A note on "Exact Solution of Bipartite Fluctuations in One-Dimensional Fermions"

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For a one-dimensional system of free fermions, we derive a connection between the full counting statistics of domain-wall and alternating occupancy states. We derive linear growth with time at the long times for the even moments in the latter case.

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The quantum evolution of highly non-equilibrium states attracts a lot of attention [1]. For one-dimensional systems, the effects of the constrained dynamics are especially pronounced, both in transport and statistical properties [2, 3]. Inspired by ultra-cold atom experiments [4] a recent work [5] exactly calculated the variance of a bipartite fluctuation in one-dimensional noninteracting fermionic dynamics starting from an alternating (A) occupancy state . This quantity grows linearly at long times, which starkly contrasts with the dynamics for the same system initialized in the domain-wall (DW) state [6, 7]. Surprisingly, the averaged number of particles [6–10] in the DW case coincides with the variance in the A case given in [5] (after some rescalings, see Eq. (6)).

In this note, we extend this relation to all even moments of the bipartite fluctuations. More precisely, we consider a lattice system of free fermions with $4L$ sites, whose dynamics is described by the hopping Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{4L}\left(a_{i+1}^{+}a_i + a_i^{+}a_{i+1}\right). \tag{1}$$

We assume hard-wall boundary conditions, $a_{4L+1} = 0$, $a_{-1} = 0$. We specify by $N_R = \sum_{i=2L+1}^{4L} a_i^{+}a_i$ the number of particles in the right part of the system. We are going to study the moment-generating function, also known as the full-counting statistics (FCS)

$$\chi_\Omega(\lambda, t) = \langle\Omega|e^{\lambda N_R(t)}|\Omega\rangle \equiv \langle\Omega|e^{itH}e^{\lambda N_R}e^{-itH}|\Omega\rangle. \tag{2}$$

The centered moments $C_\Omega^{(k)} \equiv \langle\Omega|\left(N_R(t) - C_\Omega^{(1)}\right)^k|\Omega\rangle$ can be extracted from the FCS as coefficients for the expansion at $\lambda = 0$

$$\ln\chi_\Omega(\lambda) = \lambda C_\Omega^{(1)} + \sum_{k=2}^{\infty}\frac{\lambda^k}{k!}C_\Omega^{(k)}, \tag{3}$$

with $C_\Omega^{(1)} = \langle\Omega|N_R(t)|\Omega\rangle$.

The initial state $|\Omega\rangle$ for the DW case is given by $|\mathrm{DW}\rangle = \prod_{i=1}^{2L}|i\rangle$, while for A-state (Néel) we specify the state as $|\mathrm{A}\rangle = \prod_{i=1}^{2L}|2i\rangle$. Here $|i\rangle = a_i^{+}|0\rangle$ is a single fermion state in the site i [11]. One can also use a slightly different definition of these states $|\widetilde{\mathrm{DW}}\rangle = \prod_{i=2L+1}^{4L}|i\rangle$ and $|\widetilde{\mathrm{A}}\rangle = \prod_{i=1}^{2L}|2i-1\rangle$. Because of the particle-hole symmetry the corresponding statistics are closely related

$$\frac{\chi_{\mathrm{A}}(-\lambda,t)}{\chi_{\widetilde{\mathrm{A}}}(\lambda,t)} = \frac{\chi_{\mathrm{DW}}(-\lambda,t)}{\chi_{\widetilde{\mathrm{DW}}}(\lambda,t)} = e^{-2\lambda L}. \tag{4}$$

We now formulate the main statement of this note

$$\boxed{\chi_{\mathrm{A}}(\lambda,t)\chi_{\mathrm{A}}(-\lambda,t) = \chi_{\mathrm{DW}}(2\ln\cosh(\lambda/2), 2t).} \tag{5}$$

