Logics of False Belief and Radical Ignorance

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Abstract

In the literature, the question about how to axiomatize the transitive logic of false belief is thought of as hard and left as an open problem. In this paper, among other contributions, we deal with this problem. In more details, although the standard doxastic operator is undefinable with the operator of false belief, the former is *almost definable* with the latter. On one hand, the involved almost definability schema guides us to find the desired core axioms for the transitive logic and the Euclidean logic of false belief. On the other hand, inspired by the schema and other considerations, we propose a suitable canonical relation, which can uniformly handle the completeness proof of various logics of false belief, including the transitive logic. We also extend the results to the logic of radical ignorance, due to the interdefinability of the operators of false belief and radical ignorance.

Keywords: false belief, radical ignorance, axiomatizations, expressivity, frame definability

1 Introduction

The prime motivation of this paper is to deal with an open problem about how to axiomatize the transitive logic of false belief.

The discussion about false belief dates back to Plato [1, Sec. 7]. This notion is related to a well-known distinction between knowledge and belief: knowledge must be true, but belief can be false, that is, there are false beliefs. Besides, this notion is popular in the field of cognitive science, see e.g. [12].

The first logical study on false belief is done by Steinsvold in [14], where a logic of false belief is proposed that has the operator W as a sole primitive modality. There, $W\varphi$ is read "one is wrong about φ", or "φ is a false belief", meaning that φ is false though believed. This logic is axiomatized over the class of all frames and over various frame classes, and some results of frame definability are presented. The logic is then interpreted over neighborhood semantics in [10] and [5], and analyzed in the intuitionistic setting in [15].

Although both [14, Sec. 5] and [10, Sec. 2.4] spend a whole section on discussing the problem about how to axiomatize the transitive logic of false belief, they think it is difficult. To our best knowledge, this open problem has not been solved until now.

Two difficulties are involved. One is how to find the desired core axiom, and the other is how to define a suitable canonical relation.

In this paper, we will show that although the standard doxastic operator is undefinable with the operator of false belief, the former is almost definable with the latter. On one hand, the involved almost definability schema guides us to find the desired core axioms for the transitive logic and the Euclidean logic of false belief. On the other hand, inspired by the schema and other considerations, we propose a suitable canonical relation, which can uniformly handle the completeness proof of various logics of false belief, including the transitive logic, thereby solving the open problem in [14].

Moreover, we extend the results to the logic of radical ignorance. The notion of radical ignorance is proposed in [9], to (hopefully) adequately express the important properties in the phenomenon of the Dunning–Kruger effect. This notion is formalized by using a **KT4-B4** framework, in which the epistemic accessibility relation R_K is reflexive and transitive, the doxastic accessibility relation R_B is transitive, and R_B is included in R_K. An agent is radically ignorant about φ iff the agent does not know φ and also does not know $\neg\varphi$,[1] and either the agent believes φ but it is the case that $\neg\varphi$, or it is the case that φ but the agent believes $\neg\varphi$; in symbol, $I_R\varphi =_{df} ((\neg K\varphi \wedge \neg K\neg\varphi) \wedge ((B\varphi \wedge \neg\varphi) \vee (B\neg\varphi \wedge \varphi)))$, where K and B are standard opeartors of knowledge and belief, resepctively. Under very natural assumptions, namely $K\varphi \to \varphi$, $K\varphi \to B\varphi$ and $\neg(B\varphi \wedge B\neg\varphi)$, the definition can be simplified to the following: $I_R\varphi =_{df} ((B\varphi \wedge \neg\varphi) \vee (B\neg\varphi \wedge \varphi))$.[2] However, there have been no formal systems characterizing the notion of radical ignorance.

A related work in the literature is the investigation on the notion of reliable belief in [4], with different motivations though. Note that the operator $\mathcal{R}$ of reliable belief is the negation of I_R, since $\mathcal{R}\varphi$ is equivalent to $(B\varphi \to \varphi) \wedge (B\neg\varphi \to \neg\varphi)$. The minimal logic and the serial logic of reliable belief are axiomatized there. However, the canonical model there does not apply to the transitive logic of reliable belief, thus not to the transitive logic of radical ignorance either.

As we will see, the operators of radical ignorance and false beliefs are interdefinable with each other. This may indicate that one can translate the results about false belief into those about radical ignorance via the translation induced by the definability of the operator W in terms of the operator of radical ignorance. Unfortunately, this holds for all but the minimal proof system.[3] We will illustrate this with the axiomatizations of the logic of radical ignorance over all frames and over serial and transitive frames.

The remainder of this paper is organized as follows. After introducing the lan-

[1] The original wording on page 611 of [9] is "the agent does not know both φ and $\neg\varphi$", where there is a danger of misunderstanding because of a scope ambiguity.

[2] There is an error on page 611 of [9]: it says this simplification can be done in the framework **KT4-B4**, but the serial axiom for the belief operator, viz. $\neg(B\varphi \wedge B\neg\varphi)$, is lacking. So it is **KT4-BD4** (instead of **KT4-B4**) frames, where the doxastic accessibility relation is serial as well, that are the frames which the framework of the cited paper is actually based on.

[3] Note that this is not new. Even if two operators are interdefinable with each other, it is *not* necessary that the axiomatizations of the logic with one operator as a sole modality can be obtained from those of the logic with the other operator as a sole modality via the translation induced by the interdefinability of the operators. For instance, although the necessity operator $\Box$ and the dyadic contingency operator are interdefinable with each other, the serial logic of dyadic contingency cannot be obtained from the serial system **KD** via the translation induced by the interdefinability of the operators, see [6, p. 214].

guage and semantics of the logic of false belief, we propose an almost definability schema (Sec. 2.1). Then we compare the expressive powers of the logic of false belief and standard doxastic logic, and investigate the frame definability of the former (Sec. 2.2). Sec. 2.3 axiomatizes the logic of false belief over various frame classes, including the transitive logic (Sec. 2.3.4) specially, thereby solving an open problem raised in [14]. The canonical relation here is inspired by the aforementioned almost definability schema and other considerations. Moreover, the desired core axioms for the transitive logic and the Euclidean logic of false belief are obtained from the familiar axioms via a translation induced by the almost definability schema. Last but not least, we axiomatize the logic of radical ignorance (Sec. 3). After briefly reviewing the language and semantics of the logic of radical ignorance, we note that the operators of radical ignorance and of false beliefs are interdefinable with each other. Then we axiomatize the minimal logic (Sec. 3.1) and the serial and transitive logic (Sec. 3.2) of radical ignorance.

2 False belief

2.1 Syntax and Semantics

Throughout the paper, we assume **P** to be a nonempty set of propositional variables. We first define a logical language including both false belief and belief operators. The language of the standard doxastic logic and the language of the logic of false belief can be viewed as two fragment of this language. It is the latter that is our main focus in the rest of the paper.

Definition 1. The language $\mathcal{L}(W, \Box)$ is defined recursively as follows.

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid W\varphi \mid \Box\varphi,$$

where $p \in \mathbf{P}$. The *language of the logic of false belief*, denoted $\mathcal{L}(W)$, is the fragment of $\mathcal{L}(W, \Box)$ without the construct $\Box\varphi$. The *language of standard doxastic logic*, denoted $\mathcal{L}(\Box)$, is the fragment of $\mathcal{L}(W, \Box)$ without the construct $W\varphi$.

Other connectives are defined as usual. The formula $W\varphi$ is read "φ is a false belief of the agent", or "the agent is wrong about φ".[4] The language is interpreted on models.

Definition 2. A *model* is a triple $\mathcal{M} = \langle S, R, V \rangle$, where S is a nonempty set of states, R is a binary relation over S, called 'accessibility relation', and V is a valuation. A *frame* is a model without valuations; in this case, we also say that the model is based on the frame. We use $\sim sRt$ to mean that "it does not hold that sRt".

[4]In a deontic setting, $W\neg\varphi$ (that is, $\Box\neg\varphi \wedge \varphi$) is read "$\varphi$ ought not to be done but done", which expresses some kind of vice: one did what one ought not to have done. In a metaphysical setting, $W\neg\varphi$ is read "φ is strongly accidental", c.f. [13].

Definition 3 (Semantics)**.** Given a model $\mathcal{M} = \langle S, R, V\rangle$ and a state $s \in S$, the semantics of $\mathcal{L}(W, \Box)$ is defined recursively as follows.

$\mathcal{M}, s \vDash p$	iff	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\varphi$	iff	$\mathcal{M}, s \nvDash \varphi$
$\mathcal{M}, s \vDash \varphi \wedge \psi$	iff	$\mathcal{M}, s \vDash \varphi$ and $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash W\varphi$	iff	$\mathcal{M}, s \nvDash \varphi$ and for all $t,$ if sRt then $\mathcal{M}, t \vDash \varphi$
$\mathcal{M}, s \vDash \Box\varphi$	iff	for all $t,$ if sRt then $\mathcal{M}, t \vDash \varphi$.

