Transport through a lattice with local loss: from quantum dots to lattice gases

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(Dated: April 11, 2024)

Recent work has studied fermion transport through a finite one-dimensional lattice of quantum dots, with localized particle loss from the central lattice site. The dots at each end of the lattice are connected to macroscopic leads, represented as zero-temperature reservoirs of free fermions with a given potential difference. Here we show how this model represents one limiting case of a larger class of models that can be realized with cold quantum gases in optical lattices. While quantum gas realizations allow many system parameters to be varied, we note limitations from finite size effects, and conclude that quantum dots and quantum gases offer complementary views on transport through lossy finite lattices.

I. INTRODUCTION

A. Motivation and background

The effects of localized loss on quantum transport are of high current interest both as examples of fundamental phenomena and as practical tools for controlling quantum states and processes [3–10]. In particular, recent theoretical work [1, 2] has investigated fermions traveling from one reservoir to another through a finite chain of sites, with one of the sites having a finite rate of particle loss. The finite number of sites means that the current is dramatically affected by reflection and discrete eigenfrequencies, and so losing particles from one site does not just drain away some current classically, but also modifies quantum interference effects.

Reference [2] has solved this general problem with nonequilibrium Green's functions in the Keldysh formalism extended to open quantum systems. The main result for the steady-state fermionic current through the finite chain is that conductivity shows discrete steps, as a function of voltage between the reservoirs across the chain of sites, and that the lossy site systematically smooths out these steps, even with moderate loss rates. Higher loss rates also significantly lower the steps, reducing overall conductivity.

Letting the single lossy site be the central site, with an odd number of sites in the chain, elegantly isolates the effect of loss on the conductivity steps. The conductivity steps are associated with energetic access to singleparticle eigenstates in the finite chain; with reflection symmetry, odd-parity eigenstates have nodes at the loss site, and transport through these channels is much less affected by the loss than transport through even-parity channels. This shows up clearly in [2] in conductivity steps that alternate between smooth and steep.

Although [2] explicitly raises the possibility of experimental realizations with ultracold fermi gases in optical lattices, however, its concrete results are obtained for parameter regimes that rather represent electron transport through a chain of quantum dots, with the reservoir roles played by macroscopic wire leads. In this paper we therefore look critically at a quantum gas realization of this lossy transport problem. Instead of applying an opensystem formalism, we replace the macroscopic fermion reservoirs of [2] with more lattice sites like the ones in the finite middle chain. The finite middle chain is only separated from the "reservoir" parts of the lattice by "weak links" of slower tunneling rates between sites. We will show, first of all, that this particular class of Hamiltonian realizations of the particle reservoirs can indeed reproduce exactly the results of [2]. We will then re-tune our parameters to represent experiments that will likely be more straightforward with quantum gases. We will investigate both finite-temperature and finite-size effects, and conclude that experiments with quantum gases are realistically feasible, and can provide complementary insights to those offered by quantum dots.

B. Paper structure

In Section II immediately below we will introduce our lattice Fermi gas and show how all single-particle observables can be found by solving a single-particle Schrödinger equation. Loss from the central lattice site, which is defined through a Lindblad master equation, is shown to be described exactly by an imaginary potential term in the Schrödinger equation.

In Section III we will then explain the class of initial states which let this infinite closed system exactly reproduce the observables of the open system in which a finite lattice is coupled at its ends to particle reservoirs. In particular we will present a case of our model which exactly reproduces the results of [2] for finite chains of quantum dots coupled to macroscopic leads.

In Section IV we will then re-tune our infinite model to correspond more closely to the most straightforward kind of quantum gas experiments, with chemical potential differences between the left and right reservoirs, but no onsite potential differences, and with densities of states in the middle and outer sections of the lattice all equal, instead of having much greater densities of states in the reservoirs. We predict steady-state currents that are qualitatively similar to those found in [2] for the quantum dot system with leads. There are abrupt steps in the conductivity at chemical potential differences corresponding to eigenfrequencies of the isolated middle lattice; the steps are smoothed by the loss at the central site, or remain quite sharp, depending on the parity of the corresponding eigenfunctions. Although qualitatively very similar, the straightforward quantum gas scenario is found to differ in detail from the quantum dot version of the experiment.

In Section V we will consider the experimentally important constraint of finite lattice size, which in the presence of loss means that the only true steady state of the system is absence of particles (except in cases with reflection symmetry, in which odd-parity modes may be occupied, but no transport occurs). We will show, however, that if the outer ends of the lattice contain sufficiently many sites, then there is a time interval during which the system can can sustain a quasi-steady flow of particles through the middle region. We will further show that the current during this quasi-steady era corresponds closely to the steady-state current in the infinite system, with residual finite-size effects that are small for lattices with total site numbers of order 100.

We will conclude in Section VI with a summary of our results and an outlook toward further theoretical and experimental work to address bosons, finite temperatures, interactions, and multiple loss sites. A series of Appendices supply technical details and derivations of formulas provided in the main text.

II. REDUCTION TO A ONE-PARTICLE SCHRÖDINGER EQUATION

A. Infinite lattices with weak links as finite lattices between reservoirs

We consider an infinite one-dimensional lattice of sites that can be occupied by non-interacting particles that can move between sites. In second-quantized notation the Hamiltonian operator takes the form

$$\hat{H} = \hbar \tau \sum_{m=-\infty}^{\infty} \left(-T_m (\hat{a}_{m+1}^{\dagger} \hat{a}_m + \hat{a}_m^{\dagger} \hat{a}_{m+1}) + V_m \hat{a}_m^{\dagger} \hat{a}_m \right) ,$$
(1)

where m labels each site, τ is a frequency scale, and \hat{a}_m and \hat{a}_m^{\dagger} are canonical destruction and creation operators. We will have effectively spinless fermions in mind throughout this paper, but the reduction to a singleparticle problem that we are about to describe proceeds identically for bosons as well; the only difference between fermions and bosons in this problem is whether Fermi-Dirac or Bose-Einstein distributions are appropriate as initial states. Throughout this paper we will use m and m' to refer to sites or numbers of sites, and reserve n for numbers of particles. The site-dependent parameters $T_m > 0$ and V_m do not depend arbitrarily on m, but have a simple pattern that defines a symmetrical finite chain of 2M + 1 sites $(|m| \leq M)$ centered between semi-infinite reservoir zones on the left and right:

$$T_m = \begin{cases} 1 & , & -(M+1) \le m \le M \\ \Lambda & , & m < -(M+2) \text{ or } m > M+1 \\ \tau_1 \sqrt{\Lambda} & , & m = -(M+2) \text{ or } m = M+1 \end{cases}$$

$$V_m = \begin{cases} 0 & , & |m| \le M \\ V_m = \sqrt{M} & M \end{cases}$$
(2)

$$V_m = \begin{cases} +\frac{v}{2} &, \ m < -M \\ -\frac{V}{2} &, \ m > M \end{cases}$$
(3)

Our overall frequency scale τ is thus the inter-site hopping rate in the middle chain of sites, while the dimensionless parameters Λ and τ_1 effectively define, respectively, the density of states in the infinite outer ends of the lattice, and the links between the outer ends and the finite middle section. We have included the factor of $\sqrt{\Lambda}$ with the factor τ_1 because this will turn out to leave a non-trivial model in the limit $\Lambda \to \infty$. In particular this limit will let the infinite outer ends of the lattice reproduce the effects of macroscopic leads that are treated as reservoirs, when $\tau_1 < 1$ implies a "weak link" between the finite middle lattice and the outer "reservoir" ends.

B. Localized loss at site m = 0

Following [2], we add particle loss at site m = 0 to our model by assuming the time evolution of the many-body density operator $\hat{\rho}(t)$ to be governed by the Lindblad master equation

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma\tau \left(\hat{a}_0\hat{\rho}\hat{a}_0^{\dagger} - \frac{\hat{a}_0^{\dagger}\hat{a}_0\hat{\rho} + \hat{\rho}\hat{a}_0^{\dagger}\hat{a}_0}{2}\right) .$$
(4)

With the middle-chain hopping rate τ as our basic time scale, the dimensionless rate factor γ defines the average rate $\gamma \tau$ at which particles are lost, in a probabilistic Poisson process, from site m = 0. In quantum gas realizations of our model, this loss can be tuned by ejecting particles from the central lattice site using lasers or electron beams.

