

T-CONVEXITY, WEAKLY IMMEDIATE TYPES AND T- λ -SPHERICAL COMPLETIONS OF O-MINIMAL STRUCTURES

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ABSTRACT. Let T be the theory of an o-minimal field, and T_{convex} the theory of its expansion by a predicate \mathcal{O} for a non-trivial T -convex valuation ring. For λ an uncountable cardinal, say that a unary type $p(x)$ over a model of T_{convex} is *λ -bounded weakly immediate* if its cut is defined by an empty intersection of fewer than λ many nested valuation balls. Call an elementary extension *λ -bounded wim-constructible* if it is obtained as a transfinite composition of extensions each generated by one element whose type is λ -bounded weakly immediate.

I show that λ -bounded wim-constructible extensions do not extend the residue-field sort and that any two wim-constructible extensions can be amalgamated in an extension which is again λ -bounded wim-constructible over both.

A consequence is that given a cardinal λ , every model of T_{convex} has a unique-up-to-non-unique-isomorphism λ -spherically complete λ -bounded wim-constructible extension. We call this extension the T - λ -spherical completion.

In the case T is power bounded wim-constructible extensions are just the immediate extensions. I discuss the example of power bounded theories expanded by exp.

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1. INTRODUCTION

1.1. Motivation. The broad motivation of the project underlying this paper is toward obtaining an analogue for exponential o-minimal fields of the familiar result that every real closed field has a maximal immediate extension isomorphic to the Hahn field with coefficients from the residue field and group of monomials isomorphic to the value group.

In this sense the first problem is to understand what such an analogue would be. Whereas for real closed fields maximal immediate elementary extensions are spherically complete, this is not true for maximal immediate extensions of, say, models of $T_{\text{exp}} := Th(\mathbb{R}_{\text{exp}})$: immediate elementary extensions of a model of T_{exp} must in fact be dense extensions (cf [5]). On the other hand by a theorem of Kuhlmann, Kuhlmann and Shelah [4, Thm. 1], there are no valuation-compatible exponentials on a spherically complete real closed value field.

The sought-for analogous statements should therefore require replacing both the class of extensions with respect to which we want to consider the maximal ones and the structural description of such maximal extensions.

Literature strongly suggests fields of transseries as useful structural analogues of Hahn fields in presence of an exponential, so I aim at choosing the class of extensions accordingly.

The setting of this paper will be the one of models of an o-minimal theory T endowed with a T -convex valuation and places its contributions within the model-theoretic and valuation-theoretic study of the complete weakly o-minimal theory T_{convex} of the models of T expanded by a predicate \mathcal{O} for a non-trivial T -convex valuation ring (cf [13], [12]).

Motivated by the idea that the maximal models we would like to obtain should, roughly speaking, “enjoy a certain degree of spherical completeness”, I define *weakly immediate types* as types whose cuts are given a conjunction of nested families of valuation balls (Definition 2.7) and attach to them a *cofinality* defined as the minimum number of nested valuation balls whose conjunction implies the type (Definition 3.14). Then I consider *λ -bounded wim-constructible extensions*, that is, transfinite compositions of extensions each generated by an element whose type is weakly immediate and has cofinality strictly smaller than λ (Definition 3.16).

In the case when T is power bounded, the so called residue-valuation property (rv-property, cf [16], [17], [11]), ensures that elements with weakly immediate type generates an immediate extension, thus wim-constructible extensions are just the immediate ones.

The rv-property fails for o-minimal theories defining an exponential, however even then wim-constructible extensions have good amalgamation properties and do not properly extend the residue field sort.

1.2. Setting. As proven by van den Dries and Lewenberg in [13], unary types over a model of T_{convex} are almost always completely determined by their reduct to the underlying o-minimal structure: more specifically they are unless they force the realization of the cut between the valuation ring \mathcal{O} and the elements greater than \mathcal{O} (from now on \mathcal{O} -special cut), in which case the reduct to the o-minimal structure has exactly two extensions to a type over model of T_{convex} .

This makes it convenient to rephrase the study of types over a model $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ as the study of the relationship between types over its o-minimal reduct \mathbb{E} (that is cuts over \mathbb{E}) and \mathcal{O} .

Notice that since T has definable skolem functions, all types *are* types over a model of T : namely given a set of parameters $A \subseteq \mathbb{E} \models T$, any type over A extends uniquely to a type over the definable closure of A , which is a model of T . We will therefore notationally distinguish between types for T and types for T_{convex} , by writing $\text{tp}(x/\mathbb{E})$ for the type of x over $T \models \mathbb{E}$ and $\text{tp}(x/(\mathbb{E}, \mathcal{O}))$ for the type of x over the expanded structure $(\mathbb{E}, \mathcal{O})$ whenever \mathcal{O} is a T -convex subring of \mathbb{E} .

1.3. Main results. I define a type over \mathbb{E} to be \mathcal{O} -weakly immediate if it is given by a consistent conjunction of \mathcal{O} -valuation balls (valuation balls, being order-convex, are type-definable over the o-minimal reduct \mathbb{E}) and then show that if some type $\text{tp}(x/\mathbb{E})$ over the o-minimal reduct \mathbb{E} is \mathcal{O} -weakly immediate, then $\mathbb{E}\langle x \rangle := \text{dcl}(\mathbb{E}, x)$ does not realize the \mathcal{O} -special cut (Proposition 3.4), hence $\text{tp}(x/\mathbb{E})$ extends to a unique $\text{tp}(x/(\mathbb{E}, \mathcal{O}))$.

More generally, with similar techniques and building on previous work of van den Dries and Lewenberg, I show that unary extensions of \mathbb{E} , once \mathcal{O} is fixed, can be partitioned into 5 mutually exclusive classes.

Theorem A (3.6). *If $x \in \mathbb{U} \setminus \mathbb{E}$, and \mathcal{O} is a non-trivial T -convex subring of \mathbb{E} , then one and only one of the following holds*

- (1) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle y \rangle$ for some weakly \mathcal{O} -immediate y ;
- (2) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle b \rangle$ for some b such that $\mathcal{O} < b < (\mathbb{E} \setminus \mathcal{O})^{>0}$;
- (3) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle z \rangle$ for some z such that for some (equiv. any) $\mathbb{K} \preceq \mathbb{E}$ maximal among the $\mathbb{K} \subseteq \mathcal{O}$, $\mathbb{K}\langle z \rangle$ is a cofinal extension of \mathbb{K} ;
- (4) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle d \rangle$ for some $d > \mathbb{E}$.
- (5) none of the above.

Thus, in particular, if x is \mathcal{O} -weakly immediate, then $\mathbb{E}\langle x \rangle$ with the unique extension of \mathcal{O} does not properly extend the residue field sort.

Finally I show that \mathcal{O} -wim-constructible extensions amalgamate in an extension that is wim-constructible over both (Lemma 3.22). This implies the theorem below, where a λ -bounded wim-constructible extension is understood to be a wim-constructible extension obtained by adjoining only weakly immediate elements whose type is given by an intersection of fewer than λ many valuation balls.

Theorem B (3.26). *Let $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, and λ be an uncountable cardinal. There is a unique-up-to-non-unique-isomorphism λ -spherically complete λ -bounded wim-constructible extension and it elementarily embeds in every λ -spherically complete extension of $(\mathbb{E}, \mathcal{O})$.*

The amalgamation provided by Lemma 3.22 is rather weak, so it is natural to ask if this can be strengthened (at least for some specific theories). I therefore pose the question whether wim-constructible extensions are closed under taking right (or left) factors (Question 3.29 (1) and (2) respectively). Positive answers to these questions allow for better understanding of T - λ -spherical completions (Remark 3.33).

In the case T is power bounded by the rv-property (see [16, Sec. 9 and 10] and [11, Ch. 12 and 13]) wim-constructible extensions are precisely the immediate ones and thus Questions 3.29 (1) and (2) have positive answer.

I discuss the case T is the expansion of a polynomially bounded theory by exp. In that case the following analogue of the rv-property holds: if $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, $f : \mathbb{E} \rightarrow \mathbb{E}$ is definable in \mathbb{E} , and $x \in \mathbb{U} \setminus \mathbb{E}$ is weakly \mathcal{O} -immediate, then there is an \mathbb{E} -definable composition of exponentials, translations and changes of signs g such that $g^{-1}f(x)$ is weakly \mathcal{O} -immediate (Proposition 5.12). This entails a very partial result toward a positive answer to Question 3.29 (1).

1.4. Structure of the paper. Section 2 is dedicated to the set-up: the first two subsections are dedicated to recalling some standard order- and valuation-theoretic facts, such as the relation between nested intersections of valuation balls and p.c.-sequences (Subsection 2.1) and giving a brief review of T -convexity, with particular regard of the basic facts needed throughout the note (Subsection 2.2). Subsection 2.3 is occupied by some technical Lemmas used throughout the paper.

Section 3 is occupied by the proofs of Theorems A and B (Subsections 3.1 and 3.2 respectively) and by some concluding remarks together with Question 3.29 (Subsection 3.3).

Section 4 briefly reviews the rv-property for power-bounded theories that was already established in [11, Ch. 12 and 13], giving a shorter and somewhat different proof.

Section 5 analyzes weakly-immediate types in *simply exponential theories*, that is, theories that eliminate quantifiers in some language $L_0 \cup \{\text{exp}, \log\}$ such that $T|_{L_0}$ is power bounded.

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2. SET-UP

This section is dedicated to laying up the set-up, recalling some facts relevant for our analysis of T_{convex} and to some technical lemmas.

2.1. Breadth, p.c.-sequences and weakly immediate types. In the following a *type* $p(x)$ is just a set of formulae closed under consequences in the tuple of free variables x . The set of realizations of p within a structure \mathbb{U} will be denoted by $p(\mathbb{U})$.

If \mathbb{E} is a field and $\mathcal{O} \subseteq \mathbb{E}$ is a valuation ring we will denote by \mathfrak{o} its maximal ideal, if \mathcal{O} has a subscript or superscript, we will denote the corresponding maximal ideal as \mathfrak{o} with the same subscript/superscript. We also write the associated dominance relation on $\mathbb{E} \setminus \{0\}$ as $x \preceq_{\mathcal{O}} y \Leftrightarrow x/y \in \mathcal{O}$ omitting the subscript \mathcal{O} if there is no ambiguity, similarly for $x \succ y \Leftrightarrow (x \preceq y \ \& \ y \preceq x)$, $x \prec y \Leftrightarrow (x \preceq y \ \& \ y \not\preceq x)$, $x \sim y \Leftrightarrow x - y \prec x$.

We will denote the value-group of $(\mathbb{E}, \mathcal{O})$ by $\mathbf{v}(\mathbb{E}, \mathcal{O})$ or $\mathbf{v}_{\mathcal{O}}(\mathbb{E})$ and the valuation by $\mathbf{v}_{\mathcal{O}} : \mathbb{E} \rightarrow \mathbf{v}(\mathbb{E}, \mathcal{O}) \cup \{\infty\}$. The residue field \mathcal{O}/\mathfrak{o} will be denoted by $\mathbf{r}(\mathbb{E}, \mathcal{O})$ or $\mathbf{r}_{\mathcal{O}}(\mathbb{E})$ and the quotient map by $\mathbf{r}_{\mathcal{O}} : \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{o}$. Finally the residue-value sort $\mathbb{E}^{\neq 0}/\sim_{\mathcal{O}}$ will be denoted by $\mathbf{rv}_{\mathcal{O}}(\mathbb{E})$ and the quotient map by $\mathbf{rv}_{\mathcal{O}} : \mathbb{E}^{\neq 0} \rightarrow \mathbf{rv}_{\mathcal{O}}(\mathbb{E})$. Again the subscript \mathcal{O} will be omitted if there is no ambiguity.

If \mathbb{U} is a first-order structure and $\mathbb{E} \subseteq \mathbb{U}$ is an elementary substructure we will write $\mathbb{E} \preceq \mathbb{U}$, or $\mathbb{E} \prec \mathbb{U}$ to signify that moreover $\mathbb{E} \neq \mathbb{U}$.

If T is an o-minimal theory expanding the theory of densely ordered Abelian groups, $\mathbb{U} \succ \mathbb{E} \models T$, and $x \in \mathbb{U} \setminus \mathbb{E}$, then $\mathbb{E}\langle x \rangle$ will denote the definable closure of the set $\mathbb{E} \cup \{x\}$ (which is the minimum elementary substructure of \mathbb{U} containing \mathbb{E} and x because T has definable Skolem-functions).

Throughout the rest of the paper, if \bullet is a binary relation symbol and S, C are sets, $S^{\bullet C}$ will be an abbreviation for $\{y \in S : \forall c \in C, y \bullet c\}$, if $C = \{c\}$ is a singleton we will write $S^{\bullet c}$ for $S^{\bullet C}$.

Definition 2.1. Let \mathbb{E} be an expansion of a dense linearly ordered group. We say that a partial 1-ary type $p(x)$ over \mathbb{E} is *convex* if

$$p(x_0), p(x_2), x_0 < x_1 < x_2 \vdash p(x_1),$$

or equivalently if it defines a convex subset in every elementary extension of \mathbb{E} . If $p(x)$ is a convex partial type over \mathbb{E} , we will write $\mathcal{N}(p)$ for the set of formulae

$$\mathcal{N}(p)(x) = \{a < x < b : p(x) \vdash a < x < b\}.$$

The *breadth of* p , $\text{Br}(p)(y)$ is then the partial type implied by the formulae

$$\{|y| < b - a : p(x) \vdash a < x < b\}.$$

If $p = \text{tp}(x/\mathbb{E})$ for x in some elementary extension of \mathbb{E} and p is convex, we write $\text{Br}(x/\mathbb{E})$ for $\text{Br}(p)$.

Remark 2.2. If \mathbb{E} is o-minimal, and p is any unary, complete and non-isolated type, then $p(x)$ is the set of consequences of $\mathcal{N}(p)(x)$, in particular p is convex.

Remark 2.3. For every convex type p over \mathbb{E} , $q := \text{Br}(p)$ is a convex subgroup, meaning that q is convex and

$$q(x_0), q(x_1) \vdash q(x_0 + x_1).$$

Moreover for every elementary extension \mathbb{U} of \mathbb{E} realizing p ,

$$q(\mathbb{U}) = \{y \in \mathbb{U} : y + p(\mathbb{U}) \subseteq p(\mathbb{U}) \text{ or } -y + p(\mathbb{U}) \subseteq p(\mathbb{U})\}.$$

Recall the following two standard valuation-theoretic facts.

Lemma 2.4. *If $\mathbb{U} \supseteq \mathbb{E}$ is a field extension, $\mathcal{O} \subseteq \mathbb{U}$ is a valuation subring and $x \in \mathbb{E} \setminus \mathbb{U}$, then the following are equivalent*

- (1) $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has no maximum;
- (2) $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E}) \subseteq \mathbf{v}_{\mathcal{O}}\mathbb{E}$ and $\mathbf{r}_{\mathcal{O}}(\mathcal{O} \cap \mathbb{E}) = \mathbf{r}_{\mathcal{O}}(\mathcal{O} \cap (x\mathbb{E}^{\neq 0} + \mathbb{E}))$.

Proof. Let $\mathbf{v} := \mathbf{v}_{\mathcal{O}}$, $\mathbf{r} := \mathbf{r}_{\mathcal{O}}$. $\neg(2) \Rightarrow \neg(1)$ If for some $c \in \mathbb{E}$, $\mathbf{v}(x - c) \notin \mathbf{v}\mathbb{E}$, then $\mathbf{v}(x - c)$ is maximum in $\mathbf{v}(x - \mathbb{E})$ by the ultrametric inequality. Similarly if there are $c, d \in \mathbb{E}$ such that $xc + d \in \mathcal{O}$ and $xc + d + \mathfrak{o} \cap \mathbb{E} = \emptyset$, then $\mathbf{v}(x - d/c)$ is maximum in $\mathbf{v}(x - \mathbb{E})$. (2) \Rightarrow (1), given $c \in \mathbb{E}$, since $\mathbf{v}(x - c) \in \mathbf{v}\mathbb{E}$ we can find d such that $(x - c)/d = x/d - c/d \in \mathcal{O}$, but then since $\mathbf{r}(x/d - c/d) \in \mathbf{r}(\mathbb{E})$, $(x/d - c/d + \mathfrak{o}) \cap \mathbb{E} \neq \emptyset$ and we can find $b \in \mathbb{E}$ such that $x/d - c/d - b \in \mathfrak{o}$ whence $x - c - bd \in \mathfrak{o} \cdot x$ and $\mathbf{v}(x - c - bd) > \mathbf{v}(x - c)$. \square

Lemma 2.5. *Let $\mathbb{U} \supseteq \mathbb{E}$ be fields and $\mathcal{O} \subseteq \mathcal{O}'$ be valuation subrings of \mathbb{U} such that $\mathbb{E} \cap \mathcal{O} = \mathbb{E} \cap \mathcal{O}'$. For $x \in \mathbb{U} \setminus \mathbb{E}$, the following are equivalent:*

- (1) $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has no maximum;
- (2) $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ has no maximum.

Proof. (2) \Rightarrow (1) If $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has a maximum $\gamma = \mathbf{v}_{\mathcal{O}}(x - c)$, then $\mathbf{v}_{\mathcal{O}'}(x - c)$ is maximum in $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ because $\mathbf{v}_{\mathcal{O}}(y) \mapsto \mathbf{v}_{\mathcal{O}'}(y)$ is an ordered group homomorphism between the respective value groups.