If $\mathcal{M}, s \vDash \varphi$, we say that φ is *true* in $(\mathcal{M}, s)$, and sometimes write $s \vDash \varphi$ if $\mathcal{M}$ is clear. If for all frames $\mathcal{F}$ in F, for all models $\mathcal{M}$ based on $\mathcal{F}$, for all s in $\mathcal{M}$ we have $\mathcal{M}, s \vDash \varphi$, then we say that φ is *valid on* F and write $F \vDash \varphi$; when F is the class of all frames, we say φ is *valid* and write $\vDash \varphi$. The notions for a set of formulas are defined similarly.

From the above semantics, it follows easily that W is definable in terms of $\Box$, as $\vDash W\varphi \leftrightarrow (\Box\varphi \wedge \neg\varphi)$. In contrast, as we will show in Prop. 7, the converse does not hold, since $\Box$ is *not* definable in terms of W in various classes of models.

Let $R(s) = \{t \mid sRt\}$. The semantics of W can be rewritten as follows:

$$\mathcal{M}, s \vDash W\varphi \quad \text{iff} \quad \mathcal{M}, s \nvDash \varphi \text{ and } R(s) \vDash \varphi.$$

Although $\Box$ is undefinable with W, we have the following important schema, which says that $\Box$ is *almost definable* in terms of W. We call it 'Almost Definability Schema'.

Proposition 4. $\vDash W\psi \to (\Box\varphi \leftrightarrow W(\varphi \wedge \psi))$.

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be a model and $s \in S$. Suppose that $\mathcal{M}, s \vDash W\psi$, to show that $\mathcal{M}, s \vDash \Box\varphi \leftrightarrow W(\varphi \wedge \psi)$.

First, assume that $\mathcal{M}, s \vDash \Box\varphi$, to prove that $\mathcal{M}, s \vDash W(\varphi \wedge \psi)$. By assumption, we infer that $R(s) \vDash \varphi$. By supposition, we have $\mathcal{M}, s \nvDash \psi$ and $R(s) \vDash \psi$, thus $\mathcal{M}, s \nvDash \varphi \wedge \psi$ and $R(s) \vDash \varphi \wedge \psi$. This implies that $\mathcal{M}, s \vDash W(\varphi \wedge \psi)$.

Conversely, assume that $\mathcal{M}, s \vDash W(\varphi \wedge \psi)$, then $R(s) \vDash \varphi \wedge \psi$. So $R(s) \vDash \varphi$, and therefore $\mathcal{M}, s \vDash \Box\varphi$. $\Box$

This schema is very important, since it not only guides us to find out the desired core axioms of transitive logic and Euclidean logic for $\mathcal{L}(W)$, it also motivates the canonical relation in the construction of canonical model for $\mathcal{L}(W)$. With this relation we can show the completeness of all axiomatizations uniformly, as we will see below.

2.2 Expressivity and Frame Definability

In this part, we investigate the expressive power and frame definability of $\mathcal{L}(W)$. To begin with, we have the following useful observation, which follows directly from the semantics of W.

Fact 5. For all φ, $W\varphi$ is false in each reflexive state.

Proposition 6. For any reflexive frames $\mathcal{F}$ and $\mathcal{F}'$, for any $\varphi \in \mathcal{L}(W)$, $\mathcal{F} \vDash \varphi$ iff $\mathcal{F}' \vDash \varphi$.

Proof. Let $\mathcal{F} = \langle S, R\rangle$ and $\mathcal{F}' = \langle S', R'\rangle$ be reflexive frames, and let $\varphi \in \mathcal{L}(W)$.

Suppose that $\mathcal{F} \not\models \varphi$, then there is a valuation V and a state s such that $\langle F, V\rangle, s \not\models \varphi$. Since $S' \neq \varnothing$, we may assume that $s' \in S$. Define a valuation V' on $\mathcal{F}'$ as follows: $s' \in V'(p)$ iff $s \in V(p)$ for all $p \in \mathbf{P}$. Since $\mathcal{F}$ and $\mathcal{F}'$ are both reflexive, both s and s' are reflexive. By Fact 5, this means that all $W\varphi$ are false in both states. Then by induction on $\psi \in \mathcal{L}(W)$, we can show that $\langle \mathcal{F}, V\rangle, s \models \psi$ iff $\langle \mathcal{F}', V'\rangle, s' \models \psi$. Hence $\langle \mathcal{F}', V'\rangle, s' \not\models \psi$, and therefore $\mathcal{F}' \not\models \varphi$. The converse is similar. $\square$

It turns out that $\mathcal{L}(W)$ is less expressive than $\mathcal{L}(\Box)$ on various classes of models.

Proposition 7. $\mathcal{L}(W)$ is less expressive than $\mathcal{L}(\Box)$ on the class of $\mathcal{S}5$-models. As a consequence, $\mathcal{L}(W)$ is less expressive than $\mathcal{L}(\Box)$ on the class of $\mathcal{K}$-models, $\mathcal{D}$-models, $\mathcal{T}$-models, 4-models, $\mathcal{B}$-models, 5-models, $\mathcal{D}4$-models, $\mathcal{D}45$-models.

Proof. As mentioned above, W is definable in terms of $\Box$, thus $\mathcal{L}(\Box)$ is at least as expressive as $\mathcal{L}(W)$. For the strict part, consider the following $\mathcal{S}5$-models:

$$\begin{tikzcd} \mathcal{M} & s : p \arrow[loop above] & & \mathcal{M}' & s' : p \arrow[loop above] \arrow[r, leftrightarrow] & t' : \neg p \arrow[loop above] \end{tikzcd}$$

Using Fact 5, we can show by induction that for all $\varphi \in \mathcal{L}(W)$, $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}', s' \models \varphi$. Thus $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(W)$.

However, these two models can be distingished by $\mathcal{L}(\Box)$, since $\mathcal{M}, s \models \Box p$ but $\mathcal{M}', s' \not\models \Box p$. $\square$

It may be natural to ask if there is a class of frames where $\Box$ is definable in terms of W. The answer is positive. We borrow a notion of *narcissistic* from [14, Def. 2.1]. Call s *narcissistic* if and only if s relates to itself and only to itself. Call a frame *narcissistic* if all the worlds are narcissistic; that is,

$$\forall x(xRx \wedge \forall y(xRy \to x = y)).$$

Proposition 8. On the class of narcissistic frames, $\Box$ is definable in terms of W. As a consequence, $\mathcal{L}(W)$ and $\mathcal{L}(\Box)$ are equally expressive on the class of narcissistic models.

Proof. Let F_{nar} be the class of narcissistic frames. It is straightforward to verify that $F_{nar} \models \Box\varphi \leftrightarrow \varphi$. This means that on the frame classes in question, $\Box$ is already definable in the language of propositional logic; needless to say, $\Box$ is definable in terms of W. $\square$

Remark 9. In [11, Sect. 1.4], the authors compare the expressive power of $\mathcal{L}(I)$ and $\mathcal{L}(\Box)$.[5] It turns out that neither of I and $\Box$ is, in general, definable in terms of the other. In particular, it is shown in [11, Coro. 1.31] that the indefinability of $\Box$ in terms of I applies to a wide variety of frame classes. In the meanwhile, the authors ask whether there exist any interesting classes of frames in which $\Box$ is definable in terms of I and

[5] In [11], $\mathcal{L}(I)$ is the language of the logic of factive ignorance that has the operator I of factive ignorance as a sole primitive modality. Boolean formulas are interpreted as usual, and $I\varphi$ is interpreted as follows: given a model $\mathcal{M} = \langle S, R, V\rangle$ and a state $s \in S$, $\mathcal{M}, s \models I\varphi$ iff $\mathcal{M}, s \models \varphi$ and for all $t \in S$, if sRt and $s \neq t$, then $\mathcal{M}, t \not\models \varphi$.

they think the answer is negative (see [11, p. 878]). However, the answer is actually positive, since on the class of narcissistic frames, $\Box$ is definable in terms of I. The same class of frames also establishes the definability of I in terms of $\Box$, since as one may show, $F_{nar} \vDash I\varphi \leftrightarrow \varphi$, where F_{nar} is the class of narcissistic frames.

The following result is shown in [14, Thm. 4.8], where the proof is based on a canonical model. Here we give a much simpler proof, without need of canonical models.

