# On the first $\tau$-tilting Hochschild cohomology of an algebra 

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#### Abstract

In this paper we introduce, according to one of the main ideas of $\tau$ tilting theory, the $\tau$-tilting Hochschild cohomology in degree one of a finite dimensional $k$-algebra $\Lambda$, where $k$ is a field. We define the excess of $\Lambda$ as the difference between the dimensions of the $\tau$-tilting Hochschild cohomology in degree one and the dimension of the usual Hochschild cohomology in degree one.

One of the main results is that for a zero excess bound quiver algebra $\Lambda=$ $k Q / I$, the Hochschild cohomology in degree two $H H^{2}(\Lambda)$ is isomorphic to the space of morphisms $\operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)$. This may be useful to determine when $H H^{2}(\Lambda)=0$ for these algebras.

We compute the excess for hereditary, radical square zero and monomial triangular algebras. For a bound quiver algebra $\Lambda$, a formula for the excess of $\Lambda$ is obtained. We also give a criterion for $\Lambda$ to be $\tau$-rigid.


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## 1 Introduction

Let $A$ be a finite dimensional algebra over a field $k$, that we will call an algebra for short. Let $M$ and $N$ be finitely generated left $A$-modules, henceforth called left $A$-modules. Let $\tau$ denote the Auslander-Reiten translation, see for instance [2] or [15], and denote $\mathrm{D}(-)=\operatorname{Hom}_{k}(-, k)$. We reproduce here an extract from B. Marsh's lecture notes in Cologne [17]: "the Auslander-Reiten duality suggests that in contexts where $\operatorname{Ext}_{A}^{1}(M, N)$ appears, we might investigate replacing it with $\mathrm{DHom}_{A}(N, \tau M)$ and this can be regarded as one of the main ideas of $\tau$-tilting theory." While $D$ is absent in the original text, $D$ is present in Auslander-Reiten's duality formula for it to be functorial. Of course adding $D$ does not change the dimensions. Recall that $M$ is called $\tau$-rigid if $\operatorname{Hom}_{A}(M, \tau M)=0$, see for instance [15, Subsection 4.1].

On the other hand, let $\Lambda^{e}=\Lambda \otimes_{k} \Lambda^{\text {op }}$ be the enveloping algebra of an algebra $\Lambda$. Let $X$ be a $\Lambda$-bimodule. The Hochschild cohomology of $\Lambda$ with coefficients in $X$ is $H^{n}(\Lambda, X)=\operatorname{Ext}_{\Lambda^{e}}^{n}(\Lambda, X)$, see [7, 16, 19] and it is denoted $H H^{n}(\Lambda)$ when $X=\Lambda$. Moreover, Hochschild homology is $H_{n}(\Lambda, X)=\operatorname{Tor}_{n}^{\Lambda^{e}}(\Lambda, X)$. Since left

[^0]$\Lambda^{e}$-modules are the same as $\Lambda$-bimodules, in the sequel we often replace $\Lambda^{e}$ with $\Lambda-\Lambda$.

According to the main idea of $\tau$-tilting theory mentioned above, we will investigate in this paper the replacement of $\operatorname{Ext}_{\Lambda-\Lambda}^{1}(\Lambda, X)$ by the $\tau$-tilting Hochschild cohomology in degree one ${ }^{\tau} H^{1}(\Lambda, X)=\operatorname{DHom}_{\Lambda-\Lambda}(X, \tau \Lambda)$. Note that here $\tau$ is the Auslander-Reiten translation of left $\Lambda^{e}$-modules, that is of $\Lambda$-bimodules. When $X=\Lambda$, we denote ${ }^{\tau} H H^{1}(\Lambda)={ }^{\tau} H^{1}(\Lambda, \Lambda)$. The excess $e(\Lambda)$ is defined as $\operatorname{dim}_{k}{ }^{\tau} H H^{1}(\Lambda)-\operatorname{dim}_{k} H H^{1}(\Lambda)$.

One of the main results of this paper is that for a zero excess bound quiver algebra $\Lambda=k Q / I$ we have $H H^{2}(\Lambda)=\operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)$ - see Corollary 4.8. This result will be useful in a future work to determine when an algebra with zero excess has zero Hochschild cohomology in degree 2. The algebras $\Lambda$ with $H H^{2}(\Lambda)=0$ are important since they are rigid in the following sense. Suppose that $k$ is algebraically closed and let $V$ be a $k$-vector space of dimension $n$. Let $\mathcal{A} l g_{n}$ be the affine open subscheme of algebra structures with 1 of the affine algebraic scheme defined by $\mathcal{S}_{n}(R)=\left\{\right.$ associative $R$-algebra structures on $\left.R \otimes_{k} V\right\}$, where $R$ is a commutative $k$-algebra. Corollary 2.5 of [13] states that $H H^{2}(\Lambda)=0$ if and only if the orbit of $\Lambda \in \mathcal{A l} g_{n}$ under the general linear group $\mathcal{G} L(V)$ is an open subscheme of $\mathcal{A l} g_{n}$ that is by definition, $\Lambda$ is rigid. Moreover, P. Gabriel in [13, p. 140] mentions that it should be one of the main tasks of associative algebra to determine for every $n$ the number of irreducible components of $\mathcal{A l g} g_{n}$. The determination of open orbits makes it possible to obtain lower bounds for the number of irreducible components of $\mathcal{A l} g_{n}$, as G. Mazzola did in [18 p. 100].

The paper is organised as follows. In Section 2 we give a more detailed definition, as well as properties of the $\tau$-tilting Hochschild cohomology and of the excess. Let Tr be the transpose of a bimodule, see for instance [2], and recall that $X_{\Lambda}$ denotes the coinvariants of a $\Lambda$-bimodule $X$, see Remark 2.4. We prove that ${ }^{\tau} H H^{1}(\Lambda)=(\operatorname{Tr} \Lambda)_{\Lambda}$ and we give a formula for the dimension of the vector space ${ }^{\tau} H H^{1}(\Lambda)$.