(1) \Rightarrow (2) Assume for some $c \in \mathbb{E}$, $\mathbf{v}_{\mathcal{O}'}(x - c)$ was maximum in $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$, then by Lemma 2.4 either $\mathbf{v}_{\mathcal{O}'}(x - c) \notin \mathbf{v}_{\mathcal{O}'}\mathbb{E}$ or, for some $d \in \mathbb{E}$, $(x - c)/d \in \mathcal{O}' \setminus (\mathbb{E} + \mathfrak{o}')$. In the first case $\mathbf{v}_{\mathcal{O}'}(x - c) \notin \mathbf{v}_{\mathcal{O}'}\mathbb{E}$ would imply $\mathbf{v}_{\mathcal{O}}(x - c) \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}$ again because $\mathbf{v}_{\mathcal{O}}(y) \mapsto \mathbf{v}_{\mathcal{O}'}(y)$ is an ordered group homomorphisms, but this in turn would imply that $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has maximum, contradiction. In the second case, if furthermore $(x - c)/d \notin \mathcal{O}$, then $\mathbf{v}_{\mathcal{O}}(x - c)/d \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}$ and $\mathbf{v}_{\mathcal{O}}(x - cd)$ is maximum in $\mathbf{v}_{\mathcal{O}}\mathbb{E}$ contradicting (1). If instead $(x - c)/d \in \mathcal{O}$, then it must be $(x - c)/d \in b + (\mathfrak{o} \setminus \mathfrak{o}')$ with $b \in \mathbb{E} \cap \mathcal{O}$ (because $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has

no maximum), so $x - c - bd \in d \cdot (\mathfrak{o} \setminus \mathfrak{o}')$. However, since $\mathcal{O} \cap \mathbb{E} = \mathcal{O}' \cap \mathbb{E}$, this implies that $\mathbf{v}_{\mathcal{O}}(x - c - bd) \notin \mathbf{v}_{\mathcal{O}}(\mathbb{E}) = \mathbf{v}_{\mathcal{O}'}(\mathbb{E})$, in particular $\mathbf{v}_{\mathcal{O}}(x - c - bd)$ is maximum in $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ contradicting (1) once again. \square

Lemma 2.6. *Let $\mathbb{U} \supseteq \mathbb{E}$ be ordered fields, $\mathcal{O} \subseteq \mathbb{U}$ be a convex valuation subring and $x \in \mathbb{E} \setminus \mathbb{U}$. If $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$ has no maximum, then $\mathbf{v}_{\mathcal{O}}((x - \mathbb{E})^{>0})$ and $\mathbf{v}_{\mathcal{O}}((x - \mathbb{E})^{<0})$ are cofinal in $\mathbf{v}_{\mathcal{O}}(x - \mathbb{E})$.*

Proof. Let $\mathbf{v} := \mathbf{v}_{\mathcal{O}}$. Notice that if $\mathbf{v}(x - \mathbb{E})$ has no maximum, then by Lemma 2.6, $\mathbf{v}(x - \mathbb{E}) \subseteq \mathbf{v}\mathbb{E}$. Moreover $\mathbf{v}(x - c_1) \geq \mathbf{v}(x - c_0)$ if and only if $c_1 \in c_0 + \mathcal{O}(x - c_1)$, thus it suffices to show that every valuation ball $B := c + \mathcal{O}d$ with $c, d \in \mathbb{E}$ and $x \in c + \mathcal{O}d$ contains both elements above and below x . Suppose not: then $\mathbb{E}^{>B} = \mathbb{E}^{>x}$ or $\mathbb{E}^{<B} = \mathbb{E}^{<x}$, in both cases we easily see that $\mathbf{v}(\mathbb{E} - x)$ has a maximum because if $B' := c' + \mathcal{O}d'$ with $c', d' \in \mathbb{E}$ is such that $B' \subseteq B$ and $\mathbb{E}^{>B} = \mathbb{E}^{>B'}$, then $B' = B$. \square

From now on \mathbb{E} will be an o-minimal expansion of a real closed field.

Definition 2.7. Let \mathcal{O} be a convex valuation ring on \mathbb{E} and $p(x)$ be a complete type over \mathbb{E} , we call $p(x)$ *weakly \mathcal{O} -immediate* if for a realization x of p , $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ has no maximum for some (equiv. any, by Lemma 2.5) extension \mathcal{O}' of \mathcal{O} to $\mathbb{E}\langle x \rangle$.

A *\mathcal{O} -pseudo-Cauchy sequence* in \mathbb{E} (*\mathcal{O} -p.c.-sequence* for short) is a cardinal-indexed sequence $(x_i)_{i \in \lambda} \in \mathbb{E}^\lambda$ such that for every $i < j < k < \lambda$, $x_j - x_k \prec x_i - x_j$. A *pseudolimit* for a $(x_i)_{i < \lambda}$ is any element x such that $x - x_j \prec x - x_i$ for all $i < j$ or equivalently such that $x \in \bigcap_{i < \lambda} x_i + \mathcal{O}(x_i - x_{i+1})$.

Remark 2.8. For every convex valuation ring \mathcal{O} in \mathbb{E} , valuation balls are type definable in \mathbb{E} : indeed for $a, b \in \mathbb{E}$ we have

$$\begin{aligned} xa - b \in \mathfrak{o} &\leftrightarrow |xa - b| < (\mathbb{E} \setminus \mathfrak{o})^{>0} \\ xa - b \in \mathcal{O} &\leftrightarrow |xa - b| < (\mathbb{E} \setminus \mathcal{O})^{>0} \end{aligned}$$

In particular any intersection of valuation balls is still type-definable in \mathbb{E} .

Lemma 2.9. *The following are equivalent for a complete unary type p over \mathbb{E} and a convex valuation ring $\mathcal{O} \subseteq \mathbb{E}$:*

- (1) p is weakly \mathcal{O} -immediate over \mathbb{E} ;
- (2) there are families $(x_i)_{i \in I} \in \mathbb{E}^I$, $(y_i)_{i \in I} \in \mathbb{E}^I$ such such that for every $\mathbb{U} \succ \mathbb{E}$ and every extension \mathcal{O}' of \mathcal{O} ,

$$p(\mathbb{U}) = \bigcap \{x_i + \mathcal{O}'y_i : i \in I\}$$

- (3) there is an increasing \mathcal{O} -pc-sequence $(x_i)_{i < \lambda}$ in $\mathbb{E}^{<x}$ such that for every $i < \lambda$ and for every \mathcal{O}' extending \mathcal{O} and every \mathbb{U} realizing p ,

$$p(\mathbb{U}) = \bigcap \{x_i + \mathcal{O}'(x_{i+1} - x_i) : i \in \lambda\}$$

Proof. (1) \Rightarrow (3) if p is weakly \mathcal{O} immediate over \mathbb{E} and \mathcal{O}' is the maximal extension of \mathcal{O} to \mathbb{U} , then $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ has no maximum by Lemma 2.5 and $\mathbf{v}_{\mathcal{O}'}((x - \mathbb{E})^{>0})$ is cofinal in $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ by Lemma 2.6.

So we can pick $(x_i)_{i < \lambda} \in (\mathbb{E}^{< x})^\lambda$ such that $\mathbf{v}_{\mathcal{O}'}(x - x_i)$ is strictly increasing and cofinal in $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$. This will be a \mathcal{O} -p.c. sequence in \mathbb{E} because for $i < j < k < \lambda$,

$$\mathbf{v}(x_i - x_j) = \mathbf{v}(x_i - x) < \mathbf{v}(x_j - x) = \mathbf{v}(x_j - x_k).$$

Notice that for each $i < \lambda$, $x \in x_i + \mathcal{O}' \cdot (x_i - x) = x_i + \mathcal{O}' \cdot (x_i - x_{i+1})$. So $p(\mathbb{U}) \subseteq \bigcap_{i < \lambda} x_i + \mathcal{O}' \cdot (x_i - x_{i+1})$. Notice also that $\bigcap_{i < \lambda} x_i + \mathcal{O}' \cdot (x_i - x_{i+1}) = \emptyset$ because if $y \in \bigcap_{i < \lambda} x_i + \mathcal{O}' \cdot (x_i - x_{i+1})$, then $\mathbf{v}(y - x) > \mathbf{v}(y - x_i)$ for all $i < \lambda$ and $\mathbf{v}(x - x_i)$ would not be cofinal in $\mathbf{v}(x - \mathbb{E})$. But then since $\mathcal{O}' = \{t \in \mathbb{U} : |t| < \mathbb{E}^{> \mathcal{O}'}\}$ is type-definable over \mathbb{E} and p is a complete type over \mathbb{E} , $p(\mathbb{U}) = \bigcap_{i < \lambda} x_i + \mathcal{O}'(x_{i+1} - x_i)$.

On the other hand if \mathcal{O}'' is another extension of \mathcal{O} , we have $\mathcal{O}'' \subseteq \mathcal{O}'$ and thus

$$x_{i+1} + \mathcal{O}''(x_{i+2} - x_{i+1}) \subseteq x_{i+1} + \mathcal{O}'(x_{i+2} - x_{i+1}) \subseteq x_i + \mathcal{O}''(x_{i+1} - x_i).$$

where the second inclusion is because $\mathcal{O}'(x_{i+2} - x_{i+1}) \subset \mathcal{O}''(x_{i+1} - x_i)$.

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1) for some \mathbb{U} , $p(\mathbb{U})$ is non empty, so the family of valuation balls $\{x_i + \mathcal{O}y_i : i \in I\}$ must be finitely consistent and hence, by the ultrametric inequality totally ordered by inclusion. Moreover since p is not realized in \mathbb{E} , it must be $\bigcap\{x_i + \mathcal{O}y_i : i \in I\} = \emptyset$ so $\{\mathbf{v}_{\mathcal{O}y_i} : i \in I\}$ has no maximum. Let $x \in p(\mathbb{U})$, if for some $c \in \mathbb{E}$, $\mathbf{v}_{\mathcal{O}'}(x - c)$ is maximum in $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ then $\mathbf{v}_{\mathcal{O}'}(x - c) > \mathbf{v}_{\mathcal{O}'}y_i$ for otherwise, since $\{\mathbf{v}_{\mathcal{O}y_i} : i \in I\}$ has no maximum, there there is y such that $\mathbf{v}_{\mathcal{O}'}(x - x_i) \geq \mathbf{v}_{\mathcal{O}'}y_i = \mathbf{v}_{\mathcal{O}'}(x - c)$ contradicting the maximality of $\mathbf{v}_{\mathcal{O}'}(x - c)$. But then $c \in x_i + \mathcal{O}'y_i$ for every $i \in I$, contradicting $\bigcap\{x_i + \mathcal{O}y_i : i \in I\} = \emptyset$. \square

Lemma 2.10. *Let $\mathbb{E} \prec \mathbb{U} \models T$, \mathcal{O} a convex subring of \mathbb{U} , and p a $\mathcal{O} \cap \mathbb{E}$ -weakly immediate type over \mathbb{E} , then*

- (1) for any realization $x \in p(\mathbb{U})$, $\text{Br}(p)(\mathbb{U}) = p(\mathbb{U}) - x$.
- (2) $\text{Br}(p)(\mathbb{U})$ is a \mathcal{O} -submodule of \mathbb{U} .

Proof. (1) Let $y \in p(\mathbb{U})$, then clearly $|y - x| < b - a$ for every $a, b \in \mathbb{E}$, with $a < p(\mathbb{U}) < b$, viceversa since p is weakly $\mathcal{O} \cap \mathbb{E}$ -immediate, $p(\mathbb{U}) = \bigcap\{x_i + \mathcal{O}(x_i - x_{i+1}) \in \lambda\}$ for some pc-sequence $(x_i)_{i < \lambda}$. It follows that $\text{Br}(p)(\mathbb{U}) = \bigcap_{i < \lambda} \mathcal{O}(x_i - x_{i+1})$ and by the ultrametric inequality, if $x \in p(\mathbb{U})$ and $z \in \bigcap_{i < \lambda} \mathcal{O}(x_{i+1} - x_i)$, then $z + x \in \bigcap\{x_i + \mathcal{O}(x_{i+1} - x_i) : i \in \lambda\}$ and we are done.

(2) As we just saw, $\text{Br}(p)(\mathbb{U}) = \bigcap_{i < \lambda} \mathcal{O}(x_i - x_{i+1})$, whence it is an intersection of \mathcal{O} -submodules, and thus a \mathcal{O} -submodule. \square

2.2. Review of T -convexity and tame extensions. Let T be an o-minimal theory expanding RCF in a language L . If $\mathbb{E} \models T$, recall that a T -convex subring of \mathbb{E} is a convex subring of \mathbb{E} which is closed by continuous T -definable functions $f : \mathbb{E} \rightarrow \mathbb{E}$ (by T -definable we mean \emptyset -definable in T). It is said to be *non-trivial* if $\mathbb{E} \neq \mathcal{O}$. It is not hard to see that if

$\mathbb{K} \preceq \mathbb{E}$, then the convex hull $\mathcal{O} := \text{CH}_{\mathbb{E}}(\mathbb{K})$ of \mathbb{K} in \mathbb{E} is a T -convex valuation subring.

Denote by L_{convex} the language obtained expanding L with a unary predicate \mathcal{O} and by T_{convex} the theory given by T together with an axiom scheme stating \mathcal{O} is a non-trivial T -convex valuation ring. These notions were introduced by van den Dries and Lewenberg in [13] where they proved the following

Theorem 2.11 (van den Dries - Lewenberg, (3.10)-(3.15) in [13]). *The theory T_{convex} is complete and weakly o-minimal, moreover if T eliminates quantifiers and has a universal axiomatization (resp. is model-complete) in L , then T_{convex} eliminates quantifiers (resp. is model complete) in L_{convex} .*

This has as key-ingredient the fact, which will be extremely important per se throughout this note, that if $p(x)$ is a type over an o-minimal structure \mathbb{E} , it can extend in at most two ways to a type over the expanded structure $(\mathbb{E}, \mathcal{O})$.

Lemma 2.12 (van den Dries - Lewenberg, (3.6)-(3.7) in [13]). *If $p(x)$ is a unary type over $\mathbb{E} \models T$, $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, $x \models p$ and \mathcal{O}' is a T -convex valuation subring of $\mathbb{E}\langle x \rangle$ with $\mathcal{O}' \cap \mathbb{E} = \mathcal{O}$, then $\mathcal{O}' \in \{\mathcal{O}_x^-, \mathcal{O}_x\}$ where*

$$\mathcal{O}_x^- := \text{CH}_{\mathbb{E}\langle x \rangle}(\mathcal{O}), \quad \mathcal{O}_x := \{y \in \mathbb{E}\langle x \rangle : |y| < \mathbb{E}^{\mathcal{O}}\}.$$

Remark 2.13. Of course if $\mathbb{E}\langle x \rangle$ does not contain any b with $\mathcal{O} < b < \mathbb{E}^{\mathcal{O}}$, then $\mathcal{O}_x^- = \mathcal{O}_x$. If instead $\mathbb{E}\langle x \rangle$ contains such a b , by the exchange property, $\mathbb{E}\langle x \rangle = \mathbb{E}\langle b \rangle$ and $\mathcal{O}_x = \mathcal{O}_b$. For simplicity in the following we will denote $\mathcal{O}_x^- = \text{CH}_{\mathbb{E}\langle x \rangle}(\mathcal{O})$ by \mathcal{O} if there is no ambiguity.

Recall that an elementary extension $\mathbb{K} \preceq \mathbb{E}$ of models of T is said to be *tame* (notation $\mathbb{K} \preceq_{\text{tame}} \mathbb{E}$) if \mathbb{K} is definably Dedekind-complete in \mathbb{E} in the sense that for every \mathbb{E} -definable subset X of \mathbb{E} , if $X \cap \mathbb{K}$ is bounded, then it has a supremum in \mathbb{K} . It was proven by Marker and Steinhorn in [7] that $\mathbb{K} \preceq_{\text{tame}} \mathbb{E}$ if and only if for every tuple \bar{c} of elements in \mathbb{E} , $\text{tp}(\bar{c}/\mathbb{K})$ is a definable type (see also [9]).

Tame extensions of o-minimal structures are closely related to T -convex valuation rings in fact:

Fact 2.14 (van den Dries - Lewenberg, (2.12) in [13]). *If $\mathbb{E} \models T$, \mathcal{O} is a T -convex valuation ring then $\mathbb{K} \preceq \mathbb{E}$ is maximal among the elementary substructures of \mathbb{E} contained in \mathcal{O} if and only if $\mathbb{K} \preceq_{\text{tame}} \mathbb{E}$ and $\mathbb{K} + \mathfrak{o} = \mathcal{O}$.*

To every tame extension $\mathbb{K} \preceq_{\text{tame}} \mathbb{E}$, is associated a standard part map $\text{st}_{\mathbb{K}} : \text{CH}_{\mathbb{E}}(\mathbb{K}) \rightarrow \mathbb{K}$ uniquely defined by the property that for every $x \in \text{CH}_{\mathbb{E}}(\mathbb{K})$, $|\text{st}_{\mathbb{K}}(x) - x| < \mathbb{K}^{>0}$.

Theorem 2.15 (van den Dries, Sec. 1 in [12]). *If $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, and $\mathbb{K} \preceq_{\text{tame}} \mathbb{E}$ with $\text{CH}(\mathbb{K}) = \mathcal{O}$, then $\text{st}_{\mathbb{K}} : \mathcal{O} \rightarrow \mathbb{K}$ induces an isomorphism between the induced structure on the imaginary sort \mathcal{O}/\mathfrak{o} and \mathbb{K} .*

In what follows we will also need this other result of van den Dries and Lewenberg.

