Sigma_{2}$ topological isotropy sequence".

## 1.3 $R O\left(C_{2}\right)$-graded Bredon cohomology of a point

We present our computations as modules over the $R O\left(C_{2}\right)$-graded Bredon cohomology of a point. The natural module structure is given by the fact that Betti realization induces an isomorphism of bigraded rings

$$
H_{C_{2}}^{\star, 0}(\mathbb{R}, \underline{\mathbb{Z} / n}) \cong \mathbb{M}_{n}^{C_{2}}
$$

and so $H_{C_{2}}^{\star, \star}(X, \underline{\mathbb{Z}} / n)$ is a module over $\mathrm{IM}_{n}^{C_{2}}$.
In fact, by [31], Betti realization is an isomorphism in weight zero, even with $\mathbb{Z}$-coefficients. In Section 4 , we will study a more general case $(q \in \mathbb{Z})$ than the proposition below $(q=0)$. Below, we see the reason for the above isomorphism.

Proposition 1.6. ([29]) $H^{a, b}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{a-b+b \epsilon}(p t, \mathbb{Z} / 2)$ for any $a \in \mathbb{Z}, b \geq 0$. Moreover, as rings,

$$
H^{*, *}(\mathbb{R}, \mathbb{Z} / 2)=\mathbb{Z} / 2\left[x_{2}, y_{2}\right]
$$

where the last expression is the positive cone of the Bredon cohomology of a point with deg $\left(x_{2}\right)=(1,1)$ and $\operatorname{deg}\left(y_{2}\right)=(0,1)$. The realization maps are strict monomorphisms if $b+1<a \leq 0$.

Proposition 1.7. Let $A$ be a finite abelian group and $b \geq 0$. For any $a, p \in \mathbb{Z}$, Betti realization induces an isomorphism

$$
H_{C_{2}}^{a+p \sigma, b}(\mathbb{R}, \underline{A}) \xrightarrow{\cong} H_{B r}^{a-b+p \sigma+b \epsilon}(p t, \underline{A}) .
$$

Proof. If $b \geq 0$, then $H^{a, b}(\mathbb{R}, A) \rightarrow H_{B r}^{a-b+b \epsilon}(p t, A)$ is an isomorphism for all $a \in \mathbb{Z}$ according to Voevodsky's computation in Proposition 1.6. In particular, the result holds for $p=0$. Using the comparison long exact sequence produced from the realization of the cofiber sequence 1.1 and the five lemma, the result holds for all $p$ by induction.

The $R O\left(C_{2}\right)$-graded Bredon cohomology groups of a point are shown in the diagram below. We notice that both cones of the $R O\left(C_{2}\right)$-Bredon cohomology of a point are parts of the $R O\left(C_{2}\right)$-graded Bredon cohomology of $E C_{2}$ and $\tilde{E} C_{2}$ (see for example [21]).


Figure 2: Regions of $H_{B r}^{*+* \sigma}(\mathbb{R}, \mathbb{Z} / 2)$ determined by (parts) of the Bredon cohomology of $E C_{2}$ and $\tilde{E} C_{2}$. The degrees of the displayed elements are $|\alpha|=-1+\sigma$, $|\theta|=2-2 \sigma$.

A point $(a, p)$ in Figure 2 is given by a single $\mathbb{Z} / 2$ vector space $H_{B r}^{a+p \sigma}(p t, \mathbb{Z} / 2)$. The diagonal in Figure 2 is the line $a+p=0$. The positive cone $R$ is the green region and it is a subring that can be computed as $\mathbb{Z} / 2[\sigma, \alpha]$ with $|\sigma|=\sigma$ and $|\alpha|=-1+\sigma$. The negative cone $N C$ is computed as $\mathbb{Z} / 2\left\{\frac{\theta}{\sigma^{n} \alpha^{m}}\right\}$ for $n, m \geq 0$ and $\theta$ an element divisible by $\alpha$ and $\sigma$ with the property that $\theta^{2}=\theta \alpha=\theta \sigma=0$ and $|\theta|=2-2 \sigma$. Also, the cohomology class $\alpha$ in the $R O\left(C_{2}\right)$-graded Bredon cohomology ring of $E C_{2}$ is invertible.

A detailed explanation of the above diagram is given in [21].
The $R O\left(C_{2}\right)$-graded Bredon cohomology ring of a point was computed by Stong (unpublished), but written accounts can be found in the Appendix of [4] and in Proposition 6.2 of [7]. This can be described as:

Theorem 1.8. We have a $\mathbb{M}_{2}^{C_{2}}$-algebra isomorphism

$$
H_{B r}^{*+* \sigma}(p t, \mathbb{Z} / 2) \simeq R \oplus N C=\mathbb{Z} / 2[\sigma, \alpha] \oplus \mathbb{Z} / 2\left\{\frac{\theta}{\sigma^{n} \alpha^{m}}\right\}
$$

with $R$ a subring of $H_{B r}^{*+* \sigma}(p t, \mathbb{Z} / 2)$ and $N C$ a $\mathrm{M}_{2}^{C_{2}}$-submodule with zero products.

### 1.4 Our Results

One of the main results of this paper is the additive picture of the Bredon motivic cohomology groups of the real numbers. As one can notice in Figure 3, apart from different Betti realizations in the blue region (and different computations for the graded $\mathbb{Z} / 2$-vector spaces in the green and red cone), the diagram is similar with the complex case computed in [21] and reviewed in Figure 1. This shows that in both the complex and real cases the additive groups are only determined by parts of Bredon motivic cohomologies of $\mathbf{E} C_{2}$ and $\widetilde{\mathbf{E}} C_{2}$, along with the realization.

The classical computations of the motivic cohomology of the complex numbers and real numbers (with $\mathbb{Z} / 2$-coefficients) can be understood as belonging to the blue region (more precisely to the line $q=0, b \geq 0$ ), an area where all the Betti realizations are isomorphisms and to the segment $b<0, q=0$ where the Betti realizations can be a strict monomorphism or a trivial isomorphism.


Figure 3: Regions of $H_{C_{2}}^{\star, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)$ determined by $\mathbf{E} C_{2}$, Betti realization into the $R O\left(C_{2} \times \Sigma_{2}\right.$ )-graded Bredon cohomology of a point (denoted by $H_{B r, K}^{\star}(p t$.$) ), and$ $\widetilde{\mathbf{E}} C_{2}$. The degree of the displayed element is $\left|\kappa_{2}\right|=(-2+2 \sigma,-1+\sigma)$.

A point $(b, q)$ in the Figure 1 is given by the graded $\mathbb{Z} / 2$ - vector space $\oplus_{a, p \in \mathbb{Z}^{2}} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)$. We can also view each point of the diagram as a $\mathbf{I M}_{2}^{C_{2}}$-module.

When we say that the realization is an isomorphism for a point $(b, q)$ we mean that

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is an isomorphism for any choices of $(a, p) \in \mathbb{Z}^{2}$.
In the white region of Figure 3, the Bredon motivic cohomology groups are zero. Multiplication by $\kappa_{2}$ gives the periodicity in the Bredon motivic cohomology of $\mathbf{E} C_{2}$; therefore, the multiplication by $\kappa_{2}$ is an isomorphism only in the green cone. The Bredon motivic cohomology of the real numbers is isomorphic with the Bredon motivic cohomology of $\mathbf{E} C_{2}$ in the green cone, and it is isomorphic to the reduced Bredon motivic cohomology of $\widetilde{\mathbf{E}} C_{2}$ in the red cone. The blue region includes both the computation of the motivic cohomology of $\mathbb{R}$ of V.Voevodsky [29] and the results in codimension 0,1 and $\sigma$ in the particular case $\mathbb{R}$ (or any real closed field), of the second author in [31].

Moreover, in Figure 3, we prove that the realization maps in the green cone and the red cone can be identified with maps induced by the $C_{2} \times \Sigma_{2}$-equivariant topological isotropy sequence. By studying the $C_{2} \times \Sigma_{2}$-equivariant topological isotropy sequence in sections 2.3 and 3 , we completely determine the realization maps in the green and red cones.

Based on J.Holler and I.Kriz's computation from [13], we also identify the value of the cohomology groups in Figure 3 above the line $b+q=0$. The value of the cohomology below the line $b+q=0$ can be determined from Voevodsky's computation of the motivic cohomology of $\mathbf{B} C_{2}$ over reals ([30]). Using Theorem 1.10 below (proved in Section 5), we can also compute these groups as $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology groups therefore reducing the additive computation to [13]. This unexpected link between Voevodsky's computation and $R O(G)$-graded Bredon cohomology is discussed in Remark 5.13 from Section 5 (for $G=C_{2}$ or $C_{2} \times \Sigma_{2}$ ); it is essentially a consequence of the fact that there is a lot of nontrivial information in the extra nontrivial representation indexes of Bredon motivic cohomology.

We prove the following theorem about the realization maps in Figure 3:

Theorem 1.9. The Betti realizations

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t ., \mathbb{Z} / 2)
$$

in the green cone are monomorphisms everywhere and isomorphisms if $a \leq 2 b+2$ and the Betti realizations in the blue region are isomorphisms everywhere. The realization maps in the red cone are monomorphisms everywhere.

Thus, we completely compute the Borel motivic cohomology ring of the real numbers as a proper subring of the $R O\left(C_{2} \times \Sigma_{2}\right.$ )-graded Bredon cohomology of $E_{\Sigma_{2}} C_{2}$ (which we compute in [6] and mention in section 3). The method for computing Borel motivic cohomology of the complex numbers in [21] cannot be generalized to our case, but we found a different method that computes Borel cohomology of both the real numbers and the complex numbers, therefore also reproving in a simpler and more conceptual way the results of [21]. One of the main difficulties in the case of the real numbers as opposed to the case of the complex numbers is that the $\mathbb{Z} / 2$-vector space dimensions of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology groups of a point (with $\underline{\mathbb{Z}} / 2$-coefficients) are usually higher than 1.

We prove the following theorem in Section 5:
Theorem 1.10. We have the following ring isomorphism

$$
H_{C_{2}^{\star}}^{\star, \star}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{M}_{2}^{C_{2}}\left[x_{3}, y_{3}, \kappa_{2}^{ \pm 1}\right] \hookrightarrow H_{B r}^{*+* \sigma+* \epsilon+* \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

with $x_{3}$ in bidegree $(\sigma, \sigma)$, $y_{3}$ in bidegree $(\sigma-1, \sigma)$ and $\kappa_{2}$, the invertible element, in bidegree $(2 \sigma-2, \sigma-1)$.
The realization maps

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

are isomorphic if $a \leq 2 b+2$, and injective, but not surjective if $a \geq 2 b+3$ (see Corollary 5.4). We explain in Remark 5.11 and Remark 5.13 how V.Voevodsky's computation of the motivic cohomology of $\mathbf{B} C_{2}$ [30] can be embedded, for our choices of a field, in the Borel motivic cohomology ring of a point.

We notice that the multiplication by the generator $\tau_{\sigma} \in H_{C_{2}}^{0, \sigma}\left(\mathbf{E} C_{2 \mathbb{C}}, \mathbb{Z} / 2\right)$ gives isomorphism over the complex numbers, and is therefore important in the computation of Borel motivic cohomology ring in [21]. However, in the real case this cohomology class is zero because $H_{C_{2}}^{0, \sigma}\left(\mathbf{E} C_{2 \mathbb{R}}, \mathbb{Z} / 2\right)=0$.

We computed the Bredon motivic cohomology ring of the real numbers as a subring in the $R O\left(C_{2} \times\right.$ $\Sigma_{2}$ )-graded Bredon cohomology ring of a point. We consider

$$
R=\oplus_{b \geq 0, b+q \geq 0} H_{C_{2}}^{\star, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)
$$

the subring in $H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2)$ given by the direct sum of the $\mathbb{M}_{2}^{C_{2}}$-modules in the blue region of Figure 3 (which can be considered as a subring of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point, and is discussed in the last section) and

$$
N C=\oplus_{b \geq 0, b+q<0} H_{C_{2}}^{\star, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)
$$

the $\mathrm{I}_{2}^{C_{2}}$-submodule in $H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2)$ given by the direct sum of the $\mathbb{M}_{2}^{C_{2}}$-modules in the red cone of Figure 3.

The Bredon motivic cohomology of the real numbers is computed below in terms of these objects:
Theorem 1.11. We have an isomorphism of $\mathrm{M}_{2}^{C_{2}}$-algebras

$$
H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2) \simeq\left(R, \kappa_{2}\right) \oplus N C
$$

with $\kappa_{2}$ in degree $(2 \sigma-2, \sigma-1)$ with $\left(R, \kappa_{2}\right) \hookrightarrow H_{B r}^{\star}(p t, \mathbb{Z} / 2)$ the subring of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point generated by $R$ and the cohomology class $k_{2}$ and with NC a $\mathbb{M}_{2}^{C_{2}}$-submodule having zero products.

The realization map is a monomorphism making the Bredon motivic cohomology ring of the real numbers a nontrivial proper subring of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point.

In conclusion we have the following commutative diagram of commutative rings


The horizontal maps are given by the Betti realizations and the vertical maps by the canonical inclusions.

The motivic cohomology ring of the real numbers was computed by V.Voevodsky as the positive cone of the $R O\left(C_{2}\right)$-graded Bredon cohomology of a point and it is a subring of $R$. Voevodsky's monomorphism is represented by the upper horizontal Betti realization in the above diagram.

We can conclude that Bredon and Borel motivic cohomology groups of real numbers are completely determined by the values of $R O\left(C_{2} \times \Sigma_{2}\right)$ graded Bredon cohomology groups of a point. In the same way, we can conclude that Bredon and Borel motivic cohomology groups of complex numbers are completely determined by the values of $R O\left(C_{2}\right)$ graded Bredon cohomology groups of a point.

