GENERALIZED HOMOMORPHISMS AND KK WITH EXTRA STRUCTURES

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Abstract. We develop the approach via quasihomomorphisms and the universal algebra qA to Kasparov's KK-theory, so as to cover versions of KK such as KK^{nuc}, KK^G and ideal related KK-theory.

1. Introduction

Kasparov's KK-theory is a main tool in the theory of operator algebras and noncommutative geometry. It is based on a very flexible but not easy formalism developed by Kasparov. In [5] and [6] the first named author has introduced an alternative more algebraic approach based on quasihomomorphisms and the universal algebra qA associated with an algebra A. In this picture elements of $KK(A,B)$ are represented by homomorphisms from qA to $\mathcal{K}\otimes B$ where $\mathcal{K}$ denotes the standard algebra of compact operators on $\ell^2\mathbb{N}$. One merit of this approach is a simple and universal construction of the product in KK from which in particular associativity becomes very natural. Since many important KK-elements come naturally from quasihomomorphisms, at the same time it can be used to treat KK-elements that occur in 'nature'. Note that there are possible definitions of $KK(A,B)$ that make the product and its associativity automatic but have the disadvantage that KK-elements appearing in applications never fit the definition naturally - take for instance the possible definition as homotopy classes of homomorphisms from $\mathcal{K}\otimes qA$ to $\mathcal{K}\otimes qB$. There also is the approach of [7], [9] which is based on the use of the universal algebra qA too, and works also for Banach and locally convex algebras and in fact even much more general algebras [4],[8]. The definition and especially the product however uses higher quasihomomorphisms (maps from q^nA rather than from qA). In applications to C*-algebras e.g. for classification this is not good enough because there it is usually important that a KK-element can be represented by a prequasihomomorphism instead of a Kasparov-module.

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One strength of Kasparov's formalism is the fact that by now it has been extended to define very useful versions of KK for categories of C*-algebras with additional structure such as equivariant KK-theory [11], KK^{nuc} [16] or ideal related KK-theory [12]. In this article we adapt the formalism of [6] to allow for these additional structures. We will give definitions of the various KK-theories using the approach via the universal algebra qA and establish the associative product in each case. In section 7 we will explain that our construction reproduces the KK-theories defined previously in the papers cited above. Moreover we will see there that in the case of equivariant and ideal related KK-theory we obtain a universal functor with the usual properties of split exactness, homotopy invariance and stability.
An nice feature of our approach is the fact that the ideal preserving or nuclearity condition on a homomorphism $\varphi : qA \to B$ can be characterized by a simple criterion. In fact, these conditions can already be checked on the linear map $A \ni x \mapsto \varphi(qx)$ (where qx is one of the standard generators of qA). This description of KK^{nuc} will be used in upcoming work of the second named author [3] to simplify functoriality of this functor similar to how this formalism was used in [2, Appendix B.1].

The most established and probably the most important of the KK-theories we discuss is the equivariant theory KK^G. This version of KK has been discussed on the basis of the qA approach by Ralf Meyer in [13]. In fact one basic idea in his approach appears also in our discussion. We mention however that Meyer does not touch the Kasparov product at all. Using Meyer's result we get a new description of the product in Kasparov's KK^G.
For the construction of the product we will not use Kasparov's technical theorem as in [11] or Pedersen's derivation lifting theorem as in [6] but Thomsen's somewhat simpler noncommutative Tietze extension theorem [10, 1.1.26]. In the equivariant case we will also need a new equivariant version of this theorem which we prove in section 2.

2. Preliminaries

Notation: In the following, homomorphisms between C*-algebras will always be assumed to be *-homomorphisms. By $\mathcal{K}$ we denote the standard algebra of compact operators on $\ell^2\mathbb{N}$. There is a natural isomorphism $\mathcal{K} \cong \mathcal{K} \otimes \mathcal{K}$. A C*-algebra A is called stable if $A \cong \mathcal{K} \otimes A$. Given a C*-algebra A we denote by $\mathcal{M}(A)$ its multiplier algebra. If $\varphi : A \to B$ is a σ-unital homomorphism between C*-algebras, we denote by φ° its extension to a homomorphism $\mathcal{M}(A) \to \mathcal{M}(B)$.

Let A be a C*-algebra. We denote by QA the free product $A \star A$ and by $\iota, \bar\iota$ the two natural inclusions of A into $QA = A \star A$. We denote by qA the kernel

of the natural map $A \star A \to A$ that identifies the two copies $\iota(A)$ and $\bar\iota(A)$ of A. Then qA is the closed two-sided ideal in QA that is generated by the elements $qx = \iota(x) - \bar\iota(x)$, $x \in A$.
There is the natural evaluation map $\pi_A : qA \to A$ given by the restriction to qA of the map $\mathrm{id} \star 0 : QA \to A$ that is the identity on the first copy of A and zero on the second one.

Proposition 2.1. *For $x, y \in A$ one has the identity*

$$q(xy) = \iota(x)q(y) + q(x)\bar\iota(y) = \bar\iota(x)q(y) + q(x)\iota(y)$$

Finite sums of elements of the form $\iota(x_0)qx_1 \dots qx_n$ and $qx_1, \dots qx_n$ or of the form $qx_1 \dots qx_n\iota(x_0)$ and $qx_1, \dots qx_n$ are dense in qA. In particular qA is generated as a closed left or right ideal in qA by the elements qx, $x \in A$.

Proof. The identity for $q(xy)$ is trivially checked. The other statements are consequences (for the assertion on the generation as a closed left or right ideal note that $\iota(y)qx$ is the limit of $\iota(y)u_\lambda qx$ for an approximate unit (u_λ) in qA). □

As in [6] we define a prequasihomomorphism between two C*-algebras A and B to be a diagram of the form

$$A \quad \overset{\varphi,\bar\varphi}{\rightrightarrows} \quad \mathcal{E} \rhd J \xrightarrow{\mu} B$$

i.e. two homomorphisms $\varphi, \bar\varphi$ from A to a C*-algebra $\mathcal{E}$ that contains an ideal J, with the condition that $\varphi(x) - \bar\varphi(x) \in J$ for all $x \in A$ and finally a homomorphism $\mu : J \to B$. The pair $(\varphi, \bar\varphi)$ induces a homomorphism $QA \to \mathcal{E}$ by mapping the two copies of A via $\varphi, \bar\varphi$. This homomorphism maps the ideal qA to the ideal J. Thus, after composing with μ, every such prequasihomomorphism from A to B induces naturally a homomorphism $q(\varphi, \bar\varphi) : qA \to B$. Conversely, if $\psi : qA \to B$ is a homomorphism, then we get a prequasihomomorphism by choosing $\mathcal{E} = \mathcal{M}(\psi(qA))$, $J = \psi(qA)$ and $\varphi = \psi^\circ\iota$, $\bar\varphi = \psi^\circ\bar\iota$ as well as the inclusion $\mu : \psi(qA) \hookrightarrow B$.

In this paper we will also have to use an iteration of the qA construction. We will write Q^2A for the free product $Q(QA) = QA \star QA$ and $\eta, \bar\eta$ for the two natural embeddings of QA into Q^2A. We now denote by $\varepsilon, \bar\varepsilon$ the two embeddings $A \to QA$ and get four embeddings $\eta\varepsilon, \eta\bar\varepsilon, \bar\eta\varepsilon, \bar\eta\bar\varepsilon$ of A to Q^2A. We have the ideal qA generated by the elements $\varepsilon(x) - \bar\varepsilon(x)$, $x \in A$ in QA and the ideal q^2A generated by $\eta(z) - \bar\eta(z)$, $z \in qA$ in $Q(qA)$.

In Section 6 we will use the following equivariant version of Thomsen's noncommutative Tietze extension theorem which we prove here. Recall that when G is a locally compact group, a G-C^*-algebra A is a C^*-algebra with a point-norm continuous action α of G on A. This action extends to a point-strictly continuous action α° on the multiplier algebra $\mathcal{M}(A)$, where we remark that each automorphism α°_g for $g \in G$ is strictly continuous on bounded sets. To simplify notation, we will sometimes write $g \cdot a$ instead of $\alpha_g(a)$ for $a \in A$ and $g \in G$ (or instead of $\alpha^\circ_g(a)$ if $a \in \mathcal{M}(A)$).

Proposition 2.2. *Let G be a locally compact σ-compact group, let $0 \to J \to A \xrightarrow{\pi} B \to 0$ be an extension of σ-unital G-C^*-algebras, and let $X \subset \mathcal{M}(A)$ be a norm-separable self-adjoint subspace. Let $\pi^\circ : \mathcal{M}(A) \to \mathcal{M}(B)$ be the induced homomorphism. For every z in the commutator $\mathcal{M}(B) \cap \pi^\circ(X)'$ of $\pi^\circ(X)$ in $\mathcal{M}(B)$, such that $g \cdot z = z$ for all $g \in G$ there exists $y \in \mathcal{M}(A)$ such that $\pi^\circ(y) = z$, $[y, X] \subseteq J$, $g \cdot y - y \in J$ for all $g \in G$ and $G \ni g \mapsto g \cdot y$ is norm-continuous.*

Proof. We may assume without loss of generality that z is a positive contraction. Let $h \in A$ be strictly positive, let $\mathcal{F} \subset X$ be a compact subset of contractions with dense span,[1] and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq G$ be compact neighbourhoods of the identity such that $G = \bigcup H_n$. Since B is also σ-unital, we apply [11, Lemma 1.4] and pick a (positive, increasing, contractive) approximate identity $(e_n)_{n\in\mathbb{N}}$ for B such that

$$\|(1 - e_n) z^{1/2} \pi(h)\| \leqslant 4^{-n} \tag{1}$$

$$\sup_{x \in \mathcal{F}} \|\pi^\circ(x) e_n - e_n \pi^\circ(x)\| \leqslant 4^{-n} \tag{2}$$

$$\sup_{g \in H_n} \|g \cdot e_n - e_n\| \leqslant 4^{-n} \tag{3}$$

for $n \in \mathbb{N}$. To ease notation let $e_0 = 0$. We will recursively construct positive contractions $0 = y_0 \leqslant y_1 \leqslant y_2 \leqslant \dots$ in A such that for $n \in \mathbb{N}$

$$\pi(y_n) = z^{1/2} e_n z^{1/2} \tag{4}$$

$$\|(y_{n+1} - y_n) h\| \leqslant 2^{-n} \tag{5}$$

$$\sup_{x \in \mathcal{F}} \|[y_{n+1} - y_n, x]\| \leqslant 2^{-n} \tag{6}$$

$$\sup_{g \in H_n} \|g \cdot (y_{n+1} - y_n) - (y_{n+1} - y_n)\| \leqslant 2^{-n}. \tag{7}$$

Letting $y_0 = 0$, suppose we have constructed $y_0 \leqslant \cdots \leqslant y_n$ as above. We will explain how to construct y_{n+1}.

Since $z^{1/2}(e_{n+1} - e_n) z^{1/2} \leqslant 1 - z^{1/2} e_n z^{1/2}$, we apply [14, Proposition 1.5.10] to pick $c \in A$ such that $\pi(c) = z^{1/2}(e_{n+1} - e_n) z^{1/2}$ and $0 \leqslant c \leqslant 1 - y_n$ in $\tilde{A}$. Again using [11, Lemma 1.4] we let $(v_k)_{k\in\mathbb{N}}$ be an approximate identity in J which is quasi-central relative to $\{c, y_n, h\} \cup \mathcal{F}$ and such that $\lim_{k\to\infty} \sup_{g\in H_n} \|g \cdot v_k - v_k\| = 0$. Let $y_{n+1}^{(k)} := y_n + c^{1/2}(1 - v_k) c^{1/2}$. We will show that we can pick $y_{n+1} = y_{n+1}^{(k)}$ for sufficiently large k.