C. Reduction to a Schrödinger equation with a complex potential

1. Single-particle observables

Equations (1) and (4) describe a many-body system of arbitrarily many non-interacting fermions or bosons. The most accessible experimental observables for manybody systems, however, are usually operators of the form $\hat{O} = \sum_{m'm} O_{mm'} \hat{a}^{\dagger}_{m'} \hat{a}_m$, which for appropriate matrix elements $O_{mm'}$ include particle density, momentum density, kinetic energy density, and kinetic energy current density, as well as first-order quantum coherences between arbitrary lattice sites.

Observables of this class are called single-particle observables because they do not directly probe interparticle correlations. Their expectation values in an arbitrary many-body mixed quantum state can be given in terms of the (unnormalized) single-particle density matrix $R_{m'm}$:

$$\langle \hat{O} \rangle = \operatorname{Tr}(\hat{O}\hat{\rho}) = \sum_{m',m} O_{mm'} \operatorname{Tr}(\hat{a}_{m'}^{\dagger} \hat{a}_{m} \hat{\rho})$$
$$=: \sum_{m',m} O_{mm'} R_{m'm} .$$
(5)

 $R_{m'm}$ is referred to as a single-particle density matrix because, as a Hermitian matrix with non-zero eigenvalues in the Hilbert space of lattice sites m, it is indistinguishable from the mixed-state density matrix of a single particle, apart from a normalization factor.

(This normalization factor is simply the average total particle number,

$$\sum_{m} R_{m'm} = \operatorname{Tr}\left(\hat{\rho} \sum_{m} \hat{a}_{m'}^{\dagger} \hat{a}_{m}\right).$$
 (6)

If the particles are conserved, it is often conventional to normalize $R_{m'm}$ by dividing it by its trace. Since particles can be lost in our case at site m = 0, however, the trace of $R_{m'm}$ can in general be time-dependent. The unnormalized $R_{m'm}$ that we use is the more directly relevant quantity, anyway, for the simple \hat{O} -type observables in the many-body system.)

Since no information about correlations between particles is included in $R_{m'm}$, $R_{m'm}$ is in general a very incomplete description of a many-body system. It is not an approximation, however, but rather an exact representation of a restricted class of observables—which happens to include most observables of physical interest.

2. Single-particle master equation

In the presence of interparticle interactions or nonlinear Lindblad operators, interparticle correlations can still affect the single-particle observables indirectly, even strongly, because time evolution couples $R_{m'm}$ to expectation values of higher-order combinations of $\hat{a}^{\dagger}_{m'}$ and \hat{a}_m operators, in the typically infinite BBGKY hierarchy. Since our \hat{H} from (1) includes no interactions among the particles, however, and our Lindblad operator is simply \hat{a}_0 , in our case the exact time evolution of $R_{m'm}(t)$ is given by a single-particle master equation which closes, in the sense that it involves only $R_{m'm}$ itself. Inserting our many-body Lindblad equation (4) into the definition of $R_{m'm}$ and applying canonical (anti-)commutation relations $\hat{a}_m \hat{a}^{\dagger}_m \pm \hat{a}^{\dagger}_{m'} \hat{a}_m = \delta_{m'm}$ for either fermions or bosons reveals

$$\frac{i}{\tau}\dot{R}_{m'm} = T_{m'}R_{m'+1,m} + T_{m'-1}R_{m'-1,m} - T_mR_{m',m+1} - T_{m-1}R_{m',m-1} + (V_m - V_{m'})R_{m'm} - i\frac{\gamma}{2} (\delta_{m'0}R_{0m} + \delta_{0m}R_{m'0}) .$$
(7)

3. Single-particle Schrödinger equation

Equation (7) can then be greatly simplified even further, by noting that it can be solved by a factorizing Ansatz of the form

$$R_{m'm}(t) = \sum_{j,j'} \int d\omega \, d\omega' \, A_{j'j}(\omega, \omega')$$
$$\times e^{i\tau(\omega'-\omega)t} \Psi_{m'}^{j'*}(\omega') \Psi_m^j(\omega) \tag{8}$$

as long as the $\Psi_m^j(\omega)$ are solutions $\psi_m \to \Psi_m^j(\omega)$ of timeindependent single-particle Schrödinger equation

$$\omega\psi_m = -T_m\psi_{m+1} - T_{m-1}\psi_{m-1} + (V_m - i\frac{\gamma}{2}\delta_{m0})\psi_m$$
(9)

with a complex potential at m = 0. The superscript jin $\Psi_m^j(\omega)$ is used to distinguish independent solutions to (9) having the same eigenfrequency ω , if these exist.

D. Open or closed?

This concludes our definition of our non-interacting lattice gas with weak links and loss. Although our system is open in the sense of having loss, the loss can be represented exactly with an imaginary potential. We do not include any external sources of particles or energy, or noise of any kind. We nevertheless propose to consider the middle chain of 2M + 1 sites, between the weak links, as an open system, for which the infinite outer ends of the lattice constitute reservoirs that can send particles into or through the middle chain of sites. In the next Section we will explain how this works.

III. NON-EQUILIBRIUM STATIONARY STATES

A. Non-equilibrium many-body system states from single-particle wave functions?

So how do we use solutions to our single-particle Schrödinger equation (9) to describe non-equilibrium steady states as in [2], with particle transport between the two reservoirs through the finite middle chain with its single lossy site? We will consider special cases of (8) of the form

$$R_{m'm} = \sum_{j=L,R} \int d\omega \, D_j(\omega) f_j(\omega) \Psi_{m'}^{j*}(\omega) \Psi_m^j(\omega) , \quad (10)$$

with j = L, R denoting wave functions that are incident from the left and right, respectively. Here $D_j(\omega)$ are the corresponding densities of states, and $f_j(\omega)$ are Fermi-Dirac distributions with independent temperatures and chemical potentials for j = L and j = R. (In future work it will be straightforward to use Bose-Einstein distributions instead, and describe analogous transport problems with quantum Bose gases. In our concluding Section VI we suggest that the bosonic problem will be less interesting, however, unless interactions are present.)

Since in general we allow a potential difference V between the left and right ends of our infinite lattice, as well as different equilibrium distributions for f_L and f_R , our stationary states will include transport in general.

In Appendix A we will explicitly construct the $\Psi_m^{L,R}(\omega)$ solutions to (9); in the remainder of our main text we will use reflection and transmission coefficients that are derived in that Appendix. Here we explain why an $R_{m'm}$ of the form (10) really does represent non-equilibrium steady states of our system, and then outline how we will use these $R_{m'm}$ to compute observable quantities in these states.

B. Left- and right-incident normalized eigenfunctions

The solutions $\Psi_m^{L,R}(\omega)$ that we construct explicitly in Appendix A have the following "scattering" forms for |m| > M:

$$\sqrt{2\pi}\Psi_n^L(\omega) = \begin{cases} e^{ik_Lm} + \mathcal{R}_L(\omega)e^{-ik_Lm} &, m < -M\\ \sqrt{\frac{\sin(k_L)}{\sin(k_R)}}\mathcal{T}(\omega)e^{ik_Rm} &, m > M \end{cases}$$
$$\sqrt{2\pi}\Psi_n^R(\omega) = \begin{cases} \sqrt{\frac{\sin(k_R)}{\sin(k_L)}}\mathcal{T}(\omega)e^{-ik_Lm} &, m < -M\\ e^{-ik_Rm} + \mathcal{R}_R(\omega)e^{ik_Rm} &, m > M \end{cases},$$
(11)

for transmission and reflection coefficients \mathcal{T} and $\mathcal{R}_{L,R}$ that we compute in Appendix A and will use below. The behaviors of $\Psi_m^{L,R}$ for $|m| \leq M$ are shown in Appendix A, but will not be needed for any results we will present in our main text.