Proposition 10. The following frame properties are all undefinable in $\mathcal{L}(W)$:[6]

(1) Transitivity,

(2) Euclideanness,

(3) Symmetry,

(4) weak connectedness $\forall x\forall y\forall z((xRy \wedge xRz) \to (yRz \vee y = z \vee zRy))$,

(5) weak directedness $\forall x\forall y\forall z((xRy \wedge xRz) \to \exists v(yRv \wedge zRv))$,

(6) partial functionality $\forall x\forall y\forall z((xRy \wedge xRz) \to z = y)$,

(7) narcissism $\forall x(xRx \wedge \forall y(xRy \to x = y))$,

(8) partial narcissism $\forall x\forall y(xRy \to x = y)$.

Proof. Consider the following frames:

$$\mathcal{F}: \quad \begin{tikzcd} s \arrow[loop above] \end{tikzcd} \qquad\qquad \mathcal{F}': \quad \begin{tikzcd} u' \arrow[loop above] \arrow[r, leftrightarrow] & s' \arrow[loop above] \arrow[r] & t' \arrow[loop above] \end{tikzcd}$$

One may check that for any frame property P in (1)-(8), it is *not* the case that $\mathcal{F}$ has P iff $\mathcal{F}'$ has P. However, since $\mathcal{F}$ and $\mathcal{F}'$ are both reflexive, by Prop. 6, for all $\varphi \in \mathcal{L}(W)$, we have that $\mathcal{F} \vDash \varphi$ iff $\mathcal{F}' \vDash \varphi$. This entails that the frame properties in question are all undefinable in $\mathcal{L}(W)$. $\Box$

2.3 Axiomatizations

This section presents the axiomatizations of $\mathcal{L}(W)$ over various frame classes.

2.3.1 Minimal Logic

Definition 11. The minimal logic of $\mathcal{L}(W)$, denoted $\mathbf{K^W}$, consists of the following axioms and inference rules:

A0	all instances of propositional tautologies
A1	$W\varphi \to \neg\varphi$
A2	$W\varphi \wedge W\psi \to W(\varphi \wedge \psi)$
MP	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$
R1	$\dfrac{\varphi \to \psi}{W\varphi \wedge \neg\psi \to W\psi}$

[6] To say a frame property P is *definable* in a logic $\mathcal{L}$, if there is a set Γ of $\mathcal{L}$-formulas such that for all frames $\mathcal{F}$, $\mathcal{F} \vDash \Gamma$ iff $\mathcal{F}$ has P.

The notions of theorems, provability, and derivation are defined as usual.

This system is called $\mathbf{S^W}$ in [14]. Intuitively, axiom A1 says that false beliefs are false, A2 says that false beliefs are closed under conjunction: the conjunction of two false beliefs are still a false belief. The rule R1 stipulates the almost monotonicity of the false belief operator. It is shown in [14, Thm. 3.1] that the substituition rule of equivalents for the operator W, i.e. $\varphi \leftrightarrow \psi \,/\, W\varphi \leftrightarrow W\psi$, denoted REW, is admissible in $\mathbf{K^W}$. Moreover, we have the following.

Proposition 12. The following rule is admissible in $\mathbf{K^W}$:

$$\frac{\varphi \to \psi}{W(\varphi \wedge \chi) \wedge \neg\psi \to W\psi}.$$

Proof. We have the following proof sequence in $\mathbf{K^W}$.

(1)	$\varphi \to \psi$	hypothesis
(2)	$W\varphi \wedge \neg\psi \to W\psi$	(1), R1
(3)	$\varphi \wedge \chi \to \varphi$	A0
(4)	$W(\varphi \wedge \chi) \wedge \neg\varphi \to W\varphi$	(3), R1
(5)	$W(\varphi \wedge \chi) \wedge \neg\varphi \wedge \neg\psi \to W\psi$	(2), (4)
(6)	$W(\varphi \wedge \chi) \wedge \neg\psi \to W\psi$	(1), (5)

□

We can generalize the above proposition to the following.

Proposition 13. Let $n \in \mathbb{N}$. If $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \varphi$, then $\vdash W(\chi_1 \wedge \psi) \wedge \cdots \wedge W(\chi_n \wedge \psi) \wedge \neg\varphi \to W\varphi$.

Proof. By induction on $n \in \mathbb{N}$. The case $n = 0$ is obvious. The case $n = 1$ is shown as in Prop. 12.

Now suppose that the statement holds for m (IH), to show that it also holds for $m + 1$. For this, we have the following proof sequence:

(1)	$\chi_1 \wedge \cdots \wedge \chi_{m+1} \to \varphi$	premise
(2)	$(\chi_1 \wedge \chi_2) \wedge \cdots \wedge \chi_{m+1} \to \varphi$	(1)
(3)	$W((\chi_1 \wedge \chi_2) \wedge \psi) \wedge \cdots \wedge W(\chi_{m+1} \wedge \psi) \wedge \neg\varphi \to W\varphi$	(2), IH
(4)	$W(\chi_1 \wedge \psi) \wedge W(\chi_2 \wedge \psi) \to W((\chi_1 \wedge \chi_2) \wedge \psi)$	A2
(5)	$W(\chi_1 \wedge \psi) \wedge \cdots \wedge W(\chi_{m+1} \wedge \psi) \wedge \neg\varphi \to W\varphi$	(3), (4)

□

The completeness of $\mathbf{K^W}$ is shown via the construction of a canonical model.

Definition 14. The *canonical model for* $\mathbf{K^W}$ is $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$, where

- $S^c = \{s \mid s$ is a maximal consistent set for $\mathbf{K^W}\}$,
- for all $s, t \in S^c$, R^c is defined as follows:
  - if $W\psi \in s$ for *no* ψ, then $sR^c t$ iff $s = t$, and

– if $W\psi \in s$ for *some* ψ, then sR^ct iff for all φ, if $W(\varphi \wedge \psi) \in s$, then $\varphi \in t$.

- $V^c(p) = \{s \in S^c \mid p \in s\}$.

It is worth noting that the above definition of T^c is inspired by Almost Definability Schema (Prop. 4). Recall that in the construction of the canonical model of standard doxastic logic, the canonical relation R^c is usually defined as follows: sR^ct iff for all φ, if $\Box\varphi \in s$, then $\varphi \in t$. According to Almost Definability Schema, $\Box\varphi \in s$ can be replaced by $W(\varphi \wedge \psi) \in s$ provided that $W\psi \in s$ for some ψ. This is similar to the case for minimal contingency logic [8].

However, unlike the case for minimal contingency logic [8], here "$W\psi \in s$ for some ψ" should be a precondition, instead of a conjunction, of the aforementioned replacement.[7] Moreover, if this precondition is not satisfied, then s can and only can access itself. As we will see, the case-by-case definition enables us to prove the completeness of the minimal system and its extensions, which however cannot be done if we use "$W\psi \in s$ for some ψ" as a conjunction (as the reader may verify).

Also notice that our definition differs from the canonical relation in [14, Def. 4.2] in that we have $W(\varphi \wedge \psi) \in s$ instead of $W\varphi \in s$. Besides, as already mentioned above, our definition is motivated by Almost Definability Schema. As we will see, the slight distinction enables us to show the completeness of the transitive system of $\mathcal{L}(W)$ (Sec. 2.3.4), which cannot be done with $W\varphi \in s$ instead.

The following result states that the truth lemma holds for $\mathbf{K^W}$.

Lemma 15. For all $\varphi \in \mathcal{L}(W)$ and for all $s \in S^c$, we have

$$\mathcal{M}^c, s \vDash \varphi \text{ iff } \varphi \in s.$$

Proof. By induction on φ. We only consider the case $W\varphi$.

'If': suppose that $W\varphi \in s$, to show that $\mathcal{M}^c, s \vDash W\varphi$. By supposition and axiom A1, $\neg\varphi \in s$, and thus $\varphi \notin s$. By IH, we have $\mathcal{M}^c, s \nvDash \varphi$. Now let $t \in S^c$ such that sR^ct, by IH, it suffices to show that $\varphi \in t$. By definition of T^c and supposition, we infer that for all χ, if $W(\chi \wedge \varphi) \in s$, then $\chi \in t$. By letting χ be φ, we derive that $\varphi \in t$.

'Only if': assume that $W\varphi \notin s$, to show that $\mathcal{M}^c, s \nvDash W\varphi$. If $W\psi \in s$ for *no* ψ, then we are done, since otherwise we would have $\mathcal{M}^c, s \vDash \varphi$ and $\mathcal{M}^c, s \nvDash \varphi$. Now we consider the case that $W\psi \in s$ for *some* ψ. For this, suppose that $\neg\varphi \in s$, by IH and Lindenbaum's Lemma, we only need to show that $\{\chi \mid W(\chi \wedge \psi) \in s\} \cup \{\neg\varphi\}$ (denoted Γ) is consistent.