[1] If $(x_n)_{n\in\mathbb{N}}$ is a dense sequence in the unit ball of X one could pick $\mathcal{F} = \{\frac{1}{n} x_n : n \in \mathbb{N}\} \cup \{0\}$.

That (4), (5), and (6) are satisfied is exactly as in the proof of [10], so it remains to show (7). For this we compute

$$\begin{aligned}
& \limsup_{k\to\infty} \sup_{g\in H_n} \|g\cdot(y_{n+1}^{(k)} - y_n) - (y_{n+1}^{(k)} - y_n)\| \\
= \;& \limsup_{k\to\infty} \sup_{g\in H_n} \|g\cdot((1-v_k)c) - (1-v_k)c\| \\
= \;& \limsup_{k\to\infty} \sup_{g\in H_n} \|(1-v_k)(g\cdot c - c)\| \\
= \;& \sup_{g\in H_n} \|g\cdot(z^{1/2}(e_{n+1}-e_n)z^{1/2}) - z^{1/2}(e_{n+1}-e_n)z^{1/2}\| \\
= \;& \sup_{g\in H_n} \|z^{1/2}(g\cdot(e_{n+1}-e_n) - (e_{n+1}-e_n))z^{1/2}\| \\
\overset{(3)}{\leqslant} \;& 2^{-n}.
\end{aligned}$$

Hence we may define $y_{n+1} = y_{n+1}^{(k)}$ for large k so that it satisfies (4)–(7), so we obtain our desired sequence $(y_m)_{m\in\mathbb{N}}$.

By (5) it follows that $(y_n)_n$ converges strictly to a positive contraction $y \in \mathcal{M}(A)$. Since π° is strictly continuous on bounded sets, it follows from (4) that $\pi^\circ(y) = z$ (since z is the strict limit of $z^{1/2}e_n z^{1/2}$). For $x \in \mathcal{F}$ we have by (6) that $[y_n, x]$ norm-converges to an element in A, so that $[y, x] \in A$. Moreover,

$$\pi([y,x]) = \lim_{n\to\infty} \pi^\circ([y_n,x]) \overset{(4)}{=} \lim_{n\to\infty} z^{1/2}[e_n, \pi^\circ(x)]z^{1/2} \overset{(2)}{=} 0$$

so that $[y,x] \in J$ for all $x \in \mathcal{F}$. Hence $[y,x] \in J$ for all $x \in \overline{\mathrm{span}}\mathcal{F} = X$.

As the G-action on $\mathcal{M}(A)$ is pointwise strictly continuous, it follows that $g\cdot y$ is the strict limit of $(g\cdot y_n)_{n\in\mathbb{N}}$ for any $g \in G$. By (7), $(g\cdot y_n - y_n)_{n\in\mathbb{N}}$ converges in A as $n\to\infty$ for every $g \in G$. Hence $g\cdot y - y \in A$. Moreover,

$$\begin{aligned}
\pi(g\cdot y - y) &= \lim_{n\to\infty} \pi^\circ(g\cdot y_n - y_n) \\
&\overset{(4)}{=} \lim_{n\to\infty} g\cdot(z^{1/2}e_n z^{1/2}) - z^{1/2}e_n z^{1/2} \\
&= \lim_{n\to\infty} z^{1/2}(g\cdot e_n - e_n)z^{1/2} \\
&\overset{(3)}{=} 0.
\end{aligned}$$

Hence $g\cdot y - y \in J$ for all $g \in G$.

Finally, given $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\sum_{k=N}^\infty 2^{-n} < \epsilon$. Choose an open neighbourhood $U \subseteq H_N \subseteq G$ of the identity such that $\sup_{g\in U} \|g\cdot y_N - y_N\| < \epsilon$.

Then

$$\begin{aligned}\sup_{g\in U}\|g\cdot y-y\| &= \sup_{g\in U}\Big\|\sum_{k=N}^{\infty}(g\cdot(y_{k+1}-y_k)-(y_{k+1}-y_k))+g\cdot y_N-y_N\Big\|\\ &\overset{(7)}{\leqslant} \epsilon+\sup_{g\in U}\|g\cdot y_N-y_N\|\\ &< 2\epsilon.\end{aligned}$$

Hence $G\ni g\mapsto g\cdot y\in\mathcal{M}(A)$ is norm-continuous. □

3. The product in KK

Given two homomorphisms $\varphi,\psi: X\to Y$ between C*-algebras we denote by $\varphi\oplus\psi$ the homomorphism

$$x\mapsto\begin{pmatrix}\varphi(x)&0\\0&\psi(x)\end{pmatrix}$$

from X to $M_2(Y)$. Following [6] we define

Definition 3.1. *Let A, B be C*-algebras and qA as in Section 2. We define $KK(A,B)$ as the set of homotopy classes of homomorphisms from qA to $\mathcal{K}\otimes B$.*

The set $KK(A,B)$ becomes an abelian group with the operation $\oplus$ that assigns to two homotopy classes $[\varphi],[\psi]$ of homomorphisms $\varphi,\psi: qA\to\mathcal{K}\otimes B$ the homotopy class $[\varphi\oplus\psi]$ (using an isomorphism $M_2(\mathcal{K})\cong\mathcal{K}$ to identify $M_2(\mathcal{K}\otimes B)\cong\mathcal{K}\otimes B$, which is well-defined since such an isomorphism is unique up to homotopy). In [5] it was checked that this definition of $KK(A,B)$ is equivalent to the one by Kasparov. We recapitulate now the construction in [6] of the product $KK(A,B)\times KK(B,C)\to KK(A,C)$. It is based on a functorial map $\varphi_A: qA\to M_2(q^2A)$ (which is in fact - up to stabilization by the 2×2-matrices M_2 - a homotopy equivalence). Since versions of this map and of its properties will be used in each of the subsequent sections on KK with additional structure we include complete proofs. We take this opportunity to include more details on the proofs and to arrange the arguments given in [6] in a slightly different way.
To prove the existence of the map φ_A we will use Proposition 2.2 with A in place of X. Since X in 2.2 has to be separable we will assume in this section and in later sections where we discuss the product of $KK(A,B)$ and $KK(B,C)$ to $KK(A,C)$ with extra structure that A is separable.

Given a C*-algebra A, we use the four embeddings $\eta\varepsilon,\eta\bar\varepsilon,\bar\eta\varepsilon,\bar\eta\bar\varepsilon$ of A to Q^2A from section 2. Consider the C*-algebra R generated by the matrices

$$\begin{pmatrix}R_1&R_1R_2\\R_2R_1&R_2\end{pmatrix}$$

where $R_1 = \eta(qA)$, $R_2 = \bar{\eta}(qA)$. Consider also the C*-algebra D generated by matrices of the form

$$D = \begin{pmatrix} \eta\varepsilon(x) & 0 \\ 0 & \bar{\eta}\varepsilon(x) \end{pmatrix} \quad x \in A$$

Then R is a subalgebra of $M_2(QqA)$ where QqA is the C*-subalgebra of Q^2A generated by $\eta(qA)$ and $\bar{\eta}(qA)$. Let $J = R \cap M_2(q^2A)$. Since q^2A is an ideal in QqA this is an ideal in R. One also clearly has $DR, RD \subset R$. Thus R is an ideal in $R+D$ and J is also an ideal of $R+D$ (we think of all these algebras as subalgebras of $M_2(Q^2A)$).

Because $\eta(qA)/q^2A = \bar{\eta}(qA)/q^2A \cong qA$, the quotient R/J is isomorphic to $M_2(qA)$. Moreover $(R+D)/J$ is isomorphic to the subalgebra of $M_2(Q(A))$ generated by $M_2(qA)$ together with the matrices

$$\begin{pmatrix} \iota(x) & 0 \\ 0 & \iota(x) \end{pmatrix} \quad x \in A$$

If A is separable we can use Thomsen's noncommutative Tietze extension theorem [10, 1.1.26] (see also Proposition 2.2) and lift the multiplier

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of R/J to a self-adjoint multiplier S of R that commutes mod J with D.

We can now set $F = e^{\frac{\pi i}{2} S}$ and define the automorphism σ of $\mathcal{M}(J)$ by $\operatorname{Ad} F$.

Consider the homomorphisms $A \to \mathcal{M}(J)$ given by

$$h_1 = \begin{pmatrix} \eta\varepsilon & 0 \\ 0 & \bar{\eta}\bar{\varepsilon} \end{pmatrix}, \qquad h_2 = \begin{pmatrix} \eta\bar{\varepsilon} & 0 \\ 0 & \bar{\eta}\varepsilon \end{pmatrix}$$

In the following we use the notation $\oplus$ introduced at the beginning of the section. Thus $h_1 = \eta\varepsilon \oplus \bar{\eta}\bar{\varepsilon}$ and $h_2 = \eta\bar{\varepsilon} \oplus \bar{\eta}\varepsilon$.

Definition 3.2. *We define the homomorphism $\varphi_A : qA \to J \subset M_2(q^2A)$ by the prequasihomomorphism given by the pair of homomorphisms $(h_1, \sigma h_2)$ (compare* [6]*, p.39), i.e. $\varphi_A = q(h_1, \sigma h_2)$.*

To check that the difference of h_1 and σh_2 maps to J recall that by definition σ fixes $d(x) = \eta\varepsilon(x) \oplus \bar{\eta}\varepsilon(x)$ mod J for each $x \in A$ and that $h_2(x) = d(x) - \eta(q(x)) \oplus 0$. The term $\eta q(x) \oplus 0$ is moved by σ to $0 \oplus \bar{\eta}q(x)$ mod J (note that $\eta q(x) - \bar{\eta}q(x) \in q^2A$). Since $\bar{\eta}\varepsilon(x) - \bar{\eta}(qx) = \bar{\eta}\bar{\varepsilon}(x)$ we get that $\sigma h_2(x) = h_1(x)$ mod J.

Note the φ_A is unique up to homotopy. In fact, if we picked a different operator $S_1 \in \mathcal{M}(R)$ instead of S as above, and define $S_t = (1-t)S + tS_1$ and $\sigma_t = \operatorname{Ad} e^{\frac{\pi i}{2} S_t}$, then $q(h_1, \sigma_t h_2)$ defines a homotopy from $q(h_1, \sigma h_2)$ to $q(h_1, \sigma_1 h_2)$.