In Section 3 for an hereditary algebra $\Lambda$ we prove that the dimensions of ${ }^{\tau} H H^{1}(\Lambda)$ and $H H^{1}(\Lambda)$ are equal. We say that an algebra $\Lambda$ has the $H^{2}$ cancellation properties if $H H^{2}(\Lambda)=0=H^{2}\left(\Lambda, r^{i}\right)$ for all $i>0$, where $r$ is the Jacobson radical of $\Lambda$. For instance hereditary algebras have the $H^{2}$ cancellation properties. We obtain that $e(\Lambda)=0$ whenever $\Lambda$ has the $H^{2}$ cancellation properties, based on a formula for the dimension of $H H^{1}(\Lambda)$ in [9].

In Section 3 we also consider radical square zero algebras and monomial algebras whose quiver has no oriented cycles. For those algebras $\Lambda$ we prove that $H H^{1}(\Lambda)=$ 0 if and only if ${ }^{\tau} H H^{1}(\Lambda)=0$, and this occurs precisely when $Q$ is a tree. This extends a result of [5]. We provide examples where the excess is not zero.

In Theorem 4.5 we give a formula for the excess of a bound quiver algebra. Finally we provide a criterium for the algebra $\Lambda$ to be $\tau$-rigid in terms of the dimension of its Hochschild cohomology in degree 2.

## $2 \tau$-tilting Hochschild cohomology in degree one

We begin this section by briefly recalling the definition of the Auslander-Reiten translation and the duality formula which is useful for our aims, for more details see for instance [2] or [3]. Let $A$ be an algebra and $M$ a left $A$-module.

First, the transpose $\operatorname{Tr} M$ is defined as follows. Consider a minimal projective presentation of $M$

$$
P_{1} \xrightarrow{d_{1}} P_{0} \xrightarrow{d_{0}} M \longrightarrow 0 .
$$

Applying to $d_{1}$ the functor $\operatorname{Hom}_{A}(-, A)$ which sends left $A$-modules to right $A$ modules we get

$$
\operatorname{Hom}_{A}\left(P_{0}, A\right) \xrightarrow{d_{1}^{*}} \operatorname{Hom}_{A}\left(P_{1}, A\right)
$$

By definition, $\operatorname{Tr} M=$ Coker $d_{1}^{*}$.
This gives a bijection between the isomorphism classes of indecomposable nonprojective left $A$-modules and the isomorphism classes of indecomposable nonprojective right $A$-modules.

Next, the exact functor $\mathrm{D}=\operatorname{Hom}_{k}(-, k)$ sends right $A$-modules to left $A$ modules. We obtain an exact sequence of left $A$-modules

$$
0 \longrightarrow \mathrm{D} \operatorname{Tr} M \longrightarrow \mathrm{DHom}_{A}\left(P_{1}, A\right) \xrightarrow{\mathrm{Dd}_{1}^{*}} \mathrm{DHom}_{A}\left(P_{0}, A\right)
$$

Finally by definition $\tau M=\mathrm{D} \operatorname{Tr} M$.
This gives a bijection between the isomorphism classes of indecomposable nonprojective left $A$-modules and the isomorphism classes of indecomposable noninjective left $A$-modules.

Let $M$ and $N$ be left $A$-modules. Let $\mathcal{I H o m}_{A}(M, N)$ be the $k$-subspace of $\operatorname{Hom}_{A}(M, N)$ of morphisms which factor through an injective left $A$-module. The quotient is denoted $\overline{\mathrm{Hom}}_{A}(M, N)$. The Auslander-Reiten duality formula in [1] is

$$
\operatorname{Ext}_{A}^{1}(M, N)=\mathrm{D} \overline{\operatorname{Hom}}_{A}(N, \tau M)
$$

As mentioned in the Introduction, one of the main ideas of $\tau$-tilting theory is to replace $\operatorname{Ext}_{A}^{1}(M, N)$ with $\operatorname{DHom}_{A}(N, \tau M)$, which in a sense amounts to recover the missing morphisms which factor through injectives.

Let $\Lambda$ be an algebra. To define the $\tau$-tilting Hochschild cohomology in degree one, recall that $H^{1}(\Lambda, X)=\operatorname{Ext}_{\Lambda-\Lambda}^{1}(\Lambda, X)$. Note that this concerns bimodules, hence in the following $\tau$ is the Auslander-Reiten translation for bimodules or equivalently of left $\Lambda^{e}$-modules.

Definition 2.1 Let $\Lambda$ be an algebra, and let $X$ be a $\Lambda$-bimodule. The $\tau$-tilting cohomology of $\Lambda$ with coefficients in $X$ is

$$
{ }^{\tau} H^{1}(\Lambda, X)=\operatorname{DHom}_{\Lambda-\Lambda}(X, \tau \Lambda)
$$

In this paper we will focus in the case $X=\Lambda$ :

$$
{ }^{\tau} H H^{1}(\Lambda)=\operatorname{DHom}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda)
$$

Definition 2.2 The excess of an algebra $\Lambda$ is

$$
e(\Lambda)=\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda)-\operatorname{dim}_{k} H H^{1}(\Lambda)
$$

Lemma 2.3 The excess is a non negative integer, equal to $\operatorname{dim}_{k} \mathcal{I H o m}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda)$.

Proof. By definition

$$
{ }^{\tau} H H^{1}(\Lambda)=\operatorname{DHom}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda),
$$

while

$$
H H^{1}(\Lambda)=\operatorname{Ext}_{\Lambda-\Lambda}^{1}(\Lambda, \Lambda)
$$

The Auslander-Reiten duality formula is

$$
\operatorname{Ext}_{\Lambda-\Lambda}^{1}(\Lambda, \Lambda)=\mathrm{D}\left(\frac{\operatorname{Hom}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda)}{\mathcal{I} \operatorname{Hom}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda)}\right)
$$

Next we recall some well known facts about invariants and coinvariants.
Remark 2.4 Let $\Lambda$ be an algebra and let $X$ be a $\Lambda$-bimodule.

- The subspace of invariants of $X$ is

$$
H^{0}(\Lambda, X)=X^{\Lambda}=\{x \in X \mid \forall \lambda \in \Lambda, \lambda x=x \lambda\}=\operatorname{Hom}_{\Lambda-\Lambda}(\Lambda, X)
$$

where the last canonical isomorphism sends $\varphi \in \operatorname{Hom}_{\Lambda-\Lambda}(\Lambda, X)$ to $\varphi(1)$.

