# Definability of band structures on posets 

Joel Kuperman* Alejandro Petrovich ${ }^{\dagger}$ Pedro Sánchez Terraf*

April 12, 2024


#### Abstract

The idempotent semigroups (bands) that give rise to partial orders by defining $a \leq b \Longleftrightarrow a \cdot b=a$ are the right-regular bands (RRB), which are axiomatized by $x \cdot y \cdot x=y \cdot x$. In this work we consider the class of associative posets, which comprises all partial orders underlying right-regular bands, and study to what extent the ordering determines the possible "compatible" band structures and their canonicity.

We show that the class of associative posets in the signature $\{\leq\}$ is not first-order axiomatizable. We also show that the Axiom of Choice is equivalent over $Z F$ to the fact that every tree with finite branches is associative. We also present an adjunction between the categories of RRBs and that of associative posets.

We study the smaller class of "normal" posets (corresponding to right-normal bands) and give a structural characterization.

As an application of the order-theoretic perspective on bands, we generalize results by the third author, obtaining "inner" direct product representations for RRBs having a central (commuting) element.


## 1 Introduction

Idempotent semigroups (bands) carry a natural quasiorder structure given by

$$
\begin{equation*}
a \lesssim b \Longleftrightarrow a \cdot b=a \tag{1}
\end{equation*}
$$

The associated equivalence relation, $(\approx):=(\lesssim) \cap(\gtrsim)$ is not always a congruence relation, but only for "left-semiregular" ones [3]. In particular it is known that for the variety of right-regular bands ( RRB ), axiomatized by

[^0]the equation $x \cdot y \cdot x=y \cdot x$, this congruence is the identity and hence the quasiorder (1) is actually a partial order.

In this paper, we are interested in studying the class of associative posets, which comprises all partial orders underlying right-regular bands. In doing so, we will interpret some constructions that give rise to RRBs from an order-theoretical point of view. This vantage points proves to be useful in a number of situations, e.g. the analysis of direct product decompositions carried out in Section 4. Another example is given by the results in [3], which characterizes all varieties of bands according to some particular congruences.

The class of associative posets is ample. It includes all meet-semilattices. Another source of examples is given by the duals of face posets of a hyperplane arrangement in $n$-dimensional Euclidean space [1], and more generally, any convex subset of faces of such an arrangement (which includes the case of face posets of convex polyhedra). These examples were originally presented using left-regular bands ("LRB", satisfying $x \cdot y \cdot x=x \cdot y$ ) and using the order defined by

$$
a \leq b \Longleftrightarrow a \cdot b=b
$$

Contrary to the case of commutative bands, for which the class of underlying orders (semilattices) is first-order definable, this is not the case for associative posets (Corollary 2.17). The search for a sensible (or structural) characterization of associative posets led us to the question of definability of such classes.

In the present paper, we prove (Theorem 2.25) that the fact that every tree with finite branches is associative is equivalent over $Z F$ to the Axiom of Choice, thereby showing that there is no canonical assignment of a band structure to each associative poset. Nevertheless, in Section 3 we define a left adjoint to the forgetful functor from the category of RRB to that of associative posets.

This discussion of definability permeates our whole work, and many questions remain open; some of these are gathered in Section 5. After setting up some preliminaries in the next section, we restrict ourselves in Section 2.2 to a subfamily of associative posets for which a neat structural characterization is available.

## 2 Associative posets

### 2.1 Preliminaries

For any poset $(P, \leq)$, we say that a binary operation $\cdot$ on $P$ is admissible for $(P, \leq)$ whenever $x \leq y$ if and only if

$$
\begin{equation*}
x \cdot y=x \tag{2}
\end{equation*}
$$

for all $x, y$ in $P$. In the case of an associative poset, we can moreover choose this to be an RRB operation and then we use the word posemigroup when
referring to the expanded structure $(P, \leq, \cdot)$. Conversely, given an RRB $(P, \cdot)$, we say that the partial order $x \leq y$ given by (2) is the underlying order of this RRB.

We start by introducing some useful elementary properties of posemigroups.