3.1. **The Kasparov product via the universal map φ_A.** Once the map φ_A is constructed we can define the product $KK(A,B)\times KK(B,C) \to KK(A,C)$ as follows.
Let $\alpha : qA \to \mathcal{K} \otimes B$ and $\beta : qB \to \mathcal{K} \otimes C$ represent elements $a \in KK(A,B)$ and $b \in KK(B,C)$ respectively. Since q is a functor, we can form the homomorphism $q(\alpha) : q^2A \to q(\mathcal{K}\otimes B)$. The pair of homomorphisms $(\mathrm{id}_{\mathcal{K}}\otimes\iota, \mathrm{id}_{\mathcal{K}}\otimes\bar{\iota})$ gives a natural map $\mu : q(\mathcal{K} \otimes B) \to \mathcal{K} \otimes qB$. The product of a and b is then represented by the following composition

$$qA \overset{\varphi_A}{\to} q^2A \overset{q(\alpha)}{\to} q(\mathcal{K} \otimes B) \overset{\mu}{\to} \mathcal{K} \otimes qB \overset{\mathrm{id}_{\mathcal{K}}\otimes\beta}{\longrightarrow} \mathcal{K} \otimes \mathcal{K} \otimes C \cong \mathcal{K} \otimes C \tag{8}$$

For simplicity we have left out the tensor product by the 2×2-matrices M_2 which can be absorbed in the tensor product by $\mathcal{K}$. Here and later we sometimes extend homomorphisms, such as $q(\alpha)$ here, tacitly to matrices or stabilizations. We denote the resulting homomorphism $qA \to \mathcal{K}\otimes C$ in (8) by $\beta \sharp \alpha$. This description of the product will be used in the subsequent sections in different versions.

Remark 3.3. (a) If α maps qA to $B \subset \mathcal{K} \otimes B$ then we can omit the map μ and the stabilization of β. We get that $\beta \sharp \alpha$ then is represented by $\beta q(\alpha)\varphi_A$. The same formula applies if α maps qA to $\mathcal{K} \otimes B$ and $B \cong \mathcal{K} \otimes B$.
(b) Assume that B and C are stable and let $\alpha : qA \to B$ and $\beta : qB \to C$ represent elements of $KK(A,B)$ and $KK(B,C)$. Denote by $\underline{B}$ the C*-subalgebra of B generated by $\alpha(qA)$ and by j_B the inclusion $\underline{B} \hookrightarrow B$. Let $\underline{\alpha} : qA \to \underline{B}$ and $\underline{\beta} = \beta \circ q(j_B) : q\underline{B} \to C$ denote the corestriction and restriction of α and β. Then we have $\underline{\beta} \sharp \underline{\alpha} = \beta\sharp\alpha$. In fact $\beta q(\alpha)\varphi_A$ factors as $\beta \circ q(j_B) q(\underline{\alpha})\varphi_A$ and the second expression represents $\underline{\beta} \sharp \underline{\alpha}$.
Instead of $\underline{B}$ we can just as well consider the hereditary subalgebra B_0 of B generated by $\underline{B}$ and define α_0, β_0 in analogy to $\underline{\alpha}, \underline{\beta}$. We get the formula $\beta_0 \sharp \alpha_0 = \beta\sharp\alpha$. We will use this setting below.

3.2. **Associativity.** The important point that gives associativity of the product is the existence of a homotopy inverse (up to tensoring by M_2) for φ_A. It is given by $\pi_{qA} : q^2A \to qA$. We define $\pi_{qA} : QqA \to qA$ as the homomorphism that annihilates $\bar{\eta}(qA)$ in the free product $QqA = \eta qA \star \bar{\eta} qA$, and also as in Section 2 its restriction to $q^2A \subset QqA$.

Proposition 3.4. *There is a continuous family of homomorphisms $\psi_t : q^2A \to M_2(q^2A)$, $t \in [0,1]$ such that $\psi_0 = \mathrm{id}_{q^2A} \oplus 0$ and $\psi_1 = \varphi_A \pi_{qA}$.*
There also is a continuous family of homomorphisms $\lambda_t : qA \to R \subset M_2(QqA)$ such that $\pi_{qA}\lambda_0 = \mathrm{id}_{qA} \oplus 0$ and $\pi_{qA}\lambda_1 = \pi_{qA}\varphi_A$ (here and later we extend

$\pi_{qA}: q^2A \to qA$ tacitly to a homomorphism $M_2(q^2A) \to M_2(qA)$ between 2×2-matrices).

Proof. Let S be as above a lift of the multiplier given on R/J by the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to a multiplier of R and denote by S' the multiplier of $M_2(QqA)$ given by the same matrix M. For each $t \in [0,1]$ we let σ_t denote the automorphism of R given by $\operatorname{Ad} e^{\frac{\pi i}{2}St}$ and τ_t the automorphism of $M_2(QqA)$ given by $\operatorname{Ad} e^{\frac{\pi i}{2}S't}$. Since σ_t fixes the algebra D from above pointwise mod J, the homomorphisms $\eta\varepsilon \oplus \bar\eta\bar\varepsilon$ and $\sigma_t(\eta\bar\varepsilon \oplus \bar\eta\varepsilon)$ map A to $D+R$ and their difference maps into the ideal R of $D+R$. Therefore this difference defines, for each $t \in [0,1]$ a homomorphism α_t from qA to R.
We also define a homomorphism $\bar\alpha_t : qA \to M_2(QqA)$ by the pair of homomorphisms $(\bar\eta\varepsilon \oplus \bar\eta\bar\varepsilon, \tau_t(\bar\eta\bar\varepsilon \oplus \bar\eta\varepsilon))$ from A to $M_2(Q^2A)$. Let us denote the quotient map $QqA \to QqA/q^2A$ by $x \mapsto x^\bullet$. As already remarked above, we have $R^\bullet \cong M_2(qA)$ and we also have $(M_2(\bar\eta qA))^\bullet \cong M_2(qA)$. Under the quotient map R becomes equal to $M_2(\bar\eta qA)$, σ_t becomes equal to τ_t and therefore $\alpha_t(x)^\bullet = \bar\alpha_t(x)^\bullet$ for all $x \in qA$.
It follows that the pair $(\alpha_t, \bar\alpha_t)$ defines a continuous family of homomorphisms $\psi_t : q^2A \to M_2(q^2A)$. These homomorphisms are restrictions of the maps $Q^2A \to M_2(Q^2A)$ that map $\eta\varepsilon(x)$ and $\eta\bar\varepsilon(x)$ to $\eta\varepsilon \oplus \bar\eta\bar\varepsilon$, $\sigma_t(\eta\bar\varepsilon \oplus \bar\eta\varepsilon)$ and $\bar\eta\varepsilon(x), \bar\eta\bar\varepsilon(x)$ to $\bar\eta\varepsilon \oplus \bar\eta\bar\varepsilon$, $\tau_t(\bar\eta\bar\varepsilon \oplus \bar\eta\varepsilon)$, respectively.
For $t=0$ one easily checks for $z \in qA$ that $\alpha_0(z) = \eta(z) \oplus \bar\eta(\gamma(z))$ and $\bar\alpha_0(z) = \bar\eta(z) \oplus \bar\eta(\gamma(z))$ where γ denotes the restriction of the automorphism of QA that interchanges ι and $\bar\iota$. Thus the pair $(\alpha_0, \bar\alpha_0)$ induces the homomorphism $\mathrm{id}_{q^2A} \oplus 0 : q^2A \to M_2(q^2A)$.
For $t=1$, $\alpha_1 : qA \to M_2(q^2A)$ is φ_A and $\bar\alpha_1$ is 0. This shows that $\psi_1 = \varphi_A\pi_{qA}$. It remains to show that $\pi_{qA}\varphi_A$ is homotopic to $\mathrm{id}_{qA} \oplus 0$. The map $\pi_{qA} : q^2A \to qA$ is the restriction of the homomorphism $QqA \to qA$ that annihilates $\bar\eta(qA)$. Consider $\lambda_t : qA \to R \subset M_2(QqA)$ defined by the pair $(\eta\varepsilon \oplus \bar\eta\bar\varepsilon, \sigma_t(\eta\bar\varepsilon \oplus \bar\eta\varepsilon))$. We find that $\pi_{qA}\lambda_0 = \mathrm{id}_{qA} \oplus 0$ and $\pi_{qA}\lambda_1 = \pi_{qA}\varphi_A$. □

Remark 3.5. The map φ_A is functorial (up to stable homotopy) in the following sense: If $\alpha : qA \to qB$ is a homomorphism between separable C*-algebras, then after stabilizing q^2B the homomorphisms $q(\alpha)\varphi_A$ and $\varphi_B\alpha$ are homotopic.
In fact, let $\sim$ denote stable homotopy equivalence. Using Proposition 3.4 to note that $\pi_{qA}\varphi_A \sim \mathrm{id}_{qA}$ and $\varphi_B\pi_{qB} \sim \mathrm{id}_{q^2B}$, as well as the observation $\alpha\pi_{qA} = \pi_{qB}q(\alpha)$, we get

$$q(\alpha)\varphi_A \sim \varphi_B\pi_{qB}q(\alpha)\varphi_A = \varphi_B\alpha\pi_{qA}\varphi_A \sim \varphi_B\alpha.$$

Given C*-algebras X and Y we use the standard notation $[X,Y]$ to denote the set of homotopy classes of homomorphisms from X to Y. Thus we have

$KK(X,Y) = [qX, \mathcal{K}\otimes Y]$. Given $\alpha : qX \to \mathcal{K}\otimes Y$ and $\beta : qY \to \mathcal{K}\otimes Z$ we write $\beta \sharp \alpha$ for $(\mathrm{id}_{\mathcal{K}}\otimes\beta)\mu q(\alpha)\varphi_A$, see formula (8). Thus the homotopy class $[\beta\sharp\alpha]$ represents the Kasparov product of $[\alpha]$ and $[\beta]$. One way to prove the associativity of the Kasparov product consists in identifying $KK(X,Y) = [qX, \mathcal{K}\otimes Y]$ with $[\mathcal{K}\otimes qX, \mathcal{K}\otimes qY]$ using Proposition 3.4 and to check that, under this identification the Kasparov product induced by $\sharp$ corresponds to the composition product of homomorphisms and thus is associative. This observation was stated explicitly for the first time by Skandalis in [17]. We have the following proposition.

In the following we consider qA as a subalgebra of $\mathcal{K}\otimes qA$ as the $(1,1)$-corner embedding.

Proposition 3.6. *The map $[\alpha]\mapsto[\bar\alpha]$ where $\bar\alpha = (\mathrm{id}_{\mathcal{K}}\otimes\pi_B)\alpha|_{qA}$ is an isomorphism from $[\mathcal{K}\otimes qA, \mathcal{K}\otimes qB]$ to $[qA, \mathcal{K}\otimes B]$ with inverse given by the map $[\beta]\mapsto[\beta']$ where $\beta' = \mu(\mathrm{id}_{\mathcal{K}}\otimes q(\beta)\varphi_A)$ with μ as in (8). It is multiplicative in the sense that it maps $[\beta\alpha]$ to $[\bar\beta\,\sharp\,\bar\alpha]$. In particular the product on KK induced by $\sharp$ is associative.*

For the proof of the proposition we need the following lemma.