- The vector space of coinvariants of $X$ is

$$
\left.H_{0}(\Lambda, X)=X_{\Lambda}=X /\langle\lambda x-x \lambda| \lambda \in \Lambda \text { and } x \in X\right\rangle=\Lambda \otimes_{\Lambda-\Lambda} X
$$

where the last canonical isomorphism sends $\lambda \otimes x \in \Lambda \otimes_{\Lambda-\Lambda} X$ to the class of $\lambda x$.

- It is easy to show that $D\left(X^{\Lambda}\right)=(D X)_{\Lambda}$. Observe that more generally we have in all degrees

$$
D H^{n}(\Lambda, X)=H_{n}(\Lambda, D X)
$$

Proposition 2.5 Let $\Lambda$ be an algebra. We have

$$
{ }^{\tau} H H^{1}(\Lambda)=(\operatorname{Tr} \Lambda)_{\Lambda} .
$$

Proof. Let $Y=\mathrm{D} T r \Lambda$. According to Remark 2.4 and using that $\mathrm{D}^{2}$ is the identity, we have the following chain of equalities and canonical isomorphisms of vector spaces:

$$
{ }^{\tau} H H^{1}(\Lambda)=\operatorname{DHom}_{\Lambda-\Lambda}(\Lambda, Y)=\mathrm{D}\left(Y^{\Lambda}\right)=(\mathrm{D} Y)_{\Lambda}=(\mathrm{DD} \operatorname{Tr} \Lambda)_{\Lambda}=(\operatorname{Tr} \Lambda)_{\Lambda}
$$

In this paper a quiver $Q$ is a finite oriented graph, given by a set of vertices $Q_{0}$, a set of arrows $Q_{1}$, and two maps called source and target $s, t: Q_{1} \rightarrow Q_{0}$. The quiver algebra $k Q$ is a vector space with basis the set $B$ of all oriented paths in $Q$, including those of length 0 , that is $Q_{0}$. The product of two paths is their concatenation if it is possible and 0 otherwise. The algebra structure of $k Q$ is obtained by extending linearly the product on paths. Note that $Q_{0}$ is a set of orthogonal idempotents, their sum gives the unit of $k Q$. The set of paths of strictly positive length $B^{>0}$ is
a basis of the ideal $F=\left\langle Q_{1}\right\rangle$. An ideal $I$ is admissible if there exists $n \geq 2$ such that $F^{n} \subset I \subset F^{2}$. The quotient algebra $k Q / I$ is called a bound quiver algebra.

An algebra $\Lambda$ is called sober if the endomorphism algebra of each simple left $\Lambda$-module is reduced to $k$, which is always the case if $k$ is algebraically closed. A well known result of P.Gabriel is that any sober algebra is Morita equivalent to a bound quiver algebra $k Q / I$ for a unique quiver $Q$. Note that the admissible ideal $I$ is in general not unique.

Theorem 2.6 Let $\Lambda=k Q / I$ a bound quiver algebra, and let $Z \Lambda$ be its center. We have

$$
\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda)=\operatorname{dim}_{k} Z \Lambda-\sum_{x \in Q_{0}} \operatorname{dim}_{k} x \Lambda x+\sum_{a \in Q_{1}} \operatorname{dim}_{k} t(a) \Lambda s(a)
$$

Proof. By Proposition [2.5, we have to compute $\operatorname{dim}_{k}(\operatorname{Tr} \Lambda)_{\Lambda}$. To begin with, we will consider $\operatorname{Tr} \Lambda$. Let $E=k Q_{0}$, which is a maximal commutative semisimple subalgebra of $k Q$. The projective minimal presentation of $\Lambda$ as $\Lambda$-bimodule is known to have the following form, see [6] p. 324] and [4] p. 72]

$$
\begin{equation*}
\Lambda \otimes_{E} k Q_{1} \otimes_{E} \Lambda \xrightarrow{f} \Lambda \otimes_{E} \Lambda \longrightarrow \Lambda \longrightarrow 0 \tag{2.7}
\end{equation*}
$$

where $\Lambda \otimes_{E} \Lambda \longrightarrow \Lambda$ is given by the product of $\Lambda$. For $a \in Q_{1}$ we have

$$
f(t(a) \otimes a \otimes s(a))=a \otimes s(a)-t(a) \otimes a
$$

Consequently, for $\lambda, \mu \in \Lambda$ we obtain

$$
f(\mu \otimes a \otimes \lambda))=\mu a \otimes s(a) \lambda-\mu t(a) \otimes a \lambda
$$

We write $\otimes$ instead of $\otimes_{k}$. Also note that the enveloping algebra $\Lambda^{e}$ viewed as a $\Lambda$-bimodule is isomorphic to $\Lambda \otimes \Lambda$ with action $\lambda(a \otimes b) \mu=\lambda a \otimes b \mu$.

The functor $\operatorname{Hom}_{\Lambda-\Lambda}(-, \Lambda \otimes \Lambda)$ applied to (2.7) provides the exact sequence defining $\operatorname{Tr} \Lambda$
$\operatorname{Hom}_{\Lambda-\Lambda}\left(\Lambda \otimes_{E} \Lambda, \Lambda \otimes \Lambda\right) \xrightarrow{f^{*}} \operatorname{Hom}_{\Lambda-\Lambda}\left(\Lambda \otimes_{E} k Q_{1} \otimes_{E} \Lambda, \Lambda \otimes \Lambda\right) \longrightarrow \operatorname{Tr} \Lambda \longrightarrow 0$.
Next we use that for an $E$-bimodule $U$ and a $\Lambda$-bimodule $X$ there is a canonical isomorphism

$$
\operatorname{Hom}_{\Lambda-\Lambda}\left(\Lambda \otimes_{E} U \otimes_{E} \Lambda, X\right)=\operatorname{Hom}_{E-E}(U, X)
$$

and observe that $\Lambda \otimes_{E} \Lambda=\Lambda \otimes_{E} E \otimes_{E} \Lambda$. We thus obtain the following exact sequence, where we kept the same notation for the $\Lambda$-bimodule morphism $f^{*}$

$$
\begin{equation*}
\operatorname{Hom}_{E-E}(E, \Lambda \otimes \Lambda) \xrightarrow{f^{*}} \operatorname{Hom}_{E-E}\left(k Q_{1}, \Lambda \otimes \Lambda\right) \longrightarrow \operatorname{Tr} \Lambda \longrightarrow 0 . \tag{2.8}
\end{equation*}
$$