Lemma 2.1. In every posemigroup,

1. $a \cdot b \leq b$; in particular, if $b$ is minimal, $a \cdot b=b$.
2. $a \cdot b \cdot a=b \cdot a$.
3. $a \leq b \Longrightarrow b \cdot a=a$.

Proof. 1. $(a \cdot b) \cdot b=a \cdot(b \cdot b)=a \cdot b$.
2. From Item 1 we know that $a \cdot(b \cdot a) \leq b \cdot a$ and $b \cdot a=b \cdot(a \cdot b \cdot a) \leq a \cdot b \cdot a$. By antisimmetry we obtain $a \cdot b \cdot a=b \cdot a$.
3. $a=a \cdot a=(a \cdot b) \cdot a=b \cdot a$, by Item 2 .

Corollary 2.2. Every decreasing subset of a posemigroup is a substructure.
Lemma 2.3. For every posemigroup,

1. $a \leq b$ implies $a \cdot x \leq b \cdot x$.
2. $c \leq x, y$ implies $c \leq x \cdot y, y \cdot x$. Hence if $x \cdot y=y \cdot x$, it must be the infimum of $\{x, y\}$.

Proof. 1. Using Lemma 2.1(2), $a \cdot \underline{x \cdot b \cdot x}=\underline{a \cdot b} \cdot x=a \cdot x$.
2. $c=\underline{c} \cdot x=c \cdot y \cdot x$. Hence $c \leq y \cdot x$. Symmetrically, $c \leq x \cdot y$.

In the following, we use $x \downarrow$ to denote $\{a \in P \mid a \leq x\}$.
Lemma 2.4. Let $P$ be a posemigroup. If $x \cdot y=y$ then $(y \cdot x) \downarrow$ is isomorphic to $y \downarrow$.

Proof. We prove that the function $f:(y \cdot x) \downarrow \rightarrow y \downarrow$ defined by $f(a):=a \cdot y$ is an isomorphism with inverse $b \mapsto b \cdot x$. The injectivity of $f$ follows from the fact that for all $a \leq y \cdot x$, we have $a=a \cdot(y \cdot x)=(a \cdot y) \cdot x$. Now, if $b \leq y$, take $a=b \cdot x$. We know $a \leq y \cdot x$ by Lemma 2.3(1). Now, $f(a)=(b \cdot x) \cdot y=b \cdot(x \cdot y)=b \cdot y=b$. So $f$ is surjective and hence a bijection. The fact that $f$ is order preserving is a direct consequence of Lemma 2.3(1). To see that $f^{-1}$ is order preserving, take $a \leq b \leq y$. By Lemma 2.3(1) $f^{-1}(a)=a \cdot x \leq b \cdot x=f^{-1}(b)$. So $f$ is an isomorphism.

Corollary 2.5. Let $P$ be a posemigroup. For every $x, y \in P,(x \cdot y) \downarrow$ is isomorphic to $(y \cdot x) \downarrow$.

Proof. We have $(x \cdot y) \cdot(y \cdot x)=x \cdot(y \cdot y) \cdot x=x \cdot y \cdot x=y \cdot x$. Symmetrically, $(y \cdot x) \cdot(x \cdot y)=x \cdot y$. By Lemma 2.4 we have $(x \cdot y) \downarrow \approx(y \cdot x) \downarrow$.

Corollary 2.6. Let $P$ be a posemigroup and $m \in P$ be minimal. For every $x \in P, m \cdot x \leq x$ is minimal.

### 2.2 Normal posets

It is well known [2] that there are four proper subvarieties of RRBs, each of which can be axiomatized by one extra identity besides the band axioms. These are:

- $x=y$. The trivial variety.
- $x \cdot y=y \cdot x$. The class of partial orders underlying the subvariety of commutative bands is exactly the class of (meet-)semilattices.
- $x \cdot y=y$. These are the "right-zero" bands, and the class of partial orders underlying this one is the class of antichains.
- $x \cdot y \cdot z=y \cdot x \cdot z$. These are a superset of right-zero bands called "right-normal" (RNB).

Our goal for this section is to give a characterization of normal posets, which are the orders underlying right-normal bands. Normal posets are always relative meet-semilattices (i.e., posets in which every principal ideal is a meet-semilattice) but not conversely (even assuming associativity; see Example 2.21). In fact, it is not hard to show that an associative relative meet-semilattice is a normal poset if and only if it admits an operation which acts as the meet operation in every principal ideal [4, Lemma 10].