Lemma 2.16. *Let \mathcal{O} be a T -convex subring of \mathbb{U} , $x \in \mathbb{U} \setminus \mathbb{E}$, \mathbb{K} maximal among the $\mathbb{K} \preceq \mathbb{E}$ such that $\mathbb{K} \subseteq \mathcal{O}$, and \mathbb{K}_x maximal among the $\mathbb{K} \preceq \mathbb{K}_x \preceq \mathbb{E}\langle x \rangle$ such that $\mathbb{K}_x \subseteq \mathcal{O}$, then $\mathbb{K} = \mathbb{K}_x$ or \mathbb{K}_x is a principal extension of \mathbb{K} , that is $\mathbb{K}_x = \mathbb{K}\langle y \rangle$ for some $y \in \mathbb{K}_x \setminus \mathbb{K}$.*

Proof. Follows from [13, Lemma 5.3]. \square

2.3. Some technical Lemmas. Throughout this subsection T will be any \mathfrak{o} -minimal theory expandin RCF, $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, $\mathbb{U} \succ \mathbb{E}$ an extension of \mathbb{E} and $b \in \mathbb{U} \setminus \mathbb{E}$ an element realizing the special cut $\mathcal{O} < b < \mathbb{E}^{>\mathcal{O}}$. Also $\mathbb{K} \preceq \mathbb{E}$ will be maximal such that $\mathbb{K} \subseteq \mathcal{O}$. Also, for every unary function f we write f' or ∂f for its derivative and $\dagger f = f^\dagger := f'/f$ for its logarithmic derivative (where they are defined).

Lemma 2.17. *Let $f : \mathbb{E} \rightarrow \mathbb{E}$ be a \mathbb{E} -definable function such that $f(b) \in \mathcal{O}_b$, then there are a \mathbb{K} -definable function g and $a_-, a_+ \in \mathbb{E}$ with $a_- < b < a_+$ such that*

$$(\mathbb{E}, \mathcal{O}) \models \forall t \in (a_-, a_+), f(t) - g(t) \in \mathfrak{o}$$

Moreover $f(b) - g(b) \in \mathfrak{o}_b$ and

- (1) $f(b) \in \mathcal{O}$ if and only if g is eventually bounded by some $c \in \mathbb{K}$;
- (2) $f(b) \in \mathfrak{o}$ if and only if g is infinitesimal at infinity.

Proof. Since $f(b) \in \mathcal{O}_b$, there is $a_- \in \mathbb{E}$, $a_- < b$, such that for all $t \in \mathcal{O}$ if $t > a_-$ $f(t) \in \mathcal{O}$. In particular we can write $f(t) = g(t) + \varepsilon(t)$ with g some \mathbb{K} -definable function and ε \mathbb{E} -definable such that $\varepsilon(t) \in \mathfrak{o}$ for $t > a_-$. By Proposition 3.2 ε cannot define an ordered bijection between a final segment of \mathcal{O} and an initial segment of $(\mathbb{E} \setminus \mathcal{O})^{>0}$, so there must be $a_+ > b$ such that it must be $\varepsilon(t) \in \mathfrak{o}$ for all $t \in (a_-, a_+)$.

(1) and (2) in the moreover are clear.

To see that $\varepsilon(b) \in \mathfrak{o}_b$, observe that otherwise $1/\varepsilon(b) \in \mathcal{O}_b \setminus \mathcal{O}$, but this would imply that for all big enough $t \in \mathcal{O}$, $1/\varepsilon(t) \in \mathcal{O}$, contradicting that $\varepsilon((a_-, a_+)) \subseteq \mathfrak{o}$. \square

Lemma 2.18. *Let $f : \mathbb{E} \rightarrow \mathbb{E}$ be \mathbb{E} -definable and $b \in \mathbb{U} \setminus \mathbb{E}$ be such that $\mathcal{O} < b < |\mathbb{E} \setminus \mathcal{O}|$. If $f(b) \in \mathfrak{o}_b$, then $f'(b) \in \mathfrak{o}_b$. If $f(b) \in \mathfrak{o}$ then $f'(b) \in \mathfrak{o}$.*

Proof. Assume $f(b) \in \mathcal{O}_b$, then by Lemma 2.17, there is a \mathbb{E} -definable interval J , such that $f = g + \varepsilon$ where g is \mathbb{K} -definable and $\varepsilon(J) \subseteq \mathfrak{o}$. We can also assume w.l.o.g. that ε is differentiable and monotone on J . If $I \subseteq J$ is a \mathbb{E} -definable interval such that $\varepsilon'(I) \cap \mathfrak{o} = \emptyset$, then $\sup I - \inf I \in \mathfrak{o}$ for otherwise by the mean value theorem $\varepsilon(\sup I) - \varepsilon(\inf I) > \mathfrak{o}$ contradicting $\varepsilon(I) \subseteq \mathfrak{o}$. Since $\{t \in J : \varepsilon'(t) \notin \mathfrak{o}\}$ is a finite union of convex subsets, and by the previous observation each of such subsets must be included in some translate of \mathfrak{o} , it follows that for some possibly smaller \mathbb{E} -definable interval, $\varepsilon'(J) \subseteq \mathfrak{o}$. This implies that $\varepsilon'(b) \in \mathfrak{o}_b$, by the final clause of

Lemma 2.17. Now if $f(b) \in \mathfrak{o}_b$, then up to further restricting J , $g|_J = 0$ and thus $f'(b) = \varepsilon'(b) \in \mathfrak{o}_b$, if instead $f(b) \in \mathfrak{o}$, then $g(b) \in \mathfrak{o} \cap \mathbb{K}\langle b \rangle$ and $g'(b) \in \mathfrak{o}$ as well. \square

Lemma 2.19. *Let $f : \mathbb{E}^{>0} \rightarrow \mathbb{E}^{>0}$ be a monotone \mathbb{E} -definable differentiable function. Then*

- (1) if $xf^\dagger(x(1 + \mathfrak{o})) \in \mathcal{O}$, then $f(x(1 + \mathfrak{o})) \subseteq f(x)(1 + \mathfrak{o})$;
- (2) if on some interval J and $tf^\dagger(t) \in \mathcal{O}$ for all $t \in J$ then $t_1 \asymp t_2 \Rightarrow f(t_1) \asymp f(t_2)$ for all $t_1, t_2 \in J$;

similarly for any differentiable \mathbb{E} -definable $f : \mathbb{E} \rightarrow \mathbb{E}$

- (1bis) if $f'(x + \mathfrak{o}) \in \mathcal{O}$, then $f(x + \mathfrak{o}) \subseteq f(x) + \mathfrak{o}$;
- (2bis) if on some interval J and $f'(t) \in \mathcal{O}$ for all $t \in J$, then $t_2 - t_1 \in \mathcal{O} \Rightarrow f(t_2) - f(t_1) \in \mathcal{O}$ for all $t_1, t_2 \in J$.

Proof. (1)-(2). For every $x \neq y$ by the mean value theorem, there is t between x and y such that $f(x) - f(y) = f'(t)(x - y)$ so

$$\frac{f(x) - f(y)}{f(t)} = f^\dagger(t)t \frac{x - y}{t}$$

Now notice that $x \asymp y$ if and only if $(x - y)/t \in \mathcal{O}$ and $x \sim y$ if and only if $(x - y)/t \in \mathfrak{o}$. Similarly, since f is monotone, $f(x) \sim f(y)$ (resp. $f(x) \asymp f(y)$) if and only if $(f(x) - f(y))/f(t) \in \mathfrak{o}$ (resp. $\in \mathcal{O}$).

(1bis)-(2bis) Straightforward from the mean value theorem. \square

Lemma 2.20. *If for some $c \in \mathbb{E}$, $(t - c)f^\dagger(t) \in \mathcal{O}$ for all t in some interval (a, b) , then for every weakly immediate x such that $a < x < b$, $f(x)$ is weakly immediate.*

Proof. Let $(x_i)_{i < \lambda}$ be a p.c.-sequence in $(\mathbb{E}, \mathcal{O})$ whose pseudolimit is x . Wlog we can assume that λ is a regular cardinal. It suffices to show that $(f(x_{\mu+i}))_{i < \lambda}$ is a p.c.-sequence for some $\mu < \lambda$. Notice that for $i < j < k < \lambda$, there are $\xi_{ij}, \xi_{ik} \in \mathbb{E}$ respectively with between x_i and x_j and between x_j and x_k , such that

$$\frac{f(x_j) - f(x_k)}{f(x_j) - f(x_i)} = \frac{f'(\xi_{j,k})}{f'(\xi_{i,j})} \frac{x_j - x_k}{x_j - x_i}.$$

Observe that for big enough $\mu < \lambda$, for all $k > j > i > \mu$, ξ_{ij} and ξ_{jk} are forced to be in (a, b) and be such that $\xi_{ij} - c \asymp \xi_{jk} - c$. Now set $g(t) := f'(t + c)$, and observe that by hypothesis $tg^\dagger(t) \in \mathcal{O}$ for all $t \in (a - c, b - c)$, so by Lemma 2.19, $g(\xi_{ij} - c) \asymp g(\xi_{jk} - c)$ and the thesis follows. \square

3. MAIN RESULTS

This section is occupied with the proofs of Theorem A and Theorem B.

3.1. Unary types in T -convex. Throughout the rest of the section T denotes an o-minimal theory expanding the theory RCF of real closed fields.

The goal of this subsection is to establish Theorem 3.6 which will justify Definition 3.7.

Recall the following fact about o-minimal structures.

Lemma 3.1. *Let $\mathbb{K} \models T$, $x > \mathbb{K}$ and f a \mathbb{K} -definable function, then there is a T -definable function g such that $g(x) > f(x)$.*

Proof. Without loss of generality we can assume f is continuous increasing. If f is itself T -definable there is nothing to prove. We show by induction on $n > 0$ that if we can write $f(-) = F(\bar{c}, -)$ with $\bar{c} := (c_0, \dots, c_{n-1})$ some n -tuple of parameters from \mathbb{K} and F a T -definable function, then $f(x) < g(x)$ with $g(-) = G(\bar{d}, -)$ with \bar{d} a $n - 1$ -tuple. Without loss of generality we can assume that n is minimal for f , so \bar{c} has dimension n , and whenever a T -definable cell of \mathbb{K} contains \bar{c} it is open. Up to replacing $F(-, -)$ with $F(\bar{\theta}(-), -)$ with θ a T -definable continuous parametrization of each of the finitely many open cells in question, we can also assume that $F(\bar{p}, -)$ is continuous increasing for every $\bar{p} \in \mathbb{K}^n$. Now consider a T -definable cell decomposition of \mathbb{K}^{n+1} such that on every cell F is continuous and monotone or constant in every variable. By the minimal choice of n , there will be a T -definable open cell A in \mathbb{K}^n containing \bar{c} and a continuous T -definable function $\gamma : A \rightarrow \mathbb{K}$ such that on the open $n + 1$ -cell (γ, ∞) , F is continuous, increasing in the last variable and monotone in the first n -variables. If c_{n-1} is not a definable type over $\text{dcl}_T(c_0, \dots, c_{n-2})$, there are $a, b \in \text{dcl}_T(c_0, \dots, c_{n-2})$ such that $c_{n-1} \in [a, b] \subseteq A_{c_0, \dots, c_{n-2}}$ and set

$$g(t) := 1 + \max\{F(c_0, \dots, c_{n-2}, s, t) : s \in [a, b]\},$$

then for $t > d := \max\{\gamma(c_0, \dots, c_{n-2}, s) : s \in [a, b]\} \in \text{dcl}(c_0, \dots, c_{n-2})$ we have $g(t) > f(t)$, hence $g(x) > f(x)$. If instead c_{n-1} is a definable type over $\text{dcl}(c_0, \dots, c_{n-2})$, we can without loss of generality assume that $c_{n-1} > \text{dcl}(c_0, \dots, c_{n-2})$ and in that case it suffices to choose $g(t) := 1 + \max\{F(c_0, \dots, c_{n-2}, s, t) : s \leq t\}$. \square

Proposition 3.2. *Let $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$. There are no \mathbb{E} -definable maps restricting to a continuous monotone bijection between final segments of \mathcal{O} and final segments of \mathfrak{o} , similarly for \mathcal{O} and $\mathbb{E}^{<0}$ and for \mathfrak{o} and $\mathbb{E}^{<0}$.*

Proof. Let f be an \mathbb{E} -definable map, so $f(t) = g(t, \bar{c})$ for some 0-definable g and some n -tuple \bar{c} from \mathbb{E} . The statement “for every $\bar{c} \in \mathbb{E}^n$, $g(-, \bar{c})$ does not restrict to a definable continuous monotone bijection between a final segment of \mathcal{O} and a final segment of \mathfrak{o} ” is first order in the language of T_{convex} . Since the theory of $(\mathbb{E}, \mathcal{O})$ is T_{convex} , to prove the thesis it suffices to show the statement holds in some model of T_{convex} . Therefore we can assume $\mathbb{E} = \mathbb{K}\langle d \rangle$ with $d > \mathbb{K}$, $\mathcal{O} = \text{CH}(\mathbb{K})$ and $\mathbb{K} \models T$ with cofinality at least $|T|^+$. Now notice that in such a model \mathcal{O} has the cofinality of \mathbb{K} , whereas \mathfrak{o} has at most cofinality $|T|$ by Lemma 3.1. The same argument can

be given to show that f cannot induce a monotone bijection between a final segment of \mathcal{O} and a final segment of $\mathbb{E}^{<0}$.

To show the last, it suffices to exhibit a model in which $\mathbb{E}^{<0}$ and \mathfrak{o} don't have the same cofinality. This can be constructed for example as $\mathbb{E} := \mathbb{K}\langle\{d_i : i < |T|^+\}\rangle$ where $\mathbb{K}\langle d_i : i > j \rangle < d_j$ and $\mathcal{O} = \text{CH}_{\mathbb{E}}(\mathbb{K})$: in such a model the cofinality of \mathfrak{o} is $|T|^+$, whereas the cofinality of $\mathbb{E}^{<0}$ is the cofinality of $\mathbb{K}\langle d_0 \rangle$ which is bounded by $|T|$. \square

Lemma 3.3. *Assume C is a non-empty (a priori not necessarily definable) convex subset of \mathcal{O} with $C = \mathfrak{o} + C$ and $\text{st}(C)$ without extrema, f be \mathbb{E} -definable continuous monotone and 2-fold differentiable with f' and f'^{\dagger} continuous and monotone on $f^{-1}(C)$. Assume furthermore that $f(x) = z \in C$ and $f^{-1}(C) - x$ is a \mathcal{O} -submodule of \mathbb{E} . Then $\text{st}(C)$ is an open interval.*

Proof. Let $\mu(r, t) = f(x + r(f^{-1}(t) - x))$ and $\sigma(t_1, t_2) = f(f^{-1}(t_1) + f^{-1}(t_2) - x)$. We shall first show that:

- (1) $\text{st} \sigma(\text{st}(t_1), \text{st}(t_2)) = \text{st}(\sigma(t_1, t_2))$ for all $t_1, t_2 \in \mathcal{O}$.
- (2) for every r and every t , $\text{st} \mu(\text{st}(r), t) = \text{st}(\mu(r, t))$;
- (3) there is a finite $F \subseteq C$ such that for all $t \notin F + \mathfrak{o}$ and for all $r \notin \mathfrak{o}$, $\text{st}(\mu(r, \text{st}(t))) = \text{st} \mu(r, t)$.

To see (1) observe that by Lemma 2.19 it suffices to show that for all $t_1, t_2 \in C$, $\partial_i \sigma(t_1, t_2) \in \mathcal{O}$. Notice that

$$\frac{\partial}{\partial s_1} \frac{f'(s_1 + s_2 - x)}{f'(s_1)} = \frac{f''(s_1 + s_2 - x)}{f'(s_1)} - \frac{f'(s_1 + s_2 - x)f''(s_1)}{f'(s_1)^2} = 0$$

if and only if $f'^{\dagger}(s_1 + s_2 - x) = f'^{\dagger}(s_1)$. So if $s_1, s_2 \in f^{-1}(C)$ by the hypothesis on f'^{\dagger} , this can happen only if $s_2 = x$.

Notice that

$$\partial_1 \sigma(t_1, t_2) = \frac{f'(f^{-1}(t_1) + f^{-1}(t_2) - x)}{f'(f^{-1}(t_1))}$$

so $\{t_1 \in C : \partial_1^2 \sigma(t_1, t_2) = 0\} \neq \emptyset$ if and only if $t_2 = z$. So for each t_2 either $\partial_1 \sigma(-, t_2)$ is monotone on C , or $f^{-1}(t_2) = x$ and thus $\partial_1 \sigma(t_1, t_2) = 1$ for all t_1 .

If $\partial_1 \sigma(-, t_2)$ is monotone on $C \subseteq \mathcal{O}$ and $\text{st}(C)$ has no extrema, $\partial_1 \sigma(C \times \{t_2\}) \subseteq \mathcal{O}$ because otherwise $\{t \in C : \partial_1 \sigma(t, t_2) \notin \mathcal{O}\}$ contains a segment (t_-, t_+) with $t_- + \mathfrak{o} < t_+$ and by the mean value theorem it would be $\sigma(C \times \{t_2\}) \not\subseteq \mathcal{O}$, against the hypothesis that $\sigma(C \times C) \subseteq C$. Since this holds for all t_2 , $\partial_1 \sigma(C \times C) \subseteq \mathcal{O}$. By symmetry also $\partial_2 \sigma(C \times C) \subseteq \mathcal{O}$.

Observe that this implies also that the standard part $\bar{\sigma} := \text{st} \sigma|_{\mathbb{K}}$ of σ , induces an ordered abelian group structure on $\mathbb{K} \cap C$ with 0-element $\text{st}(z)$, in particular for every $t \in C \setminus (z + \mathfrak{o})$, $\sigma(t, t) - t \notin \mathfrak{o}$.

Now observe that

$$\begin{aligned}\partial_1\mu(r, t) &= f'(r(f^{-1}(t) - x) + x) \cdot (f^{-1}(t) - x), \\ h(r, t) &:= \frac{1}{r}\partial_2\mu(r, t) = \frac{f'(r(f^{-1}(t) - x) + x)}{f'(f^{-1}(t))}.\end{aligned}$$

Therefore by the hypothesis on f , $\partial_1\mu$ is monotone continuous in the first variable. It follows that for every $t \in C$ and $r \in \mathcal{O}$, $\partial_1\mu(r, t) \in \mathcal{O}$, for otherwise $\partial_1\mu(\mathcal{O}^{\geq r}, t) \cap \mathcal{O} = \emptyset$ or $\partial_1\mu(\mathcal{O}^{\leq r}, t) \cap \mathcal{O} = \emptyset$, but both would imply by the mean value theorem that $\mu(\mathcal{O}, t) \notin \mathcal{O}$. This proves (2) by Lemma 2.19.