In the following we work out the exact sequence (2.8). Let $y, x \in Q_{0}$ and let ${ }_{y} k_{x}$ be the simple $E$-bimodule of dimension 1 given by the idempotent $y \otimes x \in E^{e}$, namely ${ }_{y} k_{x}=y E \otimes E x$. Let $U$ be an $E$-bimodule. Clearly we have a canonical isomorphism

$$
\operatorname{Hom}_{E-E}\left({ }_{y} k_{x}, U\right)=y U x
$$

Observe that as $E$-bimodules we have

$$
E=\oplus_{x \in Q_{0}} k_{x} \quad \text { and } k Q_{1}=\oplus_{a \in Q_{1} t(a)} k_{s(a)}
$$

The exact sequence (2.8) becomes, by still keeping the same notation for $f^{*}$

$$
\begin{equation*}
\oplus_{x \in Q_{0}}(x \Lambda \otimes \Lambda x) \xrightarrow{f^{*}} \oplus_{a \in Q_{1}}(t(a) \Lambda \otimes \Lambda s(a)) \longrightarrow \operatorname{Tr} \Lambda \longrightarrow 0 . \tag{2.9}
\end{equation*}
$$

Let $M$ be a right $\Lambda$-module and $N$ be a left $\Lambda$-module, $M \otimes N$ is a $\Lambda$-bimodule for the internal action $\lambda(m \otimes n) \mu=m \mu \otimes \lambda n$. On the other hand $N \otimes M$ is a $\Lambda$-bimodule for the external action $\lambda(n \otimes m) \mu=\lambda n \otimes m \mu$. Of course, these $\Lambda$-bimodules are isomorphic through the flip map $\sigma(n \otimes m)=m \otimes n$.

We rewrite 2.9 using the flips maps

$$
\sigma_{x}: x \Lambda \otimes \Lambda x \rightarrow \Lambda x \otimes x \Lambda \text { and } \sigma_{a}: t(a) \Lambda \otimes \Lambda s(a) \rightarrow \Lambda s(a) \otimes t(a) \Lambda
$$

thus getting an exact sequence for bimodules with external action. By abuse of notation we still write $f^{*}$ instead of $\left(\oplus_{a \in Q_{1}} \sigma_{a}\right) f^{*}\left(\oplus_{x \in Q_{0}} \sigma_{x}^{-1}\right)$.

$$
\begin{equation*}
\oplus_{x \in Q_{0}}(\Lambda x \otimes x \Lambda) \xrightarrow{f^{*}} \oplus_{a \in Q_{1}}(\Lambda s(a) \otimes t(a) \Lambda) \longrightarrow \operatorname{Tr} \Lambda \longrightarrow 0 . \tag{2.10}
\end{equation*}
$$

It is an easy but rather meticulous computation to track the morphism of $\Lambda$ bimodules $f^{*}$ along the previous steps. In the end, we obtain the following formula in the context of (2.10):

$$
\begin{equation*}
f^{*}(x \otimes x)=\sum_{\substack{a \in Q_{1} \\ s(a)=x}} x \otimes a-\sum_{\substack{b \in Q_{1} \\ t(b)=x}} b \otimes x \tag{2.11}
\end{equation*}
$$

Recall that our aim is to compute the dimension of the coinvariants of $\operatorname{Tr} \Lambda$, that is of $\Lambda \otimes_{\Lambda-\Lambda} \operatorname{Tr} \Lambda$ by Remark 2.4. The functor $\Lambda \otimes_{\Lambda-\Lambda}$ - is right exact and preserves direct sums, so we obtain the exact sequence

$$
\begin{equation*}
\oplus_{x \in Q_{0}}(\Lambda x \otimes x \Lambda)_{\Lambda} \xrightarrow{f_{\Lambda}^{*}} \oplus_{a \in Q_{1}}(\Lambda s(a) \otimes t(a) \Lambda)_{\Lambda} \longrightarrow(\operatorname{Tr} \Lambda)_{\Lambda} \longrightarrow 0 \tag{2.12}
\end{equation*}
$$

Moreover, as before, let $N$ (resp. $M$ ) be a left (resp. right) $\Lambda$-module. Consider the $\Lambda$-bimodule with external action $N \otimes M$. We have that $(N \otimes M)_{\Lambda}$ is isomorphic to $M \otimes_{\Lambda} N$ via the flip map. Note that this is the degree 0 instance of the graded isomorphism (see for example [7] p. 170 Corollary 4.4])

$$
H_{*}(\Lambda, N \otimes M)=\operatorname{Tor}_{*}^{\Lambda}(M, N)
$$

Thus,

$$
(\Lambda x \otimes y \Lambda)_{\Lambda}=y \Lambda \otimes_{\Lambda} \Lambda x=y \Lambda x
$$

which leads to the exact sequence

$$
\begin{equation*}
\oplus_{x \in Q_{0}} x \Lambda x \xrightarrow{f_{\Lambda}^{*}} \oplus_{a \in Q_{1}} t(a) \Lambda s(a) \longrightarrow(\operatorname{Tr} \Lambda)_{\Lambda} \longrightarrow 0 . \tag{2.13}
\end{equation*}
$$

We underline that for $y, x \in Q_{0}$, the multiplicity of the vector space $y \Lambda x$ in the second direct sum is the number of parallel arrows from $x$ to $y$.

Another easy and rather meticulous computation gives a formula for $f_{\Lambda}^{*}$ in the context of (2.13). For $\lambda \in x \Lambda x$ we have

$$
f_{\Lambda}^{*}(\lambda)=\sum_{\substack{a \in Q_{1} \\ t(a)=x}} \lambda a-\sum_{\substack{b \in Q_{1} \\ s(b)=x}} b \lambda
$$

where $\lambda a \in t(a) \Lambda s(a)$, that is the direct summand corresponding to $a$. Similarly, $b \lambda \in t(b) \Lambda s(b)$, that is the direct summand corresponding to $b$.