This relation, together with (3), immediately allows us to connect different moments for the two initial states. The first few such connections are

$$4C_{\mathrm{A}}^{(2)}(t) = C_{\mathrm{DW}}^{(1)}(2t), \tag{6}$$

$$4C_{\mathrm{A}}^{(4)}(t) = 3C_{\mathrm{DW}}^{(2)}(2t) - C_{\mathrm{DW}}^{(1)}(2t), \tag{7}$$

$$16C_{\mathrm{A}}^{(6)}(t) = 4C_{\mathrm{DW}}^{(1)}(2t) - 15C_{\mathrm{DW}}^{(2)}(2t). \tag{8}$$

The full expression for the FCS in the DW case is available in the form of a Fredholm determinant of Bessel kernel type in Refs. [6, 7]. In this form, its long-time asymptotic can be obtained analytically [12]

$$\ln\frac{\chi_{\mathrm{DW}}(\lambda,t)}{G^2\left(1+\frac{i\lambda}{2\pi}\right)G^2\left(1-\frac{i\lambda}{2\pi}\right)} = \frac{\lambda t}{\pi} + \frac{\lambda^2\ln(4t)}{2\pi} + o(1). \tag{9}$$

Here $G(x)$ is the Barnes G-function. Combining this expression with (5) and (3) one immediately gets the linear growth of all even moments in the A-case. Moreover, even moments completely define the entanglement entropy S_{A} between the two parts of the system [13–15]. This way, we obtain the linear growth of entanglement

$$S_{\mathrm{A}}(t) = \int_{-\infty}^{\infty}\frac{\ln\chi_{\mathrm{A}}(\lambda,t)}{4\sinh^2(\lambda/2)}d\lambda \overset{t\to\infty}{=} \frac{2\ln 2}{\pi}t. \tag{10}$$

To prove (5) we first introduce notations for the eigenvectors of the Hamiltonian (1) $H|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle$. We parameterize them by the same integers as the lattice sites $\alpha = 1, 2, \dots 4L$ but use Greek letters to avoid confusion.

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The corresponding energies are $\varepsilon_\alpha = \cos(\pi\alpha/(4L+1))$ and explicit expressions for $|\alpha\rangle$ reads

$$|\alpha\rangle = \sqrt{\frac{2}{4L+1}}\sum_{j=1}^{4L}\sin\left(\frac{\pi\alpha j}{4L+1}\right)|j\rangle. \tag{11}$$

The FCS can be computed employing Wick's theorem for group-like elements of the fermionic algebra [16]. In the DW case this leads to

$$\chi_{\rm DW}(\lambda,t) = \det_{1\le j,k\le 2L}\left(\delta_{jk} + z_\lambda X_{jk}\right) \tag{12}$$

where $z_\lambda = (e^\lambda - 1)$ and the matrix X after the insertions of the resolutions of unity in the energy space reads

$$X_{jk} = \sum_{\alpha,\beta}\langle j|\alpha\rangle e^{it\varepsilon_\alpha}\langle\alpha|P_R|\beta\rangle e^{-it\varepsilon_\beta}\langle\beta|k\rangle. \tag{13}$$

Here P_R is a projector on the right-hand side of the system. It can be written as $P_R = |\widetilde{\rm DW}\rangle\langle\widetilde{DW}|$ understood in the single particle sector. The corresponding matrix in the energy space is denoted as $\mathcal{R}$ with matrix elements

$$\mathcal{R}_{\alpha\beta} \equiv \langle\alpha|P_R|\beta\rangle = \sum_{k=2L+1}^{4L}\langle\alpha|k\rangle\langle k|\beta\rangle. \tag{14}$$

We can rewrite the FCS as a determinant in the energy space. Understanding sums in (13) as matrix multiplication and using the cyclic relation $\det(1+AB) = \det(1+BA)$ we arrive at

$$\chi_{\rm DW}(\lambda,t) = \det_{1\le n,m\le 4L}\left(1 + z_\lambda\mathcal{R}_t\mathcal{L}\right). \tag{15}$$

Here $\mathcal{R}_t = \Lambda_t\mathcal{R}\Lambda_t^{-1}$ stands for the time dependent projector; by Λ_t we have denoted the diagonal matrix with the entries $(\Lambda_t)_{\alpha\beta} = e^{it\varepsilon_n}\delta_{\alpha\beta}$. Additionally, we have introduced $\mathcal{L}_{\alpha\beta} = \langle\alpha|{\rm DW}\rangle\langle{\rm DW}|\beta\rangle$, understood in the same sense as (14). Obviously $\mathcal{R} + \mathcal{L} = 1$.