Notice that for each r , $\text{st}\{t \in C : \partial_2\mu(r, t) \notin \mathcal{O}\}$ must be finite, because otherwise by the mean value theorem one would get $\mu(r, C) \notin C$.

On the other hand h is monotone in the first variable if restricted to $\mathcal{O}^{\geq 0}$ or to $\mathcal{O}^{\leq 0}$, so if for some r, t with $r \notin \mathfrak{o}$, $h(r, t) \notin \mathcal{O}$, then $h(r_1, t) \notin \mathcal{O}$ for all r_1 on one side of r in $\mathcal{O}^{> \mathfrak{o}}$ or $\mathcal{O}^{< \mathfrak{o}}$. Thus there is a finite $F \subseteq C$ such that for all $t \notin F + \mathfrak{o}$, and all $r \notin \mathcal{O} \setminus \mathfrak{o}$, $h(r, t) \in \mathcal{O}$, which in turn implies that for all $t \notin F + \mathfrak{o}$ and all $r \notin \mathfrak{o}$, $\partial_2\mu(r, t) \in \mathcal{O}$. Again by Lemma 2.19 we deduce (3).

Now we prove the statement assuming for the sake of simplicity that f is increasing on $f^{-1}(C)$, the case in which f is decreasing is analogous.

Since we assumed $\text{st}(C)$ has no extrema, there is $t_0 \in C$ with $C^{\geq t_0} \cap (F \cup \{z\}) + \mathfrak{o} = \emptyset$. In particular for every $t \geq t_0 > z + \mathfrak{o}$, $\mu(2, t) = \sigma(t, t) > t + \mathfrak{o}$.

It follows that $\partial_1\mu(r, t_0) > \mathfrak{o}$ for every large enough $r \in \mathcal{O}$: if not, by weak \mathfrak{o} -minimality, there would be $r_0 \in \mathcal{O}^{\geq 1}$ such that $\partial_1\mu(\mathcal{O}^{\geq r_0}, t) \subseteq \mathfrak{o}$, but this is absurd because then the mean value theorem would give $\mu(\mathcal{O}^{\geq r_0}, t_0) \subseteq \mu(r_0, t_0) + \mathfrak{o}$ whereas $\mu(r_0, t_0) \geq t_0$ would give

$$\mu(2r_0, t_0) = \mu(2, \mu(r_0, t_0)) = \sigma(\mu(r_0, t_0), \mu(r_0, t_0)) > \mu(r_0, t_0) + \mathfrak{o},$$

contradiction.

Since $\partial_1\mu(r, t_0) > \mathfrak{o}$ for every large enough $r \in \mathcal{O}$, some final segment of $\text{st}(\mu(\mathcal{O}^{> \mathfrak{o}}, t_0))$ is an open interval in \mathbb{K} .

Now if $t_1 = \sup_{\mathbb{K}} \text{st}(\mu(\mathcal{O}^{> \mathfrak{o}}, t_0)) \in \mathbb{K}$ then it must be $t_1 \notin C$, because otherwise $t_2 := \mu(1/2, t_1) > \mu(\mathcal{O}^{> \mathfrak{o}}, t_0)$, so $t_1 = \sigma(t_2, t_2) > t_2 + \mathfrak{o}$ contradicting the fact that $t_1 = \sup \text{st}(\mu(\mathcal{O}^{> \mathfrak{o}}, t_0))$.

A similar argument shows that $\text{st}(C)$ is either unbounded below in \mathbb{K} or has an infimum in \mathbb{K} . \square

Proposition 3.4. *Let $\mathbb{U} \succ \mathbb{E}$ and $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$. If $x \in \mathbb{U} \setminus \mathbb{E}$ is weakly \mathcal{O} -immediate, then $\mathbb{E}\langle x \rangle$ does not contain any b with $\mathcal{O} < b < \mathbb{E}^{> \mathcal{O}}$.*

Proof. Let b be $\mathcal{O} < b < \mathbb{E}^{> \mathcal{O}}$ and assume toward a contradiction that $\mathbb{E}\langle x \rangle = \mathbb{E}\langle b \rangle$. So there is a \mathbb{E} -definable continuous increasing bijection $f : \mathbb{U} \rightarrow \mathbb{U}$ such that $f(x) = b$. This means that f must induce a continuous increasing bijection between $\text{tp}(x/\mathbb{E})(\mathbb{E}\langle x \rangle)$ and the set $C := \text{tp}(b/\mathbb{E})(\mathbb{E}\langle x \rangle) := \mathcal{O}_x^{> \mathcal{O}}$. Since x is weakly \mathcal{O} -immediate, $M := \text{Br}(x/\mathbb{E})(\mathbb{E}\langle x \rangle)$ is a \mathcal{O}_x -submodule and $\text{tp}(x/\mathbb{E})(\mathbb{E}\langle x \rangle) = x + M$.

Now unless f has the form $f(t) = c_0 + t$ or $f(t) = c_0 + c_1 \exp(t)$, the hypothesis of Lemma 3.3 apply, so $\text{st}_{\mathbb{K}_x}(C)$ should be an interval in \mathbb{K}_x which it is not, contradiction. On the other hand it is easy to check that if f has the form $f(t) = c_0 + t$ or $f(t) = c_0 + c_1 \exp(t)$, for some $c_0, c_1 \in \mathbb{E}$ then $f^{-1}(b)$ is not weakly immediate. \square

Proposition 3.5. *Let $\mathbb{U} \succ \mathbb{E}$, $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$, $x \in \mathbb{U} \setminus \mathbb{E}$ weakly $\mathcal{O} \cap \mathbb{E}$ -immediate over \mathbb{E} , $\mathbb{K} \prec \mathbb{E}$ maximal s.t. $\mathbb{K} \subseteq \mathcal{O}$ and $\mathbb{K}_x \preceq \mathbb{E}\langle x \rangle$ maximal s.t. $\mathbb{K} \preceq \mathbb{K}_x \subseteq \mathcal{O}$. Then $\mathbb{K} = \mathbb{K}_x$.*

Proof. By the Lemma 2.16 we have $\mathbb{K}_x = \mathbb{K}(f(x))$, for some \mathbb{E} -definable f and by Proposition 3.4 \mathbb{K} is cofinal in \mathbb{K}_x so $\text{tp}(f(x)/\mathbb{K}) \vdash \text{tp}(f(x)/\mathbb{E})$.

By Lemma 2.9 there are $(x_i^+)_{i < \lambda}$ and $(x_i^-)_{i < \lambda}$ be \mathcal{O} -p.c.-sequences for x in \mathbb{E} with $x_i^+ > x$ and decreasing and $x_i^- < x$ increasing. Assume without loss of generality that f is differentiable with $f' > 0$ and set $z_i^+ := f(x_i^+)$ and $z_i^- := f(x_i^-)$. Also up to extracting a subsequence we can assume that $z_i^+ - z_{i+1}^+ \asymp 1 \asymp z_i^- - z_{i+1}^-$. Notice that

$$x_{i+1}^+ - x_i^+ = f^{-1}(z_{i+1}^+) - f^{-1}(z_i^+) = (f^{-1})'(\zeta_i^+)(z_{i+1}^+ - z_i^+) \asymp (f^{-1})'(\zeta_i^+)$$

for some ζ_i^+ with $z_i^+ < \zeta_i^+ < z_{i+1}^+$. Hence for a sequence $(\zeta_i^+)_{i < \lambda}$ decreasing and coinital in $\mathbb{E}^{>z}$, $\mathbf{v}(f^{-1})'(\zeta_i^+)$ is an increasing sequence because $(x_i^+)_{i < \lambda}$ was a p.c.-sequence. Similarly there is a sequence $(\zeta_i^-)_{i < \lambda}$ increasing and cofinal in $\mathbb{E}^{<z}$ with $\mathbf{v}(f^{-1})'(\zeta_i^-)$ increasing. But this would imply that $(f^{-1})'$ is decreasing on a final segment $\mathbb{E}^{<z}$ and increasing on an initial segment of $\mathbb{E}^{>z}$, which contradicts the hypothesis $z \notin \mathbb{E}$. \square

Theorem 3.6. *If $x \in \mathbb{U} \setminus \mathbb{E}$, and \mathcal{O} is a non-trivial T -convex subring of \mathbb{E} , then the following are mutually exclusive*

- (1) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle y \rangle$ for some weakly \mathcal{O} -immediate y ;
- (2) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle b \rangle$ for some b such that $\mathcal{O} < b < \mathbb{E}^{>\mathcal{O}}$;
- (3) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle z \rangle$ for some z such that for some (equiv. any) $\mathbb{K} \preceq \mathbb{E}$ maximal among the $\mathbb{K} \subseteq \mathcal{O}$, $\mathbb{K}\langle z \rangle$ is a cofinal extension of \mathbb{K} ;
- (4) $\mathbb{E}\langle x \rangle = \mathbb{E}\langle d \rangle$ for some $d > \mathbb{E}$.

Proof. The fact that (1) implies that none of (2) or (3) holds is given respectively by Propsitions 3.4 and 3.5. The fact that (2) and (3) are mutually exclusive follows from Lemma 2.16. Finally (4) clearly implies that none of (1), (2), or (3) holds because then for every $h \in \mathbb{E}\langle d \rangle$, there would be $c \in \mathbb{E}$ such that $|c - h| < \mathbb{E}^{>0}$. \square

Definition 3.7. If x satisfies (1), then we call x *weakly immediately generated* with respect to \mathcal{O} (or *weakly \mathcal{O} -immediately generated*).

If x is in case (2) we call it *\mathcal{O} -special*.

If x is in case (3) we call it *\mathcal{O} -cofinally residual*.

If x is as in (4) then x is *tame*.

If $x \in \mathbb{U} \setminus \mathbb{E}$ is not in the cases above, then we call x *strictly purely \mathcal{O} -valuational*.

We say that x is *dense* over \mathbb{E} , if \mathbb{E} is order dense in $\mathbb{E}\langle x \rangle$.

Remark 3.8. If \mathbb{E} admits a non-trivial T -convex valuation ring \mathcal{O} , then x is dense if and only if x is \mathcal{O} -weakly immediate and $\text{Br}(x/\mathbb{E})(\mathbb{E}) = 0$.

Remark 3.9. If $x \in \mathbb{U} \setminus \mathbb{E}$ is strictly purely \mathcal{O} -valuational if and only if $\mathbf{v}_{\mathcal{O}}\mathbb{E}\langle x \rangle$ is a cofinal extension of $\mathbf{v}_{\mathcal{O}}\mathbb{E}$ that does not realize the type 0^+ and moreover for every $y \in \mathbb{E}\langle x \rangle$, there is $c \in \mathbb{E}$ such that $\mathbf{v}_{\mathcal{O}}(y - c) \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}$.

Theorem 3.6 can be restated in terms of unary types over $(\mathbb{E}, \mathcal{O})$.

Theorem 3.10. *Let $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$, $\mathbb{E} \prec \mathbb{U}$ with $\mathbb{E} \not\subseteq \mathcal{O}$, and $x \in \mathbb{U} \setminus \mathbb{E}$. Then one and only one of the following happens*

- (1) $\mathbf{r}(\mathbb{E}\langle x \rangle, \mathcal{O} \cap \mathbb{E}\langle x \rangle) \neq \mathbf{r}(\mathbb{E}, \mathcal{O} \cap \mathbb{E})$
- (2) *there is $y \in \mathbb{E}\langle x \rangle \setminus \mathbb{E}$, such that for every $c_0 \in \mathbb{E}$ there is $c_1 \in \mathbb{E}$ with $y - c_1 \prec y - c_0$.*
- (3) *there is $M \subseteq \mathbb{E}\langle x \rangle$ such that $\mathbf{v}(M) \cap \mathbf{v}\mathbb{E} = \emptyset$ and $\mathbb{E}\langle x \rangle = \mathbb{E} + (M \cup \{1\})$.*

Proof. Use Theorem 3.6 chasing the cases of Definition 3.7. If x is weakly \mathcal{O} -immediately generated over \mathbb{E} , then (2) happens. If x is tame or strictly purely $(\mathcal{O} \cap \mathbb{E})$ -valuational over \mathbb{E} then clearly (3) happens.

If x is $(\mathcal{O} \cap \mathbb{E})$ -special there are two cases: either $\mathbb{E}\langle x \rangle \cap \mathcal{O} = (\mathcal{O} \cap \mathbb{E})_x$ and (1) happens or $\mathbb{E}\langle x \rangle \cap \mathcal{O} = (\mathcal{O} \cap \mathbb{E})_x^-$ and (3) happens.

Finally if x is $(\mathbb{E} \cap \mathcal{O})$ -cofinally residually over \mathbb{E} , then (1) happens. \square

Definition 3.11. It may be convenient to have terminology for the types over $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ as well. If $x \in (\mathbb{U}, \mathcal{O})$ we say that x is

- (1) *residual* if (1) above happens;
- (2) *weakly immediately generated over \mathbb{E}* if (2) above happens, that is if x is weakly \mathcal{O} -immediately generated over \mathbb{E} ;
- (3) *purely valualional* if (3) above holds.

Notice that wheter x satisfies (1), (2) or (3), only depends on $\text{tp}(x/(\mathbb{E}, \mathcal{O}))$.

3.2. Wim-constructible extensions. In this subsection we define wim-constructible extensions and prove an amalgamation result. This, after some extra smallness assumptions on the extensions are imposed, allows for consideration of spherical completions of structures.

Remark 3.12. Notice that any extension $\mathbb{E}' \succ \mathbb{E}$ containing no \mathcal{O} -special elements admits a unique T -convex extension \mathcal{O}' of \mathcal{O} to \mathbb{E}' .

Remark 3.13. If $(\mathbb{E}', \mathcal{O}') \succeq (\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, if $x + \mathcal{O}'y$ is a valuation ball of $(\mathbb{E}', \mathcal{O}')$, then its trace $(x + \mathcal{O}'y) \cap \mathbb{E}$ is an intersection of valuation balls in $(\mathbb{E}, \mathcal{O})$. Notice also that if $\mathbb{E}' \succ \mathbb{E}$ contains no \mathcal{O} -special elements, then whenever $(x + \mathcal{O}'y) \cap \mathbb{E} \neq \emptyset$, the partial type of $x + \mathcal{O}'y$ is the unique extension of the partial type of $(x + \mathcal{O}'y) \cap \mathbb{E}$ to \mathbb{E}' both with respect to T and to T_{convex} .

Definition 3.14. If $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and p is a weakly immediate type over $(\mathbb{E}, \mathcal{O})$, then we define the *cofinality* of p as the cofinality of $\mathbf{v}_{\mathcal{O}'}(x - \mathbb{E})$ for

$x \in p(\mathbb{U})$, $\mathbb{U} \succ \mathbb{E}$ and \mathcal{O}' any extension of \mathcal{O} to \mathbb{U} , or equivalently the least cardinal κ such that there is a p.c. sequence $(x_i)_{i \in \lambda}$ for x .

Remark 3.15. If $y \in \mathbb{E}\langle x \rangle$ and x, y are weakly \mathcal{O} -immediate, then the types of x and y have the same cofinality.

Definition 3.16. Let $(\mathbb{E}, \mathcal{O}) \preceq (\mathbb{E}^*, \mathcal{O}^*)$ be an elementary extension of models of T_{convex} . A sequence $(x_i : i \in I)$ in \mathbb{E}^* with $(I, <)$ a well ordered set, such that for every j , x_j is weakly immediate over $\mathbb{E}_j := \mathbb{E}\langle x_i : i < j \rangle$ is called a *wim-construction* (over $(\mathbb{E}, \mathcal{O})$). $(x_i : i \in I)$ in \mathbb{E}^* is said to κ -bounded if every $\text{tp}(x_j/\langle \mathbb{E}_j, \mathcal{O}^* \cap \mathbb{E}_j \rangle)$ has cofinality $< \kappa$. We say that $(\mathbb{E}_*, \mathcal{O}_*)$ is

- (1) *wim-constructible* if there is a wim construction $(x_i : i < \lambda)$ in \mathbb{E}_1 such that $\mathbb{E}_1 = \mathbb{E}\langle x_i : i < \lambda \rangle$;
- (2) κ -bounded *wim-constructible* if there is a κ -bounded wim construction $(x_i : i < \lambda)$ in \mathbb{E}_1 such that $\mathbb{E}_1 = \mathbb{E}\langle x_i : i < \lambda \rangle$;
- (3) *strictly wim* if it has no non-weakly immediate factor, that is if every \mathbb{E}_2 with $\mathbb{E} \prec \mathbb{E}_2 \prec \mathbb{E}_1$ is such that every $x \in \mathbb{E}_1 \setminus \mathbb{E}_2$ is weakly immediate over \mathbb{E}_2 .
- (4) λ -*wim* for λ a (possibly finite) cardinal if for every $\mu < \lambda$, the definable closure of every $\mu + 1$ -tuple in \mathbb{E}_1 is wim-constructible over $(\mathbb{E}, \mathcal{O})$.

A type over $(\mathbb{E}, \mathcal{O})$ is said to be $(\kappa$ -bounded) *wim-constructible* if it is the type of a $(\kappa$ -bounded) wim-construction over $(\mathbb{E}, \mathcal{O})$.

Remark 3.17. Wim constructible types over $(\mathbb{E}, \mathcal{O})$ are uniquely determined by their reduct to \mathbb{E} .

Remark 3.18. Every strictly wim extension is wim constructible.

If strictly wim extensions are transitive (i.e. closed under composition) then the converse holds as well.