First, we prove an auxiliary result.
Lemma 2.7. Let $A$ be an RRB. Then $\theta:=\left\{(x, y) \in A^{2}: x \cdot y=y \& y \cdot x=\right.$ $x\}$ is a congruence over $A$ and $A / \theta$ is a meet-semilattice.

Proof. The relation $\theta$ is clearly symmetric and reflexive. If $x \cdot y=y$ and $y \cdot z=z$, then $x \cdot z=x \cdot(y \cdot z)=(x \cdot y) \cdot z=y \cdot z=z$. Analogously $z \cdot x=x$. Thus $\theta$ is transitive. Suppose now that $\left\langle x, x^{\prime}\right\rangle,\left\langle y, y^{\prime}\right\rangle \in \theta$. Then $(x \cdot y) \cdot\left(x^{\prime} \cdot y^{\prime}\right)=(x \cdot y) \cdot\left(y^{\prime} \cdot x^{\prime} \cdot y^{\prime}\right)=x \cdot \underline{y \cdot y^{\prime}} \cdot x^{\prime} \cdot y^{\prime}=x \cdot \underline{y^{\prime} \cdot x^{\prime} \cdot y^{\prime}}=\underline{x \cdot x^{\prime}} \cdot y^{\prime}=$ $x^{\prime} \cdot y^{\prime}$. Therefore $\theta$ is a congruence. To see that $A / \theta$ is a meet-semilattice, note that for every $x, y \in A,(x \cdot y) \cdot(y \cdot x)=x \cdot \underline{y \cdot y} \cdot x=\underline{x \cdot y \cdot x}=y \cdot x$. Analogously, $(y \cdot x) \cdot(x \cdot y)=x \cdot y$, thus $\langle x \cdot y, y \cdot x\rangle \in \bar{\theta}$. Therefore, we have that $x / \theta \cdot y / \theta=(x \cdot y) / \theta=(y \cdot x) / \theta=y / \theta \cdot x / \theta$. So $A / \theta$ is a meet-semilattice.

In view of this result, for any $\operatorname{RRB} A$, we shall call $\theta$ the semilattice congruence of $A$ and $A / \theta$ the quotient semilattice of $A$.

Theorem 2.8. Let $P$ be a poset. The following are equivalent:

1. $P$ is normal;
2. There exist a meet-semilattice $S$ and an order homomorphism $f: P \rightarrow$ $S$ which satisfies that $f_{m}:=\left.f\right|_{m \downarrow}: m \downarrow \rightarrow f(m) \downarrow$ is an isomorphism between $m \downarrow$ and $f(m) \downarrow$ for every $m \in P$.

It is immediate that in 2 , it is enough to verify the condition for each $m$ in some cofinal $M \subseteq P$.

Proof. ( $\Rightarrow$ ) Fix an admissible RNB operation • for $P$. Let $\theta$ be the semilattice congruence of $(P, \cdot)$ and $f: P \rightarrow P / \theta$ the canonical projection. First note that since each initial segment $p \downarrow$ is decreasing, it is also a substructure, so the restriction of any homomorphism to it is a homomorphism. We also have that RRB isomorphisms are isomorphisms of the underlying orders. Therefore we only need to check that $\left.f\right|_{p \downarrow}=f_{p}$ is bijective for every $p \in P$. Let $p \in P$; since $(P, \cdot)$ is an RNB, if $x, y \leq p$ and $f(x)=f(y)$, then

$$
x=x \cdot p=y \cdot x \cdot p=x \cdot y \cdot p=y \cdot p=y
$$

since $x \theta y$ and hence $f_{p}$ is injective. Also, if $x / \theta \leq p / \theta$, then $x \cdot p / \theta=x / \theta$, so $f_{p}(x \cdot p)=x / \theta$. Thus $f_{p}$ is an order isomorphism for every $p \in P$.
$(\Leftarrow)$ Assume that $S$ and $f: P \rightarrow S$ satisfy 2. Consider the antichain order $A:=(P,=)$ on $P$. Let $B \subseteq S \times A$ given by

$$
B:=\{\langle x, m\rangle \mid x \leq f(m)\}=\{\langle f(m), m\rangle \mid m \in P\} \downarrow .
$$