Let $C=\sum_{a \in Q_{1}} a \in \Lambda$. Note that for $\lambda \in \oplus_{x \in Q_{0}} x \Lambda x$ we have

$$
f_{\Lambda}^{*}(\lambda)=\lambda C-C \lambda .
$$

To show that $\operatorname{Ker} f_{\Lambda}^{*}=Z \Lambda$, it is convenient as usual to consider the $k$-category $\mathcal{C}_{\Lambda}$ associated to $\Lambda$ : its set of objects is $Q_{0}$, while the set of morphisms ${ }_{v} \mathcal{C}_{u}$ from $u$ to $v$ is $v \Lambda u$; composition is given by the product of $\Lambda$. The center of $\Lambda$ viewed in this category is

$$
\left\{\left({ }_{x} \lambda_{x}\right)_{x \in Q_{0}} \mid{ }_{v} \lambda_{v}{ }_{v} \alpha_{u}={ }_{v} \alpha_{u} \quad{ }_{u} \lambda_{u} \text { for all }{ }_{v} \alpha_{u} \in{ }_{v} \mathcal{C}_{u}\right\} .
$$

On the other hand as already observed, in case of parallel arrows there is one direct summand for each arrow in $\oplus_{a \in Q_{1}} t(a) \Lambda s(a)$. Note also that $Q_{0} \cup Q_{1}$ is a set of generators of $\mathcal{C}_{\Lambda}$ as an algebra. Using these three observations, the proof of $\operatorname{Ker} f_{\Lambda}^{*}=Z \Lambda$ is immediate.

## 3 Hereditary, radical square zero and triangular monomial algebras

In this section we compute the excess (see Definition 2.2) of some families of algebras.
3.1 Hereditary algebras and algebras with the $H^{2}$ cancellation properties

We first prove that the excess is zero for hereditary algebras. The proof is based on the fact that the set of morphisms which do not factor through injectives is zero and we believe it provides a useful method in other contexts.

Later in Theorem 3.4 we generalize the result for algebras with the $H^{2}$ cancellation properties (see the Introduction for the definition). Its proof relies on the fact that for an algebra $\Lambda$ with the $H^{2}$ cancellation properties a formula for the dimension of $H H^{1}(\Lambda)$ is known, see [9].

Theorem 3.1 Let $Q$ be a finite connected quiver without oriented cycles. Let $\Lambda=k Q$ be the corresponding hereditary algebra. We have $e(\Lambda)=0$.

## Proof.

We will show that if $I$ is an injective $\Lambda$-bimodule, then $\operatorname{Hom}_{\Lambda-\Lambda}(I, \tau \Lambda)=0 . A$ fortiori $\mathcal{I H o m}_{\Lambda-\Lambda}(\Lambda, \tau \Lambda)=0$. By Lemma 2.3, it follows that $e(\Lambda)=0$.

We have that $\operatorname{pd}_{\Lambda-\Lambda} \Lambda \leq 1$. Indeed $k Q$ is the tensor algebra $T_{k Q_{0}} k Q_{1}$. It is well known (see for instance [8. Theorem 2.3]) that there is a minimal projective resolution of $k Q$ as a $k Q$-bimodule as follows:

$$
\begin{equation*}
0 \longrightarrow k Q \otimes_{k Q_{0}} k Q_{1} \otimes_{k Q_{0}} k Q \longrightarrow k Q \otimes_{k Q_{0}} k Q \longrightarrow k Q \longrightarrow 0 . \tag{3.2}
\end{equation*}
$$

We recall [2, Proposition 1.7(a) p. 319]: let $A$ be an algebra and let $M$ be an indecomposable left $A$-module. The projective dimension of $M$ is at most 1 if
and only if $\operatorname{Hom}_{A}(\mathrm{D} A, \tau M)=0$. We will use this result for $\Lambda$-bimodules, that is replacing $A$ by the enveloping algebra of $\Lambda$. We have supposed $Q$ connected, therefore $\Lambda$ is indecomposable as a $\Lambda$-bimodule, and the aforementioned proposition of [2] applies.

It follows that $\operatorname{Hom}_{\Lambda-\Lambda}(\mathrm{D}(\Lambda \otimes \Lambda), \tau \Lambda)=0$. Of course, for an algebra $A$, every injective left $A$-module is isomorphic to a direct summand of a direct sum of copies of $\mathrm{D} A$, where $A$ is viewed as a right $A$-module and $A$ a left $A$-module. $\diamond$

Corollary 3.3 [9, 14, 11] Let $B$ the set of paths of $Q$, and let $|y B x|$ be the number of paths from $x$ to $y$. For $\Lambda=k Q$ we have

$$
\operatorname{dim}_{k} H H^{1}(\Lambda)=1-\left|Q_{0}\right|+\sum_{a \in Q_{1}}|t(a) B s(a)|=\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda) .
$$

We provide in the following a generalisation of Theorem 3.1 for algebras having the $H^{2}$ cancellation properties.

For a bound quiver algebra $\Lambda$ with the $H^{2}$ cancellation properties, the dimension of $H H^{1}(\Lambda)$ is known by [9, p. 647]. This allows to prove the following

Theorem 3.4 The excess of a bound quiver algebra $\Lambda=k Q / I$ with the $H^{2}$ cancellation properties is zero.

Proof. Let $B$ be the basis of paths of a bound quiver algebra.
We know from [9] that

$$
\operatorname{dim}_{k} H H^{1}(\Lambda)=\operatorname{dim}_{k} Z \Lambda-\sum_{x \in Q_{0}}|x B x|+\sum_{x, y \in Q_{0}}|y B x|\left|y Q_{1} x\right|
$$

Clearly $|y B x|=\operatorname{dim}_{k} y \Lambda x$. Hence by Theorem [2.6] the equality of dimensions holds.

Lemma 3.5 An hereditary algebra $k Q$ has the $H^{2}$ cancellation properties.
Proof. It follows from (3.2) that $\operatorname{pd}_{\Lambda-\Lambda} \Lambda \leq 1$. Then for any $k Q$-bimodule $X$ we have $H^{2}(k Q, X)=0$.