Lemma 3.19. *Let $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$, $\mathbb{E} \preceq \mathbb{E}_1 \prec \mathbb{U}$ with $\mathbb{E} \not\subseteq \mathcal{O}$ and $x \in \mathbb{U} \setminus \mathbb{E}$. Suppose $\text{tp}(x/\mathbb{E})(\mathbb{E}_1) = \emptyset$, then x is weakly immediate over $(\mathbb{E}, \mathbb{E} \cap \mathcal{O})$ if and only if it is weakly immediate over $(\mathbb{E}_1, \mathbb{E}_1 \cap \mathcal{O})$. If it is the case, then $\text{tp}(x/(\mathbb{E}, \mathcal{O}))$ and $\text{tp}(x/(\mathbb{E}_1, \mathcal{O} \cap \mathbb{E}_1))$ have the same cofinality.*

Proof. Let $p = \text{tp}(x/\mathbb{E})$. If x is weakly \mathcal{O} -immediate over \mathbb{E} with cofinality κ , there is a p.c.-sequence $(x_i)_{i < \kappa}$ in \mathbb{E} such that

$$p(\mathbb{U}) := \{x : x - x_i \succ x - x_j : i < j < \kappa\}$$

If $p(\mathbb{U})$ has empty intersection with \mathbb{E}_1 , it defines a weakly immediate type over $(\mathbb{E}_1, \mathcal{O} \cap \mathbb{E}_1)$ with cofinality $\leq \kappa$.

Viceversa suppose x is weakly immediate over $(\mathbb{E}_1, \mathcal{O}_1)$ and let $p_1 := \text{tp}(x/\mathbb{E}_1)$ and κ_1 its cofinality. Then $p_1(\mathbb{U})$ is an intersection of valuation balls $(B_i)_{i < \kappa_1}$ with center and radius in \mathbb{E}_1 and w.l.o.g. $B_j \subset B_i$ for all $j < i$. Each such $B_i \cap \mathbb{E}$ is either empty or an intersection of valuation

balls in $(\mathbb{E}, \mathcal{O} \cap \mathbb{E})$, however if $B_i \cap \mathbb{E} = \emptyset$ for some i , then for all $y \in B_i$, $\text{tp}(y/\mathbb{E}) = p$ and $p(\mathbb{E}_1)$ would be non empty.

Since κ_1 is regular, up to extracting a subsequence we can assume that $B_j \cap \mathbb{E} \subset B_i \cap \mathbb{E}$ for all $i < j$. But then for each i , there is a valuation ball B'_i with radius and center in \mathbb{E} such $B_{i+1} \subseteq B'_i \subseteq B_i$ and the cofinality of p is $\leq \kappa_1$. \square

Recall that two types p, q over \mathbb{E} are said to be *orthogonal* if given disjoint tuples of variables \bar{x}, \bar{y} of the appropriate length, $p(\bar{x}) \cup q(\bar{y})$ defines a complete type over \mathbb{E} .

In the following lemma given two ordered sets $(A, <)$, $(B, <)$ we denote by $A \sqcup B$ the disjoint union of A and B with the smallest *partial* order extension the orders on A and B .

Lemma 3.20. *Let $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, γ, λ be ordinals and $(x_i : i < \gamma)$ and $(y_j : j < \lambda)$ be κ -bounded wim-constructions over $(\mathbb{E}, \mathcal{O})$. If their types are orthogonal over \mathbb{E} , then for every bijective order preserving $h : \lambda \sqcup \gamma \rightarrow \beta$, $(z_k : k < \beta)$ defined by $z_k = x_{h^{-1}(k)}$ for $k \in h_*(\gamma)$ and $z_k = y_{h^{-1}(k)}$ for $k \in h_*(\lambda)$ is a κ -bounded wim-construction over \mathbb{E} .*

Proof. We need to show that z_k is wim over $\mathbb{E}\langle z_j : j < k \rangle$ with cofinality $< \kappa$. Suppose without loss of generality that $k \in h_*(\gamma)$, then z_k is wim over $\mathbb{E}\langle z_j : j \in (h_*\gamma)^{<k} \rangle$ with cofinality $< \kappa$.

On the other hand by the orthogonality hypothesis $\text{tp}(z_k/\mathbb{E}\langle z_j : j \in (h_*\gamma)^{<k} \rangle)$ is orthogonal to $\text{tp}(z_{h_*(\lambda)}/\mathbb{E}\langle z_j : j \in (h_*\gamma)^{<k} \rangle)$, so it is not realized in $\mathbb{E}\langle z_j : j < k \rangle$. Thus by Lemma 3.19, x_k is wim over $\mathbb{E}\langle z_{h(j)} : j \in \gamma, h(j) < k \rangle$ with cofinality $< \kappa$. \square

Lemma 3.21. *Let $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$ and $(\mathbb{E}_i)_{i < \lambda}$ a strictly increasing sequence of elementary substructures of \mathbb{U} and $\mathbb{E}_\lambda := \bigcup_{i < \lambda} \mathbb{E}_i$. If $y \in \mathbb{U}$ is weakly immediate over \mathbb{E}_i for all $i < \lambda$ then y is weakly immediate over \mathbb{E}_λ .*

Proof. If λ is not a limit ordinal the statement is trivial. So assume λ is a limit ordinal.

If for some $j < \lambda$, $\text{tp}(y/\mathbb{E}_j)$ is not realized in \mathbb{E}_λ , then we can invoke Lemma 3.19 and conclude.

Otherwise we may suppose that for every $j < \lambda$, there is some $h(j) \geq j$ such that $\text{tp}(y/\mathbb{E}_j)(\mathbb{E}_{h(j)+1}) \neq 0$ and $\text{tp}(y/\mathbb{E}_j)(\mathbb{E}_{h(j)}) = \emptyset$. Build a sequence $(y_j)_{j < \lambda}$ with $y_j \in \text{tp}(y/\mathbb{E}_j)(\mathbb{E}_{h(j)+1})$, so in particular $y_j \notin \mathbb{E}_{h(j)} \supseteq \mathbb{E}_j$.

Notice that $\mathcal{O}(y - y_j) \subseteq \text{Br}(y/\mathbb{E}_{h(j)})(\mathbb{E}_\lambda)$, in particular $y_j + \mathcal{O}(y - y_j) \cap \mathbb{E}_{h(j)} = \emptyset$. So for every β , given γ with $h(\gamma) > \beta$,

$$y_\gamma + (y - y_\gamma)\mathcal{O} \cap \mathbb{E}_\beta \subseteq y_\gamma + (y - y_\gamma)\mathcal{O} \cap \mathbb{E}_{h(\gamma)} = \emptyset.$$

It follows that the sequence of valuation balls $(y_\gamma + (y - y_\gamma)\mathcal{O}) \cap \mathbb{E}_\lambda$ has empty intersection in \mathbb{E}_λ and y is weakly immediate over \mathbb{E}_λ . \square

Lemma 3.22. *Let $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and $\mathbb{E}_0, \mathbb{E}_1$ be κ -bounded wim-constructible extensions of \mathbb{E} . Then there is a wim-constructible extension $\mathbb{E}_2 \supseteq \mathbb{E}_1$ and an*

elementary embedding $j : \mathbb{E}_0 \rightarrow \mathbb{E}_2$ over \mathbb{E} such that \mathbb{E}_2 is wim constructible over $j\mathbb{E}_0$.

Proof. Let \mathbb{E}_2 be a maximal wim-constructible extension within some $|\mathbb{E}_0|^+$ saturated extension of \mathbb{E}_1 . Let $(y_i : i < \beta)$ be a construction of \mathbb{E}_1 over \mathbb{E} and $(y_i : i < \alpha + \beta)$ an extension of it to a construction of \mathbb{E}_2 over \mathbb{E} .

Take a wim-construction $(x_i : i < \lambda)$ for \mathbb{E}_0 . We inductively build a sequence $(z_j : j < \lambda)$ in \mathbb{E}_2 and a decreasing sequence $(S_j)_{j \leq \lambda}$ of subsets of $\alpha + \beta$ such that for all j

- (1) $(z_i : i < j)$ is a κ -bounded wim construction over \mathbb{E} inside \mathbb{E}_2 such that $\varphi(x_i) := z_i$ for $i < j$ defines an elementary map over \mathbb{E} and thus an embedding $\varphi : \mathbb{E}_0^j := \mathbb{E}\langle x_i : i < j \rangle \rightarrow \mathbb{E}_2$;
- (2) $(y_i : i \in S_j)$ is a κ -bounded wim-construction for \mathbb{E}_2 over $\varphi(\mathbb{E}_0^j)$ with $S_j \subseteq \alpha + \beta$;
- (3) for every $k \leq j$ and $l \in S_j$, $(\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle \subseteq (\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$;
- (4) for every $k \leq j$, and every $l \in S_j$, $\text{tp}(y_l/(\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle)$ is not realized in $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$.

If $j = 0$ then just set $S_0 = \alpha + \beta$.

Suppose $(z_i : i < j)$ and S_j have been built. Notice $\varphi_*\text{tp}(x_j/\mathbb{E}_0^j)$ is realized in \mathbb{E}_2 . By Lemma 3.20 there is $\iota(j) \in S_j$, minimal such that $(\varphi_j\mathbb{E}_0^j)\langle y_i : i \leq \iota(j) \rangle$ realizes $\varphi_*\text{tp}(x_j/\mathbb{E}_0^j)$ and by Lemma 3.20 for any realization of z therein, setting $z_j = z$ and $S_{j+1} = S_j \setminus \{\iota(j)\}$ we get an extension satisfying (1) and (2).

As for (3), if $l \leq \iota(j)$, then $S_{j+1}^{<l} = S_j^{<l}$ so trivially $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle \subseteq (\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle$. On the other hand if $l > \iota(j)$, then $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle = (\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle$ by the choice of z_j and by the exchange-property. So by inductive hypothesis if $k \leq j$, then

$$(\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle \subseteq (\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle \subseteq (\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle.$$

As for (4), if $l \leq \iota(j)$, by construction $\text{tp}(z_j/\varphi\mathbb{E}_0^j)$ is not realized in $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$ and $S_{j+1}^{<l} = S_j^{<l}$, moreover if $l \in S_{j+1}$, then in fact $l < \iota(j) \notin S_{j+1}$.

It follows that $\text{tp}(y_l/(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle)$ is not realized in $(\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle$: for if it was and y'_l was a realization then by the exchange property $z_j = f(y'_l)$ for some $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$ -definable function f , but then $f(y_l) \in (\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$ would realize $\text{tp}(z_j/(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle) \vdash \text{tp}(z_j/\varphi\mathbb{E}_0^j)$, and $l = \iota(j) \notin S_{j+1}$, contradiction.

Thus in the case $l \in S_{j+1}^{<\iota(j)}$, $\text{tp}(y_l/(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle)$ extends uniquely to $\text{tp}(y_l/(\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle)$. On the other hand if $l > \iota(j)$ this is trivial as then by construction $(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle = (\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle$.

By inductive hypothesis if $k \leq j$, then $\text{tp}(y_l/(\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle)$ extends uniquely to $\text{tp}(y_l/(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle)$ which in turn extends uniquely to $\text{tp}(y_l/(\varphi\mathbb{E}_0^{j+1})\langle y_i : i \in S_{j+1}^{<l} \rangle)$.

At limit steps instead we set $S_j = \bigcap_{k < j} S_{k+1}$ and take the union of the sequences $(z_i)_{i < k+1}$ for $k < j$. To see (3) show that for $l \in S_j$,

$$(*) \quad (\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle = \bigcup_{k < j} (\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle.$$

In fact, for every $k < j$, $z_k \in (\varphi\mathbb{E}_0^{k+1})\langle y_i : i \in S_k^{<l} \rangle$ and for every $i \in S_j$, $y_i \in S_k$ for every $k < j$. For the vice-versa, it suffices to show that for every $m \in (S_0 \setminus S_j)^{<l}$, $y_m \in (\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle$.

Fix m , and set for every $k < j$

$$h(k) := \min\{s \in S_0 : y_m \in (\varphi\mathbb{E}_0^s)\langle y_i : i \in S_k^{\leq s} \rangle\}.$$

It is enough to show that for all k , there is k' with $k < k' < j$ such that the order type of $(S_k \setminus S_j)^{\leq h(k')}$ is strictly smaller than the order type of $(S_{k'} \setminus S_j)^{\leq h(k)}$.

Also observe that $(S_k \setminus S_j)^{\leq h(k)}$ must have maximum $h(k)$, because $y_m \in (\varphi\mathbb{E}_0^k)\langle y_i : i \in F \rangle$ for some finite $F \subset (S_k \setminus S_j)^{\leq h(k)}$ and for each $s \in F$, $y_s \in (\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{\leq s} \rangle$, so

$$y_m \in (\varphi\mathbb{E}_0^k)\langle y_i : i \in (S_k \setminus S_j)^{\leq \max F} \rangle$$

and by minimality of $h(k)$ it must be $h(k) \leq \max F \leq h(k)$.

By the definition of S_j there is k' such that $h(k) \notin S_{k'}$, but then

$$y_m \in (\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{\leq h(k)} \rangle \subseteq (\varphi\mathbb{E}_0^{k'})\langle y_i : i \in S_{k'}^{\leq h(k)} \rangle$$

and $S_k^{\leq h(k)} = S_{k'}^{\leq h(k)}$ so $h(k') < h(k)$ and $(S_{k'} \setminus S_j)^{\leq h(k')}$ is not cofinal in $(S_k \setminus S_j)^{\leq h(k)}$ whence it has strictly smaller order type.

To see (4) suppose $\text{tp}(y_l/(\varphi\mathbb{E}_0^k)\langle y_i : i \in S_k^{<l} \rangle)$ was realized in $(\varphi_0\mathbb{E}^j)\langle y_i : i \in S_j^{<l} \rangle$, then by (*) it would be realized in some $\text{tp}(y_l/(\varphi_0\mathbb{E}^m)\langle y_i : i \in S_m^{<l} \rangle)$ for some $j > m > k$, contradicting the inductive hypothesis.

To see (2) notice that $\text{tp}(y_l/(\varphi\mathbb{E}_0^j)\langle y_i : i \in S_j^{<l} \rangle)$ is the unique extension of $\text{tp}(y_l/\mathbb{E}\langle y_i : i \in S_0^{<l} \rangle)$, which is κ -bounded weakly immediate so we can apply Lemma 3.19. \square

Remark 3.23. If one wants to disregard the κ -bounded condition, the proof can be somewhat simplified by only showing (1)-(2)-(3) hold for the construction and using Lemma 3.21 and (3) to get (2) in the limit step.

Theorem 3.24. *For every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and every regular cardinal λ , maximal λ -bounded wim-constructible extension of $(\mathbb{E}, \mathcal{O})$ exist, are unique-up-to-non-unique isomorphism, are universal (weakly terminal) for λ -bounded wim-constructible extension, and embed into every λ -spherically complete elementary extensions of $(\mathbb{E}, \mathcal{O})$.*

Proof. Maximal λ -bounded wim-constructible extensions exist by Zorn's lemma because every λ -bounded wim-constructible extension embeds into a λ -saturated extension of \mathbb{E} . Let $\mathbb{E}_1, \mathbb{E}_2$ be two λ -bounded wim constructible extensions of \mathbb{E} with \mathbb{E}_2 maximal, by Lemma 3.22 there is an embedding of $j : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ over \mathbb{E} such that \mathbb{E}_2 is λ -bounded wim-constructible over $j\mathbb{E}_1$. If \mathbb{E}_1 was maximal as well, then j would have to be surjective. Finally if $(\mathbb{E}_3, \mathcal{O}_3)$ is a λ -spherically complete extension of $(\mathbb{E}, \mathcal{O})$, then one can use the wim-construction of \mathbb{E}_2 and the λ -spherical completeness of $(\mathbb{E}_3, \mathcal{O}_3)$ to build an elementary embedding of \mathbb{E}_2 in \mathbb{E}_3 over \mathbb{E} . \square

Remark 3.25. A λ -bounded wim-constructible extension of $(\mathbb{E}, \mathcal{O})$ is maximal (i.e. has no proper wim-constructible extension) if and only if it is λ -spherically complete. So Theorem 3.24 can be restated as

Corollary 3.26. *Let $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, and λ be an uncountable cardinal. There is a unique-up-to-non-unique-isomorphism λ -spherically complete λ -bounded wim-constructible extension and it elementarily embeds in every λ -spherically complete extension of $(\mathbb{E}, \mathcal{O})$.*

Definition 3.27. If $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, call a maximal λ -bounded weakly immediate extension the λ -bounded λ -spherical completion of $(\mathbb{E}, \mathcal{O})$.

Maximal λ -bounded wim-constructible extensions are homogeneous on wim-constructibly embedded λ -bounded wim-constructions.

Corollary 3.28. *Let $(\mathbb{E}, \mathcal{O}) \prec (\mathbb{E}_\lambda, \mathcal{O}_\lambda)$ be the λ -bounded spherical completion of $(\mathbb{E}, \mathcal{O})$, and let $(x_i : i < \alpha), (y_i : i < \alpha)$ be λ -bounded wim-construction over $(\mathbb{E}, \mathcal{O})$ with the same type over $(\mathbb{E}, \mathcal{O})$, such that $(\mathbb{E}_\lambda, \mathcal{O}_\lambda)$ is wim-constructible over $\mathbb{E}\langle x_i : i < \alpha \rangle$ and $\mathbb{E}\langle y_i : i < \alpha \rangle$. Then there is an automorphism φ of $(\mathbb{E}_\lambda, \mathcal{O}_\lambda)$ over $(\mathbb{E}, \mathcal{O})$ such that $\varphi(x_i) = y_i$.*

Proof. Let ψ be the isomorphism over \mathbb{E} between $\mathbb{E}\langle x_i : i < \alpha \rangle$ and $\mathbb{E}\langle y_i : i < \alpha \rangle$, given by $\psi(x_i) = y_i$. By Theorem 3.24, ψ extends to an automorphism φ of $(\mathbb{E}_\lambda, \mathcal{O}_\lambda)$. \square

3.3. Some Remarks and Questions. It is natural to ask whether stronger results than Lemma 3.22 hold for specific theories T .