Lemma 3.7. *The natural maps $q(\pi_A)$ and π_{qA} from q^2A to qA are homotopic as maps to $M_2(qA)$.*

Proof. Both homomorphisms from q^2A to qB are restrictions of homomorphisms from Q^2A to QB. The first one maps $\eta\varepsilon(x), \eta\bar\varepsilon(x), \bar\eta\varepsilon(x), \bar\eta\bar\varepsilon(x)$ to $\iota(x), \bar\iota(x)$, 0, 0 and the second one to $\iota(x)$, 0, $\bar\iota(x)$, 0. The homotopy between the two is obtained by rotating in the homomorphism $q^2A\to M_2(qA)$ which is the restriction of the homomorphism $Q^2A\to M_2(QA)$ mapping the generators to

$$\begin{pmatrix}\iota(x) & 0\\ 0 & 0\end{pmatrix}\quad\begin{pmatrix}\bar\iota(x) & 0\\ 0 & 0\end{pmatrix}\quad\begin{pmatrix}\bar\iota(x) & 0\\ 0 & 0\end{pmatrix}\quad\begin{pmatrix}\bar\iota(x) & 0\\ 0 & 0\end{pmatrix}$$

the second and fourth term to $\begin{pmatrix}0 & 0\\ 0 & \bar\iota(x)\end{pmatrix}$. ☐

Proof of Proposition 3.6. We use $\sim$ to mean homotopic. Up to stabilisations we have

$$(\bar\alpha)' = \mu q((\mathrm{id}_{\mathcal{K}}\otimes\pi_B)\alpha|_{qA})\varphi_A \overset{3.7}{\sim} (\mathrm{id}_{\mathcal{K}}\otimes\pi_{qB})\mu q(\alpha|_{qA})\varphi_A = \pi_{\mathcal{K}\otimes qB}q(\alpha|_{qA})\varphi_A = \alpha|_{qA}\pi_{qA}\varphi_A$$

and this is homotopic to α by Proposition 3.4. Also

$$\overline{\beta'} = (\mathrm{id}_{\mathcal{K}}\otimes\pi_B)\mu q(\beta)\varphi_A = \beta\pi_{qA}\varphi_A$$

which also is homotopic to β by 3.4 (in both cases we have used the obvious identity $\pi_Y q(\psi) = \psi\,\pi_X : qX\to Y$ for a homomophism $\psi : X\to Y$).

Concerning multiplicativity we get (omitting here for clarity the stabilizations and μ) for $\alpha : qA \to qB$ and $\beta : qB \to qC$ that

$$\overline{\beta\alpha} = \pi_C\beta\alpha \sim \pi_C\beta\alpha\pi_{qA}\varphi_A \overset{\alpha\pi_{qA}=\pi_{qB}q(\alpha)}{=} \pi_C\beta\pi_{qB}q(\alpha)\varphi_A$$
$$\overset{3.7}{\sim} \quad \pi_C\,\beta q(\pi_B)q(\alpha)\varphi_A = \pi_C\beta q(\pi_B\alpha)\varphi_A = \bar{\beta}\,\sharp\,\bar{\alpha}.$$

□

3.3. **Another description of the product.** For a prequasihomomorphism $A \rightrightarrows E \rhd J$ given by the pair of homomorphisms $\alpha, \bar{\alpha} : A \to E$ we write as above $q(\alpha,\bar{\alpha})$ for the corresponding map $qA \to J$ (i.e. the restriction of $\alpha \star \bar{\alpha}$ from QA to qA).

For the product of KK-elements $\alpha : qA \to \mathcal{K}\otimes B$ and $\beta : qB \to \mathcal{K}\otimes C$ only the restriction of β to qB_0 matters, where B_0 is the hereditary subalgebra of $\mathcal{K}\otimes B$, generated by the image $\alpha(qA)$, see Remark 3.3 (b). This observation leads to an alternative description of the product which we will also use to discuss associativity of the product in KK^{nuc} in section 5. In fact, for the purposes of this section it would suffice to use the smaller C*-subalgebra $\underline{B}$ of $\mathcal{K}\otimes B$ generated by $\alpha(qA)$ instead of B_0. But we will apply the following discussion to the product in KK^{nuc} in section 5 and there the choice of the hereditary subalgebra will be important.

With B_0 as above we define $\alpha_E, \bar{\alpha}_E : A \to \mathcal{M}(B_0)\oplus A$ by $\alpha_E(x) = (\alpha^\circ\iota_A(x)\,,\,x)$, $\bar{\alpha}_E(x) = (\alpha^\circ\bar{\iota}_A(x)\,,\,x)$ and set $E_\alpha = C^*(B_0, \alpha_E(A), \bar{\alpha}_E(A))$. This gives an exact sequence $0 \to B_0 \to E_\alpha \xrightarrow{p} A \to 0$ with two splittings given by $\alpha_E, \bar{\alpha}_E : A \to E_\alpha$. Note that the prequasihomomorphism $(\alpha_E, \bar{\alpha}_E)$ represents $\alpha : qA \to B_0$ i.e. $\alpha = q(\alpha_E, \bar{\alpha}_E)$.

Lemma 3.8. *Let α, E_α and B_0 be as above and $\beta : q(B_0) \to \mathcal{K}\otimes C$. Let $j_E : B_0 \to E_\alpha$ be the inclusion. There is $\beta' : q(E_\alpha) \to M_2(\beta(qB_0))$ such that β is homotopic to $\beta' q(j_E)$.*

Proof. Let $\kappa_\alpha : qE_\alpha \to B_0$ be the homomorphism defined by the prequasihomomorphism $(\mathrm{id}_{E_\alpha}, \alpha_E \circ p)$ (recall that $p : E_\alpha \to A$ is the quotient map) and set $\beta' = \beta\,\sharp\,\kappa_\alpha = \beta q(\kappa_\alpha)\varphi_{E_\alpha}$. It is immediately checked that $\kappa_\alpha q(j_E) = \pi_{B_0}$ (in fact $\kappa_\alpha(\iota(x)q(y)) = xy$ and $\kappa_\alpha(\bar{\iota}(x)q(y)) = 0$ for $x, y \in B_0$). Using the homotopy $\varphi_{E_\alpha} q(j_E) \sim q^2(j_E)\varphi_{B_0}$ from Remark 3.5 we get (assuming that B is stable) the following homotopy

$$\beta' q(j_E) = (\beta\sharp\kappa_\alpha)q(j_E) = \beta\,q(\kappa_\alpha)\varphi_{E_\alpha}q(j_E) \overset{3.5}{\sim} \beta\,q(\kappa_\alpha)q^2(j_E)\varphi_{B_0} = \beta q(\pi_{B_0})\varphi_{B_0} \overset{3.2}{\sim} \beta$$

□

Given a homomorphism $\mu : qA \to \mathcal{K}\otimes B$, we denote by $\breve{\mu}$ the composition $\mu\delta$ of μ with the symmetry δ of qA that exchanges the two copies of A. Then $\breve{\mu}$ is an additive homotopy inverse to μ, i.e. we have $\mu \oplus \breve{\mu} \sim 0$ (we can rotate

$\iota(x) \oplus \bar{\iota}(x)$ to $\bar{\iota}(x) \oplus \iota(x)$ in 2×2-matrices).
Note that, if ν is a second additive homotopy inverse to μ, then ν is homotopic to $\breve{\mu}$ in matrices (because $\nu \sim \nu \oplus \mu \oplus \breve{\mu} \sim 0 \oplus 0 \oplus \breve{\mu}$).

Proposition 3.9. *Let $\alpha, \beta, E_\alpha, B_0$ be as above and assume that $\beta' : qE_\alpha \to \mathcal{K} \otimes C$ extends β up to homotopy as in 3.8. If we let C_0 denote the hereditary subalgebra of $\mathcal{K} \otimes C$ generated by $\beta(qE_\alpha)$, we get two homomorphisms $\beta'_E, \bar{\beta}'_E : E_\alpha \to E_{\beta'}$ which we can compose with $\alpha_E, \bar{\alpha}_E : A \to E_\alpha$.*
The homomorphism $\beta q(\alpha) : q^2 A \to C_0 \subset \mathcal{K} \otimes C$ is homotopic to $\omega q(\pi_A)$ where $\omega : qA \to C_0 \subset \mathcal{K} \otimes C$ is given by $\omega = q(\beta'_E \alpha_E \oplus \bar{\beta}'_E \bar{\alpha}_E\,,\, \bar{\beta}'_E \alpha_E \oplus \beta_E \bar{\alpha}_E)$.

Proof. The homomorphism $\alpha = q(\alpha_E, \bar{\alpha}_E) : qA \to B_0$ extends to the homomorphism $\alpha_E \star \bar{\alpha}_E$ from QA to E_α. As a homomorphism to $M_2(E_\alpha)$ this extended map is homotopic to $(\alpha_E \oplus 0) \star (0 \oplus \bar{\alpha}_E)$. The restriction of the latter map to qA, which we denote by $\alpha^\oplus$, is described by $\alpha^\oplus = \alpha_E \pi_A \oplus \bar{\alpha}_E \breve{\pi}_A$. We have

$$\beta q(\alpha) \sim \beta' q(\alpha) \sim \beta' q(\alpha^\oplus) \sim \beta' q(\alpha_E \pi_A) \oplus \beta' q(\bar{\alpha}_E \breve{\pi}_A)$$

where we have used that β' composed with a direct sum is in 2×2-matrices homotopic to the direct sum of the two compositions. By the uniqueness of the additive homotopy inverse we have that $\beta' q(\bar{\alpha}_E \breve{\pi}_A) \sim \breve{\beta}' q(\bar{\alpha}_E \pi_A)$. The result follows since $\beta' = q(\beta'_E, \bar{\beta}'_E)$. □

Corollary 3.10. *Let $\alpha, \beta, E_\alpha, B_0$ be as above and assume that β extends up to homotopy to $\beta' : qE_\alpha \to \mathcal{K} \otimes C$. Then the KK-product $\beta \sharp \alpha$ is represented by the homomorphism $\omega : qA \to M_2(C_0) \subset \mathcal{K} \otimes C$ given by*

$$\omega = q(\beta'_E \alpha_E \oplus \bar{\beta}'_E \bar{\alpha}_E\,,\, \bar{\beta}'_E \alpha_E \oplus \beta'_E \bar{\alpha}_E).$$

Proof. By Proposition 3.9, Proposition 3.4 and Lemma 3.7 we have

$$\beta \sharp \alpha \overset{3.3}{\sim} \beta q(\alpha) \varphi_A \overset{3.9}{\sim} \omega q(\pi_A) \varphi_A \overset{3.4}{\sim} \omega.$$

□

Note that, for the formula for $\beta \sharp \alpha$ in Corollary 3.10 we don't need the universal map φ_A in full but only the product $\beta \sharp \kappa_\alpha$. One could base an alternative construction of the product in KK by reducing it to the special case of the product by κ_α.

3.4. **Another proof for associativity.** We follow here the discussion in Section 4 of [5]. Assume that we have elements in $KK(A, B)$, $KK(B, C)$, $KK(C, D)$ represented by homomorphisms $\alpha : qA \to \mathcal{K} \otimes B$, $\beta : qB \to \mathcal{K} \otimes C$, $\gamma : qC \to \mathcal{K} \otimes D$. We define successively first $E_\alpha \supset B_0$ and $\alpha_E, \bar{\alpha}_E : A \to E_\alpha$ as above, then $\beta' : qE_\alpha \to \mathcal{K} \otimes C$ such that the restriction of β' to qB_0 is homotopic to β. We let C_0 denote the hereditary subalgebra of $\mathcal{K}\otimes C$ generated by $\beta'(qE_\alpha)$. Then we define $E_{\beta'}$ as before and get homomorphisms $\beta'_E, \bar\beta'_E : E_\alpha \to E_{\beta'}$. We then take $\gamma' : qE_{\beta'} \to \mathcal{K}\otimes D$ such that its restriction to qC_0 is homotopic to γ and get homomorphisms $\gamma'_E, \bar\gamma'_E : E_{\beta'} \to E_{\gamma'}$.