Since we have neither time-reversal nor (unless V = 0) spatial reflection symmetry, the fact that the transmission coefficient $\mathcal{T}(\omega)$ is the same for both left- and rightincident waves is not obvious, but it turns out to be so in all possible cases of our model. The reflection coefficients $\mathcal{R}_{L,R}(\omega)$, in contrast, do not coincide in general. In the special cases that we will consider explicitly within our main text, however, $\mathcal{R}_{L,R} \to \mathcal{R}(\omega)$ are in fact the same, because $k_{L,R}$ are the same, because these cases will either have V = 0 or $\Lambda \to \infty$. In the |m| > M regions the wave numbers $k_{L,R}(\omega)$ can be identified immediately, by recognizing that plane waves $e^{\pm ik_L m}$ are local solutions to the single-particle Schrödinger equation (9) for m < -M, as are $e^{\pm ik_R m}$ for m > M, if

$$-2\Lambda\cos(k_{L,R}) \pm \frac{V}{2} = \omega \tag{12}$$

where the +(-) branch of \pm applies for L(R). The ratios of $\sin(k_{L,R})$ that appear as factors before $\mathcal{T}_{L,R}$ are the usual ratios of group velocities $d\omega/dk$ that appear in onedimensional transmission and reflection problems when the left and right asymptotic group velocities may be different, as here if $V \neq 0$. We can also note from (12) that the densities of states which appear in our stationary $R_{m'm}$ integral (10) are

$$D_{L,R}(\omega) = \frac{1}{\frac{d\omega}{dk_{L,R}}} = \frac{1}{2\Lambda\sin\left(k_{L,R}(\omega)\right)}, \qquad (13)$$

which as usual in one-dimensional systems are the inverse group velocities.

C. Why scattering wave functions?

The first reason for defining the left- and right-incident wave functions $\Psi_m^{L,R}$ in this way (11) is that they allow us easily to create arbitrary incident wave packets at infinity. For example, in a superposition of the form

$$\int dk_L \, e^{-\frac{1}{2}\Delta m^2 (k_L - \bar{k})^2} e^{-ik_L \bar{m}} \Psi_m^L \tag{14}$$

for arbitrary $\bar{k} \in [0, \pi]$ and large $\Delta m \gg 1$ but even larger negative $\bar{m} \ll -\Delta m$, only the incoming wave term $\propto e^{ik_Lm}$ in Ψ_m^L will contribute significantly—and it will contribute essentially the same Gaussian wave packet, centered on $m = \bar{m}$ and wave number $k_L = \bar{k}$, that we would obtain if Ψ_m^L consisted of nothing but that incoming wave because we were dealing with an infinite uniform lattice with no weak or lossy links anywhere.

It is then the implication of these incoming wave packets from infinity that is our main reason for defining the $\Psi_m^{L,R}$. An equilibrium ensemble of incoming wave packets represents the particles which approach our finite middle chain, with its lossy site, from an infinite reservoir of particles in equilibrium. Consequently we must be able to represent the single particle density matrix $R_{m'm}$ of a steady state of our system, with distinct equilibrium distributions at left and right infinity, in the form (10) that we defined at the beginning of this Section.

D. Cases with imaginary $k_{L,R}$

In general it may happen that one or the other of $k_{L,R}(\omega)$ as given by (12) is complex. No cases with complex $k_{L,R}$ will need to be considered here in our main

text, but for possible use in future work we discuss this technicality further in Appendix B.

E. Observables in the steady state

From our steady-state single-particle density matrix (10), and the explicit eigenfunctions we derive in Appendix A, it is straightforward to compute several steadystate observables as functions of the Hamiltonian parameters and the initial states of the left and right lattice ends. The most basic observable is the local occupation number at each site m, which is simply a diagonal element of the unnormalized density matrix:

$$\bar{n}_m = \operatorname{Tr}(\hat{\rho}\hat{a}_m^{\dagger}\hat{a}_m) = R_{mm} .$$
(15)

Within the finite middle chain of sites, \bar{n}_m for $|m| \leq M$ will depend on the $\Psi_m^{L,R}(\omega)$ for $|m| \leq M$, and will therefore not have any simple relation to the \mathcal{T} and $\mathcal{R}_{L,R}$ coefficients that define the wave functions for |m| > M. We will focus instead on observables that relate directly to transport.

The simplest of these is the particle current. From the single-particle master equation (7) we obtain

$$\frac{d}{dt}\bar{n}_m = -\gamma\tau\delta_{m0}\bar{n}_0 - (J_{m+1} - J_m)$$
$$J_m = i\tau T_{m-1}(R_{m,m-1} - R_{m-1,m}).$$
(16)

Since the first line of (16) is an obvious discrete version of the continuous continuity equation, we recognize J_m as the particle number current from left to right, which is also proportional to the average momentum at site m.

In a stationary state, \bar{n}_m is time-independent, and so we conclude that the particle number current is piecewise constant as a function of m:

$$J_m = \tau \begin{cases} I + \frac{\gamma \bar{n}_0}{2} &, & m \le 0\\ I - \frac{\gamma \bar{n}_0}{2} &, & m > 0 \end{cases},$$
(17)

for constant average current I. Since the difference between the piecewise constant currents on each side of m = 0 thus represents particle transport *into* the central lossy site from the two reservoirs, we can follow [2] and take I as a representation of net transport *through* the finite middle lattice from the left reservoir into the right reservoir. With our stationary $R_{m'm}$ represented in the form (10), we can evaluate $J_{L,R}$ by evaluating J_m for |m| > M, and then use our |m| > M forms for $\Psi_m^{L,R}$ in (11) to conclude

$$I = \frac{1}{4\pi} \int d\omega \left(f_L(\omega) - f_R(\omega) \right) \left(1 + |\mathcal{T}|^2 - |\mathcal{R}|^2 \right), \quad (18)$$

where $f_{L,R}(\omega)$ are equilibrium occupation distributions that can be different for the two ends of the lattice. In (18) we have assumed $\mathcal{R}_{L,R} \to \mathcal{R}$ as they do in the cases V = 0 or $\Lambda \to \infty$; a more general formula for I is given in Appendix C. The density of states factors $D_{L,R}(\omega)$ from (10) have not been omitted accidentally from (18): they are truly absent, being conveniently canceled by group velocity factors that appear in J_m .

Additional single-particle observables that may be of interest, and that can probably be measured in quantum gas experiments, include the local energy at site m, which is also simply related to the momentum current, as well as the energy current, which could offer insight into how the central lossy cite absorbs heat as well as particles. We will only consider the particle current I here in our main text, but we derive the corresponding formulas for the other observables in Appendix C.

F. Semi-infinite lattices as macroscopic leads: The $\Lambda \to \infty \mbox{ limit}$

We now confirm that our infinite lattice model, with weak links between the finite middle chain and the semiinfinite ends, does indeed represent a finite middle chain coupled to particle reservoirs.

1. Transmission and reflection coefficients \mathcal{T} and \mathcal{R} for $\Lambda \to \infty$

As we show in Appendix A, the dimensionless parameter Λ does not appear explicitly anywhere in the expressions for $\mathcal{R}_{L,R}$ and \mathcal{T} . Λ decisively affects $k_{L,R}(\omega)$ according to (12), however, and $k_{L,R}$ do both appear in $\mathcal{R}_{L,R}$ and \mathcal{T} . In the limit $\Lambda \to \infty$, in particular, the complicated general formulas of Appendix A for $\mathcal{R}_{L,R}$ and \mathcal{T} simplify dramatically.

For $\Lambda \to \infty$ the continuous bandwidth $|\omega \mp V/2| \leq 2\Lambda$ of the semi-infinite outer parts of the lattice becomes much broader than the bandwidth $|\omega| \leq 2$ of the finite middle chain of sites. Wave vectors for $|\omega| \gg 2$ decay so sharply within |m| < M that they contribute negligibly to any observables within the middle chain, including the particle current through the middle chain, so as far as any such observables are concerned, we can restrict our attention to $|\omega| \ll \Lambda$. For all these ω , the limit $\Lambda \to \infty$ means that $k_{L,R}$ are both negligibly different from $\pi/2$. This implies that the density of states in the "reservoir" ends of the lattice is uniform over the entire frequency range that matters for transport through the finite middle chain.