Remark 3.6 We will show in Subsection 3.2 that not only the hereditary algebras have the $H^{2}$ cancellation properties.

### 3.2 Radical square zero algebras

A radical square zero algebra is a bound quiver algebra of the form $k Q / F^{2}$.
Let $P$ and $P^{\prime}$ be two sets of paths of a quiver $Q$. The set of parallel paths is

$$
P / / P^{\prime}=\left\{\left(p, p^{\prime}\right) \in P \times P^{\prime} \mid s(p)=s\left(p^{\prime}\right) \text { and } t(p)=t\left(p^{\prime}\right)\right\} .
$$

For instance $Q_{1} / / Q_{0}$ is the set of loops of $Q$. We denote by $Q_{i}$ the set of paths of length $i$.

A $c$-crown is a quiver $C$ with $c$ vertices cyclically labelled and $c$ arrows, each one joining each vertex with the next one in the cyclic labelling. For instance a 1-crown is a loop, and a 2 -crown is a two-way quiver $\cdot \leftrightarrows \cdot$. The behaviour of the Hochschild cohomology of $k C / F^{2}$ is exceptional, see [10] and it will be considered separately.

Proposition 3.7 Let $Q$ be a connected quiver which is not a crown. The radical square zero algebra $\Lambda=k Q / F^{2}$ has the $H^{2}$ cancellation properties if and only if $Q_{2} / / Q_{1}=\emptyset$.

Proof. Since $r$ is a semisimple $\Lambda$-bimodule, the complex of cochains of Section 2 of [10] has zero coboundaries and $\operatorname{dim}_{k} H^{2}(\Lambda, r)=\left|Q_{2} / / Q_{1}\right|$.

Consequently if $\Lambda=k Q / F^{2}$ has the $H^{2}$ cancellation properties, then $\left|Q_{2} / / Q_{1}\right|=$ 0.

Reciprocally, note first that if $\left|Q_{2} / / Q_{1}\right|=0$ then $\left|Q_{1} / / Q_{0}\right|=0$. From [10, Theorem 3.1] we have

$$
\operatorname{dim}_{k} H H^{2}(\Lambda)=\left|Q_{2} / / Q_{1}\right|-\left|Q_{1} / / Q_{0}\right|=0
$$

Hence if $Q_{2} / / Q_{1}=\emptyset$ then $H^{2}(\Lambda, r)=0=H H^{2}(\Lambda)$.
There are zero excess algebras without the $H^{2}$ cancellation properties, as the next result shows.

Proposition 3.8 Let $Q$ be a connected quiver which is not a crown and let $\Lambda=$ $k Q / F^{2}$. We have $e(\Lambda)=0$.

Proof. Observe that $\operatorname{dim}_{k} Z \Lambda=1+\left|Q_{1} / / Q_{0}\right|$. The formula of Theorem 2.6 gives $\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda)=1+\left|Q_{1} / / Q_{0}\right|-\left|Q_{0}\right|-\left|Q_{1} / / Q_{0}\right|+\left|Q_{1} / / Q_{1}\right|=1-\left|Q_{0}\right|+\left|Q_{1} / / Q_{1}\right|$.

On the other hand we know from [10. Theorem 3.1], together with the observation in the next paragraph, that the same formula holds for $\operatorname{dim}_{k} H H^{1}(\Lambda)$.

In the proof of Theorem 3.1 in [10] it is stated that " $D$ is injective for a positive $n$ ". However for $n=0$ the kernel of $D$ has dimension one. Hence the formula for $\operatorname{dim}_{k} H H^{1}(\Lambda)$ in the statement of [10, Theorem 3.1] has to be modified by adding 1.

Proposition 3.9 Let $C$ be a c-crown, and let $\Lambda=k C / F^{2}$.

- If $c>1$, then $e(\Lambda)=0$,
- If $c=1$ and the characteristic of $k$ is not 2 , then $e(\Lambda)=1$,
- If $c=1$ and the characteristic of $k$ is 2 , then $e(\Lambda)=0$.

Proof. Observe that if $c>1$, by Theorem 2.6 we have

$$
\operatorname{dim}_{k}^{\tau} H H^{1}\left(k C / F^{2}\right)=1-c+c=1 .
$$

If $c=1$ then $k C / F^{2}=k[x] /\left(x^{2}\right)$, the algebra of dual numbers and

$$
\operatorname{dim}_{k}{ }^{\tau} H H^{1}\left(k[x] / x^{2}\right)=2-2+2=2
$$

On the other hand, it is easy to compute that if $c>1$ then $\operatorname{dim}_{k} H H^{1}(\Lambda)=1$ regardless the characteristic of $k$.

If the characteristic of $k$ is not 2 , then $\operatorname{dim}_{k} H H^{1}\left(k[x] /\left(x^{2}\right)\right)=1$ while if the characteristic of $k$ is 2 , then $\operatorname{dim}_{k} H H^{1}\left(k[x] /\left(x^{2}\right)\right)=2$.

### 3.3 Triangular monomial algebras

A monomial algebra is a bound quiver algebra $\Lambda=k Q / I$ where $I$ is generated by a minimal set of paths denoted by $Z$. The algebra $\Lambda$ is triangular if it is a quotient of a finite dimensional hereditary algebra $k Q$, that is if $Q$ has no oriented cycles. We set the following.

- $B$ is the set of paths of $Q$ which do not contain any path of $Z$. Note that $Z=\emptyset$ if and only if $B$ is the set of all the paths of $Q$. Moreover, $B$ gives a basis of $\Lambda$.
- $\left(Q_{1} / / B\right)_{u}$ is the set of pairs $(a, \epsilon) \in Q_{1} / / B$ such that for every $\gamma \in Z$, replacing each occurrence of $a$ in $\gamma$ by $\epsilon$, gives a path which is 0 in $\Lambda$. Note that $\left\{(a, a) \mid a \in Q_{1}\right\} \subset\left(Q_{1} / / B\right)_{u}$.
- $\left(Q_{1} / / B\right)_{n u}=\left(Q_{1} / / B\right) \backslash\left(Q_{1} / / B\right)_{u}$, that is the set of pairs $(a, \epsilon) \in Q_{1} / / B$ such that there exists $\gamma \in Z$ where $a$ occurs, verifying that at least one of the replacements of $a$ in $\gamma$ by $\epsilon$, gives a non zero path in $\Lambda$.

