WEIGHTS WITH MAXIMAL SYMMETRY AND FAILURES OF THE MACWILLIAMS IDENTITIES

JAY A. WOOD

In memoriam:
Melvin G. Rothenberg, 1934–2023
Erna B. Yackel, 1939–2022
Robert J. Zimmer, 1947–2023

Abstract. This paper examines the w-weight enumerators of weights w with maximal symmetry over finite chain rings and matrix rings over finite fields. In many cases, including the homogeneous weight, the MacWilliams identities for w-weight enumerators fail because there exist two linear codes with the same w-weight enumerator whose dual codes have different w-weight enumerators.

1. Introduction

The MacWilliams identities [17] reveal a relationship between the Hamming weight enumerator (hwe) of a linear code C over a finite field $\mathbb{F}_q$ and the Hamming weight enumerator of its dual code $C^\perp$:

$$\mathrm{hwe}_{C^\perp}(X,Y) = \frac{1}{|C|}\,\mathrm{hwe}_C(X+(q-1)Y, X-Y).$$

One way to try to generalize this result is to use any integer-valued weight w on a finite ring R with 1. The homogeneous weight, suitably normalized, is an example. Assume $w(0) = 0$ and $w(r) > 0$ for $r \neq 0$. Denote the maximum value of w by $w_{\max}$. For a vector $x = (x_1, x_2, \ldots, x_n) \in R^n$, set $w(x) = \sum_{i=1}^n w(x_i)$. Define the w-weight enumerator (wwe) of a left R-linear code $C \subseteq R^n$ by

$$\mathrm{wwe}_C(X,Y) = \sum_{c\in C} X^{nw_{\max}-w(c)}Y^{w(c)}.$$

Do MacWilliams identities hold for w-weight enumerators?

This paper will show that MacWilliams identities seldom hold for weights having maximal symmetry when the ring is a finite chain ring

---

2010 *Mathematics Subject Classification.* Primary 94B05.

Key words and phrases. dual code, finite Frobenius rings, homogeneous weight, MacWilliams identities, partition enumerator, counter-examples.

or a matrix ring over a finite field. For example, suppose $R = \mathbb{Z}/4\mathbb{Z}$ and w is an integer-valued weight on R with maximal symmetry (i.e., $w(1) = w(3)$ for this ring). Then the only weights for which the MacWilliams identities hold are multiples of the Hamming weight and multiples of the homogeneous weight (the Lee weight for this ring), Corollary 9.6. Similarly, let $R = M_{2\times 2}(\mathbb{F}_q)$ and w be an integer-valued weight with maximal symmetry (i.e., the value of $w(r)$ depends only on the rank of r). Then the only weights for which the MacWilliams identities hold are multiples of the Hamming weight (any q) and multiples of the homogeneous weight ($q = 2$ only), Theorem 19.5. More generally, the homogeneous weight on $R = M_{k\times k}(\mathbb{F}_q)$, $k \geq 2$, satisfies the MacWilliams identities if and only if $k = q = 2$, Theorem 19.4.

The reason the MacWilliams identities often fail is that there exist R-linear codes $C, D \subseteq R^n$ for some n such that $\mathrm{wwe}_C = \mathrm{wwe}_D$ but $\mathrm{wwe}_{C^\perp} \neq \mathrm{wwe}_{D^\perp}$. This claim presents two challenges:

- to construct linear codes $C, D \subseteq R^n$ with $\mathrm{wwe}_C = \mathrm{wwe}_D$ in such a way that it is then possible ...
- to detect differences $\mathrm{wwe}_{C^\perp} \neq \mathrm{wwe}_{D^\perp}$ in the w-weight enumerators of their dual codes.

A weight w on a finite ring R with 1 has maximal symmetry when $w(uru') = w(r)$ for all $r \in R$ and units u, u' in R. Any left linear code $C \subseteq R^n$ can be viewed as the image of an injective homomorphism $\Lambda : M \to R^n$ of left R-modules, for some finite left R-module M. The group of units of R acts on M on the left, and the maximal symmetry hypothesis implies that $x \mapsto w(x\Lambda)$ is constant on each orbit of this group action. Writing $[x]$ for the orbit of $x \in M$, we see that

$$\mathrm{wwe}_C(X, Y) = \sum_{\text{orbits } [x]} |[x]| X^{n w_{\max} - w(x\Lambda)} Y^{w(x\Lambda)}. \tag{1.1}$$

In essence, the choice of homomorphism Λ determines the 'orbit weights' $\omega(x) = w(x\Lambda)$ assigned to each orbit $[x]$ in M.

In order to find another linear code $D \subseteq R^n$ with $\mathrm{wwe}_C = \mathrm{wwe}_D$, one can try to permute the weights assigned to orbits while keeping the sum in (1.1) unchanged. For example, in $R = M_{2\times 2}(\mathbb{F}_2)$ there is one orbit of size 6 consisting of matrices of rank 2 and three orbits, each of size 3, of matrices of rank 1. One can then try to construct linear codes whose orbit weights behave as follows:

orbit	$[x_1]$	$[x_2]$	$[x_3]$	$[y]$
size of orbit	3	3	3	6
$w(x\Lambda)$	a	b	b	c
$w(x\Lambda')$	a	c	c	b

It is not obvious a priori that such constructions are possible, but Section 16 shows that constructions of this type can be carried out for all matrix rings $M_{k\times k}(\mathbb{F}_q)$ over finite fields.

When R is a finite chain ring, all the orbits have different sizes, so the permutation idea does not work. However, one can use different modules as the domains of the defining homomorphisms. For example, the ring $\mathbb{Z}/8Z$ has three modules of size 8: $M = \mathbb{Z}/8Z$ itself, $\mathbb{Z}/4Z \oplus \mathbb{Z}/2Z$, and $M' = \mathbb{Z}/2Z \oplus \mathbb{Z}/2Z \oplus \mathbb{Z}/2Z$. Considering M and M', the orbits of M are $\{1, 3, 5, 7\}$, $\{2, 6\}$, $\{4\}$, and $\{0\}$, while the orbits of M' are the 8 subsets of size 1. By choosing certain unions of orbits of M', say $\{100, 101, 110, 111\}$, $\{011, 010\}$, $\{001\}$, $\{000\}$, of the same size as the orbits of M, one can try to construct homomorphisms $\Lambda : M \to R^n$ and $\Lambda' : M' \to R^n$ achieving the same weights on corresponding orbits. While this may not seem possible at first glance, Section 8 details how such constructions exist.

In order to show that $\mathrm{wwe}_{C^\perp} \neq \mathrm{wwe}_{D^\perp}$, it is enough to show that $A_j(C^\perp) \neq A_j(D^\perp)$ for some $j > 0$; here, $A_j(C^\perp)$ is the number of codewords $v \in C^\perp$ with $w(v) = j$. The easiest case to understand is when $v \in C^\perp$ has exactly one nonzero entry; such a v is called a *singleton*. When $C \subseteq R^n$ is the image of a homomorphism $\Lambda : M \to R^n$, the components of $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are elements $\lambda_i \in \mathrm{Hom}_R(M, R)$. By understanding how many elements $r \in R$ annihilate any given λ_i, i.e., $\lambda_i r = 0$, one can write down formulas for the contributions of singletons to $A_j(C^\perp)$, Proposition 5.2. When j is sufficiently small, only singletons can contribute to $A_j(C^\perp)$, Corollary 5.4. This technique turns out to be surprisingly effective in allowing one to prove that $A_j(C^\perp) \neq A_j(D^\perp)$ in a large number of situations.

This paper is divided into three parts. The first establishes notation and ideas that can apply to any finite ring with 1. In particular, codes will usually be linear codes over a finite ring with 1. A weight w on R will be assumed to have maximal symmetry and have positive integer values, and w will be extended additively to R^n. The second part examines the construction of linear codes and the analysis of singleton dual codewords over a finite chain ring, while the third part does the same for matrix rings over finite fields. An appendix provides a short outline of a proof of the MacWilliams identities over finite Frobenius rings using the Fourier transform and the Poisson summation formula.

Part 1. Generalities

2. Preliminaries

This section will review without proof some terminology and results from [28] about characters of finite abelian groups, finite Frobenius rings, additive and linear codes and their dual codes, symmetry groups, and weights.

Let A be a finite abelian group. A *character* of A is a group homomorphism $\pi : A \to (\mathbb{C}^\times, \cdot)$ from A to the multiplicative group of nonzero complex numbers. Denote by $\widehat{A}$ the set of all characters of A; $\widehat{A}$ is a finite abelian group under pointwise multiplication of functions. The groups A and $\widehat{A}$ are isomorphic, but not naturally so; $|\widehat{A}| = |A|$. The double character group is naturally isomorphic to the group: $(\widehat{A})\widehat{\ } \cong A$; $a \in A$ corresponds to evaluation at a of $\pi \in \widehat{A}$, i.e., $\pi \mapsto \pi(a)$.

For a subgroup $B \subseteq A$, define its *annihilator* by

$$(\widehat{A} : B) = \{\pi \in \widehat{A} : \pi(b) = 1, \text{for all } b \in B\}.$$

Then $(\widehat{A} : B)$ is a subgroup of $\widehat{A}$, $(\widehat{A} : B) \cong (A/B)\widehat{\ }$, and $|(\widehat{A} : B)| = |A/B| = |A|/|B|$. Identifying $A \cong (\widehat{A})\widehat{\ }$, we have $(A : (\widehat{A} : B)) = B$.

Throughout this paper R will denote a finite (associative) ring with 1; R may be noncommutative. The group of units (invertible elements) of R is denoted $\mathcal{U} = \mathcal{U}(R)$. The *Jacobson radical* $J(R)$ of R is the intersection of all maximal left ideals of R; $J(R)$ is itself a two-sided ideal of R. The left/right *socle* $\operatorname{soc}({}_RR)$, $\operatorname{soc}(R_R)$ of R is the left/right ideal generated by the minimal left/right ideals of R. A ring R (perhaps infinite) is *Frobenius* if ${}_RJ(R) \cong \operatorname{soc}({}_RR)$ and $J(R)_R \cong \operatorname{soc}(R_R)$ [15, Theorem (16.14)]; a theorem of Honold [13] says that one of these isomorphisms suffices for finite rings.

Every finite ring R has an underlying additive abelian group. Its character group $\widehat{R}$ is a bimodule over R. The two scalar multiplications are written in exponential form, with $\pi \in \widehat{R}$, $r, s \in R$:

$$({}^r\pi)(s) = \pi(sr), \quad \pi^r(s) = \pi(rs).$$

A finite ring R is Frobenius if and only if $R \cong \widehat{R}$ as left (resp., right) R-modules, [25, Theorem 3.10]. This implies that a finite Frobenius ring admits a character χ, called a *generating character*, such that $r \mapsto {}^r\chi$ is an isomorphism of left R-modules (resp., $r \mapsto \chi^r$ is an isomorphism of right R-modules). A generating character has the property that any one-sided ideal of R that is contained in $\ker \chi$ must be the zero ideal.

An *additive code* of length n over R is an additive subgroup $C \subseteq R^n$. If $C \subseteq R^n$ is a left, resp., right, R-submodule, then C is a left (resp., right) *R-linear code*. One way to present a left R-linear code is as the image $C = \operatorname{im} \Lambda$ of a homomorphism $\Lambda : M \to R^n$ of left R-modules, and similarly for right linear codes.

We will write homomorphisms of left R-modules with inputs on the left, so that preservation of scalar multiplication is $(rx)\phi = r(x\phi)$, where $r \in R$, $x \in M$, M a left R-module, and ϕ a homomorphism of left R-modules with domain M

Define the *standard dot product* on R^n by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \in R,$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in R^n$. Given an additive code $C \subseteq R^n$, define dual codes by

$$\begin{aligned} \mathcal{L}(C) &= \{y \in R^n : y \cdot x = 0, \text{for all } x \in C\}, \\ \mathcal{R}(C) &= \{y \in R^n : x \cdot y = 0, \text{for all } x \in C\}. \end{aligned} \tag{2.1}$$

When R is Frobenius, with generating character χ, also define

$$\begin{aligned} \mathfrak{L}(C) &= \{y \in R^n : \chi(y \cdot x) = 0, \text{for all } x \in C\}, \\ \mathfrak{R}(C) &= \{y \in R^n : \chi(x \cdot y) = 0, \text{for all } x \in C\}. \end{aligned} \tag{2.2}$$

Using the isomorphisms $r \mapsto {}^r\chi$ and $r \mapsto \chi^r$ of R to $\widehat{R}$, there are isomorphisms $R^n \to \widehat{R}^n$ of left, resp., right, R-modules given by $x \mapsto {}^x\chi$ and $x \mapsto \chi^x$, where ${}^x\chi(y) = \chi(y \cdot x)$ and $\chi^x(y) = \chi(x \cdot y)$, for $x, y \in R^n$. Under the isomorphism $x \mapsto {}^x\chi$, $\mathfrak{R}(C)$ is taken to $(\widehat{R}^n : C)$, while under the isomorphism $x \mapsto \chi^x$, $\mathfrak{L}(C)$ is taken to $(\widehat{R}^n : C)$.

Lemma 2.3. *Suppose R is Frobenius and $C \subseteq R^n$ is an additive code. Then*

- $|C| \cdot |\mathfrak{L}(C)| = |C| \cdot |\mathfrak{R}(C)| = |R^n|$;
- $\mathfrak{L}(\mathfrak{R}(C)) = C = \mathfrak{R}(\mathfrak{L}(C))$.

Remark 2.4. Note that $\mathcal{R}(C) \subseteq \mathfrak{R}(C)$ and $\mathcal{L}(C) \subseteq \mathfrak{L}(C)$. In general these containments will be proper. However, if C is a left R-linear code, then $\mathcal{R}(C) = \mathfrak{R}(C)$. Similarly, $\mathcal{L}(C) = \mathfrak{L}(C)$ if C is right R-linear.

A *weight* on R is a function $w : R \to \mathbb{C}$ from R to the complex numbers $\mathbb{C}$ with $w(0) = 0$. In most of this paper we will study weights having positive integer values, except for $w(0) = 0$. A weight w will be extended additively to R^n, so that $w(v) = \sum_{i=1}^{n} w(v_i) \in \mathbb{C}$, where $v = (v_1, v_2, \ldots, v_n) \in R^n$.

Every weight w on R has two *symmetry groups*, left and right:

$$
\begin{aligned}
G_{\mathrm{lt}}(w) &= \{u \in \mathcal{U} : w(ur) = w(r) \text{ for all } r \in R\}, \\
G_{\mathrm{rt}}(w) &= \{u \in \mathcal{U} : w(ru) = w(r) \text{ for all } r \in R\}.
\end{aligned} \tag{2.5}
$$

A weight w has *maximal symmetry* when $G_{\mathrm{lt}}(w) = G_{\mathrm{rt}}(w) = \mathcal{U}$.

Example 2.6. The most well-known weight on R is the *Hamming weight* H, defined by $\mathrm{H}(0) = 0$ and $\mathrm{H}(r) = 1$ for $r \neq 0$. The Hamming weight has maximal symmetry.

Example 2.7. Another well-known weight on R having maximal symmetry is the *homogeneous weight* $\mathrm{W} : R \to \mathbb{R}$. The homogeneous weight was first introduced in [5] over integer residue rings and generalized to all finite rings and modules in [14] and [10]. A homogeneous weight is characterized by the choice of a real number $\zeta > 0$ and the following properties [10]:

- $\mathrm{W}(0) = 0$;
- $G_{\mathrm{lt}}(\mathrm{W}) = \mathcal{U}$; and
- $\sum_{x \in Rr} \mathrm{W}(x) = \zeta |Rr|$ for nonzero principal left ideals $Rr \subseteq R$.

The last property says that all nonzero left principal ideals of R have the same average weight ζ. In fact, the average weight property holds for all nonzero left ideals of R if and only if R is Frobenius [10, Corollary 1.6].

When $R = \mathbb{F}_q$, all the nonzero elements are units, so $\mathrm{W}(u) = \mathrm{W}(1)$ for all units u. Thus W is a constant multiple (namely, $\mathrm{W}(1)$) times the Hamming weight. Note that $\zeta = (q-1)\mathrm{W}(1)/q$ over $\mathbb{F}_q$.

Greferath and Schmidt [10, Theorem 1.3] prove that homogeneous weights exist on any R by giving an explicit formula for W in terms of ζ and the Möbius function μ (see [22, §5.5]) of the poset of principal left ideals of R; namely:

$$
\mathrm{W}(r) = \zeta \left(1 - \frac{\mu(0, Rr)}{|\mathcal{U} r|} \right), \quad r \in R. \tag{2.8}
$$

This formula implies that all the values of W are rational multiples of ζ. By choosing ζ appropriately, one can produce a homogeneous weight on R with integer values. Another consequence of the formula is that any two homogeneous weights on R are scalar multiples of each other: if W and W' are homogeneous weights on R with average weights ζ and ζ', respectively, then $\mathrm{W}' = (\zeta'/\zeta)\mathrm{W}$.

Example 2.9. Let $R = M_{k \times k}(\mathbb{F}_q)$ be the ring of $k \times k$ matrices over the finite field $\mathbb{F}_q$. The *rank weight* R is defined by $\mathrm{R}(r) = \mathrm{rk}(r)$, the usual rank of the matrix $r \in R$. The rank weight has maximal symmetry.

3. MacWilliams identities

In her 1962 doctoral dissertation, Florence Jessie MacWilliams gave a formula relating the Hamming weight enumerator of a linear code over a finite field to the Hamming weight enumerator of its dual code [16, 17]. In this section we will describe this work of MacWilliams as well as some of its generalizations.

For any linear code $C \subseteq R^n$ over a finite ring R, the *Hamming weight enumerator* is the following homogeneous polynomial of degree n:

$$\mathrm{hwe}_C(X,Y) = \sum_{x \in C} X^{n-\mathrm{H}(x)} Y^{\mathrm{H}(x)}, \tag{3.1}$$

where H is the Hamming weight, as in Example 2.6. The formula relating hwe_C and $\mathrm{hwe}_{C^\perp}$ is quoted next.

Theorem 3.2 (MacWilliams identities [16, 17])**.** *If $C \subseteq \mathbb{F}_q^n$ is a linear code over the finite field $\mathbb{F}_q$, then*

$$\mathrm{hwe}_{C^\perp}(X,Y) = \frac{1}{|C|}\,\mathrm{hwe}_C(X+(q-1)Y, X-Y).$$

Remark 3.3. Note in particular that the formula for $\mathrm{hwe}_{C^\perp}$ depends only on hwe_C and not on a more detailed knowledge of the code C. By applying the MacWilliams identities to $C^\perp$ and $C = (C^\perp)^\perp$, the roles of C and $C^\perp$ can be reversed.

We isolate one consequence of the MacWilliams identities.

Corollary 3.4. *If C and D are two linear codes over $\mathbb{F}_q$ with $\mathrm{hwe}_C = \mathrm{hwe}_D$, then $\mathrm{hwe}_{C^\perp} = \mathrm{hwe}_{D^\perp}$.*

The MacWilliams identities for the Hamming weight enumerator can be generalized in several ways. One way is to generalize the algebraic structure of the codes. There are versions of the MacWilliams identities with the Hamming weight enumerator for additive codes over finite abelian groups [6], as well as for left (or right) linear codes over a finite Frobenius ring [25, Theorem 8.3]. In the latter, one replaces q with $|R|$ and $C^\perp$ with $\mathcal{R}(C)$ (with $\mathcal{L}(C)$ if C is right linear).

Another way to generalize the MacWilliams identities is to generalize the enumerator. There are two broad ways of doing this, stemming from two interpretations of the exponents in (3.1). Following Gluesing-Luerssen [8], one of the generalizations will be called *partition enumerators*; the other will be called *w-weight enumerators*. These enumerators will be defined below, and the Hamming weight enumerator will be an example of both. While most of the following material can be formulated for additive codes over finite abelian groups, the discussion here will be restricted to linear codes over finite rings.

Suppose S is a finite set. A *partition* of S is a collection $\mathcal{P} = \{P_i\}$ of nonempty subsets of S such that the subsets are pairwise disjoint and cover S, i.e., $S = \uplus_i P_i$. The subsets P_i are called the *blocks* of the partition.

Suppose a finite ring R has a partition $\mathcal{P} = \{P_i\}_{i=1}^m$. Define counting functions $n_i : R^n \to \mathbb{N}$, $i = 1, 2, \ldots, m$, by $n_i(x) = |\{j : x_j \in P_i\}|$, for $x = (x_1, x_2, \ldots, x_n) \in R^n$. The counting functions count how many entries of x belong to each block of the partition. For a linear code $C \subseteq R^n$ define the *partition enumerator* associated to C and the partition $\mathcal{P}$ to be the following homogeneous polynomial of degree n in the variables $Z_1, Z_2, \ldots, Z_m$:

$$\mathrm{pe}_C^{\mathcal{P}}(Z_1, \ldots, Z_m) = \sum_{x \in C} \prod_{i=1}^{m} Z_i^{n_i(x)}. \tag{3.5}$$

Examples of such partition enumerators include:

- the *complete enumerator* (ce) based on the singleton partition $\mathcal{P} = \{\{r\}\}_{r \in R}$;
- a *symmetrized enumerator* (se) based on a partition consisting of the orbits of a group action on R;
- the Hamming (weight) enumerator based on the partition with blocks $\{0\}$ and the set difference $R - \{0\}$.

While the literature refers to the examples above as weight enumerators, the first two do not involve weights, so I will use the shorter names indicated.

Suppose R has two partitions $\mathcal{P} = \{P_i\}_{i=1}^m$ and $\mathcal{Q} = \{Q_j\}_{j=1}^{m'}$. Also suppose $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$, i.e., each block P_i is contained in some (unique) block Q_j; write $j = f(i)$. Write the partition enumerators of a linear code $C \subseteq R^n$, using variables Z_i, $i = 1, 2, \ldots, m$, for $\mathcal{P}$, and $\mathcal{Z}_j$, $j = 1, 2, \ldots, m'$, for $\mathcal{Q}$:

$$\mathrm{pe}_C^{\mathcal{P}}(Z_1, \ldots, Z_m) \quad \text{and} \quad \mathrm{pe}_C^{\mathcal{Q}}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{m'}).$$

The specialization of variables $Z_i \rightsquigarrow \mathcal{Z}_{f(i)}$ allows us to write the $\mathcal{Q}$-enumerator in terms of the $\mathcal{P}$-enumerator:

$$\mathrm{pe}_C^{\mathcal{Q}}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{m'}) = \mathrm{pe}_C^{\mathcal{P}}(Z_1, \ldots, Z_m)\big|_{Z_i \rightsquigarrow \mathcal{Z}_{f(i)}}. \tag{3.6}$$

The MacWilliams identities are known to generalize to the complete enumerator and certain symmetrized enumerators, over finite fields [18] and finite Frobenius rings [25]. The MacWilliams identities generalize to so-called reflexive partition enumerators over finite Frobenius rings; see [8] for details. For the symmetrized enumerator, see Theorem 11.6. A short review of the main arguments used for proving the MacWilliams identities over finite Frobenius rings is in Appendix A.

For the other type of enumerator, suppose R is a finite ring with 1, and w is a weight on R with positive integer values (except $w(0) = 0$). Denote the largest value of w by $w_{\max}$. For any left R-linear code $C \subseteq R^n$, define the *w-weight enumerator* of C by

$$\mathrm{wwe}_C(X, Y) = \sum_{x \in C} X^{n w_{\max} - w(x)} Y^{w(x)}. \tag{3.7}$$

The w-weight enumerator is a homogeneous polynomial of degree $nw_{\max}$ in X and Y. Different codewords in C may have the same weight. Collecting terms in (3.7) leads to

$$\mathrm{wwe}_C(X, Y) = \sum_{j=0}^{n w_{\max}} A_j^w(C) X^{n w_{\max} - j} Y^j, \tag{3.8}$$

where $A_j^w(C)$ is the number of codewords of C having weight j:

$$A_j^w(C) = |\{x \in C : w(x) = j\}|. \tag{3.9}$$

We write $A_j(C)$ when w is clear from context. To save space in examples in later sections we will often write wwe with $X = 1$ and $Y = t$, so that $\mathrm{wwe}_C = \sum_j A_j(C) t^j$. When $w = \mathrm{H}$, the Hamming weight, we recover the Hamming weight enumerator hwe.

Remark 3.10. A disadvantage of using the notation $\mathrm{wwe}_C = \sum_j A_j(C) t^j$ is that information about the length n of the code is lost. Of course, if the length of C is known, then the homogeneous form (3.7) of wwe is easily recovered. For example, suppose C is a linear code of length n, and let D be the linear code of length $n + 1$ obtained by appending a zero to each codeword of C. Since $w(0) = 0$, there are no changes in the weights of the codewords, so that $A_j(C) = A_j(D)$ for all j. However, $\mathrm{wwe}_D(X, Y) = X\, \mathrm{wwe}_C(X, Y)$.

The partition enumerators and the w-weight enumerators are related. Given a weight w on R, let $\mathcal{Q}$ be the partition of R into the orbits $\mathrm{orb}(r)$ of $G_{\mathrm{lt}}(w)$ acting on R on the left, and let $\mathcal{P}$ be the complete partition of R; $\mathcal{P}$ is a refinement of every partition, hence a refinement of $\mathcal{Q}$. Use variables Z_r, $r \in R$, for $\mathcal{P}$, and $\mathcal{Z}_{\mathrm{orb}(r)}$ for $\mathcal{Q}$. Then the specialization of variables $\mathcal{Z}_{\mathrm{orb}(r)} \rightsquigarrow X^{w_{\max} - w(r)} Y^{w(r)}$ allows us to write wwe_C in terms of $\mathrm{se}_C^{\mathcal{Q}}$ for any R-linear code $C \subseteq R^n$:

$$\mathrm{wwe}_C(X, Y) = \mathrm{se}_C^{\mathcal{Q}}(\mathcal{Z}_{\mathrm{orb}(r)})\big|_{\mathcal{Z}_{\mathrm{orb}(r)} \rightsquigarrow X^{w_{\max} - w(r)} Y^{w(r)}}. \tag{3.11}$$

This specialization is well-defined by the definition of the symmetry group G_{lt}: the value of w is constant on every left orbit of G_{lt}. Section 11 gives details of this situation over finite chain rings.

One way to view the MacWilliams identities is in terms of the diagram in Figure 1 below. In the diagram, the map $\mathcal{R}$ sends a left

$$\begin{tikzcd}
\{(n,M)\text{-linear codes}\} \arrow[r, "\mathcal{R}"] \arrow[d, "\mathrm{ce}"] \arrow[dd, bend right, "\mathrm{se}"'] \arrow[ddd, bend right=60, "\mathrm{wwe}"'] & \{(n,|R|^n/M)\text{-linear codes}\} \arrow[d, "\mathrm{ce}"'] \arrow[dd, bend left, "\mathrm{se}"] \arrow[ddd, bend left=60, "\mathrm{wwe}"] \\
\mathbb{C}[Z_r]_n \arrow[r, "MW"] \arrow[d, squiggly, "\mathrm{spec}"] & \mathbb{C}[Z_r]_n \arrow[d, squiggly, "\mathrm{spec}"'] \\
\mathbb{C}[\mathcal{Z}_{\mathrm{orb}(r)}]_n \arrow[r, "MW"] \arrow[d, squiggly, "\mathrm{spec}"] & \mathbb{C}[\mathcal{Z}_{\mathrm{orb}(r)}]_n \arrow[d, squiggly, "\mathrm{spec}"'] \\
\mathbb{C}[X,Y]_{nw_{\max}} \arrow[r, dotted, "?"] & \mathbb{C}[X,Y]_{nw_{\max}}
\end{tikzcd}$$

FIGURE 1. Relations among enumerators

R-linear code of size M to its right R-linear dual code $\mathcal{R}(C)$. Under favorable circumstances (e.g., R Frobenius), the dual code has size $|\mathcal{R}(C)| = |R|^n/|C|$. The vertical maps $\mathrm{ce}, \mathrm{se}, \mathrm{wwe}$ associate to a linear code its complete enumerator, $\mathcal{Q}$-symmetrized enumerator, and w-weight enumerator, respectively. The other vertical maps (both called spec) are the specializations of variables described in (3.6) and (3.11).

Because the MacWilliams identities hold for ce and se, the horizontal maps MW are the MacWilliams transforms that provide the linear changes of variables. The solid arrows in the diagram commute.

The big question under study in this paper is whether there is a horizontal map '?' that makes the diagram commute for the w-weight enumerator. If such a map exists, then the following property holds: if $\mathrm{wwe}_C = \mathrm{wwe}_D$, then $\mathrm{wwe}_{\mathcal{R}(C)} = \mathrm{wwe}_{\mathcal{R}(D)}$. We refer to this property by saying *the weight w respects duality.* To formalize:

Definition 3.12. A weight w on a finite ring R *respects duality* if $\mathrm{wwe}_C = \mathrm{wwe}_D$ implies $\mathrm{wwe}_{\mathcal{R}(C)} = \mathrm{wwe}_{\mathcal{R}(D)}$ for all left R-linear codes $C, D \subseteq R^n$, $n \geq 1$.

Corollary 3.4 says that the Hamming weight on $\mathbb{F}_q$ respects duality. If a weight w does not respect duality, then the MacWilliams identities cannot hold for wwe.

As an example, let us see what happens when a weight is multiplied by a positive constant. Suppose $w : R \to \mathbb{Z}$ is a weight on R, and let $\tilde{w} = cw$, with c a positive integer. Denote the w-weight enumerators for w and $\tilde{w}$ by wwe and wwe^c respectively.

Lemma 3.13. *Let $w : R \to \mathbb{Z}$ be a weight on R, and let $\tilde{w} = cw$, with c a positive integer. Then*

$$\mathrm{wwe}^c_C(X, Y) = \mathrm{wwe}_C(X^c, Y^c).$$

Proof. For any element $r \in R$, we have

$$X^{\tilde{w}_{\max} - \tilde{w}(r)} Y^{\tilde{w}(r)} = (X^c)^{w_{\max} - w(r)} (Y^c)^{w(r)}. \qquad \square$$

Proposition 3.14. *Let $w : R \to \mathbb{Z}$ be a weight on R, and let $\tilde{w} = cw$, with c a positive integer. Then $\tilde{w}$ respects duality if and only if w respects duality.*

Even if a weight w does not respect duality (and hence wwe does not satisfy the MacWilliams identities), it is still possible to determine both wwe_C and $\mathrm{wwe}_{\mathcal{R}(C)}$ by calculating se_C, using the MacWilliams identities for se to find $\mathrm{se}_{\mathcal{R}(C)}$, and then specializing variables to get wwe_C and $\mathrm{wwe}_{\mathcal{R}(C)}$. But one cannot go directly from wwe_C to $\mathrm{wwe}_{\mathcal{R}(C)}$.

The main objective of this paper is to show that it is rare for a weight having maximal symmetry to respect duality, at least over finite chain rings or matrix rings over finite fields. In addition to Theorem 3.2 for the Hamming weight enumerator and its generalization to finite Frobenius rings, the MacWilliams identities for w-weight enumerators are known to hold for the Lee weight on $\mathbb{Z}/4\mathbb{Z}$ [12] (and see Theorem 6.5) and the homogeneous weight on the matrix ring $M_{2\times 2}(\mathbb{F}_2)$, Theorem 12.12. The MacWilliams identities for w-weight enumerators are known to fail for the Rosenbloom-Tsfasman weight on matrices [7], the Lee weight on $\mathbb{Z}/m\mathbb{Z}$, $m \geq 5$ [1], and the homogeneous weight on $\mathbb{Z}/m\mathbb{Z}$ for composite $m \geq 6$ [30]. In all cases, the failure is proved by showing that the weight does not respect duality: there exist linear codes C and D with $\mathrm{wwe}_C = \mathrm{wwe}_D$, yet $\mathrm{wwe}_{\mathcal{R}(C)} \neq \mathrm{wwe}_{\mathcal{R}(D)}$, by virtue of $A^w_j(\mathcal{R}(C)) \neq A^w_j(\mathcal{R}(D))$ for some j.

The hypothesis that w has maximal symmetry is important. There are results about 'Lee weights' of different types that can be valid because they secretly tap into the Hamming weight or the Lee weight on $\mathbb{Z}/4\mathbb{Z}$; cf., [31].

4. Linear codes via multiplicity functions

In later sections linear codes will be presented as images of homomorphisms of left R-modules. In turn, the homomorphisms will be described in terms of multiplicity functions. In this section, the use of multiplicity functions to describe linear codes will be summarized briefly. Addititonal information can be found in [26, §3] or [28, §7].

As throughout this paper, let R be a finite ring with 1. Suppose M is a finite unital left R-module; *unital* means $1x = x$ for all $x \in M$. We call any homomorphism $\lambda : M \to R$ of left R-modules a *linear functional* on M. Define $M^\sharp = \operatorname{Hom}_R(M, R)$ to be the set of all left linear functionals on M. We will write inputs to linear functionals on the left, so that $(rx)\lambda = r(x\lambda)$ for $r \in R$, $x \in M$, and $\lambda \in M^\sharp$; $M^\sharp$ is a right R-module, with λr given by $x(\lambda r) = (x\lambda)r$ for $r \in R$, $x \in M$, and $\lambda \in M^\sharp$. The left R-module M admits a left action by the group of units $\mathcal{U}$ of R using left scalar multiplication. Denote the orbit of $x \in M$ by $\operatorname{orb}(x)$ or by $[x]$. Similarly, the right R-module $M^\sharp$ admits a right $\mathcal{U}$-action, with orbits denoted $\operatorname{orb}(\lambda)$ or $[\lambda]$.

A left R-linear code of length n *parametrized* by M is the image $C = \operatorname{im} \Lambda$ of a homomorphism $\Lambda : M \to R^n$ of left R-modules. The module M is the *information module* of the linear code C. Denote the components of Λ by $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, with each $\lambda_i : M \to R$ being linear functional on M. We call the λ_i the *coordinate functionals* of the linear code C.

Suppose w is a weight on R. Then the weight w and a parametrized code C given by $\Lambda : M \to R^n$ define a *weight function* $W_\Lambda : M \to \mathbb{C}$ by $W_\Lambda(x) = w(x\Lambda) = \sum_{i=1}^n w(x\lambda_i)$, $x \in M$.