We can now apply Proposition 3.9 to determine the two products $\gamma' \sharp (\beta'\sharp\alpha)$ and $(\gamma'\sharp\beta') \sharp \alpha$. They will be homotopic to $\gamma\sharp(\beta\sharp\alpha)$ and $(\gamma\sharp\beta)\sharp\alpha$. By Remark 3.3 and Corollary 3.10 the previous products can be described as $\gamma'\sharp\omega_1$ and $\omega_2\sharp\alpha$ with

$$\begin{aligned}\omega_1 &= q(\beta'_E\alpha_E \oplus \bar\beta'_E\bar\alpha_E\, ,\, \bar\beta'_E\alpha_E \oplus \beta'_E\bar\alpha_E)\\ \omega_2 &= q(\gamma'_E\beta'_E \oplus \bar\gamma'_E\bar\beta'_E\, ,\, \bar\gamma'_E\beta'_E \oplus \gamma_E\bar\beta'_E)\end{aligned}$$

We can now apply Proposition 3.9 to both products. By the special form of ω_1, the homomorphisms $\gamma'_E, \bar\gamma'_E$ can be composed with the homomomorphisms occuring in the two components of ω_1. Therefore γ' extends to E_{ω_1} and we are in the situation of 3.9. Second, the two homomorphisms defining ω_2 can be composed with $\alpha_E, \bar\alpha_E$ and therefore ω_2 extends to E_α. When we apply Proposition 3.9 to $\gamma' \sharp (\beta'\sharp\alpha)$ and $(\gamma'\sharp\beta') \sharp \alpha$ and use the special form of ω_1, ω_2 we find that in both cases the triple product is given by

$$q(\gamma'_E\beta'_E\alpha_E\oplus\bar\gamma'_E\bar\beta'_E\alpha_E\oplus\gamma'_E\bar\beta'_E\bar\alpha_E\oplus\bar\gamma'_E\beta'_E\bar\alpha_E\, ,\, \bar\gamma'_E\beta'_E\alpha_E\oplus\gamma'_E\bar\beta'_E\alpha_E\oplus\bar\gamma'_E\bar\beta'_E\bar\alpha_E\oplus\gamma'_E\beta'_E\bar\alpha_E)$$

4. The ideal related case

All ideals in C*-algebras in this section will be closed and two-sided.

Definition 4.1. *Let X be a topological space and $\mathcal{O}(X)$ its lattice of open subsets. An action of X on a C*-algebra A with ideal lattice $\mathcal{I}(A)$ is an order preserving map $\mathcal{O}(X) \ni U \mapsto A(U) \in \mathcal{I}(A)$.*

Let A, B be C*-algebras with an action of X.

A homomorphism (or also a linear map) $\psi : A \to B$ is said to be X-equivariant if ψ maps $A(U)$ to $B(U)$ for each $U \in \mathcal{O}(X)$.

A homomorphism φ from qA to B is said to be weakly X-equivariant, if the maps $A \ni x \mapsto \varphi(\iota(x)z), x \mapsto \varphi(\bar\iota(x)z)$ are X-equivariant for each $z \in qA$.

We say that $\varphi : qA \to B$ is q_X-equivariant if the map $A \ni x \mapsto \varphi(qx)$ is X-equivariant.

Finally, given X and a C*-algebra A with an action of X, we can define actions of X on QA and qA by letting $QA(U)$ and $qA(U)$ be the closed ideals generated by $Q(A(U))$ in QA and by $Q(A(U))qA + +qA\,Q(A(U))$ in qA, respectively (these are the kernels of the natural maps $QA \to Q(A/A(U))$ and $qA \to$

$q(A/A(U)))$. We denote QA, qA with these actions by Q_XA, q_XA. Then

$$0 \to q_XA \to Q_XA \to A \to 0$$

is an X-equivariant exact sequence with equivariant splitting $\iota : A \to Q_XA$.

Proposition 4.2. *Let A, B be C^*-algebras with an action of X and φ a homomorphism $qA \to B$. The following are equivalent*

- *φ is weakly X-equivariant*
- *φ is q_X-equivariant*
- *φ is X-equivariant as a homomorphism $q_XA \to B$*

Proof. Assume that φ is q_X-equivariant. By Proposition 2.1, qA is the closed span of elements $qy\,w$ for $y \in A$ and $w \in qA$. Then $\varphi(\iota(x)qy\,w) = \varphi(q(xy)w) - \varphi(qx\,\bar{\iota}(y)w)$ is in $B(U)$ whenever x is in $A(U)$ for all $y \in A$, $w \in qA$. Similarly for $\varphi(\bar{\iota}(x)qy\,w)$, which shows that φ is weakly X-equivariant.
Conversely, assume that φ is weakly X-equivariant. Let $x \in A(U)$ and (u_λ) an approximate unit for qA. Then $\varphi(qx) = \lim_\lambda \varphi(qx\,u_\lambda) = \lim_\lambda \varphi((\iota(x) - \bar{\iota}(x))u_\lambda) \in B(U)$.
If φ is weakly X-equivariant then $\varphi(qA\,\iota(x)qA)$ and $\varphi(qA\,\bar{\iota}(x)qA)$ are contained in $B(U)$ for all $x \in A(U)$ and thus, by definition of $q_XA(U)$ we get that $\varphi(q_XA(U)) \subset B(U)$.
Finally, if $\varphi : q_XA \to B$ is X-equivariant, then $\varphi(Q(A(U))qA) \subset B(U)$ which means that φ is weakly X-equivariant. □

Definition 4.3. *Let A, B be C^*-algebras with an action of X. We define $KK(X; A, B)$ as the set of homotopy classes of weakly X-equivariant homomorphisms (or equivalently of q_X-equivariant morphisms) $qA \to \mathcal{K} \otimes B$ (with homotopy in the category of such morphisms).*
Equivalently this is the set of equivariant homotopy classes of X-equivariant homomorphisms $q_XA \to \mathcal{K} \otimes B$.

In the X-equivariant case the construction of the product actually carries over directly from section 3. We can apply the arguments from there basically verbatim to q_XA in place of qA because all the maps and homotopies occuring in the discussion are naturally X-equivariant. In particular, the automorphism σ used in the construction of φ_A is inner and therefore respects ideals and is X-equivariant. This in turn implies that φ_A also is X-equivariant as a map from q_XA to $M_2(q_X^2A)$ with $q_X^2A = q_X(q_XA)$. Moreover, the homotopies used in the proofs of Propositions 3.4 and 3.6 are manifestly X-equivariant. We obtain

Proposition 4.4. *Let A, B, C be C^*-algebras with an action of the topological space X. There is a natural bilinear and associative product $KK(X; A, B) \times KK(X; B, C) \to KK(X; A, C)$ which extends the composition product of X-equivariant homomorphisms.*

5. KK^{nuc} via the qA formalism

We start with a discussion of nuclear and weakly nuclear linear maps between C*-algebras. While nuclearity is most often studied in the context of completely positive maps, Pisier considered the case for more general linear maps in [15, Chapter 12]. Since we think that these notions have some independent interest we do this in more detail than what is actually needed for our purposes.

Definition 5.1. *Let $\rho\colon A \to B$ be a linear map between C*-algebras. We let $\|\rho\|_{\mathrm{nuc}}$ (the nuclear norm) denote the infimum over all $K \geqslant 0$ for which*

$$\rho \otimes \mathrm{id}\colon A \otimes_{\mathrm{alg}} D \to B \otimes_{\max} D$$

is bounded by K for all C-algebras D, if we equip $A \otimes_{\mathrm{alg}} D$ with the minimal C^*-tensor norm. We say that ρ is* nuclear *if $\|\rho\|_{\mathrm{nuc}}$ is finite.*

In comparison, a linear map $\phi\colon A \to B$ between C^*-algebras is *completely bounded* (resp. *weakly decomposable*[2]) if there is a constant K such that the map $\phi \otimes \mathrm{id}\colon A \otimes_{\mathrm{alg}} D \to B \otimes_{\mathrm{alg}} D$ is bounded in norm by K when both tensor products are equipped with the minimal (resp. maximal) C^*-tensor product.

Since it suffices to check complete boundedness for D being matrix algebras, it follows that weakly decomposable maps are completely bounded.

Note that if $\rho\colon A \to B$ is nuclear (or weakly decomposable) and ρ takes values in a C^*-subalgebra $B_0 \subseteq B$, the corestriction $\rho|^{B_0}$ is not necessarily nuclear (or weakly decomposable) since the map $B_0 \otimes_{\max} D \to B \otimes_{\max} D$ is not necessarily faithful. However, the map $B_0 \otimes_{\max} D \to B \otimes_{\max} D$ is faithful if B_0 is a hereditary C^*-algebra so in that case $\rho|^{B_0}$ is still nuclear (or weakly decomposable). This explains why we often consider hereditary C^*-subalgebras, instead of just ordinary subalgebras, in the theory below.

If E is a C^*-algebra with closed ideal B, a linear map $\psi\colon A \to E$ is called *weakly nuclear* (relative to B) if $\psi b\colon A \to B$ (i.e. the map $x \mapsto \psi(x)b$) is nuclear for all $b \in B$. We address in Remark 5.3 why this notion agrees with the more traditional notion of weak nuclearity.

Here are some easy observations on nuclear linear maps. If X is a C*-subalgebra of a C*-algebra Y, we denote in the following by $\overline{X}^{Y}$ the hereditary subalgebra $\overline{XYX}$ of Y generated by X.

Lemma 5.2. *Let A, B, C, D be C^*-algebras.*

[2] This name is motivated by the result from [15, Chapter 14] (which is due to Kirchberg) where this definition is shown to be equivalent to the map $\phi\colon A \to B \subseteq B^{**}$ being decomposable, i.e. a linear combination of completely positive maps.

(1) *For a fixed $K \geqslant 0$, the set of linear maps $\rho\colon A \to B$ with $\|\rho\|_{\mathrm{nuc}} \leqslant K$ is closed in the point-norm topology.*
(2) *The set of nuclear linear maps $A \to B$ is a Banach space with respect to the nuclear norm.*
(3) *If $\rho\colon A \to B$ is nuclear and D is a nuclear C^*-algebra, then $\mathrm{id}_D \otimes \rho$ extends canonically to a nuclear map $D \otimes A \to D \otimes B$.*
(4) *If $\rho\colon A \to B$ is completely positive and nuclear then $\|\rho\|_{\mathrm{nuc}} = \|\rho\|$.*
(5) *If $\phi\colon A \to B$, $\rho\colon B \to C$ and $\psi\colon C \to D$ are linear maps such that ϕ is completely bounded, ρ is nuclear, and ψ is weakly decomposable, then $\psi\rho\phi$ is nuclear.*
(6) *If $\psi\colon A \to E$ is a homomorphism with an ideal $B \lhd E$, and if $b \in B$ such that ψb is nuclear, then $\|\psi b\|_{\mathrm{nuc}} \leqslant \|b\|$.*
(7) *If $\psi\colon A \to E$ is a homomorphism with an ideal $B \lhd E$, and if $X \subseteq B$ is a subset such that B is generated as a closed right ideal by X, then ψ is weakly nuclear relative to B provided ψb is nuclear for all $b \in X$.*

Proof. (1), (2), and (5) are immediate to verify, while (4) is classical, see for instance [1, Theorem 3.5.3].

(3): That $\mathrm{id}_D \otimes \rho$ extends is immediate from the definition of nuclearity of ρ, and nuclearity of $\mathrm{id}_D \otimes \rho$ follows since $\mathrm{id}_E \otimes \mathrm{id}_D \otimes \rho$ extends to a linear map

$$E \otimes_{\min} (D \otimes A) = (E \otimes D) \otimes_{\min} A \to (E \otimes D) \otimes_{\max} B = E \otimes_{\max} (D \otimes B)$$

bounded by $\|\rho\|_{\mathrm{nuc}}$ for any C^*-algebra E by nuclearity of D and ρ.