With the replacement $k_{L,R} \to \pi/2$, then, our expressions from Appendix A simplify—in particular $\mathcal{R}_{L,R} \to \mathcal{R}$ because $k_L = k_R$. Defining $k(\omega)$ without subscript to be the wave number in the middle chain,

$$k(\omega) = \cos^{-1}\left(-\frac{\omega}{2}\right),\qquad(19)$$

we obtain (see Appendix A)

$$1 + |\mathcal{T}|^{2} - |\mathcal{R}|^{2} = \frac{2}{|Z(\omega)|^{2}} \left[\sin^{2}(k) + \frac{\gamma}{4} \left(\frac{\sin^{2}[k(M+1)]}{\tau_{1}^{2}} + \tau_{1}^{2} \sin^{2}(kM) \right) \right] |Z(\omega)|^{2} = \left(\frac{\sin[k(M+1)] \cos[k(M+1)]}{\tau_{1}^{2}} + \tau_{1}^{2} \sin(kM) \cos(kM) \right)^{2} + \left[\sin(k) + \frac{\gamma}{4\sin(k)} \left(\frac{\sin^{2}[k(M+1)]}{\tau_{1}^{2}} + \tau_{1}^{2} \sin^{2}(kM) \right) \right]^{2}.$$
(20)

Since we consider small τ_1 , to make weak links between the outer ends and the middle chain, the important feature in (20) is that the denominator $|Z(\omega)|^2$ is large (of order τ_1^{-4}), making the current contribution from a given ω small, unless either $\sin[(M+1)k(\omega)]$ or $\cos[(M+1)k(\omega)]$ is small. These are precisely the conditions $\sin[2k(\omega)(M+1)] = 0$ that define the 2M + 1discrete eigenfrequencies

$$\omega_n = -2\cos(k_n) k_n = \frac{n\pi}{2(M+1)} , \quad n \in [1, 2M+1]$$
(21)

of the middle chain when it is isolated $(\tau_1 \rightarrow 0 \text{ so that})$ the outer ends of the lattice can be ignored). Transport through the weakly linked middle chain will therefore be small except for particles with frequencies close to one of these resonant frequencies ω_n .

It can further be noted, however, that the numerator in $1 + |\mathcal{T}|^2 - |\mathcal{R}|^2$ in (20) can also somewhat large of order $\gamma \tau_1^{-2}$ —unless $\sin[(M+1)k(\omega)]$ is small. For moderate γ this broadens the transmission resonances at $\cos[(M+1)k_n] = 0$, although for large γ the denominator term of order $\gamma^2 \tau_1^{-4}$ also lowers the contributions of these resonances.

At the resonances with $\sin[(M+1)k_n] = 0$, in contrast, all the terms with negative powers of τ_1 vanish, and small τ_1 makes the transmission factor $(1+|\mathcal{T}|^2-|\mathcal{R}|^2)/2$ close to one. This is easy to understand: the isolated-chain eigenfrequencies ω_n with $\sin[(M+1)k_n] = 0$ are those for which the isolated-chain eigenfunctions are odd functions of m and so have nodes at the lossy site m = 0. Transport through these channels is protected against loss by parity symmetry, as reported in [2]. Transport at their resonant frequencies is only affected by loss insofar as the weak links provide non-resonant coupling to other modes of the finite chain that do not have nodes at m = 0 and so do have loss.

2. Current I through the chain for $\Lambda \to \infty$

To confirm that we are reproducing the results of [2], we follow it in considering fermions at zero temperature, with a non-zero voltage difference V ranging through the finite chain bandwidth $-2 < \omega < 2$. Since the density of



FIG. 1. The dimensionless current I through a finite middle chain of five sites (M = 2), divided by τ_1^2 as in [2], as a function of potential difference V between the left and right semi-infinite ends of the infinite lattice, for four different values of γ (from top to bottom at the left side of the plot: $\gamma = 0$, 0.5, 5, 100.) In all cases $\tau_1 = 0.1$. The thin, dark curves that only extend to V = 4.5 are computed by numerical integration of our (18) with (20) and (22), while the broader, fainter curves that extend to the right edge of the plot are taken from Figure 5a) of Reference [2]. The thin, dark curves lie directly on top of the broad, faint curves, confirming the equivalence of the present model for $\Lambda \to \infty$ and the open system treated in [2].

conduction-band electrons in a metal lead is a nearly constant property of the material regardless of electrostatic potential, we keep the chemical potentials (*i.e.* Fermi energies) $\mu_{L,R}$ in the left and right reservoir $f_{L,R}(\omega)$ equal to the voltages $\pm V/2$, so that the reservoirs differ in potential but have equal fermion densities:

$$f_{L,R}(\omega) = \theta \left(\pm \frac{V}{2} - \omega \right) \,. \tag{22}$$

As we see in Fig. 1, our infinite lattice model exactly reproduces the results from [2] for a finite lattice between two macroscopic leads, treated with Kheldysh path integrals. Figure 1 differs only in plotting resolution from Figure 5 of [2]. Scaling the lattice weak link as $\tau_1 \tau \sqrt{\Lambda}$ while the outer lattice ends have hopping rate $\Lambda \tau$, and then taking $\Lambda \to \infty$ to make the reservoir density of states effectively constant, maps the infinite lattice gas model onto the macroscopic leads problem of [2].

IV. A QUANTUM GAS VERSION

A. The quantum gas set-up

Having confirmed that the infinite lattice model with weak links can indeed describe the finite chain of quantum dots coupled to leads as studied in [2], we now examine a version of the lattice problem that would be more straightforward to realize with an ultracold Fermi gas in a one-dimensional optical lattice (or a three-dimensional array of many one-dimensional lattices, allowing singleshot measurements of ensembles of the one-dimensional system). While the limit of large Λ could be achieved by making τ small, with high potential barriers between the sites in the middle chain, this would slow the system's approach to a steady state, possibly to the point where steady-state behavior could not be seen within the sample lifetime. We therefore focus on the experimentally more straightforward case $\Lambda = 1$.

The on-site potential in the lattice can also be made non-uniform in cold atom experiments, but it may be challenging to make the potential piecewise uniform with abrupt steps between the middle and the outer ends of the lattice. We therefore set $V \rightarrow 0$, leaving the two weak links as the only nonuniformity in the lattice. With no analogous constraint to the fixed fermion density of metal leads, however, we are free to let the chemical potentials $\mu_{L,R}$ in the two lattice ends be arbitrary. Varying $\mu_L - \mu_R$ while keeping V = 0 will turn out to have similar effects to varying V with Fermi energies locked to $\mu_{L,R} = \pm V/2$; the mechanism of opening more transmission channels as the Fermi level rises is exactly the same, after all.

With no need to maintain fixed Fermion densities in both reservoirs, moreover, we are free in the quantum gas scenario to let μ_R lie below the ground state energy, so that no particles at all are incident from the right, and the entire current is from left reservoir to right reservoir, through the middle chain with its central site loss, minus reflection from the weak links back into the left reservoir. This means that instead of (22) we now take

$$f_L(\omega) = \theta(\mu_L - \omega) \tag{23}$$

and $f_R = 0$.