Lemma 4.1. *Suppose w is a weight on R with symmetry groups (2.5), and suppose C is an R-linear code parametrized by $\Lambda : M \to R^n$. Then*

(1) *the weight function $W_\Lambda : M \to \mathbb{C}$ is constant on each left $G_{\mathrm{lt}}(w)$-orbit* $\operatorname{orb}(x) \subseteq M$*;*
(2) $W_{\Lambda'} = W_\Lambda$*, if $\Lambda' = (\lambda_{\tau(1)}u_1, \ldots, \lambda_{\tau(n)}u_n)$, where τ is a permutation of $\{1, 2, \ldots, n\}$ and $u_1, \ldots, u_n \in G_{\mathrm{rt}}(w)$.*

Proof. These follow directly from (2.5). □

Remark 4.2. If the weight w has nonnegative integer values, the weight function W_Λ determines the w-weight enumerator of the linear code C:

$$\mathrm{wwe}_C = \frac{1}{|\ker \Lambda|} \sum_{x \in M} t^{W_\Lambda(x)}.$$

When Λ is injective, the w-weight enumerator can be written in terms of the sizes of orbits:

$$\mathrm{wwe}_C = \sum_{[x] \subseteq M} |[x]| t^{W_\Lambda(x)}. \tag{4.3}$$

Lemma 4.1 shows that the weight function W_Λ depends only on the numbers of coordinate functionals belonging to different $G_{\mathrm{rt}}(w)$-orbits in $M^\sharp$. We formalize this observation next. Write $F(X,Y)$ for the set of all functions from X to Y.

Given an information module M, define the orbit spaces:

$$\mathcal{O} = G_{\mathrm{lt}}(w)\backslash M, \quad \mathcal{O}^\sharp = M^\sharp/G_{\mathrm{rt}}(w).$$

Lemma 4.1 shows that W_Λ depends only on a function $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$, where $\eta([\lambda])$ is the number of coordinate functionals that belong to the orbit $\mathrm{orb}(\lambda) = [\lambda]$. (In [21, §3.6], Peterson and Weldon call η the *modular representation* of the linear code C.)

The weight w induces an additive map $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(\mathcal{O}, \mathbb{C})$. For $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$, define $\omega = W(\eta)$, the *list of orbit weights*, by

$$\omega([x]) = \sum_{[\lambda]\in\mathcal{O}^\sharp} w(x\lambda)\eta([\lambda]). \tag{4.4}$$

By (2.5), the value of $w(x\lambda)$ is well-defined. If w has values in $\mathbb{Z}$ or $\mathbb{Q}$, then W has values in $F(\mathcal{O}, \mathbb{Z})$ or $F(\mathcal{O}, \mathbb{Q})$, accordingly. In these latter cases, tensoring with $\mathbb{Q}$ yields a linear transformation $W : F(\mathcal{O}^\sharp, \mathbb{Q}) \to F(\mathcal{O}, \mathbb{Q})$.

By ordering the elements of $\mathcal{O}$ and $\mathcal{O}^\sharp$, one can define a matrix W whose rows are indexed by $\mathcal{O}$, whose columns are indexed by $\mathcal{O}^\sharp$, and whose entry at position $([x],[\lambda])$ is the well-defined value $w(x\lambda)$:

$$W_{[x],[\lambda]} = w(x\lambda), \quad [x] \in \mathcal{O}, [\lambda] \in \mathcal{O}^\sharp. \tag{4.5}$$

Treating η and ω as column vectors, (4.4) is just matrix multiplication: $\omega = W\eta$.

Any element $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$, called a *multiplicity function*, determines an R-linear code as the image of the homomorphism $\Lambda_\eta : M \to R^N$ of left R-modules given by sending $x \in M$ to the N-tuple $(\ldots, x\lambda, \ldots)$, where a representative of each orbit $[\lambda] \in \mathcal{O}^\sharp$ is repeated $\eta([\lambda])$ times; $N = \sum_{[\lambda]\in\mathcal{O}^\sharp} \eta([\lambda])$. Said another way, treat elements $\lambda \in M^\sharp$ as columns of a generator matrix, with representatives of $[\lambda]$ repeated $\eta([\lambda])$ times. The resulting linear code C_η is well-defined up to monomial equivalence; its list $\omega = W\eta$ of orbit weights is well-defined. Using ω, one can write down the w-weight enumerator of C_η, as in Remark 4.2:

$$\mathrm{wwe}_{C_\eta} = \sum_{x\in M} t^{w(x\Lambda_\eta)} = \sum_{[x]\in\mathcal{O}} |[x]| t^{\omega([x])},$$

assuming that Λ_η is injective. (If Λ_η is not injective, we must divide by $|\ker \Lambda_\eta|$.)

Remark 4.6. The definition of $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(\mathcal{O}, \mathbb{C})$ is valid whether or not $w(0) = 0$. When $w(0) = 0$, then $W(\eta)([0]) = \omega([0]) = 0$ for any $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$. Also, when $w(0) = 0$, $W([\delta_{[0]}]) = 0$, where $\delta_{[0]}$ is the indicator function of the orbit $[0] \in \mathcal{O}^\sharp$ of the zero-functional.

More formally, define

$$F_0(\mathcal{O}^\sharp, \mathbb{N}) = \{\eta \in F(\mathcal{O}^\sharp, \mathbb{N}) : \eta([0]) = 0\},$$
$$F_0(\mathcal{O}, \mathbb{C}) = \{\omega \in F(\mathcal{O}, \mathbb{C}) : \omega([0]) = 0\}.$$

When $w(0) = 0$, the image of W is contained in $F_0(\mathcal{O}, \mathbb{C})$. We denote the restriction of W to $F_0(\mathcal{O}^\sharp, \mathbb{N})$ by $W_0 : F_0(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{C})$; W_0 is an additive map. The map W_0 is represented by a matrix whose rows are indexed by the nonzero orbits $[x] \in \mathcal{O}$, whose columns are indexed by the nonzero orbits $[\lambda] \in \mathcal{O}^\sharp$, and with entries given as in (4.5). When $w(0) = 0$, the map W can never be injective (because $\delta_{[0]} \in \ker W$), but the map W_0 often is injective. When w has values in $\mathbb{Q}$ and $w(0) = 0$, the linear transformation $W_0 : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is often invertible. This will be an important tool in later sections.

We conclude this section with a short discussion of the effective length of codes. The next lemma is a variant of [26, (6.1)].

Lemma 4.7. *Suppose $C \subseteq R^n$ is an R-linear code. Then*

$$\sum_{c \in C} w(c) = \sum_{i=1}^{n} |\ker \lambda_i| \sum_{b \in \operatorname{im} \lambda_i} w(b). \tag{4.8}$$

Proof. Write $\lambda_1, \ldots, \lambda_n$ for the coordinate functionals of C. Then, $\sum_{c \in C} w(c) = \sum_{c \in C} \sum_{i=1}^{n} w(c_i) = \sum_{c \in C} \sum_{i=1}^{n} w(c\lambda_i)$. Now interchange the finite sums, and use that λ_i is a homomorphism, so $\sum_{c \in C} w(c\lambda_i) = |\ker \lambda_i| \sum_{b \in \operatorname{im} \lambda_i} w(b)$. □

There are situations where (4.8) simplifies. Define a weight w on R to be *egalitarian* if there exists a constant γ such that, for any nonzero left ideal $B \subseteq R$, $\sum_{b \in B} w(b) = \gamma |B|$. This definition is due to [14]. The homogeneous weight on a finite Frobenius ring is an example of an egalitarian weight; see Example 2.7.

Define the *effective length* of a linear code C to be $\operatorname{efflng}(C) = |\{i : \lambda_i \neq 0\}|$. If C is given by a generator matrix, the effective length counts the number of nonzero columns of the generator matrix.

Proposition 4.9. *Suppose $C \subseteq R^n$ is an R-linear code and w is an egalitarian weight on R. Then*

$$\sum_{c \in C} w(c) = \gamma |C| \operatorname{efflng}(C).$$

Proof. Use Lemma 4.7. For $\lambda_i \neq 0$, set $B = \operatorname{im} \lambda_i$ and note that $|C| = |\ker \lambda_i| \cdot |\operatorname{im} \lambda_i|$. ☐

Corollary 4.10. *Let w be an egalitarian weight on R. Suppose $C, D \subseteq R^n$ are two R-linear codes. If $\mathrm{wwe}_C = \mathrm{wwe}_D$, then the effective lengths of C and D are equal.*

Proof. The hypothesis means that $A_j^w(C) = A_j^w(D)$ for all j. But

$$|C| = \sum_{j=0}^{nw_{\max}} A_j^w(C) \quad \text{and} \quad \sum_{c \in C} w(c) = \sum_{j=0}^{nw_{\max}} j A_j^w(C),$$

so $|C| = |D|$ and $\sum_{c \in C} w(c) = \sum_{d \in D} w(d)$. Apply Proposition 4.9. ☐

5. Singletons in dual codes

In this section we describe some general results on the contributions to $\mathrm{wwe}_{C^\perp}$ coming from singleton vectors.

As usual, suppose R is a finite ring with 1 and $C \subseteq R^n$ is a left R-linear code. Its dual code $C^\perp$ is the right dual code $\mathcal{R}(C)$ of (2.1). We assume w is an integer-valued weight on R with $w(r) > 0$ for $r \neq 0$ and $w(0) = 0$. Let $\mathring{w} = \min\{w(r) : r \neq 0\}$, so that $\mathring{w} > 0$.

We say that a vector $v \in R^n$ is a *singleton* if v has exactly one nonzero entry. Given a vector $v \in R^n$, recall that the weight of the vector is $w(v) = \sum_{j=1}^n w(v_j)$. The smallest possible nonzero weight of a vector is $\mathring{w}$, which is attained by any singleton whose nonzero entry r has $w(r) = \mathring{w}$.

We want to write down the contributions of singletons to the w-weight enumerator of a linear code, especially to a dual code. As in (3.9), recall that $A_j(C) = |\{x \in C : w(x) = j\}|$. To track the contributions of singletons we write

$$A_j^{\mathrm{sing}}(C) = |\{x \in C : x \text{ is a singleton and } w(x) = j\}|.$$

Of course, $A_j^{\mathrm{sing}}(C) \leq A_j(C)$. Equality will be addressed in Corollary 5.4 below.

For any $\lambda \in M^\sharp$ and positive integer j, define $\mathrm{ann}_{\mathrm{rt}}(\lambda, j) = \{r \in R : \lambda r = 0 \text{ and } w(r) = j\}$, the set of elements in R of weight j that annihilate λ.

Lemma 5.1. *Suppose w has maximal symmetry. For any $\lambda \in M^\sharp$ and $u \in \mathcal{U}$, $\mathrm{ann}_{\mathrm{rt}}(\lambda u, j) = u^{-1} \mathrm{ann}_{\mathrm{rt}}(\lambda, j)$. In particular, $|\mathrm{ann}_{\mathrm{rt}}(\lambda u, j)| = |\mathrm{ann}_{\mathrm{rt}}(\lambda, j)|$ for any $u \in \mathcal{U}$, $\lambda \in M^\sharp$.*

Proof. Suppose $w(r) = j$. By maximal symmetry, $w(ur) = w(r) = j$ for all $u \in \mathcal{U}$. Because $(\lambda u)(u^{-1} r) = \lambda r$, we see that $r \in \mathrm{ann}_{\mathrm{rt}}(\lambda, j)$ if and only if $u^{-1} r \in \mathrm{ann}_{\mathrm{rt}}(\lambda u, j)$. ☐

Proposition 5.2. *Assume w is an integer-valued weight on R with maximal symmetry. If C is a linear code determined by a multiplicity function η, then, for any positive integer j,*

$$A_j^{\text{sing}}(C^{\perp}) = \sum_{[\lambda]\in\mathcal{O}^{\sharp}} |\text{ann}_{\text{rt}}(\lambda, j)|\eta([\lambda]).$$

Proof. Suppose C has coordinate functionals $\lambda_1, \ldots, \lambda_n$. Let v be a singleton vector with nonzero entry $r \in R$ appearing in position i. Then $v \in C^{\perp}$ if and only if $\lambda_i r = 0$. Thus

$$A_j^{\text{sing}}(C^{\perp}) = \sum_{i=1}^{n} |\text{ann}_{\text{rt}}(\lambda_i, j)|,$$

which reduces to the stated formula because of Lemma 5.1. □

In later sections, formulas for $|\text{ann}_{\text{rt}}(\lambda, j)|$ will be very specific, depending on the nature of the ring R.

Lemma 5.3. *Suppose $v \in R^n$ has weight $w(v)$ satisfying $\mathring{w} \le w(v) < 2\mathring{w}$. Then v must be a singleton.*

Proof. Suppose v has at least two nonzero entries, say in positions j_1, j_2. Then $w(v) \ge w(v_{j_1}) + w(v_{j_2}) \ge 2\mathring{w}$. □

Corollary 5.4. *If $\mathring{w} \le d < 2\mathring{w}$, then $A_d(C) = A_d^{\text{sing}}(C)$.*

In later sections, Corollary 5.4 will be applied mostly to dual codes, in tandem with Proposition 5.2.

Remark 5.5. In order that $A_j^{\text{sing}}(C)$ be nonzero, it is necessary that $j = w(r)$ for some $r \in R$.

It is possible that $A_j(C) = A_j^{\text{sing}}(C)$ even when $j \ge 2\mathring{w}$. For example: when $j = w(r)$ is not equal to a linear combination of the form $\sum_{s:w(s)<j} c_s w(s)$ with c_s being nonnegative integers.

Part 2. Finite Chain Rings

6. Definitions and a positive result

A finite ring R with 1 is a *chain ring* if its left ideals form a chain under set inclusion. In particular, R has a unique maximal left ideal, denoted $\mathfrak{m}$, so that R is a local ring. Examples of chain rings include finite fields, $\mathbb{Z}/p^m\mathbb{Z}$ with p prime, Galois rings, $\mathbb{F}_q[X]/(X^m)$; cf., [19]. Every finite chain ring is Frobenius [24, Lemma 14].

From [4, Lemma 1] we know that $\mathfrak{m}$ is a principal ideal, say $\mathfrak{m} = R\theta = \theta R$, that $\theta^m = 0$ for some (smallest) $m \ge 1$, and that every left or right ideal of R is a two-sided ideal of the form $R\theta^j = \theta^j R$, $j = 0, 1, \ldots, m$. In particular, $\mathfrak{m}$ is a two-sided ideal, so that $R/\mathfrak{m}$ is a finite field, say $R/\mathfrak{m} \cong \mathbb{F}_q$, of order q, a prime power. Write (θ^j) for $R\theta^j = \theta^j R$. Thus, all the ideals of R are displayed here:

$$R = (\theta^0) \supset (\theta) \supset (\theta^2) \supset \cdots \supset (\theta^{m-1}) \supset (\theta^m) = (0). \tag{6.1}$$

Each quotient $(\theta^j)/(\theta^{j+1})$ is a one-dimensional vector space over $R/\mathfrak{m}$, with basis element $\theta^j + (\theta^{j+1})$. It follows that

$$|(\theta^j)| = q^{m-j}, \quad j = 0, 1, \ldots, m. \tag{6.2}$$

In particular, $|R| = q^m$. The group of units of R, denoted $\mathcal{U} = \mathcal{U}(R)$, equals the set difference $R - \mathfrak{m}$. The group $\mathcal{U}$ acts of R on the left and on the right by multiplication. The orbits of the actions are exactly the set differences $\operatorname{orb}(\theta^j) = (\theta^j) - (\theta^{j+1})$, which have size

$$|\operatorname{orb}(\theta^j)| = q^{m-j-1}(q-1), \quad \text{for } j < m. \tag{6.3}$$

In particular, the left orbits of $\mathcal{U}$ equal the right orbits: $\mathcal{U}\theta^j = \theta^j\mathcal{U}$.

From (6.1), we see that every element $r \in R$ has the form $r = u\theta^j$ where u is a unit of R and j is uniquely determined by r (the largest i such that $r \in (\theta^i)$). Note that the annihilator of (θ^j) is (θ^{m-j}).

Let w be a weight on R with positive integer values for $r \neq 0$ in R. Assume that w has maximal symmetry, so that $w(ur) = w(ru) = w(r)$ for all $r \in R$ and units $u \in \mathcal{U}$. This means that w is constant on the $\mathcal{U}$-orbits $\operatorname{orb}(\theta^j) = (\theta^j) - (\theta^{j+1})$. Define w_j as the common value of w on $\operatorname{orb}(\theta^j)$, so that $w_j = w(u\theta^j) = w(\theta^j u)$ for all units $u \in \mathcal{U}$. Then $w_0, w_1, \ldots, w_{m-1}$ are positive integers, and $w_m = 0$.

Example 6.4. Choosing $\zeta = q-1$, we see from Example 2.7 and (6.1) that the homogeneous weight w on a chain ring R has the following integer values:

$$\mathrm{w}(r) = \begin{cases} 0, & r = 0, \\ q, & r \in (\theta^{m-1}) - (0), \\ q-1, & r \in R - (\theta^{m-1}). \end{cases}$$

Then $\mathrm{w}_0 = \cdots = \mathrm{w}_{m-2} = q-1$, $\mathrm{w}_{m-1} = q$, and $\mathrm{w}_m = 0$.

Do the MacWilliams identities hold for the homogeneous weight enumerator over a finite chain ring R? We will see that the answers depend on q and m.

When $m = 1$, then $\theta = 0$, so that R is a finite field $\mathbb{F}_q$. As we saw in Example 2.7, the homogeneous weight w on $\mathbb{F}_q$ equals a multiple of the Hamming weight. By Theorems 3.2 and Lemma 3.13, the homogeneous weight over finite fields respects duality.

For $m \geq 2$, there is one special case ($m = q = 2$, Theorem 6.5, below) where the MacWilliams identities hold for the homogeneous weight. In the remaining cases, we will see in Theorem 10.2 that the homogeneous weight does not respect duality.

The results just described apply to the chain rings $\mathbb{Z}/p^m\mathbb{Z}$, so that the MacWilliams identities hold for the homogeneous weight over $\mathbb{Z}/p\mathbb{Z}$, p prime, and over $\mathbb{Z}/4\mathbb{Z}$ [12, Equation (9)], but not for other prime powers. More generally, over $\mathbb{Z}/m\mathbb{Z}$, m not a prime power (so that $\mathbb{Z}/m\mathbb{Z}$ is not a chain ring), the homogeneous weight does not respect duality [30, Theorem 6.2].

Theorem 6.5. *The MacWilliams identities hold for the homogeneous weight enumerator over a finite chain ring R with $q = 2$ and $m = 2$. If $C \subseteq R^n$ is a linear code and $C^\perp$ is its dual code, then*

$$\mathrm{howe}_{C^\perp}(X, Y) = \frac{1}{|C|}\,\mathrm{howe}_C(X + Y, X - Y).$$

Proof. Appendix A outlines of a proof of the MacWilliams identities over finite Frobenius rings and describes the Fourier transform. Here, we provide details relevant to the chain rings appearing in this theorem.

We know that $|R| = 4$, with $R = \{0, 1, \theta, 1 + \theta\}$. The values of the homogeneous weight, with $\zeta = 1$, are:

r	0	1	θ	$1+\theta$
$\mathrm{w}(r)$	0	1	2	1

The additive group of R could be a cyclic group of order 4 (in which case $\theta = 2$, in order that (θ) be a maximal ideal) or a Klein 4-group. In either case, there exists a generating character χ of R with the following values. (What is crucial is that $\chi(\theta) = -1$.)

	r	0	1	θ	$1+\theta$
cyclic	$\chi(r)$	1	i	-1	$-i$
Klein	$\chi(r)$	1	1	-1	-1

Define $f : R \to \mathbb{C}[X, Y]$ by $f(r) = X^{2-\mathrm{w}(r)}Y^{\mathrm{w}(r)}$. For either choice of χ, the Fourier transform $\widehat{f}$ of (A.2) is the same:

r	0	1	θ	$1+\theta$
$f(r)$	X^2	XY	Y^2	XY
$\widehat{f}(r)$	$(X+Y)^2$	X^2-Y^2	$(X-Y)^2$	X^2-Y^2

Note that the values of $\widehat{f}$ have the form $\widehat{f}(r) = (X + Y)^{2-\mathrm{w}(r)}(X - Y)^{\mathrm{w}(r)}$. I.e., $\widehat{f}(r) = f(r)|_{X\leftarrow X+Y, Y\leftarrow X-Y}$. The rest of the argument in Appendix A now carries through. $\square$

7. Modules over a chain ring

We continue to assume that R is a finite chain ring with maximal ideal $\mathfrak{m} = (\theta)$ such that $R/\mathfrak{m} \cong \mathbb{F}_q$ and $\theta^m = 0$, for some integer $m \geq 2$. By forming quotients from (6.1), we define cyclic R-modules $Z_k = R/(\theta^k)$, of order q^k, $k = 1, 2, \ldots, m$. The module Z_1 is the unique simple R-module. Also define semisimple modules $S_k = Z_1 \oplus \cdots \oplus Z_1$ with k summands, $k = 2, 3, \ldots, m$; $|S_k| = q^k$.

In the next two sections, we will construct two families of linear codes with the same w-weight enumerators. Here is a brief sketch of the construction. We will build examples using Z_k and S_k, $2 \leq k \leq m$, as the underlying information modules. On Z_k, choose a multiplicity function. The weights of elements will be constant along nonzero orbits $\mathcal{O}$, which have sizes $q^k - q^{k-1}, q^{k-1} - q^{k-2}, \ldots, q^2 - q, q - 1$. We then need to build a linear code based on S_k with the same w-weight enumerator. This entails choosing subsets of S_k with sizes matching the sizes $q^k - q^{k-1}, q^{k-1} - q^{k-2}, \ldots, q^2 - q, q - 1$. We do this by fixing a filtration of S_k using linear subspaces. We can then solve for a multiplicity function for S_k. If the original multiplicity function on Z_k calls for one functional from each class, then the multiplicity function on S_k is reasonably nice. We then calculate the common weight enumerator for these multiplicity functions.

In this section we study the orbit structure of Z_k as well as the subsets arising from a filtration of S_k. In Section 8, the multiplicity functions are described and analyzed.

In order to define linear codes over Z_k and S_k, let us examine their linear functionals. Recall first that the left linear functionals of R itself are given by right multiplications by elements of R. That is, $R^\sharp \cong R$ as right R-modules, with $r \in R$ corresponding to the left linear functional $\rho_r \in R^\sharp$ defined by $r'\rho_r = r'r$, $r' \in R$.

If $\lambda \in Z_k^\sharp$, then the composition with the natural quotient map must equal ρ_r for some $r \in R$.

$$\begin{tikzcd} R \arrow[rr, dotted, "\rho_r"] \arrow[d] & & R \\ Z_k = R/(\theta^k) \arrow[urr, "\lambda"'] & & \end{tikzcd}$$

Conversely, $\rho_r : R \to R$ factors through Z_k if and only if $(\theta^k) \subseteq \ker \rho_r$; i.e., if and only if $\theta^k r = 0$. This occurs when $r \in (\theta^{m-k})$. Thus $Z_k^\sharp \cong (\theta^{m-k})$ as right R-modules. In particular, $Z_1^\sharp \cong (\theta^{m-1})$ as right R-modules.

As for $S_k = Z_1 \oplus \cdots \oplus Z_1$, we have $S_k^\sharp \cong (\theta^{m-1}) \oplus \cdots \oplus (\theta^{m-1})$ as right R-modules. For $s = (s_1, \ldots, s_k) \in S_k$ and $\mu = \langle \mu_1\theta^{m-1}, \ldots, \mu_k\theta^{m-1}\rangle \in S_k^\sharp$, $s\mu = \sum_{i=1}^k s_i\mu_i\theta^{m-1} \in R$. Both S_k and $S_k^\sharp$ are k-dimensional vector spaces over $R/\mathfrak{m} \cong \mathbb{F}_q$.

In order to exploit maximal symmetry in Lemma 4.1, we want to understand the orbit structures of the left actions of $\mathcal{U}$ on Z_k and S_k, and, to a lesser extent, the orbit structures of the right actions of $\mathcal{U}$ on $Z_k^\sharp$ and $S_k^\sharp$.

Because (θ^k) is a two-sided ideal of R, Z_k is itself a chain ring. Its left R-submodules are the same as its left ideals:

$$Z_k \supset R\theta \supset R\theta^2 \supset \cdots \supset R\theta^{k-1} \supset \{0\}. \tag{7.1}$$

Note that (7.1) can be viewed as a filtration of Z_k by R-submodules.

Lemma 7.2. *The orbits of the left action of $\mathcal{U}$ on Z_k are:*

$$\mathrm{orb}(1) = \mathrm{orb}(\theta^0), \mathrm{orb}(\theta), \ldots, \mathrm{orb}(\theta^{k-1}), \{0\} = \mathrm{orb}(0) = \mathrm{orb}(\theta^k).$$

The sizes of the orbits are: $|\mathrm{orb}(0)| = 1$ *and*

$$|\mathrm{orb}(\theta^i)| = q^{k-i-1}(q-1), \quad i = 0, 1, \ldots, k-1.$$

Proof. Apply (6.3) to the chain ring Z_k. □

Similar to Lemma 7.2, the orbits of the right action of $\mathcal{U}$ on $Z_k^\sharp$ are $\mathrm{orb}(\theta^{m-k}), \mathrm{orb}(\theta^{m-k+1}), \ldots, \mathrm{orb}(\theta^{m-1}), \{0\} = \mathrm{orb}(\theta^m)$.

Because S_k is a vector space over $\mathbb{F}_q \cong R/\mathfrak{m}$, the action of $\mathcal{U}$ on S_k reduces to the action of the multiplicative group $\mathbb{F}_q^\times$. The $\mathcal{U}$-orbits are $\{0\}$ and $L - \{0\}$, for every 1-dimensional subspace $L \subseteq S_k$. The same structure applies to the dual vector space $S_k^\sharp$; the $\mathcal{U}$-orbits are $\{0\}$ and the nonzero elements of 1-dimensional subspaces.

For later use in Section 8, we will identify subsets (say, $\mathcal{S}_1, \ldots, \mathcal{S}_k$) of S_k that will be the counterparts to the orbits $\mathrm{orb}(1), \ldots, \mathrm{orb}(\theta^{k-1})$ of Z_k (in reverse order). In particular, we want two features:

- each $\mathcal{S}_i$ is a union of $\mathcal{U}$-orbits in S_k;
- $|\mathcal{S}_i| = |\mathrm{orb}(\theta^{k-i})| = q^{i-1}(q-1)$ for $i = 1, \ldots, k$.

To define the $\mathcal{S}_i$, we define a filtration on S_k. Recall that S_k is a k-dimensional vector space over $R/\mathfrak{m} \cong \mathbb{F}_q$. Write elements of $S_k = Z_1 \oplus \cdots \oplus Z_1$ as row vectors of length k over $\mathbb{F}_q$. (Row vectors will be written as $(x_1, \ldots, x_k)$, while column vectors will be written as $\langle x_1, \ldots, x_k\rangle$.) Define vector subspaces of S_k: $V_0 = \{0\}$ and $V_i = \{(0, \ldots, 0, s_{k-i+1}, \ldots, s_k) \in S_k : s_j \in \mathbb{F}_q\}$, for $i = 1, 2, \ldots, k$. In V_i, the first $k - i$ entries are zero; the last i entries vary over $\mathbb{F}_q$. Then,

$\dim_{\mathbb{F}_q} V_i = i$ for $i = 0, 1, \ldots, k$, so that $|V_i| = q^i$, and

$$S_k = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}. \tag{7.3}$$

Set $\mathcal{S}_i = V_i - V_{i-1}$, for $i = 1, \ldots, k$; then $|\mathcal{S}_i| = q^{i-1}(q-1)$. Set $\mathcal{S}_0 = \{0\}$. Because the V_i are vector subspaces, the $\mathcal{S}_i$ are unions of $\mathcal{U}$-orbits.

We also want to understand the linear functionals on S_k in terms of the filtration (7.3). To that end, we examine the dual filtration of $S_k^\sharp$ defined by annihilators $\mathcal{V}_i = \operatorname{ann}(V_i) = \{\mu \in S_k^\sharp : V_i \mu = 0\}$. Then, $\dim_{\mathbb{F}_q} \mathcal{V}_i = k - \dim_{\mathbb{F}_q} V_i = k - i$, $|\mathcal{V}_i| = q^{k-i}$, and

$$\{0\} = \mathcal{V}_k \subset \mathcal{V}_{k-1} \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{V}_0 = S_k^\sharp. \tag{7.4}$$

If we view elements of $S_k^\sharp$ as column vectors $\mu = \langle \mu_1 \theta^{m-1}, \ldots, \mu_k \theta^{m-1} \rangle$, then $\mathcal{V}_i$ consists of those μ whose last i entries equal zero. The zero entries of $\mathcal{V}_i$ align with the nonzero entries of elements of V_i.

The set differences $\mathcal{V}_i - \mathcal{V}_{i+1}$ consist of all $\mu = \langle \mu_1 \theta^{m-1}, \ldots, \mu_k \theta^{m-1} \rangle$ with $\mu_{k-i+1} = \cdots = \mu_k = 0$, $\mu_{k-i} \neq 0$, and $\mu_1, \ldots, \mu_{k-i-1} \in \mathbb{F}_q$. The $\mathcal{U}$-orbit of μ is the set of all nonzero scalar multiples of μ. In each orbit there is exactly one 'normalized' representative with $\mu_{k-i} = 1$. For $i = 0, 1, \ldots, k-1$, let B_i be the subset of $\mathcal{V}_i - \mathcal{V}_{i+1}$ consisting of all the normalized representatives; i.e.,

$$B_i = \{\mu \in S_k : \mu_{k-i} = 1, \mu_{k-i+1} = \cdots = \mu_k = 0\}; \tag{7.5}$$

as a special case, set $B_k = \{0\}$. Then $|B_i| = q^{k-i-1}$, except $|B_k| = 1$.

Lemma 7.6. *Let $s \in \mathcal{S}_i$. Then, for $j = 0, 1, \ldots, k-1$,*

$$|B_j \cap \operatorname{ann}(s)| = \begin{cases} q^{k-j-1}, & i \leq j \leq k-1, \\ 0, & j = i-1, \\ q^{k-j-2}, & 0 \leq j \leq i-2. \end{cases}$$

Proof. The case $\mathcal{S}_0 = \{0\}$ has $\operatorname{ann}(0) = S_k^\sharp$. Then $|B_j \cap \operatorname{ann}(0)| = |B_j| = q^{k-j-1}$ for all $0 \leq j \leq k-1$.

Now let $1 \leq i \leq k$. The element $s \in \mathcal{S}_i = V_i - V_{i-1}$ is nonzero and has the form $s = (0, \ldots, 0, s_{k-i+1}, \ldots, s_k)$, with $s_{k-i+1} \neq 0$. Any $\mu \in \mathcal{V}_{i-1} - \mathcal{V}_i$ has the form $\mu = \langle \mu_1 \theta^{m-1}, \ldots, \mu_{k-i+1} \theta^{m-1}, 0, \ldots, 0 \rangle$ with $\mu_{k-i+1} \neq 0$. Thus $s\mu = s_{k-i+1} \mu_{k-i+1} \theta^{m-1} \neq 0$, so that $|B_{i-1} \cap \operatorname{ann}(s)| = 0$. For $i \leq j \leq k-1$, use the definition of $\mathcal{V}_i$ to see that $\mathcal{V}_j \subseteq \mathcal{V}_i = \operatorname{ann}(V_i) \subseteq \operatorname{ann}(s)$. So $B_j \subseteq \operatorname{ann}(s)$ and $|B_j \cap \operatorname{ann}(s)| = |B_j| = q^{k-j-1}$.

Because $s \neq 0$, $\operatorname{ann}(s)$ is a vector subspace of $S_k^\sharp$ with $\dim_{\mathbb{F}_q} \operatorname{ann}(s) = k-1$. By dimension counting, $\dim(\mathcal{V}_j \cap \operatorname{ann}(s))$ equals $k-j-1$ or $k-j$. The case $\dim(\mathcal{V}_j \cap \operatorname{ann}(s)) = k-j$ occurs when $\mathcal{V}_j \subseteq \operatorname{ann}(s)$. This is the case $i \leq j \leq k-1$ above. When $j = i-1$, $\mathcal{V}_{i-1} \cap \operatorname{ann}(s) = \mathcal{V}_i$, so that $|B_{i-1} \cap \operatorname{ann}(s)| = 0$, as we saw above. Finally, let $0 \leq j \leq i-2$. Then $\dim(\mathcal{V}_j \cap \operatorname{ann}(s)) = k-j-1$ and $\dim(\mathcal{V}_{j+1} \cap \operatorname{ann}(s)) = k-j-2$. This implies $|(\mathcal{V}_j - \mathcal{V}_{j+1}) \cap \operatorname{ann}(s)| = |\mathcal{V}_j \cap \operatorname{ann}(s)| - |\mathcal{V}_{j+1} \cap \operatorname{ann}(s)| = q^{k-j-2}(q-1)$. Taking normalized representatives implies $|B_j \cap \operatorname{ann}(s)| = q^{k-j-2}$. □

8. Two families of linear codes with the same wwe

We continue to assume that R is a finite chain ring with maximal ideal $\mathfrak{m} = (\theta)$ such that $R/\mathfrak{m} \cong \mathbb{F}_q$ and $\theta^m = 0$, for some integer $m \geq 2$. Let w be an integer-valued weight on R with maximal symmetry. Denote the common value of w on $\operatorname{orb}(\theta^i)$ by $w_i > 0$, $i = 0, 1, \ldots, m-1$, and $w_m = w(0) = 0$.