Theorem 3.10 Let $\Lambda=k Q /\langle Z\rangle$ be a triangular monomial algebra. We have

$$
e(\Lambda)=\left|\left(Q_{1} / / B\right)_{n u}\right| .
$$

Proof. When $Q$ has no oriented cycles, the formula for the dimension of $H H^{1}(\Lambda)$ given in [12] is as follows:

$$
\begin{equation*}
\operatorname{dim}_{k} H H^{1}(\Lambda)=\operatorname{dim}_{k} Z \Lambda-\left|Q_{0}\right|+\left|\left(Q_{1} / / B\right)_{u}\right| . \tag{3.11}
\end{equation*}
$$

Note that since $Q$ has no oriented cycles, for all $x \in Q_{0}$ we have $x \Lambda x=k$. Hence Theorem 2.6 gives

$$
\begin{equation*}
\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda)=\operatorname{dim}_{k} Z \Lambda-\left|Q_{0}\right|+\left|\left(Q_{1} / / B\right)\right| \tag{3.12}
\end{equation*}
$$

Theorem 3.13 Let $Q$ be a connected quiver without oriented cycles and let $\Lambda=$ $k Q /\langle Z\rangle$ be a triangular monomial algebra. The following are equivalent:
(1) $H H^{1}(\Lambda)=0$.
(2) $Q$ is a tree.
(3) ${ }^{\tau} H H^{1}(\Lambda)=0$.

Remark 3.14 The equivalence between (1) and (2) is proved without the triangular hypothesis in [5] Theorem 2.2].

## Proof.

For (1) implies (2) the formula (3.11) gives

$$
1-\left|Q_{0}\right|+\left|\left(Q_{1} / / B\right)_{u}\right|=0
$$

We have $\left\{(a, a) \mid a \in Q_{1}\right\} \subset\left(Q_{1} / / B\right)_{u}$ hence $1-\left|Q_{0}\right|+\left|Q_{1}\right| \leq 0$. The Euler characteristic of the underlying graph of $Q$ is $\chi(Q)=\left|Q_{0}\right|-\left|Q_{1}\right|$, hence $\chi(Q) \geq 1$.

Any finite graph has the homotopy type of a graph with 1 vertex and $n$ loops, which fundamental group is free on $n=1-\chi(Q)$ generators. We infer $n \leq 0$, hence $n=0$ and $Q$ is a tree.

Concerning (2) implies (3) since $Q$ is a tree we have $\chi(Q)=\left|Q_{0}\right|-\mid\left(Q_{1} \mid=1\right.$ On the other hand $\left(Q_{1} / / B\right)=\left\{(a, a) \mid a \in Q_{1}\right\}$, and we have $\left|Q_{1} / / B\right|=\mid\left(Q_{1} \mid\right.$. The formula 3.12 gives ${ }^{\tau} H H^{1}(\Lambda)=0$.

The implication (3) $\Rightarrow(1)$ follows from Lemma 2.3

Example 3.15 Let $Q$ and $R$ respectively denote the quivers

and


The following table lists the results for the corresponding oriented monomial algebras:

| Quiver | Z | $\left(Q_{1} / / B\right)_{n u}$ | $e(\Lambda)$ | $\operatorname{dim}_{k} H H^{1}(\Lambda)$ | $\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $Q$ | $\{c a\}$ | $\{(a, b)\}$ | 1 | 2 | 3 |
| $R$ | $\{b a\}$ | $\emptyset$ | 0 | 2 | 2 |
| $R$ | $\{d a\}$ | $\{(d, b c)\}$ | 1 | 1 | 2 |

## 4 The excess

The proof of the next result relies on the calculation of the dimensions of two vector spaces and the observation that they are equal. An explicit isomorphism between these two vector spaces remains unknown to us.

Proposition 4.1 Let $\Lambda=k Q / I$ be a bound quiver algebra. We have

$$
\operatorname{dim}_{k} H^{1}(k Q, k Q / I)=\operatorname{dim}_{k}^{\tau} H H^{1}(\Lambda) .
$$

Proof. Let $X$ be a $k Q$-bimodule. We assert that

$$
\begin{equation*}
\operatorname{dim}_{k} H^{1}(k Q, X)=\operatorname{dim}_{k} X^{k Q}-\sum_{x \in Q_{0}} \operatorname{dim}_{k} x X x+\sum_{a \in Q_{1}} \operatorname{dim}_{k} t(a) X s(a) . \tag{4.2}
\end{equation*}
$$

Recall the projective resolution of $k Q$ as a $k Q$-bimodule (3.2)

$$
\begin{equation*}
0 \longrightarrow k Q \otimes_{k Q_{0}} k Q_{1} \otimes_{k Q_{0}} k Q \xrightarrow{g} k Q \otimes_{k Q_{0}} k Q \longrightarrow k Q \longrightarrow 0 . \tag{4.3}
\end{equation*}
$$

The functor $\operatorname{Hom}_{k Q-k Q}(-, X)$ gives the complex of cochains
$0 \longrightarrow \operatorname{Hom}_{k Q-k Q}\left(k Q \otimes_{E} k Q, X\right) \xrightarrow{g^{*}} \operatorname{Hom}_{k Q-k Q}\left(k Q \otimes_{E} k Q_{1} \otimes_{E} k Q, X\right) \longrightarrow 0$
where $\operatorname{Ker} g^{*}=H^{0}(k Q, X)$ and Cokerg* $=H^{1}(k Q, X)$. The same way we have obtained (2.8) and (2.9) leads to an exact sequence
$0 \longrightarrow H^{0}(k Q, X) \longrightarrow \oplus_{x \in Q_{0}} x X x \xrightarrow{g^{*}} \oplus_{a \in Q_{1}} t(a) X s(a) \longrightarrow H^{1}(k Q, X) \longrightarrow 0$.
which gives the equality (4.2).
We assert that if $X$ is a $k Q / I$-bimodule, that is $I X=X I=0$, then $X^{k Q}=$ $X^{k Q / I}$. Indeed, we have

$$
X^{k Q}=\operatorname{Hom}_{k Q-k Q}(k Q, X)=\operatorname{Hom}_{k Q / I-k Q / I}(k Q / I, X)=X^{k Q / I}
$$