At the end of the paper, we show that all the computations and results of this paper are valid if we replace $\mathbb{R}$ by an arbitrary real closed field.

A brief outline of the paper is as follows. Sections 1 and 2 are devoted to the introduction and preliminaries. The main computations of the Bredon motivic cohomology of a point are carried out in Sections 3, 4 and 6. In Section 5 we discuss the Bredon motivic cohomology of $\mathbf{E} C_{2}$ over the real numbers, which is usually called the Borel motivic cohomology of a point (or of real numbers).

## Notation.

- We write $X_{k}$ for a smooth scheme over $k$.
- $K:=C_{2} \times \Sigma_{2}$ is the Klein four-group. We denote by $\Sigma_{2}$ the second copy of $C_{2}$.
- $H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \underline{A})$ is the Bredon motivic cohomology of a $C_{2}$-smooth scheme, with coefficients $A$. All cohomology that appears in this paper is understood to be with $\mathbb{Z} / 2$
- $H^{n, q}(X, A)$ is the motivic cohomology of a smooth scheme $X$. We only consider the case where $A=\mathbb{Z} / 2$.
- $H_{B r}^{a+p \sigma}(X, \underline{A})$ is the Bredon cohomology of a $C_{2}$-topological space $X$ with coefficients in the constant Mackey functor $\underline{A}$ (generated by the classes $x_{1}, y_{1}, \theta_{1}$ ). If instead of $\sigma$ we write $\epsilon$, we mean the same cohomology generated by $x_{2}, y_{2}, \theta_{2}$ (viewed as being embedded in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point). We only consider the case where $\underline{A}=\underline{Z} / 2$.
- $H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}(X, \underline{A})$ is the Bredon cohomology of a $C_{2} \times \Sigma_{2}$-topological space $X$ with coefficients in the constant Mackey functor $\underline{A}$. We only consider the case where $\underline{A}=\mathbb{Z} / 2$. Sometimes we write $H_{B r, K}^{\star}(X, \underline{A})$ for the same cohomology, where $K$ is the Klein four-group.
- We have the convention that $\star$ denotes a $R O(G)$-grading, while $*$ denotes an integer grading. For example, $H_{C_{2}}^{\star, \star}(X)=\oplus_{a, b, p, q} H_{C_{2}}^{a+p \sigma, b+q \sigma}(X), H^{*, *}(X)=\oplus_{a, b} H^{a, b}(X)$ or
$H_{B r}^{*, *, *, *}(X)=\oplus_{a, b, p, q} H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}(X)$.
- We write $\underline{H}_{V}^{B r}, \underline{H}_{B r}^{V}$ for the usual Mackey functors associated to $H_{V}^{B r}$ or $H_{B r}^{V}$, with $V$ a $G$-representation.
- $S^{V}$ is the topological sphere associated to the $C_{2} \times \Sigma_{2}$ representation $V$. For example, $V$ can be $\sigma, \epsilon$ or $\sigma \otimes \epsilon$.
- All $C_{2}$-varieties are over $\mathbb{R}$, and we view $C_{2}$ as the group scheme $C_{2}=\operatorname{Spec}(\mathbb{R}) \sqcup \operatorname{Spec}(\mathbb{R})$.
- $\mathbb{M}_{n}^{C_{2}}:=H_{B r}^{*+* \sigma}(p t, \underline{\mathbb{Z}} / n)$.
- We denote by $\kappa_{2}$ the invertible element in the Bredon motivic cohomology of $\mathbf{E} C_{2}$ as well as its Betti realization in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of $E_{\Sigma_{2}} C_{2}$ (which also gives $E_{\Sigma_{2}} C_{2}$ its unique periodicity in its $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology).
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## $2 R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point

## $2.1 R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology groups of a point

The $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point with $\mathbb{Z} / 2$-coefficients was computed in [23]. The results were given in terms of Poincare series of graded $C_{2}$-spaces and reproduced also in [10] (however see the modification in Proposition 2.3 from [10]). We have the following:

Proposition 2.1. ([23]) Let $l, n \geq 0$ and $i, j \geq 0$. Let $\alpha$ and $\beta$ two 1-dimensional irreducible $C_{2} \times \Sigma_{2}$ representations. The Poincare series for $H_{B r}^{-*+V}(p t, \mathbb{Z} / 2)$ is
a) If $V=0$ then 1 .
b) If $V=n \alpha$ then $1+x+x^{2}+\ldots+x^{n}$.
c) If $V=-j \alpha$ then $x^{-n}+\ldots+x^{-3}+x^{-2}$.
d) If $V=n \alpha+j \beta$ then $\left(1+x+\ldots+x^{n}\right)\left(1+x+\ldots+x^{l}\right)$.
e) If $V=n \alpha-j \beta$ then $\left(1+x+\ldots+x^{n}\right)\left(x^{-j}+\ldots+x^{-2}\right)$.
f) If $V=-i \alpha-j \beta$ then $\left(x^{-n}+\ldots+x^{-2}\right)\left(x^{-j}+\ldots+x^{-2}\right)$.

When all three 1-dimensional irreducible representations are involved the answer is more complicated. The following is the description of the Bredon cohomology groups of a point in the positive cone.
Proposition 2.2. ([23]) Let $l, m, n \geq 0$. The Poincare series for $H_{B r}^{-*+l \alpha+m \beta+n \gamma}(p t, \mathbb{Z} / 2)$ is

$$
\left(1+x+\ldots+x^{l}\right)\left(1+x+\ldots+x^{m}\right)+x\left(1+x+\ldots+x^{l+m}\right)\left(1+\ldots+x^{n-1}\right) .
$$

The following is the description of the Bredon cohomology groups of a point in the mixed cone of type I:
Proposition 2.3. [23] Let $k, l, m \geq 1$. If $k \leq l, m$ then the Poincare series for $H_{B r}^{-*+l \alpha+m \beta-k \gamma}(p t, \mathbb{Z} / 2)$ is

$$
\left(\frac{1}{x^{k}}+\ldots+\frac{1}{x}\right)\left(1+x+\ldots+x^{k-2}\right)+x^{k}\left(1+\ldots+x^{l-k}\right)\left(1+\ldots+x^{m-k}\right)
$$

In the case $k>l$ the Poincare series for $H_{B r}^{-*+l \alpha+m \beta-k \gamma}(p t, \mathbb{Z} / 2)$ is

$$
\frac{1}{x^{l+1}}\left(1+\ldots+x^{l}\right)\left(1+\ldots+x^{l-1}\right)+\frac{1}{x^{k}}\left(1+\ldots+x^{k-l-2}\right)\left(1+\ldots+x^{l+m}\right)
$$

Swapping the role of $l$ and $m$ gives the case $k>m$.
The following is the description of the Bredon cohomology groups of a point in the mixed cone of type II:
Proposition 2.4. [23] Let $j, k, l \geq 1$. Then the Poincare series for $H_{B r}^{-*+l \alpha-j \beta-k \gamma}(p t, \mathbb{Z} / 2)$ is

$$
\frac{1}{x^{j+k-l}}\left(1+\ldots+x^{j-l-2}\right)\left(1+\ldots+x^{k-l-2}\right)+\frac{1}{x^{l+1}}\left(1+\ldots+x^{l}\right)\left(1+\ldots+x^{l-1}\right)
$$

if $j, k \geq l+1$ or

$$
\frac{1}{x^{j}}\left(1+\ldots+x^{j-2}\right)\left(1+\ldots+x^{l-k}\right)+\frac{1}{x^{k}}\left(1+\ldots+x^{l-1}\right)\left(1+\ldots+x^{k-1}\right)
$$

if $l \geq k$. Swapping the role of $j$ and $k$ gives the case $l \geq j$.
The following is the description of the Bredon cohomology groups of a point in the negative cone:
Proposition 2.5. [23] Let $i, k, j \geq 1$. Then the Poincare series for $H_{B r}^{-*-i \alpha-j \beta-k \gamma}(p t, \mathbb{Z} / 2)$ is

$$
\frac{1}{x^{i+j+k}}\left[\left(1+x+\ldots+x^{j+k-2}\right)\left(1+\ldots+x^{i-2}\right)+x^{i-1}\left(1+\ldots+x^{k-1}\right)\left(1+\ldots x^{j-1}\right)\right] .
$$

We conclude with the following vanishing propositions. From Proposition 2.4 we have
Proposition 2.6. If $a>b \geq-q>0$ then $H_{B r}^{a+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)=0$.
From Proposition 2.2 we have
Proposition 2.7. If $b, q \geq 0$ and $a \geq 1$, then $H_{B r}^{a+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)=0$.

## $2.2 R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point

In this section we describe the positive cone of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point $H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)$, as well as some cohomology classes that appear in this ring following [8].

Positive Cone; i.e. $p, b, q \geq 0$.
We have for $V$ an actual $C_{2} \times \Sigma_{2}$-representation the following equality of Mackey functors:

$$
\underline{\pi}_{p}\left(S^{V} \wedge H \underline{\mathbb{Z}} / 2\right) \cong \underline{H}_{p}^{B r}\left(S^{V} ; \underline{\mathbb{Z}} / 2\right) \cong \underline{H}_{p-V}^{B r}(p t ; \underline{\mathbb{Z}} / 2) \cong \underline{H}_{B r}^{-p+V}(p t ; \underline{\mathbb{Z}} / 2)
$$

Then we have the generators of the positive cones $\mathbb{Z} / 2\left[x_{i}, y_{i}\right], i=1,2,3$, corresponding to the three nontrivial one-dimensional $C_{2} \times C_{2}$ representations.

We denote by $\pi_{*}^{G}$ the top level of the Mackey functor given by the equivariant stable homotopy group. The computations below follow from Proposition 2.2 and Proposition 2.1.

$$
\begin{aligned}
& x_{1} \in \pi_{0}^{G}\left(S^{0,1,0,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{0,1,0,0}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& y_{1} \in \pi_{1}^{G}\left(S^{0,1,0,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{-1,1,0,0}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& x_{2} \in \pi_{0}^{G}\left(S^{0,0,1,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{0,0,1,0}(p t ; \underline{Z} / 2) \cong \mathbb{Z} / 2, \\
& y_{2} \in \pi_{1}^{G}\left(S^{0,0,1,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{-1,0,1,0}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& x_{3} \in \pi_{0}^{G}\left(S^{0,0,0,1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{0,0,0,1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& y_{3} \in \pi_{1}^{G}\left(S^{0,0,0,1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{-1,0,0,1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \theta_{1} \in \pi_{-2}^{G}\left(S^{0,-2,0,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{2,-2,0,0}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \theta_{2} \in \pi_{-2}^{G}\left(S^{0,0,-2,0} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{2,0,-2,0}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \theta_{3} \in \pi_{-2}^{G}\left(S^{0,0,0,-2} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{2,0,0,-2}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2 .
\end{aligned}
$$

with the cohomological classes given by the only non-trivial element in each of the above abelian groups. From Proposition 2.1 and Theorem 1.8, we have that

$$
H_{B r}^{*+* \sigma}(p t, \mathbb{Z} / 2)=\mathbb{Z} / 2\left[x_{1}, y_{1}\right] \oplus \mathbb{Z} / 2\left\{\frac{\theta_{1}}{x_{1}^{n_{1}} y_{1}^{m_{1}}}\right\}
$$

and

$$
H_{B r}^{*+* \epsilon}(p t, \mathbb{Z} / 2)=\mathbb{Z} / 2\left[x_{2}, y_{2}\right] \oplus \mathbb{Z} / 2\left\{\frac{\theta_{2}}{x_{2}^{n_{1}} y_{2}^{m_{1}}}\right\}
$$

or

$$
H_{B r}^{*+* \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)=\mathbb{Z} / 2\left[x_{3}, y_{3}\right] \oplus \mathbb{Z} / 2\left\{\frac{\theta_{3}}{x_{3}^{n_{1}} y_{3}^{m_{1}}}\right\}
$$

are all nontrivial proper subrings in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point.
Theorem 2.8. ([8]) The Mackey functor structure of the positive cone in $\underline{\pi}_{\star} H \underline{\mathbb{Z}} / 2$ is given by the Mackey functor of $R O\left(C_{2} \times \Sigma_{2}\right)$-graded rings

where each restriction map is the identity on a generator of the domain that is also a generator of the codomain and is zero on a generator otherwise. For example, the restriction of the $x_{1}$ in the top level is zero in $\frac{\mathbb{Z} / 2\left[y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right]}{\left(x_{2} y_{3}+y_{2} x_{3}\right)}$ and is $x_{1}$ in $\frac{\mathbb{Z} / 2\left[x_{1}, y_{1}, y_{2}, x_{3}, y_{3}\right]}{\left(x_{1} y_{3}+y_{1} x_{3}\right)}$ and $\frac{\mathbb{Z} / 2\left[x_{1}, y_{1}, x_{2}, y_{2}, y_{3}\right]}{\left(x_{1} y_{2}+y_{1} x_{2}\right)}$. The transfer maps are always zero.

To describe the other parts of the $R O\left(C_{2} \times \Sigma_{2}\right)$ Bredon cohomology ring of a point we need more cohomology classes than those described above.