To begin, we use $w_0, \ldots, w_{m-1}$ to define several numerical quantities. For $i = 0, 1, \ldots, m-1$, define

$$a_i = \sum_{j=0}^{i} q^j w_{m-j-1}. \tag{8.1}$$

Also define, for $k = 2, 3, \ldots, m$,

$$\Delta_k = k q^{k-1} w_{m-1} - \sum_{i=0}^{k-1} q^{k-i-1} a_i. \tag{8.2}$$

Recall the cyclic R-module $Z_k = R/(\theta^k)$ and the semisimple R-module $S_k = Z_1 \oplus \cdots \oplus Z_1$ (k summands) from Section 6. We will construct R-linear codes with Z_k and S_k as their underlying information modules. The linear codes will be images of homomorphisms $\Lambda : Z_k \to R^n$ and $\Gamma : S_k \to R^n$ of left R-modules. As explained in Section 4, the linear codes are determined by their multiplicity functions.

Definition 8.3. Define a left R-linear code C_k parametrized by Z_k by using the linear functionals given by right multiplication by each of $\theta^{m-k}, \ldots, \theta^{m-1}$, each repeated $q^{k-1} w_{m-1}$ times, and the zero functional, repeated $\max\{0, -\Delta_k\}$ times; cf., (8.2).

Equivalently, C_k has a generator matrix of size $1 \times (k q^{k-1} w_{m-1} + \max\{0, -\Delta_k\})$, with entries $\theta^{m-k}, \ldots, \theta^{m-1}$, each repeated $q^{k-1} w_{m-1}$ times, plus entries of 0, repeated $\max\{0, -\Delta_k\}$ times. Thus, 0 does not appear if $\Delta \geq 0$, and 0 appears $-\Delta_k$ times when $\Delta_k < 0$.

Proposition 8.4. *The linear code C_k of Definition 8.3 has length $k q^{k-1} w_{m-1} + \max\{0, -\Delta_k\}$. Its weight function W_Λ has values*

$$W_\Lambda(\theta^i) = q^{k-1} w_{m-1} (w_{m-k+i} + \cdots + w_{m-1}),$$

for $i = 0, 1, \ldots, k-1$, and the w-weight enumerator of C_k is

$$\begin{aligned}\mathrm{wwe}_{C_k} &= 1 + \sum_{i=0}^{k-1} |\mathrm{orb}(\theta^i)| t^{W_\Lambda(\theta^i)} \\ &= 1 + \sum_{i=0}^{k-1} q^{k-i-1}(q-1) t^{q^{k-1} w_{m-1} \sum_{j=i}^{k-1} w_{m-k+j}}.\end{aligned}$$

Proof. The formula for the length follows directly from Definition 8.3. In calculating $W_\Lambda(\theta^i)$, remember that $w_m = w(\theta^m) = w(0) = 0$.

$$\begin{aligned}W_\Lambda(\theta^i) &= \sum_{j=m-k}^{m-1} q^{k-1} w_{m-1} w(\theta^i \theta^j) \\ &= q^{k-1} w_{m-1} \left(w_{i+m-k} + \cdots + w_{m-1} \right).\end{aligned}$$

The functional in $Z_k^\sharp$ given by right multiplication by θ^{m-k} is injective, so that Λ is also injective. Then wwe_{C_k} follows from (4.3). □

We now want to define linear codes D_k parametrized by $\Gamma : S_k \to R^n$ such that $\mathrm{wwe}_{C_k} = \mathrm{wwe}_{D_k}$. The form of wwe_{C_k} was determined by (4.3), in particular by the sizes of the orbits $\mathrm{orb}(\theta^i)$ and the value of W_Λ on those orbits. In order to be able to match terms in the equation $\mathrm{wwe}_{C_k} = \mathrm{wwe}_{D_k}$, we make use of the subsets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ of S_k defined following (7.3). Also recall the sets B_i defined in (7.5). We will design Γ so that W_Γ is constant on each $\mathcal{S}_i$, with value equal to the value of W_Λ on $\mathrm{orb}(\theta^{k-i})$.

Definition 8.5. Define a left R-linear code D_k parametrized by $\Gamma : S_k \to R^n$ by using the linear functionals in $\cup_i B_i$, with each $\mu \in B_i$ repeated a_i times (cf., (8.1)), $i = 0, 1, \ldots, k-1$, and the zero-functional in B_k repeated $\max\{\Delta_k, 0\}$ times (cf., (8.2)).

Equivalently, D_k has a generator matrix of size $k \times (\sum_{i=0}^{k-1} q^{k-i-1} a_i + \max\{\Delta_k, 0\})$, with columns given by $\mu \in \cup_i B_i$, with each $\mu \in B_i$ repeated a_i times, $i = 0, 1, \ldots, k$, the zero-column in B_k repeated $\max\{\Delta_k, 0\}$ times. Thus, zero-columns do not appear if $\Delta \leq 0$, and the zero-column appears Δ_k times when $\Delta_k > 0$.

We now express the values of W_Γ in terms of $a_0, a_1, \ldots, a_{k-1}$.

Proposition 8.6. *The weight function $W_\Gamma : S_k \to \mathbb{Z}$ is constant on each $\mathcal{S}_i$, with $W_\Gamma(\mathcal{S}_0) = 0$ and, for $i = 1, \ldots, k$,*

$$W_\Gamma(\mathcal{S}_i) = \left(q^{k-i} a_{i-1} + \sum_{j=0}^{i-2} q^{k-j-2}(q-1) a_j \right) w_{m-1}.$$

Proof. If $i = 0$, then $\mathcal{S}_0 = \{0\}$, so that $W_\Gamma(\mathcal{S}_0) = 0$. Now let $i = 1, \ldots, k$, and let $s \in \mathcal{S}_i$. Using Lemma 7.6, we see that

$$\begin{aligned} W_\Gamma(s) &= \sum_{j=0}^{k-1} a_j \sum_{\mu \in B_j} w(s\mu) = \sum_{j=0}^{i-1} a_j \sum_{\mu \in B_j} w(s\mu) \\ &= \left(q^{k-i} a_{i-1} + \sum_{j=0}^{i-2} q^{k-j-2}(q-1) a_j \right) w_{m-1}. \end{aligned}$$

This formula depends only on i, so W_Γ is constant on each $\mathcal{S}_i$. ☐

In order that $\mathrm{wwe}_{C_k} = \mathrm{wwe}_{D_k}$, we need $W_\Lambda(\mathrm{orb}(\theta^{k-i})) = W_\Gamma(\mathcal{S}_i)$, for $i = 1, \ldots, k$. That is, canceling a common factor, we need

$$q^{k-1}(w_{m-i} + \cdots + w_{m-1}) = q^{k-i} a_{i-1} + \sum_{j=0}^{i-2} q^{k-j-2}(q-1) a_j, \tag{8.7}$$

for $i = 1, \ldots, k$. The number of additive terms on each side of the equation is i. Starting with $i = 1$ and working upwards, we solve a triangular system recursively for $a_0, a_1, \ldots, a_{k-1}$.

Lemma 8.8. *The solutions of* (8.7) *are*

$$a_i = w_{m-1} + q w_{m-2} + \cdots + q^i w_{m-i-1},$$

for $i = 0, 1, \ldots, k-1$. *This formula matches* (8.1).

Proof. Exercise, by induction. The terms simplify by telescoping. ☐

Theorem 8.9. *For each* $k = 2, 3, \ldots, m$, *the codes* C_k *and* D_k *have the same length and satisfy* $\mathrm{wwe}_{C_k} = \mathrm{wwe}_{D_k}$.

Proof. The definition of Δ_k in (8.2) guarantees that the codes have the same length. The a_i of (8.1) were defined so that Lemma 8.8 holds. Thus $W_\Lambda(\mathrm{orb}(\theta^{k-i})) = W_\Gamma(\mathcal{S}_i)$, for $i = 1, \ldots, k$. Because $|\mathrm{orb}(\theta^{k-i})| = |\mathcal{S}_i|$, the equality of the w-weight enumerators follows from (4.3). ☐

Remark 8.10. It is possible to generalize the constructions of C_k and D_k by allowing more general expressions for the multiplicities of the linear functionals appearing in C_k. In Definition 8.3, $\theta^{m-k}, \ldots, \theta^{m-1}$ could be repeated $q^{k-1} w_{m-1} b_{m-k}, \ldots, q^{k-1} w_{m-1} b_{m-1}$ times, respectively. One can then mimic (8.7) and Lemma 8.8 to solve for the multiplicities a_i used in defining D_k. One must be careful in choosing the b's in order that the a's come out nonnegative. Sufficiently large b's should work. The factors of $q^{k-1} w_{m-1}$ are present so that the a's are integers. The present work does not need this level of generality.

In later sections, we will use the formula below for Δ_k, which is expressed in terms of some numerical quantities defined next.

Definition 8.11. For $i = 1, 2, \ldots, m$, define $\epsilon_i = w_i - w_{i-1}$. For example, $\epsilon_1 = w_1 - w_0$, while $\epsilon_m = -w_{m-1}$ (as $w_m = 0$). For $i = 1, \ldots, m-2$, define $\epsilon'_i = \epsilon_i$, and define $\epsilon'_{m-1} = (q-1)w_{m-1} - qw_{m-2} = q\epsilon_{m-1} + \epsilon_m$.

Also define these polynomial expressions in q: $p_0 = 0$, and

$$p_i = 1 + 2q + 3q^2 + \cdots + iq^{i-1}, \quad i = 1, 2, \ldots. \tag{8.12}$$

Then for $i = 1, 2, \ldots$, the p_i are positive and satisfy the formulas

$$p_i - qp_{i-1} = 1 + q + q^2 + \cdots + q^{i-1}. \tag{8.13}$$

Proposition 8.14. *For $k = 2, 3, \ldots, m$,*

$$\Delta_k = p_{k-1}\epsilon'_{m-1} + \sum_{j=2}^{k-1} q^j p_{k-j}\epsilon'_{m-j}.$$

Proof. Start with (8.2), and replace a_i using (8.1):

$$\Delta_k = kq^{k-1}w_{m-1} - \sum_{i=0}^{k-1} q^{k-i-1} \sum_{j=0}^{i} q^j w_{m-1-j}.$$

Interchange the order of summation and use (8.13):

$$\begin{aligned}
\Delta_k &= kq^{k-1}w_{m-1} - \sum_{j=0}^{k-1}\sum_{i=j}^{k-1} q^{k-i+j-1}w_{m-1-j} \\
&= kq^{k-1}w_{m-1} - \sum_{j=0}^{k-1} q^j(p_{k-j} - qp_{k-j-1})w_{m-1-j} \\
&= kq^{k-1}w_{m-1} - \sum_{j=0}^{k-1} q^j p_{k-j}w_{m-1-j} + \sum_{j=0}^{k-1} q^{j+1}p_{k-j-1}w_{m-1-j}.
\end{aligned}$$

Re-index the last summation with $\ell = j+1$ (the $\ell = k$ term vanishes), separate some initial terms, and combine the rest using Definition 8.11:

$$\begin{aligned}
\Delta_k &= kq^{k-1}w_{m-1} - \sum_{j=0}^{k-1} q^j p_{k-j}w_{m-1-j} + \sum_{\ell=1}^{k} q^\ell p_{k-\ell}w_{m-\ell} \\
&= kq^{k-1}w_{m-1} - p_k w_{m-1} + qp_{k-1}\epsilon_{m-1} + \sum_{j=2}^{k-1} q^j p_{k-j}\epsilon_{m-j}.
\end{aligned}$$

Simplify the coefficient of w_{m-1} and again use Definition 8.11:

$$\begin{aligned}\Delta_k &= p_{k-1}\epsilon_m + qp_{k-1}\epsilon_{m-1} + \sum_{j=2}^{k-1} q^j p_{k-j}\epsilon_{m-j} \\ &= p_{k-1}\epsilon'_{m-1} + \sum_{j=2}^{k-1} q^j p_{k-j}\epsilon'_{m-j}. \qquad \square\end{aligned}$$

9. Analysis of dual codewords of low weight

Our ultimate objective is to prove that for some $k = 2, 3, \ldots, m$, the codes C_k and D_k of Theorem 8.9 have dual codes with different weight enumerators: $\mathrm{wwe}_{C_k^\perp} \neq \mathrm{wwe}_{D_k^\perp}$. We will try to do this in the most direct way—by showing that $C_k^\perp$ and $D_k^\perp$ have different numbers of codewords of the smallest possible weight. With that in mind, let's develop some notation.

Recall that we are assuming the chain ring R is equipped with an integer-valued weight w of maximal symmetry. The common value on $\mathrm{orb}(\theta^i)$ is denoted $w_i > 0$, and $w_m = w(0) = 0$. Let $\mathring{w} = \min\{w_i : i = 0, 1, \ldots, m-1\}$, so that $\mathring{w} > 0$. Define $\mathring{I} = \{i : w_i = \mathring{w}\}$, the set of exponents of θ that achieve the minimum value of the weight; $\mathring{I}$ is nonempty.

We now turn our attention to the linear codes C_k and D_k of Theorem 8.9 and codewords of weight $d < 2\mathring{w}$ in their dual codes. All such codewords must be singletons by Lemma 5.3. We will abuse notation slightly by using the phrase 'singleton θ^i' to mean a singleton whose nonzero entry is a unit multiple of θ^i.

Lemma 9.1. *Suppose an integer d satisfies $\mathring{w} \leq d < 2\mathring{w}$. Let $I_d = \{i : w_i = d\}$. If $0 \in I_d$, then, for $k = 2, 3, \ldots, m$,*

$$A_d(C_k^\perp) - A_d(D_k^\perp) = -|\mathrm{orb}(1)|\Delta_k - \sum_{\substack{i \in I_d \\ 0 < i < k}} (k-i) q^{k-1} w_{m-1} |\mathrm{orb}(\theta^i)|.$$

If $0 \notin I_d$ and I_d is nonempty, then, for $k = 2, 3, \ldots, m$,

$$A_d(C_k^\perp) - A_d(D_k^\perp) = -\sum_{\substack{i \in I_d \\ i < k}} (k-i) q^{k-1} w_{m-1} |\mathrm{orb}(\theta^i)|.$$

Proof. Nonzero contributions to $A_d(C_k^\perp) - A_d(D_k^\perp)$ are made by singletons of weight d in $C_k^\perp$ or $D_k^\perp$. The nonzero entry of a singleton of weight d must be a unit multiple of θ^i with $i \in I_d$. In order for a singleton to belong to a dual code, its nonzero entry—located in position

j, say—must annihilate column j of the generator matrix of the primal code.

If $0 \in I_d$, a singleton 1 annihilates only zero-columns. (Remember that the nonzero entry of a singleton 1 is a unit.) The number of zero-columns is determined by Δ_k: if $\Delta_k > 0$, then D_k has Δ_k zero-columns; if $\Delta_k < 0$, then C_k has $-\Delta_k$ zero-colums. The net contribution to $A_d(C_k^\perp) - A_d(D_k^\perp)$ is $-|\mathrm{orb}(1)|\Delta_k$, the number of units times the number of zero-columns.

If $0 < i \in I_d$, then a singleton θ^i annihilates all the columns of D_k. Such singletons contribute $-|\mathrm{orb}(\theta^i)| \cdot \mathrm{length}(D_k)$ to $A_d(C_k^\perp) - A_d(D_k^\perp)$. On the other hand, when $i \geq k$, a singleton θ^i annihilates all columns of C_k. Such singletons contribute $|\mathrm{orb}(\theta^i)| \cdot \mathrm{length}(C_k)$ to $A_d(C_k^\perp) - A_d(D_k^\perp)$. When $i < k$, a singleton θ^i annihilates all columns of C_k except those with entries $\theta^{m-k}, \ldots, \theta^{m-i-1}$. Such singletons contribute $|\mathrm{orb}(\theta^i)|(\mathrm{length}(C_k) - (k-i)q^{k-1}w_{m-1})$ to $A_d(C_k^\perp) - A_d(D_k^\perp)$. Because C_k and D_k have the same length, the total contribution by singleton θ^i's is 0 when $i \geq k$ and $-(k-i)q^{k-1}w_{m-1}|\mathrm{orb}(\theta^i)|$ when $i < k$. Summing over $i \in I_d$ completes the proof. $\square$

Our main interest is dual codewords of weight $\mathring{w}$. For $k = 2, 3, \ldots, m$, define

$$\delta_k = A_{\mathring{w}}(C_k^\perp) - A_{\mathring{w}}(D_k^\perp). \tag{9.2}$$

Our aim is to show, whenever possible, for a given weight w, that $\delta_k \neq 0$ for some k. We restate Lemma 9.1 for the case where $d = \mathring{w}$.

Lemma 9.3. *Fix $k = 2, 3, \ldots, m$. If $0 \in \mathring{I}$, then*

$$\delta_k = -|\mathrm{orb}(1)|\Delta_k - \sum_{\substack{i \in \mathring{I} \\ 0<i<k}} (k-i)q^{k-1}w_{m-1}|\mathrm{orb}(\theta^i)|.$$

If $0 \notin \mathring{I}$, then

$$\delta_k = -\sum_{\substack{i \in \mathring{I} \\ i<k}} (k-i)q^{k-1}w_{m-1}|\mathrm{orb}(\theta^i)|.$$

We draw three corollaries, using notation from Definition 8.11.

Corollary 9.4. *If $0 \notin \mathring{I}$, then $\delta_k < 0$ for all $k = 1 + \min \mathring{I}, \ldots, m$. Weights w on R with $w_0 > \mathring{w}$ do not respect duality.*

Corollary 9.5. *Suppose $\mathring{I} = \{0\}$. Then the following hold:*

1. *If $m \geq 3$ and j, $1 \leq j \leq m-1$, is the largest index with $\epsilon'_j \neq 0$, then $\delta_{m-j+1} \neq 0$.*
2. *If $m = 2$ and $\epsilon'_1 = (q-1)w_1 - qw_0 \neq 0$, then $\delta_2 \neq 0$.*

(3) *If* $m=2$, $\epsilon'_1=(q-1)w_1-qw_0=0$, *and* $q>2$, *then* $A_{w_1}(C_2^\perp)<A_{w_1}(D_2^\perp)$.

Weights w *on* R *with* $w_0<w_i$ *for all* $i=1,2,\ldots,m-1$ *do not respect duality, except when* $m=2$, $q=2$, *and* $w_1=2w_0$*; cf., Theorem 6.5.*

Proof. Suppose $m\geq 3$. By the hypothesis on $\mathring{I}$, $\epsilon'_1=w_1-w_0>0$, so there exists a maximal index j, $1\leq j\leq m-1$, with $\epsilon'_j\neq 0$. Thus, $\epsilon'_\ell=0$ for $\ell=j+1,\ldots,m-1$. From Proposition 8.14, $\Delta_{m-j+1}=q^{m-j}\epsilon'_j\neq 0$. From Lemma 9.3 and the hypothesis, $\delta_{m-j+1}=-|\mathrm{orb}(1)|\Delta_{m-j+1}\neq 0$.

When $m=2$, $\epsilon'_1=(q-1)w_1-qw_0$, not w_1-w_0. If $\epsilon'_1\neq 0$, the proof proceeds as above: $\Delta_2\neq 0$, and $\delta_2\neq 0$.

When $m=2$ and $\epsilon'_1=0$, then $\Delta_2=\delta_2=0$. However, $\epsilon'_1=0$ means that $(q-1)w_1=qw_0$, i.e., $w_1=(q/(q-1))w_0>w_0$. But note, for integers $q\geq 2$, that $q/(q-1)\leq 2$ with equality holding if and only if $q=2$. Assuming $q>2$, we have $\mathring{w}=w_0<w_1<2\mathring{w}$. By Lemma 9.1 applied to $d=w_1$ and $0\notin I_d$, we see that $A_{w_1}(C_2^\perp)<A_{w_1}(D_2^\perp)$.

When $m=2$, $q=2$, and $w_1=2w_0$, we are in the situation of Theorem 6.5, where the MacWilliams identities hold. $\square$

Corollary 9.6. *Let* R *be a finite chain ring with* $m=2$*. Then every weight* w *on* R *having maximal symmetry does not respect duality, except for multiples of the Hamming weight* (*any* q) *or the homogeneous weight* ($q=2$ *only*).

Proof. Because $m=2$, there are only w_0 and w_1. If $w_0>w_1$, Corollary 9.4 implies w does not respect duality. If $w_0=w_1$, w is a multiple of the Hamming weight, and the MacWilliams identities hold [25, Theorem 8.3]. If $w_0<w_1$, then Corollary 9.5 applies: w does not respect duality, except for multiples of the homogeneous weight if $q=2$. $\square$

10. Weak monotonicity and final arguments

When $\{0\}\subsetneq\mathring{I}$, the formula for δ_k in Lemma 9.3 is difficult to exploit systematically; the combinatorics can be formidable. (But not always: see Example 11.9.) In order to make progress, we will assume that the weight w on the chain ring R is *weakly monotone*; i.e., we assume

$$\mathring{w}=w_0\leq w_1\leq\cdots\leq w_{m-2}\leq w_{m-1}. \tag{10.1}$$

This hypothesis implies that $\epsilon_i\geq 0$ for $i=1,2,\ldots,m-1$ in Definition 8.11. However, $\epsilon_m=-w_{m-1}<0$. The weakly monotone hypothesis allows us to state our main result.

Theorem 10.2. *Let* R *be a finite chain ring with a weakly monotone weight* w*. Then* w *does not respect duality, except when*

- *w is a multiple of the Hamming weight, or*
- *$m = 2$, $q = 2$, and $w_1 = 2w_0$.*

Theorem 10.2 will follow from Theorem 10.10, which we will prove after we establish some technical lemmas.

Equalities are possible in (10.1). Given a weakly monotone weight w, define j_0 to be the largest index such that

$$\mathring{w} = w_0 = \cdots = w_{j_0} < w_{j_0+1}. \tag{10.3}$$

Similarly, define j_1 to be the smallest index such that

$$w_{j_1-1} < w_{j_1} = \cdots = w_{m-1}. \tag{10.4}$$

There are three situations to highlight, depending upon how many nonzero values w takes.

- If w has only one nonzero value, then $w_0 = \cdots = w_{m-1}$, so that $j_0 = m-1$ and $j_1 = 0$. The weight w is a multiple of the Hamming weight.
- If w has exactly two nonzero values, then $w_0 = \cdots = w_{j_0} < w_{j_0+1} = \cdots = w_{m-1}$, so that $j_1 = j_0 + 1$. The homogeneous weight is an example of this, with $j_0 = m-2$, $j_1 = m-1$.
- If w has three or more values, define j_2 so that

  $$\cdots w_{j_2-1} < w_{j_2} = \cdots = w_{j_1-1} < w_{j_1} = \cdots = w_{m-1}.$$

  Said another way, j_1 is the largest index less than m with $\epsilon_{j_1} > 0$, and j_2 is the second-largest index less than m with $\epsilon_{j_2} > 0$.

The weight w has a least two nonzero values if and only if $j_1 > 0$. In that case, $j_0 < j_1 \leq m-1$, with

$$\epsilon_{j_1} > 0 \text{ and } \epsilon_{j_1+1} = \cdots = \epsilon_{m-1} = 0. \tag{10.5}$$

If w has at least three values, then $j_0 < j_2 < j_1 \leq m-1$ and, in addition to (10.5), we have

$$\epsilon_{j_2} > 0 \text{ and } \epsilon_{j_2+1} = \cdots = \epsilon_{j_1-1} = 0. \tag{10.6}$$

The key to our analysis is a simplified expression for δ_k; cf., (8.12).

Lemma 10.7. *Suppose $\{0, 1, \ldots, j_0\} = \mathring{I}$, with $j_0 \geq 1$. If k is an integer, $2 \leq k \leq j_0 + 1$, then*

$$\delta_k = -q^m(q-1) \sum_{j=1}^{k-1} q^{j-1} p_{k-j} \epsilon_{m-j}.$$

If k is an integer, $j_0+2 \le k \le m$, then

$$\delta_k = -((k-j_0-1)q^{m+k-j_0-2} - q^{m+k-j_0-3} - \cdots - q^{m-1})\epsilon_m \\ -q^m(q-1)\sum_{j=1}^{k-1} q^{j-1}p_{k-j}\epsilon_{m-j}.$$

Proof. For any integer k, $2 \le k \le m$, Lemma 9.3 and (6.3) imply

$$\begin{aligned}\delta_k &= -|\mathrm{orb}(1)|\Delta_k - \sum_{i=1}^{\min\{j_0,k-1\}} (k-i)q^{k-1}w_{m-1}|\mathrm{orb}(\theta^i)| \\ &= -q^{m-1}(q-1)\Delta_k - \sum_{i=1}^{\min\{j_0,k-1\}} (k-i)q^{m+k-i-2}(q-1)w_{m-1}.\end{aligned}$$

In the formula for Δ_k given in Proposition 8.14, use $\epsilon'_{m-1} = q\epsilon_{m-1} + \epsilon_m = q\epsilon_{m-1} - w_{m-1}$. Because of telescoping sums, the w_{m-1}-terms cancel completely when $2 \le k \le j_0+1$ or cancel partially when $j_0+2 \le k \le m$. The terms that remain are as stated. □

In the case of $j_0+2 \le k \le m$, note that the ϵ_m-term is positive, as $-\epsilon_m = w_{m-1} > 0$ and the numerical sum is positive. The other terms are nonpositive. The balance between the terms appears to be problematic. When $2 \le k \le j_0+1$, $\delta_k \le 0$.

We will also need information about how δ_k changes when $k \ge j_0+2$.

Lemma 10.8. *If $k \ge j_0+2$ and $k \ge m-j_1$, then*

$$\delta_{k+1} - \delta_k = -q^{m+k-j_0-2}(q-1) \\ \times \left\{ (k-j_0)\epsilon_m + q^{j_0+1}\sum_{i=m-j_1}^{k}(k-i+1)\epsilon_{m-i} \right\}.$$

Proof. Because $k \ge j_0+2$, the second formula in Lemma 10.7 applies to both δ's. All but the highest order terms cancel, leaving

$$\delta_{k+1} - \delta_k = -(k-j_0)q^{m+k-j_0-2}(q-1)\epsilon_m \\ - q^{m+k-1}(q-1)\sum_{i=1}^{k}(k-i+1)\epsilon_{m-i}.$$

Because $j_1 \ge m-k$, (10.5) implies that terms vanish in the summation for $i = 1, 2, \ldots, m-j_1-1$. □

Corollary 10.9. *Suppose $j_0 + j_1 = m - 1$. If $k \geq j_0 + 2$, then*

$$\begin{aligned}\delta_{k+1} - \delta_k = &-q^{m+k-j_0-2}(q-1)(k-j_0)\left(\epsilon_m + q^{j_0+1}\epsilon_{j_1}\right)\\ &- q^{m+k-1}(q-1)\sum_{i=m-j_1+1}^{k}(k-i+1)\epsilon_{m-i}.\end{aligned}$$

Proof. Note first that $j_0 + j_1 = m - 1$ implies $j_0 + 2 = m - j_1 + 1$. Thus, if $k \geq j_0 + 2$, then $k \geq m - j_1$ is automatic. Apply Lemma 10.8 and notice that the term inside the summation with $i = m - j_1$ is $(k - m + j_1 + 1)\epsilon_{j_1} = (k - j_0)\epsilon_{j_1}$. □

The next theorem gives a more detailed description of the claims in Theorem 10.2.

Theorem 10.10. *Let R be a finite chain ring with a weakly monotone weight w. Let j_0 and j_1 be as defined in (10.3) and (10.4). Then the following statements hold.*

1. *If $j_0 = m - 1$, then w is a multiple of the Hamming weight.*
2. *If $j_0 = 0$, then Corollary 9.5 applies.*

In the following statements, assume $1 \leq j_0 < m - 1$.

3. *If $j_0 + j_1 \geq m$, then $\delta_k < 0$ for $m - j_1 + 1 \leq k \leq j_0 + 1$.*
4. *If $j_0 + j_1 \leq m - 2$, then $\delta_k > 0$ for $j_0 + 2 \leq k \leq m - j_1$.*
5. *If $j_0 + j_1 = m - 1$ and $w_{m-1} \neq q^{j_0+1}\epsilon_{j_1}$, then $\delta_{j_0+2} \neq 0$.*
6. *Suppose $j_0 + j_1 = m - 1$ and $w_{m-1} = q^{j_0+1}\epsilon_{j_1}$. If w has at least three nonzero values, then $\delta_k < 0$ for $k \geq m - j_2 + 1$.*
7. *Suppose $j_0 + j_1 = m - 1$, $w_{m-1} = q^{j_0+1}\epsilon_{j_1}$, and w has two nonzero values. Then $j_1 = j_0 + 1$, $m = 2j_0 + 2$ is even, and*

$$A_{w_{m-1}}(C_k^{\perp}) - A_{w_{m-1}}(D_k^{\perp}) < 0$$

for $k > j_1$.

Remark 10.11. The length of the 'run' $w_{j_1} = \cdots = w_{m-1}$ is $m - j_1$. The length of the 'run' $w_0 = \cdots = w_{j_0}$ is $j_0 + 1$. Their difference is

$$(m - j_1) - (j_0 + 1) = \begin{cases} +, & j_0 + j_1 \leq m - 2, \\ 0, & j_0 + j_1 = m - 1, \\ -, & j_0 + j_1 \geq m. \end{cases}$$

Proof of Theorem 10.10. The first two claims were explained after (10.4). If $1 \leq j_0 < m - 1$ and $j_0 + j_1 \geq m$, then any k satisfying $m - j_1 + 1 \leq k \leq j_0 + 1$ has $\delta_k < 0$ by Lemma 10.7 and (10.5). Namely, the first formula in Lemma 10.7 applies, and $\epsilon_{j_1} > 0$ appears in the formula because $j_1 \geq m - k + 1$.

If $1 \leq j_0 < m-1$ and $j_0 + j_1 \leq m-2$, then any k satisfying $j_0 + 2 \leq k \leq m - j_1$ has $\delta_k > 0$. Now the second formula in Lemma 10.7 applies, and only the (positive) ϵ_m-term survives, because $j_1 < m - k + 1$.

If $1 \leq j_0 < m-1$ and $j_0 + j_1 = m-1$, then $j_0 + 2 = m - j_1 + 1$. Setting $k = j_0 + 2$, we see from the second formula of Lemma 10.7 that

$$\begin{aligned} \delta_{j_0+2} &= -(q^m - q^{m-1})\epsilon_m - q^m(q-1)q^{j_0}\epsilon_{j_1} \\ &= -q^{m-1}(q-1)\left(\epsilon_m + q^{j_0+1}\epsilon_{j_1}\right). \end{aligned}$$

As $\epsilon_m = -w_{m-1}$, we see that $w_{m-1} \neq q^{j_0+1}\epsilon_{j_1}$ implies $\delta_{j_0+2} \neq 0$.

Next, suppose $1 \leq j_0 < m-1$, $j_0 + j_1 = m-1$, and $w_{m-1} = q^{j_0+1}\epsilon_{j_1}$. We just saw that this implies $\delta_{j_0+2} = 0$. Applying Corollary 10.9, first with $k = j_0 + 2$, and then recursively, we see, for any $k \geq j_0 + 2$, that δ_k has the form:

$$\delta_k = -\sum_{i=m-j_1+1}^{k-1} c(k)_i \epsilon_{m-i},$$

where each $c(k)_i$ is a positive integer depending on k. Because each $\epsilon_{m-i} \geq 0$, each $\delta_k \leq 0$, and $\delta_k < 0$ if at least one $\epsilon_{m-i} > 0$ in the interval of summation. Remember (10.6). If $k \geq m - j_2 + 1$, then $m - j_2 + 1 > m - j_1 + 1 = j_0 + 2$, $m - (k-1) \leq j_2$, and $\epsilon_{j_2} > 0$ appears in the expression for δ_k. Thus $\delta_k < 0$.

Finally, suppose $1 \leq j_0 < m-1$, $j_0 + j_1 = m-1$, $w_{m-1} = q^{j_0+1}\epsilon_{j_1}$, and w has exactly two nonzero values. That means that $j_1 = j_0 + 1$, so that $m = 2j_0 + 2$. In addition, we must have $\epsilon_{j_1} = w_{m-1} - w_0$. Using this in the equation $w_{m-1} = q^{j_0+1}\epsilon_{j_1}$ yields $q^{j_0+1}w_0 = (q^{j_0+1} - 1)w_{m-1}$.

Because the coefficients in this last equation are relatively prime, there exists a positive integer s such that $w_0 = (q^{j_0+1} - 1)s$ and $w_{m-1} = q^{j_0+1}s$. Calculate: $2w_0 - w_{m-1} = (q^{j_0+1} - 2)s > 0$, because $j_0 \geq 1$ and $q \geq 2$. (But see Remark 10.12.) Thus $\mathring{w} = w_0 < w_{m-1} < 2\mathring{w}$. Using the second formula of Lemma 9.1, with $d = w_{m-1}$ and $0 \notin I_d$, we see that $A_{w_{m-1}}(C_k^\perp) - A_{w_{m-1}}(D_k^\perp) < 0$ for any $k > j_1 = \min I_d$. $\square$

Remark 10.12. In general, for positive integers $q \geq 2$, $q^{j_0+1} - 2 \geq 0$, with equality if and only if $j_0 = 0$ and $q = 2$. Equality, again, points to Theorem 6.5.

Example 10.13. When $m = 3$, the only situation not covered by Corollaries 9.4, 9.5, or Theorem 10.2 is when $w_0 = w_2 < w_1$. In this case, $\epsilon'_1 = w_1 - w_0 > 0$ and $\epsilon'_2 = (q-1)w_2 - qw_1 < 0$. Then Proposition 8.14 and Lemma 9.3 imply that

$$\delta_2 = -|\mathrm{orb}(1)|\Delta_2 = -q^2(q-1)((q-1)w_2 - qw_1) > 0.$$

Thus, this w does not respect duality. We conclude that, for $m = 3$, the only weights that respect duality are multiples of the Hamming weight.

11. Symmetrized enumerators and examples

In this section there are details of the MacWilliams identities for the symmetrized enumerator for the action of the full group $\mathcal{U}$ of units on a finite chain ring R, followed by several examples. The details supplement the general outline provided in Appendix A.

Suppose R is a finite chain ring with $\mathfrak{m} = (\theta)$, $R/\mathfrak{m} \cong \mathbb{F}_q$, and $\theta^m = 0$. As we have seen earlier, the group of units $\mathcal{U}$ acts on R on the left, with orbits $\operatorname{orb}(1), \operatorname{orb}(\theta), \ldots, \operatorname{orb}(\theta^{m-1}), \operatorname{orb}(\theta^m) = \{0\}$. The dual action of $\mathcal{U}$ on $R^\sharp \cong R_R$ has the same orbit structure.