In Eq. (15) one immediately sees that the factor $\mathcal{R}_t$ is responsible for the observable and $\mathcal{L}$ for the initial state. For the A-state instead of the matrix $\mathcal{L}$ we have a much simpler expressions: $\langle\alpha|{\rm A}\rangle\langle{\rm A}|\beta\rangle = (\delta_{\alpha\beta} - \mathcal{P}_{\alpha\beta})/2$ with $\mathcal{P}_{\alpha\beta} = \delta_{\alpha,4L+1-\beta}$. This can be obtained using the exact form of the eigenvectors (11) or making use of the chiral symmetry present in the system. In the single-particle basis the symmetry is given by $C = \mathbb{1}_{2L}\otimes\sigma_z$, satisfying $\{H, C\} = 0$. One can easily see that C distinguishes odd and even sites. Moreover, the way we have parameterized energy allows us to conclude $\varepsilon_{4L+1-\alpha} = -\varepsilon_\alpha$, which is equivalent to $\mathcal{P}\Lambda_t = \Lambda_t^{-1}\mathcal{P}$. This relation together with $\mathcal{P}^2 = 1$ and $\mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P}$ leads to

$$(\mathcal{R}_t\mathcal{P})^2 = \Lambda_t^{-1}\mathcal{R}_{2t}\mathcal{R}\Lambda_t. \tag{16}$$

Taking into account that $\mathcal{R}_t$ is a projector i.e. $\mathcal{R}_t^2 = \mathcal{R}_t$ we can present

$$\begin{aligned}\chi_{\rm DW}(\lambda,t) = \det\left(1 + z_\lambda\mathcal{R}_t(1-\mathcal{R})\right) = \\ \det\left(1 + z_\lambda\mathcal{R}_t\right)\det\left(1 + z_{-\lambda}\mathcal{R}_t\mathcal{R}\right)\end{aligned} \tag{17}$$

Notice that the first determinant can be easily computed $\det(1 + z_\lambda\mathcal{R}_t) = (1+z_\lambda)^{2L} = e^{2\lambda L}$. Similary for the A-state $|{\rm A}\rangle$ we obtain

$$\begin{aligned}\chi_{\rm A}(\lambda,t) = \det\left(1 + z_\lambda\mathcal{R}_t\frac{1-\mathcal{P}}{2}\right) = \\ \det\left(1 + z_\lambda\mathcal{R}_t/2\right)\det\left(1 - \mu_\lambda\mathcal{R}_t\mathcal{P}\right)\end{aligned} \tag{18}$$

where $\mu_\lambda = (e^\lambda - 1)/(e^\lambda + 1)$. We notice that

$$\begin{aligned}&\det\left(1 - \mu_\lambda\mathcal{R}_t\mathcal{P}\right)\det\left(1 + \mu_\lambda\mathcal{R}_t\mathcal{P}\right) \\ &\quad = \det\left(1 - \mu_\lambda^2(\mathcal{R}_t\mathcal{P})^2\right) = \det(1 - \mu_\lambda^2\mathcal{R}_{2t}\mathcal{R}),\end{aligned} \tag{19}$$

where in the last step we have used (16) and eliminated the matrices Λ_t under the determinant. At the final step we use

$$\det\left(1 + z_\lambda\mathcal{R}_t/2\right) = e^{\lambda L}\left(\cosh(\lambda/2)\right)^{2L}, \tag{20}$$

to arrive that the result (17). Moreover, from the relations (17) and (18), Eq. (4) follows immediately.

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