The computations below follow from Proposition 2.3, Proposition 2.4 and Proposition 2.5. We have the following seven new nontrivial cohomology classes:

$$
\begin{aligned}
& \Theta \in \pi_{-3}^{G}\left(S^{0,-1,-1,-1} \wedge H \underline{\mathbb{Z}} / 2\right) \cong H_{B r}^{3,-1,-1,-1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \kappa_{1} \in \pi_{1}^{G}\left(S^{0,-1,1,1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{-1,-1,1,1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \kappa_{2} \in \pi_{1}^{G}\left(S^{0,1,-1,1} \wedge H \underline{\mathbb{Z}} / 2\right) \cong H_{B r}^{-1,1,-1,1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \kappa_{3} \in \pi_{1}^{G}\left(S^{0,1,1,-1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{-1,1,1,-1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \iota_{1} \in \pi_{-1}^{G}\left(S^{0,1,-1,-1} \wedge H \underline{\mathbb{Z}} / 2\right) \cong H_{B r}^{1,1,-1,-1}(p t ; \underline{\mathbb{Z}} / 2) \cong \mathbb{Z} / 2, \\
& \iota_{2} \in \pi_{-1}^{G}\left(S^{0,-1,1,-1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{1,-1,1,-1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2, \\
& \iota_{3} \in \pi_{-1}^{G}\left(S^{0,-1,-1,1} \wedge H \underline{\mathbb{Z} / 2}\right) \cong H_{B r}^{1,-1,-1,1}(p t ; \underline{\mathbb{Z} / 2}) \cong \mathbb{Z} / 2 .
\end{aligned}
$$

These cohomological classes satisfy the following relationships. For each $\{i, j, k\}=\{1,2,3\}$ (i.e. they are all distinct), we have

$$
\begin{aligned}
\iota_{i} \theta_{i} & =\Theta \text { and } \kappa_{i} \theta_{j}=\iota_{k}, \\
\theta_{j} \theta_{k} & \neq 0 \\
\iota_{i} \theta_{j} & =0, \\
\iota_{i} \kappa_{i} & =0, \\
\Theta^{2} & =\theta_{i} \Theta=\kappa_{i} \Theta=\iota_{i} \Theta=0, \\
\iota_{i} \iota_{j} & =0,
\end{aligned}
$$

and we can think of $\Theta$ as being divisible by $\theta_{1}, \theta_{2}$ and $\theta_{3}$, where

$$
\iota_{i}=\frac{\Theta}{\theta_{i}} \text { and } \kappa_{i}=\frac{\Theta}{\theta_{j} \theta_{k}} .
$$

Then we can think of $\Theta$ as being infinitely divisible by $x_{i}, y_{i}$ for $i=1,2,3$.
By degree reasons, we have

$$
\Theta x_{i}=\Theta y_{i}=0
$$

for all $i \in\{1,2,3\}$ and similarly

$$
\iota_{i} x_{j}=\iota_{i} y_{j}=0
$$

for all $i, j \in\{1,2,3\}$ with $i \neq j$. However, it is not true that $\kappa_{i} x_{i}=\kappa_{i} y_{i}=0$ for all $i \in\{1,2,3\}$. For each $\{i, j, k\}=\{1,2,3\}$, we have the following relations

$$
\begin{aligned}
\kappa_{i} x_{i} & =x_{j} y_{k}+y_{j} x_{k}, \\
\kappa_{i} y_{i} & =y_{j} y_{k}, \\
\kappa_{i}^{2} & \neq 0, \\
\kappa_{i} \kappa_{j} & =y_{k}^{2} .
\end{aligned}
$$

We cannot express $\kappa_{i}^{2}$ in terms of $x_{i}, y_{i}$ and $\theta_{i}$.
It is proved in [8] that the entire $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point can be expressed in terms of the above cohomology classes. See also [6] for a more explicit description of the cohomology classes in this ring.

### 2.3 Topological $C_{2} \times \Sigma_{2}$-isotropy cofiber sequence

We define $E_{\Sigma_{2}} C_{2}$ to be the $\Sigma_{2}$-equivariant universal free $C_{2}$-space (see [[22],VII.1]). By construction we have that

$$
E_{\Sigma_{2}} C_{2}=\operatorname{colim}_{n} S(n \sigma+n \sigma \otimes \epsilon)
$$

and we have a cofiber sequence

$$
S(n \sigma+n \sigma \otimes \epsilon)_{+} \rightarrow p t_{+} \rightarrow S^{n \sigma+n \sigma \otimes \epsilon}
$$

which, by taking colimits, gives the following topological isotropy $C_{2} \times \Sigma_{2}$-sequence

$$
E_{\Sigma_{2}} C_{2+} \rightarrow p t_{+} \rightarrow \tilde{E}_{\Sigma_{2}} C_{2} .
$$

According to [20] we have that $E_{\Sigma_{2}} C_{2}$ is the topological realization of $\mathbf{E} C_{2}$. This implies that the Bredon cohomology of the $C_{2} \times \Sigma_{2}$-space $E_{\Sigma_{2}} C_{2}$ has a $-1+\sigma-\epsilon+\sigma \otimes \epsilon$ periodicity because the Bredon motivic cohomology of $\mathbf{E} C_{2}$ is $(2 \sigma-2, \sigma-1)$ periodic [[20], Theorem 5.4]. According to the topological realization, we also have that $\tilde{E}_{\Sigma_{2}} C_{2}$ has, in its reduced Bredon cohomology, periodicities $\sigma$ and $\sigma \otimes \epsilon$ because the reduced Bredon motivic cohomology of $\widetilde{\mathbf{E}} C_{2}$ is $(0, \sigma)$ and $(\sigma, 0)$ periodic [[20], Proposition 5.7]. This is because $\tilde{E}_{\Sigma_{2}} C_{2}$ is the Betti realization of $\widetilde{\mathbf{E}} C_{2}$. The above isotropy cofiber sequence is given by the realization of the motivic isotropy sequence

$$
\mathbf{E} C_{2+} \rightarrow p t_{+} \rightarrow \widetilde{\mathbf{E}} C_{2}
$$

obtained from taking the colimits over the cofiber sequence

$$
(\mathbb{A}(n \sigma) \backslash\{0\})_{+} \rightarrow p t_{+} \rightarrow T^{n \sigma}=S^{\sigma} \wedge S_{t}^{\sigma}
$$

We prove the following vanishing theorem for the above topological $C_{2} \times \Sigma_{2}$-isotropy sequence:
Theorem 2.9. The $H_{B r}^{\star}(p t, \mathbb{Z} / 2)$-module map induced by the above topological isotropy sequence

$$
\tilde{H}_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right) \rightarrow H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is the zero map for all $a, b \in \mathbb{Z}$ and $p, q \geq 0$. The $H_{B r}^{\star}(p t, \mathbb{Z} / 2)$-module map

$$
\tilde{H}_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right) \rightarrow H_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is the zero map for all $a$ and $p, q$ not both negative or $a \leq 3$ and $p, q$ are arbitrary.
Proof. Let $V=a+b \epsilon$ and take $W=a^{\prime}+b^{\prime} \epsilon$ with $a^{\prime}, b^{\prime} \geq 0$ such that $V+W$ is an actual $\Sigma_{2^{-}}$ representation. Denote by $F(X, Y)$ the $\Sigma_{2}$-space of nonequivariant maps with the conjugacy action for two pointed $\Sigma_{2}$-spaces $X$ and $Y$. We study the map

$$
\widetilde{H}_{B r}^{V}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=\left[\tilde{E}_{\Sigma_{2}} C_{2}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K} \rightarrow \widetilde{H}_{B r}^{V}\left(p t_{+}\right)=\left[p t_{+}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K}
$$

induced by the isotropy map $p t_{+} \rightarrow \tilde{E}_{\Sigma_{2}} C_{2}$. But this last map factors through the map

$$
p t_{+} \hookrightarrow S^{\sigma} \hookrightarrow S^{\sigma+\sigma \otimes \epsilon} \hookrightarrow \tilde{E}_{\Sigma_{2}} C_{2}
$$

so the above cohomology map factors through $\left[S^{\sigma}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K}$. But the target has trivial $C_{2}$-action so the cohomology map factors through $\left[S^{\sigma} / C_{2}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K}$. But $S^{\sigma} / C_{2}=I$, which is contractible, implying that

$$
\begin{gathered}
\widetilde{H}_{B r}^{V}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=\left[\tilde{E}_{\Sigma_{2}} C_{2}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K} \rightarrow \widetilde{H}_{B r}^{V}(I)=\left[S^{\sigma} / C_{2}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K}=0 \rightarrow \\
\rightarrow \widetilde{H}_{B r}^{V}\left(p t_{+}\right)=\left[p t_{+}, F\left(S^{W}, M \otimes S^{V+W}\right)\right]_{K}
\end{gathered}
$$

so the topological isotropy map is zero in cohomology for indexes $a+b \epsilon, a, b \in \mathbb{Z}$. The first statement of the theorem is implied by the periodicity of $\tilde{E}_{\Sigma_{2}} C_{2}$; i.e. through multiplication by the $x_{1}$ and $x_{3}$
classes from the Bredon cohomology ring of a point described above. For example, multiplication with $x_{1}$ gives


The last case of the theorem is symmetric in $p$ and $q$. Suppose for example that $p=-n<0$ and $q \geq 0$. By multiplication with $x_{3}$ as above we can assume that $q=0$. Let $a>3$ (by periodicity and the fact that $\widetilde{H}_{B r}^{a+0 \sigma+0 \epsilon+0 \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=0$ if $a \leq 3$, we conclude that $\widetilde{H}_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=0$ for any $p, q \in \mathbb{Z}$, so the map is zero). Then

$$
\widetilde{H}_{B r}^{a-n \sigma}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=\left[\tilde{E}_{\Sigma_{2}} C_{2}, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K} \rightarrow \widetilde{H}_{B r}^{V}\left(p t_{+}\right)=\left[p t_{+}, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K}
$$

We describe the action of $C_{2} \times \Sigma_{2}$ on $S^{\sigma+\sigma \otimes \epsilon}=S(1 \oplus \sigma \oplus \sigma \otimes \epsilon)$. Denote by $a$ the nontrivial element of $C_{2}$ and $b$ the nontrivial element of $\Sigma_{2}$. Then $a b$ is the nontrivial element of the diagonal subgroup $\Delta$. Then $a(x, y, z)=(x,-y,-z), b(x, y, z)=(x, y,-z)$ and $a b(x, y, z)=(x,-y, z)$. When we restrict to the action on $S^{\sigma \otimes \epsilon}=S(1 \oplus \sigma \otimes \epsilon)$ then $a(x, 0, z)=(x, 0,-z), b(x, y, z)=(x, 0,-z)$ and $a b(x, 0, z)=(x, 0, z)$. In conclusion $\Delta$ acts trivially on $S^{\sigma \otimes \epsilon}$. Also

$$
I=S^{\sigma \otimes \epsilon} / C_{2}=S^{\sigma \otimes \epsilon} / \Sigma_{2}=S^{\sigma \otimes \epsilon} / K
$$

because $C_{2}$ and $\Sigma_{2}$ act in the same way on $S^{\sigma \otimes \epsilon}$. Because $\Sigma_{2}$ acts trivially on $F\left(S^{n \sigma}, M \otimes S^{a}\right)$ it implies that $C_{2}$ acts as $\Delta$ on this space.

Now the map $p t_{+} \rightarrow \tilde{E}_{\Sigma_{2}} C_{2}$ factors as

$$
p t_{+} \hookrightarrow S^{\sigma \otimes \epsilon} \hookrightarrow S^{\sigma \oplus \sigma \otimes \epsilon} \hookrightarrow \tilde{E}_{\Sigma_{2}} C_{2}
$$

and consequently we have a factorization of the topological isotropy map

$$
\tilde{H}_{B r}^{a-n \sigma}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right) \rightarrow\left[S^{\sigma \otimes \epsilon}, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K} \rightarrow \widetilde{H}_{B r}^{a-n \sigma}\left(p t_{+}\right)
$$

But the homotopy group of pointed maps is zero i.e.

$$
\left[S^{\sigma \otimes \epsilon}, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K}=\left[S^{\sigma \otimes \epsilon} / \Sigma_{2}, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K}=\left[I, F\left(S^{n \sigma}, M \otimes S^{a}\right)\right]_{K}=0
$$

It implies the second statement of the theorem.

## $2.4 \quad \Sigma_{2}$-equivariant classifying spaces

Let $B_{\Sigma_{2}} C_{2}$ be the $\Sigma_{2}$-equivariant classifying space. It is constructed as

$$
B_{\Sigma_{2}} C_{2}=E_{\Sigma_{2}} C_{2} / C_{2}=\operatorname{colim}_{n} S(n \sigma+n \sigma \otimes \epsilon) / C_{2} .
$$

It is the realization of

$$
\mathbf{B} C_{2}=\mathbf{E} C_{2} / C_{2}=\operatorname{colim}_{n}(\mathbb{A}(n \sigma) \backslash 0) / C_{2},
$$

the classifying space of $C_{2}$ over the field of real numbers. The $R O\left(\Sigma_{2}\right)$-graded Bredon cohomology of $B_{\Sigma_{2}} C_{2}$ is given below:
Theorem 2.10. ([24],[16]) We have that

$$
H_{B r}^{*+* \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=H_{B r}^{*+* \epsilon}(p t)[c, b] /\left(c^{2}=x_{2} c+y_{2} b\right)
$$

where $|c|=\epsilon$ and $|b|=1+\epsilon$ and $x_{2}, y_{2} \in H_{B r}^{\star}(p t, \mathbb{Z} / 2)$ are the usual classes in the degrees $\epsilon$ and $\epsilon-1$ respectively.

The motivic cohomology of $\mathbf{B} C_{2}$ over the field of real numbers is computed below:

Theorem 2.11. ([30]) We have that

$$
H^{*, *}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)=H^{*, *}(\mathbb{R}, \mathbb{Z} / 2)[s, t] /\left(s^{2}=\tau t+\rho s\right)
$$

where $|s|=(1,1)$ and $|t|=(2,1), \tau \in H^{0,1}(\mathbb{R}, \mathbb{Z} / 2)=\mathbb{Z} / 2$ and $\rho$ is the class of $[-1] \in H^{1,1}(\mathbb{R}, \mathbb{Z} / 2)=$ $\mathbb{Z} / 2$.