For an element $r \in R$, define $\nu(r)$ via $r \in \operatorname{orb}(\theta^{\nu(r)})$; i.e., $\nu(r)$ is the exponent i of θ such that $r = u\theta^i$ for some unit $u \in \mathcal{U}$. Define $f : R \to \mathbb{C}[Z_0, \ldots, Z_m]$ by $f(r) = Z_{\nu(r)}$, $r \in R$. Then define $F : R^n \to \mathbb{C}[Z_0, \ldots, Z_m]$ by

$$F(x) = \prod_{\ell=1}^{n} f(x_\ell) = \prod_{\ell=1}^{n} Z_{\nu(x_\ell)}, \quad x = (x_1, x_2, \ldots, x_n) \in R^n. \tag{11.1}$$

The *symmetrized enumerator* of an additive code $C \subseteq R^n$ is the following element $\operatorname{se}_C \in \mathbb{C}[Z_0, \ldots, Z_m]$:

$$\operatorname{se}_C = \operatorname{se}_C(Z_0, \ldots, Z_m) = \sum_{x \in C} F(x) = \sum_{x \in C} \prod_{\ell=1}^{n} Z_{\nu(x_\ell)}.$$

Any finite chain ring R is a Frobenius ring, with a generating character χ. We will use the following properties of χ [25]:

(1) for any nonzero ideal I of R, $\sum_{r \in I} \chi(r) = 0$;
(2) every $\pi \in \widehat{R}$ has the form $\pi = \chi^r$ for some unique $r \in R$;
(3) $\chi^r(0) = 1$ for all $r \in R$.

We first calculate the sum of a character over the orbits $\operatorname{orb}(\theta^i)$.

Lemma 11.2. *Suppose $i, j = 0, 1, \ldots, m$ and $r \in \operatorname{orb}(\theta^j)$. Then*

$$\sum_{s \in \operatorname{orb}(\theta^i)} \chi^r(s) = \begin{cases} 0, & i + j \leq m - 2, \\ -q^{m-i-1}, & i + j = m - 1, \\ q^{m-i-1}(q-1), & i + j \geq m, i < m, \\ 1, & i + j \geq m, i = m. \end{cases}$$

Proof. When $i = m$, $\theta^m = 0$. Then $\sum_{s \in \mathrm{orb}(\theta^m)} \chi^r(s) = 1$, as $\chi^r(0) = 1$. For $i = 0, 1, \ldots, m-1$, $\mathrm{orb}(\theta^i) = (\theta^i) - (\theta^{i+1})$, so, for χ itself,

$$\begin{aligned} \sum_{s \in \mathrm{orb}(\theta^i)} \chi(s) &= \sum_{s \in (\theta^i)} \chi(s) - \sum_{s \in (\theta^{i+1})} \chi(s) \\ &= \begin{cases} 0, & i = 0, 1, \ldots, m-2, \\ -1, & i = m-1, \end{cases} \end{aligned} \tag{11.3}$$

using property (1) of χ.

Now suppose $r \in \mathrm{orb}(\theta^j)$, so that $r = u\theta^j$, $u \in \mathcal{U}$. Left multiplication by r maps $\mathrm{orb}(\theta^i)$ onto $\mathrm{orb}(\theta^{i+j})$, with each element in $\mathrm{orb}(\theta^{i+j})$ being hit $|\mathrm{orb}(\theta^i)|/|\mathrm{orb}(\theta^{i+j})|$ times. (Because $\mathcal{U}\theta^i = \theta^i\mathcal{U}$, units can be moved across powers of θ; for any unit u, $u\theta^i = \theta^i u'$ for some unit u'.) This implies that

$$\sum_{s \in \mathrm{orb}(\theta^i)} \chi^r(s) = \sum_{s \in \mathrm{orb}(\theta^i)} \chi(rs) = \frac{|\mathrm{orb}(\theta^i)|}{|\mathrm{orb}(\theta^{i+j})|} \sum_{t \in \mathrm{orb}(\theta^{i+j})} \chi(t).$$

Using (6.3) and (11.3), we get the stated result. ☐

Remark 11.4. The formulas in Lemma 11.2 depend only on the orbit of r, not r itself. This is a general feature of character sums over the blocks of a partition coming from a group action [8, Theorem 2.6].

Define the *generalized Kravchuk matrix* K by

$$K_{ij} = \sum_{s \in \mathrm{orb}(\theta^i)} \chi^r(s), \quad r \in \mathrm{orb}(\theta^j).$$

We calculate the Fourier transform of f and F as in (A.2).

Lemma 11.5. *For any $r \in R$, the Fourier transform of f is*

$$\widehat{f}(r) = \sum_{i=0}^{m} Z_i K_{ij}, \quad r \in \mathrm{orb}(\theta^j).$$

The Fourier transform of F is

$$\widehat{F}(r_1, r_2, \ldots, r_n) = \prod_{\ell=1}^{n} \widehat{f}(r_\ell) = \prod_{\ell=1}^{n} \left(\sum_{i=0}^{m} Z_i K_{i,\nu(r_\ell)} \right).$$

Proof. The identity $\widehat{F}(r_1, r_2, \ldots, r_n) = \prod_{\ell=1}^n \widehat{f}(r_\ell)$ is well-known, [25, Proposition A.5], so it is enough to calculate $\widehat{f}(r)$:

$$\begin{aligned}\widehat{f}(r) &= \sum_{s \in R} \chi(rs) f(s) = \sum_{s \in R} \chi^r(s) Z_{\nu(s)} \\ &= \sum_{i=0}^m \sum_{s \in \mathrm{orb}(\theta^i)} \chi^r(s) Z_i = \sum_{i=0}^m Z_i \sum_{s \in \mathrm{orb}(\theta^i)} \chi^r(s). \qquad \square\end{aligned}$$

We now have the MacWilliams identities for the symmetrized enumerator over a finite chain ring; cf., [25, Theorem 8.4], [8, Theorem 3.5].

Theorem 11.6. *Suppose R is a finite chain ring and $C \subseteq R^n$ is a left R-linear code. Then, using $C^\perp = \mathcal{R}(C)$,*

$$\mathrm{se}_{C^\perp}(Z_0, \ldots, Z_m) = \frac{1}{|C|} \mathrm{se}_C(\mathcal{Z}_0, \ldots, \mathcal{Z}_m)\Big|_{\mathcal{Z}_j = \sum_{i=0}^m Z_i K_{ij}}.$$

Proof. Follow the outline in Appendix A, apply Lemma 11.5, and note that $\mathcal{R}(C) = \mathfrak{R}(C)$ for left R-linear codes. $\square$

A version of this theorem, valid for the partition determined by the homogeneous weight, appears in [20, Theorem 2.1]

Example 11.7. Let $R = \mathbb{Z}/8\mathbb{Z}$. Then $\mathcal{U} = \{1, 3, 5, 7\}$. The $\mathcal{U}$-orbits are $\mathrm{orb}(1) = \mathcal{U}$, $\mathrm{orb}(2) = \{2, 6\}$, $\mathrm{orb}(4) = \{4\}$, and $\mathrm{orb}(0) = \{0\}$. The generalized Kravchuk matrix is

$$K = \begin{bmatrix} 0 & 0 & -4 & 4 \\ 0 & -2 & 2 & 2 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let w be the homogeneous weight, so that $\mathrm{w}_0 = \mathrm{w}_1 = 1$, $\mathrm{w}_2 = 2$, and $\mathrm{w}_3 = 0$. The linear codes C_3 and D_3 of Theorem 8.9 have the following codewords, with multiplicities listed above the horizontal line, telling how many times the given entry is repeated.

8	8	8
0	0	0
1	2	4
2	4	0
3	6	4
4	0	0
5	2	4
6	4	0
7	6	4

2	2	2	2	4	4	8
0	0	0	0	0	0	0
4	4	4	4	0	0	0
0	0	4	4	4	4	0
4	4	0	0	4	4	0
0	4	0	4	0	4	4
4	0	4	0	0	4	4
0	4	4	0	4	0	4
4	0	0	4	4	0	4

Then the symmetrized enumerators of the codes are:

$$\begin{aligned}\mathrm{se}_{C_3} &= 4Z_0^8Z_1^8Z_2^8 + 2Z_1^8Z_2^8Z_3^8 + Z_2^8Z_3^{16} + Z_3^{24},\\ \mathrm{se}_{D_3} &= 4Z_2^{16}Z_3^8 + 2Z_2^{12}Z_3^{12} + Z_2^8Z_3^{16} + Z_3^{24}.\end{aligned}$$

One can compute the symmetrized enumerators of the dual codes (via Theorem 11.6 and SageMath [23], say), but the results have too many terms to include here.

Specializing $Z_i \rightsquigarrow t^{\mathrm{w}_i}$ (and taking a Taylor expansion for the dual codes) yields the homogeneous weight enumerators:

$$\begin{aligned}\mathrm{howe}_{C_3} &= 1 + t^{16} + 2t^{24} + 4t^{32},\\ \mathrm{howe}_{D_3} &= 1 + t^{16} + 2t^{24} + 4t^{32},\\ \mathrm{howe}_{C_3^\perp} &= 1 + 16t + 1848t^2 + 60400t^3 + \cdots,\\ \mathrm{howe}_{D_3^\perp} &= 1 + 48t + 1832t^2 + 64656t^3 + \cdots.\end{aligned}$$

The computed value $\delta_3 = A_1(C_3^\perp) - A_1(D_3^\perp) = -32$ matches the value given in the second formula of Lemma 10.7.

Example 11.8. Still use $R = \mathbb{Z}/8\mathbb{Z}$, but change the weight to $w_0 = 1, w_1 = w_2 = 2, w_3 = 0$. The multiplicities of both codes (call them $C_{3'}$ and $D_{3'}$) change accordingly:

8	8	8	6
0	0	0	0
1	2	4	0
2	4	0	0
3	6	4	0
4	0	0	0
5	2	4	0
6	4	0	0
7	6	4	0

2	2	2	2	6	6	10
0	0	0	0	0	0	0
4	4	4	4	0	0	0
0	0	4	4	4	4	0
4	4	0	0	4	4	0
0	4	0	4	0	4	4
4	0	4	0	0	4	4
0	4	4	0	4	0	4
4	0	0	4	4	0	4

Now the symmetrized enumerators are:

$$\begin{aligned}\mathrm{se}_{C_{3'}} &= 4Z_0^8Z_1^8Z_2^8Z_3^6 + 2Z_1^8Z_2^8Z_3^{14} + Z_2^8Z_3^{22} + Z_3^{30},\\ \mathrm{se}_{D_{3'}} &= 4Z_2^{20}Z_3^{10} + 2Z_2^{16}Z_3^{14} + Z_2^8Z_3^{22} + Z_3^{30}.\end{aligned}$$

The w-weight enumerators are

$$\begin{aligned}\mathrm{wwe}_{C_{3'}} &= 1 + t^{16} + 2t^{32} + 4t^{40},\\ \mathrm{wwe}_{D_{3'}} &= 1 + t^{16} + 2t^{32} + 4t^{40},\\ \mathrm{wwe}_{C_{3'}^\perp} &= 1 + 24t + 1074t^2 + 36584t^3 + \cdots,\\ \mathrm{wwe}_{D_{3'}^\perp} &= 1 \qquad\quad + 1354t^2 + 34304t^3 + \cdots.\end{aligned}$$

For this w, $\mathring{I} = \{0\}$. Lemma 9.3 implies that $\delta_{3'} = -4\Delta_{3'} = 24$, which matches the computed value.

Example 11.9. For a final example, still use $R = \mathbb{Z}/8\mathbb{Z}$, but change the weight to $w_0 = 1, w_1 = 2, w_2 = 1, w_3 = 0$, which lies outside the scope of the main results given in previous sections; cf., Example 10.13. The multiplicities of both codes (call them $C_{3''}$ and $D_{3''}$) change accordingly:

4	4	4	11
0	0	0	0
1	2	4	0
2	4	0	0
3	6	4	0
4	0	0	0
5	2	4	0
6	4	0	0
7	6	4	0

1	1	1	1	5	5	9
0	0	0	0	0	0	0
4	4	4	4	0	0	0
0	0	4	4	4	4	0
4	4	0	0	4	4	0
0	4	0	4	0	4	4
4	0	4	0	0	4	4
0	4	4	0	4	0	4
4	0	0	4	4	0	4

Now the symmetrized enumerators are:

$$\begin{aligned}
\mathrm{se}_{C_{3''}} &= 4Z_0^4 Z_1^4 Z_2^4 Z_3^{11} + 2Z_1^4 Z_2^4 Z_3^{15} + Z_2^4 Z_3^{19} + Z_3^{23}, \\
\mathrm{se}_{D_{3''}} &= 4Z_2^{16} Z_3^7 + 2Z_2^{12} Z_3^{11} + Z_2^4 Z_3^{19} + Z_3^{23}.
\end{aligned}$$

The w-weight enumerators are

$$\begin{aligned}
\mathrm{wwe}_{C_{3''}} &= 1 + t^4 + 2t^{12} + 4t^{16}, \\
\mathrm{wwe}_{D_{3''}} &= 1 + t^4 + 2t^{12} + 4t^{16}, \\
\mathrm{wwe}_{C_{3''}^\perp} &= 1 + 63t + 2111t^2 + 51635t^3 + \cdots, \\
\mathrm{wwe}_{D_{3''}^\perp} &= 1 + 23t + 1195t^2 + 38431t^3 + \cdots.
\end{aligned}$$

In this example $0 \in \mathring{I}$, and Lemma 9.3 implies that $\delta_{3''} = 44 - 4 = 40$, which matches the computed value. The weight w does not satisfy the hypotheses of Corollaries 9.4, 9.5 or Theorem 10.2; nonetheless, we see that w does not respect duality. Example 10.13 uses $k = 2$ to reach the same conclusion.

Part 3. Matrix Rings over Finite Fields

12. Matrix modules, their orbits, and a positive result

We begin our study of matrix rings by describing certain matrix modules, the orbits of the group of units, the homogeneous weight, and the MacWilliams identities for $M_{2\times 2}(\mathbb{F}_2)$.

Fix integers k, m with $2 \le k \le m$. Let $R = M_{k\times k}(\mathbb{F}_q)$ be the ring of $k \times k$ matrices over a finite field $\mathbb{F}_q$, and let $M = M_{k\times m}(\mathbb{F}_q)$ be the left

R-module of $k \times m$ matrices over $\mathbb{F}_q$. Both R and M are vector spaces over $\mathbb{F}_q$. The scalar multiplication of R on M is the multiplication of matrices. The group $\mathrm{GL}(k, \mathbb{F}_q)$ of invertible $k \times k$ matrices is the group of units $\mathcal{U} = \mathcal{U}(R)$ of R; $\mathcal{U}$ acts on M on the left via matrix multiplication. The size of $\mathrm{GL}(k, \mathbb{F}_q)$ is:

$$|\mathrm{GL}(k, \mathbb{F}_q)| = (q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}). \tag{12.1}$$

Our first objective is to understand the cyclic left R-submodules of M and the $\mathcal{U}$-orbits in M.

Given a $k \times m$ matrix $x \in M$, denote by $\mathrm{rowsp}(x)$ the row space of x, i.e., the $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^m$ spanned by the rows of x. A left R-module is *cyclic* if it is generated by one element; i.e., it has the form Rx for some element x in the module. Denote the orbit of $x \in M$ under the action of $\mathcal{U}$ by $\mathrm{orb}(x)$ or $[x]$; denote the rank of a matrix x by $\mathrm{rk}\, x$.

Lemma 12.2. *Let $x \in M = M_{k\times m}(\mathbb{F}_q)$. Then*

- *for $y \in M$, $y \in Rx$ if and only if $\mathrm{rowsp}(y) \subseteq \mathrm{rowsp}(x)$;*
- *for $y \in M$, $Ry = Rx$ if and only if $\mathrm{rowsp}(y) = \mathrm{rowsp}(x)$ if and only if $\mathrm{orb}(y) = \mathrm{orb}(x)$;*
- *if $y \in \mathrm{orb}(x)$, then $\mathrm{rk}\, y = \mathrm{rk}\, x$.*

Proof. If $y = rx$, then the rows of y are linear combinations of the rows of x. This implies $\mathrm{rowsp}(y) \subseteq \mathrm{rowsp}(x)$. Conversely, if $\mathrm{rowsp}(y) \subseteq \mathrm{rowsp}(x)$, then each row of y is a linear combination of the rows of x, say $y_i = \sum_{j=1}^k r_{ij} x_j$, for some $r_{ij} \in \mathbb{F}_q$, where the rows of x and y are denoted with subscripts. Define $r \in R$ by $r = (r_{ij})$; then $y = rx$.

For the second item, apply the first item twice, symmetrically in y and x. When $\mathrm{rowsp}(y) = \mathrm{rowsp}(x)$, both x and y row reduce to the same row-reduced echelon form, which means they are in the same $\mathcal{U}$-orbit. □

Let $\mathcal{P}_M$ be the partially ordered set (poset) of all cyclic left R-submodules of $M = M_{k\times m}(\mathbb{F}_q)$, and let $\mathcal{P}_{k,m}$ be the poset of all linear subspaces of dimension at most k in $\mathbb{F}_q^m$. Define $\rho : \mathcal{P}_M \to \mathcal{P}_{k,m}$ by $\rho(Rx) = \mathrm{rowsp}(x)$; ρ is well-defined by Lemma 12.2. Conversely, given a linear subspace $V \subseteq \mathbb{F}_q^m$, define

$$\psi(V) = \{x \in M : \mathrm{rowsp}(x) \subseteq V\}.$$

Proposition 12.3. *When $\dim V \leq k$, $\psi(V)$ is a cyclic left R-submodule of M. The map $\rho : \mathcal{P}_M \to \mathcal{P}_{k,m}$ is an isomorphism of posets, with inverse given by ψ.*

Proof. Suppose $\dim V \le k$. Choose a basis of V, and define $x \in M$ to have the chosen basis of V as its first $\dim V$ rows, followed by rows of zeros. Then $\mathrm{rowsp}(x) = V$. By Lemma 12.2, $\psi(V) = Rx$ is a cyclic module. The argument also shows that ρ is surjective. Lemma 12.2 implies that ρ is injective and preserves inclusion. $\square$

Corollary 12.4. *The orbits* $\mathrm{orb}(x)$*,* $x \in M$*, of the left action of* $\mathcal{U} = \mathrm{GL}(k, \mathbb{F}_q)$ *on* $M = M_{k\times m}(\mathbb{F}_q)$ *are in one-to-one correspondence with the linear subspaces of dimension at most k contained in $\mathbb{F}_q^m$. More precisely, the orbits of matrices of rank j are in one-to-one correspondence with linear subspaces of dimension j in $\mathbb{F}_q^m$. The linear subspace corresponding to* $\mathrm{orb}(x)$ *is* $\mathrm{rowsp}(x)$*.*

There are similar results for linear functionals on M. A *linear functional* on M is a homomorphism $\lambda : M \to R$ of left R-modules; inputs will be written on the left, so that $(rx)\lambda = r(x\lambda)$ for $r \in R$ and $x \in M$. The collection of all linear functionals is denoted $M^\sharp = \mathrm{Hom}_R(M, R)$; $M^\sharp$ is a right R-module, with addition defined point-wise and λr defined by $x(\lambda r) = (x\lambda)r$, where $\lambda \in M^\sharp$, $r \in R$, $x \in M$. When $M = M_{k\times m}(\mathbb{F}_q)$, then $M^\sharp = M_{m\times k}(\mathbb{F}_q)$, with the evaluation $x\lambda \in R$, $x \in M$, $\lambda \in M^\sharp$, being matrix multiplication.

The orbits $\mathrm{orb}(\lambda)$, $\lambda \in M^\sharp$, of the right action of $\mathcal{U} = \mathrm{GL}(k, \mathbb{F}_q)$ on $M^\sharp = M_{m\times k}(\mathbb{F}_q)$ are in one-to-one correspondence with the linear subspaces of dimension at most k contained in $\mathbb{F}_q^m$. More precisely, the orbits of linear functionals of rank j are in one-to-one correspondence with linear subspaces of dimension j in $\mathbb{F}_q^m$. The linear subspace corresponding to $\mathrm{orb}(\lambda)$ is the column space $\mathrm{colsp}(\lambda)$.

Remark 12.5. Given a linear functional $\lambda \in M^\sharp$, i.e., $\lambda : M \to R$, its kernel consists of $x \in M$ such that $\mathrm{rowsp}(x) \subseteq \mathrm{colsp}(\lambda)^\perp$. Here, for a linear subspace $Y \subseteq \mathbb{F}_q^m$, denote by $Y^\perp$ its orthogonal with respect to the standard dot product on $\mathbb{F}_q^m$. Given $\lambda_i \in M^\sharp$, $i = 1, 2, \dots, n$, define $\Lambda : M \to R^n$ by $x\Lambda = (x\lambda_1, \dots, x\lambda_n) \in R^n$. Then $\ker \Lambda$ consists of $x \in M$ with $\mathrm{rowsp}(x) \subseteq \cap\, \mathrm{colsp}(\lambda_i)^\perp = (\mathrm{colsp}(\lambda_1) + \cdots + \mathrm{colsp}(\lambda_n))^\perp$. This latter uses the fact that $(X+Y)^\perp = X^\perp \cap Y^\perp$, for linear subspaces $X, Y \subseteq \mathbb{F}_q^m$. In particular, if $\mathrm{colsp}(\lambda_1), \dots, \mathrm{colsp}(\lambda_n)$ span $\mathbb{F}_q^m$, then Λ is injective.

We record the number of $\mathcal{U}$-orbits and their sizes, depending on their rank. The *q-binomial coefficient* $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q$ for $1 \le j \le m$ is defined by

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{(q^m - 1)(q^{m-1} - 1)\cdots(q^{m-j+1} - 1)}{(q^j - 1)(q^{j-1} - 1)\cdots(q - 1)}.$$

For $m \ge 0$, $\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right]_q = 1$. If $j < 0$ or $j > m$, then $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q = 0$.

Proposition 12.6. *There is one orbit of rank* 0*, of size* $\mathscr{S}_0 = 1$*, in* M*. For any integer* j*,* $1 \le j \le k$*, all orbits of rank* j *matrices in* M *have the same size. The number and size* $\mathscr{S}_j$ *of orbits of rank* j *matrices in* M *are:*

number	size
$\begin{bmatrix} m \\ j \end{bmatrix}_q$	$\mathscr{S}_j = \prod_{i=0}^{j-1}(q^k - q^i)$

The ratio $\mathscr{S}_{j+1}/\mathscr{S}_j$ *satisfies* $\mathscr{S}_{j+1}/\mathscr{S}_j = q^k - q^j$*, for* $0 \le j \le k-1$*.*

The size of a cyclic submodule Rr *depends only on* $\operatorname{rk} r$*:* $|Rr| = q^{k \operatorname{rk} r}$*.*

We note that the sizes $\mathscr{S}_j$ and $|Rr|$ do not depend on m.

Proof. It is well-known (e.g., [22, Theorem 3.2.6]) that the q-binomial coefficient $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q$ counts the number of j-dimensional linear subspaces in $\mathbb{F}_q^m$, so the number of orbits follows from Corollary 12.4.

In addition to the left action of $\mathcal{U} = \mathrm{GL}(k, \mathbb{F}_q)$ on $M = M_{k\times m}(\mathbb{F}_q)$, there is also a right action of $\mathrm{GL}(m, \mathbb{F}_q)$ on M via matrix multiplication; both actions preserve rank. As matrix multiplication is associative, these two actions commute. Thus right multiplication by $P \in \mathrm{GL}(m, \mathbb{F}_q)$ maps the $\mathcal{U}$-orbit $\operatorname{orb}(x)$, $x \in M$, to $\operatorname{orb}(xP)$, and the two orbits have the same size.

Suppose the integer j satisfies $1 \le j \le k$. Choose $x_0 \in M$ to have the first j standard basis vectors (i.e., $(1, 0, \ldots, 0)$, etc.) as its first j rows, with the remaining rows being all zeros. Pick any $y \in M$ with $\operatorname{rk} y = j$. Choose a basis of $\operatorname{rowsp}(y)$, and extend it to a basis of $\mathbb{F}_q^m$. Use this basis of $\mathbb{F}_q^m$ as the rows of a matrix $P \in \mathrm{GL}(m, \mathbb{F}_q)$. Then the rows of x_0P consist of the chosen basis of $\operatorname{rowsp}(y)$, followed by $k-j$ zero-rows. Thus we have $\operatorname{rowsp}(x_0P) = \operatorname{rowsp}(y)$, so that $\operatorname{orb}(y) = \operatorname{orb}(x_0P)$, by Lemma 12.2. We conclude that $|\operatorname{orb}(y)| = |\operatorname{orb}(x_0P)| = |\operatorname{orb}(x_0)|$, so that all orbits of rank j matrices have the same size.

As for the size $\mathscr{S}_j$ of an orbit of rank j matrices, it is enough to calculate $|\operatorname{orb}(x_0)|$ using $|\operatorname{orb}(x_0)| = |\mathcal{U}|/|\operatorname{stab}(x_0)|$, where $\operatorname{stab}(x_0) = \{u \in \mathcal{U} : ux_0 = x_0\}$ is the stabilizer subgroup of x_0. Then $u \in \operatorname{stab}(x_0)$ has the form

$$u = \left[\begin{array}{c|c} I_j & B \\ \hline 0 & D \end{array}\right],$$

with I_j the $j \times j$ identity matrix, B arbitrary, and D invertible. Then

$$|\operatorname{orb}(x_0)| = |\mathrm{GL}(k, \mathbb{F}_q)|/(q^{j(k-j)}|\mathrm{GL}(k-j, \mathbb{F}_q)|),$$

which simplifies as claimed.

The same argument using the right action of $\mathrm{GL}(m, \mathbb{F}_q)$ implies that the size of a cyclic submodule $Rr \subseteq M$ depends only on $\operatorname{rk} r$. Indeed, suppose $\operatorname{rk} r_1 = \operatorname{rk} r_2$. By row and column operations, there are units $u_1 \in \mathrm{GL}(k, \mathbb{F}_q)$ and $u_2 \in \mathrm{GL}(m, \mathbb{F}_q)$ such that $r_2 = u_1 r_1 u_2$. By Lemma 12.2, $Rr_2 = Rr_1 u_2$. Right multiplication by u_2 maps Rr_1 isomorphically to $Rr_1 u_2$. We conclude that $|Rr_2| = |Rr_1 u_2| = |Rr_1|$. For $j = 1, 2, \ldots, k$, let $r \in M$ be the following matrix of rank j:

$$r = \left[\begin{array}{c|c} I_j & 0 \\ \hline 0 & 0 \end{array}\right].$$

Then Rr consists of all matrices in M whose last $m - j$ columns are zero. The cyclic submodule Rr has size $|Rr| = q^{kj} = q^{k \operatorname{rk} r}$. $\square$

We now turn our attention to the homogeneous weight on $R = M_{k\times k}(\mathbb{F}_q)$. Because of Proposition 12.3, the Möbius function for the poset of principal left ideals of R equals the Möbius function for the poset of linear subspaces of $\mathbb{F}_q^k$, which, following [11, (2.7)], is

$$\mu(V_1, V_2) = (-1)^c q^{\binom{c}{2}}, \quad V_1 \subseteq V_2 \subseteq \mathbb{F}_q^k, \tag{12.7}$$

where $c = \dim V_2 - \dim V_1$ is the codimension of V_1 in V_2.

Equation (2.8) yields the following formula for the homogeneous weight w on R; this formula also appears in [14, Proposition 7]. We write $\rho = \operatorname{rk}(r)$, $r \in R$:

$$\mathrm{w}(r) = \begin{cases} 0, & \rho = 0, \\ \zeta\left(1 - \frac{(-1)^\rho}{(q^k-1)(q^{k-1}-1)\cdots(q^{k-\rho+1}-1)}\right), & \rho > 0. \end{cases} \tag{12.8}$$

Note that $\mathrm{w}(r)$ depends only on ρ, which is consistent with w being constant on left $\mathcal{U}$-orbits. Write w_ρ for the common value of $\mathrm{w}(r)$ where $\operatorname{rk} r = \rho$. By choosing ζ appropriately, namely

$$\zeta = (q^k - 1)(q^{k-1} - 1) \cdots (q - 1)/q,$$

the homogeneous weight will have integer values.

Example 12.9. For $M_{2\times 2}(\mathbb{F}_q)$, the homogeneous weight is

	w_0	w_1	w_2
general q	0	$q^2 - q$	$q^2 - q - 1$
$q = 2$	0	2	1
$q = 3$	0	6	5

with average weight $\zeta = (q^2 - 1)(q - 1)/q$ in general, so that $\zeta = 3/2$ for $q = 2$, and $\zeta = 16/3$ for $q = 3$.

Example 12.10. For $M_{3\times 3}(\mathbb{F}_q)$, the homogeneous weight is:

	w_0	w_1	w_2	w_3
q	0	$q^5 - q^4 - q^3 + q^2$	$q^5 - q^4 - q^3 + q$	$q^5 - q^4 - q^3 + q + 1$
$q = 2$	0	12	10	11
$q = 3$	0	144	138	139

,

with average weight $\zeta = (q^3 - 1)(q^2 - 1)(q - 1)/q$ in general, so that $\zeta = 21/2$ for $q = 2$, and $\zeta = 416/3$ for $q = 3$.

Lemma 12.11. *The homogeneous weight on $M_{k\times k}(\mathbb{F}_q)$ satisfies*

$$0 = \mathrm{w}_0 < \mathrm{w}_2 < \mathrm{w}_4 < \cdots < \zeta < \cdots < \mathrm{w}_3 < \mathrm{w}_1.$$

Moreover, $2\mathrm{w}_2 - \mathrm{w}_1 > 0$ *for all* $k \geq 2$, $q \geq 2$, except *for* $k = q = 2$, *where* $2\mathrm{w}_2 - \mathrm{w}_1 = 0$.

Proof. The denominator in (12.8) is an increasing function of $\rho = \mathrm{rk}(r)$. This, together with the alternating sign of $(-1)^\rho$, yields the inequalities among the w_ρ. By moving ζ to the left side of (12.8), one sees that $\mathrm{w}_\rho - \zeta = -(-1)^\rho \zeta/((q^k - 1) \cdots (q^{k-\rho+1} - 1))$. This implies that $\mathrm{w}_\rho - \zeta$ is positive when ρ is odd and negative when ρ is even.

For $k \geq 2$, one calculates that

$$2\mathrm{w}_2 - \mathrm{w}_1 = \zeta \frac{(q^k - 2)(q^{k-1} - 1) - 2}{(q^k - 1)(q^{k-1} - 1)}.$$

Using $k \geq 2$ and $q \geq 2$, the numerator satisfies

$$(q^k - 2)(q^{k-1} - 1) - 2 \geq (q^2 - 2)(q - 1) - 2 = q(q + 1)(q - 2).$$

This last expression is positive when $q > 2$ and vanishes when $q = 2$. Even for $q = 2$, the earlier inequality is strict when $k > 2$. Thus, $2\mathrm{w}_2 - \mathrm{w}_1 > 0$, except for $k = q = 2$, where $2\mathrm{w}_2 - \mathrm{w}_1 = 0$. $\square$

Using Lemma 5.3, we see that any nonzero vector v with $\mathrm{w}(v) < \mathrm{w}_1$ must be a singleton. Any nonzero vector with $\mathrm{w}(v) = \mathrm{w}_1$ must be a singleton (or a doubleton, i.e., two nonzero entries, only for $M_{2\times 2}(\mathbb{F}_2)$).

Theorem 12.12. *The MacWilliams identities hold for the homogeneous weight over $R = M_{2\times 2}(\mathbb{F}_2)$. For a linear code $C \subseteq R^n$,*

$$\mathrm{howe}_{C^\perp}(X, Y) = \frac{1}{|C|} \mathrm{howe}_C(X + 3Y, X - Y).$$

Proof. As in the proof of Theorem 6.5, we provide details to be used in the argument outlined in Appendix A.

From Example 12.9, we have that $\mathrm{w}_0 = 0$, $\mathrm{w}_1 = 2$, and $\mathrm{w}_2 = 1$. A generating character for R is $\chi(r) = (-1)^{\mathrm{tr}\, r}$, $r \in R$, where tr is the matrix trace. Define $f : R \to \mathbb{C}[X,Y]$ by $f(r) = X^{2-\mathrm{w}(r)}Y^{\mathrm{w}(r)}$. The value of $f(r)$ depends only on $\operatorname{rk} r$:

$$\begin{array}{c|ccc} \operatorname{rk} r & 0 & 1 & 2 \\ \hline f(r) & X^2 & Y^2 & XY \end{array}.$$

A calculation shows the Fourier transform (A.2) depends only on $\operatorname{rk} r$:

$$\widehat{f}(r) = \begin{cases} X^2 + 9Y^2 + 6XY = (X+3Y)^2, & \operatorname{rk} r = 0, \\ X^2 + Y^2 - 2XY = (X-Y)^2, & \operatorname{rk} r = 1, \\ X^2 - 3Y^2 + 2XY = (X+3Y)(X-Y), & \operatorname{rk} r = 2. \end{cases}$$

Note that $\widehat{f}(r)$ has the form of $f(r)$ with a linear substitution: $X \leftarrow X+3Y$, $Y \leftarrow X-Y$. Applying these details, the rest of the argument in Appendix A carries through. □

There is a Gray map from $M_{2\times 2}(\mathbb{F}_2)$ equipped with the homogeneous weight to $\mathbb{F}_4^2$ equipped with the Hamming weight (of $\mathbb{F}_4$) [2].

13. W-matrix

In this section we determine the W-matrix of (4.5) for a weight w on $R = M_{k\times k}(\mathbb{F}_q)$ that has maximal symmetry.