This less symmetrical but simpler set-up also has the advantage of showing more conductivity steps. In the $\pm V \neq 0$ scenario with equal fermion densities in leads, the currents from right and from left exactly cancel in the frequency range $\omega < -V/2$, so that the integral (18) over ω is effectively over the symmetrical middle range $|\omega| < V/2$. Since the resonant frequencies of the transmission channels through the finite middle chain are in symmetrical pairs $\pm \omega_n$, raising V effectively only opens new channels in pairs, producing only M + 1 conductivity steps. In the quantum gas scenario with particles coming only from the left reservoir, in contrast, raising μ_L through the system's frequency band sweeps through each of the 2M + 1 transmission resonances separately. 7



FIG. 2. Dimensionless fermion current I at zero temperature. as a function of left-side chemical potential μ_L , for two different central-site loss rates γ . Both plots show $\tau_1 = 0.1$ as in Fig. 1, but with V = 0, $\Lambda = 1$, and no fermions incident from the right. Although the middle chains here have seven sites (M = 3), the leftmost and rightmost current steps are small and gentle in a), and completely flattened in b), leaving only five current steps clearly visible. In a) we have $\gamma = 0.5$, while b) shows $\gamma = 5$. As well as suppressing the outermost of the seven current steps, the moderate loss in a) has noticeably smoothed out two of the five more visible steps in the current, while the stronger loss in b) has smoothed them completely into gradual slopes. The other three visible current steps are not significantly affected by loss in either case, since they are due to transport through odd-parity channels having nodes at the lossy site m = 0. Although $I(\mu_L)$ differs in detail from analogous plots of I(V) for quantum dots between leads, the same phenomena of conductivity steps with parity-dependent smoothing by loss are revealed.

Figure 2 shows two quantum gas analogs of Fig. 1, plotting the dimensionless steady-state current I as a function of left-side chemical potential μ_L , with no fermions incoming from the right. The current plotted in Fig. 2 is the same integral (18) as in Fig. 1, but with the different $f_{L,R}(\omega)$ of (23) as well as the more complicated \mathcal{R} and \mathcal{T} coefficients that are given by inserting $V \to 0$ and $\Lambda \to 1$ in the general formulas of Appendix A.

Five steplike rises in conductivity through the finite middle chain of seven sites (M = 3) are clearly visible in

Fig. 2; looking closely at a), one can also discern two more small, smooth steps at the left and right edges of the plot. Also apparent is the effect of loss in making some steps noticeably steeper than others. In Fig. 2a) the tiny first and seventh steps, as well as the more visible third and fifth steps, are all due to even-parity channels that are affected by the loss at m = 0, and so more gently sloped than the three other steeper steps, which are due to oddparity channels with nodes at m = 0. The ten-timeshigher loss rate in Fig. 2b) is enough to suppress the first and seventh steps entirely, and eliminate the shoulders of what were the third and fifth steps in a), leaving them as gradual slopes, while still leaving the three parityprotected transport resonances as sharp steps.

Many different parameter regimes of this kind of lossy transport can be investigated in quantum gas experiments, which are highly tunable. While quantum gases in optical lattices may in this sense be a more flexible platform than quantum dots, quantum gases do have their own constraints. While room temperature can easily be effectively zero temperature for conduction electrons in metals, even nanokelvin temperatures may not be negligible for ultracold Fermi gases. And while macroscopic leads are effectively infinite in size, optical lattices in experiments are bound to be finite. We will examine finite temperature effects next, and treat the conceptually greater difference of finite system size in the following Section.

B. Finite temperature effects

Finite temperature effects are incorporated straightforwardly by replacing the step function form (22) of the zero-temperature Fermi-Dirac function with the general case

$$f_L(\omega) = \frac{1}{e^{\frac{\omega+\mu_L}{T}} + 1} \tag{24}$$

for temperature $T_L \to T$, leaving $f_R(\omega) = 0$. The effect of finite temperature is likewise simple: it smears out sharp features of frequency.

Since the main phenomena to be studied in this kind of transport are the conductivity steps and their parityand γ -dependent steepness, thermal smearing of the conductivity curve is a serious issue for experiments. Since the frequency spacing of our transmission resonances is of order 2/(2M+1), temperatures higher than this spacing (in units of $\hbar \tau/k_B$) will wash out the conductivity steps. Even if T/τ is below the 2/(2M+1) threshold, moreover, thermal smoothing of the conductivity steps can reduce their dramatic signature of discrete transmission channels to only a slight waviness in $I(\mu_L)$, as seen in Fig. 3a); in this regime, moreover, the additional smoothing of even-parity channels due to the m = 0 loss can be hard to discern. Both the steps and the effect of parity can still be seen at sufficiently low temperatures, as in Fig. 3b), but such low temperatures may be experimentally challenging. If the basic $T < 2\tau/(2M+1)$ limit can be beaten, the problem of losing visibility of the parity effect for $T \lesssim \gamma$ can be solved by increasing γ , as in Fig. 3c).

In condensed matter systems with Fermi temperatures in thousands of Kelvin, even room temperature is negligibly different from zero temperature. In quantum gas experiments, in contrast, the ratio $k_B T/(\hbar \tau)$ of temperature to hopping rate cannot be made arbitrarily low. While it is easy to make τ small by increasing laser power in the optical lattice to reduce tunnelling rates exponentially, reducing the lattice strength in order to accelerate tunnelling will also weaken the tight binding approximation and turn on excitations out of the lowest band, so that our single-band lattice model (1)breaks down. A realistic upper bound on τ is probably in the range of 10^4 s^{-1} , corresponding to a temperature range in hundreds of nanoKelvin[11]. With the currently achievable temperatures for lattice fermions in tens of nK, reaching $k_B T/(\hbar \tau) < 0.1$ will likely require future technical progress. As Fig. 3 shows, a lower ratio like $k_B T/(\hbar \tau) = 0.01$ may be needed to compete with the current step sharpness of quantum dot experiments, but a ratio of 0.1 can still be sufficient to show the steps as well as the smoothing effect of loss in dependence on parity. As in many potentially interesting cold atom experiments, reaching significantly lower temperatures will be an important challenge.

We now turn to the issue of finite size effects in quantum gas realizations, which requires some new concepts but probably poses less serious problems than finite temperature.

V. FINITE SIZE EFFECTS

A. Finite size and finite time

Cold atom experiments can flexibly tune many parameters, but the number of lattice sites cannot really be tuned to infinity. Although the finite sizes of cold atom "reservoirs" can limit the effects that can be seen in cold atom experiments, we will now show how measurements of finite time evolution can in fact reproduce the steadystate behavior of infinite systems.

In the finite system with a lossy impurity there can be no steady state current: currents can only be transient, strictly speaking, because in the limit of infinite time all initial particles will have simply been lost. Technically transient currents do not have to be short-lived phenomena, however. They can easily be quasi-steady over intermediate time scales.



FIG. 3. Ultracold Fermi gas current I through the lossy impurity, as a function of left-side chemical potential μ_L , at finite left-side temperature (thin black curves), with zerotemperature results for comparison (thick gray curves). All three plots show V = 0, $\tau_1 = 0.1$, and M = 3 as in Fig. 2 (seven sites in the middle chain, although at most five current steps are large enough to see clearly), with no fermions incoming from the right. In a) we have $\gamma = 0.5$ and $k_B T_L = 0.1 \hbar \tau$ (black curve); the comparatively high temperature smooths out all the abrupt zero-temperature steps into slopes that are similarly gradual. In b) we lower the temperature to $k_B T = 0.01\hbar\tau$; the steepest zero-temperature steps are only slightly less steep at this low temperature, and the effect of loss in smoothing out the second and fourth visible steps is still noticeable. In c) we raise the loss rate γ to 5; with stronger loss, the same temperature $k_B T = 0.1\hbar\tau$ as in a) no longer buries the smoothing effect of the loss on the second and fourth visible steps.

B. Quasi-stationary currents

For example, a natural initial state to prepare experimentally is a finite-system analog of the current-fromthe-left scenarios shown for infinite lattices in Fig. 2. To prepare the initial state we set $\tau_1 \rightarrow 0$, fully breaking the weak link and isolating the middle chain of our lattice from its outer ends. We then populate the isolated left end of the lattice with fermions in equilibrium, leaving the middle and right-end lattices empty. At t = 0we then restore non-zero τ_1 and let the left-end population of fermions flow through the weakly linked middle chain, with its loss at m = 0, and into the right lattice end. At different times t > 0 we measure the decreasing population difference between left and right sides,

$$\frac{\Delta \mathcal{N}}{2} = \frac{1}{2} \sum_{m=M+1}^{M+N} (n_{-m} - n_m) , \qquad (25)$$

which is due to the same left-to-right current that we have examined in previous Sections, albeit no longer in a steady state. For clarity we return to the zerotemperature scenarios; the frequency smoothing effect of finite temperature can be added straightforwardly to the finite-size and finite-time effects that we now examine.