It is obvious that the realization map

$$
H^{a, b}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

is an isomorphism for any $a \in \mathbb{Z}$ such that $a \leq 2 b$. The realization map sends

$$
s \rightarrow c
$$

$$
t \rightarrow b
$$

$$
\tau \rightarrow y_{2}
$$

$$
\rho \rightarrow x_{2}
$$

The way we choose the generator $s$ in Theorem 2.11 is the following: it is the unique element $s \in$ $H^{1,1}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)$ such that the restriction to $H^{1,1}(p t, \mathbb{Z} / 2)$ is zero and the Bockstein homomorphism $\delta: \widetilde{H}^{1,1}(-, \mathbb{Z} / 2) \rightarrow \widetilde{H}^{2,1}(-, \mathbb{Z})$ sends $\delta(s)=t$. We choose $t$ to be the Euler class of the line bundle on $\mathbf{B} C_{2}$ corresponding to the tautological representation of $\mu_{2}$.

Lemma 2.12. We have that

$$
H_{B r}^{a+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=0
$$

when $b=a-1$ or $b=a-2$.

Proof. We know that

$$
H_{B r}^{a+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=\frac{H_{B r}^{a+b \epsilon}(p t, \mathbb{Z} / 2)[c, b]}{\left(c^{2}=y_{2} b+x_{2} c\right)}
$$

so every generator can be obtained by multiplying an element $\alpha \in H_{B r}^{\star}(p t, \mathbb{Z} / 2)$ with either $b^{n}$ or $c b^{n}$. If $\alpha$ is in the positive cone, the possibilities are bounded by the products $b^{n}$ along the line $a=b$, and if $\alpha$ is in the negative cone, the possibilities are bounded by the products $\theta_{2} c b^{n}$, along the line $b=a-3$. This is shown in the diagram below.


Figure 4

Corollary 2.13. If $a \leq b+2, b<0$ then

$$
H_{B r}^{a, b}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=0
$$

If $b=0$ then

$$
\widetilde{H}_{B r}^{a}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=0
$$

if $a \leq 2$.
In conclusion we have
Corollary 2.14. The realization maps are isomorphisms

$$
H^{a, b}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-b+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

for any $a, b \in \mathbb{Z}$ such that $a \leq 2 b$.
We have that

$$
H^{*, *}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2[\tau, \rho][s, t] /\left(s^{2}=\tau t+\rho s\right)
$$

because the motivic cohomology of the real numbers agrees with the positive cone of the $R O\left(C_{2}\right)$-graded Bredon cohomology of a point (see Proposition 1.6).

Looking to Figure 4, we see that this ring is represented by elements in the upper cone starting from the diagonal line given by the powers of $b$. The lower cone that is in the lower half plane $a>b$ is not included in the image of the realization maps from Corollary 2.14.

Considering $W_{q}=A(q \sigma) \backslash\{0\}, q>0$, which has a free $C_{2}$-action, we have that $H_{C_{2}}^{*, *}\left(W_{q}, \mathbb{Z} / 2\right)$ is generated over the motivic cohomology of a point by $s t^{i}, t^{i}$ with $0 \leq i \leq q-1$ because of [30] and Proposition 2.15. Thus we have over a field $k$ of characteristic zero

$$
H_{C_{2}}^{*, *}\left(W_{q}, \mathbb{Z} / 2\right) \simeq H^{*, *}(k)[s, t] /\left(s^{2}=\tau t+\rho s, t^{q}\right)
$$

and the target of the realization maps over reals is in

$$
H_{B r}^{*, *}\left(P\left(\mathbb{R}^{q+q \sigma}\right), \mathbb{Z} / 2\right) \simeq H_{B r}^{*, *}(p t)[c, b] /\left(c^{2}=x_{2} c+y_{2} b, b^{q}=0\right)
$$

computed in [[24], Theorem 4.11]. In conclusion the realization maps are isomorphisms

$$
H_{C_{2}}^{a, b}\left(W_{q}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-b+b \epsilon}\left(S(q \sigma+q \sigma \otimes \epsilon) / C_{2}, \mathbb{Z} / 2\right)
$$

for any $a, b \in \mathbb{Z}$ such that $a \leq 2 b$.


Figure 5
In conclusion, the Bredon motivic cohomology in integer indexes of $W_{q}$ is represented in Figure 4 as the truncation of the upper half plane along the line $(q, b)$ with $b \geq 0$ with the realization described in Figure 5.

The following proposition implies $t^{q}=0$ for dimension reasons. Notice that if $k<n$ then $t^{k} \in$ $H_{C_{2}}^{2 k, k}\left(W_{q}, \mathbb{Z} / 2\right) \simeq H_{C_{2}}^{2 k, k}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)$ is nonzero from [30].

## Proposition 2.15.

$$
H_{C_{2}}^{2 b, b}\left(W_{-q}, \mathbb{Z} / 2\right)=0
$$

for $b \geq-q>0$ and any field $k$ of characteristic zero.
Proof. Let $x_{1} \in H^{\sigma, 0}(k) \simeq \mathbb{Z} / 2$ and $x_{3} \in H^{\sigma, \sigma}(k) \simeq \mathbb{Z} / 2$ the non-zero generators ([31], Proposition 4.3) and $x_{1} x_{3} \neq 0$ from the cofiber sequence 1.1. Write $k_{2}$ for the invertible element of degree $(2 \sigma-2, \sigma-1)$ in the Bredon motivic cohomology of $\mathbf{E} C_{2}$.

We have that

$$
0 \rightarrow H_{C_{2}}^{2 b, b}(k, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{2 b, b}\left(W_{-q}, \mathbb{Z} / 2\right) \rightarrow \widetilde{H}_{C_{2}}^{2 b+1, b}\left(T^{-q \sigma}, \mathbb{Z} / 2\right) \rightarrow 0
$$

and the first term is zero if $b \geq-q>0$. It implies that $H_{C_{2}}^{2 b, b}\left(W_{-q}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{C_{2}}^{2 b+1, b}\left(T^{-q \sigma}, \mathbb{Z} / 2\right)$. We have that $k_{2} t \in H_{C_{2}}^{2 \sigma, \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=H_{C_{2}}^{2 \sigma, \sigma}(k, \mathbb{Z} / 2) \simeq \mathbb{Z} / 2$ from [[31],Proposition 3.3, Proposition 4.2] and 1.1, with $k_{2} t$ nonzero. Because $x_{1} x_{3} \in H_{C_{2}}^{2 \sigma, \sigma}(k, \mathbb{Z} / 2)$ is nonzero it implies $k_{2} t=x_{1} x_{3}$. Then we have the following commutative diagram:

$$
\begin{gathered}
H_{C_{2}}^{2 b+2 q \sigma, b+q \sigma}(k, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{2 b+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=\mathbb{Z} / 2\left(k_{2}^{q} t^{b+q}\right) \\
\downarrow\left(k_{2} t\right)^{-q} \\
H_{C_{2}}^{2 b, b}(k, \mathbb{Z} / 2)=0 \xrightarrow{0} H_{C_{2}}^{2 b, b}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)=\mathbb{Z} / 2\left(t^{b}\right)
\end{gathered}
$$

It implies that the right vertical map is an isomorphism and then the upper horizontal map is zero. From the isotropy sequence we have then
$0 \rightarrow \mathbb{Z} / 2=H_{C_{2}}^{2 b+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \xrightarrow{\widetilde{ }} \widetilde{H}_{C_{2}}^{2 b+1+2 q \sigma, b+q \sigma}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right)=\mathbb{Z} / 2 \rightarrow H_{C_{2}}^{2 b+1+2 q \sigma, b+q \sigma}(k, \mathbb{Z} / 2) \rightarrow 0$ which implies that $H_{C_{2}}^{2 b+1+2 q \sigma, b+q \sigma}(k, \mathbb{Z} / 2)=0=\widetilde{H}_{C_{2}}^{2 b+1, b}\left(T^{-q \sigma}, \mathbb{Z} / 2\right)$. This concludes the proof.

## 3 The case $b+q<0$.

The following proposition is used in the red cone of Figure 3:
Proposition 3.1. If $b \geq 0$ and $b+q<0$ then $H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq \widetilde{H}_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right)$.
Proof. The proposition follows from the motivic isotropy sequence and the fact that

$$
H^{a+p \sigma, b}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=0
$$

if $b<0$. We also use the periodicity of the Bredon motivic cohomology of $\mathbf{E} C_{2}$ to obtain

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=H_{C_{2}}^{a+2 q+(p-2 q) \sigma, b+q}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=0
$$

if $b+q<0$.
Proposition 3.2. ([[21], Prop. 2.8]) If $b \leq 0$ and $b+q<0$ then $H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=0$.
Proof. This follows from the motivic isotropy sequence and the known periodicities. This proposition is true for any $C_{2}$-equivariant scheme $X$.

Proposition 3.3. a) The realization maps

$$
\widetilde{H}_{C_{2}}^{a, b}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{B r}^{a-b+b \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

are isomorphisms for $a \leq 2 b+1$ and any $b \in \mathbb{Z}$. Moreover

$$
\widetilde{H}_{C_{2}}^{a, b}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right)=0
$$

if $a>2 b+1$ or $b \leq 0$.
b) The red cone of Figure 3 is a $\mathbb{M}_{2}^{C_{2}}$-submodule of

$$
\widetilde{H}_{C_{2}}^{*, *}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{C_{2}}^{*, *}\left(\Sigma^{1} \mathbf{B} C_{2}, \mathbb{Z} / 2\right)\left[\sigma^{ \pm 1}, \epsilon^{ \pm 1}\right]
$$

with invertible cohomology classes $\sigma$ of degree $(\sigma, 0)$ and $\epsilon$ of degree $(0, \sigma)$.
Proof. We have that $\mathbf{B} C_{2} \rightarrow p t$ admits a section, which makes the isotropy motivic sequence split (in integer indexes), giving a short exact sequence

$$
0 \rightarrow H_{C_{2}}^{*, *}(p t, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{*, *}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right) \rightarrow \widetilde{H}_{C_{2}}^{*+1, *}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \rightarrow 0
$$

This gives an isomorphism $\widetilde{H}_{C_{2}}^{*, *}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{C_{2}}^{*, *}\left(\Sigma^{1} \mathbf{B} C_{2}, \mathbb{Z} / 2\right)$. We also have a commutative diagram

$$
\begin{aligned}
& \widetilde{H}_{C_{2}}^{a+1, b}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \xrightarrow{\cong} \widetilde{H}_{C_{2}}^{a, b}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)
\end{aligned}
$$

where the right vertical map is an isomorphism if $a \leq 2 b$ (see Corollary 2.14). When $a \geq 2 b+2$ or $b \leq 0$ we have that

$$
\widetilde{H}_{C_{2}}^{a, b}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right)=0
$$

from [[21], Prop. 2.7 and Prop. 2.9].
According to Proposition 3.1, we have that the groups in the red cone of Figure 3 are isomorphic to the reduced Bredon motivic cohomology of $\widetilde{\mathbf{E}} C_{2}$. It is obvious that

$$
\widetilde{H}_{C_{2}}^{* *}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{C_{2}}^{*, *}\left(\Sigma^{1} \mathbf{B} C_{2}, \mathbb{Z} / 2\right)\left[\sigma^{ \pm 1}, \epsilon^{ \pm 1}\right]
$$

according to the discussion above and the periodicities of the latest cohomology.

We know that over any field $k$ of characteristic zero we have $\widetilde{H}_{C_{2}}^{a, b}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right)=0$ for $a \leq 1[[20]$, Lemma 4.2].

We have for $a \leq 2 b+1, b+q<0$ the diagram

$$
\begin{gathered}
\widetilde{H}_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \xrightarrow{\downarrow \cong} \underset{\underline{\cong}}{\cong} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R} ., \mathbb{Z} / 2) \\
\widetilde{H}_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
\end{gathered}
$$

If $a>2 b+1, b+q<0$ then the upper horizontal map is zero because it is an isomorphism of two groups that are zero. It implies that the right vertical realization maps are either trivial monomorphisms (the domain is zero) or they coincide with an induced map in the topological isotropy sequence.

According to the periodicity of the Bredon cohomology of $E_{\Sigma_{2}} C_{2}$ and the proof of Proposition 3.3 we have

$$
\widetilde{H}_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{B r}^{a-b+b \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right) \simeq \widetilde{H}_{B r}^{a-b-1+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

The latest group is reviewed in Theorem 2.10. From Corollary 2.13 we conclude that

$$
\widetilde{H}_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=0
$$

if $a \leq 3$ and any $p, q \in \mathbb{Z}$.
In conclusion, we have the following corollary about the realization maps in the red cone of Figure 3:

Corollary 3.4. Let $b+q<0, p \in \mathbb{Z}$. The realization maps

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

are trivial monomorphisms (the domain is zero) if $a>2 b+1$ or $a \leq 1$ or $b \leq 0$. When $a \leq 2 b+1$, the displayed realization maps coincide with the $C_{2} \times \Sigma_{2}$-isotropy sequence maps

$$
\widetilde{H}_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

The next theorem says that all realization maps from Corollary 3.4 are monomorphisms.
ncinj Theorem 3.5. The connecting map in the topological $C_{2} \times \Sigma_{2}$-isotropy sequence

$$
\delta: H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}\right) \rightarrow \tilde{H}_{B r}^{a+1+p \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)
$$

vanishes when both $1-b \leq a \leq b+1$ and $b+q<0$.
Proof. We know that all the generators in $H_{B r, K}^{\star}\left(E_{\Sigma_{2}} C_{2}\right)$ are of the form

$$
\alpha=\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} x_{1}^{n_{1}} x_{3}^{n_{3}} y_{1}^{m_{1}} y_{3}^{m_{3}} \kappa_{2}^{n} \quad \text { or } \quad x_{2}^{n_{2}} y_{2}^{m_{2}} x_{1}^{n_{1}} x_{3}^{n_{3}} y_{1}^{m_{1}} y_{3}^{m_{3}} \kappa_{2}^{n}
$$

where $n_{i}, m_{i} \geq 0$ and $n \in \mathbb{Z}$. This is because, using Theorem 2.9 and the topological $C_{2} \times \Sigma_{2}$ cofiber sequence one has

$$
H_{B r, K}^{\star}\left(E_{\Sigma_{2}} C_{2}\right)=\frac{H_{B r}^{*+* \epsilon}(p t)\left[x_{3}, y_{3}, x_{1}, y_{1}, \kappa_{2}^{ \pm 1}\right]}{\left(\kappa_{2} y_{2}=y_{1} y_{3}, \kappa_{2} x_{2}=x_{1} y_{3}+x_{3} y_{1}\right)}
$$

as in [6].
We have from the topological $C_{2} \times \Sigma_{2}$-isotropy sequence that if $n>0$

$$
H_{B r}^{n-n \sigma+n \epsilon-n \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=\mathbb{Z} / 2\left(\kappa_{2}^{-n}\right) \stackrel{\delta}{\simeq} \widetilde{H}_{B r}^{n+1-n \sigma+n \epsilon-n \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)=\mathbb{Z} / 2\left(\frac{\Sigma b^{n}}{x_{1}^{n} x_{3}^{n}}\right)
$$

so $\delta\left(\kappa_{2}^{-n}\right)=\frac{\Sigma b^{n}}{x_{1}^{n} x_{3}^{n}}$. The isomorphism follows from the fact that $H_{B r}^{n+1-n \sigma+n \epsilon-n \sigma \otimes \epsilon}(p t)=0$ (see Proposition 2.4). Here, $b$ is the class appearing in Theorem 2.10 and $\Sigma b$ is the corresponding class in the cohomology of the suspension.