(6): Note that $\theta\colon A \to B$ given by $\theta(x) = b^*\psi(x)b$ is both completely positive and nuclear (it is the nuclear map ψb multiplied by b^*), and thus $\|\theta\|_{\mathrm{nuc}} \leqslant \|b\|^2$ by (4). Let D be a non-zero C^*-algebra and $x = \sum_{j=1}^N a_j \otimes d_j \in A \otimes_{\mathrm{alg}} D$ with minimal tensor norm $\|x\|_{\min} = 1$. Then

$$\begin{aligned}
\|(\psi b \otimes \mathrm{id}_D)(x)\|_{\max} &= \|\sum_{j=1}^N \psi(a_j)b \otimes d_j\|_{\max} \\
&= \|\sum_{i,j=1}^N \theta(a_i^* a_j) \otimes d_i^* d_j\|_{\max}^{1/2} \\
&= \|(\theta \otimes \mathrm{id}_D)(x^*x)\|_{\max}^{1/2} \\
&\leqslant \|\theta\|_{\mathrm{nuc}}^{1/2} \\
&\leqslant \|b\|.
\end{aligned}$$

(7): This is an easy consequence of parts (2) and (6). □

Remark 5.3. Classically a homomorphism (or completely positive map) $\psi\colon A \to E$ being weakly nuclear relative to a closed ideal B means that $b^*\psi b\colon A \to B$ is nuclear for all $b \in B$. We will show that this agrees with our definition above.

If ψb is nuclear then clearly so is $b^*\psi b$ so one implication is obvious. Conversely, suppose $c^*\psi c$ is nuclear for all $c \in B$, so that we should show that ψb is nuclear for all $b \in B$. Let $(e_\lambda)_\lambda$ be an approximate identity in B. By Lemma 5.2(1) it suffices to show that there is an upper bound on the nuclear norms of the maps $e_\lambda \psi b$. By the polarisation identity we have

$$e_\lambda \psi b = \frac{1}{4} \sum_{j=0}^{3} i^j (i^j e_\lambda + b)^* \psi(.)(i^j e_\lambda + b)$$

and by Lemma 5.2(4) we obtain

$$\|e_\lambda \psi b\|_{\text{nuc}} \leqslant \frac{1}{4} \sum_{j=0}^{3} \|(i^j e_\lambda + b)^* \psi(.)(i^j e_\lambda + b)\| \leqslant (1 + \|b\|)^2 \|\psi\|.$$

Hence ψb is nuclear.

If X is a C*-subalgebra of the multiplier algebra $\mathcal{M}(Y)$, we denote by $\overline{X}^Y$ the hereditary subalgebra XYX of Y generated by X (note that XYX is a C^*-algebra by the Cohen–Hewitt factorisation theorem).

Proposition 5.4. *Let $\psi\colon qA \to B$ be a homomorphism. The following are equivalent:*

(i) *The map $A \ni x \mapsto \psi(qx) \in B$ is nuclear;*
(ii) *The maps $A \to B$ given by $x \mapsto \psi(\iota(x)y)$ and $x \mapsto \psi(\bar{\iota}(x)y)$ are nuclear for all $y \in qA$;*
(iii) *ψ is represented by a prequasihomomorphism*

$$(\psi_1, \psi_2)\colon A \rightrightarrows E \rhd J \hookrightarrow B$$

where ψ_1, ψ_2 are weakly nuclear relative to J;
(iv) *If $\psi^\circ\colon QA \to \mathcal{M}(\overline{\psi(qA)}^B)$ is the canonical extension of ψ, then $\psi^\circ\iota$ and $\psi^\circ\bar{\iota}$ are weakly nuclear.*
(v) *If $E = \psi(qA)B$ is considered as a Hilbert B-module, the Kasparov module*

$$\left(\psi^\circ\iota \oplus \psi^\circ\bar{\iota}\colon A \to \mathcal{B}(E \oplus E^{op}), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

is nuclear in the sense of Skandalis.

Proof. With E as in (v), $\mathcal{B}(E)$ is canonically isomorphic to $\mathcal{M}(\overline{\psi(qA)}^B)$ and hence (iv) and (v) are equivalent by [16, 1.5].

(iv) implies (iii) is immediate since ψ is induced by

$$(\psi^\circ\iota, \psi^\circ\bar{\iota})\colon A \rightrightarrows \mathcal{M}(\overline{\psi(qA)}^B) \rhd \overline{\psi(qA)}^B \hookrightarrow B.$$

For (iii) $\Rightarrow$ (ii) we have $x \mapsto \psi(\iota(x)y) = \psi_1(x)\psi(y)$ is nuclear for all $y \in qA$, and similarly $x \mapsto \psi(\bar{\iota}(x)y)$ is nuclear.

For (ii) $\Rightarrow$ (i), let for $y \in qA$ $\psi_y, \bar{\psi}_y \colon A \to B$ be the completely positive maps given by $\psi_y(x) = \psi(y^*\iota(x)y)$ and $\bar{\psi}_y(x) = \psi(y^*\bar{\iota}(x)y)$ which are nuclear by (ii). As these maps are completely positive, their nuclear norm $\|\psi_y\|_{\mathrm{nuc}} = \|\psi_y\| \leqslant \|y\|^2$ (Lemma 5.2(4)), and similarly $\|\bar{\psi}_y\|_{\mathrm{nuc}} \leqslant \|y\|^2$. Hence

$$x \mapsto \psi(y^*\, qx\, y) = \psi_y(x) - \bar{\psi}_y(x)$$

has nuclear norm bounded by $2\|y\|^2$. Letting y range through an approximate identity for qA, these nuclear maps converge point-norm to $x \mapsto \psi(qx)$ and have nuclear norm bounded by 2, so $\|x \mapsto \psi(qx)\|_{\mathrm{nuc}} \leqslant 2$ by Lemma 5.2(1).

(i) $\Rightarrow$ (iv): By Proposition 2.1, $\overline{\psi(qA)}^B$ is generated as a closed left ideal by $\{\psi(qa) : a \in A\}$. So to check that $\psi^\circ\iota$ is weakly nuclear it suffices by Lemma 5.2(7) to check that

$$x \mapsto \psi^\circ\iota(x)\,\psi(qa) = \psi(\iota(x)qa) \overset{2.1}{=} \psi(q(xa)) - \psi(q(x))\psi^\circ\bar{\iota}(a)$$

is nuclear, which holds by Lemma 5.2(5) (applied to the weakly decomposable maps given by right multiplication by a fixed element). Similarly $\psi^\circ\bar{\iota}$ is weakly nuclear. □

Definition 5.5. *We say that a homomorphism $\psi \colon qA \to B$ is q-*nuclear *if it satisfies the equivalent conditions in the above proposition.*

Definition 5.6. *We define $KK^{nuc}(A,B)$ as the abelian group $[qA, \mathcal{K}\otimes B]_{\mathrm{nuc}}$ of homotopy classes (in the same category of maps) of q-nuclear homomorphisms $qA \to \mathcal{K} \otimes B$.*

Remark 5.7. The definition of $KK^{nuc}(A,B)$ from [16] for A separable and B σ-unital uses the original definition of Kasparov but assuming all Kasparov modules and homotopies are nuclear. The argument from [5] combined with Proposition 5.4 shows that the obvious map from Skandalis' KK^{nuc}-group to $[qA, \mathcal{K} \otimes B]_{\mathrm{nuc}}$ is an isomorphism. This map, in particular, takes a Kasparov module induced by a prequasihomomorphism as in Proposition 5.4(iii) (with $\mathcal{K}\otimes B$ instead of B) to the induced q-nuclear homomorphism $\phi \colon qA \to \mathcal{K}\otimes B$.

Remark 5.8. A C*-algebra A is K-nuclear in the sense of Skandalis, if and only if the natural projection $\pi_A : qA \to A$ composed with the inclusion $A \to \mathcal{K} \otimes A$ is homotopic to a q-nuclear homomorphism $qA \to \mathcal{K} \otimes A$.

We now discuss the product of elements in KK^{nuc} by elements in KK. We want to see that our formula in Subsection 3.1 for the product of two KK-elements represented by $\rho : qA \to \mathcal{K} \otimes B$ and $\psi : qB \to \mathcal{K} \otimes C$ gives a well defined element in $KK^{nuc}(A, C)$ if ρ or ψ is q-nuclear. The product, as we defined it, depends only on the restriction of ψ to $q(\rho(qA))$. But if $\rho : qA \to B$ is q-nuclear then we don't know if $\rho : qA \to \rho(qA)$ is too. Therefore we apply the formula for the product from Section 3 to the corestrictions/restrictions $\rho_0 : qA \to B_0$ and $\psi_0 : qB_0 \to C_0$ of ρ and ψ, where $B_0 = \overline{\rho(qA)}^B$, and

$C_0 = \overline{\psi(qB_0)}^C$ are the hereditary subalgebras generated by $\rho(qA)$ and $\psi(qB_0)$. Then ρ_0 is q-nuclear iff ρ is and $\rho = j_{B_0} \circ \rho_0$ for the embedding $j_{B_0} : B_0 \to \mathcal{K} \otimes B$ (and the same for ψ and ψ_0). Similarly we denote by $(\psi_0 \sharp \rho_0)_0$ the corestriction of $\psi_0 \sharp \rho_0$ to the hereditary subalgebra C_0 generated by the image of $\psi_0 \sharp \rho_0$. The product in KK without nuclearity condition of ψ and ρ will be the same as the product $(\psi_0 \sharp \rho_0)_0$ composed with the embedding $j_{C_0} : C_0 \hookrightarrow \mathcal{K} \otimes C$ (see Remark 3.3 (b)). We call ρ_0, ψ_0 the completed form of ρ, ψ and $(\psi_0 \sharp \rho_0)_0$ the completed product.
We consider the two maps $\eta^\psi, \bar{\eta}^\psi : B_0 \to \mathcal{M}(C_0)$ given by $\eta^\psi = \psi_0^\circ \iota_{B_0}$, $\bar{\eta}^\psi = \psi_0^\circ \bar{\iota}_{B_0}$ (with $\iota_{B_0}, \bar{\iota}_{B_0} : B_0 \to QB_0$ the natural inclusions) and set $R_1^\psi = \eta^\psi(B_0)$, $R_2^\psi = \bar{\eta}^\psi(B_0)$ and let R^ψ be the C*-algebra generated in $M_2(\mathcal{M}(C_0))$ by the matrices in

$$\begin{pmatrix} R_1^\psi & R_1^\psi R_2^\psi \\ R_2^\psi R_1^\psi & R_2^\psi \end{pmatrix}$$

We also denote by J_0 the intersection of R^ψ with $M_2(C_0)$.
We can extend $\eta^\psi, \bar{\eta}^\psi$ to maps from the multipliers of B_0 to the multipliers of R_1^ψ, R_2^ψ respectively. By composing these extended maps with the natural maps $\varepsilon^\rho, \bar{\varepsilon}^\rho : A \to \mathcal{M}(B_0)$ (given by $\rho_0^\circ \iota$ and $\rho_0^\circ \bar{\iota}$) we obtain maps $\eta^\psi \varepsilon^\rho, \eta^\psi \bar{\varepsilon}^\rho : A \to \mathcal{M}(R_1^\psi)$ and $\bar{\eta}^\psi \varepsilon^\rho, \bar{\eta}^\psi \bar{\varepsilon}^\rho : A \to \mathcal{M}(R_2^\psi)$.
This means that the maps

$$h_1^{\psi\rho} = \begin{pmatrix} \eta^\psi \varepsilon^\rho & 0 \\ 0 & \bar{\eta}^\psi \bar{\varepsilon}^\rho \end{pmatrix} \qquad h_2^{\psi\rho} = \begin{pmatrix} \eta^\psi \bar{\varepsilon}^\rho & 0 \\ 0 & \bar{\eta}^\psi \varepsilon^\rho \end{pmatrix}$$

are homomorphisms from A to the multipliers of R^ψ.