Since $W\psi \in s$, $\{\chi \mid W(\chi \wedge \psi) \in s\}$ is nonempty. If Γ is not consistent, then there are $\chi_1, \dots, \chi_n$ such that $W(\chi_i \wedge \psi) \in s$ for all $i = 1, \dots, n$ and

$$\vdash \chi_1 \wedge \dots \wedge \chi_n \to \varphi.$$

By Prop. 13,

$$\vdash W(\chi_1 \wedge \psi) \wedge \dots \wedge W(\chi_n \wedge \psi) \wedge \neg\varphi \to W\varphi.$$

[7]In other words, in the case of $W\psi \in s$ for some ψ, we replace $\Box\varphi \in s$ with $W(\varphi \wedge \psi) \in s$ in the definition of the canonical relation of the canonical model for standard doxastic logic.

As $W(\chi_i \wedge \psi) \in s$ for all $i = 1, \ldots, n$ and $\neg\varphi \in s$, we infer that $W\varphi \in s$, which contradicts the assumption, as desired. $\square$

It is now routine to show the following.

Theorem 16. $\mathbf{K^W}$ is sound and strongly complete with resepct to the class of all frames.

2.3.2 Serial Logic

Let $\mathbf{KD^W}$ denote $\mathbf{K^W}$ + AD, where AD is $\neg W\bot$.

Theorem 17. $\mathbf{KD^W}$ is sound and strongly complete with respect to the class of serial frames.

Proof. For soundness, by Thm. 16, it remains only to show the validity of the axiom AD.

If there is a serial model $\mathcal{M} = \langle S, T, V \rangle$ and a state $s \in S$ such that $\mathcal{M}, s \vDash W\bot$. Then $\mathcal{M}, s \vDash \top$ and for all t, if sRt then $\mathcal{M}, t \vDash \bot$. This is impossible since R is serial. Hence $\neg W\bot$ is valid over the class of serial models.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{KD^W}$ as in Def. 14. By Thm. 16, it suffices to prove that R^c is serial. For this, assume that $s \in S^c$. We consider two cases. If there is no ψ such that $W\psi \in s$, then by definition of R^c, sR^cs. The remainder is the case that there is some ψ such that $W\psi \in s$. In this case, the set $\{\varphi \mid W(\varphi \wedge \psi) \in s\}$ is nonempty. By definition of R^c and Lindenbaum's Lemma, we only need to show that this set is consistent.

If not, then there are $\varphi_1, \ldots, \varphi_n$ such that $W(\varphi_i \wedge \psi) \in s$ for all $i = 1, \ldots, n$, and

$$\vdash \varphi_1 \wedge \cdots \wedge \varphi_n \to \bot.$$

By Prop. 13,

$$\vdash W(\varphi_1 \wedge \psi) \wedge \cdots \wedge W(\varphi_n \wedge \psi) \wedge \neg\bot \to W\bot.$$

Since $W(\varphi_i \wedge \psi) \in s$ for all $i = 1, \ldots, n$, and $\neg\bot \in s$, we infer that $W\bot \in s$, which contradicts the fact that $\vdash \neg W\bot$. $\square$

2.3.3 Reflexive Logic

Let $\mathbf{T^W}$ denote $\mathbf{K^W}$ + $\neg W\varphi$.

Theorem 18. $\mathbf{T^W}$ is sound and strongly complete with respect to the class of reflexive frames.

Proof. For soundness, by Thm. 16, it remains only to show the validity of $\neg W\varphi$, which can be obtained from Fact 5.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{T^W}$ as in Def. 14. By Thm. 16, it suffices to show that R^c is reflexive. For this, let $s \in S^c$. Since $\vdash \neg W\varphi$, there is no ψ such that $W\psi \in s$. By definition of R^c, we derive that sR^cs, as desired. $\square$

2.3.4 Transitive Logic

Let $\mathbf{K4^W}$ denote the extension of $\mathbf{K^W}$ with the following axiom:

$$\text{A4} \quad W\psi \wedge W(\varphi \wedge \psi) \to W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi).$$

As mentioned in the introduction, a difficult thing in axiomatizing $\mathcal{L}(W)$ over transitive frames is how to find the desired core axiom. Here the axiom A4 is obtained from the modal axiom 4 (i.e. $\Box\varphi \to \Box\Box\varphi$) via a translation induced by Almost Definability Schema.

$$\begin{array}{lll} & W\psi \to (\Box\varphi \to \Box(W\chi \to \Box\varphi)) & (1) \\ \iff & W\psi \to (W(\varphi \wedge \psi) \to W((W\chi \to \Box\varphi) \wedge \psi) & (2) \\ \iff & W\psi \to (W(\varphi \wedge \psi) \to W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)) & (3) \\ \iff & W\psi \wedge W(\varphi \wedge \psi) \to W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi) & (4) \end{array}$$

We write $W\psi \to (\Box\varphi \to \Box(W\chi \to \Box\varphi))$ rather than $\Box\varphi \to \Box\Box\varphi$, since $\Box$ is definable in terms of W under the condition $W\psi$ for some ψ. Note that every transformation is equivalent. The above transitions from (1) to (2) and from (2) to (3) follow from Prop. 4. By using propositional calculus (axiom A0), we then obtain the axiom (4), that is, A4.

Proposition 19. A4 is valid on the class of transitive frames.

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be a transitive model and $s \in S$. Suppose, for reductio, that $\mathcal{M}, s \vDash W\psi \wedge W(\varphi \wedge \psi)$ but $\mathcal{M}, s \nvDash W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)$. From $\mathcal{M}, s \vDash W\psi$ it follows that $R(s) \vDash \psi$ and $\mathcal{M}, s \nvDash \psi$, thus $\mathcal{M}, s \nvDash (W\chi \to W(\varphi \wedge \chi)) \wedge \psi$. This plus $\mathcal{M}, s \nvDash W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)$ implies that $R(s) \nvDash (W\chi \to W(\varphi \wedge \chi)) \wedge \psi$, that is, there exists t such that sRt and $\mathcal{M}, t \nvDash (W\chi \to W(\varphi \wedge \chi)) \wedge \psi$. Since $R(s) \vDash \psi$, we infer that $\mathcal{M}, t \vDash \psi$, thus $\mathcal{M}, t \nvDash W\chi \to W(\varphi \wedge \chi)$, namely $t \vDash W\chi$ and $t \nvDash W(\varphi \wedge \chi)$. This entails that $t \nvDash \chi$ and thus $t \nvDash \varphi \wedge \chi$, hence there is a u such that tRu and $\mathcal{M}, u \nvDash \varphi \wedge \chi$. However, we have also $\mathcal{M}, u \vDash \varphi \wedge \chi$: $\mathcal{M}, u \vDash \chi$ follows from $t \vDash W\chi$ and tRu, whereas $\mathcal{M}, u \vDash \varphi$ is due to the fact that sRu (this is because sRt and tRu and R is transitive) and $s \vDash W(\varphi \wedge \psi)$. A contradiction. $\square$

With the previous preparation in hand, we can show the following.

Theorem 20. $\mathbf{K4^W}$ is sound and strongly complete with respect to the class of transitive frames.

Proof. The soundness follows immediately from Thm. 16 and Prop. 19.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{K4^W}$ as in Def. 14. By Thm. 16, it suffices to show that R^c is transitive. Let $s, t, u \in S^c$. Suppose that sR^ct and tR^cu, to prove that sR^cu. We consider the following cases.

- $W\psi \in s$ for no ψ. In this case, by definition of R^c, $s = t$. Then sR^cu.
- $W\psi \in t$ for no ψ. Similar to the first case, we can show that sR^cu.

- $W\psi \in s$ for some ψ and $W\psi' \in t$ for some ψ'. In this case, assume for all φ that $W(\varphi \wedge \psi) \in s$, to show that $\varphi \in u$. Using axiom A4, we derive that $W((W\psi' \to W(\varphi \wedge \psi')) \wedge \psi) \in s$. By $sR^c t$ and definition of R^c, we infer that $W\psi' \to W(\varphi \wedge \psi') \in t$, thus $W(\varphi \wedge \psi') \in t$. Now using $tR^c u$ and definition of R^c, we conclude that $\varphi \in u$, as desired.

$\square$

We have thus solved an open problem raised in [14]. By Thm. 17 and Thm. 20, we have the following.

Theorem 21. $\mathbf{KD4^W}$ is sound and strongly complete with respect to the class of $\mathcal{D}4$-frames.

This also answers another open problem raised in [14, Sect. 5].