Question 3.29. *Let $\mathbb{E} \prec \mathbb{E}_1$ be a wim-constructible extension.*

- (1) *Are all \mathbb{E}_0 with $\mathbb{E} \prec \mathbb{E}_0 \prec \mathbb{E}_1$ wim-constructible over \mathbb{E} ?*
- (1w) *If $\mathbb{E} \prec \mathbb{E}_0 \prec \mathbb{E}_1$ and \mathbb{E}_1 is wim constructible over \mathbb{E}_0 , is \mathbb{E}_0 necessarily wim-constructible over \mathbb{E} ?*
- (2) *Is \mathbb{E}_1 necessarily wim-constructible over all \mathbb{E}_0 with $\mathbb{E} \prec \mathbb{E}_0 \prec \mathbb{E}_1$?*
- (2w) *If $\mathbb{E} \prec \mathbb{E}_0 \prec \mathbb{E}_1$ and \mathbb{E}_0 is wim constructible over \mathbb{E} , is \mathbb{E}_1 necessarily wim-constructible over \mathbb{E}_0 ?*

Remark 3.30. Notice that a positive answer to (i) implies a positive answer to (iw).

Remark 3.31. Question (1) can be equivalently restated as:

(1e) are wim-constructible extension λ -wim for al λ ?

Remark 3.32. The answer to both (1) and (2) is affirmative in the case T is power-bounded (see Remark 4.15). We will give a very partial result toward (1) for some exponential theories in Section 5.

Remark 3.33. Let $(\mathbb{E}_\lambda, \mathcal{O}_\lambda) \succeq (\mathbb{E}, \mathcal{O})$ be the λ -spherical completion of $(\mathbb{E}, \mathcal{O})$. Suppose Question 3.29 (1) has affirmative answer, then every λ -spherically complete $(\mathbb{E}_1, \mathcal{O}_1)$ such that $(\mathbb{E}_\lambda, \mathcal{O}_\lambda) \succeq (\mathbb{E}_1, \mathcal{O}_1) \succeq (\mathbb{E}, \mathcal{O})$ is isomorphic to $(\mathbb{E}_\lambda, \mathcal{O}_\lambda)$ over $(\mathbb{E}, \mathcal{O})$. If furthermore Question 3.29 (2) has affirmative answer, then $\mathbb{E}_1 = \mathbb{E}_\lambda$.

In particular an affirmative answer to Question 3.29 (1), would imply that a λ -spherical completion for $(\mathbb{E}, \mathcal{O})$ could be obtained as a union $\bigcup_{i < \lambda} \mathbb{E}_i$ within a sufficiently saturated extension $(\mathbb{E}^*, \mathcal{O}^*) \succ (\mathbb{E}, \mathcal{O})$ of a sequence of \mathbb{E}_i such that $\mathbb{E}_i = \bigcup_{j < i} \mathbb{E}_{j+1}$ and \mathbb{E}_{i+1} is the definable closure (for T) of a λ -spherical completion of \mathbb{E}_i *qua real closed valued field* in $(\mathbb{E}^*, \mathcal{O}^*)$.

Remark 3.34. Adapting some results from [8] and [10] (see also [1]), it can be shown that for every T and every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ there is big enough λ , such that the λ -bounded λ -spherical completion $(\mathbb{E}_\lambda, \mathcal{O}_\lambda)$ of $(\mathbb{E}, \mathcal{O})$ is isomorphic as a real closed valued field (RCVF) to a λ -bounded Hahn field in the sense of [6] with coefficients from the residue field of $(\mathbb{E}, \mathcal{O})$. Moreover the isomorphism can be choosen so that the image of $\mathbb{E} \subseteq \mathbb{E}_\lambda$ is truncation-closed and - in the case T defines an exponential - also so that the logarithm of the monomial group is the set of purely infinite elements. This will be presented in another paper and won't be used here.

The goal of the subsequent sections is to start the study of maximal λ -bounded wim-constructible models of T_{convex} . For power bounded theories this is essentially known, because if T is power bounded, all weakly immediately generated elements over some $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ are in fact weakly immediate, hence if x is weakly \mathcal{O} -immediate over \mathbb{E} the $(\mathbb{E}\langle x \rangle, \mathcal{O}_x)$ is in fact *immediate* over $(\mathbb{E}, \mathcal{O})$ and a maximal λ -bounded extension of $(\mathbb{E}, \mathcal{O})$ is an expansion of the λ -spherical completion of the reduct of $(\mathbb{E}, \mathcal{O})$ to RCVF. This follows from results in [13], [12], and [11], and we report more precisely on this in Section 4.

For exponential o-minimal theories, the situation is not as well known and we commence its study in Section 5 by focusing on theories arising as expansions of power bounded theories with an exponential function.

4. THE POWER-BOUNDED CASE

In this section we review the known results about weakly immediate types in the case T is power bounded. All the results are virtually all included in [13], [12], and [11] but we give somewhat different and shorter proofs which will adapt better to the exponential case.

4.1. Introductory remarks. Let T be an o-minimal theory and $\mathbb{K} \models T$. Recall that a power of \mathbb{K} is a \mathbb{K} -definable endomorphism $\theta : \mathbb{K}^{>0} \rightarrow \mathbb{K}^{>0}$, of the group $(\mathbb{K}^{>0}, \cdot)$. By o-minimality such endomorphism must be monotone and differentiable on its whole domain. The *exponent* of θ is defined as the element $\theta'(1) \in \mathbb{K}$. It is easy to see that the set $\text{Exponents}(\mathbb{K})$ of exponents of \mathbb{K} forms a subfield of \mathbb{K} . The standard notation for powers is to write $t \mapsto \theta(t)$ as $t^\alpha := \theta(t)$ where $\alpha = \theta'(1) \in \text{Exponents}(\mathbb{K})$.

There is a fundamental dichotomy established by Miller (see [2, Ch. 16]) regarding the subfield $\text{Exponents}(\mathbb{K})$: it is either \mathbb{K} or is such that every \mathbb{K} definable $S \subseteq \text{Exponents}(\mathbb{K})$ is finite. More specifically

Theorem 4.1 (Miller). *The following are equivalent for an o-minimal expansion \mathbb{K} of a real closed field.*

- (1) \mathbb{K} does not define an exponential, that is, an isomorphism $\exp : (\mathbb{K}, +) \rightarrow (\mathbb{K}^{>0}, \cdot)$;
- (2) $\text{Exponents}(\mathbb{K}) \neq \mathbb{K}$;
- (3) $\text{Exponents}(\mathbb{K})$ does not contain any interval of \mathbb{K} ;
- (4) for every $\mathbb{K}_1 \succeq \mathbb{K}$, $\text{Exponents}(\mathbb{K}) = \text{Exponents}(\mathbb{K}_1)$;
- (5) every \mathbb{K} -definable function $f : \mathbb{K} \rightarrow \mathbb{K}$ is eventually bounded by some power, that is there is $\beta \in \text{Exponents}(\mathbb{K})$, such that for big enough $t \in \mathbb{K}$, $f(t) < t^\beta$.
- (6) for every \mathbb{K} -definable function $f : \mathbb{K} \rightarrow \mathbb{K}$ there is an exponent $\beta \in \text{Exponents}(\mathbb{K})$, such that $\lim_{s \rightarrow \infty} f(s)/s^\beta \in \mathbb{K}$.

Definition 4.2 (Miller). A theory T is said to be *power bounded* if one (or equivalently all) of its models are power bounded. For a power bounded theory T , $\text{Exponents}(T)$ is defined to be the field of exponents in its prime model (this is then by (4) above the field of exponents of every $\mathbb{K} \models T$).

The main thing we'll need besides Miller's Theorem above is the following Lemma, also due to Miller.

Lemma 4.3 (Miller). *If T is power bounded, $\mathbb{K} \models T$ and $f : \mathbb{K} \rightarrow \mathbb{K}$ is \mathbb{K} -definable and γ is an exponent of T then the following are equivalent*

- (1) $\lim_{t \rightarrow \infty} t f^\dagger(t) = \gamma$;
- (2) there is $k \in \mathbb{K}^{\neq 0}$ such that $\lim_{t \rightarrow \infty} \frac{f(t)}{k t^\gamma} = 1$.

Proof. Recall that for every \mathbb{K} -definable f there is an exponent γ such that (2) holds.

If $\lim_{t \rightarrow \infty} f(t) = \pm\infty$, by power-boundedness, there is a power t^γ such that $\lim_{t \rightarrow \infty} f(t)t^{-\gamma} \in \mathbb{K}^{\neq 0}$ and by L'Hopital's rule

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t^\gamma} = \lim_{t \rightarrow \infty} \frac{f'(t)}{\gamma t^{\gamma-1}}$$

So the limit in (1) is γ . If $\lim_{t \rightarrow \infty} f(t) = 0$ the argument is the same. If $\lim_{t \rightarrow \infty} f(t) = c \in \mathbb{K}^{\neq 0}$, then $f(t) = c + g(t)$ where $\lim_{t \rightarrow \infty} g(t) = 0$. Then

it suffices to observe that

$$tf^\dagger(t) = tg^\dagger(t)g(t)/f(t),$$

so the limit in (1) is in fact 0. \square

4.2. The rv-property. As observed in [12] the hypothesis of being power-bounded has strong consequences on T_{convex} , in particular with regard to the structure induced on the value-group sort (cf [12, Thm. 4.4]). The starting point of this analysis is [12, Prop. 4.2] which is here convenient to condensate with some of its consequences in the following

Lemma 4.4. *Let T be an o-minimal theory, then the following are equivalent*

(1) *T is power bounded;*

(2) *for every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and every unary function $f : \mathbb{E} \rightarrow \mathbb{E}^{>0}$*

$$(\mathbb{E}, \mathcal{O}) \models \exists a \in \mathbb{E}, \exists M \in \mathcal{O}, \forall t \in \mathcal{O}, t > M \rightarrow f(t)/a \in \mathcal{O} \setminus \mathfrak{o}$$

(3) *for every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and every unary function $f : \mathbb{E} \rightarrow \mathbb{E}^{>0}$*

$$(\mathbb{E}, \mathcal{O}) \models \exists M \in \mathcal{O}, \forall t \in \mathcal{O}, t > M \rightarrow f^\dagger(t) \in \mathcal{O}$$

(4) *for every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and every unary function $f : \mathbb{E} \rightarrow \mathbb{E}^{>0}$ there is a natural number k such that*

$$(\mathbb{E}, \mathcal{O}) \models \exists t_0, \dots, t_{k-1}, \forall t, \bigwedge_{i \in k} |t - t_i| > \mathfrak{o} \rightarrow f^\dagger(t) \in \mathcal{O}.$$

(5) *for every $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and every unary function $f : \mathbb{E}^{>0} \rightarrow \mathbb{E}^{>0}$ there is a natural number k such that*

$$(\mathbb{E}, \mathcal{O}) \models \exists t_0, \dots, t_{k-1}, \forall t, \bigwedge_{i \in k} t \not\sim t_i \rightarrow tf^\dagger(t) \in \mathcal{O}.$$

Proof. (1) \Rightarrow (2), this is essentially [12, Proposition 4.2], we repeat the argument for the convenience of the reader. Since T_{convex} is complete it suffices to prove it holds for every \mathbb{E} -definable function $f : \mathbb{E} \rightarrow \mathbb{E}$ in some model $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$. Take $\mathbb{K} \models T$ of cofinality greater than $|T|$, let $d > \mathbb{K}$ and set $(\mathbb{K}\langle d \rangle, \mathcal{O}) \models T_{\text{convex}}$. Since

$$\mathcal{O} \ni t \mapsto \mathbf{v}_{\mathcal{O}} f(t) \in \mathbf{v}_{\mathcal{O}} \mathbb{E} = \mathbf{v}_{\mathcal{O}} d^{\text{Exponents}(T)},$$

is eventually a weakly monotone function by o-minimality of \mathbb{E} , and the cofinality \mathcal{O} is greater than the cardinality of $\mathbf{v}_{\mathcal{O}}(\mathbb{E})$, such function must be eventually constant.

(2) \Rightarrow (3), suppose (3) fails, then for some $M \in \mathcal{O}$, for every $t > M$, $|f^\dagger(t)| > \mathcal{O}$. We can also assume without loss of generality that for every $t_1, t_2 \in \mathcal{O}$, with $t_2 > t_1 > M$, $f(t_2) > f(t_1) > 0$, so in particular for $t > M$, $f^\dagger(t) > \mathcal{O}$. Now take $M < t_1 < t_2 + \mathfrak{o}$, and take t_0 such that $t_1 < t_0 + \mathfrak{o} < t_2$, then for some $\xi \in (t_0, t_2)$

$$\frac{f(t_2)}{f(t_1)} = \frac{f(t_0) + f(t_2) - f(t_0)}{f(t_1)} = \frac{f(t_0)}{f(t_1)} + f^\dagger(\xi)(t_2 - t_0) \frac{f(\xi)}{f(t_1)} > f^\dagger(\xi)$$

by the mean value theorem and because f is increasing and positive. Since by hypothesis $f^\dagger(\xi) > \mathcal{O}$, it follows that $f(t_2)/f(t_1) > \mathcal{O}$. Since t_1 and t_2 were arbitrary with $t_2 > t_1 + \mathfrak{o}$, this contradicts (2).

(3) \Rightarrow (4), assume not, then by weak \mathfrak{o} -minimality of $(\mathbb{E}, \mathcal{O})$ there would be $t_0, t_1 \in \mathbb{E}$, $t_1 > t_0 + \mathfrak{o}$ such that for every $t \in (t_0, t_1)$, $f^\dagger(t) > \mathcal{O}$. Consider $t_2 := (t_0 + t_1)/2$ and define $g(t) = f(t_2 - 1/t)$, then for $t > 2/(t_1 - t_0)$ and $t \in \mathcal{O} \setminus \{0\}$,

$$g^\dagger(t) = t^{-2} f^\dagger(t_2 - 1/t) > \mathcal{O}$$

contradicting (3).

(4) \Rightarrow (5), assume (5) fails, then again by weak \mathfrak{o} -minimality, there are some $t_1 > t_0 > 0$, $t_1 \not\asymp t_0$ such that for every $t \in (t_0, t_1)$, $|tf^\dagger(t)| > \mathcal{O}$. Without loss of generality, we can assume $t_1 \asymp t_0$, so considering $g(t) := f(t_0 t)$ we would have that for $t \in (1, t_1/t_0)$, $g^\dagger(t) = t_0 f^\dagger(t_0 t) \asymp t_0 t f^\dagger(t_0 t) \succ \mathcal{O}$ and this contradicts (4) as $t_1/t_0 > 1 + \mathfrak{o}$.

(5) \Rightarrow (3) is clear.

(3) \Rightarrow (1) follows from Miller's dichotomy, because if T is not power bounded it defines an exponential $\exp : \mathbb{E} \rightarrow \mathbb{E}$, satisfying $\exp^\dagger(t) = t$, but then for $d > \mathcal{O}$, the definability of $f(t) := \exp(dt)$ contradicts (3). \square

Corollary 4.5. *If T is power bounded and $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, and x is \mathcal{O} -cofinally residual over \mathbb{E} then $\mathbf{v}_{\mathcal{O}}(\mathbb{E}\langle x \rangle) = \mathbf{v}_{\mathcal{O}}(\mathbb{E})$.*

Proof. By hypothesis there is $z \in \mathbb{E}\langle x \rangle$ such that $\mathbb{K}\langle z \rangle = \mathbb{K}_x$ and $\mathbb{K}\langle z \rangle$ is a cofinal extension of \mathbb{K} . By (5) in Lemma 4.4, there are $t_1, t_2 \in \mathbb{E}$ such that $\mathfrak{o} < t_1 < z < t_2 + \mathfrak{o} \subseteq \mathcal{O}$, f is increasing on (t_1, t_2) and for every $t \in (t_1, t_2)$, $tf^\dagger(t) \in \mathcal{O}$. Now invoke Lemma 2.19 to conclude that $f(t) \asymp f((t_2 + t_1)/2)$. \square

Lemma 4.6. *Let f be \mathbb{E} -definable, differentiable and never 0. If $f^\dagger(t)t \sim \beta$ for all t in some interval J , then on that interval $f(t) \sim ct^\beta$ for some c .*

Proof. Let $h(t) := f(t)/t^\beta$ and assume $r < s \in J$, then there is $r < \eta < s$ s.t.

$$\frac{h(s) - h(r)}{h(\eta)} = h^\dagger(\eta)\eta \cdot \frac{s - r}{\eta} = \frac{s - r}{\eta} \left(\eta f^\dagger(\eta) - \beta \right)$$

So if $r \asymp s$, then $h(r) \sim h(s)$ and by continuity of h it has to be $h(r) \sim h(s)$ for every $r, s \in J$ (otherwise h^{-1} could be used to define a bijection between an initial segment of $|\mathcal{O} \setminus \mathfrak{o}|$ and an initial segment of $|\mathbb{E} \setminus \mathcal{O}|$, contradicting Proposition 3.2). \square

Lemma 4.7. *Let T be power-bounded, $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and $b \in \mathbb{U} \setminus \mathbb{E}$ be such that $\mathcal{O} < b < |\mathbb{E} \setminus \mathcal{O}|$. For every definable $f : \mathbb{E} \rightarrow \mathbb{E}$ not constant around b , there is a unique $\beta_f \in \text{Exponents}(T) \setminus \{0\}$ and some $c_0, c_1 \in \mathbb{E}$ such that $f(b) - c_0 \sim_{\mathcal{O}} c_1 b^{\beta_f}$. Moreover if $\{f_s : s \in D\}$ is a \mathbb{E} -definable family of unary functions, $\{\beta_{f_s} : s \in D\}$ is finite.*

Proof. Since f is not constant around b , $f(b) \notin \mathbb{E}$ and by Theorem 3.6, $f(b)$ is neither cofinally residual nor weakly immediate, hence for some c_0 , $\mathbf{v}_{\mathcal{O}}(f(b) - c_0) \notin \mathbf{v}_{\mathcal{O}}(\mathbb{E})$. Since the theory is power bounded, by (2) of Lemma 4.4, there is $a \in \mathbb{E}$ and $\sigma \in \{\pm 1\}$ such that $a(f(b) - c_0)^\sigma \in |\mathcal{O}_b \setminus \mathcal{O}|$. Now $g(t) := a(f(t) - c_0)^\sigma$ is such that $|\mathbb{E} \setminus \mathcal{O}| > g(b) > \mathcal{O}$, so by Lemma 2.17 and by power boundedness, for some positive exponent $|\beta|$, $g(b)/b^{|\beta|} \in \mathcal{O} \setminus \mathfrak{o}$. So $g(b)/b^{|\beta|} \in k + \mathfrak{o}$ for some $k \in \mathcal{O} \setminus \mathfrak{o}$ and setting $c_1 = k/a$ and $\text{sgn}(\beta) = \sigma$, yields $f(b) - c_0 \sim c_1 b^\beta$.