$B$ is a subalgebra of the direct product RNB structure on $S \times A$. Let $h: B \rightarrow P$ given by $h(\langle x, m\rangle)=\left(f_{m}\right)^{-1}(x)$. Note that $h$ is surjective.
Claim 1. $\delta:=\operatorname{ker} h=\{\langle x, y\rangle \mid h(x)=h(y)\}$ is a congruence over $B$.
Indeed, if $\langle x, m\rangle \delta\left\langle x^{\prime}, m^{\prime}\right\rangle$ and $\langle y, n\rangle \delta\left\langle y^{\prime}, n^{\prime}\right\rangle$, then $\left(f_{m}\right)^{-1}(x)=\left(f_{m^{\prime}}\right)^{-1}\left(x^{\prime}\right)$, and so $x=f\left(\left(f_{m}\right)^{-1}(x)\right)=f\left(\left(f_{m^{\prime}}\right)^{-1}\left(x^{\prime}\right)\right)=x^{\prime}$. Analogously, using that $\left(f_{n}\right)^{-1}(y)=\left(f_{n^{\prime}}\right)^{-1}\left(y^{\prime}\right)$ we see that $y=y^{\prime}$. Now we have

$$
h(\langle x, m\rangle \cdot\langle y, n\rangle)=h(\langle x \wedge y, n\rangle)=\left(f_{n}\right)^{-1}(x \wedge y)=: p,
$$

while

$$
h\left(\left\langle x^{\prime}, m^{\prime}\right\rangle \cdot\left\langle y^{\prime}, n^{\prime}\right\rangle\right)=\left(f_{n^{\prime}}\right)^{-1}\left(x^{\prime} \wedge y^{\prime}\right)=\left(f_{n^{\prime}}\right)^{-1}(x \wedge y)=: q .
$$

As $q \leq\left(f_{n^{\prime}}\right)^{-1}(y)=\left(f_{n}\right)^{-1}(y) \leq n$, and $f(q)=x \wedge y=f(p)$, we must have $p=q$ as both $p$ and $q$ belong to $n \downarrow$.
Claim 2. $\varphi: B / \operatorname{ker} h \rightarrow P$ given by $\varphi([\langle x, m\rangle])=h(\langle x, m\rangle)$ is an order isomorphism.

The map $\varphi$ is clearly bijective so we only have to check that it is order preserving and that is has an order preserving inverse. Given $[\langle x, m\rangle],\left[\left\langle y, m^{\prime}\right\rangle\right] \in$ $B /$ ker $h$ such that $[\langle x, m\rangle] \cdot\left[\left\langle y, m^{\prime}\right\rangle\right]=[\langle x, m\rangle]$, we have $[\langle x, m\rangle]=[\langle x \wedge$ $\left.\left.y, m^{\prime}\right\rangle\right]$, so

$$
\varphi([\langle x, m\rangle])=\varphi\left[\left\langle x \wedge y, m^{\prime}\right\rangle\right]=\left(f_{m^{\prime}}\right)^{-1}(x \wedge y) \leq\left(f_{m^{\prime}}\right)^{-1}(y)=\varphi\left(\left[\left\langle y, m^{\prime}\right\rangle\right]\right) .
$$

We also have that $\varphi^{-1}(p)=[\langle f(p), p\rangle]$. If $p \leq q$, then

$$
[\langle f(p), p\rangle] \cdot[\langle f(q), q\rangle]=[\langle f(p), q\rangle]=[\langle f(p), p\rangle]=\varphi^{-1}(p)
$$

Then $\varphi$ is a poset isomorphism. Therefore, the RNB structure of $B / \operatorname{ker} h$ is admissible for $P$.

Example 2.9. Applying the previous theorem we can see that the poset depicted in Figure 1 is normal.


Figure 1: A normal poset

Example 2.10. For the leftmost poset from Figure 2, different homomorphisms can be chosen. Each one gives rise to a different compatible rightnormal band operation.