2.3.5 Euclidean Logic

Let $\mathbf{K5^W}$ denote the extension of $\mathbf{K^W}$ with the following axiom:

$$\text{A5}\quad W\psi \wedge \neg W(\varphi \wedge \psi) \to W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi)$$

Again, the axiom A5 is obtained from the modal axiom 5 (i.e. $\neg\Box\varphi \to \Box\neg\Box\varphi$) via a translation induced by Almost Definability Schema.

$$\begin{array}{lll} & W\psi \to (\neg\Box\varphi \to \Box(W\chi \to \neg\Box\varphi)) & (1') \\ \Longleftrightarrow & W\psi \to (\neg W(\varphi \wedge \psi) \to W((W\chi \to \neg\Box\varphi) \wedge \psi) & (2') \\ \Longleftrightarrow & W\psi \to (\neg W(\varphi \wedge \psi) \to W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi)) & (3') \\ \Longleftrightarrow & W\psi \wedge \neg W(\varphi \wedge \psi) \to W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi) & (4') \end{array}$$

Here, we write $W\psi \to (\neg\Box\varphi \to \Box(W\chi \to \neg\Box\varphi))$ instead of $\neg\Box\varphi \to \Box\neg\Box\varphi$, since $\Box$ is definable in terms of W provided that $W\psi$ for some ψ. Again, every transformation is equivalent. The above transitions from $(1')$ to $(2')$ and from $(2')$ to $(3')$ follow from Prop. 4. Then by using axiom A0, we get the axiom $(4')$, that is, A5.

Different from our $\mathbf{K5^W}$, the Euclidean system in [14], denoted $\mathbf{S^W} \oplus \mathrm{A}^Q$ there, is defined as the extension of $\mathbf{S^W}$ (that is, our $\mathbf{K^W}$) with A^Q, where A^Q is $W\varphi \to W(\neg W\psi \wedge \varphi)$. It is shown in [14, Thm. 4.15] that $\mathbf{S^W} \oplus \mathrm{A}^Q$ is sound and complete with respect to the class of Euclidean (and transitive) frames. In what follows, we show that A^Q is provable in $\mathbf{K5^W}$.

Proposition 22. A^Q is provable in $\mathbf{K5^W}$.

Proof. We have the following proof sequence in $\mathbf{K5^W}$.

(1)	$W\psi \wedge \neg W(\neg W\psi \wedge \varphi) \to W((W\psi \to \neg W(\neg W\psi \wedge \psi)) \wedge \varphi)$	A5
(2)	$\psi \to \neg W\psi$	A1
(3)	$(\neg W\psi \wedge \psi) \leftrightarrow \psi$	(2), A0
(4)	$W(\neg W\psi \wedge \psi) \leftrightarrow W\psi$	(3), REW
(5)	$(W\psi \to \neg W(\neg W\psi \wedge \psi)) \leftrightarrow (W\psi \to \neg W\psi)$	(4), A0
(6)	$(W\psi \to \neg W(\neg W\psi \wedge \psi)) \leftrightarrow \neg W\psi$	(5), A0
(7)	$((W\psi \to \neg W(\neg W\psi \wedge \psi)) \wedge \varphi) \leftrightarrow (\neg W\psi \wedge \varphi)$	(6), A0
(8)	$W((W\psi \to \neg W(\neg W\psi \wedge \psi)) \wedge \varphi) \leftrightarrow W(\neg W\psi \wedge \varphi)$	(7), REW
(9)	$W\psi \wedge \neg W(\neg W\psi \wedge \varphi) \to W(\neg W\psi \wedge \varphi)$	(1), (8)
(10)	$W\psi \to W(\neg W\psi \wedge \varphi)$	(9), A0

□

Below, we will demonstrate that our axiom A5 is valid over the class of Euclidean frames.

Proposition 23. A5 is valid on the class of Euclidean frames.

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be an Euclidean model and $s \in S$.

Suppose, for reductio, that $\mathcal{M}, s \vDash W\psi \wedge \neg W(\varphi \wedge \psi)$ but $\mathcal{M}, s \nvDash W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi)$. Then $\mathcal{M}, s \nvDash \psi$, thus $\mathcal{M}, s \nvDash \varphi \wedge \psi$ and $\mathcal{M}, s \nvDash (W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi$. It follows that there exists t such that sRt and $\mathcal{M}, t \nvDash \varphi \wedge \psi$, and there exists u such that sRu and $\mathcal{M}, u \nvDash (W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi$. Using $s \vDash W\psi$ again, we derive that $t \vDash \psi$ and $u \vDash \psi$, and then $t \nvDash \varphi$ and $u \nvDash W\chi \to \neg W(\varphi \wedge \chi)$, that is, $u \vDash W\chi \wedge W(\varphi \wedge \chi)$. By sRu and sRt and the Euclideanness of R, we have uRt. Then it follows from $u \vDash W(\varphi \wedge \chi)$ that $t \vDash \varphi$, as desired. □

Proposition 24. $\mathbf{K5^W}$ is sound with respect to the class of Euclidean frames.

Now we demonstrate the completeness of $\mathbf{K5^W}$ over Euclidean frames. Our proof is different from that used in [14, Thm. 4.15]. The proof is nontrivial. This is because the canonical model is secondarily reflexive (defined later), not Euclidean. Thus we need to transform the secondarily reflexive model into an Euclidean model, and the truth values of $\mathcal{L}(W)$-formulas have to be preserved during the transformation. This is our strategy. To begin with, we need a notion of secondary reflexivity.

We say that a model $\mathcal{M} = \langle S, R, V\rangle$ is *secondarily reflexive*, if for all $s, t \in S$, sRt implies tRt. We have the following general result, which will be used in the proof of the completeness of $\mathbf{K5^W}$ (Thm. 26).

Proposition 25. For every secondarily reflexive model $\mathcal{M} = \langle S, R, V\rangle$, there exists an Euclidean model $\mathcal{M}' = \langle S, R', V\rangle$ such that for all $s \in S$, for all $\varphi \in \mathcal{L}(W)$, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s \vDash \varphi$.

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be a secondarily reflexive model. Construct a model $\mathcal{M}' = \langle S, R', V\rangle$ such that $R' = R \cup \{(y, z) \mid xRy \text{ and } x'Rz \text{ for some } x, x' \in S\}$.

First, $\mathcal{M}'$ is Euclidean. Let $s, t, u \in S$ such that $sR't$ and $sR'u$. The goal is to show $tR'u$. By definition of R', we consider the following cases.

- sRt and sRu. Then $tR'u$.
- $\sim sRt$ and sRu. Then xRs and $x'Rt$ for some $x, x' \in S$. Then $tR'u$.
- sRt and $\sim sRu$. Similar to the second case, we can show that $tR'u$.
- $\sim sRt$ and $\sim sRu$. Then xRs and $x'Rt$ for some $x, x' \in S$, and yRs and $y'Ru$ for some $y, y' \in S$. Then $tR'u$.

It remains only to show that for all $s \in S$, for all $\varphi \in \mathcal{L}(W)$, we have

$$\mathcal{M}, s \vDash \varphi \text{ iff } \mathcal{M}', s \vDash \varphi.$$

We proceed by induction on φ. The nontrivial case is $W\varphi$.

Suppose that $\mathcal{M}, s \nvDash W\varphi$, to show that $\mathcal{M}', s \nvDash W\varphi$. By supposition, either $\mathcal{M}, s \vDash \varphi$ or for some t such that sRt we have $\mathcal{M}, t \nvDash \varphi$. By induction hypothesis and $R \subseteq R'$, $\mathcal{M}', s \vDash \varphi$ or for some t such that $sR't$ and $\mathcal{M}', t \nvDash \varphi$. Thus $\mathcal{M}', s \nvDash W\varphi$.

Conversely, assume that $\mathcal{M}', s \nvDash W\varphi$, to prove that $\mathcal{M}, s \nvDash W\varphi$. By assumption, either $\mathcal{M}', s \vDash \varphi$ or for some t such that $sR't$ we have $\mathcal{M}', t \nvDash \varphi$. If the first case holds, by induction hypothesis, we derive that $\mathcal{M}, s \vDash \varphi$, thus $\mathcal{M}, s \nvDash W\varphi$. If the second case holds, according to the definition of R', we consider the following two cases.

- sRt. By $\mathcal{M}', t \nvDash \varphi$ and induction hypothesis, $\mathcal{M}, t \nvDash \varphi$. Thus $\mathcal{M}, s \nvDash W\varphi$.
- xRs and $x'Rt$ for some $x, x' \in S$. Since xRs and $\mathcal{M}$ is secondarily reflexive, it follows that sRs. Then using Fact 5, we conclude that $\mathcal{M}, s \nvDash W\varphi$.

□

The reader may ask if the above statement can be extended to the case of transitivity and serial. That is, do we have the following: Every (serial,) transitive and secondarily reflexive model is $\mathcal{L}(W)$-equivalent to a (serial,) transitive and Euclidean model? We do not the answer. As we check, the construction $\mathcal{M}'$ in the proof of Prop. 25 does not preserve transitivity. We will come back to this issue.

Theorem 26. $\mathbf{K5^W}$ is sound and strongly complete with respect to the class of Euclidean frames.