Now we show that if $f(b) - c_0' \sim c_1' b^{\beta'}$ then $c_0 \sim c_0'$, $c_1 \sim c_1'$ and $\beta = \beta'$. In fact clearly it must be $\mathbf{v}_{\mathcal{O}}(c_0 - c_0') > \mathbf{v}_{\mathcal{O}}(c_1 b^\beta)$ and $c_1 b^\beta \sim c_1' b^{\beta'}$, but then $b^{\beta' - \beta} \sim c_1/c_1'$ so $\mathbf{rv}_{\mathcal{O}}(b^{\beta' - \beta}) \in \mathbf{rv}_{\mathcal{O}}\mathbb{E}$ and this is possible only if $\beta = \beta'$.

For the last bit it suffices to show that $\{\beta_{f_s} : s \in D\}$ is definable in \mathbb{E} . For this observe that by Lemma 2.18, $b f_s^{\dagger}(b) \sim \beta_s - 1$ if $\beta_s \neq 1$ and $b f_s^{\dagger}(b) \prec 1$ if and only if $\beta_s = 1$.

Now $\{b f_s^{\dagger}(b) : s \in D\}$ is definable in $\mathbb{E}\langle b \rangle$ thus if $\pi : \mathcal{O} \rightarrow \mathbb{K}$ is a standard part map $\{\beta_{f_s} : s \in D\} = \{1 + \pi b f_s^{\dagger}(b) : s \in D\}$ is definable, in fact over \mathbb{K} . \square

Lemma 4.8. *Let T be power-bounded, $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$. For every \mathbb{E} -definable $f : \mathbb{E} \rightarrow \mathbb{E}$, and for every a there is a finite set $F_a \subseteq \mathbb{E}$ such that*

$$\forall t \in \mathbb{E} \setminus F_a + a\mathcal{O}, \exists c, c - t \notin a\mathcal{O}, (t - c)f^{\dagger}(t) \in \mathcal{O}$$

or equivalently

$$\forall t \in \mathbb{E} \setminus F_a + a\mathcal{O}, f^{\dagger}(t) \prec 1/a$$

Proof. Observe that it suffices to prove the statement for $a = 1$, as the statement for general a follows applying the statement with $a = 1$ to the function $f_a(t) := f(at)$.

For every x , by (5) of Lemma 4.4 applied to $t \mapsto f'(x + t)$ there are $x_- < x + \mathcal{O} < x_+$ such that $(t - x)f^{\dagger}(t) \in \mathcal{O}$ for all $t \in (x_-, x_+) \setminus x + \mathcal{O}$. In particular $f^{\dagger}(t) \prec 1$ for all $t \in (x_-, x_+) \setminus x + \mathcal{O}$.

Consider the $(\mathbb{E}, \mathcal{O})$ -definable set $D = \{t \in \mathbb{E} : f^{\dagger}(t) \notin \mathcal{O}\}$, by weak \mathfrak{o} -minimality of $(\mathbb{E}, \mathcal{O})$ this set must be a finite union of order-convex sets, but by the observation above, any order convex subset $C \subseteq D$, must be contained in a single coset of \mathcal{O} hence the thesis. \square

Corollary 4.9. *Let T be power bounded $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ and $f : \mathbb{E} \rightarrow \mathbb{E}$, \mathbb{E} -definable. There is a natural number n such that whenever M is a \mathcal{O} -submodule of \mathbb{E} , there is a finite $F_M \subseteq \mathbb{E}$ with $|F_M| < n$ such that for every $x \in \mathbb{E}$, if $x \notin F_M + M$, then $f^{\dagger}(x)M \subseteq \mathfrak{o}$.*

Proof. Let $(\mathbb{E}_1, \mathcal{O}_1)$ be an extension of \mathbb{E} such that $M = z\mathcal{O}_1 \cap \mathbb{E}$ with $z \in \mathbb{E}_1$. There is a finite $F \subseteq \mathbb{E}_1$ such that whenever $x \notin F + z\mathcal{O}_1$, $z f^{\dagger}(x)\mathcal{O}_1 \subseteq \mathfrak{o}_1$.

Let $F_M \subseteq \mathbb{E}$ be a finite subset such that $F_M + M = (F + z\mathcal{O}_1) \cap \mathbb{E}$. Such set can be obtained picking representatives of $(t + z\mathcal{O}_1) \cap \mathbb{E}$ for every $t \in F$ such that $(t + z\mathcal{O}_1) \cap \mathbb{E} \neq \emptyset$.

Now if $x \in \mathbb{E}$ and $x \notin F_M + M$, then $x \notin F + z\mathcal{O}_1$ so $Mf^\dagger(x) \subseteq z\mathcal{O}_1 f^\dagger(x) \subseteq \mathfrak{o}$. \square

We are now ready to give a short proof of the residue-valuation property. This was proven first by van den Dries and Spaissegger in [16, Proposition 9.2] for polynomially bounded structures over \mathbb{R} and later in full for power-bounded structures in [11].

Proposition 4.10 (Tyne). *Let T be power bounded, $x \in \mathbb{U} \succ \mathbb{E} \models T$ and \mathcal{O} a non-trivial T -convex valuation subring of \mathbb{E} . If x is weakly \mathcal{O} -immediate then every $y \in \mathbb{E}\langle x \rangle$ is weakly \mathcal{O} -immediate.*

Proof. Assume $(x_i)_{i < \lambda}$ is a \mathcal{O} -p.c. sequence in \mathbb{E} for x and let $f : \mathbb{E} \rightarrow \mathbb{E}$ be an increasing \mathbb{E} -definable function. Without loss of generality we can assume that $\text{Br}(x/\mathbb{E})(\mathbb{E}) \neq 0$, for otherwise x is dense and $f(x)$ is still dense by continuity of f at generic points.

Let $p := \text{tp}(x/\mathbb{E})$, $M := \text{Br}(p)(\mathbb{E})$ and $M_x := \text{Br}(p)(\mathbb{E}\langle x \rangle)$. By Corollary 4.9 there is a finite $F_M \subseteq \mathbb{E}$ such that for every $t \notin F_M + M$, $f^\dagger(t)M \subseteq \mathfrak{o}$.

Since $x \notin F_M + M$, there are $a_0, a_1 \in \mathbb{E}$ such that $a_0 < x < a_1$, $(a_0, a_1) \cap F_M + M = \emptyset$ and $f^\dagger|_{[a_0, a_1]}$ is continuous monotone or constant. It follows that $|f^\dagger(x)| \leq \max\{|f^\dagger(t)| : t \in [a_0, a_1]\} =: d$.

Thus $M = \text{Br}(p)(\mathbb{E}) \subseteq \mathfrak{o}/d$, that is $\text{Br}(p)(\mathbb{E})(t) \vdash d|t| < r$ for all $r \in \mathbb{E}^{>\mathfrak{o}}$. Since by Theorem 3.6, $\mathbb{E}\langle x \rangle$ does not contain any y with $\mathfrak{o} < y < \mathbb{E}^{>\mathfrak{o}}$, $M_x = \text{Br}(p)(\mathbb{E}\langle x \rangle) \subseteq \mathfrak{o}/d$ as well and $f^\dagger(x)M_x \subseteq \mathfrak{o}$.

It follows that there is $\tilde{c} \in \mathbb{E}\langle x \rangle$ such that $|x - \tilde{c}| > M_x$ and $(x - \tilde{c})f^\dagger(x) \in \mathcal{O}$: since $f^\dagger(x)M_x \subseteq \mathfrak{o}$, $1/(f^\dagger(x)) \notin M_x$, and $\tilde{c} := x - (f^\dagger(x))$ does the job. On the other hand since $p(\mathbb{E}\langle x \rangle) = x + M_x$, and $\tilde{c} \notin x + M_x$, there is $c \in \mathbb{E}$ between x and \tilde{c} . For such c , $|x - c| < |x - \tilde{c}|$, so $|x - c|f^\dagger(x) \in \mathcal{O}$ and $f(x)$ is weakly immediate by Lemma 2.20. \square

As a Corollary we get the following which to the knowledge of the author was first stated in [3]

Corollary 4.11. *If T is a power bounded theory, and $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$, then there is $(\mathbb{E}_1, \mathcal{O}_1) \succeq (\mathbb{E}, \mathcal{O})$ such that $\mathbf{rv}_{\mathcal{O}_1}(\mathbb{E}_1) = \mathbf{rv}_{\mathcal{O}}(\mathbb{E})$ and $(\mathbb{E}_1, \mathcal{O}_1)$ is spherically complete.*

Proof. Use Theorem 3.24 with a cardinal λ greater than the cofinality of any subset of $\mathbf{v}_{\mathcal{O}}(\mathbb{E})$ to get a λ -spherically complete weakly immediate $(\mathbb{E}_1, \mathcal{O}_1) \succeq (\mathbb{E}, \mathcal{O})$ and observe that by Proposition 4.10 $\mathbf{rv}_{\mathcal{O}_1}(\mathbb{E}_1) = \mathbf{rv}_{\mathcal{O}}(\mathbb{E})$. \square

In conjunction with the result [12, Thm. 4.4], that for power-bounded T , the induced theory of $\mathbf{v}_{\mathcal{O}}\mathbb{E}$ is the (\mathfrak{o} -minimal) theory of an ordered vector space over $\text{Exponents}(T)$, Proposition 4.10 implies

Corollary 4.12. *If T is power bounded, $x \in \mathbb{U} \succ \mathbb{E} \models T$, and \mathcal{O} is a non-trivial T -convex valuation subring of \mathbb{E} , then if for some $y \in \mathbb{E}\langle x \rangle$, $\mathbf{v}_{\mathcal{O}}(y) \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}$, then there are $c_1, c_2 \in \mathbb{E}$ and $\beta \in \text{Exponents}(T)$, such that $\mathbf{v}_{\mathcal{O}}(y) = \mathbf{v}_{\mathcal{O}}c_2 + \beta\mathbf{v}_{\mathcal{O}}(c_1 - x)$.*

Proof. If x is \mathcal{O} -special then the thesis follows from by Lemma 4.7. So we may assume x is not \mathcal{O} -special.

If for some $y \in \mathbb{E}\langle x \rangle$, $\mathbf{v}_{\mathcal{O}}(y) \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}$, then y is not weakly-immediate over $(\mathbb{E}, \mathcal{O})$, so by Proposition 4.10 it is also not weakly immediately *generated* and x is not weakly immediate. It is also not cofinally residual by Corollary 4.5. It follows that there is c_1 such that $\mathbf{v}_{\mathcal{O}}(x - c_1) \notin \mathbf{v}_{\mathcal{O}}(\mathbb{E})$. Now $y = f(x)$ for some \mathbb{E} -definable f .

Set $g(t) := f(c_1 + t)$, since $\mathbf{v}_{\mathcal{O}}(x - c_1) \notin \mathbf{v}_{\mathcal{O}}(\mathbb{E})$, by (5) in Lemma 4.4, $tg^{\dagger}(t) \in \mathcal{O}$, for all t in some \mathbb{E} -definable neighborhood of $x - c_1$, in particular g by Lemma 2.19, induces a function $G : J_0 \rightarrow J_1$, with J_0, J_1 $(\mathbb{E}, \mathcal{O})$ -definable intervals in $\mathbf{v}_{\mathcal{O}}(\mathbb{E}\langle x \rangle)$ containing $\mathbf{v}_{\mathcal{O}}(x - c_1)$, such that $G(\mathbf{v}_{\mathcal{O}}(t)) = \mathbf{v}_{\mathcal{O}}(g(t))$. Moreover by Lemma 4.6 G is affine. \square

As showed in [17] in the case of polinomially bounded theories over \mathbb{R} , this implies a preparation theorem. The argument here is the same as [17, Lemma 2.2], with a little extra care needed due to the fact that the theory may not have an Archimedean prime model.

Proposition 4.13. *Let T be a power bounded o -minimal theory, $T \models \mathbb{K}$ and $f : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ be a definable function, then there is a covering of \mathbb{K}^{n+1} into finitely many definable cells $\{C_i : i \in M\}$ such that for every i , there are $\theta_i : \mathbb{K}^n \rightarrow \mathbb{K}$, $a_i : \mathbb{K}^n \rightarrow \mathbb{K}$, $\lambda_i \in \text{Exponents}(T)$ and 0-definable constants, $0 < u_i^- < u_i^+$ such that for every C_i either f is identically 0 on C_i or*

$$\mathbb{K} \models \forall (\bar{c}, t) \in C_i, \theta_i(\bar{c}) \neq t \ \& \ u_i^- < \left| \frac{f(\bar{c}, t)}{a_i(\bar{c})|t - \theta_i(\bar{c})|^{\lambda_i}} \right| < u_i^+$$

Proof. By compactness it suffices to show that for every $\mathbb{E} \models T$, and every 1-ary \mathbb{E} -definable function, there is a partition of \mathbb{E} into finitely many \mathbb{E} -definable sets C_i such that for every i either $f(C_i) = \{0\}$ or there are $a_i, \theta_i \in \mathbb{E}$ with $\theta_i \notin C_i$, $\lambda_i \in \text{Exponents}(T)$ and 0-definable $0 < u_i^- < u_i^+$ such that

$$\mathbb{E} \models \forall t \in C_i, u_i^- < \left| \frac{f(t)}{a_i(t - \theta_i)^{\lambda_i}} \right| < u_i^+$$

this in turn is equivalent to show that for every $\mathbb{U} \succ \mathbb{E}$ and $x \in \mathbb{U} \setminus \mathbb{E}$, such that $f(x) \neq 0$, there are $a, \theta \in \mathbb{E}$, $\lambda \in \text{Exponents}(T)$, and u^-, u^+ 0-definable such that $u^- < |f(x)/(a(x - \theta)^\lambda)| < u^+$. Let \mathcal{O} be the minimal T -convex subring of $\mathbb{E}\langle x \rangle$ (that is \mathcal{O} is the convex hull of the prime model of T). Let $\mathbb{K} \prec \mathbb{E}$ be maximal such that $\mathbb{K} \subseteq \mathcal{O}$. If $\mathbf{v}_{\mathcal{O}}f(x) \in \mathbf{v}_{\mathcal{O}}(\mathbb{E}\langle x \rangle)$, then $f(x)/a \asymp_{\mathcal{O}} 1$. On the other hand if $\mathbf{v}_{\mathcal{O}}f(x) \notin \mathbf{v}_{\mathcal{O}}\mathbb{E}\langle x \rangle$, then by Corollary 4.12, there are $a, \theta \in \mathbb{E}$ and λ such that $\mathbf{v}_{\mathcal{O}}a + \lambda \mathbf{v}_{\mathcal{O}}(x - \theta) = \mathbf{v}_{\mathcal{O}}(f(x))$. \square

4.3. Immediate extensions. An easy consequence of Proposition 4.10 and Lemma 2.4 is that an extension of models of T_{convex} is wim-constructible if and only if it is immediate.

Corollary 4.14. *If T is power bounded then $(\mathbb{E}, \mathcal{O}) \preceq (\mathbb{E}_1, \mathcal{O}_1) \models T_{\text{convex}}$ is wim-constructible if and only if $\mathbf{v}(\mathbb{E}) = \mathbf{v}(\mathbb{E}_1)$ and $\mathbf{r}(\mathbb{E}) = \mathbf{r}(\mathbb{E}_1)$.*

Proof. If $\mathbf{v}(\mathbb{E}, \mathcal{O}) = \mathbf{v}(\mathbb{E}_1, \mathcal{O}_1)$ and $\mathbf{r}(\mathbb{E}, \mathcal{O}) = \mathbf{r}(\mathbb{E}_1, \mathcal{O}_1)$, then by Lemma 2.4 all elements of $(\mathbb{E}_1, \mathcal{O}_1)$ must be weakly immediate over $(\mathbb{E}, \mathcal{O})$. On the other hand since immediate extensions are closed under left factors, $(\mathbb{E}\langle x \rangle, \mathcal{O}_1 \cap \mathbb{E}\langle x \rangle)$ is again immediate over $(\mathbb{E}, \mathcal{O})$. A transfinite inductive argument thus shows that $(\mathbb{E}_1, \mathcal{O}_1)$ must be constructible.

Viceversa Proposition 4.10 implies that if x is weakly immediate over $(\mathbb{E}, \mathcal{O})$ then $(\mathbb{E}\langle x \rangle, \mathcal{O}_x)$ is immediate over $(\mathbb{E}, \mathcal{O})$ and again a transfinite induction shows that constructible extensions are immediate. \square

Remark 4.15. This implies that both Question 3.29, (1) and (2) have positive answer because immediate extensions are closed both under left and right factors. It also follows that for any $(\mathbb{E}, \mathcal{O})$, for all big enough λ the λ -spherical completions of $(\mathbb{E}, \mathcal{O})$ are spherically complete contrary to what happens when T is exponential.