If $n \geq 0$, then $\alpha \in \operatorname{ker} \delta$ from Theorem 2.9. This is because it belongs to the image of the injective map (in this range) given by $H_{B r}^{\star}(p t) \rightarrow H_{B r}^{\star}\left(E_{\Sigma_{2}} C_{2}\right)$.

Let $n=-N$ where $N>0$. If

$$
\alpha=\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} x_{1}^{n_{1}} x_{3}^{n_{3}} y_{1}^{m_{1}} y_{3}^{m_{3}} \kappa_{2}^{-N}
$$

then, because $\delta$ is a $H_{B r, K}^{\star}(p t, \mathbb{Z} / 2)$-module map, we have

$$
\begin{aligned}
\delta(\alpha) & =\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} x_{1}^{n_{1}} x_{3}^{n_{3}} y_{1}^{m_{1}} y_{3}^{m_{3}} \frac{\Sigma b^{N}}{x_{1}^{N} x_{3}^{N}} \\
& =x_{1}^{n_{1}-N} x_{3}^{n_{3}-N} \cdot y_{1}^{m_{1}} y_{3}^{m_{3}} \cdot\left(\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right)
\end{aligned}
$$

If $\alpha$ lives in the range $1-b \leq a \leq b+1$, then we must have that

$$
y_{1}^{m_{1}} y_{3}^{m_{3}} \cdot\left(\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right)=0
$$

since multiplication with $y_{1}$ and $y_{3}$ must eventually cross the gap along the line $b=a-3$ in $\tilde{H}_{K}^{\star}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)$. Indeed, we have that

$$
\left|\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right|=(A, 0, B, 0)=\left(N+m_{2}+3,0, N-m_{2}-n_{2}-2,0\right)
$$

where $B \leq A-5<A-3$. But we know that $\alpha$ lives in the range $a \leq b+1$, so $A-3-m_{1}-m_{3}<a=$ $A-m_{1}-m_{3} \leq b+1=B+1$. Then we have

$$
A-3>B \quad \text { and } \quad A-3 \leq B+m_{1}+m_{3}
$$

so $A-m_{1}^{\prime}-m_{3}^{\prime}-3=B$ for some $m_{1}^{\prime} \in\left[0, m_{1}\right]$ and $m_{3}^{\prime} \in\left[0, m_{3}\right]$. Then

$$
y_{1}^{m_{1}^{\prime}} y_{3}^{m_{3}^{\prime}} \cdot\left(\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right) \in \tilde{H}_{K}^{A-m_{1}^{\prime}-m_{3}^{\prime}, m_{1}^{\prime}, B, m_{3}^{\prime}}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=0
$$

since by Lemma $2.12 \tilde{H}_{K}^{\star}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right)=0$ along the line $a-3=b$. Thus,
$\delta(\alpha)=x_{1}^{n_{1}-N} x_{3}^{n_{3}-N} \cdot y_{1}^{m_{1}} y_{3}^{m_{3}} \cdot\left(\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right)=x_{1}^{n_{1}-N} x_{3}^{n_{3}-N} \cdot y_{1}^{m_{1}-m_{1}^{\prime}} y_{3}^{m_{3}-m_{3}^{\prime}} y_{1}^{m_{1}^{\prime}} y_{3}^{m_{3}^{\prime}} \cdot\left(\frac{\theta_{2}}{x_{2}^{n_{2}} y_{2}^{m_{2}}} \Sigma b^{N}\right)=0$.
If

$$
\alpha=x_{2}^{n_{2}} y_{2}^{m_{2}} x_{1}^{n_{1}} x_{3}^{n_{3}} y_{1}^{m_{1}} y_{3}^{m_{3}} \kappa_{2}^{-N}
$$

then the condition $b+q<0$ implies that

$$
n_{2}+m_{2}+n_{3}+m_{3}<0,
$$

which is false. Hence there are no non-vanishing $\alpha$ in the given range.

Remark 3.6. We use the vanishing result of Proposition 3.5 in the last section, but we can notice from the above proof that there is also a vanishing for the larger range $a \leq b+2, b+q<0$. The fact that there is no lower bound for $a$ in Proposition 3.5 follows also from Corollary 3.4.

## 4 The case $b \geq 0, b+q \geq 0$.

We will prove that all the realization maps in the range $b \geq 0, b+q \geq 0$ are isomorphisms. Notice that this is a generalization of the fact that the motivic cohomology of the real numbers has isomorphic realization maps into the Bredon cohomology of a point when the weight is greater than or equal to zero (Proposition 1.6).

Theorem 4.1. We have that if $b \geq 0, b+q \geq 0$ then

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

If moreover $a>2 b$ then $H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=0$ and the codomain is also zero in this range for any $q \in \mathbb{Z}$.

Proof. We have that for an actual $C_{2}$-representation $V=b+q \sigma, b, q \geq 0$ (see [[31], Proposition 3.4 and Proposition 3.5]),

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=H_{G N i s}^{a-2 b}\left(\mathbb{R}, C_{*} z(V)^{\mathbb{Z} / 2}\right)
$$

From [27] we have a decomposition in motivic complexes in $D M^{-}(\mathbb{R})$

$$
C_{*} z(V)^{\mathbb{Z} / 2}=\oplus_{j=0}^{n-1}(\mathbb{Z} / 2(j)[2 j] \oplus \mathbb{Z} / 2(j)[2 j+1]) \oplus \mathbb{Z} / 2(n)[2 n]
$$

Applying cohomology as above we obtain that

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq \oplus_{j=0}^{q-1} H^{a+2 j, j+b}(\mathbb{R}, \mathbb{Z} / 2) \oplus H^{a+2 j+1, j+b}(\mathbb{R}, \mathbb{Z} / 2) \oplus H^{a+2 q, q+b}(\mathbb{R}, \mathbb{Z} / 2)
$$

In particular, if $a>b, q \geq 0$ then $H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=0$; its target is zero in this case from Proposition 2.7.

Applying the realization functor to the above decomposition of complexes we obtain a decomposition in the topological target, giving a decomposition for the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point:
$H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2) \simeq \oplus_{j=0}^{q-1} H_{B r}^{a-b+j, j+b}(p t, \mathbb{Z} / 2) \oplus H_{B r}^{a-b+j+1, j+b}(p t, \mathbb{Z} / 2) \oplus H_{B r}^{a-b+q, b+q}(p t, \mathbb{Z} / 2)$.
Notice that the realization from Proposition 1.6 applies to each term of the direct sum because $b, q \geq 0$. The statement of the theorem follows now in the case $b, q \geq 0$.

We have that

$$
W_{n}=A(n \sigma) \backslash 0_{+} \rightarrow S^{0} \rightarrow T^{n \sigma}=\frac{A(n \sigma)}{A(n \sigma) \backslash 0}
$$

is an equivariant $C_{2}$-motivic cofiber sequence that gives a long exact sequence in Bredon motivic cohomology.

Notice that $T^{n \sigma}=S^{n \sigma} \wedge S_{t}^{n \sigma}=S^{2 n \sigma, n \sigma}$ ([20]) and for $q<0$ we have by definition that

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=\widetilde{H}_{C_{2}}^{a, b}\left(T^{-q \sigma}, \mathbb{Z} / 2\right)
$$

Because the realization maps

$$
H_{C_{2}}^{a, b}\left(W_{-q}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-b+b \epsilon}(S(-q \sigma \oplus-q \sigma \otimes \epsilon), \mathbb{Z} / 2)
$$

are isomorphisms when $a \leq 2 b$ it implies that

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is an isomorphism if $q<0$ and $b+q \geq 0$ and $a \leq 2 b$ from the following diagram and 5-lemma:

Notice that if $a>2 b$ then $H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)=0$ if $q<0$ and $b \geq-q$. This follows from [[21], Proposition 2.9] if $a \geq 2 b+2$ and from Proposition 2.15 in the case $a=2 b+1$. But

$$
H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)=0
$$

if $a-b>b \geq-q>0$ (see Proposition 2.6) so the realization maps are isomorphisms in this range. We conclude that

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is an isomorphism if $q<0$ and $b+q \geq 0$ which concludes the proof.
The next proposition settles the blue range of Figure 3.
Theorem 4.2. Let $b \geq 0$ and $b+q \geq 0$. Then the realization map is an isomorphism

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2) .
$$

Proof. We have the following diagram for $p+1=2 q$ :

$$
\begin{aligned}
& H_{C_{2}}^{a+(p+1) \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)>H^{a+p+1, b+q}(\mathbb{R}, \mathbb{Z} / 2)>H_{C_{2}}^{a+1+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)>H_{C_{2}}^{a+1+(p+1) \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)>H^{a+p+2, b+q}(\mathbb{R}, \mathbb{Z} / 2
\end{aligned}
$$

and conclude from five lemma, Proposition 4.1 and Proposition 1.6 the isomorphism of the realization maps for $p=2 q-1$ and arbitrary $a$. Downward induction concludes the theorem for $p<2 q$.

For the case $p=2 q$ we use the diagram:

$$
\begin{aligned}
& H^{a-1+p, b+q}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{a+(p+1) \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H^{a+p+1, b+q}(\mathbb{R}, \mathbb{Z} / 2) \rightarrow H_{C_{2}}^{a+1+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)
\end{aligned}
$$

Using five lemma, Proposition 4.1 and Proposition 1.6 we conclude that the realization maps are isomorphic for $p=2 q+1$ and arbitrary $a$. Upward induction concludes the theorem for $p>2 q$.

### 4.1 The case $b \geq 0$ and $b+q<0$ revisited

Consider the case $b \geq 0$ and $b+q<0$. We notice from the proof of Theorem 4.1 that if $q<0$ and $b \geq 0$ we have a diagram

$$
\begin{array}{cc}
H_{C_{2}}^{a-1, b}(\mathbb{R}, \mathbb{Z} / 2) \longrightarrow H_{C_{2}}^{a-1, b}\left(W_{-q}, \mathbb{Z} / 2\right) \longrightarrow \widetilde{H}_{C_{2}}^{a, b}\left(T^{-q}, \mathbb{Z} / 2\right) \longrightarrow H_{C_{2}}^{a, b}(\mathbb{R}, \mathbb{Z} / 2) \longrightarrow H_{C_{2}}^{a, b}\left(W_{-q}, \mathbb{Z} / 2\right) \\
\downarrow & \downarrow \\
\downarrow & \downarrow \\
H_{B r}^{a-b-1, b}(p t) \longrightarrow H_{B r}^{a-b-1, b}(S(-q \sigma-q \sigma \otimes \epsilon)) \rightarrow \widetilde{H}_{B r}^{a-b, b}\left(S^{-q \sigma-q \sigma \otimes \epsilon}\right) \longrightarrow H_{B r}^{a-b, b}(p t) \longrightarrow H_{B r}^{a-b, b}(S(-q \sigma-q \sigma \otimes \epsilon)) .
\end{array}
$$

It implies from 5-lemma that the realization maps

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}(\mathbb{R}) \rightarrow H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t .),
$$

given by the middle vertical map above, are isomorphisms if $a \leq 2 b+1$.
If $a \geq 2 b+2$ the maps are trivially injective from [[21], Proposition 2.9] because the domain is zero.
Because of the periodicity $(\sigma, 0)$ in this range for Bredon motivic cohomology of a point we conclude the following:
ncinj2 Proposition 4.3. Let $b \geq 0$ and $b+q<0$. Then the realization maps

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t .)
$$

are monomorphisms if $p \leq 2 q$.

In the case $a \geq 2 b+2$ they are trivially monomorphisms. Proposition 4.3 is equivalent with part of Proposition 3.5 in the view of Corollary 3.4. The above realization maps are monomorphisms in the case $p>2 q, a \leq 2 b+1$ from Proposition 3.5. In particular, it shows that the realization maps in the red cone of Figure 3 are all monomorphisms.