As we can see in Fig. 4a), there is a brief initial period in which the fermions first begin to move through the middle chain of sites, via the weak links. Within a time of order M/τ , however, this initial onset phase ends and we can see a remarkably steady fall of $\Delta \mathcal{N}$. The initial equilibrium state in the left end of the lattice can be viewed as an ensemble of wave packets moving in both directions; these packets move to and through the middle chain from further back in the left reservoir, just as they would if the reservoir were infinite. Even the packets which begin moving to the left, and then approach the middle chain after reflecting from the left end of the lattice, simply reproduce the effect of incoming packets from beyond the lattice end. The first effect which does not look just like an infinite system, in fact, is the return to the weak link of packets which have been reflected from it, and then reflected again from the left end of the lattice. This occurs for packets with group velocity v_G at time $t = 2N/v_G$. The maximum group velocity in our system is 2τ , occurring at $\omega = 0$, and so the earliest finite size effects are those that appear in Fig. 4 at $t = N/\tau$. Between $t \lesssim M/\tau$ and $t < N/\tau$ there can be plenty of time to observe a quasi-stationary current, as long as N is sufficiently larger than M. And as Fig. 4 shows, the rate at which $\Delta \mathcal{N}$ falls, during this quasi-stationary epoch between the initial transient and the onset of finite-size effects, can closely match the steady-state current of the infinite system.



FIG. 4. Finite-time evolution in finite systems. The initial state is prepared in a finite lattice with N = 20, 40, or 80 sites in the outer lattice on each side. With the weak links between reservoir ends and middle chain initially turned off completely $(\tau_1 \rightarrow 0)$, the N-site left end lattice is initially half-filled at zero temperature, with the right end and central chain of seven sites (M = 3) both initially empty. At t = 0 the weak links are turned on $\tau_1 \rightarrow 0.1$ and the system's single-particle density matrix is evolved in time under (7) numerically, for a duration $2.5N/\tau$. The difference in total left- and rightside populations $\Delta N/2 = (N_L - N_R)/2$ is plotted in each case of N. The dashed line has the slope -I, where I is the steady-state current for the $N \to \infty$ limit as shown in Fig. 2. The early transient epoch is so brief that it can only be seen clearly with shorter time axis of the N = 20 plot. The onset of finite-size effects at $t = N/\tau$ is clear in all cases: with group velocity 2τ for half filling, this is how long it takes for particles that initially reflect from the weak link to return to the weak link after reflecting again from the end of the lattice. For times after the short initial transient and before $t = N/\tau$, the finite system's particle difference closely reproduces the infinite system's steady-state current—in the case $\mu_L = 0$ of initial half-filling. See Fig. 5 for other cases of μ_L .

C. Extended transient effects near transmission resonances

In fact Fig. 4 provides an overly optimistic view of how easy it is to see steady-state current with finite-time measurements: although $\mu_L = 0$ is the worst case for finitesize effects, since v_G is highest there, it turns out to be the best case for short duration of the initial transient period. While $\mu_L = 0$ is not so much better than typical μ_L that are not close to resonances, the slope of $\Delta \mathcal{N}$ can take longer to settle down steadily for μ_L near transmission resonances other than the one at $\mu_L = 0$; see Fig. 5. The curves of $\Delta \mathcal{N}(t)$ appear quite similar for different outer-lattice sizes N, when both t and μ_L are rescaled with N appropriately, indicating that the extended transient epochs near transmission resonances are probably not really finite-size effects, but rather finite-time effects. We cannot avoid these problems by letting the system evolve for longer times, however, because finite-size effects will appear if we wait too long.

D. Quasi-stationary current versus μ_L

While finite lattices may thus force finite-time problems upon us, though, these problems may not actually be too severe. Fig. 6 shows finite-system analogs of Fig. 2a), with three different cases of N (20, 40, and 80 in Figs. 6a), 6b), and 6c), respectively). Each of these three plots further shows three different sim-



FIG. 5. Zero-temperature fermion flow over time in finite systems with different N as in Fig. V B, for μ_L near the first conductivity step at $\mu_L = -\sqrt{2}$, and times less than t = 2.5N. Top row: N = 20 sites in the lattice ends, with initial Fermion numbers 3,4,5,6 from left plot to right. Middle row: N = 40, initial numbers 8,9,10,11. Bottom row: N = 40, initial numbers 19, 20, 21, 22. The different initial particle numbers all correspond to μ_L near $-\sqrt{2}$; in particular the second column of plots have μ_L just below $-\sqrt{2}$ and the third column have μ_L just above it. Since the group velocities at frequencies near these μ are near $\sqrt{2}$ rather than 2 as at $\omega = 0$, finite-size effects are not yet apparent. Finite-time effects, however, are clear: the initial transient epoch lasts much longer near the conductivity resonance.

ple attempts to extract the quasi-steady current from the growth of $\Delta \mathcal{N}(t)$ before the onset of finite-size effects at $t = N/\tau$. These attempts represent I as local slopes $[\Delta \mathcal{N}(t_m + 1/\tau) - \Delta \mathcal{N}(t_m - 1/\tau)]/(2\tau)$, for three different choices of the measurement time t_m : the early choice $t_m = N/(4\tau)$, the late choice $t_m = 3N/(4\tau)$, and the intermediate choice $t_m = N/(2\tau)$. If $\Delta \mathcal{N}(t)$ were as straight as it is in Fig. 4, at $\mu_L = 0$, these three choices of t_m would all yield very similar I estimates. Discrepancies between the three attempts to find I show the non-stationary nature of the fermion flow at other μ_L .

As Fig. 6 shows, it is possible to resolve the steadystate current $I(\mu_L)$ from Fig. 2a) fairly well with finite lattices as small as N = 20, and lattices as large as N =80 allow essentially duplicating the steady-state current curve of the infinite system, with clear conductivity steps that are noticeably smoother for the even-parity channels that are affected by loss. Even though extended transient effects near the steps still exist even for N = 80, as shown in Fig. 5, this issue only really affects the two data points on either side of the step, and for N = 80 there are many other points to map the curve out accurately.

Ultimately the main limitation from finite lattices turns out to be the fundamentally limited frequency resolution from only having an N-dimensional single-particle Hilbert space in an N-site lattice. As long as N is larger than M by enough both to delay finite-size effects until the initial transient epoch is mostly over and to resolve the 2M + 1 conductivity steps in the frequency band $-2 < \omega < 2$, experiments with ultracold Fermi gases in finite optical lattices can indeed reveal the same interesting transport phenomena with a lossy site that have been predicted for finite chains of quantum dots coupled



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FIG. 6. Zero-temperature current I versus initial left-side chemical potential $\mu_L = \mu$ computed at three different times before the onset of finite-size effects, for each of three different finite lattice sizes. All cases have $\tau_1 = 0.1, \gamma = 0.5, M =$ 3, and initial $\mu_R < -2$ (no particles in the right reservoir section). All three plots a), b), c) show the same thick, gray curve, which is $I(\mu)$ for the steady state of the infinite lattice with $\Lambda = 1$ and V = 0. In plots a), b), and c) we show cases with N = 20, 40, 80. Each plot shows I computed as $([n_R(t +$ 1) $-n_L(t+1)$] $- [n_R(t-1) - n_L(t-1)]/2$ for t = 0.25N (o markers), t = 0.5N (+ markers), and t = 0.75N (x markers). The markers are discrete points rather than continuous lines because zero-temperature fermions in a finite lattice simply occupy the n lowest modes, for integer n between 1 and N.

0.004

0.002



FIG. 7. Current I through the finite chain of seven sites (M = 3) as in Fig. IVA, but for non-interacting bosons at maximal chemical potential $\mu_L = -2$, as a function of temperature. No conductivity steps are apparent: while the sharp edge of the zero-temperature Fermi-Dirac distribution rises with increasing μ , increasing temperature in the Bose-Einstein distribution merely spreads the distribution out flatter, steadily overlapping more and more with all seven conductivity resonances.

to leads.