Proof. By Prop. 24, it suffices to show the completeness of $\mathbf{K5^W}$. For this, define $\mathcal{M}^c$ w.r.t. $\mathbf{K5^W}$ as in Def. 14. Firstly, we show that $\mathcal{M}^c$ is secondarily reflexive, that is, the following holds:

$$(*) \text{ for all } s, t \in S^c, \text{ if } sR^ct \text{ then } tR^ct.$$

Let $s, t \in S^c$. Suppose that sR^ct, to show that tR^ct. According to the definition of R^c, we consider the following cases.

- There is no ψ such that $W\psi \in s$. Then $s = t$. Thus tR^ct.
- There is no ψ' such that $W\psi' \in t$. Then as $t = t$, we also have tT^ct.

- $W\psi \in s$ and $W\psi' \in t$ for some ψ and ψ'. If it fails that $tR^c t$, according to the definition of R^c, it follows that for some φ, $W(\varphi \wedge \psi') \in t$ and $\varphi \notin t$. As $sR^c t$, we must have $W(\varphi \wedge \psi) \notin s$, thus $\neg W(\varphi \wedge \psi) \in s$. Using axiom A5, we infer that $W((W\psi' \to \neg W(\varphi \wedge \psi')) \wedge \psi) \in s$. Using $sR^c t$ again, we derive that $W\psi' \to \neg W(\varphi \wedge \psi') \in t$, thus $\neg W(\varphi \wedge \psi') \in t$, which is contrary to $W(\varphi \wedge \psi') \in t$ and the consistency of t.

We have thus shown (*). This implies that $\mathcal{M}^c$ is a secondarily reflexive model. That is to say, every consistent set is satisfiable in a secondarily reflexive model.

Now by Prop. 25, we obtain that every consistent set is satisfiable in an Euclidean model, as desired. $\square$

It may be worth remarking that axiom A4 is provable in $\mathbf{K5^W}$, because it is valid on the class of Euclidean frames.

Proposition 27. A4 is valid on the class of Euclidean frames.

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be an Euclidean model and $s \in S$. Suppose, for a contradiction, that $\mathcal{M}, s \vDash W\psi \wedge W(\varphi \wedge \psi)$ and $\mathcal{M}, s \nvDash W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)$. Then $\mathcal{M}, s \nvDash \psi$, thus $s \nvDash (W\chi \to W(\varphi \wedge \chi)) \wedge \psi$. It then follows that there exists t such that sRt and $\mathcal{M}, t \nvDash (W\chi \to W(\varphi \wedge \chi)) \wedge \psi$. Moreover, as $s \vDash W\psi$, $t \vDash \psi$, thus $t \nvDash W\chi \to W(\varphi \wedge \chi)$, and hence $t \vDash W\chi$. However, since sRt and R is Euclidean, tRt. By Fact 5, we should have also $t \nvDash W\chi$. A contradiction. $\square$

So $\mathbf{K4^W} \subseteq \mathbf{K5^W}$. Note that in the above proof, $s \vDash W(\varphi \wedge \psi)$ is not needed. This means that a stronger version of A4, that is, $W\psi \to W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)$ is valid over the class of Euclidean frames, thus provable in $\mathbf{K5^W}$. In contrast, this formula is not valid over the class of transitive frames (as one may verify), thus not provable in $\mathbf{K4^W}$. This establishes that $\mathbf{K4^W} \subset \mathbf{K5^W}$.

Moreover, $\mathbf{K45^W} = \mathbf{K5^W}$, where $\mathbf{K45^W}$ is the extension of $\mathbf{K5^W}$ with the axiom A4. As a consequence, we have another completeness result.

Theorem 28. $\mathbf{K45^W}$ is sound and strongly complete with respect to the class of Euclidean frames.

Theorem 29. $\mathbf{K5^W} (= \mathbf{K45^W})$ is sound and strongly complete with respect to the class of transitive and Euclidean frames.

Proof. The soundness is direct from Thm. 26.

For the completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{K5^W}$ as in Def. 14. We have shown that $\mathcal{M}^c$ is transitive (Thm. 20) and secondarily reflexive (Thm. 26). This entails that every consistent set, say Γ, is satisfiable in a transitive and secondarily reflexive model, say $(\mathcal{M}, s)$. Let $\mathcal{M}' = \langle S, R, V\rangle$ is the submodel of $\mathcal{M}$ generated by s. By the generated submodel theorem for standard modal logic $\mathcal{L}(\Box)$, we have $\mathcal{M}', s \vDash \Gamma$. Now construct a new model $\mathcal{N} = \langle S, R', V\rangle$ such that $R' = R \cup (Z(s) \times Z(s))$, where $Z(s) = \{x \mid sRx\}$. We can see that $\mathcal{N}$ is transitive and Euclidean.

It remains only to show that for all $x \in S$, for all $\varphi \in \mathcal{L}(W)$, $\mathcal{M}', x \vDash \varphi$ iff $\mathcal{N}, x \vDash \varphi$. We proceed by induction on φ. The only nontrivial case is $W\varphi$.

Suppose that $\mathcal{M}', x \nvDash W\varphi$. Then $\mathcal{M}', x \vDash \varphi$ or for some y such that xRy and $\mathcal{M}', y \nvDash \varphi$. By induction hypothesis and $R \subseteq R'$, $\mathcal{N}, x \vDash \varphi$ or for some y such that $xR'y$ and $\mathcal{M}', y \nvDash \varphi$. Thus $\mathcal{N}, x \nvDash W\varphi$.

Conversely, assume that $\mathcal{N}, x \nvDash W\varphi$. Then $\mathcal{N}, x \vDash \varphi$ or for some y such that $xT'y$ and $\mathcal{N}, y \nvDash \varphi$. If $\mathcal{N}, x \vDash \varphi$, by induction hypothesis, $\mathcal{M}', x \vDash \varphi$, thus $\mathcal{M}', x \nvDash W\varphi$. If for some y such that $xR'y$ and $\mathcal{N}, y \nvDash \varphi$, according to the defintion of R', we consider two cases.

- xRy. Then by induction hypothesis and $\mathcal{N}, y \nvDash \varphi$, we have $\mathcal{M}', y \nvDash \varphi$, and then $\mathcal{M}', x \nvDash W\varphi$.
- $(x, y) \in Z(s) \times Z(s)$. Then $x \in Z(s)$, that is, sRx. Since R is secondarily reflexive (note that the property of secondary reflexivity is preserved under generated submodels), it follows that xRx. By Fact 5, $\mathcal{M}', x \nvDash W\varphi$.

Since $\mathcal{M}', s \vDash \Gamma$, we infer that $\mathcal{N}, s \vDash \Gamma$. Thus Γ is satisfiable in a transitive and Euclidean model, as desired. $\square$

Similarly, we can show the following. Let $\mathbf{KD5^W}$ is the extension of $\mathbf{K5^W}$ with the axiom $\neg W\bot$.

Theorem 30. $\mathbf{KD5^W} = \mathbf{KD45^W}$ is sound and strongly complete with respect to the class of serial, transitive and Euclidean frames.

Going back to the discussion after Prop. 25, although the construction $\mathcal{M}'$ in the proof of Prop. 25 does not preserve transitivity, this property is indeed preserved under generated submodels and the construction of $\mathcal{N}$ in Thm. 29 and also Thm. 30.

In a similar vein, by translating axiom B (viz. $\neg\varphi \to \Box\neg\Box\varphi$) via the translation induced by Almost Definability Schema, we can obtain an axiom $W\psi \wedge \neg\varphi \to W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi)$ of $\mathcal{L}(W)$ (denoted AB) over symmetric frames. One may verify that AB is valid over the class of symmetric frames.

3 Radical Ignorance

Definition 31 (Language)**.** The language of the logic of radical ignorance, denoted $\mathcal{L}(I_R)$, is defined recursively as follows:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid I_R\varphi.$$

Intuitively, $I^R\varphi$ is read "one is *Rumsfeld ignorant of* φ". Other connectives are defined as usual.

The notions of models and frames are defined as in Def. 2, and the semantics of $\mathcal{L}(I_R)$ is defined as in Def. 3, except that

$$\begin{array}{lll} \mathcal{M}, s \vDash I_R\varphi & \text{iff} & \textit{either } (R(s) \vDash \varphi \text{ and } \mathcal{M}, s \nvDash \varphi) \\ & & \textit{or } (R(s) \vDash \neg\varphi \text{ and } \mathcal{M}, s \vDash \varphi). \end{array}$$

Note that the semantics of $I_R\varphi$ is equivalent to the following one.

$$\mathcal{M}, s \vDash I_R\varphi \quad \text{iff} \quad \begin{array}{l}(R(s) \vDash \varphi \text{ or } \mathcal{M}, s \vDash \varphi) \text{ and} \\ (R(s) \vDash \neg\varphi \text{ or } \mathcal{M}, s \vDash \neg\varphi)\end{array}$$

We will use the two semantics of I_R interchangeably.