Remark 4.4. Some of the realization maps in the red cone of Figure 3 are strictly monomorphisms. For example, in the case $a=2 b+1$, according to Proposition 3.5, we have the following short exact sequence:

$$
0 \rightarrow \widetilde{H}_{B r}^{2+\epsilon-3 \sigma \otimes \epsilon}\left(\tilde{E}_{C_{2}} C_{2}\right)=\mathbb{Z} / 2 \rightarrow H_{B r}^{2+\epsilon-3 \sigma \otimes \epsilon}(p t)=\mathbb{Z} / 2 \oplus \mathbb{Z} / 2 \rightarrow H_{B r}^{2+\epsilon-3 \sigma \otimes \epsilon}\left(E_{C_{2}} C_{2}\right)=\mathbb{Z} / 2 \rightarrow 0
$$

thus the realization map in bidegree $(3-3 \sigma, 1-3 \sigma)$ is a nontrivial monomorphism, not surjective.
If we compute the realization map in bidegree $(3-2 \sigma, 1-2 \sigma)$ we see that it is an isomorphism. Indeed, from Proposition 3.5, we have the following short exact sequence:

$$
0 \rightarrow \widetilde{H}_{B r}^{2+\epsilon-2 \sigma \otimes \epsilon}\left(\tilde{E}_{C_{2}} C_{2}\right)=\mathbb{Z} / 2 \rightarrow H_{B r}^{2+\epsilon-2 \sigma \otimes \epsilon}(p t)=\mathbb{Z} / 2 \rightarrow H_{B r}^{2+\epsilon-2 \sigma \otimes \epsilon}\left(E_{C_{2}} C_{2}\right)=0 \rightarrow 0
$$

## 5 Bredon motivic cohomology of $\mathrm{E} C_{2}$

In this section we completely compute the Bredon motivic cohomology groups and ring of $\mathbf{E} C_{2}$ over the real numbers. The methods we use will also reprove in a different way the computation of Borel motivic cohomology ring of the complex numbers given in [21].

2qEG Proposition 5.1. The realization map

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+q \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

is an isomorphism for any $a \leq 2 b+2, b+q \geq 0$. For $b+q<0$ or $a \geq 2 b+1$ the realization map is zero because the domain is zero.

Proof. According to the periodicity of the Bredon motivic cohomology of $\mathbf{E} C_{2}$ we have that

$$
H_{C_{2}}^{a+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H_{C_{2}}^{a+2 q, b+q}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H^{a+2 q, b+q}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)
$$

Because of the vanishing of motivic cohomology (see [25]) we have that the above isomorphisms are zero if $b+q<0$ or if $a \geq 2 b+1$. Also, because of Lemma 2.12 and the isomorphisms below, the codomains are also zero when $a=2 b+1$ or $a=2 b+2$.

Looking to the realization maps for $b+q \geq 0$ and Corollary 2.14, we have the following diagram

The right vertical map is an isomorphism if $a \leq 2 b+2$ from Corollary 2.14 and 2.13. We notice that in the case $a=2 b+1$ or $a=2 b+2$ both cohomologies of the right vertical map are zero. The horizontal maps are isomorphisms from periodicity and properties of Borel motivic cohomology (see [20]). It implies that the left vertical map is an isomorphism for $a \leq 2 b+2$ and $b+q \geq 0$.

Proposition 5.2. The realization map gives an isomorphism

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

for any $a \leq 2 b+2$.

Proof. We have the following diagram:

$$
\begin{aligned}
& H_{C_{2}}^{a-1+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right) \rightarrow H^{a-1+2 q, b+q}(\mathbb{R}) \rightarrow H_{C_{2}}^{a+(2 q-1) \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right) \rightarrow H_{C_{2}}^{a+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right) \rightarrow H^{a+2 q, b+q}(\mathbb{R})
\end{aligned}
$$

The upper sequence is induced by the motivic $C_{2}$-cofiber sequence

$$
\mathbf{E} C_{2+} \wedge C_{2+} \simeq C_{2+} \rightarrow \mathbf{E} C_{2+} \rightarrow \mathbf{E} C_{2+} \wedge S^{\sigma}
$$

The lower sequence is induced by the $C_{2} \times \Sigma_{2}$-equivariant cofiber sequence induced by the above cofiber sequence through realization

$$
E_{\Sigma_{2}} C_{2+} \wedge C_{2+} \simeq C_{2+} \rightarrow E_{\Sigma_{2}} C_{2+} \rightarrow E_{\Sigma_{2}} C_{2+} \wedge S^{\sigma}
$$

Here we used that $\mathbf{E} C_{2}$ is non-equivariantly contractible and that the isomorphism from the motivic cofiber sequence commutes with the realization (see [20]).

The middle map of the diagram is an isomorphism for $a \leq 2 b+2$ by five lemma. Now downward induction concludes that the middle map is an isomorphism for $p \leq 2 q, a \leq 2 b+2$. Upward induction in the diagram below concludes the case $p>2 q, a \leq 2 b+2$.

$$
\begin{aligned}
& H_{C_{2}}^{a+2 q, b+q}(\mathbb{R}) \rightarrow H_{C_{2}}^{a+2 q \sigma, b+q \sigma}\left(\mathbf{E C}_{2}\right) \rightarrow H_{C_{2}}^{a+(2 q+1) \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right) \rightarrow H_{C_{2}}^{a+1+2 q, b+q}(\mathbb{R}) \rightarrow H_{C_{2}}^{a+1+2 q \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right)
\end{aligned}
$$

The following theorem computes all of the Borel motivic cohomology of the real numbers.
Theorem 5.3. $H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)$ for any $b+q \geq 0$. It is zero otherwise.

Proof. The proof follows from the following diagram for $b+q \geq 0$. The upper box in Figure 6 is commutative because the realization is a ring map, and the second box is commutative because it is given by the realization applied to the isotropy sequences. The top left vertical and right vertical isomorphisms follow from the corresponding periodicities. The bottom horizontal map follows from Proposition 4.2. The left lower vertical map is an isomorphism from the motivic isotropy sequence together with [[31], Proposition 4.3] which says that $\widetilde{H}^{a+p \sigma, 0}\left(\widetilde{\mathbf{E}} C_{2}\right)=0$ for any $a, p \in \mathbb{Z}$, and the $(0, \sigma)$ periodicity of the Bredon motivic cohomology $\widetilde{\mathbf{E}} C_{2}$.


Corollary 5.4. The realization map gives an isomorphism

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

for any $a \leq 2 b+2$ and a monomorphism (but not surjective) if $a \geq 2 b+3$. Moreover if $b<0$ and $a \leq 2 b+2$ there is an identification of realization maps (in addition to the identification of groups) i.e.


Figure 7
The horizontal maps are induced by the isotropy sequences. Notice that if $b<0$ the Figure 7 shows that the realization maps

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \hookrightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

are isomorphisms for $a \leq 2 b+2$ and monomorphisms for $a \geq 2 b+3$.
Proof. Because $b<0$, from the motivic isotropy sequence we have that

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}) \simeq H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}\right)
$$

are isomorphisms.
The lower horizontal map in the Figure 7 is an isomorphism for $a \leq 2 b+2$ and $b<0$ from the isotropy sequence and Corollary 2.13.

We can see from Figure 6 that the lower right vertical map is an isomorphism if $a \leq 2 b+2$ because $\widetilde{H}_{B r}^{a}\left(\tilde{E}_{\Sigma} C_{2}, \mathbb{Z} / 2\right)=0$ if $a \leq 3$ (Corollary 2.13). The monomorphism of the realization maps when $a \geq 2 b+3$ follows from Theorem $2.9(b+q \geq 0)$ and Figure 6. It follows that the realization maps for the Bredon motivic cohomology of $\mathbf{E} C_{2}$ are isomorphisms if $a \leq 2 b+2$ and monomorphisms for $a \geq 2 b+3$. We conclude that the realization maps

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t)
$$

are isomorphisms if $a \leq 2 b+2$ and $b<0$. When $a>2 b+3$ and $b<0$ these maps are monomorphisms because in Figure 7 the right vertical maps are monomorphisms and the upper horizontal maps are isomorphisms.

We notice that we have the following exact sequence:

$$
\widetilde{H}_{B r}^{a-2 b, 0}\left(\tilde{E}_{C_{2}} C_{2}\right) \rightarrow H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}(p t) \rightarrow H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}\left(E_{\Sigma} C_{2}\right) \rightarrow \widetilde{H}_{B r}^{a-2 b+1,0}\left(\tilde{E}_{\Sigma} C_{2}\right) .
$$

Moreover $\widetilde{H}_{B r}^{n, 0}\left(\tilde{E}_{\Sigma} C_{2}\right)=0$ if $n \leq 3$ and all the elements for $n>0$ from

$$
\widetilde{H}_{B r}^{n+1,0}\left(\tilde{E}_{C_{2}} C_{2}\right) \simeq \widetilde{H}_{B r}^{n, 0}\left(B_{C_{2}} C_{2}\right)
$$

are either zero or of the form $\frac{\theta_{2}}{x_{2}^{n^{\prime}} y_{2}^{m}} c^{p} b^{q}$ with bidegree $\left(2+m+q,-2-n^{\prime}-m+p+q\right)$ where $p+q=2+m+n^{\prime}$ and $2+m+q \geq 2$ and $p=0,1$. One can notice that $\Sigma \theta_{2} c b \in \widetilde{H}_{B r}^{4,0}\left(\tilde{E}_{\Sigma} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2$. Notice that $\widetilde{H}_{B r}^{n, 0}\left(\tilde{E}_{\Sigma} C_{2}, \mathbb{Z} / 2\right) \neq 0$ if $n \geq 4$ because we can always choose $m, n^{\prime}, p, q \geq 0$ such that $m+q=n-2$ and $p=2+n^{\prime}+m-q$ and $p=0,1$. Because the middle maps in the above long exact sequence are injective it implies that

$$
\widetilde{H}_{B r}^{a-2 b, 0}\left(\tilde{E}_{C_{2}} C_{2}\right) \rightarrow H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}(p t)
$$

is the zero map. This gives a split short exact sequence

$$
0 \rightarrow H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}(p t) \rightarrow H_{B r}^{a-2 b+(p-q+b) \sigma+(b+q) \sigma \otimes \epsilon}\left(E_{\Sigma} C_{2}\right) \rightarrow \widetilde{H}_{B r}^{a-2 b+1,0}\left(\tilde{E}_{\Sigma} C_{2}\right) \rightarrow 0
$$

This implies from Figure 6 that the realization maps of $\mathbf{E} C_{2}$ are not surjective if $a \geq 2 b+3$.
Corollary 5.4 describes the realization maps in the green cone of Figure 3. Theorem 5.3 computes all of the groups in the green cone of Figure 3.

Remark 5.5. The realization homomorphisms

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \hookrightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

are monomorphisms for $a \geq 2 b+3, b<0, b+q \geq 0$ from Theorem 5.4. There are examples where these realizations are strictly monomorphisms. For example

$$
H_{C_{2}}^{-1,-2+2 \sigma}(\mathbb{R}) \simeq \mathbb{Z} / 2 \hookrightarrow H_{B r}^{1-2 \sigma-2 \epsilon+2 \sigma \otimes \epsilon}(p t) \simeq \mathbb{Z} / 2 \oplus \mathbb{Z} / 2
$$

as well as

$$
H_{C_{2}}^{0,-2+2 \sigma}(\mathbb{R}) \simeq \mathbb{Z} / 2 \hookrightarrow H_{B r}^{2-2 \sigma-2 \epsilon+2 \sigma \otimes \epsilon}(p t) \simeq \mathbb{Z} / 2 \oplus \mathbb{Z} / 2
$$

with the computations following from Theorem 5.3 and Proposition 2.4. On the other hand

$$
H_{C_{2}}^{1,-2+2 \sigma}(\mathbb{R})=0 \simeq H_{B r}^{3-2 \sigma-2 \epsilon+2 \sigma \otimes \epsilon}(p t)=0
$$

so we can also have isomorphisms in this range. For non-trivial isomorphisms, we have for example

$$
H_{C_{2}}^{1,-2+3 \sigma}(\mathbb{R})=\mathbb{Z} / 2 \simeq H_{B r}^{3-3 \sigma-2 \epsilon+3 \sigma \otimes \epsilon}(p t)=\mathbb{Z} / 2
$$

Theorem 5.6. We have the following diagram over the complex numbers


Figure 8

Here $u$ is the cohomology class in degree $(2 \sigma-2, \sigma-1)$ that gives the periodicity in Borel motivic cohomology over the complex numbers.

Proof. The arguments for Figure 8 are similar to those for Figure 6. The cohomology class $u$ gives the periodicity in the Bredon motivic cohomology of $\mathbf{E} C_{2}$ and its image through the realization map induces the periodicity in the Bredon cohomology of $E C_{2}$.

We see from Figure 8 that over the complex numbers $\tau_{\sigma} \in H_{C_{2}}^{0, \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)$ (the only non-trivial element in this group) gives a non-zero multiplication on Borel motivic cohomology of a point if $b+q \geq 0$. This is because it induces multiplication by 1 in the $R O\left(C_{2}\right)$ graded Bredon cohomology of a point in the bottom right corner. Thus, we compute the Borel motivic cohomology ring over the complex numbers to be

$$
H_{C_{2}}^{\star, \star}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=\mathbb{M}_{2}^{C_{2}}\left[\tau_{\sigma}, u^{ \pm 1}\right]
$$

Remark 5.7. We see from Figure 8 that the gap at $a=2 b+1$ in Borel motivic cohomology of a point noticed in [21] corresponds with the well known gap at $a=1$ in the $R O\left(C_{2}\right)$ graded Bredon cohomology of a point.

Over the real numbers the situation is different. If the weight $b+q=0$ then $a=2 b+1$ is a gap corresponding to the gap in the Bredon cohomology of a point at $a=1$ i.e.

$$
H_{C_{2}}^{2 b+1+p \sigma, b-b \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=0
$$

for any $b \in \mathbb{Z}$. In the case where $a \geq 2 b+1$, it vanishes if $p-q+b \geq-1$ and it can be nontrivial if $p \leq q-b-2$ and $b+q \geq 1$ (see Theorem 5.3).