2 ^µ

1

DISCUSSION AND OUTLOOK VI.

In conclusion, we have constructed a class of fermionic lattice gas models which can reproduce the results of [2] in the limit $\Lambda \to \infty$, and then represent quantum gas experiments in the limit $\Lambda \to 1$. Different but similar effects are predicted for quantum gases in optical lattice, in comparison with the parity-dependent conductivity steps that are predicted to be seen in quantum dot chains between leads. Finite-temperature effects seem likely to be a practical challenge in near-future experiments with quantum gases. While these effects will not prevent observation of the most interesting conductivity features, they will likely prevent quantum gas experiments from matching the sharpness of conductivity steps that can be seen in condensed matter systems.

One new possibility that can appear with quantum gases in lattices is transport of bosons instead of fermions. In the absence of interactions, however, bosons do not readily display the dramatic step-like conductivity of zero-temperature fermions, because Bose-Einstein distributions cannot replicate the moveable sharp edge of a low-temperature Fermi-Dirac distribution with variable Fermi energy. For non-interacting bosons, the chemical potential can rise no higher than the bottom of the single-particle energy band; the only way to have varying exposure to the different discrete transport channels through the finite middle chain is to vary the reservoir temperatures.

As Fig. 7 disappointingly shows, increasing temperature simply mixes in higher-frequency channels continuously. No conductivity steps can be seen—at least not in scenarios like those we have considered in this paper, and not without interactions among the bosons. Perhaps different kinds of experiments will be able to show more abrupt unlocking of discrete transport channels for bosons. Further studies may also reveal more interesting transport behavior in interacting bosons, which can, among other complications, have higher chemical potentials.

ACKNOWLEDGMENTS

The author thanks Corinna Kollath and A.-M. Vi-

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VII. APPENDIX A: THE SINGLE-PARTICLE WAVE FUNCTIONS $\Psi_m^{L,R}(\omega)$

A. Unnormalized eigenfunctions of definite parity for $|m| \le M$

For $V \neq 0$ our entire system lacks reflection symmetry, but there is always reflection symmetry within the middle chain $|m| \leq M$. This makes it straightforward to construct ψ_m eigenvectors that are either odd or even in m, for $|m| \leq M$. Once we have these solutions, we will be able to combine them into left-incident and right-incident $\Psi_m^{L,R}$, and finally normalize them.

The odd solutions are the simplest, since they vanish at m = 0 and are therefore unaffected by the imaginary potential term γ . They can simply be written as

$$\psi_{m}^{-}(\omega) = \lim_{|m| \le M} \sin\left(k(\omega)m\right)$$
$$\omega = -2\cos(k)$$
$$k(\omega) = \cos^{-1}\left(-\frac{\omega}{2}\right).$$
(26)

Note that we may need to consider $|\omega| > 2$, for which $k(\omega)$ is complex, because for $\Lambda > 1$ the outer parts of the lattice support eigenmodes in a wider bandwidth than that of the middle chain.

The modes that are even in the middle chain are a bit less trivial, because of the complex potential at n = 0:

$$\psi_m^+(\omega) \stackrel{=}{\underset{|m| \le M}{=}} \cos\left(k(\omega)m\right) - \frac{i\gamma}{4\sin\left(k(\omega)\right)} \sin\left(k(\omega)|m|\right) \,.$$
(27)

How these eigenvectors behave for |m| > M is then determined firstly by solving (9) for |m| > (M+1), which implies

$$\psi_{n}^{\pm}(\omega) =_{|m|>M} \alpha_{L,R}^{\pm} e^{ik_{L,R}(\omega)|m|} + \beta_{L,R}^{\pm} e^{-ik_{L,R}(\omega)|m|} ,$$
(28)

suri for very useful discussions, and acknowledges support from the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR185 (OSCAR), Project No. 277625399. In particular the essential ideas of representing lead contacts as weak links, and addressing finite-size effects, are due to C. Kollath.

for |m| > M, with the subscript L applying for m < -Mand R for m > M, and with $k_{L,R}(\omega)$ defined in (12) above.

To fix the separate left and right coefficients $\alpha_{L,R}^{\pm}$ and $\beta_{L,R}^{\pm}$, we finally solve (9) for |m| = M and |m| = M + 1, which provide two conditions at each end of the finite middle chain that are analogous to the continuity conditions on the wave function and its first derivative in continuous Schrödinger problems. After some algebra we obtain

$$\psi_{m}^{-} = \sup_{|m| > M} \operatorname{sgn}(m) \left[\frac{A_{L,R}}{2} e^{ik_{L,R}|m|} + \frac{A_{L,R}^{*}}{2} e^{-ik_{L,R}|m|} \right]$$

$$\psi_{m}^{+} = \left[B_{L,R} - \frac{i\gamma}{4\sin(k)} A_{L,R} \right] \frac{e^{ik_{L,R}|m|}}{2}$$

$$+ \left(B_{L,R}^{*} - \frac{i\gamma}{24\sin(k)} A_{L,R}^{*} \right) \frac{e^{-ik_{L,R}|m|}}{2} \quad (29)$$

where the L subscripts apply for m < -M and the R subscripts for m > M. The coefficients here are

$$A_{L,R}(\omega) = \frac{e^{-ik_{L,R}M}}{i\sin(k_{L,R})} \times \left(\frac{1}{\tau_1}\sin[k(M+1)] - \tau_1\sin(kM)e^{-ik_{L,R}}\right) \\B_{L,R}(\omega) = \frac{e^{-ik_{L,R}M}}{i\sin(k_{L,R})} \times \left(\frac{1}{\tau_1}\cos[k(M+1)] - \tau_1\cos(kM)e^{-ik_{L,R}}\right),$$
(30)

where k without L, R subscripts is the middle-chain $k(\omega)$ from (19), which is reproduced in (26). Although we have the same Λ for both m > M and m < -M, if $V \neq 0$ then $k_L \neq k_R$ and so ψ_m^{\pm} do not actually have definite parity for |m| > M.

B. Left- and right-incident wave functions

We then simply construct linear combinations of ψ_m^{\pm} which contain no incoming wave on one side, and in which the amplitude of the incoming wave on the other side is $1/\sqrt{2\pi}$. The result is

$$\Psi_{m}^{L}(\omega) = \frac{A_{R}^{*}\psi_{m}^{+}(\omega) - \left(B_{R}^{*} - \frac{i\gamma}{4\sin(k)}A_{R}^{*}\right)\psi_{m}^{-}(\omega)}{\sqrt{2\pi}Z}$$
$$\Psi_{m}^{R}(\omega) = \frac{A_{L}^{*}\psi_{m}^{+}(\omega) + \left(B_{L}^{*} - \frac{i\gamma}{4\sin(k)}A_{L}^{*}\right)\psi_{m}^{-}(\omega)}{\sqrt{2\pi}Z}$$
$$Z(\omega) = \frac{A_{R}^{*}B_{L}^{*} + A_{L}^{*}B_{R}^{*}}{2} - \frac{i\gamma}{4\sin(k)}A_{L}^{*}A_{R}^{*} \qquad (31)$$

which further implies

$$\mathcal{T} = \frac{i}{Z} \frac{\sin(k)}{\sqrt{\sin(k_L)\sin(k_R)}}$$
$$\mathcal{R}_L = \frac{1}{2Z} \left(A_R^* B_L + B_R^* A_L - \frac{i\gamma}{2\sin(k)} A_R^* A_L \right)$$
$$\mathcal{R}_R = \frac{1}{2Z} \left(A_L^* B_R + B_L^* A_R - \frac{i\gamma}{2\sin(k)} A_L^* A_R \right) . \quad (32)$$

As we note in Section III of our main text, the dimensionless parameter Λ does not appear explicitly in the results for $\mathcal{R}_{L,R}$ and \mathcal{T} . Λ does appear in Equation (12), however, which defines $k_{L,R}(\omega)$, and so Λ actually does affect $\mathcal{R}_{L,R}$ and \mathcal{T} substantially. In particular both for $\Lambda = 1$ and for $\Lambda \to \infty$ we obtain $k_L = k_R$ and the distinctions between $A_{L,R}$, $B_{L,R}$ disappears, so that we can take $\mathcal{R}_{L,R} \to \mathcal{R}$ as in our main text.