As usual, let $R = M_{k\times k}(\mathbb{F}_q)$, and suppose w is a weight on R with maximal symmetry. Suppose $r \in R$. By row and column reduction there exist units $u_1, u_2 \in \mathcal{U} = \mathrm{GL}(k, \mathbb{F}_q)$ such that

$$u_1 r u_2 = \left[\begin{array}{c|c} I_\rho & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $\rho = \operatorname{rk} r$. Thus $w(r) = w(u_1 r u_2) = w(\left[\begin{smallmatrix} I_\rho & 0 \\ 0 & 0 \end{smallmatrix}\right])$, which says that the value of $w(r)$ depends only on the rank of r. Write $w_0, w_1, \ldots, w_k$ for the value of w on matrices of rank $0, 1, \ldots, k$, respectively.

Remark 13.1. While $w(0) = 0$ is part of the definition of a weight, some of the results of this section will be more natural to state if we allow w_0 to be viewed as an indeterminate. We will proceed with w_0 as an indeterminate, and later show, in Theorem 13.16 and Corollary 13.18, how the general results are affected when we set $w_0 = 0$.

The information module M will be $M = M_{k\times m}(\mathbb{F}_q)$ with $m \geq k$. Then $\operatorname{Hom}_R(M, R) = M_{m\times k}(\mathbb{F}_q)$, achieved by right multiplication against M; i.e., the evaluation pairing $M \times \operatorname{Hom}_R(M,R) \to R$ sends $x \in M$ and $\lambda \in \operatorname{Hom}_R(M,R)$ to $x\lambda \in R$.

Because of maximal symmetry, the symmetry groups of w are $G_{\mathrm{lt}} = G_{\mathrm{rt}} = \mathrm{GL}(k, \mathbb{F}_q)$. The orbit space $\mathcal{O} = G_{\mathrm{lt}} \backslash M$ is represented by row-reduced echelon matrices of size $k \times m$, and $\mathcal{O}^\sharp = \operatorname{Hom}_R(M,R)/G_{\mathrm{rt}}$

is represented by column-reduced echelon matrices of size $m \times k$. The matrix transpose maps $\mathcal{O} \leftrightarrow \mathcal{O}^\sharp$ bijectively. The sets $\mathcal{O}$ and $\mathcal{O}^\sharp$ are partitioned by rank, and elements of $\mathcal{O}$ and $\mathcal{O}^\sharp$ correspond to linear subspaces of $\mathbb{F}_q^m$ of dimension at most k, by Corollary 12.4. (Left orbits in $\mathcal{O}$ are viewed in terms of row spaces; their transposes, right orbits in $\mathcal{O}^\sharp$, are viewed in terms of column spaces.)

The rows of W are indexed by elements $[x] \in \mathcal{O}$, ordered so that ranks go from 0 to k. Similarly, the columns of W are indexed by elements $[\lambda] \in \mathcal{O}^\sharp$, ordered to match $\mathcal{O}$ under the bijection $\mathcal{O} \leftrightarrow \mathcal{O}^\sharp$. The $[x], [\lambda]$-entry of W is simply $w(x\lambda)$, i.e., the value of w at the evaluation $x\lambda \in R$. The value $w(x\lambda)$ is well-defined by the definition of the symmetry groups. By maximal symmetry, the value $w(x\lambda)$ depends only on the rank $\operatorname{rk}(x\lambda)$. The matrix W is square of size $N_{k,m}$, the number of linear subspaces of dimension at most k in $\mathbb{F}_q^m$.

Suppose $[x] \in \mathcal{O}$ corresponds to the linear subspace $X \subseteq \mathbb{F}_q^m$ and $[\lambda] \in \mathcal{O}^\sharp$ corresponds to Y. We seek to express $\operatorname{rk}(x\lambda)$, and hence $w(x\lambda)$, in terms of X and Y.

For a linear subspace $X \subseteq \mathbb{F}_q^m$, denote by $X^\perp$ its orthogonal with respect to the standard dot product on $\mathbb{F}_q^m$. Then $\dim X^\perp = m - \dim X$ and $(X^\perp)^\perp = X$, for all linear subspaces $X \subseteq \mathbb{F}_q^m$. Also, $(X \cap Y)^\perp = X^\perp + Y^\perp$, for linear subspaces $X, Y \subseteq \mathbb{F}_q^m$.

Lemma 13.2. *For linear subspaces $X, Y \subseteq \mathbb{F}_q^m$ representing $[x] \in \mathcal{O}$ and $[\lambda] \in \mathcal{O}^\sharp$, respectively:*

(1) $\dim X - \dim(X \cap Y^\perp) = \dim Y - \dim(Y \cap X^\perp)$, *and*
(2) $\operatorname{rk}(x\lambda) = \dim X - \dim(X \cap Y^\perp) = \dim Y - \dim(Y \cap X^\perp)$.

Proof. Consider $(X \cap Y^\perp)^\perp = X^\perp + Y$, and compare dimensions:

$$\begin{aligned} m - \dim(X \cap Y^\perp) &= \dim X^\perp + \dim Y - \dim(X^\perp \cap Y) \\ &= m - \dim X + \dim Y - \dim(X^\perp \cap Y). \end{aligned}$$

We conclude that $\dim X - \dim(X \cap Y^\perp) = \dim Y - \dim(X^\perp \cap Y)$.

Choose as a representative x a $k \times m$ matrix whose first $\dim(X \cap Y^\perp)$ rows form a basis for $X \cap Y^\perp$, whose next $\dim X - \dim(X \cap Y^\perp)$ rows complete to a basis of X, and whose remaining rows are zeros. Choose a representative λ by reversing the roles of X and Y: its first $\dim(Y \cap X^\perp)$ columns form a basis for $Y \cap X^\perp$, its next $\dim Y - \dim(Y \cap X^\perp)$ columns complete to a basis of Y, and its remaining columns are zeros. Then $x\lambda$ has the form

$$x\lambda = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & Z & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

where Z is a square matrix of size $(\dim X - \dim(X \cap Y^\perp)) \times (\dim Y - \dim(X^\perp \cap Y)$. The matrix Z is nonsingular. If not, there exists a nonzero vector v with $vZ = 0$. Then $[0|v|0]x \in Y^\perp$, which violates the choice of basis of $X \cap Y^\perp$ in the construction of x. The formula for $\operatorname{rk}(x\lambda)$ now follows. $\square$

The matrix W is symmetric when we use the bijection $\mathcal{O} \leftrightarrow \mathcal{O}^\sharp$ to align the indexing.

Because some of our later results depend upon inverting W, we need to understand when the matrix W is invertible. We will be able to transform W into a block diagonal format by making use of the Möbius function of the poset $\mathcal{P}_{k,m}$ of linear subspaces of dimension at most k in $\mathbb{F}_q^m$. Versions of this block diagonal format can be found in [9, §4], [28, Theorem 9.6], and [29, §6].

Recall that we index the rows and columns of W by linear subspaces of dimension at most k in $\mathbb{F}_q^m$, with ranks increasing from 0 to k.

Define a matrix P, with rows and columns indexed by linear subspaces of dimension $\leq k$ in $\mathbb{F}_q^m$, using the same ordering as for W. The entry $P_{\alpha,\beta}$ is given by

$$P_{\alpha,\beta} = \begin{cases} \mu(0,\beta) = (-1)^{\dim \beta} q^{\binom{\dim \beta}{2}}, & \text{if } \beta \subseteq \alpha, \\ 0, & \text{if } \beta \not\subseteq \alpha. \end{cases} \tag{13.3}$$

Because we are ordering rows and columns so that ranks increase, we see that P is lower triangular. Its diagonal entries are $P_{\alpha,\alpha} = (-1)^{\dim \alpha} q^{\binom{\dim \alpha}{2}} \neq 0$. Thus P is invertible over $\mathbb{Q}$.

For $j = 0, 1, \ldots, k$, define an incidence matrix $\mathscr{I}_j$ over $\mathbb{Q}$, square of size $\left[{m \atop j} \right]_q$, with rows and columns indexed by linear subspaces of dimension j in $\mathbb{F}_q^m$, using the dimension j portion of the ordering used for W and P. The α, δ-entry of $\mathscr{I}_j$ is given by

$$(\mathscr{I}_j)_{\alpha,\delta} = \begin{cases} 1, & \alpha \cap \delta^\perp = 0, \\ 0, & \alpha \cap \delta^\perp \neq 0. \end{cases}$$

The incidence matrices $\mathscr{I}_j$ are invertible by [29, Proposition 6.7].

Our main objective in this section is to prove the next theorem.

Theorem 13.4. *For positive integers $2 \leq k \leq m$ and a weight w on $M_{k\times k}(\mathbb{F}_q)$ having maximal symmetry and w_0 indeterminate, we have*

$$PWP^\top = \begin{bmatrix} c_0\mathscr{I}_0 & 0 & \cdots & 0 \\ 0 & c_1\mathscr{I}_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c_k\mathscr{I}_k \end{bmatrix},$$

where, for $j = 0, 1, \ldots, k$*,*

$$c_j = (-1)^j q^{\binom{j}{2}} \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q w_\ell. \tag{13.5}$$

Before we prove Theorem 13.4, we prove some preliminary lemmas and propositions that will be used in the proof. We begin by quoting the well-known Cauchy Binomial Theorem, e.g., [22, Theorem 3.2.4].

Theorem 13.6 (Cauchy Binomial Theorem)**.** *For a positive integer* k*,*

$$\prod_{i=0}^{k-1}(1 + xq^i) = \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q x^j.$$

In particular, using $x = -1$*, for* k *positive,*

$$\sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q = 0. \tag{13.7}$$

In a vector space, a *frame* is an ordered set of linearly independent vectors; if there are b such vectors, we call the frame a b-frame.

Lemma 13.8. *Let* V *be a vector space over* $\mathbb{F}_q$ *with* $\dim V = v$*, and let* D *be a linear subspace of* V *with* $\dim D = d$*. Then*

$$|\{B \subseteq V : \dim B = b, B \cap D = 0\}| = q^{bd} \begin{bmatrix} v-d \\ b \end{bmatrix}_q.$$

Proof. We count the number of b-frames outside of D, and divide by $|\mathrm{GL}(b, \mathbb{F}_q)|$, (12.1). Then, factoring out b factors of q^d from

$$\frac{(q^v - q^d)(q^v - q^{d+1}) \cdots (q^v - q^{d+b-1})}{(q^b - 1)(q^b - q) \cdots (q^b - q^{b-1})}$$

yields the stated result. $\square$

Lemma 13.9. *Let* V *be a vector space over* $\mathbb{F}_q$ *with* $\dim V = v$*, and let* D *be a linear subspace of* V *with* $\dim D = d$*. For any* $j = 1, 2, \ldots, d$*,*

$$|\{B \subseteq V : \dim B = b, \dim(B \cap D) = j\}| = q^{(b-j)(d-j)} \begin{bmatrix} d \\ j \end{bmatrix}_q \begin{bmatrix} v-d \\ b-j \end{bmatrix}_q.$$

Proof. The count equals the number of j-dimensional subspaces $J \subseteq D$ times the number of B's of dimension b with $B \cap D = J$. The number of j-dimensional subspaces of D is $\left[\begin{smallmatrix} d \\ j \end{smallmatrix}\right]_q$. The set of b-dimensional subspaces $B \subseteq V$ with $B \cap D = J$ is in one-to-one correspondence with the set of $(b-j)$-dimensional subspaces of V/J that intersect D/J trivially. By Lemma 13.8, the number of such subspaces is $q^{(b-j)(d-j)} \left[\begin{smallmatrix} v-d \\ b-j \end{smallmatrix}\right]_q$. $\square$

Lemma 13.9 sharpens [29, Lemma 6.9]; the latter's $C_1(b)$ is $q^{b(m-b)}$.

Lemma 13.10. *Let V be a vector space over $\mathbb{F}_q$ with $\dim V = v$, and let A and D be linear subspaces of V with $\dim A = a$ and $\dim D = d$. If $\dim(A \cap D) = i$, then, for any $j = 0, 1, \ldots, i$,*

$$|\{B \subseteq A : \dim B = b, \dim(B \cap D) = j\}| = q^{(b-j)(i-j)} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} a-i \\ b-j \end{bmatrix}_q .$$

Proof. Note that $B \cap D = B \cap (A \cap D)$. Apply Lemmas 13.8 and 13.9 with ambient space A and subspace $A \cap D$. $\square$

Lemma 13.11. *Suppose $\alpha, \delta \subseteq \mathbb{F}_q^m$ are linear subspaces. Then $\dim(\alpha \cap \delta^\perp) \geq \dim \alpha - \dim \delta$. If $\dim \alpha > \dim \delta$, then $\dim(\alpha \cap \delta^\perp) > 0$.*

Proof. Using $\alpha + \delta^\perp \subseteq \mathbb{F}_q^m$, compare dimensions:

$$m \geq \dim(\alpha + \delta^\perp) = \dim \alpha + m - \dim \delta - \dim(\alpha \cap \delta^\perp),$$

from which the result follows. $\square$

Proposition 13.12. *If $\dim(\alpha \cap \delta^\perp) > 0$, then $(PW)_{\alpha,\delta} = 0$. If $\alpha \cap \delta^\perp = 0$, then, writing $a = \dim \alpha$,*

$$(PW)_{\alpha,\delta} = \sum_{\beta \subseteq \alpha} \mu(0, \beta) w_{\dim \beta} = \sum_{r=0}^{a} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} a \\ r \end{bmatrix}_q w_r .$$

In particular, the matrix PW is block upper triangular.

Likewise, if $\dim(\gamma \cap \beta^\perp) > 0$, then $(WP^\top)_{\beta,\gamma} = 0$. If $\gamma \cap \beta^\perp = 0$, then, writing $c = \dim \gamma$,

$$(WP^\top)_{\beta,\gamma} = \sum_{\epsilon \subseteq \gamma} \mu(0, \epsilon) w_{\dim \epsilon} = \sum_{s=0}^{c} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} c \\ s \end{bmatrix}_q w_s .$$

In particular, the matrix $WP^\top$ is block lower triangular.

Proof. From the definition of the matrix P and Lemma 13.2,

$$(PW)_{\alpha,\delta} = \sum_{\beta \subseteq \alpha} \mu(0, \beta) w_{\dim \beta - \dim(\beta \cap \delta^\perp)} .$$

In this formula, the subscript $\dim \beta - \dim(\beta \cap \delta^\perp) = \dim \delta - \dim(\delta \cap \beta^\perp) \leq \dim \delta - \dim(\delta \cap \alpha^\perp) = \dim \alpha - \dim(\alpha \cap \delta^\perp)$, as $\beta \subseteq \alpha$, so that $\alpha^\perp \subseteq \beta^\perp$. Writing $i = \dim(\alpha \cap \delta^\perp)$, we see that the subscript

$\dim\beta - \dim(\beta\cap\delta^\perp) \le a-i$. Thus

$$\begin{aligned}(PW)_{\alpha,\delta} &= \sum_{r=0}^{a-i} \sum_{\substack{\beta\subseteq\alpha \\ \dim\beta-\dim(\beta\cap\delta^\perp)=r}} \mu(0,\beta) w_r \\ &= \sum_{r=0}^{a-i}\sum_{j=0}^{i} \sum_{\substack{\beta\subseteq\alpha \\ \dim\beta=r+j \\ \dim(\beta\cap\delta^\perp)=j}} \mu(0,\beta) w_r .\end{aligned}$$

By Lemma 13.10 and (12.7), the coefficient C_r of w_r is

$$\begin{aligned}C_r &= \sum_{j=0}^{i} q^{r(i-j)} \begin{bmatrix} i \\ j\end{bmatrix}_q \begin{bmatrix} a-i \\ r\end{bmatrix}_q (-1)^{r+j} q^{\binom{r+j}{2}} \\ &= (-1)^r q^{\binom{r}{2}} q^{ir} \begin{bmatrix} a-i \\ r\end{bmatrix}_q \left(\sum_{j=0}^{i} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} i \\ j\end{bmatrix}_q\right) \\ &= \begin{cases} (-1)^r q^{\binom{r}{2}} \left[\begin{smallmatrix} a \\ r\end{smallmatrix}\right]_q, & i=0, \\ 0, & i>0.\end{cases}\end{aligned}$$

To simplify, we used the identity $\binom{r+j}{2} = \binom{r}{2} + jr + \binom{j}{2}$ and (13.7).

If $\dim\delta < \dim\alpha$, Lemma 13.11 implies $i = \dim(\alpha\cap\delta^\perp) > 0$. Using dimension to create blocks, we see that PW is block upper triangular.

Essentially the same arguments yield the results about $WP^\top$. $\square$

Recall that the matrix P is lower triangular, so that $P^\top$ is upper triangular. Thus $PWP^\top$ will be both lower and upper block triangular, hence block diagonal. The exact form of $PWP^\top$ is the next result.

Proposition 13.13. *If* $\dim\alpha \neq \dim\delta$, *then* $(PWP^\top)_{\alpha,\delta} = 0$. *If* $\dim\alpha = \dim\delta = a$, *then*

$$(PWP^\top)_{\alpha,\delta} = \begin{cases} (-1)^a q^{\binom{a}{2}} \sum_{j=0}^{a} (-1)^j q^{\binom{j}{2}} \left[\begin{smallmatrix} a \\ j\end{smallmatrix}\right]_q w_j, & \alpha\cap\delta^\perp = 0, \\ 0, & \alpha\cap\delta^\perp \neq 0.\end{cases}$$

Proof. We first show that $\alpha\cap\delta^\perp\neq 0$ implies $(PWP^\top)_{\alpha,\delta}=0$. Assume $\alpha\cap\delta^\perp \neq 0$. For any $\gamma\subseteq\delta$, we have $\delta^\perp\subseteq\gamma^\perp$, so that $\alpha\cap\delta^\perp\subseteq\alpha\cap\gamma^\perp$. Thus $\alpha\cap\gamma^\perp\neq 0$ for all $\gamma\subseteq\delta$.

Using the definition of P, we see that

$$(PWP^\top)_{\alpha,\delta} = \sum_{\gamma\subseteq\delta} (PW)_{\alpha,\gamma}\mu(0,\gamma). \tag{13.14}$$

By Proposition 13.12, all the $(PW)_{\alpha,\gamma}$-terms in (13.14) vanish, so that $(PWP^\top)_{\alpha,\delta} = 0$, as claimed.

Essentially the same argument using

$$(PWP^\top)_{\alpha,\delta} = \sum_{\beta \subseteq \alpha} \mu(0,\beta)(WP^\top)_{\beta,\delta}$$

shows that $\delta \cap \alpha^\perp \neq 0$ implies $(PWP^\top)_{\alpha,\delta} = 0$.

From Lemma 13.2 we have the equation $\dim \alpha - \dim \delta = \dim(\alpha \cap \delta^\perp) - \dim(\delta \cap \alpha^\perp)$. If $\dim \alpha \neq \dim \delta$, at least one of $\alpha \cap \delta^\perp$ or $\delta \cap \alpha^\perp$ is nonzero. Thus $(PWP^\top)_{\alpha,\delta} = 0$.

At last, suppose $\dim \alpha = \dim \delta$. In this situation, note that $\alpha \cap \delta^\perp = 0$ if and only if $\delta \cap \alpha^\perp = 0$. As above, if $\alpha \cap \delta^\perp \neq 0$, then $(PWP^\top)_{\alpha,\delta} = 0$. If $\alpha \cap \delta^\perp = 0$, then the equality of dimensions yields $\mathbb{F}_q^m = \alpha \oplus \delta^\perp$. For any $\gamma \subsetneq \delta$, we have $\delta^\perp \subsetneq \gamma^\perp$, so that $\alpha \cap \gamma^\perp \neq 0$. By Proposition 13.12, $(PW)_{\alpha,\gamma} = 0$ for $\gamma \subsetneq \delta$. Thus, (13.14) implies that

$$(PWP^\top)_{\alpha,\delta} = (PW)_{\alpha,\delta}\mu(0,\delta),$$

which gives the stated formula. □

Proof of Theorem 13.4. The nonzero entries in Proposition 13.13 depend only on $\dim \alpha$ and equal the c_j in the statement of Theorem 13.4. Whether $\alpha \cap \delta^\perp = 0$ is marked by the incidence matrix $\mathscr{I}_j$. □

Corollary 13.15. *The matrix W is invertible over $\mathbb{Q}$ if and only if*

$$c_j = (-1)^j q^{\binom{j}{2}} \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q w_\ell \neq 0,$$

for all $j = 0, 1, \ldots, k$.

Proof. The matrices P, $P^\top$, and $\mathscr{I}_j$ are all invertible over $\mathbb{Q}$. □

Recall from Remark 13.1 that we have been treating w_0 as an indeterminate. When w_0 is set equal to 0, the first row and first column of the matrix W consist of 0's. Then W cannot be invertible. Equivalently, $c_0 = 0$ in Corollary 13.15.

To get around this lack of invertibility, we make the following adjustments, as in Remark 4.6. Define a matrix W_0 with rows and columns indexed by the *nonzero* elements of $\mathcal{O}$ and $\mathcal{O}^\sharp$, respectively, ordered so that ranks go from 1 to k. The $[x], [\lambda]$-entry is again $w(x\lambda)$, with $w(0) = 0$ now. Similarly, define a matrix P_0 with rows and columns indexed by the nonzero elements of $\mathcal{O}$ and (α, β)-entry given by (13.3). Then the counterparts of Theorem 13.4 and Corollary 13.15 are the following.

Theorem 13.16. *For positive integers $2 \leq k \leq m$ and a weight w on $M_{k\times k}(\mathbb{F}_q)$ having maximal symmetry and $w(0) = 0$, we have*

$$P_0 W_0 P_0^\top = \begin{bmatrix} c_1 \mathscr{I}_1 & 0 & \cdots & 0 \\ 0 & c_2 \mathscr{I}_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c_k \mathscr{I}_k \end{bmatrix},$$

where, for $j = 1, \dots, k$,

$$c_j = (-1)^j q^{\binom{j}{2}} \sum_{\ell=1}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q w_\ell. \tag{13.17}$$

Proof. The relationships between the matrices P and P_0 and between W and W_0, when $w_0 = 0$, are given below. The notations $\operatorname{row}(a)$ and $\operatorname{col}(a)$ mean a row, resp., column, vector, all of whose entries are a.

$$P = \left[\begin{array}{c|c} 1 & \operatorname{row}(0) \\ \hline \operatorname{col}(1) & P_0 \end{array}\right], \quad W|_{w_0=0} = \left[\begin{array}{c|c} 0 & \operatorname{row}(0) \\ \hline \operatorname{col}(0) & W_0 \end{array}\right].$$

Then

$$(PWP^\top)|_{w_0=0} = \left[\begin{array}{c|c} 0 & \operatorname{row}(0) \\ \hline \operatorname{col}(0) & P_0 W_0 P_0^\top \end{array}\right],$$

and the result follows from Theorem 13.4, with $w_0 = 0$. □

Corollary 13.18. *The matrix W_0 is invertible over $\mathbb{Q}$ if and only if*

$$c_j = (-1)^j q^{\binom{j}{2}} \sum_{\ell=1}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q w_\ell \neq 0,$$

for all $j = 1, \dots, k$.

Remark 13.19. The extension property (EP) for w holds when the W_0 map is injective (zero right null space) for all information modules. We see that EP holds if and only if all $c_j \neq 0$ for $j = 1, 2, \dots, k$. See [28, Theorem 9.5]. In particular, when W_0 is invertible, then $W_0 : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is an isomorphism, Remark 4.6.

Remark 13.20. It is possible to generalize Propositions 13.12 and 13.13 to the context of an alphabet $A = M_{k\times\ell}(\mathbb{F}_q)$ and information module $M = M_{k\times m}(\mathbb{F}_q)$, with $m \geq \ell \geq k$. This paper does not require such a level of generality.

14. Locally constant functions

In this section we examine a type of multiplicity function that will feature prominently in the construction of linear codes in Section 16.

As in previous sections, $R = M_{k\times k}(\mathbb{F}_q)$ and $M = M_{k\times m}(\mathbb{F}_q)$, with $k \le m$. The ring R is equipped with a weight w having maximal symmetry and positive integer values. As in Remark 13.1, we will treat w_0 as an indeterminate. Recall that the orbit spaces $\mathcal{O}$ and $\mathcal{O}^\sharp$ are ordered by rank, from 0 to k. Write $\mathcal{O}_i$ and $\mathcal{O}_i^\sharp$ for the collections of orbits of rank equal to i. Recall that orbits are denoted by $\operatorname{orb}(x)$ or $[x]$. Given sets $B \subseteq A$, we denote the indicator function of B by $1_B : A \to \mathbb{Z}$, with

$$1_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

The type of multiplicity function to be considered has the form $1_{\mathcal{O}_i^\sharp}$ or a linear combination of such indicator functions.

Given a multiplicity function $\eta : \mathcal{O}^\sharp \to \mathbb{N}$ (or, more generally, $\eta : \mathcal{O}^\sharp \to \mathbb{Q}$), we refer to $\omega = W\eta$ as the *list of orbit weights* of η. The list ω is a function $\omega : \mathcal{O} \to \mathbb{Q}$. We say that η, resp., ω, is *locally constant* if $\eta = \sum_{j=0}^k a_j 1_{\mathcal{O}_j^\sharp}$, resp., $\omega = \sum_{j=0}^k b_j 1_{\mathcal{O}_j}$, for rational constants a_j, b_j. Said another way, η is locally constant if $\operatorname{rk}\lambda_1 = \operatorname{rk}\lambda_2$ implies $\eta([\lambda_1]) = \eta([\lambda_2])$; i.e., η is constant on each $\mathcal{O}_j^\sharp$. Similar comments apply to ω.

Let $\eta_j = 1_{\mathcal{O}_j^\sharp}$, and $\omega_j = W\eta_j$. We show that ω_j is locally constant.

Proposition 14.1. *Let $\eta_j = 1_{\mathcal{O}_j^\sharp}$. Then $\omega_j = W\eta_j$ is locally constant. If $i = \operatorname{rk} x$, then*

$$\omega_j([x]) = \sum_{d=0}^{\min\{j,m-i\}} q^{(m-i-d)(j-d)} \begin{bmatrix} m-i \\ d \end{bmatrix}_q \begin{bmatrix} i \\ j-d \end{bmatrix}_q w_{j-d}.$$

In particular, when $j = 1$, $i = \operatorname{rk} x = 1, 2, \ldots, k$, and $w_0 = 0$,

$$\omega_1([x]) = q^{m-i} \begin{bmatrix} i \\ 1 \end{bmatrix}_q w_1 = (q^{m-i} + q^{m-i+1} + \cdots + q^{m-1})w_1,$$

is an increasing function of i.

Proof. Because $\omega_j = W\eta_j$, we have

$$\omega_j([x]) = \sum_{[\lambda]\in\mathcal{O}^\sharp} w(x\lambda)\eta_j([\lambda]) = \sum_{[\lambda]\in\mathcal{O}_j^\sharp} w(x\lambda).$$

Using Lemma 13.2, represent $\operatorname{orb}(x) = [x]$ by a linear subspace $X \subseteq \mathbb{F}_q^m$, $\dim X = i$, and $\operatorname{orb}(\lambda) = [\lambda]$ by Y, $\dim Y = j$. Then $w(x\lambda) =$

$w_{i-\dim(X\cap Y^\perp)} = w_{j-\dim(Y\cap X^\perp)}$. Lemma 13.9 now implies

$$\begin{aligned}\omega_j([x]) &= \sum_{d=0}^{\min\{j,m-i\}} |\{Y : \dim Y = j, \dim(Y\cap X^\perp) = d\}| w_{j-d} \\ &= \sum_{d=0}^{\min\{j,m-i\}} q^{(m-i-d)(j-d)} \begin{bmatrix} m-i \\ d \end{bmatrix}_q \begin{bmatrix} i \\ j-d \end{bmatrix}_q w_{j-d}.\end{aligned}$$

When $j = 1$, the formula simplifies as stated. □

Corollary 14.2. *If η is locally constant, then so is $\omega = W\eta$.*

15. $\overline{W}$-matrix

There is an 'averaged' version $\overline{W}$ of the W-matrix that will be useful in our analysis of dual codewords in Section 18. We describe $\overline{W}$ in this section. We continue to assume that the information module is $M = M_{k\times m}(\mathbb{F}_q)$ and that w_0 is indeterminate.

Lemma 15.1. *For $i = 0, 1, \ldots, k$ and $[\lambda] \in \mathcal{O}_j^\sharp$, the value of*

$$\sum_{[x]\in\mathcal{O}_i} W_{[x],[\lambda]} = \sum_{d=0}^{i} q^{(i-d)(m-j-d)} \begin{bmatrix} m-j \\ d \end{bmatrix}_q \begin{bmatrix} j \\ i-d \end{bmatrix}_q w_{i-d}$$

depends only on i and $j = \operatorname{rk}\lambda$.

Proof. The proof of Proposition 14.1 applies, interchanging the roles of $i = \operatorname{rk} x$ and $j = \operatorname{rk}\lambda$. □

Definition 15.2. Define $\overline{W}$ to be the integer $(k+1)\times(k+1)$ matrix

$$\overline{W}_{ij} = \sum_{[x]\in\mathcal{O}_i} W_{[x],[\lambda]}, \quad [\lambda] \in \mathcal{O}_j^\sharp, \tag{15.3}$$

for $i, j = 0, 1, \ldots, k$. The definition is well-defined by Lemma 15.1.

Example 15.4. When $k = 2$ and $m = 3$, we see that

$$\overline{W} = \begin{bmatrix} w_0 & w_0 & w_0 \\ (1+q+q^2)w_0 & (1+q)w_0 + q^2 w_1 & w_0 + (q+q^2)w_1 \\ (1+q+q^2)w_0 & w_0 + (q+q^2)w_1 & (1+q)w_1 + q^2 w_2 \end{bmatrix}.$$

We next formalize the relationship between $\overline{W}$ and W, in order to determine when $\overline{W}$ is invertible.

Recall that W is square of size $N_{k,m}$, the number of linear subspaces of dimension at most k in $\mathbb{F}_q^m$. Define B to be a $(k+1)\times N_{k,m}$ matrix. For $i = 0, 1, \ldots, k$, row i of B is the indicator function $1_{\mathcal{O}_i}$ for the collection of linear subspaces of dimension i in $\mathbb{F}_q^m$. Similarly, define an $N_{k,m} \times (k+1)$ matrix E. For $j = 0, 1, \dots, k$, column j of E is $(1/\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q) 1_{\mathcal{O}_j^\sharp}$. Notice that $BE = I_{k+1}$, and EB is block diagonal, with the block indexed by rank j being $1/\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q$ times the square all-one matrix of size $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q$.

Lemma 15.5. *We have:* $\overline{W} = BWE$.

Proof. Left multiplication by B givens the sums of Lemma 15.1. Right multiplication by E averages those (equal!) sums over $[\lambda] \in \mathcal{O}_j^\sharp$. □

Recall the P-matrix of (13.3).

Lemma 15.6. *We have:* $BPEB = BP$ *and* $EBP^\top E = P^\top E$.

Proof. The matrix BP has rows that are locally constant, and EB acts as the identity when it right multiplies matrices with rows that are locally constant. Similarly, $P^\top E$ has columns that are locally constant, and EB acts as the identity when it left multiplies matrices with columns that are locally constant. □

Our next result is the counterpart of Theorem 13.4. Set $Q_1 = BPE$ and $Q_2 = BP^\top E$. One verifies that Q_1 is lower triangular and Q_2 is upper triangular. Their i,j-entries are

$$(Q_1)_{i,j} = (-1)^j q^{\binom{j}{2}} \begin{bmatrix} m-j \\ i-j \end{bmatrix}_q, \quad (Q_2)_{i,j} = (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q. \tag{15.7}$$

In particular, the diagonal entries of both are $(-1)^j q^{\binom{j}{2}}$, $j = 0, 1, \dots, k$, so that both Q_1 and Q_2 are invertible.

Theorem 15.8. *We have* $Q_1 \overline{W} Q_2 = BPWP^\top E$ *and*

$$Q_1 \overline{W} Q_2 = \begin{bmatrix} c_0 & 0 & \dots & 0 & \dots & 0 \\ 0 & q^{m-1} c_1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & q^{j(m-j)} c_j & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & q^{k(m-k)} c_k \end{bmatrix},$$

where $c_0, c_1, \dots, c_k$ *are in* (13.5).

Proof. The first equation follows from Lemma 15.6. The second equation follows from Proposition 13.13. The factor $q^{j(m-j)}$ arises from counting the number of α of dimension j in $\mathbb{F}_q^m$ that satisfy $\alpha \cap \delta^\perp = 0$ for a fixed δ of dimension j. That count uses Lemma 13.8. □

Corollary 15.9. *The matrix* W *is invertible if and only if* $\overline{W}$ *is invertible.*

When $w_0 = 0$ one can define a $k \times k$ matrix $\overline{W}_0$ using (15.3), but only for $i, j = 1, 2, \ldots, k$. By defining smaller versions of B, E, Q_1, Q_2, one can prove the following results using a proof similar to that of Theorem 13.16.

Theorem 15.10. *Suppose $w_0 = 0$. Then $Q_{0,1}\overline{W}_0 Q_{0,2} = B_0 P_0 W_0 P_0^\top E_0$ and*

$$Q_{0,1}\overline{W}_0 Q_{0,2} = \begin{bmatrix} q^{m-1}c_1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \ldots & q^{j(m-j)}c_j & \ldots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & q^{k(m-k)}c_k \end{bmatrix},$$

where $c_1, \ldots, c_k$ are in (13.5), but with $w_0 = 0$.

Corollary 15.11. *When $w_0 = 0$, the matrix W_0 is invertible if and only if $\overline{W}_0$ is invertible.*

16. Constructions

In this section we construct two linear codes C and D over $R = M_{k\times k}(\mathbb{F}_q)$ with $\mathrm{wwe}_C = \mathrm{wwe}_D$, assuming that w has maximal symmetry, that $w_0 = 0$, and that the associated W_0 matrix is invertible, Corollary 13.18. Both linear codes will have information module $M = M_{k\times m}(\mathbb{F}_q)$, with $m > k$.