Lemma 5.9. *If ρ or ψ is q-nuclear, then $h_1^{\psi\rho}$ and $h_2^{\psi\rho}$ are weakly nuclear relative to J_0.*

Proof. Assume that ρ is weakly nuclear. Then the map $A \ni x \mapsto v\varepsilon^\rho(x)v^*$ is nuclear for each $v \in B_0$ and the same for $\bar{\varepsilon}^\rho$. If we apply η^ψ to this map we see that $A \ni x \mapsto w\eta^\psi\varepsilon^\rho(x)w^*$ is nuclear for each $w \in \eta^\psi(B_0)$. If we multiply w in this map by $y \in C_0$ on the left we find that $A \ni x \mapsto yw\eta^\psi\varepsilon^\rho(x)w^*y^*$ is nuclear for each $w \in \eta^\psi(B_0)$ and $y \in C_0$ and the same for $\bar{\eta}^\psi$ and $\bar{\varepsilon}^\rho$ in place of η^ψ and/or ε^ρ. By matrix multiplication this shows that the maps $A \ni x \mapsto zh_i^{\psi\rho}z^*$ are nuclear for $i = 1, 2$ and each $z \in J_0$.
Assume now that ψ is q-nuclear.
If (u_λ) is an approximate unit for B_0, then, by the special definition of R^ψ, we have that $zh_1^{\psi\rho}(u_\lambda)$ and $zh_2^{\psi\rho}(u_\lambda)$ tend to z for each $z \in R^\psi$.
By q-nuclearity of ψ, for each $z \in J_0$ the map $A \ni x \mapsto z\eta^\psi(u_\lambda\varepsilon^\rho(x)u_\lambda^*)z^*$ is nuclear for each λ and the same for $\bar{\eta}^\psi$ and $\bar{\varepsilon}^\rho$. In the limit over λ we get that the map $A \ni x \mapsto z\eta^\psi\varepsilon^\rho(x)z^*$ is nuclear as well (as the set of nuclear c.p. maps is point-norm closed) as the corresponding maps with η^ψ and ε^ρ replaced with $\bar{\eta}^\psi$ and/or $\bar{\varepsilon}^\rho$. This shows that for $i = 1, 2$ and $y \in J_0$ the maps

$A \ni x \mapsto yh_i^{\psi\rho}(x)y^*$ are nuclear and thus that $h_1^{\psi\rho}, h_2^{\psi\rho}$ are weakly nuclear relative to J_0. $\square$

We now examine the product of the bivariant elements represented by ρ_0 and ψ_0. As in the universal case we have that $R^\psi/J_0 \cong M_2(B_0)$ and we can lift the multiplier $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to a multiplier S_0 of J_0 that commutes mod J_0 with $\eta\varepsilon(x) \oplus \bar{\eta}\varepsilon(x)$ for $x \in A$. We set $F_0 = e^{\frac{\pi i}{2} S_0}$ and $\sigma_t^\psi = \operatorname{Ad} e^{\frac{\pi i}{2} S_0}$ and $\sigma^\psi = \sigma_1^\psi$. If $h_2^{\psi\rho}$ is weakly nuclear relative to J_0, so is the composition $\sigma^\psi h_2^{\psi\rho}$. The homomorphism $(\psi_0 \sharp \rho_0)_0 = q(h_1^{\psi\rho}, \sigma^\psi h_2^{\psi\rho}) : qA \to M_2(C_0)$ represents the product and defines an element of $KK(A, C_0)$ which, by Lemma 5.9, is q-nuclear whenever ρ or ψ is. We get

Proposition 5.10. *The pairing $(\psi_0, \rho_0) \mapsto j_{C_0}(\psi_0 \sharp \rho_0)_0$ induces well defined bilinear products $KK^{nuc}(A, B) \times KK(B, C) \to KK^{nuc}(A, C)$ and $KK(A, B) \times KK^{nuc}(B, C) \to KK^{nuc}(A, C)$.*

Proof. The product $j_{C_0} \circ \psi_0 \sharp \rho_0$ represents an element of $KK^{nuc}(A, C)$ by Lemma 5.9 and the discussion after the lemma. It is well defined since q-nuclear homotopies on the side of $[qA, \mathcal{K} \otimes B_0]_{\mathrm{nuc}}$ or $[qB_0, \mathcal{K} \otimes C_0]_{\mathrm{nuc}}$ induce elements of $KK^{nuc}(A, B_0[0,1])$ or $KK^{nuc}(B_0, C_0[0,1])$. The product with such an element gives q-nuclear homotopies of the product. $\square$

5.1. **Associativity.** Assume that we have elements in $KK(A, B)$, $KK(B, C)$, $KK(C, D)$ represented by homomorphisms $\alpha : qA \to \mathcal{K} \otimes B$, $\beta : qB \to \mathcal{K} \otimes C$, $\gamma : qC \to \mathcal{K} \otimes D$ and assume that one of those is q-nuclear. In order to show that the two different products $\gamma \sharp (\beta \sharp \alpha)$ and $(\gamma \sharp \beta) \sharp \alpha$ are homotopic via a q-nuclear homotopy and are themselves both q-nuclear we can proceed exactly as in subsection 3.4. Using the notation from there we obtain modified homomorphisms α, β', γ'. By Proposition 5.10, β', γ' will be q-nuclear if β resp. γ is. According to subsection 3.4 the product is given for both choices of parentheses by the homomorphism $qA \to D_0 \subset \mathcal{K} \otimes D$ given by

$$q(\gamma'_E\beta'_E\alpha_E \oplus \bar{\gamma}'_E\bar{\beta}'_E\alpha_E \oplus \gamma'_E\bar{\beta}'_E\bar{\alpha}_E \oplus \bar{\gamma}'_E\beta'_E\bar{\alpha}_E \,,\, \bar{\gamma}'_E\beta'_E\alpha_E \oplus \gamma'_E\bar{\beta}'_E\alpha_E \oplus \bar{\gamma}'_E\bar{\beta}'_E\bar{\alpha}_E \oplus \gamma'_E\beta'_E\bar{\alpha}_E)$$

It is q-nuclear by Proposition 5.10.

Remark 5.11. (a)In the situation above it follows from Proposition 5.10 that the two products with different choice of parentheses are q-nuclear, if one of the α, β, γ is. But if we have already established that the product is given by the long expression above and that β' or γ' is q-nuclear once β or γ is q-nuclear, then the q-nuclearity of the product is obvious. In fact we get the chain of ideals

$$\gamma'_E\beta'_E\alpha_E A \rhd \gamma'_E\beta'_E B_0 \rhd \gamma'_E C_0 \rhd D_0$$

and an analogous chain of ideals for each composition $\gamma'_E\beta'_E\alpha_E, \bar{\gamma}'_E\bar{\beta}'_E\alpha_E \ldots$. This shows that each of these compositions is weakly nuclear relative to D_0 as soon as one of the α, β, γ is q-nuclear.
(b) For the proof of associativity of the product in KK^{nuc} we could also adapt the arguments from subsection 3.2 or from [6], but the proof in subsection 3.4 is particularly well suited for the situation in KK^{nuc}.

6. The equivariant case

Let G be a locally compact σ-compact group. A G-C*-algebra is a C*-algebra with an action of G by automorphisms $\alpha_g, g \in G$ such that the map $G \ni g \mapsto \alpha_g(x)$ is continuous for each $x \in A$. We denote by $\mathcal{K} = \mathcal{K}_{\mathbb{N}}$ the algebra of compact operators on $\ell^2\mathbb{N}$ and by $\mathcal{K}_G$ the algebra $\mathcal{K}(L^2G)$ of compact operators on L^2G. They are G-algebras with the trivial action and with the adjoint action $\mathrm{Ad}\,\lambda$ of G, respectively, where $\lambda\colon G \to \mathcal{U}(L^2G)$ is the left regular representation. We also denote by $\mathcal{K}_{\mathbb{N}G}$ their tensor product with the tensor product action and will later use the fact that $\mathcal{K}_{\mathbb{N}G}$ is equivariantly isomorphic to $\mathcal{K}_{\mathbb{N}G} \otimes \mathcal{K}_{\mathbb{N}G}$ (by Fell's absorption principle the tensor product of λ by any unitary representation of G is equivalent to a multiple of λ).

Given a G-C*-algebra (A, α) we consider the Hilbert A-module $L^2(G, A)$ with the natural action of G given by $\lambda\alpha$ where λ is the action by translation on G. The algebra of compact operators on $L^2(G, A)$ in the sense of Kasparov is isomorphic to $\mathcal{K}_G \otimes A$. The induced action of G on $\mathcal{K}_G \otimes A$ is $\mathrm{Ad}\,\lambda \otimes \alpha$.

Since $A \mapsto QA$ is a functor, the action α induces actions of G on QA, qA and on Q^2A, q^2A, R, J (see Section 3) which we still denote by α.

Definition 6.1. *Given G-C*-algebras (A, α) and (B, β) where A is separable, define $KK^G(A, B)$ as the set of homotopy classes (in the category of equivariant homomorphisms) of equivariant *-homomorphisms from $\mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes A)$ to $\mathcal{K}_{\mathbb{N}G} \otimes B$.*

Remark 6.2. (a) The pair of homomorphisms $(\mathrm{id} \otimes \iota, id \otimes \bar{\iota})$ gives an equivariant homomorphism from $q(\mathcal{K}_{\mathbb{N}G} \otimes A)$ to $\mathcal{K}_{\mathbb{N}G} \otimes qA$. Therefore every equivariant homomorphism $qA \to \mathcal{K}_{\mathbb{N}G} \otimes B$ (or equivalently every equivariant prequasihomomorphism $A \to \mathcal{K}_{\mathbb{N}G} \otimes B$) induces by stabilization an element of $KK^G(A, B)$.
(b) It is a consequence of Definition 6.1 that the so defined KK^G is the universal functor satisfying the usual properties of homotopy invariance, stability and split exactness, see Section 7. Using the characterization of KK^G by these properties in [18] our KK^G is the same as the one of Kasparov [11]. Ralf Meyer has shown in [13] by direct comparison that Definition 6.1 gives the same functor as the one of [11].

(c) Using Meyer's result our construction of the product below gives an alternative definition of the product in Kasparov's KK^G.

In order to describe the composition product for KK^G we will use an equivariant version of the map φ_A in Section 3 this time from $q(\mathcal{K}_{\mathbb{N}G}\otimes A)$ to $M_2(q^2(\mathcal{K}_{\mathbb{N}G}\otimes A))$. As a first step we are now going to construct an equivariant map φ_0 from $q(\mathcal{K}_G\otimes A)$ to $M_2(\mathcal{K}_G\otimes q^2A)$.