Figure 2: Multiple RNB structures

Note that the quotient semilattice of the RNB obtained in the last part of the proof is isomorphic to the semilattice $S$ with which we started. Also the canonical projection of the semilattice congruence is the homomorphism $f$. It is not hard to see that in the case of right-normal bands, one can completely determine the product using two equationally definable binary relations: the underlying partial order and the semilattice congruence. This tells us, according to Beth's Theorem, that there is a first order formula $\varphi(x, y, z)$ in the language $\{\leq, \theta\}$ which is equivalent to the formula $x \cdot y=z$. In fact, Theorem 2.8 provides information for defining it. We simply need a formula which roughly says " $z$ is the representative of the class $x / \theta \wedge y / \theta$
which is below $y$ ". For this, we define $\psi(x, y)$ which states " $x / \theta \leq y / \theta$ ", as $\psi(x, y):=\exists c, c \leq y \wedge x \theta c$. Then we set

$$
\varphi(x, y, z):=z \leq y \wedge \psi(z, x) \wedge \forall d,(\psi(d, x) \wedge \psi(d, y)) \rightarrow \psi(d, z)
$$

Example 2.11. The non-normal poset in Figure 3 admits two non isomorphic right-regular band operations. The semilattice congruence is the same in both right-regular bands.


Figure 3: An associative poset with a single semilattice congruence.

A few questions regarding definability remain open. One of them is the following:
Question 2.12. Is the Axiom of Choice needed to define a right-normal band operation for every normal poset?

### 2.3 Constructions

In this section we discuss closure under certain model-theoretic operations of the class of associative posets. These are obviously correlated to algebraic constructions on the RRB side.

Given two posets $\left(P, \leq^{P}\right)$ and $\left(Q, \leq^{Q}\right)$ with $P$ disjoint from $Q$, we define their disjoint union $\left(P \sqcup Q,\left(\leq^{P}\right) \sqcup\left(\leq^{Q}\right)\right)$, in which the copies of both posets are unrelated. We extend this concept in the obvious way to define the disjoint union of a family of posets, $\bigsqcup_{i \in I} P_{i}$. We also define their ordered sum $P+Q$ as the poset

$$
\left(P \sqcup Q,\left(\leq^{P}\right) \sqcup\left(\leq^{Q}\right) \sqcup(P \times Q)\right),
$$

and analogously for a family of posets, $\sum_{i \in I} P_{i}$, for any linearly ordered index set $I$.

Lemma 2.13. Let $\left\{P_{i}: i \in I\right\}$ be a family of associative posets. Then $\sum_{i \in I} P_{i}$ is associative.
Proof. First, we prove that the sum of two associative posets $P$ and $Q$ is associative. We define a posemigroup operation over $P+Q$ in the following manner:

$$
x \cdot y= \begin{cases}\min \{x, y\} & x, y \text { comparable } \\ x \cdot{ }^{Q} y & x, y \in Q \\ x \cdot{ }^{P} y & x, y \in P\end{cases}
$$

A straightforward case analysis shows associativity. By induction, a sum of a finite family of associative posets is associative.

Now let $\left\{P_{i}: i \in I\right\}$ be a family of associative posets and let ${ }_{i}$ denote an admissible RRB structure over $P_{i}$. Consider the following product over $\sum_{i \in I} P_{i}:$

$$
x \cdot y= \begin{cases}\min \{x, y\} & x \text { and } y \text { comparable } \\ x \cdot_{i} y & x, y \in P_{i}\end{cases}
$$

Let $x, y, z$ be arbitrary elements in $\bigsqcup_{i \in I} P_{i}$ and $i, j, k \in I$ be such that $x \in P_{i}$, $y \in P_{j}, z \in P_{k}$. Now we have, by associativity of $P_{i}+P_{j}+P_{k}$ and the fact that this finite sum is a subalgebra of $\sum_{i \in I} P_{i}$, that $(x \cdot y) \cdot z=x \cdot(y \cdot z)$.

Lemma 2.14. Let $P$ be an associative (resp., normal) poset, $I$ a set and $P_{i}$ an isomorphic copy of $P$ for each $i \in I$. Then $\bigsqcup_{i \in I} P_{i}$ is associative (normal).