Here is a sketch of the main idea behind the construction. Suppose C is the image of $\Lambda : M \to R^n$. Recall that the maximal symmetry hypothesis means that W_Λ is constant on any left $\mathcal{U}$-orbit in the information module M. Suppose Λ has the property that N chosen orbits of rank s have the same value v_1 of W_Λ. Pick one orbit of rank $s+1$, and denote by v_2 its value of W_Λ. Now try to swap values on those orbits: try to find a linear code D, the image of $\Gamma : M \to R^n$, so that W_Γ has value v_2 on the N chosen orbits of rank s, value v_1 on the chosen orbit of rank $s+1$, and $W_\Gamma = W_\Lambda$ on all other orbits. If N times the size of a rank s orbit equals the size of a rank $s+1$ orbit, then $\mathrm{wwe}_C = \mathrm{wwe}_D$, provided such a D exists and provided that C and D have the same length; cf., Remark 3.10. If the codes have different lengths, we append enough zero-functionals to the shorter code so that the lengths become equal. Because $w_0 = 0$, the additional zero-functionals have no effect on the weights.

We first need a few facts about q-binomial coefficients.

Lemma 16.1. *For integers $0 \leq s \leq m$,*

$$\begin{bmatrix} m \\ s \end{bmatrix}_q = \begin{bmatrix} m \\ m-s \end{bmatrix}_q; \quad \begin{bmatrix} m \\ s+1 \end{bmatrix}_q = \frac{q^{m-s}-1}{q^{s+1}-1} \begin{bmatrix} m \\ s \end{bmatrix}_q;$$

$$\begin{bmatrix} m \\ s \end{bmatrix}_q \leq \begin{bmatrix} m \\ s+1 \end{bmatrix}_q, \quad 0 \leq s < m/2;$$

$$\begin{bmatrix} m \\ s \end{bmatrix}_q = q^{m-s} \begin{bmatrix} m-1 \\ s-1 \end{bmatrix}_q + \begin{bmatrix} m-1 \\ s \end{bmatrix}_q, \quad 0 \leq s < m;$$

$$\begin{bmatrix} m \\ s \end{bmatrix}_q = \begin{bmatrix} m-1 \\ s-1 \end{bmatrix}_q + q^s \begin{bmatrix} m-1 \\ s \end{bmatrix}_q, \quad 0 \leq s < m;$$

Proof. Most of the identities are in [22, Chapter 3] or its exercises. The inequality follows from the preceding identity, because the multiplying fraction is at least 1 for $s < m/2$. ☐

Lemma 16.2. *Suppose integers k, m, s satisfy $0 < s < k < m$. Then*

$$q^k \leq \begin{bmatrix} m \\ s \end{bmatrix}_q - \begin{bmatrix} m-1 \\ s \end{bmatrix}_q.$$

Proof. The result is true for $s = 1$: $\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} m-1 \\ 1 \end{smallmatrix}\right]_q = q^{m-1} \geq q^k$. Now suppose $s \geq 2$, so that $2 \leq s \leq m-2$. By symmetry and monotonicity in Lemma 16.1, we see that $\left[\begin{smallmatrix} m-1 \\ s-1 \end{smallmatrix}\right]_q \geq \left[\begin{smallmatrix} m-1 \\ 1 \end{smallmatrix}\right]_q = q^{m-2} + q^{m-3} + \cdots + q + 1 \geq q^{m-2}$. Use Lemma 16.1 again to see that

$$\begin{bmatrix} m \\ s \end{bmatrix}_q - \begin{bmatrix} m-1 \\ s \end{bmatrix}_q = q^{m-s} \begin{bmatrix} m-1 \\ s-1 \end{bmatrix}_q \geq q^2 q^{m-2} = q^m > q^k. \qquad ☐$$

For the constructions, assume $k \geq 2$, $m > k$, and set $w_0 = 0$. We assume a weight w on R has maximal symmetry, and we assume the associated W_0-matrix is invertible. There will be a construction for each value of $s = 1, 2, \ldots, k-1$. The integer s determines the ranks of the orbits that will be swapped.

Fix an integer s with $1 \leq s < k < m$, and choose an orbit $\operatorname{orb}(\lambda_0) \in \mathcal{O}^\sharp$ with $\operatorname{rk} \lambda_0 = 1$. As in Corollary 12.4, this orbit corresponds to the linear subspace $L_0 = \operatorname{colsp}(\lambda_0) \subseteq \mathbb{F}_q^m$, with $\dim L_0 = 1$. As $\dim L_0^\perp = m-1$, there are $\left[\begin{smallmatrix} m-1 \\ s \end{smallmatrix}\right]_q$ linear subspaces of dimension s contained in $L_0^\perp$, leaving $\left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} m-1 \\ s \end{smallmatrix}\right]_q$ linear subspaces of dimension s in $\mathbb{F}_q^m$ that are not contained in $L_0^\perp$. By Lemma 16.2, there are at least $q^k - q^s < q^k$ linear subspaces of dimension s in $\mathbb{F}_q^m$ that are not contained in $L_0^\perp$. Similarly, using $m - s \geq 2$, there are $\left[\begin{smallmatrix} m-1 \\ s+1 \end{smallmatrix}\right]_q \geq 1$ linear subspaces of dimension $s+1$ contained in $L_0^\perp$.

Choose distinct s-dimensional linear subspaces $X_1, X_2, \dots, X_{q^k-q^s}$ of $\mathbb{F}_q^m$ that are not contained in $L_0^\perp$. Also choose one $(s+1)$-dimensional linear subspace $Y \subseteq L_0^\perp$. By Corollary 12.4, these linear subspaces correspond to distinct orbits $[x_1] = \operatorname{orb}(x_1), \dots, [x_{q^k-q^s}] = \operatorname{orb}(x_{q^k-q^s})$ and $[y] = \operatorname{orb}(y)$ in $\mathcal{O}$, with $\operatorname{rk} x_i = s$ and $\operatorname{rk} y = s+1$. Then $|\operatorname{orb}(y)| = \sum_{i=1}^{q^k-q^s} |\operatorname{orb}(x_i)|$, by Proposition 12.6.

We now consider several indicator functions on $\mathcal{O}^\sharp$ and find the weights they determine at the orbits $[x_1], \dots, [x_{q^k-q^s}]$ and $[y]$.

Let $1_{[\lambda_0]} : \mathcal{O}^\sharp \to \mathbb{Z}$ be the indicator function of $[\lambda_0] = \operatorname{orb}(\lambda_0) \in \mathcal{O}^\sharp$:

$$1_{[\lambda_0]}([\lambda]) = \begin{cases} 1, & [\lambda] = [\lambda_0], \\ 0, & [\lambda] \neq [\lambda_0]. \end{cases}$$

From Lemma 13.2, the weights of $1_{[\lambda_0]}$ at the orbits $[x_i]$, $1 \le i \le q^k - q^s$, and $[y]$ are $(W_0 1_{[\lambda_0]})([x]) = w(x\lambda_0) = w_{\dim X - \dim(X \cap L_0^\perp)}$:

$$(W_0 1_{[\lambda_0]})([x]) = \begin{cases} w_1, & [x] = [x_i], \\ 0, & [x] = [y]. \end{cases}$$

The exact values of $W_0 1_{[\lambda_0]}$ at other inputs will not be relevant. What is crucial is that the value at $[y]$ is 0 and that the values at the $[x_i]$ are equal and positive.

Recall that $\mathcal{O}_1^\sharp = \{\operatorname{orb}(\lambda) \in \mathcal{O}^\sharp : \operatorname{rk} \lambda = 1\}$; let $1_{\mathcal{O}_1^\sharp}$ be its indicator function. The orbit weights $\omega_1 = W_0 1_{\mathcal{O}_1^\sharp}$ are found in Proposition 14.1. The indicator function $1_{\mathcal{O}_+^\sharp}$ of the set of all nonzero orbits $\mathcal{O}_+^\sharp$ is locally constant, so $W_0 1_{\mathcal{O}_+^\sharp}$ is also locally constant, Corollary 14.2. Set $\alpha_1 = (W_0 1_{\mathcal{O}_+^\sharp})(\operatorname{orb}(x))$, when $\operatorname{rk} x = 1$, and $\alpha_2 = (W_0 1_{\mathcal{O}_+^\sharp})(\operatorname{orb}(x))$ when $\operatorname{rk} x = 2$. Both α_1, α_2 are positive integers.

Define a function $\varsigma^{(s)} \in F_0(\mathcal{O}, \mathbb{Z})$ by

$$\varsigma^{(s)}(\operatorname{orb}(x)) = \begin{cases} -1, & \operatorname{orb}(x) = \operatorname{orb}(x_i), \\ 1, & \operatorname{orb}(x) = \operatorname{orb}(y), \\ 0, & \text{otherwise}. \end{cases}$$

Because we are assuming W_0 is invertible, Remark 13.19 says that $W_0 : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is an isomorphism. Thus $W_0^{-1}\varsigma^{(s)} \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$ exists, but it has rational values of both signs. To clear denominators, choose a positive integer c sufficiently large so that $\sigma^{(s)} = c w_1 W_0^{-1} \varsigma^{(s)}$ has integer values. Then $\sigma^{(s)}(\operatorname{orb}(0)) = 0$, as $\sigma^{(s)} \in F_0(\mathcal{O}^\sharp, \mathbb{Z})$, and $W_0 \sigma^{(s)} = c w_1 \varsigma^{(s)}$, so that $(W_0 \sigma^{(s)})(\operatorname{orb}(x))$ equals 0 or $\pm c w_1$.

We collect the values of $W_0\eta$ at the $[x_i]$ and $[y]$, for various η.

$$\begin{array}{c|c|c} \eta & (W_0\eta)([x_i]) & (W_0\eta)([y]) \\ \hline 1_{[\lambda_0]} & w_1 & 0 \\ 1_{\mathcal{O}^\sharp_+} & \alpha_1 & \alpha_2 \\ 1_{\mathcal{O}^\sharp_1} & (q^{m-s}+\cdots+q^{m-1})w_1 & (q^{m-s-1}+\cdots+q^{m-1})w_1 \\ \sigma^{(s)} & -cw_1 & cw_1 \end{array} \tag{16.3}$$

For later use, write $B_{m,s} = q^{m-s-1}+\cdots+q^{m-1}$; $B_{m,s}>0$.

Some of the values of $\sigma^{(s)}$ will be negative. Choose a positive integer a sufficiently large so that all the values of $aw_1 1_{\mathcal{O}^\sharp_+} + \sigma^{(s)}$ are nonnegative. Choose an integer $b \geq 1$ large enough that $a(\alpha_1-\alpha_2) < bq^{m-s-1}$. (If $\alpha_1 < \alpha_2$, then $b=1$ suffices.) Set

$$\Delta = \sum_{\operatorname{orb}(\lambda)\in\mathcal{O}^\sharp_+} \sigma^{(s)}(\operatorname{orb}(\lambda)), \tag{16.4}$$

$$\varepsilon = aw_1 1_{\mathcal{O}^\sharp_+} + b1_{\mathcal{O}^\sharp_1} + (c + a(\alpha_2-\alpha_1) + bq^{m-s-1})1_{[\lambda_0]}.$$

Define two multiplicity functions $\eta_C, \eta_D \in F(\mathcal{O}^\sharp, \mathbb{Q})$ by setting

$$\begin{aligned} \eta_C &= \varepsilon + \max(\Delta,0)1_{[0]}, \\ \eta_D &= \varepsilon + \sigma^{(s)} - \min(\Delta,0)1_{[0]}. \end{aligned} \tag{16.5}$$

Theorem 16.6. *Let $R = M_{k\times k}(\mathbb{F}_q)$, $k\geq 2$, and $M = M_{k\times m}(\mathbb{F}_q)$, $k<m$. Let w be a weight on R with maximal symmetry, positive integer values, and $w(0)=0$. Assume the associated W_0-matrix is invertible.*

Then, for any integer s, $1\leq s<k$, the multiplicity functions η_C and η_D of (16.5) have nonnegative integer values, i.e., $\eta_C,\eta_D \in F(\mathcal{O}^\sharp,\mathbb{N})$, and they define left R-linear codes C and D, respectively. The two codes have the same length. Their weights at the orbits $\operatorname{orb}(x_i)$, $i = 1,2,\ldots,q^k-q^s$, and $\operatorname{orb}(y)$ are

$$\begin{array}{c|c|c} \eta & (W\eta)(\operatorname{orb}(x_i)) & (W\eta)(\operatorname{orb}(y)) \\ \hline \eta_C & (c+a\alpha_2+bB_{m,s})w_1 & (a\alpha_2+bB_{m,s})w_1 \\ \eta_D & (a\alpha_2+bB_{m,s})w_1 & (c+a\alpha_2+bB_{m,s})w_1 \end{array}.$$

At all other orbits, their weights agree. In particular, $\operatorname{wwe}_C = \operatorname{wwe}_D$.

Proof. All of $a, w_1, b, c, \alpha_1, \alpha_2, \max(\Delta,0)$, and $-\min(\Delta,0)$ are nonnegative integers. The integer $b\geq 1$ was chosen so that $a(\alpha_2-\alpha_1) + bq^{m-s-1} > 0$, so the values of η_C are nonnegative integers. The positive integer a was chosen so that $aw_1 1_{\mathcal{O}^\sharp_+} + \sigma^{(s)}$ has nonnegative values, which implies that the values of η_D are nonnegative integers.

Both η_C and η_D contain the term $b1_{\mathcal{O}_1^\sharp}$. Because the linear subspace spanned by $\{\mathrm{colsp}(\lambda) : \lambda \in \mathcal{O}_1^\sharp\}$ is all of $\mathbb{F}_q^m$, Remark 12.5 implies that Λ_C and Λ_D are both injective. The value of Δ was chosen to satisfy $\Delta = \mathrm{efflng}(D) - \mathrm{efflng}(C)$. If $\Delta > 0$, then D is longer, and we add Δ zero-functionals to C; if $\Delta < 0$, then C is longer, and we add $-\Delta$ zero-functionals to D. Thus the codes have the same length.

The weights $(W\eta)(\mathrm{orb}(x_i))$ and $(W\eta)(\mathrm{orb}(y))$ follow from (16.3) and (16.5). Because $W\eta_D - W\eta_C = W\sigma^{(s)} = cw_1\varsigma^{(s)}$, the form of $\varsigma^{(s)}$ implies that the weights for C and D are equal at all other orbits. As noted earlier, $|\mathrm{orb}(y)| = \sum_{i=1}^{q^k-q^s} |\mathrm{orb}(x_i)|$, by Proposition 12.6. We conclude that $\mathrm{wwe}_C = \mathrm{wwe}_D$. □

Remark 16.7. Here is a summary of the motivations for various parts of the construction. The integer c was chosen to clear denominators so that $\sigma^{(s)}$ would have integer values. However, $\sigma^{(s)}$ has both positive and negative values, because W has nonnegative entries and $W\sigma^{(s)} = cw_1\varsigma^{(s)}$ has mixed signs. So, a was chosen so that $aw_1 1_{\mathcal{O}_+^\sharp} + \sigma^{(s)}$ has nonnegative integer values. The function $1_{\mathcal{O}_+^\sharp}$ (resp., $1_{\mathcal{O}_1^\sharp}$) is used because its weights are the same at all the $[x_i]$'s. The integer $b \geq 1$ was chosen so that $(W_0(aw_1 1_{\mathcal{O}_+^\sharp} + b1_{\mathcal{O}_1^\sharp}))(\mathrm{orb}(x_i)) < (W_0(aw_1 1_{\mathcal{O}_+^\sharp} + b1_{\mathcal{O}_1^\sharp}))(\mathrm{orb}(y))$ and also to guarantee that Λ_C and Λ_D are injective. Then the coefficient of $1_{[\lambda_0]}$ was chosen so that the weights interchange when $\sigma^{(s)}$ is added to η_C. The function $1_{[\lambda_0]}$ is used because it changes the weights at the $[x_i]$'s in the same way but not the weight at $[y]$.

17. Degeneracies

The constructions of Theorem 16.6 made use of the invertibility of the matrix W_0. In this section, we describe a construction in the situation where W_0 is not invertible. We continue to assume information module $M = M_{k\times m}(\mathbb{F}_q)$ and weight w with maximal symmetry, positive integer values, and $w_0 = 0$.

Corollary 13.18 says that W_0 is singular when at least one

$$c_j = (-1)^j q^{\binom{j}{2}} \sum_{s=1}^{j} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q w_s,$$

$j = 1, 2, \ldots, k$, vanishes. By hypothesis on w, $w_1 > 0$, so that $c_1 \neq 0$.

Lemma 17.1. *Suppose $c_j = 0$ for some $j = 2, 3, \ldots, k$. Then any row of the matrix P_0 indexed by a linear subspace $\gamma \subseteq \mathbb{F}_q^m$ with* $\dim\gamma = j$ *belongs to* $\ker W_0$.

Proof. The second part of Proposition 13.12 shows that, if $\dim\gamma = j$, then $(WP^\top)_{\beta,\gamma} = 0$. This holds for any β and any γ with $\dim\gamma = j$. This means the columns of $P^\top$ indexed by γ with $\dim\gamma = j$ belong to $\ker W$. Those columns are the same as the rows of P indexed by γ with $\dim\gamma = j$. Because $w_0 = 0$, the same analysis applies with W_0 and P_0. ☐

Suppose $c_j = 0$ for some $j = 2, 3, \ldots, k$. Pick a row v_γ of P_0 indexed by a linear subspace $\gamma \subseteq \mathbb{F}_q^m$ with $\dim\gamma = j$. Define two multiplicity functions $\eta_\pm \in F_0(\mathcal{O}^\sharp, \mathbb{N})$ based on the positive, resp., negative, parts of v_γ:

$$\begin{aligned} \eta_+ &= (v_\gamma + |v_\gamma|)/2 + 1_{\mathcal{O}_1^\sharp}, \\ \eta_- &= -(v_\gamma - |v_\gamma|)/2 + 1_{\mathcal{O}_1^\sharp}, \end{aligned}$$

where $|v_\gamma|$ means the vector obtained from v_γ by taking the absolute value of each entry. The terms $1_{\mathcal{O}_1^\sharp}$ are included so that the associated homomorphisms $\Lambda_{\eta_\pm}$ are injective; see Remark 12.5. Note that $\eta_+ - \eta_- = v_\gamma$, so that $W_0\eta_+ = W_0\eta_-$. Modify η_+ by setting $\eta_+([0]) = 1$; set $\eta_-([0]) = 0$.

Proposition 17.2. *Let $C_\pm$ be the linear codes determined by $\eta_\pm$. Then C_+ and C_- have the same length, and $\mathrm{wwe}_{C_+} = \mathrm{wwe}_{C_-}$.*

Proof. In (13.3), there are $\left[\begin{smallmatrix} j \\ r \end{smallmatrix}\right]_q$ nonzero terms in rank r positions of the row v_γ. Then the difference in lengths of C_+ and C_- is

$$\begin{aligned} \mathrm{length}(C_+) - \mathrm{length}(C_-) &= \sum_{\substack{r=0 \\ r \text{ even}}}^{j} q^{\binom{r}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q - \sum_{\substack{r=0 \\ r \text{ odd}}}^{j} q^{\binom{r}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \\ &= \sum_{r=0}^{j} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q = 0, \end{aligned}$$

by (13.7). Zero-functionals don't change the value of $W\eta_+$. We then have $W\eta_+ = W\eta_-$, so that the w-weight enumerators are equal. ☐

This same construction was used (with the Hamming weight) in [27, p. 703] and [28, p. 145].

Remark 17.3. The swapping idea in Theorem 16.6 does not always work in the degenerate case because $\varsigma^{(s)}$ is not always in $\operatorname{im} W_0$. See Remark 18.13 for more details.

18. Analysis of singleton dual codewords

In this section we analyze dual codewords that are singleton vectors. We will then apply this analysis to the codes constructed using (16.5). The key result, Theorem 18.12, shows how the contributions of singletons of rank i to $A^{\text{sing}}_{w_i}(D^\perp) - A^{\text{sing}}_{w_i}(C^\perp)$ depend on the parameter s used in (16.5).

Given a left R-linear code $C \subseteq R^n$, recall the right dual code $\mathcal{R}(C)$ from (2.1). We will often denote $\mathcal{R}(C)$ by $C^\perp$.

When an R-linear code C is given by a multiplicity function η, a singleton vector v belongs to $C^\perp$ when the nonzero entry r of v right-annihilates the coordinate functional λ in that position: $\lambda r = 0$. Remember that $\lambda \in \operatorname{Hom}_R(M, R) = M_{m\times k}(\mathbb{F}_q)$ and $r \in R = M_{k\times k}(\mathbb{F}_q)$.

Given a functional λ with $\operatorname{rk}\lambda = j$, we will determine how many elements $r \in R$ with $\operatorname{rk} r = i$ satisfy $\lambda r = 0$. For $\lambda \in \operatorname{Hom}_R(M, R)$, define

$$\operatorname{ann}(i, \lambda) = \{r \in R : \operatorname{rk} r = i \text{ and } \lambda r = 0\}.$$

Recall that the sizes $\mathscr{S}_j$ of $\mathcal{U}$-orbits were given in Proposition 12.6.

Lemma 18.1. *Suppose* $\lambda \in \operatorname{Hom}_R(M, R)$. *Then the size of* $\operatorname{ann}(i, \lambda)$ *depends only on* $\operatorname{rk}\lambda$. *If* $\operatorname{rk}\lambda = j$, *then*

$$|\operatorname{ann}(i, \lambda)| = \begin{cases} \mathscr{S}_i \begin{bmatrix} k-j \\ i \end{bmatrix}_q, & i \le k - j, \\ 0, & i > k - j. \end{cases}$$

Proof. View $M_{m\times k}(\mathbb{F}_q)$ and $M_{k\times k}(\mathbb{F}_q)$ as spaces of $\mathbb{F}_q$-linear transformations $\mathbb{F}_q^m \to \mathbb{F}_q^k$ and $\mathbb{F}_q^k \to \mathbb{F}_q^k$, respectively, with inputs written on the left. Then $\lambda r = 0$ means that $\operatorname{im}\lambda \subseteq \ker r$. Given that $\operatorname{rk}\lambda = j$ and $\operatorname{rk} r = i$, we see that a necessary condition for $\lambda r = 0$ is that $j = \dim\operatorname{im}\lambda \le \dim\ker r = k - i$.

Given $\operatorname{im}\lambda \subseteq \mathbb{F}_q^k$, the number of linear subspaces K (candidates for $\ker r$) satisfying $\operatorname{im}\lambda \subseteq K \subseteq \mathbb{F}_q^k$ and $\dim K = k - i$ equals the number of linear subspaces of dimension $k - i - j$ in $\mathbb{F}_q^k/\operatorname{im}\lambda$, a vector space of dimension $k - j$. That number is $\begin{bmatrix} k-j \\ k-i-j \end{bmatrix}_q = \begin{bmatrix} k-j \\ i \end{bmatrix}_q$. For a given K, there is a $\mathcal{U}$-orbit's worth of matrices r with $\ker r = K$, and hence $\operatorname{rk} r = i$. The size of that orbit is $\mathscr{S}_i$. □

We will abuse notation and write $|\operatorname{ann}(i, j)|$ for the common value $|\operatorname{ann}(i, \lambda)|$ when $\operatorname{rk}\lambda = j$.

Example 18.2. Let $k = 2$. We display the values of $|\mathrm{ann}(i,j)|$ in a matrix with row index i and column index j, $i, j = 0, 1, 2$:

$$\begin{bmatrix} 1 & 1 & 1 \\ (q^2-1)(q+1) & q^2-1 & 0 \\ (q^2-1)(q^2-q) & 0 & 0 \end{bmatrix}.$$

Suppose a left R-linear code C is given by a multiplicity function $\eta : \mathcal{O}^\sharp \to \mathbb{N}$. For $j = 0, 1, \ldots, k$, define

$$\overline{\eta}_j = \sum_{[\lambda] \in \mathcal{O}^\sharp_j} \eta([\lambda]).$$

Then $\overline{\eta}_j$ counts the number of coordinate functionals having rank j. Set $\overline{\eta} = \langle \overline{\eta}_0, \overline{\eta}_1, \ldots, \overline{\eta}_k \rangle \in \mathbb{N}^{k+1}$. We call the $\overline{\eta}_j$ 'rank-sums'.

Proposition 18.3. *Suppose a left R-linear code C is given by a multiplicity function η. Then the contribution of singletons of rank i to $A^{\mathrm{sing}}_{w_i}(C^\perp)$ is*

$$\sum_{j=0}^{k-i} |\mathrm{ann}(i,j)| \overline{\eta}_j.$$

In particular, the contribution of singletons of rank k to $A^{\mathrm{sing}}_{w_k}(C^\perp)$ is $|\mathrm{GL}(k, \mathbb{F}_q)| \overline{\eta}_0$.

Proof. There are a total of $\overline{\eta}_j$ coordinate functionals of rank j. For each one, apply Lemma 18.1. ☐

The larger i is, the fewer terms there are in the summation.

Corollary 18.4. *For a linear code C given by multiplicity function η,*

$$A^{\mathrm{sing}}_d(C^\perp) = \sum_{i : w_i = d} \sum_{j=0}^{k-i} |\mathrm{ann}(i,j)| \overline{\eta}_j.$$

Having seen the importance of the rank-sums $\overline{\eta}_j$, we next see how they behave with respect to the $\overline{W}$-matrix. Given a function $\omega : \mathcal{O} \to \mathbb{Q}$, define

$$\overline{\omega}_i = \sum_{[x] \in \mathcal{O}_i} \omega([x]), \quad i = 0, 1, \ldots, k.$$

Set $\overline{\omega} = \langle \overline{\omega}_0, \overline{\omega}_1, \ldots, \overline{\omega}_k \rangle$.

Proposition 18.5. *Suppose $\eta : \mathcal{O}^\sharp \to \mathbb{Q}$. If $\omega = W\eta$, then $\overline{\omega} = \overline{W}\overline{\eta}$.*

Proof. Sum the rows indexed by rank i elements of $\mathcal{O}$ in $\omega = W\eta$, change the order of summation, and use Lemma 15.1 and (15.3). ☐

Because our ultimate objective is to find linear codes C and D of the same length with $\mathrm{wwe}_C = \mathrm{wwe}_D$ and $\mathrm{wwe}_{C^\perp} \neq \mathrm{wwe}_{D^\perp}$, we now apply Proposition 18.3 and Corollary 18.4 to two linear codes C and D of the same length. Write $\delta\eta = \eta_D - \eta_C$ and $\delta A_d^{\mathrm{sing}} = A_d^{\mathrm{sing}}(D^\perp) - A_d^{\mathrm{sing}}(C^\perp)$. The net contribution of rank i singletons to $\delta A_{w_i}^{\mathrm{sing}}$ then simplifies to

$$\sum_{j=0}^{k-i} |\mathrm{ann}(i,j)| \delta\overline{\eta}_j, \tag{18.6}$$

and $\delta A_d^{\mathrm{sing}} = \sum_{i:w_i=d} \sum_{j=0}^{k-i} |\mathrm{ann}(i,j)| \delta\overline{\eta}_j$.

Remark 18.7. The last two formulas will be the main tools for showing that two dual codes have different w-weight enumerators. But the fomulas cut both ways. If all the $\delta\overline{\eta}_j$ vanish, then singletons cannot detect differences between wwe_C and wwe_D. See Example 21.5.

Now consider specifically the linear codes C and D constructed by (16.5). In this case, we see that

$$\begin{aligned} \delta\eta &= \sigma^{(s)} - \Delta 1_{[0]}, \\ \overline{\eta}_D - \overline{\eta}_C &= \langle -\Delta, \bar{\sigma}_1^{(s)}, \ldots, \bar{\sigma}_k^{(s)} \rangle, \\ \overline{\omega}_D - \overline{\omega}_C &= \overline{W} \bar{\sigma}^{(s)} = \langle 0, \ldots, 0, -cw_1(q^k - q^s), cw_1, 0, \ldots, 0 \rangle, \end{aligned}$$

where the nonzero entries of $\overline{\omega}_D - \overline{\omega}_C$ are in positions s and $s+1$.

Recall from Theorem 16.6 that C and D have the same length. This is reflected in the fact that the sum of the entries of $\overline{\eta}_D - \overline{\eta}_C$ is $-\Delta + \sum_j \bar{\sigma}_j^{(s)} = 0$, from (16.4). This allows us to re-write the net contribution (18.6) of rank i singletons to $\delta A_{w_i}^{\mathrm{sing}}$ as $\sum_{j=1}^{k} \left(|\mathrm{ann}(i,j)| - |\mathrm{ann}(i,0)|\right) \bar{\sigma}_j^{(s)}$. To write this equation in matrix form, define a $k \times k$ matrix $\overline{\mathrm{Ann}}$ by

$$\overline{\mathrm{Ann}}_{i,j} = |\mathrm{ann}(i,j)| - |\mathrm{ann}(i,0)| \quad i,j = 1,2,\ldots,k.$$

In summary, the net contributions of singletons of rank i to $\delta A_{w_i}^{\mathrm{sing}}$ are given by the entries of

$$\overline{\mathrm{Ann}}\, \bar{\sigma}^{(s)}. \tag{18.8}$$

Lemma 18.9. *The matrix* $\overline{\mathrm{Ann}}$ *is invertible over* $\mathbb{Q}$.

Proof. Define a $(k+1) \times (k+1)$ matrix $\mathcal{A}$ by

$$\mathcal{A}_{i,j} = |\mathrm{ann}(i,j)|, \quad i,j = 0,1,2,\ldots,k.$$

Then $\mathcal{A}$ is upper 'anti-triangular' by Lemma 18.1, i.e., $\mathcal{A}_{i,j} = 0$ when $i + j > k$. Then $\det \mathcal{A} = \pm \prod_{i=0}^{k} \mathscr{S}_i$ is nonzero. Thus $\mathcal{A}$ is invertible.

Subtract the zeroth column of $\mathcal{A}$ from every other column. The resulting matrix is still invertible and has the form

$$\left[\begin{array}{c|c} 1 & \mathrm{row}(0) \\ \hline \mathrm{col}(*) & \overline{\mathrm{Ann}} \end{array}\right],$$

where $\mathrm{col}(*)$ is a column of nonzero entries. Thus $\overline{\mathrm{Ann}}$ is invertible. $\square$

The matrices $\overline{W}_0$ and $\overline{\mathrm{Ann}}$ are both $k \times k$, so they both define $\mathbb{Q}$-linear transformations $\mathbb{Q}^k \to \mathbb{Q}^k$. We next explore how these linear transformations behave with respect to certain filtrations of $\mathbb{Q}^k$.

Define a $k \times k$ matrix T over $\mathbb{Q}$:

$$T_{i,j} = (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q, \quad i, j = 1, 2, \dots, k.$$

By standard conventions, $\begin{bmatrix} j \\ i \end{bmatrix}_q = 0$ when $i > j$. This implies that T is upper-triangular and invertible. (By (15.7), T is just $Q_{0,2}$.)

Lemma 18.10. *The matrices $\overline{W}_0 T$ and $\overline{\mathrm{Ann}}\, T$ are lower triangular.*

The proof of Lemma 18.10 will utilize the next lemma.

Lemma 18.11. *For integers $0 \le i < j \le k$,*

$$\sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i \end{bmatrix}_q = 0.$$

Proof. We first prove the edge cases $i = 0$ or $j = k$, and then prove the remaining cases by induction on k.

Suppose $i = 0$. Then the term $\begin{bmatrix} k-\ell \\ 0 \end{bmatrix}_q = 1$ for all $\ell = 0, \dots j$. The sum reduces to (13.7), which vanishes, as $j > 0$.

Suppose $j = k$. We observe that

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i \end{bmatrix}_q = \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k-i \\ \ell \end{bmatrix}_q,$$

and that $\begin{bmatrix} k-\ell \\ i \end{bmatrix}_q = 0$ for $\ell > k - i$. Then

$$\begin{aligned} \sum_{\ell=0}^{k} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i \end{bmatrix}_q &= \sum_{\ell=0}^{k-i} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k-i \\ \ell \end{bmatrix}_q \\ &= \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{\ell=0}^{k-i} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} k-i \\ \ell \end{bmatrix}_q = 0, \end{aligned}$$

by (13.7), as $k - i > 0$.

We prove the remaining cases, $0 < i < j < k$, by induction on k. The first case is when $k = 3$, with $i = 1 < j = 2$. By direct calculation,

$$\sum_{\ell=0}^{2} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} 2 \\ \ell \end{bmatrix}_q \begin{bmatrix} 3-\ell \\ 1 \end{bmatrix}_q = (q^2+q+1) - (q+1)^2 + q = 0.$$

For the induction step, suppose $0 < i < j < k+1$. Apply an identity from Lemma 16.1, so that

$$\begin{aligned} \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k+1-\ell \\ i \end{bmatrix}_q &= \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i-1 \end{bmatrix}_q \\ &\quad + q^i \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i \end{bmatrix}_q = 0, \end{aligned}$$

because both sums on the right side vanish by the induction hypothesis or the edge cases. □

Proof of Lemma 18.10. We need to show that the i,j-entry of each product vanishes when $i < j$. For $\overline{W}_0 T$, the result is the $w_0 = 0$ analog of Proposition 13.12, as T is $Q_{0,2}$. For $\overline{\operatorname{Ann}}\, T$,

$$(\overline{\operatorname{Ann}}\, T)_{i,j} = \sum_{\ell=1}^{j} \mathscr{S}_i \left(\begin{bmatrix} k-\ell \\ i \end{bmatrix}_q - \begin{bmatrix} k \\ i \end{bmatrix}_q \right) (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q .$$

Note that the sum does not change if we include $\ell = 0$. Applying the distributive law and (13.7), as $j > 0$, we see from Lemma 18.11 that

$$(\overline{\operatorname{Ann}}\, T)_{i,j} = \mathscr{S}_i \sum_{\ell=0}^{j} (-1)^\ell q^{\binom{\ell}{2}} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k-\ell \\ i \end{bmatrix}_q = 0. \qquad \square$$

We now define two filtrations of $\mathbb{Q}^k$. Let $e_1 = (1,0,\ldots,0), \ldots, e_k = (0,\ldots,0,1)$ be the standard basis vectors of $\mathbb{Q}^k$. For $i = 1, 2, \ldots, k$, define linear subspaces of $\mathbb{Q}^k$:

$$\begin{aligned} \mathcal{V}_i &= \operatorname{span}\{e_i, e_{i+1}, \ldots, e_k\}, \\ \mathcal{T}_i &= \operatorname{span}\{\text{columns } i, i+1, \ldots, k \text{ of the matrix } T\}. \end{aligned}$$

Then $\dim \mathcal{V}_i = \dim \mathcal{T}_i = k - i + 1$, and

$$\begin{aligned} \mathbb{Q}^k &= \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots \supset \mathcal{V}_k \supset \{0\}, \\ \mathbb{Q}^k &= \mathcal{T}_1 \supset \mathcal{T}_2 \supset \cdots \supset \mathcal{T}_k \supset \{0\}. \end{aligned}$$

Lemma 18.10 shows that the matrices $\overline{\operatorname{Ann}}$ and $\overline{W}_0$ (when invertible) map the $\mathcal{T}$-filtration isomorphically to the $\mathcal{V}$-filtration.