C. Infinite Λ limit

As we note in Section IV of our main text, the $\Lambda \to \infty$ limit which makes the reservoir density of states constant over the middle chain bandwidth lets us replace $k_{L,R} \to \pi/2$. Our expressions for Z, \mathcal{T} , and $\mathcal{R}_{L,R}$ then simplify greatly. We obtain

$$\begin{aligned} A_{L,R} &\longrightarrow_{\Lambda \to \infty} (-i)^{M+1} \left(\frac{\sin[k(M+1)]}{\tau_1} + i\tau_1 \sin(kM) \right) \\ B_{L,R} &\longrightarrow_{\Lambda \to \infty} (-i)^{M+1} \left(\frac{\cos[k(M+1)]}{\tau_1} + i\tau_1 \cos(kM) \right) \\ &|Z|^2 &\longrightarrow_{\Lambda \to \infty} \left[\frac{\sin[k(M+1)]\cos[k(M+1)]}{\tau_1^2} + \tau_1^2 \sin(kM)\cos(kM) \right]^2 \\ &+ \left[\sin(k) + \frac{\gamma}{4\sin(k)} \left(\frac{\sin^2[k(M+1)]}{\tau_1^2} + \tau_1^2 \sin^2(kM) \right) \right]^2 \\ &|\mathcal{T}|^2 &\longrightarrow_{\Lambda \to \infty} \frac{\sin^2(k)}{|Z|^2} \\ \mathcal{R}_{L,R}|^2 &\longrightarrow_{\Lambda \to \infty} \frac{1}{|Z|^2} \left[\left(\frac{\sin[k(M+1)]\cos[k(M+1)]}{\tau_1^2} + \tau_1^2 \sin^2(kM)\cos(kM) \right)^2 \\ &+ \frac{\gamma^2}{16\sin^2(k)} \left(\frac{\sin^2[k(M+1)]}{\tau_1^2} + \tau_1^2 \sin^2(kM) \right)^2 \right], \end{aligned}$$
(33)

as used in our main text.

VIII. APPENDIX B

As we noted in Section III of our main text, for the class of models that we have defined it can in general happen that one of $k_{L,R}$ is complex—or even that both of them are. This can happen for one of $k_{L,R}$, for V > 0, because the bandwidth $4\Lambda\tau$ of the outer lattice ends is finite, and so a relative shift of V between the bands of the outer ends means that they do not fully overlap. Wherever an ω is in the continuous part of the infinite model's spectrum, because it is in the band of one lattice end, $k_{L,R}$ will be purely imaginary for $\omega < -2\Lambda \pm V/2$, below the corresponding lattice end's frequency band, and it will be π plus an imaginary term for $\omega > 2\Lambda \pm V/2$, above the band.

It can also happen that both $k_{L,R}$ are complex, for a discrete set of complex ω that are in the spectrum of our non-Hermitian Hamiltonian which includes the imaginary potential at m = 0, if $\Lambda < 1$ so that the middle chain possesses eigenstates outside the bands of both the outer lattice ends.

We have only considered $\Lambda \geq 1$ in this paper, leaving other cases for future work. In fact we have only explicitly considered cases either with V > 0 but $\Lambda \to \infty$, so that $k_{L,R}$ are both real for all ω that are involved in transport through the middle chain, or else with $\Lambda = 1$ but V = 0, so that all $k_{L,R}$ are real. For the benefit of future work, however, we note that our formulas extend straightforwardly to include complex $k_{L,R}$, when they are taken in the general forms shown in Appendix A above, and when the observables include the $X_{L,R}$ factors as in Appendix C below. If for example $k_R(\omega)$ has an imaginary part for some ω but $k_L(\omega)$ is real, we can keep our $\Psi_n^L(\omega)$ as described above, with the stipulation that we take the positive imaginary branch of k_R , but there will simply be no Ψ_n^R for this ω , because the "incoming" wave would be exponentially growing. It will also be important to note that the evanescent transmitted waves in these cases of $\Psi_n^L(\omega)$ carry no particle or energy currents to $n \to +\infty$ even though $|\mathcal{T}| > 0$.

IX. APPENDIX C

1. Particle current with complex $k_{L,R}$

In the general case where $k_{L,R}$ are not equal, and may be complex, it is necessary to modify our main text's formula (18) for the particle current through the middle chain, by distinguishing \mathcal{R}_L and \mathcal{R}_R , and by including additional factors $X_{L,R}(\omega)$:

$$I = \frac{1}{4\pi\tau_1^2} \int d\omega \left[f_L(\omega) \left(1 + X_L |\mathcal{T}|^2 - |\mathcal{R}_L|^2 \right) - f_R(\omega) \left(1 + X_R |\mathcal{T}|^2 - |\mathcal{R}_R|^2 \right) \right] .$$
 (34)

The factor X_L here is equal to one if k_R is real, and is zero if k_R has a non-zero imaginary part; X_R depends in the same way on whether k_L is purely real. These factors do not have to be inserted by hand, but come out simply in evaluating J_m in the cases where k_L or k_R is either imaginary or imaginary plus π .

The subscripts are not accidentally switched: the factor X_L depends on k_R , and *vice versa*, because the issue is whether or not the wave which is transmitted through to the right from the left, or *vice versa*, is evanescent or propagating.

2. Local energy at site m

The Hamiltonian (1) is a sum over sites m, and apart from site m = 0, where there is an imaginary potential, each term in the *m*-sum in (1) is an observable that can be interpreted as the local energy associated with site m; we can treat each site's two neighbors symmetrically by averaging their contributions. The expectation value of this local energy is then

$$E_{m} = \hbar \tau \left(-\frac{T_{m}}{2} \left(R_{m+1,m} + R_{m,m+1} \right) - \frac{T_{m-1}}{2} \left(R_{m,m-1} + R_{m-1,m} \right) + V_{m} R_{mm} \right)$$
(35)

according to (1).

3. Energy current

Just as the particle number current J_m could be inferred from the time derivative of the average local number \bar{n}_m , we can also compute from (7) that

$$\frac{d}{dt}E_m = -(Q_{m+1} - Q_m) - \frac{\gamma\tau}{2}E_0(\delta_{m,-1} + \delta_{m0})$$

$$Q_m = -i\frac{\hbar\tau^2}{2}T_{m-1}\Big[T_m(R_{m+1,m-1} - R_{m-1,m+1})$$

$$+ T_{m-2}(R_{m,m-2} - R_{m-2,m})$$

$$- (V_m + V_{m-1})(R_{m,m-1} - R_{m-1,m})\Big], (36)$$

and recognize Q_m as the local energy current. In the steady state $\dot{E}_m = 0$ then implies that Q_m is piecewise constant,

$$Q_m = \frac{\tau}{4\pi} \left(Q - 2\pi\gamma \operatorname{sgn}(m) E_0 \right) \,, \tag{37}$$

with $Q_0 = Q$ also being the average energy current through the finite middle chain from left to right.

Evaluating $Q = \lim_{m\to\infty} (Q_{-m} + Q_m)/2$ in (10) and using (9) then reveals a result which we might well have guessed without even thinking much about what it meant: the average energy current in the stationary state is a similar integral to the integral (18) for the average particle current I, with the integrand simply multiplied by $\hbar\tau\omega$.

$$Q = \frac{\hbar\tau^2}{4\pi} \int d\omega \,\omega \left[f_L(\hbar\tau\omega) \left(1 + X_L |\mathcal{T}|^2 - |\mathcal{R}_L|^2 \right) - f_R(\hbar\tau\omega) \left(1 + X_R |\mathcal{T}|^2 - |\mathcal{R}_R|^2 \right) \right] \,. \tag{38}$$