As we will see below in Theorem 5.8, all the nontrivial elements in these groups are nilpotent, and from Theorem 5.4 the realization maps are monomorphisms in this range.

We start analyzing the ring structure of Borel motivic cohomology of a point. First, we decide which topological cohomological classes come from corresponding algebraic cohomological classes via the realization maps.

We have that

$$
x_{3} \in H_{C_{2}}^{\sigma, \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{\sigma \otimes \epsilon}(p t, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\sigma, \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2
$$

is the unique nontrivial class. Also

$$
y_{3} \in H_{C_{2}}^{\sigma-1, \sigma}(\mathbb{R} ., \mathbb{Z} / 2) \simeq H_{B r}^{-1+\sigma \otimes \epsilon}(p t, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\sigma-1, \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2
$$

is the unique nontrivial class. It is also obvious that

$$
y_{1} \in H_{C_{2}}^{\sigma-1,0}(\mathbb{R} ., \mathbb{Z} / 2) \simeq H_{B r}^{-1+\sigma}(p t, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\sigma-1,0}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2
$$

and

$$
x_{1} \in H_{C_{2}}^{\sigma, 0}(\mathbb{R} ., \mathbb{Z} / 2) \simeq H_{B r}^{\sigma}(p t, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\sigma, 0}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{Z} / 2
$$

are the unique nontrivial classes.
The cohomology groups of $E_{\Sigma_{2}} C_{2}$ in the above indexes are also isomorphic through the realization because they fulfill the condition $a \leq 2 b+2$.

For example we have


We also have the following diagram:


The top left corner is generated by the class $\kappa_{2}$ and the bottom right corner is generated by its image, which we also denote by $\kappa_{2}$. In conclusion $\kappa_{2}$ becomes invertible in the cohomology ring $H_{B r}^{\star}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)$ because the realization is a ring map. We have

$$
\theta_{1} \kappa_{2}=\iota_{3} \in H_{C_{2}}^{0, \sigma-1}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{B r}^{1-\sigma-\epsilon+\sigma \otimes \epsilon}(p t, \mathbb{Z} / 2) \simeq \mathbb{Z} / 2
$$

The product is nontrivial because its realization is nontrivial.
We also have


The Bredon motivic cohomology ring of $\mathbf{E} C_{2}$ over the reals is computed in the following theorem as a subring of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of $E_{\Sigma_{2}} C_{2}$.

Theorem 5.8. We have the following diagram of commutative rings


Thus,

$$
H_{C_{2}}^{\star, \star}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq \mathbb{M}_{2}^{C_{2}}\left[x_{3}, y_{3}, \kappa_{2}^{ \pm 1}\right]
$$

Moreover if $p \leq q-b-2$ the Bredon motivic cohomology of $\mathbf{E} C_{2}$ has all nonzero elements nilpotent.
Proof. Firstly, we have that $H_{C_{2}}^{\star, 0}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\star, 0}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)$. Secondly, according to Proposition 5.3, the Borel motivic cohomology ring, after multiplication with $\kappa_{2}^{ \pm 1}$, can be reduced to a cohomological ring that is isomorphic (as rings) to the topological cohomology subring $H_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)$ for $a, p \in \mathbb{Z}$ and $q \geq 0$.

We distinguish two cases: Case 1: $p \geq 0$, and Case 2: $p<0$.
Case 1: Let $p, q \geq 0$. Then the groups are computed by the Poincare series $\left(1+x+x^{2}+\ldots+\right.$ $\left.x^{p}\right)\left(1+x+\ldots+x^{q}\right)$ (see Theorem 2.1). It implies that the groups are zero if $a>0$. We will prove that all their generators are given by elements of the form $x_{1}^{n_{1}} y_{1}^{m_{1}} x_{3}^{n_{2}} y_{3}^{m_{2}}$ with $n_{1}, m_{1}, n_{2}, m_{2} \geq 0$. Since $\left|x_{1}\right|=(0,1,0,0),\left|x_{3}\right|=(0,0,0,1),\left|y_{1}\right|=(-1,1,0,0)$ and $\left|y_{3}\right|=(-1,0,0,1)$, the monomial $x_{1}^{n_{1}} y_{1}^{m_{1}} x_{3}^{n_{2}} y_{3}^{m_{2}}$ has degree $\left(-m_{1}-m_{2}, n_{1}+m_{1}, 0, n_{2}+m_{2}\right)$, and belongs to $H_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)$. We have that $0 \leq m_{1} \leq p$ and $0 \leq m_{2} \leq q$. Notice that if we fix a pair ( $m_{1}, m_{2}$ ), we have a unique element in the Poincare sum containing $y_{1}^{m_{1}} y_{3}^{m_{2}}$ such that $a=-m_{1}-m_{2}$. Furthermore, a term $y_{1}^{m_{1}} y_{3}^{m_{2}}$ in the Poincare sum gives a unique pair $\left(m_{1}, m_{2}\right)$ (order sensitive) in $H_{B r}^{a+p \sigma+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)$ with $n_{1}=p-m_{1}$ and $n_{2}=q-m_{2}$.

Case 2: Let $q \geq 0$ and $p<0$. Then the groups are computed by the Poincare series $\left(x^{p}+\ldots+\right.$ $\left.x^{-2}\right)\left(1+x+\ldots+x^{q}\right)$ (see Theorem 2.1). We will prove that all the elements in this case are given by the generators $\frac{\theta_{1}}{x_{1}^{n_{1}} y_{1}^{m_{1}}} x_{3}^{n_{2}} y_{3}^{m_{2}}$. Here $\theta_{1}$ has degree $(2,-2,0,0)$. A monomial $\frac{\theta_{1}}{x_{1}^{n_{1}} y_{1}^{m_{1}}} x_{3}^{n_{2}} y_{3}^{m_{2}}$ has degree $\left(2+m_{1}-m_{2},-2-n_{1}-m_{1}, 0, n_{2}+m_{2}\right)$. It implies that $0 \leq m_{2} \leq q$ and $0 \leq m_{1} \leq-p-2$. Any choice of a pair $\left(m_{1}, m_{2}\right)$ in these intervals gives a unique monomial $y_{1}^{-m_{1}-2} y_{3}^{m_{2}}$ and a unique $\frac{\theta_{1}}{x_{1}^{n_{1}} y_{1}^{m_{1}}} x_{3}^{n_{2}} y_{3}^{m_{2}}$ for a fixed choice of $p \leq-2, q \geq 0$. For $p=-1$ the groups are zero. The group is nonzero if $2-q \leq a \leq-p$.

The following corollary describes the $\mathbb{M}_{2}^{C_{2}}$-module structure of the green cone in Figure 3.
Corollary 5.9. We have the following $\mathrm{IM}_{2}^{C_{2}}$-module

$$
\oplus_{b<0} H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)=\kappa_{2}\left(\mathbb{M}_{2}^{C_{2}}\left[x_{3}, y_{3}, \kappa_{2}\right]\right)
$$

Remark 5.10. Over the complex numbers or over the real numbers, all the elements in $H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right)$ for $a \geq 2 b+1$ are either zero or nilpotent. In particular over the complex numbers, all the realization maps for $a \geq 2 b+1$ are zero because there are no nontrivial nilpotents in the $R O\left(C_{2}\right)$-graded Bredon cohomology of $E C_{2}$, and the realizations are ring maps. This was proved with other methods in [21].
Vcomp Remark 5.11. We have the following ring structure on the motivic cohomology of $\mathbf{B} C_{2}$ (over the reals, [30], reviewed in Theorem 2.11),

$$
H^{*, *}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)=H^{*, *}(\mathbb{R}, \mathbb{Z} / 2)[s, t] /\left(s^{2}=\tau t+\rho s\right)
$$

According to Figure 6, we can identify $\tau=\frac{y_{1} y_{3}}{\kappa_{2}}, \rho=\frac{x_{1} y_{3}+y_{1} x_{3}}{\kappa_{2}}, s=\frac{y_{1} x_{3}}{\kappa_{2}}$ and $t=\frac{x_{1} x_{3}}{\kappa_{2}}$. We notice that $\tau$ and $t$ are the unique generators of their groups; for $s, \rho$ we have 3 distinct choices of elements in

$$
H_{B r}^{-1+\sigma+\sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)=\mathbb{Z} / 2\left(y_{1} x_{3}\right) \oplus \mathbb{Z} / 2\left(x_{1} y_{3}\right)
$$

and according to the relation $s^{2}=\tau t+\rho s$, the choice of $\rho$ is unique.
There are two equally good choices for $s: s=\frac{y_{1} x_{3}}{\kappa_{2}}$ and $s=\frac{y_{3} x_{1}}{\kappa_{2}}$. Notice that according to the relations in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point, we have that the realization
takes $\rho$ into $x_{2}$ and $\tau$ into $y_{2}$ (therefore justifying our previous notation in Lemma 2.12). This also implies the following relations in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of $E_{\Sigma_{2}} C_{2}$

$$
\begin{gathered}
\kappa_{2} y_{2}=y_{1} y_{3}, \\
\kappa_{2} x_{2}=x_{1} y_{3}+y_{1} x_{3}
\end{gathered}
$$

and because in this range the ring map $H_{B r, K}^{\star}(p t, \mathbb{Z} / 2) \hookrightarrow H_{B r, K}^{\star}\left(E_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)$ is injective from Theorem 2.9, it implies that the above relations also hold in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point, giving an alternate proof of these relations from [8].

Remark 5.12. We have that the motivic cohomology over $\mathbb{C}$ of $\mathbf{B} C_{2}$ ([30]) is

$$
H^{*, *}\left(\mathbf{B} C_{2}, \mathbb{Z} / 2\right)=H^{*, *}(\mathbb{C}, \mathbb{Z} / 2)[s, t] /\left(s^{2}=\tau t\right)=\mathbb{Z} / 2[s, t, \tau] /\left(s^{2}=\tau t\right)
$$

Here $s$ is in degree $(1,1), t$ is in degree $(2,1)$ and $\tau \in H^{0,1}(\mathbb{C}, \mathbb{Z} / 2)$. According to Figure 8, in the complex case we can identify $s=\frac{\sigma \alpha}{u}$ (because $\left.s u=\sigma \alpha \in H_{B r}^{-1+2 \sigma}(p t)\right), t=\frac{\sigma^{2}}{u}$ (because $t u=\sigma^{2} \in H_{B r}^{2 \sigma}(p t)$ ), and $\tau=\frac{\alpha^{2}}{u}$ (because $\left.u \tau=\alpha^{2} \in H_{B r}^{-2+2 \sigma}(p t)\right)$ as the unique generators of their respective groups. Here $\sigma$ is in degree $\sigma$, and $\alpha$ is in degree $\sigma-1$. Notice that in this case the relation $s^{2}=\tau t$ becomes obvious.

The complex realization of $\tau$ is 1 , so the relation $u \tau=\alpha^{2}$ becomes $\operatorname{Re}(u)=\alpha^{2}$, which is obvious in the $R O\left(C_{2}\right)$-graded Bredon cohomology of $E C_{2}$.

Remark 5.13. The computation of the Borel cohomology of the complex and real numbers can be used to independently compute the motivic cohomology of $\mathbf{B} C_{2}$ over the complex or real numbers. From the above we identify the motivic cohomology ring of $\mathbf{B} C_{2}$ over the reals with a subring of

$$
\left(\oplus_{a \in \mathbb{Z}, b \geq 0} H_{B r}^{a+b \sigma+b \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)\right)\left[k_{2}^{-1}\right],
$$

and therefore it is given by the following elements:

$$
\sum \frac{x_{1}^{n_{1}} y_{1}^{m_{1}} x_{3}^{n_{2}} y_{3}^{m_{2}}}{\kappa_{2}^{b}}
$$

with $n_{i}, m_{i} \geq 0$ and $n_{1}+m_{1}=b=n_{2}+m_{2}$. The non-trivial generators are chosen from those elements with $b=1$. See Remark 5.11.

In the complex case, the computation is simpler. The graded motivic cohomology group of $\mathbf{B} C_{2}$ over the complex numbers can be identified with

$$
\oplus_{a \in \mathbb{Z}, b \geq 0} H_{B r}^{a+2 b \sigma}(p t, \mathbb{Z} / 2)
$$

and is therefore given by the elements

$$
\sum \frac{\alpha^{n} \sigma^{m}}{u^{b}}
$$

with $n+m=2 b, n, m \geq 0$. The non-trivial generators are those elements with $b=1$. See Remark 5.12.

## 6 Bredon motivic cohomology of the real numbers

In this section, we show that the Bredon motivic cohomology ring of the real numbers is a subring in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded cohomology of a point, and find a decomposition into $\mathbb{I}_{2}^{C_{2}}$-modules.

Let

$$
R:=\oplus_{b \geq 0, b+q \geq 0} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \hookrightarrow H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2)
$$

which is a cohomology subring on which the realization maps give an isomorphism (see the blue region of Figure 3).