Theorem 18.12. *Let C and D be the linear codes constructed using (16.5) and $\varsigma^{(s)}$ for some s, $1 \leq s < k$, with $\overline{W}_0$ invertible. Then the contribution by rank i singletons to $\delta A^{\text{sing}}_{w_i} = A^{\text{sing}}_{w_i}(D^\perp) - A^{\text{sing}}_{w_i}(C^\perp)$ is zero if $i < s$. The contribution of rank s singletons to $\delta A^{\text{sing}}_{w_s}$ is nonzero.*

Proof. The vector $\bar{\varsigma}^{(s)}$ belongs to $\mathcal{V}_s - \mathcal{V}_{s+1}$. Then $\bar{\sigma}^{(s)} = \overline{W}_0^{-1} \bar{\varsigma}^{(s)} \in \mathcal{T}_s - \mathcal{T}_{s+1}$. This, in turn, implies that $\overline{\operatorname{Ann}}\, \bar{\sigma}^{(s)} \in \mathcal{V}_s - \mathcal{V}_{s+1}$. By (18.8), singletons of rank i, $i < s$, make zero contribution to $\delta A^{\text{sing}}_{w_i}$, while the singletons of rank s make a nonzero contribution to $\delta A^{\text{sing}}_{w_s}$. □

We point out that Theorem 18.12 makes no claims about the contribution of singletons of rank k to $\delta A^{\text{sing}}_{w_k}$.

Remark 18.13. If W_0 is degenerate because $c_j = 0$, then $\overline{W}_0$ is also degenerate, and $\overline{W}_0$ maps $\mathcal{T}_j$ into $\mathcal{V}_{j+1}$, and $\mathcal{T}_i$, $i < j$, will map to a proper linear subspace of $\mathcal{V}_i$. If j is the largest index such that $c_j = 0$, then $\bar{\sigma}^{(s)}$ will be in the image of $\overline{W}_0$ provided $j < s$. Recalling that $s < k$ and that $c_1 \neq 0$, we see that $\bar{\sigma}^{(s)} \in \operatorname{im} \overline{W}_0$ when $2 \leq j < s < k$. For example, when $k = 3$, no such s can exist.

19. Main results

We are now in a position to prove that a large number of weights with maximal symmetry, including the homogeneous weight, do not respect duality.

As usual in this part of the paper, let $R = M_{k\times k}(\mathbb{F}_q)$ with $k \geq 2$. Suppose w is a weight on R that has maximal symmetry, positive integer values, and $w(0) = 0$. The value $w(r)$, $r \in R$, depends only on the rank $\operatorname{rk} r$ of r. Write w_i for the common value $w(r)$ when $\operatorname{rk} r = i$.

We will say that a weight w is *nondegenerate* if the expressions c_i of (13.5) are nonzero for all $i = 1, 2, \ldots, k$. (Note that $c_0 = 0$ because $w(0) = 0$, and $c_1 = w_1 > 0$.) If at least one of $c_2, \ldots, c_k$ vanishes, we say that w is *degenerate*.

Let $\mathring{w} = \min\{w_1, w_2, \ldots, w_k\}$; $\mathring{w}$ is a positive integer. Write $\mathring{I} = \{i : w_i = \mathring{w}\}$ for set of indices i where w_i achieves the minimum positive value $\mathring{w}$. In general, for an integer $d \geq \mathring{w}$, set $I_d = \{i : w_i = d\}$; depending on d, I_d may be empty. Of course, $\mathring{I} = I_{\mathring{w}}$ is nonempty.

Theorem 19.1. *Assume w is a nondegenerate weight on $R = M_{k\times k}(\mathbb{F}_q)$, with maximal symmetry, positive integer values, and $w(0) = 0$. Suppose there is an integer d such that $\mathring{w} \leq d < 2\mathring{w}$, I_d is nonempty, and $k \notin I_d$. Then w does not respect duality: there exist linear codes C and D over R of the same length such that $\mathrm{wwe}_C = \mathrm{wwe}_D$ and $A_d(C^\perp) \neq A_d(D^\perp)$. In particular, if $k \notin \mathring{I}$, then w does not respect duality.*

Proof. Suppose d satisfies the stated hypotheses, and let $s = \max I_d$; then $s < k$. Construct linear codes C and D over R using (16.5) using $s = \max I_d$. Because $s < k$, Theorem 16.6 implies that the codes have the same length and that $\mathrm{wwe}_C = \mathrm{wwe}_D$.

As for the dual codes, Corollary 5.4 says that $A_d(D^\perp) - A_d(C^\perp) = A_d^{\mathrm{sing}}(D^\perp) - A_d^{\mathrm{sing}}(C^\perp) = \delta A_d^{\mathrm{sing}}$. The only singletons that can contribute to $\delta A_d^{\mathrm{sing}}$ are those of rank i with $i \in I_d$, Corollary 18.4. Theorem 18.12 says that singletons of rank $i < s$ make zero contributions to $\delta A_d^{\mathrm{sing}}$, while singletons of rank s make a nonzero contribution to $\delta A_d^{\mathrm{sing}}$. We conclude that $A_d(D^\perp) - A_d(C^\perp) = \delta A_d^{\mathrm{sing}} \neq 0$. □

Remark 19.2. If $k \in I_d$, the arguments given above are not conclusive. In Theorem 18.12 there is always the possibility that the contributions of singletons of rank k could cancel the contributions of singletons of lower rank $i \in I_d$. Even if $I_d = \{k\}$, it is possible that singletons of rank k make zero contributions. This happens for the homogeneous weight, for example; see Corollary 4.10 and Proposition 18.3.

Corollary 19.3. *Suppose a nondegenerate w satisfies $\mathring{w} < w_i < 2\mathring{w}$ for some i, $1 \leq i \leq k$. Then w does not respect duality.*

Proof. By the definition of $\mathring{w}$, there is some index j so that $w_j = \mathring{w}$. As $w_j < w_i$, we have $i \neq j$. The index k can equal at most one of i or j. Apply Theorem 19.1 to the other one. □

We can, at long last, prove that the homogeneous weight on $M_{k\times k}(\mathbb{F}_q)$ does not respect duality, provided $k > 2$ or $q > 2$. The homogeneous weight on any finite Frobenius ring has the Extension Property [10, Theorem 2.5]. Since the matrix ring $M_{k\times k}(\mathbb{F}_q)$ is Frobenius, it follows that W_0 and $\overline{W}_0$ are invertible for the homogeneous weight and any information module M, Remark 13.19. This says that the homogeneous weight is nondegenerate.

Theorem 19.4. *Let $R = M_{k\times k}(\mathbb{F}_q)$, $k \geq 2$, with the homogeneous weight* w. *Then* w *respects duality if and only if $k = 2$ and $q = 2$.*

Proof. The 'if' portion is Theorem 12.12. For the 'only if' portion, Lemma 12.11 says $\mathrm{w}_2 = \mathring{\mathrm{w}}$ is the smallest nonzero value of w, while w_1 is the largest value, with $\mathrm{w}_2 < \mathrm{w}_1 \leq 2\mathrm{w}_2$. There is equality $\mathrm{w}_1 = 2\mathrm{w}_2$ if and only if $k = 2$ and $q = 2$. For all other values of k and q, there is strict inequality: $\mathrm{w}_2 < \mathrm{w}_1 < 2\mathrm{w}_2$. Now apply Theorem 19.1 with $d = \mathrm{w}_1$. □

The last result of this section determines, for $k = 2$, all the weights with maximal symmetry (nondegenerate or not) that respect duality.

The list is a short one: the Hamming weight (any q) and the homogeneous weight (only when $q = 2$).

Theorem 19.5. *Let w be a weight on $R = M_{2\times 2}(\mathbb{F}_q)$ having maximal symmetry, positive integer values, and $w(0) = 0$. Assume w is neither a multiple of the Hamming weight ($w_1 = w_2$) nor, when $q = 2$, a multiple of the homogeneous weight ($w_1 = 2w_2$ when $q = 2$). Then w does not respect duality.*

Proof. Using $m = k + 1 = 3$ and $s = 1$ (the only possible value of s), Example 15.4 gives the $\overline{W}_0$-matrix:

$$\overline{W}_0 = \begin{bmatrix} q^2 w_1 & (q^2+q)w_1 \\ (q^2+q)w_1 & (q+1)w_1 + q^2 w_2 \end{bmatrix}.$$

Then $\det \overline{W}_0 = q^3 w_1(-(q+1)w_1 + qw_2)$. As $w_1 > 0$ by hypothesis, $\det \overline{W}_0$ vanishes only when $-(q+1)w_1 + qw_2 = 0$.

First consider the nondegenerate case, where $-(q+1)w_1 + qw_2 \neq 0$. Use the construction of (16.5) with $m = 3$ and $s = 1$ to produce linear codes C and D with $\mathrm{wwe}_C = \mathrm{wwe}_D$. The net counts of singleton vectors in the dual codes depends only on $\bar\sigma$. A calculation shows that

$$\begin{aligned} \overline{W}_0 \langle (q+1)w_1 + (q^2-q)w_2, -q^2 w_1\rangle & \\ &= (q(-(q+1)w_1 + qw_2))\langle -(q^2-q), 1\rangle \\ &= (q(-(q+1)w_1 + qw_2))\bar\varsigma^{(1)}. \end{aligned}$$

Thus, up to scaling, we take $\bar\sigma^{(1)} = \langle (q+1)w_1 + (q^2-q)w_2, -q^2 w_1\rangle$.

Using Example 18.2, we see that the matrix $\overline{\mathrm{Ann}}$ is

$$\overline{\mathrm{Ann}} = \begin{bmatrix} -q(q^2-1) & -(q^2-1)(q+1) \\ -(q^2-1)(q^2-q) & -(q^2-1)(q^2-q) \end{bmatrix}.$$

By (18.8), the net contributions to $\delta A^{\mathrm{sing}}_{w_j} = A^{\mathrm{sing}}_{w_j}(D^\perp) - A^{\mathrm{sing}}_{w_j}(C^\perp)$ by rank j singletons are given by the entries of $\overline{\mathrm{Ann}}\,\bar\sigma$:

$$\overline{\mathrm{Ann}}\,\bar\sigma = q(q-1)(q^2-1)\begin{bmatrix} (q+1)w_1 - qw_2 \\ (q^2-q-1)w_1 - (q^2-q)w_2 \end{bmatrix}. \tag{19.6}$$

The contribution for $j = 1$ is nonzero because w is nondegenerate. The contribution for $j = 2$ is nonzero, provided w is not a multiple of the homogeneous weight; see Example 12.9.

Because $k = 2$, there are just a few (nondegenerate) cases:

- If $w_1 < w_2$, then $\delta A_{w_1} = \delta A^{\mathrm{sing}}_{w_1} \neq 0$, by (19.6).
- If $w_2 < w_1$ and w is not homogeneous, then $\delta A_{w_2} = \delta A^{\mathrm{sing}}_{w_2} \neq 0$, by (19.6).
- If w is homogeneous, apply Theorem 19.4 (except when $q = 2$).

- If $w_1 = w_2$, w is a multiple of the Hamming weight.

Now suppose w is degenerate, so that $qw_2 = (q+1)w_1$. Because q and $q+1$ are relatively prime, there exists a positive integer τ such that $w_1 = q\tau$ and $w_2 = (q+1)\tau$. Then $w_1 < w_2 < 2w_1$, as $q \geq 2$. The degenerate matrix $\overline{W}_0$ becomes

$$\overline{W}_{\text{degen}} = \begin{bmatrix} q^2 w_1 & q(q+1)w_1 \\ q(q+1)w_1 & (q+1)^2 w_1 \end{bmatrix}.$$

A basis for $\ker \overline{W}_{\text{degen}}$ is $\langle -(q+1), q \rangle$.

Use the linear codes $C_\pm$ of Proposition 17.2. They have the same length and w-weight enumerators. Their net rank-sums $\overline{\eta}_i(C_+) - \overline{\eta}_i(C_-)$ are $1, -(q+1), q$, respectively. Then the contributions of singletons are:

$$\overline{\text{Ann}} \begin{bmatrix} -(q+1) \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ (q^2-1)(q^2-q) \end{bmatrix}.$$

Because $w_1 < w_2 < 2w_1$, Corollary 5.4 applies to w_2. Then $A_{w_2}(C_+^\perp) - A_{w_2}(C_-^\perp) = A_{w_2}^{\text{sing}}(C_+^\perp) - A_{w_2}^{\text{sing}}(C_-^\perp) = (q^2-1)(q^2-q) > 0$. □

In Section 22, the case of $R = M_{3\times 3}(\mathbb{F}_2)$ is discussed in detail.

20. Rank partition enumerators

Section 3 described various enumerators including the complete enumerator and symmetrized enumerators. In this section we focus on a particular enumerator, the rank partition enumerator, over the matrix ring $M_{k\times k}(\mathbb{F}_q)$. The rank partition enumerator is a partition enumerator associated to rank, and it is coarser than the symmetrized enumerator associated to the group action of $\mathrm{GL}(k, \mathbb{F}_q)$ acting on $M_{k\times k}(\mathbb{F}_q)$.

On $R = M_{k\times k}(\mathbb{F}_q)$, define the *rank partition* $\mathcal{RK} = \{R_i\}_{i=0}^k$, with

$$R_i = \{s \in R : \operatorname{rk} s = i\}.$$

As in Section 3, define counting functions $n_i : R^n \to \mathbb{N}$, $i = 0, 1, \ldots, k$, by $n_i(x) = |\{j : x_j \in R_i\}|$, for $x = (x_1, x_2, \ldots, x_n) \in R^n$. For a linear code $C \subseteq R^n$, define the *rank partition enumerator* rpe_C associated to C to be the homogeneous polynomial of degree n in the variables $Z_0, Z_1, \ldots, Z_k$ given by

$$\mathrm{rpe}_C(Z) = \sum_{x\in C} \prod_{j=1}^n Z_{\operatorname{rk} x_j} = \sum_{x\in C} \prod_{i=0}^k Z_i^{n_i(x)}.$$

If w is a weight on R with maximal symmetry and positive integer values, then the value of $w(r)$, $r \in R$, depends only on the rank $\operatorname{rk} r$ of r. Write w_i for $w(r)$ when $\operatorname{rk} r = i$, and denote by $w_{\max}$ the largest value of w. Then the specialization of variables $Z_i \rightsquigarrow X^{w_{\max}-w_i}Y^{w_i}$ allows one to write wwe_C in terms of rpe_C:

$$\mathrm{wwe}_C(X,Y) = \mathrm{rpe}_C(Z)|_{Z_i \rightsquigarrow X^{w_{\max}-w_i}Y^{w_i}}. \tag{20.1}$$

As an example of some of the results of [8], we will show that the rank partition enumerator satisfies the MacWilliams identities. Then (20.1) will allow us to calculate $\mathrm{wwe}_{C^\perp}$ for many examples. This will be one way to illustrate the main results of Section 19 (and to prove additional results).

In order to show that the MacWilliams identities hold for the rank partition enumerator, we refer to the argument outlined in Appendix A and provide details on the relevant Fourier transforms.

It is well-known ([25, Example 4.4]) that $R = M_{k\times k}(\mathbb{F}_q)$ is a Frobenius ring with a generating character χ. To describe the standard generating character χ, we first recall the standard generating character θ_q of $\mathbb{F}_q$: $\theta_q(a) = \zeta^{\mathrm{Tr}_{q\to p}(a)}$, $a \in \mathbb{F}_q$, where $q = p^e$, p prime, $\zeta = \exp(2\pi i/p) \in \mathbb{C}^\times$, and $\mathrm{Tr}_{q\to p}$ is the absolute trace from $\mathbb{F}_q \to \mathbb{F}_p$. Then define $\chi(s) = \theta_q(\operatorname{tr} s)$, $s \in R$, where tr is the matrix trace. Because the matrix trace over $\mathbb{F}_q$ satisfies $\operatorname{tr}(rs) = \operatorname{tr}(sr)$, we see that $\chi(rs) = \chi(sr)$ for all $r, s \in R$. As is the case for all generating characters, χ has the property that $\ker\chi$ contains no nonzero left or right ideal of R, [25, Lemma 4.1].

Lemma 20.2. *Each partition block R_i of $\mathcal{RK}$ is invariant under left or right multiplication by units. If $\operatorname{rk} r_1 = \operatorname{rk} r_2$, then $\sum_{s\in R_i}\chi(sr_1) = \sum_{s\in R_i}\chi(sr_2)$ for all i.*

Proof. The rank of a matrix is invariant under multiplication by units. If $\operatorname{rk} r_1 = \operatorname{rk} r_2$, then, using row and column operations, there are units u_1, u_2 such that $r_2 = u_1 r_1 u_2$. Thus,

$$\sum_{s\in R_i}\chi(sr_2) = \sum_{s\in R_i}\chi(su_1r_1u_2) = \sum_{s\in R_i}\chi(u_2su_1r_1) = \sum_{s\in R_i}\chi(sr_1),$$

using the property $\chi(rs) = \chi(sr)$ and the bi-invariance of R_i. ☐

Define the *Kravchuk matrix* K of size $(k+1)\times(k+1)$ for the rank partition $\mathcal{RK}$ by

$$K_{i,j} = \sum_{s\in R_i}\chi(sr), \quad i, j = 0, 1, \ldots, k,$$

where $r \in R$ has $\operatorname{rk} r = j$. This sum is well-defined by Lemma 20.2.

In order to develop an explicit formula for $K_{i,j}$, we first remark that R_i, being invariant under left multiplication by units, equals the disjoint union of the left $\mathcal{U}$-orbits it contains. We already know the number and sizes of the $\mathcal{U}$-orbits in R, Proposition 12.6. So, we turn our attention to sums of the form $\sum_{t \in \operatorname{orb}(s)} \chi(tr)$, for $r \in R$.

Let $\mathcal{P}_R$ be the poset of all principal left ideals of R under set containment; all the left ideals of R are principal. (This is the type of poset used in Proposition 12.3, with $M = R$ and $m = k$.) Fix an element $r \in R$, and define two functions $f_r, g_r : \mathcal{P}_R \to \mathbb{C}$ by

$$f_r(Rs) = \sum_{t \in Rs} \chi(tr), \quad g_r(Rs) = \sum_{t \in \operatorname{orb}(s)} \chi(tr).$$

The definition of g_r is well-defined by Lemma 12.2. We collect some facts about f_r and g_r in the next lemma. For $r \in R$, its left *annihilator* is $\operatorname{ann}_{\mathrm{lt}}(r) = \{s \in R : sr = 0\}$; the left annihilator is a left ideal of R.

Lemma 20.3. *For any $r, s \in R$, we have*

$$f_r(Rs) = \sum_{Rt \subseteq Rs} g_r(Rt).$$

The values of f_r are

$$f_r(Rs) = \begin{cases} |Rs|, & \text{if } Rs \subseteq \operatorname{ann}_{\mathrm{lt}}(r), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first equality reflects the fact that the left ideal Rs is invariant under left multiplication by units, and hence Rs equals the disjoint union of the left $\mathcal{U}$-orbits it contains.

As in (2.2), denote $\mathfrak{R}(Rs) = \{r \in R : \chi(tr) = 1, \text{for all } t \in Rs\}$. We claim that $r \in \mathfrak{R}(Rs)$ if and only if $Rs \subseteq \operatorname{ann}_{\mathrm{lt}}(r)$. The 'if' direction is clear: if $Rs \subseteq \operatorname{ann}_{\mathrm{lt}}(r)$, then $tr = 0$, and hence $\chi(tr) = 1$, for all $t \in Rs$. Conversely, suppose $r \in \mathfrak{R}(Rs)$. Then $Rsr \subseteq \ker \chi$. We conclude that $Rsr = 0$, because any left ideal in $\ker \chi$ must be zero. Thus $Rs \subseteq \operatorname{ann}_{\mathrm{lt}}(r)$. The second formula now follows from Lemma A.1. $\square$

Proposition 20.4. *Let μ be the Möbius function of the poset $\mathcal{P}_R$. Then*

$$g_r(Rs) = \sum_{Rt \subseteq Rs} \mu(Rt, Rs) f_r(Rt), \quad Rs \in \mathcal{P}_R.$$

Simplifying, we have

$$g_r(Rs) = \sum_{j=0}^{\operatorname{rk} s - \operatorname{rk}(sr)} (-1)^{\operatorname{rk} s - j} q^{\binom{\operatorname{rk} s - j}{2}} q^{kj} \begin{bmatrix} \operatorname{rk} s - \operatorname{rk}(sr) \\ j \end{bmatrix}_q.$$

Proof. The first equation comes from Möbius inversion, [22, Theorem 5.5.5], because of the first equation in Lemma 20.3.

By Proposition 12.3, the poset $\mathcal{P}_R$ is isomorphic to the poset $\mathcal{P}_{k,k}$ of linear subspaces of $\mathbb{F}_q^k$. This allows us to translate the equation for $g_r(Rs)$ into geometric language. Let $r \in R$ correspond to a linear subspace $Y \subseteq \mathbb{F}_q^k$, with $\dim Y = \operatorname{rk} r$. Then $|Rr| = q^{k \dim Y} = q^{k \operatorname{rk} r}$, by Proposition 12.6. Similarly, let $s, t \in R$ correspond to X, T, with $\dim X = \operatorname{rk} s$ and $\dim T = \operatorname{rk} t$. The Möbius function of $\mathcal{P}_{k,k}$ is in (12.7). The condition $Rs \subseteq \operatorname{ann}_{\mathrm{lt}}(r)$ becomes $X \subseteq Y^\perp$.

Using Lemma 20.3, the formula for $g_r(Rs)$ simplifies:

$$\begin{aligned} g_r(Rs) &= \sum_{Rt \subseteq Rs \cap \operatorname{ann}_{\mathrm{lt}}(r)} \mu(Rt, Rs)|Rt| \\ &= \sum_{T \subseteq X \cap Y^\perp} (-1)^{\dim X - \dim T} q^{\binom{\dim X - \dim T}{2}} q^{k \dim T} \\ &= \sum_{j=0}^{\dim(X \cap Y^\perp)} (-1)^{\operatorname{rk} s - j} q^{\binom{\operatorname{rk} s - j}{2}} q^{kj} \begin{bmatrix} \dim(X \cap Y^\perp) \\ j \end{bmatrix}_q . \end{aligned}$$

Finally, Lemma 13.2 implies $\dim(X \cap Y^\perp) = \operatorname{rk} s - \operatorname{rk}(sr)$. □

To simplify notation slightly, define

$$B(i, \ell) = \sum_{j=0}^{\ell} (-1)^{i-j} q^{\binom{i-j}{2}} q^{kj} \begin{bmatrix} \ell \\ j \end{bmatrix}_q , \tag{20.5}$$

for $i = 0, 1, \ldots, k$ and $0 \leq \ell \leq i$. Then $\sum_{t \in \operatorname{orb}(s)} \chi(tr) = g_r(Rs) = B(\operatorname{rk} s, \operatorname{rk} s - \operatorname{rk}(sr))$. In addition, suppose there are linear subspaces $C \subseteq A \subseteq \mathbb{F}_q^k$ with $\dim A = a$ and $\dim C = c$. Then define $I(a, b, c, d) = |\{B \subseteq A : \dim B = b, \text{ and } \dim(B \cap C) = d\}|$. By Lemma 13.10 (with $D = A \cap C$),

$$I(a, b, c, d) = q^{(b-d)(c-d)} \begin{bmatrix} c \\ d \end{bmatrix}_q \begin{bmatrix} a - c \\ b - d \end{bmatrix}_q .$$

Proposition 20.6. *The Kravchuk matrix K has entries*

$$K_{i,j} = \sum_{\ell=0}^{i} I(k, i, k - j, \ell) B(i, \ell),$$

for $i, j = 0, 1, \ldots, k$.

Proof. As mentioned earlier, if $j = \operatorname{rk} r$, the sum in $K_{i,j} = \sum_{s \in R_i} \chi(sr)$ can be split up into sums over the left $\mathcal{U}$-orbits contained in R_i. The individual sums over orbits depend upon $\operatorname{rk} s$ and $\operatorname{rk}(sr)$, so we need to count the number of orbits $\operatorname{orb}(s)$ with $\operatorname{rk} s = i$ for various values of $\operatorname{rk}(sr)$.

In terms of linear subspaces, we need to count the number of linear subspaces X of $\mathbb{F}_q^k$ with $\dim X = i$ and $\dim(X \cap Y^\perp) = \ell$. Because $\dim Y^\perp = k - \operatorname{rk} r = k - j$, this count is $I(k, i, k-j, \ell)$. □

Example 20.7. For $k = 2$, the Kravchuk matrix is:

$$K = \begin{bmatrix} 1 & 1 & 1 \\ (q^2-1)(q+1) & q^2-q-1 & -q-1 \\ (q^2-q)(q^2-1) & -q^2+q & q \end{bmatrix}.$$

Suppose $C \subseteq R^n$ is an additive code. The annihilators $\mathfrak{L}(C)$ and $\mathfrak{R}(C)$ were defined in (2.2). The MacWilliams identities for the rank partition enumerator are next.

Theorem 20.8. *Let $R = M_{k\times k}(\mathbb{F}_q)$ with Kravchuk matrix K. If C is an additive code in R^n, then*

$$\operatorname{rpe}_{\mathfrak{L}(C)}(Z_i) = \frac{1}{|C|} \operatorname{rpe}_C(\mathcal{Z}_j)|_{\mathcal{Z}_j = \sum_i K_{i,j} Z_i},$$
$$\operatorname{rpe}_{\mathfrak{R}(C)}(Z_i) = \frac{1}{|C|} \operatorname{rpe}_C(\mathcal{Z}_j)|_{\mathcal{Z}_j = \sum_i K_{i,j} Z_i}.$$

The formulas are reversible in C and $\mathfrak{L}(C)$, resp., C and $\mathfrak{R}(C)$.

If $C \subseteq R^n$ is a left, resp., right, R-linear code, then $\mathcal{R}(C) = \mathfrak{R}(C)$, resp., $\mathcal{L}(C) = \mathfrak{L}(C)$.

Proof. We add details to the outline provided in Appendix A. Let $V = \mathbb{C}[Z_0, Z_1, \ldots, Z_k]$, and define $f : R \to V$ by $f(s) = Z_{\operatorname{rk} s}$, $s \in R$. We calculate the Fourier transform of f, as in (A.2). Write $j = \operatorname{rk} r$.

$$\begin{aligned} \widehat{f}(r) &= \sum_{s\in R} \chi(rs) Z_{\operatorname{rk} s} = \sum_{s \in R} \chi(sr) Z_{\operatorname{rk} s} \\ &= \sum_{i=0}^{k} \sum_{s \in R_i} \chi(sr) Z_i = \sum_{i=0}^{k} K_{i,j} Z_i. \end{aligned}$$

Note that $\widehat{f}(r)$ depends only on $\operatorname{rk} r$, so that $\widehat{f}(r)$ equals $f(r)$ after applying the linear substitution $Z_j \leftarrow \sum_{i=0}^k K_{i,j} Z_i$.

To reverse roles, use Lemma 2.3 and apply the formulas to the pair $\mathfrak{L}(C)$ and $\mathfrak{R}(\mathfrak{L}(C)) = C$ and the pair $\mathfrak{R}(C)$ and $\mathfrak{L}(\mathfrak{R}(C)) = C$. For the case of linear codes, see Remark 2.4. □

21. Examples

In this section we calculate a number of examples over $R = M_{2\times 2}(\mathbb{F}_2)$. Set $m = 3$, so that the information module is $M = M_{2\times 3}(\mathbb{F}_2)$. The orbit spaces $\mathcal{O}_1$ and $\mathcal{O}_2$ have representatives in reduced row-echelon form and are ordered as follows:

$$\begin{aligned} \mathcal{O}_1 &= [\left[\begin{smallmatrix}1&0&1\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix}\right]], \\ \mathcal{O}_2 &= [\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0&1\\0&1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&1&0\\0&0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0&0\\0&1&1\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}\right]]. \end{aligned} \tag{21.1}$$

The rank 1 orbits have size 3, and the rank 2 orbits have size 6, as in Proposition 12.6. The representatives of $\mathcal{O}_1^\sharp$ and $\mathcal{O}_2^\sharp$ are the transposes of the representatives of $\mathcal{O}_1$ and $\mathcal{O}_2$, using the same orderings, with rank 1 coming before rank 2. The W_0-matrix has size 14×14, while $\overline{W}_0$ is 2×2:

$$W_0 = \begin{bmatrix} 0 & 0 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 \\ 0 & w_1 & 0 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 0 & w_1 & w_1 & w_1 & 0 & 0 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 \\ 0 & w_1 & w_1 & w_1 & 0 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & 0 \\ w_1 & w_1 & w_1 & 0 & 0 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 \\ w_1 & 0 & 0 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & 0 & w_1 \\ w_1 & w_1 & 0 & 0 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 \\ w_1 & w_1 & 0 & w_1 & w_1 & w_1 & w_1 & w_2 & w_2 & w_1 & w_1 & w_2 & w_1 & w_2 \\ 0 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_2 & w_1 & w_1 & w_2 & w_2 & w_2 & w_1 \\ w_1 & 0 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_2 & w_2 & w_2 & w_1 & w_2 \\ w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & 0 & w_1 & w_2 & w_2 & w_1 & w_2 & w_2 & w_1 \\ w_1 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_2 & w_2 & w_2 & w_2 & w_1 & w_1 & w_1 \\ w_1 & w_1 & w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_2 & w_1 & w_2 & w_1 & w_2 & w_2 \\ w_1 & w_1 & w_1 & 0 & w_1 & w_1 & w_1 & w_2 & w_1 & w_2 & w_1 & w_1 & w_2 & w_2 \end{bmatrix},$$

$$\overline{W}_0 = \begin{bmatrix} 4w_1 & 6w_1 \\ 6w_1 & 3w_1 + 4w_2 \end{bmatrix}.$$

Akin to Figure 1 on page 10, the rank partition enumerator of a linear code specializes to the w-weight enumerator of the code under the specialization $Z_0 \rightsquigarrow 1$, $Z_1 \rightsquigarrow t^{w_1}$ and $Z_2 \rightsquigarrow t^{w_2}$. Using the Kravchuk matrix K from Example 20.7, with $q = 2$:

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 9 & 1 & -3 \\ 6 & -2 & 2 \end{bmatrix},$$

the MacWilliams identities for the rank partition enumerator yield the rank partition enumerator for the dual code, Theorem 20.8. Using the same specialization, the w-weight enumerator of the dual code is obtained.

In all of the examples that follow, multiplicity functions η and lists ω of orbit weights are written in terms of the ordering of $\mathcal{O}$ given in (21.1), with ranks separated by vertical lines. All calculations were performed in SageMath [23]. Rank partition enumerators of dual codes are not listed because they would use too much space. Only the lowest order terms in the w-weight enumerators of dual codes are displayed.