Recall that the operator W of false belief is interpreted as follows:

$$\mathcal{M}, s \vDash W\varphi \quad \text{iff} \quad R(s) \vDash \varphi \text{ and } \mathcal{M}, s \nvDash \varphi.$$

One may check that the operators of radical ignorance and of false belief are interdefined with each other, as $\vDash I_R\varphi \leftrightarrow (W\varphi \vee W\neg\varphi)$ and $\vDash W\varphi \leftrightarrow (I_R\varphi \wedge \neg\varphi)$. This may indicate that one can translate the results about false belief into those about radical ignorance via the translation induced by the interdefinability of the operators. Unfortunately, this holds for all but proof systems. Here we illustrate this with the axiomatizations of the logic of radical ignorance over all frames and over serial and transitive frames.

3.1 Minimal logic

The minimal logic of $\mathcal{L}(I_R)$, denoted $\mathbf{K^{RI}}$, consists of the following axioms and inference rules:

A0	all instances of propositional tautologies
RI-Equ	$I_R\varphi \leftrightarrow I_R\neg\varphi$
RI-Con	$I_R\varphi \wedge \neg\varphi \wedge I_R\psi \wedge \neg\psi \to I_R(\varphi \wedge \psi)$
MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
RI-R	$\dfrac{\varphi \to \psi}{I_R\varphi \wedge \neg\varphi \to I_R\psi \vee \psi}$

Note that the above axioms and inference rules can be obtained from those of $\mathbf{K^W}$ by a translation induced by the definability of W in terms of I_R, that is $\vDash W\varphi \leftrightarrow (I_R\varphi \wedge \neg\varphi)$, except for axiom RI-Equ. Actually, the translation only gives us an incomplete proof system, since the axiom RI-Equ is valid, but not provable in the translated system. To see this, consider an auxiliary semantics in which all formulas of the form $I_R\varphi$ are interpreted as φ. Under this semantics, the translated system are sound, but RI-Equ is not valid.

Proposition 32. The following rule is admissible in $\mathbf{K^{RI}}$:

$$\frac{\varphi \to \psi}{I_R(\varphi \wedge \chi) \wedge \neg(\varphi \wedge \chi) \to I_R\psi \vee \psi}.$$

Proof. We have the following proof sequence in $\mathbf{K^{RI}}$.

(1)	$\varphi \to \psi$	premise
(2)	$I_R\varphi \wedge \neg\varphi \to I_R\psi \vee \psi$	(1), RI-R
(3)	$I_R\varphi \to I_R\psi \vee \psi \vee \varphi$	(2)
(4)	$\varphi \wedge \chi \to \varphi$	A0
(5)	$I_R(\varphi \wedge \chi) \wedge \neg(\varphi \wedge \chi) \to I_R\varphi \vee \varphi$	(4), RI-R
(6)	$I_R(\varphi \wedge \chi) \wedge \neg(\varphi \wedge \chi) \to I_R\psi \vee \psi \vee \varphi$	$(3), (5)$
(7)	$I_R(\varphi \wedge \chi) \wedge \neg(\varphi \wedge \chi) \to I_R\psi \vee \psi$	$(1), (6)$

□

We can generalize the above result to the following.

Proposition 33. The following rule is admissible: for all $n \in \mathbb{N}$,

$$\frac{\chi_1 \wedge \cdots \wedge \chi_n \to \varphi}{I_R(\chi_1 \wedge \psi) \wedge \neg(\chi_1 \wedge \psi) \wedge \cdots \wedge I_R(\chi_n \wedge \psi) \wedge \neg(\chi_n \wedge \psi) \to I_R\varphi \vee \varphi}.$$

Proof. By induction on $n \in \mathbb{N}$. The case $n = 0$ is obvious. The case $n = 1$ is shown as in Prop. 32.

Now suppose that the statement holds for the case $n = m$ (IH), to show it also holds for the case $n = m + 1$. For this, we have the following proof sequence:

(1)	$\chi_1 \wedge \cdots \wedge \chi_{m+1} \to \varphi$	premise
(2)	$(\chi_1 \wedge \chi_2) \wedge \cdots \wedge \chi_{m+1} \to \varphi$	(1)
(3)	$I_R(\chi_1 \wedge \chi_2 \wedge \psi) \wedge \neg(\chi_1 \wedge \chi_2 \wedge \psi) \wedge \cdots \wedge$ $I_R(\chi_{m+1} \wedge \psi) \wedge \neg(\chi_{m+1} \wedge \psi) \to I_R\varphi \vee \varphi$	(2), IH
(4)	$I_R(\chi_1 \wedge \psi) \wedge \neg(\chi_1 \wedge \psi) \wedge I_R(\chi_2 \wedge \psi) \wedge \neg(\chi_2 \wedge \psi) \to$ $I_R(\chi_1 \wedge \chi_2 \wedge \psi) \wedge \neg(\chi_1 \wedge \chi_2 \wedge \psi)$	RI-Con
(5)	$I_R(\chi_1 \wedge \psi) \wedge \neg(\chi_1 \wedge \psi) \wedge \cdots \wedge I_R(\chi_{m+1} \wedge \psi)$ $\wedge \neg(\chi_{m+1} \wedge \psi) \to I_R\varphi \vee \varphi$	(3), (4)

□

By Def. 14 and the definability of W in terms of I_R, we obtain the canonical model for $\mathbf{K}^{\mathbf{RI}}$ as follows.

Definition 34. The *canonical model for* $\mathbf{K}^{\mathbf{RI}}$ is $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$, where

- $S^c = \{s \mid s$ is a maximal consistent set for $\}$
- if $I_R\psi \wedge \neg\psi \in s$ for *no* ψ, then sR^ct iff $s = t$, and
  if $I_R\psi \wedge \neg\psi \in s$ for *some* ψ, then sR^ct iff for all φ, if $I_R(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \psi) \in s$, then $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Lemma 35. For all $\varphi \in \mathcal{L}(I_R)$, for all $s \in S^c$, we have

$$\mathcal{M}^c, s \vDash \varphi \text{ iff } \varphi \in s.$$

Proof. By induction on $\varphi \in \mathcal{L}(I_R)$. The nontrivial case is $I_R\varphi$.

Suppose that $I_R\varphi \in s$ (thus $I_R\neg\varphi \in s$), to show that $\mathcal{M}^c, s \vDash I_R\varphi$. By induction hypothesis, we show that

(*) if $\varphi \notin s$, then for all $x \in S^c$ such that sR^cx, we have $\varphi \in x$, and if $\varphi \in s$, then for all $y \in S^c$ such that sR^cy, we have $\varphi \notin y$.

Firstly, we assume that $\varphi \notin s$, then $\neg\varphi \in s$. By supposition, $I_R\varphi \wedge \neg\varphi \in s$. Let $x \in S^c$ such that sR^cx. Then according to the definition of R^c, we have: for all χ, if $I_R(\chi \wedge \varphi) \wedge \neg(\chi \wedge \varphi) \in s$, then $\chi \in x$. By letting χ be φ, we can show that $\varphi \in x$. A similar argument applies to the second conjunct of (*).

Conversely, suppose that $I_R\varphi \notin s$ (thus $I_R\neg\varphi \notin s$), to prove that $\mathcal{M}^c, s \nvDash I_R\varphi$. By induction hypothesis, it suffices to show the following fails:

(a) *either* $\varphi \notin s$ and for all $x \in S^c$ such that sR^cx, we have $\varphi \in x$, *or* $\varphi \in s$ and for all $y \in S^c$ such that sR^cy, we have $\varphi \notin y$.

This amounts to showing the following (a1) and (a2) hold.

(a1) if $\varphi \notin s$, then for some $x \in S^c$ such that sR^cx, we have $\varphi \notin x$, *and*

(a2) if $\varphi \in s$, then for some $y \in S^c$ such that sR^cy, we have $\varphi \in y$.

For (a1), assume that $\varphi \notin s$. If $I_R\psi \wedge \neg\psi \in s$ for no ψ, then according to the definition of R^c, we have sR^cs. In this case, s is a desired x. If $I_R\psi \wedge \neg\psi \in s$ for some ψ, by definition of R^c and Lindenbaum's Lemma, it remains only to show that the set $\{\chi \mid I_R(\chi \wedge \psi) \wedge \neg(\chi \wedge \psi) \in s\} \cup \{\neg\varphi\}$ (denoted Γ) is consistent.