We know from Theorem 4.2 that $R$ is ring isomorphic to a subring in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point. We have that the subring

$$
R \simeq \oplus_{a, p \in \mathbb{Z}, b \geq 0, b+q \geq 0} H_{B r}^{a+p \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
$$

is given by a direct sum of four distinct pieces depending on the signs of $p$ and $q$.
One piece that is contained above is simply the positive cone (see Theorem 2.8), which corresponds to the case where $p, q \geq 0$ (topological; the motivic relation is $p \geq q \geq 0$ ) i.e.

$$
\frac{\mathbb{Z} / 2\left[x_{1}, y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right]}{\left(x_{1} y_{2} y_{3}+y_{1} x_{2} y_{3}+y_{1} y_{2} x_{3}\right)} .
$$

Notice that $f=x_{1} y_{2} y_{3}+y_{1} x_{2} y_{3}+y_{1} y_{2} x_{3}=0$ is a trivial relation on $R$ because $R$ also contains the cohomological class $\kappa_{1} \in H_{C_{2}}^{0,1+\sigma}(\mathbb{R}, \mathbb{Z} / 2)$. This is because one can write, according to the relations in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point [8] reviewed in Section 2.2, that

$$
f=x_{1} \kappa_{1} y_{1}+y_{1} x_{2} y_{3}+y_{1} y_{2} x_{3}=x_{2} y_{1} y_{3}+x_{3} y_{2} y_{1}+x_{2} y_{1} y_{3}+x_{3} y_{1} y_{2}=0
$$

We used the relations

$$
\begin{gathered}
\kappa_{1} x_{1}=x_{2} y_{3}+y_{2} x_{3} \\
\kappa_{1} y_{1}=y_{2} y_{3}
\end{gathered}
$$

which are also available in the subring $R$. In conclusion, the subring $R$ contains the subring $\mathbb{Z} / 2\left[x_{1}, y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right]$, and, as a subring, the motivic cohomology ring of $\mathbb{R}$, which is $\mathbb{Z} / 2\left[x_{2}, y_{2}\right]$ (see Proposition 1.6). Besides these classes, other important classes belong to $R$, including, for example, $\theta_{1}, \kappa_{1}, \kappa_{3}, \iota_{2}$.

From Corollary 5.4 we know that

$$
M:=\oplus_{b<0, b+q \geq 0} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)
$$

is a $\mathrm{IM}_{2}^{C_{2}}$-submodule in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point with $\kappa_{2} \in M$. Corollary 5.9 shows that

$$
M \simeq \kappa_{2}\left(\mathrm{IM}_{2}^{C_{2}}\left[x_{3}, y_{3}, \kappa_{2}\right]\right) \hookrightarrow H_{B r}^{\star}(p t, \mathbb{Z} / 2)
$$

It implies that the direct sum $P:=R \oplus M$ is a subring in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology of a point. Obviously, from the previous section's discussion on motivic cohomology classes,

$$
\mathrm{IM}_{2}^{C_{2}}\left[x_{3}, y_{3}\right] \subset R
$$

and

$$
\kappa_{2} \notin R .
$$

We obtain:
Proposition 6.1. We have the following isomorphism of rings

$$
P \simeq\left(R, \kappa_{2}\right) \subset H_{B r}^{\star}(p t, \mathbb{Z} / 2)
$$

where $\left(R, \kappa_{2}\right)$ is the subring in $H_{B r}^{\star}(p t, \mathbb{Z} / 2)$ generated by $R$ and $\kappa_{2}$.
We notice that $\theta_{1}, \iota_{3}, \kappa_{i} \in P$ are nontrivial cohomology classes and $\theta_{2}, \theta_{3} \notin P$. For a review of these cohomological classes see Section 2.2.

Let

$$
N C:=\oplus_{b \geq 0, b+q<0} H_{C_{2}}^{\star, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2)
$$

which is a $\mathbb{M}_{2}^{C_{2}}$-submodule of the Bredon motivic cohomology of $\mathbb{R}$, and also fits into an inclusion of multiplicative maps

$$
N C \hookrightarrow \widetilde{H}_{C_{2}}^{\star, \star}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \hookrightarrow \widetilde{H}_{B r}^{\star}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)
$$

We have that the quotient map

$$
\tilde{E}_{\Sigma_{2}} C_{2} \rightarrow \tilde{E}_{\Sigma_{2}} C_{2} / C_{2} \stackrel{h t p y}{\sim} \Sigma B_{\Sigma_{2}} C_{2}
$$

induces an isomorphism of (non-unital) commutative rings

$$
\widetilde{H}_{B r}^{*+* \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2} / C_{2}\right) \simeq \tilde{H}_{B r}^{*+* \epsilon}\left(\Sigma B_{\Sigma_{2}} C_{2}\right)
$$

Therefore, $N C$ has zero products, because $\Sigma$-suspension gives the commutative ring

$$
\tilde{H}_{B r}^{\star}\left(\tilde{E}_{\Sigma_{2}} C_{2}\right) \simeq \tilde{H}_{B r}^{*+* \epsilon}\left(\Sigma B_{\Sigma_{2}} C_{2}\right)\left[x_{1}^{ \pm 1}, x_{3}^{ \pm 1}\right]
$$

with zero multiplication.
The image of $N C$ in $\widetilde{H}_{B r}^{\star}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)$ is

$$
N C \subset \oplus_{2 \leq a \leq 2 b+1} \widetilde{H}_{B r}^{a-b+b \epsilon}\left(\Sigma B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)\left[x_{1}^{ \pm 1}, x_{3}^{ \pm 1}\right] \simeq \oplus_{1-b \leq a \leq b} \widetilde{H}_{B r}^{a+b \epsilon}\left(B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)\left[x_{1}^{ \pm 1}, x_{3}^{ \pm 1}\right]
$$

This implies, using the Bredon cohomology of $B_{\Sigma_{2}} C_{2}$, that

$$
N C \subset\left\{x_{2}^{n} y_{2}^{m} \Sigma\left(b^{p} c\right), x_{2}^{n} y_{2}^{m} \Sigma\left(b^{p}\right)\right\}\left[x_{1}^{ \pm 1}, x_{3}^{-1}\right]
$$

the subset with the degree of $x_{3}^{-1}$ given by $q \leq-b \leq 0$, where $b$ is the degree of $\epsilon$ i.e.

$$
\begin{gathered}
N C=\oplus_{a \leq 2 b+1} x_{3}^{-b-1} \widetilde{H}_{B r}^{a+b \epsilon}\left(\Sigma B_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right)\left[x_{1}^{ \pm 1}, x_{3}^{-1}\right]= \\
=\left\{x_{3}^{-n-m-p-2} x_{2}^{n} y_{2}^{m} \Sigma\left(b^{p} c\right), x_{3}^{-n-m-p-1} x_{2}^{n} y_{2}^{m} \Sigma\left(b^{p}\right)\right\}\left[x_{1}^{ \pm 1}, x_{3}^{-1}\right]= \\
=\left\{\left(\frac{x_{2}}{x_{3}}\right)^{n}\left(\frac{y_{2}}{x_{3}}\right)^{m} \frac{\Sigma\left(b^{p} c\right)}{\left(x_{3}\right)^{p+2}},\left(\frac{x_{2}}{x_{3}}\right)^{n}\left(\frac{y_{2}}{x_{3}}\right)^{m} \frac{\Sigma\left(b^{p}\right)}{\left(x_{3}\right)^{p+1}}\right\}\left[x_{1}^{ \pm 1}, x_{3}^{-1}\right] .
\end{gathered}
$$

Moreover, for $a \leq 2 b+1, b \geq 0, b+q<0$ we have that

$$
\begin{aligned}
& \widetilde{H}_{C_{2}}^{a+p \sigma, b+q \sigma}\left(\widetilde{\mathbf{E}} C_{2}, \mathbb{Z} / 2\right) \xrightarrow{\cong} H_{C_{2}}^{a+p \sigma, b+q \sigma}(\mathbb{R}, \mathbb{Z} / 2) \\
& \downarrow \cong \quad \downarrow \\
& \widetilde{H}_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}\left(\tilde{E}_{\Sigma_{2}} C_{2}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(p t, \mathbb{Z} / 2)
\end{aligned}
$$

and from Proposition 3.5 (see also Proposition 4.3) we obtain an injective multiplicative map

$$
N C \hookrightarrow H_{B r, K}^{\star}(p t, \mathbb{Z} / 2),
$$

with zero products in the domain.
We can now compute the Bredon motivic cohomology ring of $\mathbb{R}$ in the following theorem:
Theorem 6.2. We have an isomorphism of $\mathrm{I}_{2}^{C_{2}}$-algebras

$$
H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2) \simeq\left(R, \kappa_{2}\right) \oplus N C
$$

with $\kappa_{2}$ in degree $(2 \sigma-2, \sigma-1)$, and $\left(R, \kappa_{2}\right) \hookrightarrow H_{B r}^{\star}(p t, \mathbb{Z} / 2)$ a subring, and $N C$ is a $\mathbb{M}_{2}^{C_{2}}$-module with zero products. In particular, there is a ring monomorphism

$$
H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2) \hookrightarrow H_{B r, K}^{\star}(p t, \mathbb{Z} / 2)
$$

Notice that the negative cone of the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded cohomology ring of a point (where $b, p, q<0$, see Section 2.1) is completely outside the image of the realization map from Theorem 6.2 because of the vanishing range of Bredon motivic cohomology from Proposition 3.2. The cohomological class $\Theta$ and its quotients $\frac{\Theta}{x_{1}^{n_{1}} y_{1}^{m T_{1}} x_{2}^{n_{2}} y_{2}^{m m_{2}} x_{3}^{n_{3}} y_{3}^{m 3}}$ belong to this cone (see Section 2.1) and they are not in the image of the realization map.

Also, according to Remark 5.5 and Remark 4.4, there are other topological classes outside the negative cone that are not in the image of this realization map. Moreover the topological cohomological classes $\iota_{1}$ and $\theta_{2}$ are not in the image of the realization map and do not belong to the negative cone or to the examples in Remarks 5.5 and 4.4.

In [6] we prove a detailed description in terms of generators and relations of the decomposition of Theorem 6.2.

We obtain from Theorem 6.2 or Theorem 1.7 the following interesting subrings:
Corollary 6.3. We have that

$$
\begin{aligned}
& H_{C_{2}}^{*+* \sigma, *}(\mathbb{R}, \mathbb{Z} / 2) \simeq \mathbb{M}_{2}^{C_{2}}\left[x_{2}, y_{2}\right] \\
& H_{C_{2}}^{*+* \sigma, * \sigma}(\mathbb{R}, \mathbb{Z} / 2) \simeq \mathbb{M}_{2}^{C_{2}}\left[x_{3}, y_{3}\right]
\end{aligned}
$$

### 6.1 Real closed fields

Theorem 6.4. Let $k$ be a real closed field. Then

$$
\begin{gathered}
H_{C_{2}}^{\star, \star}(\mathbb{R}, \mathbb{Z} / 2) \simeq H_{C_{2}}^{\star, \star}(k, \mathbb{Z} / 2), \\
H_{C_{2}}^{\star, \star}\left(\mathbf{E} C_{2}, \mathbb{Z} / 2\right) \simeq H_{C_{2}}^{\star, \star}\left(\mathbf{E} C_{2 k}, \mathbb{Z} / 2\right) .
\end{gathered}
$$

Proof. Bredon motivic cohomology

$$
F(U)=H_{C_{2}}^{a+p \sigma, b+q \sigma}(X \times U)
$$

is a homotopy invariant presheaf with equivariant transfers for any $X$ a smooth $C_{2}$-scheme ([19], [20]) over a field $k$ of characteristic zero. In particular, the restriction of $F$ to $S m / k$ is a pseudo pretheory on $S m / k$. From the main result of [1] (with the comment of [18], Theorem 4.18) we know that if $k \subset \mathbb{R}$ is a real closed subfield then

$$
F(U) \simeq F\left(U_{\mathbb{R}}\right)
$$

for any smooth $C_{2}$-scheme $U$ over $k$, implying the result in the theorem. Let $k$ be an arbitrary real closed field and $L=k[i]$ the corresponding algebraic closed field. We can write $L=\cup L_{\alpha}$ with $L_{\alpha}$ a subfield of $L$ of finite transcendence degree over $\mathbb{Q}$ and $\alpha \in A$ a well-order set and $k=\cup k_{\alpha}$ where $k_{\alpha}=L_{\alpha} \cap k$. In Theorem 2.20 [18] it is proved that $k_{\alpha}$ is isomorphic to a real closed field embedded in $\mathbb{R}$. Using the above considerations, we conclude that for a $C_{2}$-smooth scheme $X$ we have

$$
H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(X_{k}\right)=\operatorname{colim}_{\alpha} H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(X_{k_{\alpha}}\right)=H_{C_{2}}^{a+p \sigma, b+q \sigma}\left(X_{\mathbb{R}}\right)
$$

For a real closed field $k \subset \mathbb{R}$ and $X$ a $C_{2}$-smooth scheme over $k$ there is a cycle map

$$
c y c_{k}: H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \mathbb{Z} / 2) \rightarrow H^{a+p \sigma, b+q \sigma}\left(X_{\mathbb{R}}, \mathbb{Z} / 2\right) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(X(\mathbb{C}), \mathbb{Z} / 2)
$$

Therefore, according to the proof of Theorem 6.4 and Theorem 6.2, we conclude that the Bredon motivic cohomology ring of a real closed field embedded in $\mathbb{R}$ is, up to isomorphism, a nontrivial proper subring in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of a point.

We also conclude that the Borel motivic cohomology ring of a real closed field embedded in $\mathbb{R}$ is a nontrivial subring in the $R O\left(C_{2} \times \Sigma_{2}\right)$-graded Bredon cohomology ring of $E_{\Sigma_{2}} C_{2}$.

In the case of a $C_{2}$-smooth scheme over a real closed field $k \subset \mathbb{R}$ the range of isomorphism for $c y c_{k}$ is in general much more restricted. As a generalization of [[18], Theorem 4.18] and of [[11], Corollary $5.13]$ and according to [[20], Theorem 7.10] we obtain the following:

Corollary 6.5. Let $X$ be a smooth scheme over a real closed field $k$ embedded in $\mathbb{R}$. Then the cycle map cyck

$$
c y c_{k}: H_{C_{2}}^{a+p \sigma, b+q \sigma}(X, \mathbb{Z} / 2) \rightarrow H_{B r}^{a-b+(p-q) \sigma+b \epsilon+q \sigma \otimes \epsilon}(X(\mathbb{C}), \mathbb{Z} / 2)
$$

is an isomorphism if $a+p \leq b+q$ and $a \leq \min \{b-q, b\}$ and a monomorphism if $a+p \leq b+q+1$ and $a \leq \min \{b-q, b\}+1$.

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