It is differences such as $A_j(D^\perp) - A_j(C^\perp)$ that ultimately matter, so we will write $\delta\overline{\eta} = \overline{\eta}(D) - \overline{\eta}(C)$. By (18.6) and Example 18.2 with $q = 2$, the net contributions of rank i singletons to $\delta A^{\text{sing}}_{w_i} = A^{\text{sing}}_{w_i}(D^\perp) - A^{\text{sing}}_{w_i}(C^\perp)$ appear as the entries in the vector

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 0 \\ 6 & 0 & 0 \end{bmatrix} \delta\overline{\eta}. \tag{21.2}$$

Example 21.3. Set $w_1 = 2$ and $w_2 = 3$. Then $\mathring{w} = 2$, and $\mathring{w} \le w_i < 2\mathring{w}$ for $i = 1, 2$. This is a degenerate case, as $\det W_0 = 0$. Use the codes in Proposition 17.2, with an extra rank 1 functional so that the associated homomorphisms Λ are injective; see Remark 12.5. The multiplicity functions and lists of orbit weights are:

$$\begin{aligned} \eta_+ &= \langle 1|1,0,0,0,0,0,0|2,0,0,0,0,0,0\rangle, \\ \omega_+ &= \langle 0|4,4,2,4,6,6,6|8,6,6,6,8,6,8\rangle, \\ \eta_- &= \langle 0|1,1,0,0,0,1,1|0,0,0,0,0,0,0\rangle, \\ \omega_- &= \langle 0|4,4,2,4,6,6,6|8,6,6,6,8,6,8\rangle. \end{aligned}$$

The equation $\omega_+ = \omega_-$ is a feature of the construction in Proposition 17.2. Both codes have length 4, and $\delta\overline{\eta} = \langle -1, 3, -2\rangle$. Then (21.2) implies that $\delta A_2 = \delta A_2^{\text{sing}} = 0$, while $\delta A_3 = \delta A_3^{\text{sing}} = -6$. The enumerators are

$$\begin{aligned} \operatorname{se}_{C_+} &= Z_0^4 + 3Z_0^3 Z_1 + 9Z_0^2 Z_1^2 + 27Z_0 Z_1^3 + 6Z_0^2 Z_2^2 + 18Z_0 Z_1 Z_2^2, \\ \operatorname{se}_{C_-} &= Z_0^4 + 3Z_0^3 Z_1 + 9Z_0^2 Z_1^2 + 33Z_0 Z_1^3 + 18Z_1^4; \\ \operatorname{wwe}_{C_+} &= 1 + 3t^2 + 9t^4 + 33t^6 + 18t^8, \\ \operatorname{wwe}_{C_-} &= 1 + 3t^2 + 9t^4 + 33t^6 + 18t^8; \\ \operatorname{wwe}_{C_+^\perp} &= 1 + 12t^2 + 6t^3 + 36t^4 + \cdots, \\ \operatorname{wwe}_{C_-^\perp} &= 1 + 12t^2 \qquad\quad + 54t^4 + \cdots. \end{aligned}$$

Example 21.4. Set $w_1 = 1$ and $w_2 = 2$, so the weight of a matrix equals its rank. Then $\mathring{w} = 1$, but $w_2 = 2\mathring{w}$, so Corollary 5.4 applies only to $w_1 = 1$. Use the codes given by (16.5), with ς and σ given here:

$$\begin{aligned} \varsigma &= \langle 0,-1,-1,0,0,0,0|0,0,0,0,0,0,1\rangle, \\ \sigma &= \langle -1,-1,-1,-1,-3,1,-1|0,0,0,0,0,2,2\rangle. \end{aligned}$$

The scaling is such that $W_0\sigma = 2\varsigma$, so that $c = 2$. One calculates that $\alpha_1 = 10w_1 = 10$ and $\alpha_2 = 9w_1 + 4w_2 = 17$. It suffices to take $a = 3$ and $b = 0$. Then $\Delta = -3$. The multiplicity functions and lists of orbit weights are

$$\begin{aligned}
\eta_{C_2} &= \langle 0|3,3,3,26,3,3,3|3,3,3,3,3,3,3\rangle, \\
\eta_{D_2} &= \langle 3|2,2,2,25,0,4,2|3,3,3,3,3,5,5\rangle; \\
\omega_{C_2} &= \langle 0|30,53,53,53,30,53,30|74,74,74,74,74,74,51\rangle, \\
\omega_{D_2} &= \langle 0|30,51,51,53,30,53,30|74,74,74,74,74,74,53\rangle.
\end{aligned}$$

Both codes have length 65, and $\delta\overline{\eta} = \langle 3,-7,4\rangle$. Then, using (21.2), $\delta A_1 = \delta A_1^{\mathrm{sing}} = 6$ and $\delta A_2^{\mathrm{sing}} = 18$. The enumerators are

$$\begin{aligned}
\mathrm{se}_{C_2} &= Z_0^{65} + 9Z_0^{35}Z_1^{30} + 12Z_0^{12}Z_1^{53} + 6Z_0^{26}Z_1^{27}Z_2^{12} + 36Z_0^{3}Z_1^{50}Z_2^{12}, \\
\mathrm{se}_{D_2} &= Z_0^{65} + 9Z_0^{35}Z_1^{30} + 6Z_0^{14}Z_1^{51} + 6Z_0^{12}Z_1^{53} + 6Z_0^{3}Z_1^{50}Z_2^{12} \\
&\quad + 24Z_0^{5}Z_1^{46}Z_2^{14} + 6Z_0^{28}Z_1^{21}Z_2^{16} + 6Z_0^{7}Z_1^{42}Z_2^{16}; \\
\mathrm{wwe}_{C_2} &= 1 + 9\,t^{30} + 6\,t^{51} + 12\,t^{53} + 36\,t^{74}, \\
\mathrm{wwe}_{D_2} &= 1 + 9\,t^{30} + 6\,t^{51} + 12\,t^{53} + 36\,t^{74}; \\
\mathrm{wwe}_{C_2^\perp} &= 1 + 132\,t + 15762\,t^2 + 1674894\,t^3 + \cdots, \\
\mathrm{wwe}_{D_2^\perp} &= 1 + 138\,t + 16176\,t^2 + 1695210\,t^3 + \cdots.
\end{aligned}$$

Even though $\delta A_2^{\mathrm{sing}} = 18$, $\delta A_2 = 414$; there are dual codewords of weight 2 coming from vectors with two nonzero entries, both of rank 1, that account for the difference. This illustrates the importance of the strict inequality $\mathring{w} \le d < 2\mathring{w}$ in Corollary 5.4.

Example 21.5. Set $w_1 = 4$ and $w_2 = 5$. Then $\mathring{w} = 4$ and $\mathring{w} \le w_i < 2\mathring{w}$, $i = 1, 2$, so Corollary 5.4 applies to both w_1 and w_2.

In this example, the codes given by (16.5) will be used; call then C_3 and D_3. In addition, two other codes C_4, D_4 will be given. They are designed so that their lists of orbit weights have just three different values. One of the values applies to all the rank 2 orbits. The other two values apply to three, resp., four, rank 1 orbits. For the code C_4, the three rank 1 orbits are linearly independent, while for D_4 they are linearly dependent.

For C_3 and D_3, ς and σ are given here:

$$\begin{aligned}
\varsigma &= \langle 0,-1,-1,0,0,0,0|0,0,0,0,0,0,1\rangle, \\
\sigma &= \langle 2,2,2,-1,3,1,2|0,0,0,0,0,-4,-4\rangle.
\end{aligned}$$

The scaling is such that $W_0\sigma = 8\varsigma$, so that $c = 2$. One calculates that $\alpha_1 = 10w_1 = 40$ and $\alpha_2 = 9w_1 + 4w_2 = 56$. It suffices to take $a = 1$ and $b = 0$. Then $\Delta = 3$. The multiplicity functions and lists of orbit weights are

$$\begin{aligned}
\eta_{C_3} &= \langle 3|4,4,4,22,4,4,4|4,4,4,4,4,4,4\rangle, \\
\eta_{D_3} &= \langle 0|6,6,6,21,7,5,6|4,4,4,4,4,0,0\rangle; \\
\omega_{C_3} &= \langle 0|160,232,232,232,160,232,160 \\
&\qquad |296,296,296,296,296,296,224\rangle, \\
\omega_{D_3} &= \langle 0|160,224,224,232,160,232,160 \\
&\qquad |296,296,296,296,296,296,232\rangle.
\end{aligned}$$

Both codes have length 77, and $\delta\overline{\eta} = \langle -3, 11, -8\rangle$. Then, using (21.2), $\delta A_4 = \delta A_4^{\text{sing}} = 6$ and $\delta A_5 = \delta A_5^{\text{sing}} = -18$. The enumerators are

$$\begin{aligned}
\mathrm{se}_{C_3} &= Z_0^{77} + 9Z_0^{37}Z_1^{40} + 12Z_0^{19}Z_1^{58} + 6Z_0^{25}Z_1^{36}Z_2^{16} + 36Z_0^{7}Z_1^{54}Z_2^{16}, \\
\mathrm{se}_{D_3} &= Z_0^{77} + 9Z_0^{37}Z_1^{40} + 6Z_0^{21}Z_1^{56} + 6Z_0^{19}Z_1^{58} + 6Z_0^{21}Z_1^{48}Z_2^{8} \\
&\quad + 6Z_0^{5}Z_1^{64}Z_2^{8} + 24Z_0^{6}Z_1^{59}Z_2^{12} + 6Z_0^{7}Z_1^{54}Z_2^{16}; \\
\mathrm{wwe}_{C_3} &= 1 + 9\,t^{160} + 6\,t^{224} + 12\,t^{232} + 36\,t^{296}, \\
\mathrm{wwe}_{D_3} &= 1 + 9\,t^{160} + 6\,t^{224} + 12\,t^{232} + 36\,t^{296}; \\
\mathrm{wwe}_{C_3^\perp} &= 1 + 165\,t^4 + 18\,t^5 + 21186\,t^8 + \cdots, \\
\mathrm{wwe}_{D_3^\perp} &= 1 + 171\,t^4 \qquad\quad + 21918\,t^8 + \cdots.
\end{aligned}$$

The codes C_4, D_4 have multiplicity functions and lists of orbit weights:

$$\begin{aligned}
\eta_{C_4} &= \langle 0|2,3,3,1,2,3,2|2,6,2,6,6,2,6\rangle, \\
\eta_{D_4} &= \langle 0|2,2,4,2,2,2,2|6,6,2,2,6,2,6\rangle; \\
\omega_{C_4} &= \langle 0|136,144,144,136,136,144,136 \\
&\qquad |192,192,192,192,192,192,192\rangle, \\
\omega_{D_4} &= \langle 0|136,144,136,136,136,144,144 \\
&\qquad |192,192,192,192,192,192,192\rangle.
\end{aligned}$$

Both codes have length 46, and $\delta\overline{\eta} = \langle 0,0,0\rangle$. Then, using (21.2), $\delta A_4 = \delta A_4^{\text{sing}} = 0$ and $\delta A_5 = \delta A_5^{\text{sing}} = 0$. The enumerators are

$$
\begin{aligned}
\mathrm{se}_{C_4} &= Z_0^{46} + 12Z_0^{12}Z_1^{34} + 9Z_0^{10}Z_1^{36} + 6Z_0Z_1^{33}Z_2^{12} \\
&\quad + 18Z_0^2Z_1^{28}Z_2^{16} + 18Z_0^3Z_1^{23}Z_2^{20}, \\
\mathrm{se}_{D_4} &= Z_0^{46} + 12Z_0^{12}Z_1^{34} + 9Z_0^{10}Z_1^{36} + 36Z_0^2Z_1^{28}Z_2^{16} + 6Z_0^4Z_1^{18}Z_2^{24}; \\
\mathrm{wwe}_{C_4} &= 1 + 12\,t^{136} + 9\,t^{144} + 42\,t^{192}, \\
\mathrm{wwe}_{D_4} &= 1 + 12\,t^{136} + 9\,t^{144} + 42\,t^{192}; \\
\mathrm{wwe}_{C_4^\perp} &= 1 + 48\,t^4 + 4059\,t^8 + 1440\,t^9 + 522\,t^{10} + 290160\,t^{12} + \cdots, \\
\mathrm{wwe}_{D_4^\perp} &= 1 + 48\,t^4 + 4059\,t^8 + 1440\,t^9 + 522\,t^{10} + 290112\,t^{12} + \cdots.
\end{aligned}
$$

The calculation shows that $\delta A_{12} \neq 0$, but this would be difficult to detect theoretically.

Example 21.6. Set $w_1 = 3$ and $w_2 = 7$. Then $\mathring{w} = 3$, but $w_2 > 2\mathring{w}$, so Corollary 5.4 applies only to $w_1 = 3$. Use the codes given by (16.5), with ς and σ given here:

$$
\begin{aligned}
\varsigma &= \langle 0,-1,-1,0,0,0,0|0,0,0,0,0,0,1\rangle, \\
\sigma &= \langle -3,-3,-3,-5,-11,5,-3|0,0,0,0,0,6,6\rangle.
\end{aligned}
$$

The scaling is such that $W_0\sigma = 30\varsigma$, so that $c = 10$. One calculates that $\alpha_1 = 10w_1 = 30$ and $\alpha_2 = 9w_1 + 4w_2 = 55$. It suffices to take $a = 4$ and $b = 0$. Then $\Delta = -11$. The multiplicity functions and lists of orbit weights are

$$
\begin{aligned}
\eta_{C_5} &= \langle 0|12,12,12,122,12,12,12|12,12,12,12,12,12,12\rangle, \\
\eta_{D_5} &= \langle 11|9,9,9,117,1,17,9|12,12,12,12,12,18,18\rangle; \\
\omega_{C_5} &= \langle 0|360,690,690,690,360,690,360 \\
&\qquad |990,990,990,990,990,990,660\rangle, \\
\omega_{D_5} &= \langle 0|360,660,660,690,360,690,360 \\
&\qquad |990,990,990,990,990,990,690\rangle.
\end{aligned}
$$

Both codes have length 278, and $\delta\overline{\eta} = \langle 11, -23, 12\rangle$. Then, using (21.2), $\delta A_3 = \delta A_3^{\text{sing}} = 30$ and $\delta A_7^{\text{sing}} = 66$. The enumerators are

$$\begin{aligned}
\mathrm{se}_{C_5} &= Z_0^{278} + 9Z_0^{158}Z_1^{120} + 12Z_0^{48}Z_1^{230} + 6Z_0^{122}Z_1^{108}Z_2^{48} \\
&\quad + 36Z_0^{12}Z_1^{218}Z_2^{48}, \\
\mathrm{se}_{D_5} &= Z_0^{278} + 9Z_0^{158}Z_1^{120} + 6Z_0^{58}Z_1^{220} + 6Z_0^{48}Z_1^{230} + 6Z_0^{12}Z_1^{218}Z_2^{48} \\
&\quad + 24Z_0^{20}Z_1^{204}Z_2^{54} + 6Z_0^{128}Z_1^{90}Z_2^{60} + 6Z_0^{28}Z_1^{190}Z_2^{60}; \\
\mathrm{wwe}_{C_5} &= 1 + 9\,t^{360} + 6\,t^{660} + 12\,t^{690} + 36\,t^{990}, \\
\mathrm{wwe}_{D_5} &= 1 + 9\,t^{360} + 6\,t^{660} + 12\,t^{690} + 36\,t^{990}; \\
\mathrm{wwe}_{C_5^\perp} &= 1 + 582\,t^3 + 316947\,t^6 \qquad\qquad + 152382900\,t^9 + \cdots, \\
\mathrm{wwe}_{D_5^\perp} &= 1 + 612\,t^3 + 326649\,t^6 + 66\,t^7 + 154592448\,t^9 + \cdots.
\end{aligned}$$

Even though $w_2 = 7 > 2\mathring{w} = 6$, we still have $\delta A_7 = \delta A_7^{\text{sing}}$. The reason is that w_2 is not an integer multiple of w_1: a vector can have weight 7 only if it is a singleton with nonzero entry of rank 2. See Remark 5.5.

22. The case of $M_{3\times 3}(\mathbb{F}_q)$

In this section we show that most weights of maximal symmetry on $M_{3\times 3}(\mathbb{F}_q)$ do not respect duality. There is one situation that remains unsettled.

Theorem 22.1. *Let $R = M_{3\times 3}(\mathbb{F}_q)$, and suppose w in a weight on R with maximal symmetry, positive integer values, and $w(0) = 0$. If w is nondegenerate and not a multiple of the Hamming weight, then w does not respect duality. If w is degenerate because $-(q+1)w_1 + qw_2 = 0$, then w does not respect duality.*

The situation where w is degenerate because $-(q^2+q+1)w_1 + q(q^2+q+1)w_2 - q^3w_3 = 0$ is unsettled.

Proof. In order to display the matrices $\overline{W}_0$ and $\overline{\mathrm{Ann}}$ in a way that fits on the page, we name certain polynomials in q:

$$f_+ = q+1, \quad f_- = q-1, \quad f_2 = q^2+q+1.$$

The $\overline{W}_0$-matrix is

$$\overline{W}_0 = \begin{bmatrix} q^3w_1 & q^2f_+w_1 & qf_2w_1 \\ q^2f_2w_1 & qf_+^2w_1 + q^4w_2 & f_2w_1 + q^2f_2w_2 \\ qf_2w_1 & f_+w_1 + q^2f_+w_2 & f_2w_2 + q^3w_3 \end{bmatrix}.$$

Consistent with Theorem 15.10, the determinant of $\overline{W}_0$ factors as

$$\det \overline{W}_0 = -q^6w_1(-f_+w_1 + qw_2)(-f_2w_1 + qf_2w_2 - q^3w_3).$$

The annihilator matrix $\overline{\mathrm{Ann}}$ is

$$\overline{\mathrm{Ann}} = \begin{bmatrix} -q^2 f_- f_2 & -q f_+ f_- f_2 & -f_- f_2^2 \\ -q^2 f_+^2 f_-^2 f_2 & -q f_+ f_-^2 f_2^2 & -q f_+ f_-^2 f_2^2 \\ -q^3 f_+ f_-^3 f_2 & -q^3 f_+ f_-^3 f_2 & -q^3 f_+ f_-^3 f_2 \end{bmatrix}.$$

Suppose w is nondegenerate, i.e., $\overline{W}_0$ is invertible. We first collect the results of some calculations of (18.8) made using SageMath. The leading terms of $\overline{\mathrm{Ann}}\,\bar{\sigma}^{(s)}$, where $\bar{\sigma}^{(s)} = \overline{W}_0^{-1}\bar{\varsigma}^{(s)}$, for the swaps $\bar{\varsigma}^{(1)} = \langle -q^3+q, 1, 0\rangle$ and $\bar{\varsigma}^{(2)} = \langle 0, -q^3+q^2, 1\rangle$ are:

$$\overline{\mathrm{Ann}}\,\bar{\sigma}^{(1)} = \left\langle \frac{q^5-q^3-q^2+1}{w_1}, *, \frac{\alpha_1}{\beta_1} \right\rangle, \tag{22.2}$$

$$\overline{\mathrm{Ann}}\,\bar{\sigma}^{(2)} = \left\langle 0, \frac{q^7-q^6-q^5+q^3+q^2-q}{f_+w_1-qw_2}, \frac{\alpha_2}{\beta_2} \right\rangle. \tag{22.3}$$

Note that the denominators are nonzero because $\overline{W}_0$ is invertible.

The term at rank 3 of $\overline{\mathrm{Ann}}\,\bar{\sigma}^{(1)}$ has numerator:

$$\begin{aligned} \alpha_1 = {} & f_+f_-^3f_2(q^6+q^5-q^4-2q^3+q+1)w_1^2 \\ & - qf_+f_-^3f_2(q^6+2q^5-3q^3-2q^2+1)w_1w_2 + q^3f_+^2f_-^4f_2^2w_2^2 \\ & + q^3f_+f_-^3f_2(q^4-2q^2-q+1)w_1w_3 - q^5f_+^2f_-^4f_2w_2w_3, \end{aligned}$$

and nonzero denominator

$$\beta_1 = w_1(-f_+w_1+qw_2)(-f_2w_1+qf_2w_2-q^3w_3).$$

The term at rank 3 of $\overline{\mathrm{Ann}}\,\bar{\sigma}^{(2)}$ has numerator:

$$\begin{aligned} \alpha_2 = {} & qf_+f_-^3f_2(q^3-q^2-q-1)w_1 \\ & - q^2f_+f_-^3f_2(q^3-q-1)w_2 + q^4f_+f_-^4f_2w_3 \end{aligned}$$

and nonzero denominator

$$\beta_2 = (-f_+w_1+qw_2)(-f_2w_1+qf_2w_2-q^3w_3).$$

Using SageMath, one can solve for those nonzero weights with $\alpha_1 = \alpha_2 = 0$, namely (up to uniform scale factors):

$$w_1 = 1, \qquad w_2 = \frac{q^3-q-1}{q^3-q}, \qquad w_3 = \frac{q^5-q^4-q^3+q+1}{q^5-q^4-q^3+q^2};$$

$$w_1 = 1, \qquad w_2 = \frac{q+1}{q}, \qquad w_3 = \frac{q^2+q+1}{q^2}.$$

In the first case, one confirms that w is nondegenerate and that $w_2 < w_3 < w_1$. In the second case, one notes that w is degenerate (both $-f_+w_1+qw_2$ and $-f_2w_1+qf_2w_2-q^3w_3$ vanish in the factorization of $\det\overline{W}_0$) and that $w_1 < w_2 < w_3$.

One can also solve for those nonzero weights where the sum of the entries of $\overline{\mathrm{Ann}}\,\bar{\sigma}^{(2)}$ vanishes:

$$w_1 = (q+1)w_2 - qw_3, \tag{22.4}$$

or where $w_1 = w_3$ and $\alpha_2 = 0$:

$$w_2 = (q^4 - q^2 - q - 1)w_1/(q^4 - q^2 - q) < w_1 = w_3. \tag{22.5}$$

Now apply the results of the calculations, still assuming that w is nondegenerate.

- If $w_1 < \min\{w_2, w_3\}$, then $\delta A_{w_1} = \delta A^{\mathrm{sing}}_{w_1} \neq 0$, using (22.2).
- If $w_2 < \min\{w_1, w_3\}$, then $\delta A_{w_2} = \delta A^{\mathrm{sing}}_{w_2} \neq 0$, using (22.3).
- If $w_3 < \min\{w_1, w_2\}$, then $\delta A_{w_3} = \delta A^{\mathrm{sing}}_{w_3} \neq 0$. At least one of α_1, α_2 is nonzero (because $w_3 < w_2$), so use the corresponding (22.2) or (22.3).
- If $\mathring{w} = w_1 = w_2 < w_3$, then $\delta A_{\mathring{w}} = \delta A^{\mathrm{sing}}_{\mathring{w}} \neq 0$, using (22.3).
- If $\mathring{w} = w_1 = w_3 < w_2$, then $\delta A_{\mathring{w}} = \delta A^{\mathrm{sing}}_{\mathring{w}} \neq 0$, using (22.3). Here, $\alpha_2 \neq 0$ by (22.5).
- If $\mathring{w} = w_2 = w_3 < w_1$, then $\delta A_{\mathring{w}} = \delta A^{\mathrm{sing}}_{\mathring{w}} \neq 0$, using (22.3). The sum of the rank 2 and rank 3 contributions does not vanish. If it did, (22.4) and $w_2 = w_3$ would imply $w_1 = w_2 = w_3$. This contradicts the hypothesis that $w_2 = w_3 < w_1$.
- If $w_1 = w_2 = w_3$, then w is a multiple of the Hamming weight.

If w is degenerate with $-(q+1)w_1 + qw_2 = 0$, then $w_1 < w_2 < 2w_1$. The construction of Proposition 17.2, with $j = 2$, has $\delta\overline{\eta} = \langle 1, -(q+1), q, 0\rangle$. Then, dropping the initial term of $\delta\overline{\eta}$, we have

$$\overline{\mathrm{Ann}}\,\delta\overline{\eta} = q^3 f_+ f_-^2 f_2 \langle 0, 1, f_-\rangle.$$

- If $w_1 < w_3$, then $\delta A_{w_2} = \delta A^{\mathrm{sing}}_{w_2} \neq 0$, using Corollary 5.4. This still works if $w_2 = w_3$ because $1 + f_- = q \neq 0$.
- If $w_3 \leq w_1$, then $\delta A_{w_3} = \delta A^{\mathrm{sing}}_{w_3} \neq 0$. □

In the situation where w is degenerate with $-(q^2+q+1)w_1 + q(q^2+q+1)w_2 - q^3 w_3 = 0$, the construction of Proposition 17.2, with $j = 3$, has $\delta\overline{\eta} = \langle 1, -(q^2+q+1), q(q^2+q+1), -q^3\rangle$. After dropping the rank 0 term, we have

$$\overline{\mathrm{Ann}}\,\delta\overline{\eta} = q^3 f_+ f_-^3 f_2 \langle 0, 0, 1\rangle.$$

If $\mathring{w} \leq w_3 < 2\mathring{w}$, then $\delta A_{w_3} = \delta A^{\mathrm{sing}}_{w_3} \neq 0$, and w does not respect duality.

From the degeneracy equation we have $(q^2+q+1)(qw_2 - w_1) = q^3 w_3$. Because q^3 and q^2+q+1 are relatively prime, there exists a positive integer a such that $qw_2 - w_1 = q^3 a$ and $w_3 = (q^2+q+1)a$. Similarly, $w_1 = q(w_2 - q^2 a)$, so there exists a positive integer b such that $w_1 = qb$

and $w_2 - q^2 a = b$. In all, $w_1 = qb$, $w_2 = q^2 a + b$, and $w_3 = (q^2+q+1)a$, for some positive integers a, b.

Because of Proposition 3.14, only the ratio $\rho = a/b \in \mathbb{Q}$, $\rho > 0$, matters. One can show that $\mathring{w} = w_3$ when $0 < \rho \leq q/(q^2+q+1)$ and that $\mathring{w} = w_1 < w_3 < 2w_1$ when $q/(q^2+q+1) < \rho < 2q/(q^2+q+1)$. Thus, for $0 < \rho < 2q/(q^2+q+1)$, w does not respect duality. For $\rho \geq 2q/(q^2+q+1)$, singletons alone are not enough to decide whether w respects duality.

For example, when the weight w over $\mathbb{F}_2$ has $w_1 = 2$, $w_2 = 5$, and $w_3 = 7$ (so $\rho = 1 > 4/7$), calculations like those in Example 21.5 show that $\delta A_6 \neq 0$. The dual codewords involved have three nonzero entries of rank 1. The combinatorics of dual codewords of this type can be very complicated (see [1, §3] for the situation over finite fields), and we will not pursue the matter further.

Appendix A. Fourier transform

This appendix will be a brief review without proof of the use of the Fourier transform and the Poisson summation formula in proving the MacWilliams identities for additive codes over a finite Frobenius ring. The main ideas go back to Gleason (see [3]) and can be generalized to additive codes over a finite abelian group. Details can be found [28].

In this appendix, R is a finite Frobenius ring with generating character χ. The annihilators $\mathfrak{L}(C)$ and $\mathfrak{R}(C)$ were defined in (2.2).

Lemma A.1. *Suppose $C \subseteq R^n$ is an additive code. If $y \in R^n$, then*

$$\sum_{x \in C} \chi(x \cdot y) = \begin{cases} |C|, & \text{if } y \in \mathfrak{R}(C), \\ 0, & \text{otherwise}, \end{cases}$$

$$\sum_{x \in C} \chi(y \cdot x) = \begin{cases} |C|, & \text{if } y \in \mathfrak{L}(C), \\ 0, & \text{otherwise}. \end{cases}$$

Let V be a vector space over the complex numbers $\mathbb{C}$. If $f : R^n \to V$ is any function, define its *Fourier transform* $\widehat{f} : R^n \to V$ by

$$\widehat{f}(x) = \sum_{y \in R^n} \chi(x \cdot y) f(y), \quad x \in R^n. \tag{A.2}$$

There is an inversion formula:

$$f(x) = \frac{1}{|R^n|} \sum_{y \in R^n} \chi(-y \cdot x) \widehat{f}(y), \quad x \in R^n.$$

This version of the Fourier transform differs from that in [28] in that the isomorphism $x \mapsto {}^x\chi$ has been incorporated into the definition.

Suppose, in addition, that V is an algebra over $\mathbb{C}$. If $f_i : R \to V$ for $i = 1, 2, \ldots, n$, and $F : R^n \to V$ is given by $F(r_1, r_2, \ldots, r_n) = \prod_{i=1}^n f_i(r_i)$, then $\widehat{F}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n \widehat{f}_i(x_i)$.

Theorem A.3 (Poisson summation formula)**.** *Suppose R is Frobenius and $C \subseteq R^n$ is an additive code. If $f : R^n \to V$, then*

$$\sum_{y \in \mathfrak{R}(C)} f(y) = \frac{1}{|C|} \sum_{x \in C} \widehat{f}(x).$$

Remark A.4. There is another version of the Fourier transform, with $\widehat{f}(x) = \sum_{y \in R^n} \chi(y \cdot x) f(y)$, that incorporates the isomorphism $x \mapsto \chi^x$ instead. Then the Poisson summation formula has the form

$$\sum_{y \in \mathfrak{L}(C)} f(y) = \frac{1}{|C|} \sum_{x \in C} \widehat{f}(x).$$

If χ has the property that $\chi(rs) = \chi(sr)$, $r, s \in R$, then the two versions of the Fourier transform agree and both forms of the Poisson summation formula are valid. This situation occurs over $M_{k \times k}(\mathbb{F}_q)$.

In order to prove the MacWilliams identities over a finite Frobenius ring R for a partition enumerator pe or a w-weight enumerator wwe as described in Section 3, here is the standard argument. There are generalizations of this argument in [8]. For a partition enumerator whose partition $\mathcal{P} = \{P_i\}_{i=1}^m$ of R has m blocks, set $V = \mathbb{C}[Z_1, \ldots, Z_m]$, with one variable for each block. Define $[r] = i$ when $r \in P_i$. For a w-weight enumerator, whose weight w has positive integer values with maximum value $w_{\max}$, set $V = \mathbb{C}[X, Y]$. Define $f : R \to V$ by

$$f(a) = \begin{cases} Z_{[a]}, & \text{for pe,} \\ X^{w_{\max} - w(a)} Y^{w(a)}, & \text{for wwe.} \end{cases}$$

On R^n, define $F : R^n \to V$ by $F(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n f(x_i)$. For an additive code $C \subseteq R^n$, note that

$$\sum_{x \in C} F(x) = \begin{cases} \mathrm{pe}_C(Z_1, \ldots, Z_m), & \text{for pe,} \\ \mathrm{wwe}_C(X, Y), & \text{for wwe.} \end{cases}$$

The next steps depend on the specific ring, partition, and weight w:

- calculate the Fourier transform of $f : R \to V$;
- find $\widehat{F}$ by the product formula above;
- recognize the form of $\widehat{F}$ as an enumerator (if possible);
- apply the Poisson summation formula.

Some care must be taken to show that the form of $\widehat{F}$ is that of an enumerator. Care must also be taken to check if one can reverse the roles of the code and its annihilator. See [8] for details.

References

[1] N. Abdelghany and J. A. Wood, *Failure of the MacWilliams identities for the Lee weight enumerator over $\mathbb{Z}_m$, $m \geqslant 5$*, Discrete Math. **343** (2020), no. 11, 112036, 12. MR 4119402

[2] A. Alahmadi, H. Sboui, P. Solé, and O. Yemen, *Cyclic codes over $M_2(\mathbb{F}_2)$*, J. Franklin Inst. **350** (2013), no. 9, 2837–2847. MR 3146951

[3] E. F. Assmus, Jr. and H. F. Mattson, Jr., *Coding and combinatorics*, SIAM Rev. **16** (1974), 349–388. MR 0359982 (50 #12432)

[4] A. E. Clark and D. A. Drake, *Finite chain rings*, Abh. Math. Sem. Univ. Hamburg **39** (1973), 147–153. MR 0332875 (48 #11200)

[5] I. Constantinescu and W. Heise, *A metric for codes over residue class rings of integers*, Problemy Peredachi Informatsii **33** (1997), no. 3, 22–28. MR 1476368 (99a:94058)

[6] P. Delsarte, *Bounds for unrestricted codes, by linear programming*, Philips Res. Rep. **27** (1972), 272–289. MR 0314545 (47 #3096)

[7] S. T. Dougherty and M. M. Skriganov, *MacWilliams duality and the Rosenbloom-Tsfasman metric*, Mosc. Math. J. **2** (2002), no. 1, 81–97, 199. MR 1900585

[8] H. Gluesing-Luerssen, *Fourier-reflexive partitions and MacWilliams identities for additive codes*, Des. Codes Cryptogr. **75** (2015), no. 3, 543–563. MR 3336966

[9] M. Greferath, Th. Honold, C. Mc Fadden, J. A. Wood, and J. Zumbrägel, *MacWilliams' extension theorem for bi-invariant weights over finite principal ideal rings*, J. Combin. Theory Ser. A **125** (2014), 177–193. MR 3207471

[10] M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams's equivalence theorem*, J. Combin. Theory Ser. A **92** (2000), no. 1, 17–28. MR 1783936 (2001j:94045)

[11] P. Hall, *The Eulerian functions of a group*, Quart. J. Math. **7** (1936), 134–151.

[12] A. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, *The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes*, IEEE Trans. Inform. Theory **40** (1994), no. 2, 301–319. MR 1294046 (95k:94030)

[13] T. Honold, *Characterization of finite Frobenius rings*, Arch. Math. (Basel) **76** (2001), no. 6, 406–415. MR 1831096 (2002b:16033)

[14] T. Honold and A. A. Nechaev, *Weighted modules and representations of codes*, Problems Inform. Transmission **35** (1999), no. 3, 205–223. MR 1730800 (2001f:94016)

[15] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR 1653294 (99i:16001)

[16] F. J. MacWilliams, *Combinatorial problems of elementary abelian groups*, ProQuest LLC, Ann Arbor, MI, 1962, Thesis (Ph.D.)–Radcliffe College. MR 2939359

[17] ———, *A theorem on the distribution of weights in a systematic code*, Bell System Tech. J. **42** (1963), 79–94. MR 0149978 (26 #7462)

[18] F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland Publishing Co., Amsterdam, 1977, North-Holland Mathematical Library, Vol. 16. MR 0465509 (57 #5408a)

[19] B. R. McDonald, *Finite rings with identity*, Pure and Applied Mathematics, Vol. 28, Marcel Dekker, Inc., New York, 1974. MR 0354768

[20] M. Moeini, R. Rezaei, and K. Samei, *MacWilliams identity for linear codes over finite chain rings with respect to homogeneous weight*, Bull. Korean Math. Soc. **58** (2021), no. 5, 1163–1173. MR 4319450

[21] W. W. Peterson and E. J. Weldon, Jr., *Error-correcting codes*, 2 ed., The M.I.T. Press, Cambridge, Mass.-London, 1972. MR 49 #12164

[22] B. E. Sagan, *Combinatorics: the art of counting*, Graduate Studies in Mathematics, vol. 210, American Mathematical Society, Providence, RI, 2020. MR 4249619

[23] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.5)*, 2022, `https://www.sagemath.org`.

[24] J. A. Wood, *Extension theorems for linear codes over finite rings*, Applied algebra, algebraic algorithms and error-correcting codes (Toulouse, 1997), Lecture Notes in Comput. Sci., vol. 1255, Springer, Berlin, 1997, pp. 329–340. MR 1634126

[25] ———, *Duality for modules over finite rings and applications to coding theory*, Amer. J. Math. **121** (1999), no. 3, 555–575. MR 1738408

[26] ———, *The structure of linear codes of constant weight*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1007–1026. MR 1867370 (2002k:94042)

[27] ———, *Code equivalence characterizes finite Frobenius rings*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 699–706. MR 2358511 (2008j:94062)

[28] ———, *Foundations of linear codes defined over finite modules: the extension theorem and the MacWilliams identities*, Codes over rings (Ankara, 2008) (P. Solé, ed.), Ser. Coding Theory Cryptol., vol. 6, World Sci. Publ., Hackensack, NJ, 2009, pp. 124–190. MR 2850303

[29] ———, *Isometry groups of additive codes over finite fields*, J. Algebra Appl. **17** (2018), no. 10, 39 pages, paper 1850198. MR 3866771

[30] ———, *Homogeneous weight enumerators over integer residue rings and failures of the MacWilliams identities*, Rev. Un. Mat. Argentina **64** (2023), no. 2, 333–353.

[31] B. Yildiz and S. Karadeniz, *Linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$: MacWilliams identities, projections, and formally self-dual codes*, Finite Fields Appl. **27** (2014), 24–40. MR 3168981

Western Michigan University
Email address: `jay.wood@wmich.edu`