We consider first, as in Section 1, the algebras

$$R = \begin{pmatrix} R_1 & R_1R_2 \\ R_2R_1 & R_2 \end{pmatrix} \qquad D = C^*\left\{\begin{pmatrix} \eta\varepsilon(x) & 0 \\ 0 & \bar\eta\varepsilon(x) \end{pmatrix} \quad x\in A\right\}$$

where $R_1 = \eta(qA)$, $R_2 = \bar\eta(qA)$ as well as the ideal $J = R\cap M_2(q^2A)$.

As in Section 3 we have that $(R+D)/J$ is isomorphic to the subalgebra of $M_2(Q(A))$ generated by $M_2(qA)$ together with the matrices

$$\begin{pmatrix} \iota(x) & 0 \\ 0 & \iota(x) \end{pmatrix} \quad x\in A.$$

Using the equivariant version of Proposition 2.2 (Thomsen's noncommutative Tietze extension theorem) we can lift the multiplier $S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of R/J to a self-adjoint multiplier S of J that commutes mod J with D and which satisfies $\alpha_g(S)-S\in J$ for all $g\in G$.

This multiplier S can be extended to a G-invariant self-adjoint element S' of $\mathcal{B}(L^2(G,J))$ by setting $S'(\xi)(s) = S_s\xi(s)$ for $s\in G$ where $S_s = \alpha_s(S) = \alpha_s S\alpha_s^{-1}$ and where $\xi\in C_c(G,A)\subset L^2(G,A)$. It is immediate that S' is invariant for the action $\lambda\alpha$ of G on $L^2(G,J)$. Thus S' defines a G-invariant multiplier of $\mathcal{K}_G\otimes J$.

The important point now is that moreover S' commutes mod $\mathcal{K}_G\otimes J$ with $D' = \mathcal{K}_G\otimes D$. In fact, for a typical rank 1 element of the form $|\,f_1\,\rangle\langle\, f_2\,|$ in $\mathcal{K}_G$ with $f_1,f_2\in C_c(G,\mathbb{C})$, $x\in D$ and $\xi\in C_c(G,J)\subset L^2(G,J)$ we get

$$\begin{aligned}
([S',(|f_1\rangle\langle f_2|\otimes x)]\ \xi)\,(s) &= f_1(s)\int(\overline{f_2(t)}(S_sx - xS_t))\xi(t)dt \\
&= f_1(s)\int(\overline{f_2(t)}(S_sx - S_tx))\xi(t)dt - f_1(s)\int(\overline{f_2(t)}(S_tx - xS_t))\xi(t)dt
\end{aligned}$$

where $S_tx - xS_t$, $S_sx - S_tx$ are in J and continuous in t. In fact, S was chosen, using 2.2 to commute mod J with D and such that $S_s - S$, $S_t - S$ are in J and continuous in s,t.

As in Section 3 we can now choose $F' = e^{\frac{\pi i}{2}S'}$. Then $\mathrm{Ad}\, F'$ defines an automorphism σ' of the multipliers of $\mathcal{K}_G \otimes J$. Tensoring by $\mathrm{id}_{\mathcal{K}_G}$ we extend the maps $\eta\varepsilon, \eta\bar{\varepsilon}, \bar{\eta}\varepsilon, \bar{\eta}\bar{\varepsilon} : A \to Q^2A$ to homomorphisms from $\mathcal{K}_G \otimes A$ to $\mathcal{K}_G \otimes Q^2A$, still denoted by $\eta\varepsilon, \eta\bar{\varepsilon}, \bar{\eta}\varepsilon, \bar{\eta}\bar{\varepsilon}$. Then the pair of homomorphisms

$$\left(\begin{pmatrix} \eta\varepsilon & 0 \\ 0 & \bar{\eta}\bar{\varepsilon} \end{pmatrix}, \sigma' \begin{pmatrix} \bar{\eta}\varepsilon & 0 \\ 0 & \eta\bar{\varepsilon} \end{pmatrix} \right)$$

defines an equivariant homomorphism $\varphi_0 : q(\mathcal{K}_G \otimes A)$ to $\mathcal{K}_G \otimes J$ (note that, by definition of R, both $\begin{pmatrix} \eta\varepsilon & 0 \\ 0 & \bar{\eta}\bar{\varepsilon} \end{pmatrix}$ and $\begin{pmatrix} \bar{\eta}\varepsilon & 0 \\ 0 & \eta\bar{\varepsilon} \end{pmatrix}$ map $\mathcal{K}_G \otimes A$ to the multipliers of $\mathcal{K}_G \otimes R$).

We can now stabilize the algebras involved in the definition of φ_0 by $\mathcal{K}_{\mathbb{N}G}$. Setting $A' = \mathcal{K}_{\mathbb{N}G} \otimes A$ and using the fact that $\mathcal{K}_{\mathbb{N}G} \otimes \mathcal{K}_{\mathbb{N}G} \cong \mathcal{K}_{\mathbb{N}G}$ we obtain the stabilized equivariant map

$$\varphi'_A : \mathcal{K}_{\mathbb{N}G} \otimes qA' \to \mathcal{K}_{\mathbb{N}G} \otimes J'$$

where $J' = R' \cap q^2(A')$. As in the non-equivariant case, the map φ'_A induces the associative product $KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C)$ as follows: let elements of $KK^G(A, B)$ and of $KK^G(B, C)$ be represented by equivariant maps

$$\mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes A) \xrightarrow{\mu} \mathcal{K}_{\mathbb{N}G} \otimes B \quad \text{and} \quad \mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes B) \xrightarrow{\nu} \mathcal{K}_{\mathbb{N}G} \otimes C$$

respectively. Using the fact that $\mathcal{K}_{\mathbb{N}G} \cong \mathcal{K}_{\mathbb{N}G} \otimes \mathcal{K}_{\mathbb{N}G}$, we get a map

$$q^2(\mathcal{K}_{\mathbb{N}G} \otimes A) \cong q^2(\mathcal{K}_{\mathbb{N}G} \otimes \mathcal{K}_{\mathbb{N}G} \otimes A) \xrightarrow{\kappa} q(\mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes A)$$

and, using this, we can form the following composition

$$\begin{aligned} \mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes A) \xrightarrow{\varphi'_A} \mathcal{K}_{\mathbb{N}G} \otimes q^2(\mathcal{K}_{\mathbb{N}G} \otimes A) \xrightarrow{\kappa} \mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes A)) \\ \xrightarrow{\mathrm{id} \otimes q(\mu)} \mathcal{K}_{\mathbb{N}G} \otimes q(\mathcal{K}_{\mathbb{N}G} \otimes B) \xrightarrow{\nu} \mathcal{K}_{\mathbb{N}G} \otimes C \end{aligned}$$

which represents the product in $KK^G(A, C)$.

6.1. **Associativity.** Associativity of the product in KK^G follows as in Subsection 3.2 since all the isomorphisms and homotopies used there are manifestly G-equivariant once the automorphisms σ_t are chosen to be equivariant.

7. Universality and connection to the usual definitions

We show now that the functors $KK(X; \cdot)$ and KK^G that we have studied in Sections 4 and 6 are characterized - just like ordinary KK - by split exactness together with homotopy invariance and stability in their respective category. It seems that KK^{nuc} could also be characterized by a suitable more involved split exactness property for exact sequences with a weakly nuclear splitting.

We leave that open - partly also because we think that such a characterization would be of minor interest.
Split exactness on equivariant, equivariantly split exact sequences does in fact follow for the functors $KK(X;\,\cdot\,)$ and KK^G that we have studied in Sections 4 and 6 from the existence of the product, by the simple argument in [6, 2.1].

7.1. **The case of ideal related KK-theory.** Let X be a topological space.

Proposition 7.1. *$KK(X;\,\cdot\,,\,\cdot\,)$ is the universal functor from the category of separable C*-algebras with an action of X to an additive category which is stable, homotopy invariant and split exact on exact sequences of algebras in the category with an X-equivariant homomorphism splitting.*

Proof. Given a C*-algebra A with an action of X, consider the exact sequence

$$0 \to q_X A \to Q_X A \to A \to 0$$

with the equivariant splitting $\iota : A \to Q_X A$. The usual argument showing that a free product of C*-algebras is KK-equivalent to the direct sum (see [6] Proposition 3.1) is compatible with the action of X, so that $Q_X A$ is equivalent in $KK(X;\cdot,\cdot)$ to $A \oplus A$ with the natural action of X - just by homotopy invariance and stability. Let now F be a functor from the category of separable C*-algebras with an X-action to an additive category which is stable, homotopy invariant and equivariantly split exact. Then $F(Q_X A)$ is isomorphic, via the natural map, to $F(A \oplus A) = F(A) \oplus F(A)$ and by split exactness consequently $F(q_X A) \cong F(A)$. By Definition 4.3 every element of $KK(X; A, B)$ is represented by an X-equivariant homomorphism $q_X A \to \mathcal{K} \otimes B$. Applying F to the homotopy class of such a homomorphism we get a morphism $F(A) \cong F(q_X A) \to F(\mathcal{K} \otimes B) \cong F(B)$. Since the isomorphisms involved are natural this morphism is uniquely determined.
Conversely $KK(X;\cdot)$ is homotopy invariant, stable and splits on X-equivariantly split exact sequences. □

7.2. **The case of KK^G.** If G is a locally compact σ-compact group we also have

Proposition 7.2. *(cf.*[13]*) KK^G is the universal functor on the category of separable G-C*-algebras which is homotopy invariant, stable under tensor product by $\mathcal{K}_{\mathbb{N}G}$ and split exact on extensions $0 \to I \to E \to A \to 0$ of G-C*-algebras with an equivariant splitting homomorphism $s : A \to E$.*

Proof. Let F be a functor with the given properties from the category of G-C*-algebras to an additive category and set $A' = \mathcal{K}_{\mathbb{N}G} \otimes A$. Homotopy invariance and stability of F imply that $F(QA') \cong F(A' \oplus A')$ (by the argument in [6] Proposition 3.1 which is compatible with the action of G). Split exactness

implies that $F(QA') \cong (F(qA') \oplus F(A')$ and finally that $F(qA') \cong F(A')$ naturally. Since also $F(A') \cong F(A)$ for all A by stability, the assertion then follows from the definition of KK^G, see 6.1.
Conversely, KK^G is equivariantly split exact by the remark at the beginning of the section. □

7.3. **Connection to the usual definitions.** The usual definitions of the different versions of $KK(A,B)$ are based on A-B Kasparov modules (E,F) with additional structure. In such a Kasparov module one can always assume that $F = F^*$ and $F^2 = 1$. Conjugation of the (first component for the $\mathbb{Z}/2$-grading of the) left action φ of A on E by F gives a second homomorphism $\bar{\varphi} : A \to \mathcal{B}(E)$. Depending on the situation, φ will 'weakly' respect the additional structure (X-equivariance, G-equivariance or nuclearity respectively). Now in order to get a homomorphism from qA to $\mathcal{K}(E)$ respecting the additional structure we need to know that $\bar{\varphi}$ also respects the structure 'weakly'. Since $\bar{\varphi} = \operatorname{Ad} F\varphi$, and $\operatorname{Ad} F$ is inner, this is automatic for X-equivariance. In the case of KK^G this has been established in the paper by Ralf Meyer. In the case of KK^{nuc} the equivalence between q-nuclear homomorphisms $qA \to \mathcal{K}(E)$ and nuclear Kasparov modules has been shown in Proposition 5.4. In the case of KK^G and $KK(X)$ we get the equivalence then from the universality of our definition.

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