Note that for $X=\Lambda$ we have $\Lambda^{\Lambda}=Z \Lambda$. We obtain the following

$$
\operatorname{dim}_{k} H^{1}(k Q, k Q / I)=\operatorname{dim}_{k} Z \Lambda-\sum_{x \in Q_{0}} \operatorname{dim}_{k} x \Lambda x+\sum_{a \in Q_{1}} \operatorname{dim}_{k} t(a) \Lambda s(a)
$$

which is the same formula than the one for $\operatorname{dim}_{k}{ }^{\tau} H H^{1}(\Lambda)$ in Theorem 2.6

Next we recall Corollary 2.4 of [8].
Proposition 4.4 [8] Let $\Lambda=k Q / I$ be a bound quiver algebra and let $X$ be a $\Lambda$-bimodule. There is an exact sequence
$0 \longrightarrow H^{1}(\Lambda, X) \longrightarrow H^{1}(k Q, X) \longrightarrow \operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, X\right) \longrightarrow H^{2}(\Lambda, X) \longrightarrow 0$.
An immediate consequence of the above is a formula for the excess of an algebra, which involves the dimension of the Hochschild cohomology in degree 2.

Theorem 4.5 Let $\Lambda=k Q / I$ be a bound quiver algebra. We have

$$
e(\Lambda)=\operatorname{dim}_{k} \operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)-\operatorname{dim}_{k} H H^{2}(\Lambda)
$$

Remark 4.6 If $I=0$, then $H H^{2}(k Q)=0$ and so $e(k Q)=0$. This confirms Theorem 3.1] as well as Theorem 3.4 for an hereditary algebra.

We infer three corollaries for a bound quiver algebra $\Lambda=k Q / I$.
Corollary 4.7 If $\Lambda$ verifies the $H^{2}$ cancelling properties then

$$
\operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)=0
$$

Corollary 4.8 If $e(\Lambda)=0$ then

$$
H H^{2}(\Lambda)=\operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)
$$

Corollary 4.9 The algebra $\Lambda$ is $\tau$-rigid as a $\Lambda$-bimodule if and only if

- $H H^{1}(\Lambda)=0$,
- $\operatorname{dim}_{k} H H^{2}(\Lambda)=\operatorname{dim}_{k} \operatorname{Hom}_{k Q-k Q}\left(I / I^{2}, \Lambda\right)$.


## References

[1] Auslander, M.; Reiten, I. Representation theory of artin algebras III, Comm. Algebra 3, (1975) 239-294.
[2] Auslander, M.; Reiten, I.; Smalø, S.O. Representation Theory of Artin Algebras. Cambridge University Press; 1995.
[3] Angeleri Hügel, L. An Introduction to Auslander-Reiten Theory. Lecture Notes Advanced School on Representation Theory and related Topics, 2006, ICTP Trieste. http://docplayer.net/178390377-Advanced-school-and-conference-on-representation-theory-and-related-topics.html
[4] Bardzell, M. J. The alternating syzygy behavior of monomial algebras, J. AIgebra 188, (1997) 69-89.
[5] Bardzell, M. J.; Marcos, E. N. Induced boundary maps for the cohomology of monomial and Auslander algebras. Reiten, Idun (ed.) et al., Algebras and modules II. Eighth international conference on representations of algebras, Geiranger, Norway, August 4-10, 1996. Providence, RI: American Mathematical Society. CMS Conf. Proc. 24, (1998) 47-54.
[6] Butler, M.C.R.; King A.D. Minimal resolutions of algebras, J. Algebra 212, (1999) 323-362.
[7] Cartan, H.; Eilenberg, S. Homological algebra. Princeton University Press, Princeton, N. J., 1956.
[8] Cibils, C. Rigid monomial algebras. Math. Ann. 289, (1991) 95-109.
[9] Cibils, C. On the Hochschild cohomology of finite dimensional algebras. Commun. Algebra 16, (1988) 645-649.
[10] Cibils, C. Hochschild cohomology algebra of radical square zero algebras. Reiten, Idun (ed.) et al., Algebras and modules II. Eighth international conference on representations of algebras, Geiranger, Norway, August 4-10, 1996. Providence, RI: American Mathematical Society. CMS Conf. Proc. 24, (1998) 93-101.
[11] Cibils, C. On $H^{1}$ of finite dimensional algebras. Bol. Acad. Nac. Cienc., Córdoba 65, (2000) 73-80.
[12] Cibils, C; Saorín, M. The first cohomology group of an algebra with coefficients in a bimodule. J. Algebra 237, (2001) 121-141.
[13] Gabriel, P. Finite representation type is open. Represent. Algebr., Proc. int. Conf., Ottawa 1974, Lect. Notes Math. 488, (1975) 132-155.
[14] Happel, D. Hochschild cohomology of finite-dimensional algebras. Séminaire d'algèbre P. Dubreil et M.-P. Malliavin, Proc., Paris/Fr. 1987/88, Lect. Notes Math. 1404, (1989) 108-126.
[15] Iyama, O.; Reiten, I. Introduction to $\tau$-tilting theory. Proc. Natl. Acad. Sci. USA 111, (2014) 9704-9711.
[16] Hochschild, G. On the cohomology groups of an associative algebra, Ann. Math. 46 (1945), 58-67.
[17] Marsh, B. $\tau$-tilting theory and $\tau$-exceptional sequences, mini-course at the $\tau$ Research School, 5-8 September 2023, University of Cologne.
https://sites.google.com/view/tau-tilting-school-cologne/schedule-abstracts http://www1.maths.leeds.ac.uk/~marsh/MarshTauTiltingParts1to4.pdf
[18] Mazzola, G. The algebraic and geometric classification of associative algebras of dimension five. Manuscr. Math. 27 (1979), 81-101.
[19] Witherspoon, S. Hochschild cohomology for algebras. Graduate Studies in Mathematics 204. Providence, RI: American Mathematical Society (AMS), 2019.

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