5. THE SIMPLY EXPONENTIAL CASE

5.1. Introductory remarks. This subsection is dedicated to some basic observations on exponential o-minimal structures.

Definition 5.1. An o-minimal theory T expanding RCF is *exponential* if for some model $\mathbb{K} \models T$, there is a \mathbb{K} -definable $\exp : \mathbb{K} \rightarrow \mathbb{K}^{>0}$ such that $\exp(\mathbb{K}) \neq \{1\}$ and $\exp(x+y) = \exp(x)\exp(y)$ for every $x, y \in \mathbb{K}$.

Remark 5.2. Observe that if $\exp : \mathbb{K} \rightarrow \mathbb{K}^{>0}$ is such that $\exp(x+y) = \exp(x)\exp(y)$ for every $x, y \in \mathbb{K}$, then $t \mapsto \exp(-t)$ has the same property. Hence if T is exponential we may as well assume that it defines an exponential \exp such that $\exp(1) > 1$. On the other hand $\{t \in \mathbb{K}^{\geq 0} : \exp(t) \geq 1\}$ is a submonoid of $(\mathbb{K}, +)$, hence $\{t \in \mathbb{K} : t(\exp(t) - 1) \geq 0\}$ is a subgroup of $(\mathbb{K}, +)$, thus if $\exp(1) > 1$, by o-minimality $\{t \in \mathbb{K} : t(\exp(t) - 1) \geq 0\} = \mathbb{K}$ and \exp is increasing.

Remark 5.3. Given two exponential functions \exp and \exp^* defined in a model $\mathbb{K} \models T$, there is a constant $k \in \mathbb{K}$ such that $\exp^*(t) = \exp(kt)$: indeed $\log \circ \exp^* : \mathbb{K} \rightarrow \mathbb{K}$ would be a definable non-zero endomorphism of $(\mathbb{K}, +)$ hence, by o-minimality, an omotethy. In particular, together with the previous remark, we get that if T is exponential, then it defines a \emptyset -definable increasing exponential function \exp .

Remark 5.4. It follows from the defining property of an exponential, that every exponential \exp definable in any model of an o-minimal theory T is differentiable at a point t if and only if it is differentiable in 0, hence it must be everywhere differentiable, moreover it must satisfy $\exp^\dagger(t) = \exp'(0)$. By the same argument of Remark 5.3, for a model $\mathbb{K} \models T$, for every constant $a \neq 0$ there is a unique exponential \exp^a such that $(\exp^a)^\dagger(t) = a$, in particular there is a unique one such that $\exp^\dagger(t) = 1$: this specific \exp is 0-definable and for every $a \in \mathbb{K}$ it must be $\exp^a(t) = \exp(at)$. In the following we will refer to the unique \exp definable in T such that $\exp^\dagger(t) = t$ as *the exponential of T* .

Remark 5.5. If T is an exponential theory and $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$ then $\exp(\mathcal{O}) = \mathcal{O}^{>0}$ (by T -convexity and because $\exp : \mathbb{E} \rightarrow \mathbb{E}^{>0}$ is 0-definable continuous). It follows that for every $a \in \mathbb{E}$, $\exp(\mathcal{O} + a) = \mathbf{v}_{\mathcal{O}}^{-1}(\exp(a))$.

The first expansions of \mathbb{R} defining an exponential that have been proven to be o-minimal were \mathbb{R}_{exp} the ordered field of the reals expanded with the natural exponential [18], and its expansion by all restricted analytic functions $\mathbb{R}_{\text{an,exp}}$ [15]. In fact $\mathbb{R}_{\text{an,exp}}$ also eliminates quantifiers in the language of restricted analytic functions together with \exp and \log [14].

In [17, Thm 3.2] it was shown more generally that if T_0 is any theory of a power-bounded expansion \mathbb{R}_0 of the field of reals and T_0 defines the restricted natural exponential $\exp|_{[0,1]}$ and eliminates quantifiers, then the expansion $\mathbb{R}_{0,\text{exp}}$ of \mathbb{R}_0 by the unrestricted exponential function is o-minimal and its theory T eliminates quantifiers in the language $L = L_0 \cup \{\exp, \log\}$ where L_0 is the language of T_0 . We will focus mainly on this situation.

Definition 5.6. We will call an exponential o-minimal theory T *simply exponential* if it has a power-bounded reduct $T_0 := T|_{L_0}$ to a language L_0 such that T is an expansion by definition of $T' := T|_{(L_0 \cup \{\exp\})}$ and T' eliminates quantifiers in $L_0 \cup \{\exp, \log\}$.

5.2. Weakly immediate types in simply exponential theories. This section is dedicated to the analysis of weakly immediate types in a simply exponential theory T . This is meant as a step toward understanding λ -spherical completions for models of T_{convex} . In particular it will allow to show that wim-constructible extensions are 1-wim (Corollary 5.15) which is a step toward answering Question 3.29 (1) for these theories.

Definition 5.7. Let T be an o-minimal exponential theory and $(\mathbb{E}, \mathcal{O}) \models T_{\text{convex}}$. A *generalized nested exponential over \mathbb{E}* (gne for short) $g : \mathbb{E} \rightarrow \mathbb{E}$, is an \mathbb{E} -definable a composition of translations, changes of signs and exponentials. More specifically:

- (1) $g(t) := t + c$ is a gne of height 0 for every $c \in \mathbb{E}$;
- (2) if h is a gne of height m , then $g(t) := c + \sigma \exp(h(t))$ for $c \in \mathbb{E}$ and $\sigma \in \{\pm 1\}$ is a gne of height $m + 1$.

Let $(\mathbb{U}, \mathcal{O}) \succeq (\mathbb{E}, \mathcal{O})$ and $x \in \mathbb{U}$, we say inductively that that

- (1) g is *normal* at x if $g(x) = x + c$ with $c \succ x$ or $c = 0$, or if $g(t) = c + \sigma \exp(h(t))$ with h normal at x and, again, $c = 0$ or $c \succ \exp(h(x))$.
- (2) g is *essential* at x if $g(t) = t + c$ or if $g(t) = c + \sigma \exp(h(t))$ with h essential at x and $(h(x) + \mathcal{O}) \cap \mathbb{E} = \emptyset$.

Lemma 5.8. *Let g be a gne and assume it is essential at x , then there is a gne g_1 , normal and essential at x such that $\text{tp}(g(x)/\mathbb{E}) = \text{tp}(g_1(x)/\mathbb{E})$.*

Proof. By induction on the height of g . If $g(t) = t + c$, the statement is clear as if $\mathbf{v}(x) \notin \mathbf{v}(\mathbb{E})$ and $c \preceq x$, then $\text{tp}(c + x/\mathbb{E}) = \text{tp}(x/\mathbb{E})$.

Let $g(t) = c + \exp(h(t))$. By the inductive hypothesis, there is a gne h_1 normal and essential such that $\text{tp}(h_1(x)/\mathbb{E}) = \text{tp}(h(x)/\mathbb{E})$. Now if $c \succ$

$\exp(h(x))$, then $c \succ \exp(h_1(x))$ and we are done. Instead if $c \neq 0$ and $c \prec \exp(h(x))$, we have $\text{tp}(g(x)/\mathbb{E}) = \text{tp}(\exp h(x)/\mathbb{E}) = \text{tp}(\exp h_1(x)/\mathbb{E})$. \square

Lemma 5.9. *Let g be a gne and assume x is weakly \mathcal{O} -immediate, then there is a gne g_1 , essential at $g_1^{-1}g(x)$ such that $g_1^{-1}g(x)$ is weakly \mathcal{O} -immediate.*

Proof. By induction on the height of g . If $g(t) = t + c$, then it suffices to take $g_1(t) = t - c$. Now assume $g(t) = c + \exp(h(t))$, we can find by inductive hypothesis h_1 such that $h_1(z) = h(x)$, h_1 is essential at z and z is \mathcal{O} -immediate. If $(h(x) + \mathcal{O}) \cap \mathbb{E} = \emptyset$, then $g_1(t) := c + \exp(h_1(t))$ has the required property.

If instead there is $c_1 \in (h(x) + \mathcal{O}) \cap \mathbb{E}$, either $h(x)$ is \mathcal{O} -weakly immediate and we are done, or c_1 can be chosen so that $\mathbf{v}(h(x) - c_1) \notin \mathbf{v}(\mathbb{E})$, so we set $\bar{h}(x) = h(x) - c_1$. Now \bar{h} is by construction a gne of height strictly lower than g and we can apply the inductive hypothesis, to get a gne \bar{h}_1 essential at some \mathcal{O} -weakly immediate z with $\bar{h}_1(z) = \bar{h}(x)$, moreover by Lemma 5.8 without loss of generality we can assume $\bar{h}_1(z) = \exp(h_2(z))$ for some other gne h_2 normal at z . Since $\mathbf{v}(\bar{h}(x)) \notin \mathbf{v}\mathbb{E}$, and $\mathbf{v}(\bar{h}(x)) > 0$, $\text{tp}(\exp(\bar{h}(x))/\mathbb{E}) = \text{tp}(1 + \bar{h}(x)/\mathbb{E})$, so

$$g(x) = c + \exp(c_1) + \exp(c_1 + \log \bar{h}(x)) = c + \exp(c_1) + \exp(c_1 + h_2(z))$$

and $g_1(t) = c + \exp(c_1) + \exp(c_1 + h_2(t))$ has the required property. \square

Lemma 5.10. *Let $g : \mathbb{E} \rightarrow \mathbb{E}$ be a \mathbb{E} -definable gne essential at x . Then there is a gne g_1 essential at $z := g_1^{-1} \lg g(x)$ such that $\text{tp}(z/\mathbb{E}) = \text{tp}(x/\mathbb{E})$.*

Proof. Assume $g(t) = c + \exp(h(t))$. Now $\lg g(t) = \lg c + \log(1 + \exp(h(t) - \lg c))$. Notice $\varepsilon := \exp(h(x) - \lg c) \prec 1$ by normality of g at x and moreover $\mathbf{v}(\varepsilon) \notin \mathbf{v}(\mathbb{E})$ because g is essential at x . Observe that

$$\lg(\lg g(x) - \lg(c)) = \lg \log(1 + \varepsilon) = \lg \varepsilon + \lg \left(1 + \frac{\log(1 + \varepsilon) - \varepsilon}{\varepsilon} \right)$$

so $\lg(\lg(g(x) - \lg(c))) \in \lg \varepsilon + \varepsilon/2 + \varepsilon^2 \mathcal{O}_x$. Since $\lg(\varepsilon) + \mathcal{O}_x \cap \mathbb{E} = \emptyset$ this implies $\text{tp}(\lg \varepsilon/\mathbb{E}) = \text{tp}(\lg \log(1 + \varepsilon)/\mathbb{E})$.

Notice also $\lg \varepsilon = h(x) - \lg c$, so

$$\text{tp}(\lg(\lg g(x) - \lg(c)) + \lg(c)/\mathbb{E}) = \text{tp}(\lg \varepsilon + \lg(c)/\mathbb{E}) = \text{tp}(h(x)/\mathbb{E}).$$

It follows that if we set $g_1(t) := \lg(c) + \exp(h(t) - \lg(c))$, then

$$g_1^{-1}(\lg g(x)) = h^{-1}(\lg(\lg g(x) - \lg(c)) + \lg(c))$$

therefore $\text{tp}(g_1^{-1}(\lg g(x))/\mathbb{E}) = \text{tp}(h^{-1}h(x)/\mathbb{E}) = \text{tp}(x/\mathbb{E})$.

Notice that g_1 is essential by construction as $h(x) - \lg c = \lg \varepsilon$ and $\lg(\varepsilon) + \mathcal{O}_x \cap \mathbb{E} = \emptyset$. \square

Lemma 5.11. *Let $g : \mathbb{E} \rightarrow \mathbb{E}$ be a \mathbb{E} -definable gne essential at x . Then there is g_1 essential at $z := g_1^{-1} \exp g(x)$ such that $\text{tp}(z/\mathbb{E}) = \text{tp}(x/\mathbb{E})$.*

Proof. If $g(x) + \mathcal{O}_x \cap \mathbb{E} = \emptyset$, then $g_1 := \exp g$ does the job. So assume that $g(t) = c + \exp(h(t))$ with $c \succ 1 \succ \exp(h(x))$. Since g is essential $\mathbf{v}(h(x)) \notin \mathbf{v}(\mathbb{E})$, therefore $\text{tp}(\exp(\exp(h(x)))/\mathbb{E}) = \text{tp}(1 + \exp(h(x))/\mathbb{E})$ and $\text{tp}(\exp(g(x))/\mathbb{E}) = \text{tp}(\exp(c)(1 + \exp(h(x)))/\mathbb{E})$. It follows that $g_1(t) := \exp(c) + \exp(c + h(t))$ has the required properties in this case. \square

Proposition 5.12. *Let T_0 be a power bounded \mathcal{O} -minimal theory which eliminates quantifiers in a language L and T be an exponential \mathcal{O} -minimal theory in $L_{\exp, \log} := L \cup \{\exp, \log\}$ which eliminates quantifiers and expands T_0 . Assume x is weakly \mathcal{O} -immediate but not dense over \mathbb{E} and $f : \mathbb{E} \rightarrow \mathbb{E}$ a \mathbb{E} -definable function. Then there is a weakly immediate $z \in \mathbb{E}\langle x \rangle$ and a \mathbb{E} -definable gne g , essential at z such that $f(x) = g(z)$.*

Proof. We know T eliminates quantifiers in $L \cup \{\exp, \log\}$. So without loss of generality f is given by a term in $L \cup \{\exp, \log\}$. If f is a term in L , then $f(x)$ is \mathcal{O} -weakly immediate by the rv-property because T_0 is power bounded. If $f(x) = \exp(t(x))$, then by inductive hypothesis we have a gne g essential at x such that $z := g^{-1}t(x)$ is \mathcal{O} -weakly immediate, but then by Lemma 5.11 there is g_1 essential at z such that $\text{tp}(g_1(z)/\mathbb{E}) = \text{tp}(\exp(g_1(z))/\mathbb{E})$. If $f(x) = \lg(t(x))$, then again by inductive hypothesis we have a gne g essential at x such that $z := g^{-1}t(x)$ is \mathcal{O} -weakly immediate and we can conclude similarly by Lemma 5.10. \square

Lemma 5.13. *Let $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$ and $\mathbb{E} \prec \mathbb{E}_1 \prec \mathbb{U}$ with $\mathbb{E} \not\subseteq \mathcal{O}$. Assume z is weakly \mathcal{O} -immediate over \mathbb{E} , $g : \mathbb{E}_1 \rightarrow \mathbb{E}_1$ is a \mathbb{E}_1 -definable gne essential and normal at z and that $\text{tp}(g(z)/\mathbb{E})$ is not realized in \mathbb{E}_1 . Then there is a \mathbb{E} -definable gne g_1 and a z_1 \mathcal{O} -weakly immediate over \mathbb{E} such that $\text{tp}(g_1(z_1)/\mathbb{E}) = \text{tp}(g(z)/\mathbb{E})$.*

Proof. By induction on the height of g . If $g(t) = t + c$, then since $\text{tp}(z + c/\mathbb{E})$ is not realized in \mathbb{E}_1 and $z + c$ is weakly \mathcal{O} -immediate over \mathbb{E}_1 , $z_1 := z + c$ is weakly \mathcal{O} -immediate over \mathbb{E} .

Let $g(t) = c + \sigma \exp(h(t))$. By inductive hypothesis, there is h_1 , \mathbb{E} -definable, and z_1 weakly immediate over \mathbb{E} with $\text{tp}(h(z)/\mathbb{E}) = \text{tp}(h_1(z_1)/\mathbb{E})$.

If $\mathbb{E} \cap (c + \exp(h(z))\mathfrak{o}) = \emptyset$ then $\text{tp}(g(z)/\mathbb{E})$ would be realized in \mathbb{E}_1 by any element in $c + \mathbb{E}_1^{\prec \exp(h(z))}$ contradicting that $\text{tp}(g(z)/\mathbb{E})$ is not realized in \mathbb{E}_1 .

Thus we can find $c_1 \in \mathbb{E}$ such that $c - c_1 \prec \exp h(z)$. But then since $\mathbf{v}(\exp h(z)) \notin \mathbf{v}\mathbb{E}_1$, $\text{tp}(c + \exp(h(z))/\mathbb{E}) = \text{tp}(c_1 + \exp(h(z))/\mathbb{E}) = \text{tp}(c_1 + \exp(h_1(z_1))/\mathbb{E})$. \square

Corollary 5.14. *Let T be simply exponential, $(\mathbb{U}, \mathcal{O}) \models T_{\text{convex}}$ and $\mathbb{E} \prec \mathbb{E}_1 \prec \mathbb{E}_2 \prec \mathbb{U}$ with $\mathbb{E} \not\subseteq \mathcal{O}$. Suppose \mathbb{E}_1 is 1-wim over $(\mathbb{E}, \mathcal{O} \cap \mathbb{E})$ and \mathbb{E}_2 is 1-wim over $(\mathbb{E}_1, \mathcal{O} \cap \mathbb{E}_1)$, then \mathbb{E}_2 is 1-wim over $(\mathbb{E}, \mathcal{O} \cap \mathbb{E})$.*

Proof. Let $z \in \mathbb{E}_2$. If $\text{tp}(z/\mathbb{E})$ is realized in \mathbb{E}_1 , then z is weakly immediately generated over \mathbb{E} and there is nothing to show. So assume $\text{tp}(z/\mathbb{E})$ is not realized in \mathbb{E}_1 , by Proposition 5.12 $z = g(y)$ for some \mathbb{E}_1 -definable gne

essential at y and y weakly immediate over \mathbb{E}_1 . By Lemma 5.13, there is \mathbb{E} -definable gne g_1 and a y_1 weakly immediate over \mathbb{E} such that $\text{tp}(g(y)/\mathbb{E}) = \text{tp}(g_1(y_1)/\mathbb{E})$, so z is weakly immediately generated over \mathbb{E} once again. \square

Corollary 5.15. *If T is simply exponential then every wim-constructible extension of models of T_{convex} is 1-wim.*

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