If Γ is *not* consistent, then there exist $\chi_1, \ldots, \chi_n$ such that $I_R(\chi_i \wedge \psi) \wedge \neg(\chi_i \wedge \psi) \in s$ for $i = 1, \ldots, n$ and

$$\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \varphi.$$

By Prop. 33, we infer that

$$\vdash I_R(\chi_1 \wedge \psi) \wedge \neg(\chi_1 \wedge \psi) \wedge \cdots \wedge I_R(\chi_n \wedge \psi) \wedge \neg(\chi_n \wedge \psi) \to I_R\varphi \vee \varphi.$$

As $I_R(\chi_i \wedge \psi) \wedge \neg(\chi_i \wedge \psi) \in s$ for $i = 1, \ldots, n$, we derive that $I_R\varphi \vee \varphi \in s$, which contradicts the supposition and the assumption. Thus we complete the proof of (a1).

Similarly, we can prove (a2), by using $I_R\neg\varphi \notin s$ and $\neg\varphi \notin s$ instead. □

Theorem 36. $\mathbf{K^{RI}}$ is sound and strongly complete with respect to the class of all frames.

3.2 Serial and transitive logic

In this section, we consider the extension of $\mathbf{K^{RI}}$ over serial and transitive frames. This is in line with the frames that the framework of [9] is actually based on, where the doxastic accessibility relation is serial and transitive, see Fn. 2 for the remark.

Define $\mathbf{KD4^{RI}}$ to be the extension of $\mathbf{K^{RI}}$ with the axiom RI-D ($\neg I^R\bot$) and the following axiom (denoted RI-4):

$$\begin{aligned} I_R\psi \wedge \neg\psi \wedge I_R(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \psi) \to I_R((I_R\chi \wedge \neg\chi \to I_R(\varphi \wedge \chi) \\ \wedge\neg(\varphi \wedge \chi)) \wedge \psi) \wedge \neg((I_R\chi \wedge \neg\chi \to I_R(\varphi \wedge \chi) \wedge \neg(\varphi \wedge \chi)) \wedge \psi) \end{aligned}$$

Again, the above axioms RI-D and RI-4 are obtained from, respectively, axioms AD and A4 via a translation induced by the interdefinability of W in terms of I_R.

Theorem 37. $\mathbf{KD4^{RI}}$ is sound and strongly complete with respect to the class of serial and transitive frames.

Proof. For soundness, by Thm. 36, it suffices to show the validity of axioms RI-D and RI-4 over serial and transitive frames. This follows directly from the validity of AD and A4 over the frames under discussion (Thm. 17 and Prop. 19) with the definability of W in terms of I_R.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{KD4^{RI}}$ as in Def. 34. By Thm. 36, it remains only to show that R^c is serial and transitive.

For seriality, suppose that $s \in S^c$. If $I_R\psi \wedge \neg\psi \in s$ for no ψ, then according to the definition of R^c, we derive that $sR^c s$. If $I_R\psi \wedge \neg\psi \in s$ for some ψ, by definition of R^c and Lindenbaum's Lemma, it suffices to prove that $\{\chi \mid I_R(\chi \wedge \psi) \wedge \neg(\chi \wedge \psi) \in s\}$ is consistent.

Since $I_R\psi \wedge \neg\psi \in s$, the set $\{\chi \mid I_R(\chi \wedge \psi) \wedge \neg(\chi \wedge \psi) \in s\}$ is nonempty. If the set is not consistent, then there are $\chi_1, \ldots, \chi_n$ such that $I_R(\chi_i \wedge \psi) \wedge \neg(\chi_i \wedge \psi) \in s$ for $i = 1, \ldots, n$ and

$$\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \bot.$$

By Prop. 33,

$$\vdash I_R(\chi_1 \wedge \psi) \wedge \neg(\chi_1 \wedge \psi) \wedge \cdots \wedge I_R(\chi_n \wedge \psi) \wedge \neg(\chi_n \wedge \psi) \to I_R\bot \vee \bot.$$

As $I_R(\chi_i \wedge \psi) \wedge \neg(\chi_i \wedge \psi) \in s$ for $i = 1, \ldots, n$, we conclude that $I_R\bot \vee \bot \in s$. As $\bot \notin s$, $I_R\bot \in s$. However, by axiom RI-D, $\neg I_R\bot \in s$. This contradicts the consistency of s.

For transitivity, let $s, t, u \in S^c$. Assume that $sR^c t$ and $tR^c u$, to show that $sR^c u$. We consider the following three cases.

- $I_R\psi \wedge \neg\psi \in s$ for no ψ. In this case, by definition of R^c and $sR^c t$, it follows that $s = t$, thus $sR^c u$ by assumption that $tR^c u$.
- $I_R\psi \wedge \neg\psi \in t$ for no ψ. In this case, by definition of R^c and $tR^c u$, it follows that $t = u$, thus $sR^c u$ by assumption that $sR^c t$.
- $I_R\psi \wedge \neg\psi \in s$ for some ψ, and $I_R\psi' \wedge \neg\psi' \in t$ for some ψ'. In this case, suppose that for any φ we have $I_R(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \psi) \in s$, we need to show that $\varphi \in u$. Since $I_R\psi \wedge \neg\psi \in s$, by supposition and axiom RI-4, we derive that $I_R((I_R\psi' \wedge \neg\psi' \to I_R(\varphi \wedge \psi') \wedge \neg(\varphi \wedge \psi')) \wedge \psi) \wedge \neg((I_R\psi' \wedge \neg\psi' \to I_R(\varphi \wedge \psi') \wedge \neg(\varphi \wedge \psi')) \wedge \psi) \in s$. As $sR^c t$, it follows that $I_R\psi' \wedge \neg\psi' \to I_R(\varphi \wedge \psi') \wedge \neg(\varphi \wedge \psi') \in t$, thus $I_R(\varphi \wedge \psi') \wedge \neg(\varphi \wedge \psi') \in t$. As $tR^c u$, we conclude that $\varphi \in u$, as desired.

□

Remark 38. In the introduction, we note that the canonical model in [4] does not apply to the transitive logic of reliable belief, thus not apply to the transitive logic of radical ignorance. Recall from [4, Def. 6.3] that the canonical model for the logic of reliable belief is defined such that $sR^c t$ iff for all φ, $\neg\mathcal{R}\varphi \wedge \neg\varphi \in s$ implies $\varphi \in t$. As observed in the introduction, I_R is equivalent to the negation of $\mathcal{R}$. Accordingly, in the case of radical ignorance, sR^ct iff for all φ, $I_R\varphi \wedge \neg\varphi \in s$ implies $\varphi \in t$. As the reader check, R^c is not transitive. In contrast, our R^c in Def. 34 is indeed transitive, as shown in Thm. 37.

4 Conclusion and Discussions

In this paper, we investigated the logics of false belief and radical ignorance. We proposed an almost definability schema, called 'Almost Definability Schema', which guides us to find the desired core axioms for the transitive logic and the Euclidean logic of false belief, and (with other considerations) also inspires us to propose a suitable canonical relation in the construction of the canonical model for the minimal logic of false belief. The canonical relation can uniformly handle the completeness proof of various logics of false belief, including the transitive logic, thereby solving an open problem in [14]. We explored the expressivity and frame definability of the logic of false belief. Moreover, due to the interdefinability of the operators of radical ignorance and false belief, we also axiomatized the logic of radical ignorance over the class of all frames and the class of serial and transitive frames. When translating the minimal logic of false belief to that of radical ignorance, we need to be cautious, since the translation only gives us an incomplete proof system, and one special axiom needed to be considered as well.

The almost definability schema is an important and useful tool in finding the suitable canoical relation and the desired core axioms for the special systems. Such usage has been made in the literature, see [2, 3, 7, 8]. This seems to be incomparable with other methods. We can try to extend such almost definability schema to other logics.

Coming back to the logics involved in this paper, one can explore the bisimulation notion for the logics of false belief and radical ignorance. Note that the almost definability schema is not enough for the notion of the bisimulation here, as in the case of the canonical relation. More things are needed to be taken account of. This is unlike the bisimulation of the contingency logic in the literature [7].

Another future work is to axiomatize the logic of false belief over symmetric frames. As remarked before Sec. 3, Almost Definability Schema also guides us to find an axiom AB, that is, $W\psi \wedge \neg\varphi \to W((W\chi \to \neg W(\varphi \wedge \chi)) \wedge \psi)$ of $\mathcal{L}(W)$, which is valid over the class of symmetric frames. If we define the canonical model $\mathcal{M}^c$ for the system $\mathbf{B^W}$ (that is, the extension of $\mathbf{K^W}$ with the axiom AB) as in Def. 14, we can show that $\mathcal{M}^c$ is *almost* symmetric: let $s, t \in S^c$, if sR^ct and $W\psi \in t$ for some ψ, then tR^cs. However, $\mathcal{M}^c$ is not symmetric, since if $W\psi \in t$ for no ψ and $t \neq s$, then according to the definition of R^c, $\sim tR^cs$, even if sR^ct. Therefore, in order to axiomatize the symmetric logic of false belief, more work needs to be done.

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