Proof. Observe that $\bigsqcup_{i \in I} P_{i}$ is the underlying order of the right-regular (resp., normal) given by $(P, \cdot) \times\left(I, \pi_{2}\right)$, where $\cdot$ is an admissible right-regular (normal) band structure for $P$ and $\pi_{2}(x, y)=y$ for any $x, y \in I$.

Similar arguments also yield:
Lemma 2.15. Let $\left\{Q_{i}: i \in I\right\}$ be a family of associative posets and let $P$ an associative poset with top element 1 . Let us define $R_{i}=P_{i}+Q_{i}$, with $P_{i}$ an isomorphic copy of $P$, then $\bigsqcup_{i \in I} R_{i}$ is associative.

Theorem 2.16. The class of normal posets is not axiomatizable by firstorder sentences.

Proof. Consider the disjoint union $P=\mathbb{R} \sqcup \mathbb{R}$, which is normal by Lemma 2.14. It can be shown by using Ehrenfeucht-Fraïssé games that $\mathbb{R} \sqcup \mathbb{Q}$ is elementarily equivalent to $P$. But by Lemma 2.4 this poset is even not associative. Therefore, the class of normal posets is not closed under elementary equivalence and therefore is not a first-order class.

Corollary 2.17. The class of associative posets is not axiomatizable by first-order sentences.

### 2.4 Examples

Every poset $P$ admits a binary operation • in the sense of (2). This operation can be chosen to be commutative if $P$ is not the two-element antichain. As it is well-known, meet-semilattices are characterized by the fact that • can be (uniquely) chosen to be commutative and associative; therefore every meet-semilattice is immediately a normal poset and hence associative.

The following is the smallest example of a non-associative poset.

Example 2.18 (The hummingbird). The poset in Figure 4 is not associative. Assume by way of contradiction that it admits an RRB structure $\cdot$. By Lemma 2.3(2), $b \cdot x=x$ and $x \cdot b=b$; but this contradicts Corollary 2.5.


Figure 4: The hummingbird.

Example 2.19 (The 3-crown). The poset depicted in Figure 5 is also not associative, but for totally different reasons (see Appendix A for a proof). It is noteworthy that analogous ( $2 n$ )-crowns of even width are all associative.


Figure 5: The crown poset.

Example 2.20 (The puppy). The poset depicted in Figure 6 is an example of an associative poset in which there is no $\operatorname{RRB}$ operation for which $x \cdot y=$ $x \wedge y$ holds for every pair of elements $x, y$ such that $x \wedge y$ exists. In the only admissible RRB operation for this poset, $a \cdot b=b$ and $b \cdot a=a$.


Figure 6: The puppy.

Example 2.21 (The tulip). The poset depicted in Figure 7 is an example of a non-normal associative relative meet-semilattice (every initial segment is a meet-semilattice). The situation is analogous to the previous example; we must have $a \cdot b=b$ and $b \cdot a=a$.

We will end this section by showing that the order dual of $\mathbb{N} \times \mathbb{N}$ admits only one RRB structure. It is straightforward to check that the only RRB operation admissible for a chain is the infimum. The next lemma extends this observation.


Figure 7: The tulip.

Lemma 2.22. Let $C_{1}$ and $C_{2}$ be chains, and at least one of them has no minimum. Then the only admissible $R R B$ on the direct product $C_{1} \times C_{2}$ is given by the infimum.

Proof. Let $\leq^{i}$ denote the order in $C_{i}$, and $\leq$ the direct product order. Now assume, by way of contradiction, that there are $a, b \in L$ such that $a \cdot b \neq a \wedge b$. Without loss of generality, assume that $a \cdot b=b$ and $b \cdot a=a$.

Assume that $c<a$ and $c \not \leq b$ for a fixed $c$. Then we have

$$
(c \cdot b) \cdot a=c \cdot(b \cdot a)=c \cdot a=c .
$$

Hence we deduce that $c \cdot b<b, c \cdot b \neq c$, and $c \cdot b \not \leq a$. From the last one we have $c \wedge b<c \cdot b$.

Again without loss of generality, we may assume $a=\left\langle a_{1}, a_{2}\right\rangle, b=\left\langle b_{1}, b_{2}\right\rangle$ with $a_{1}<^{1} b_{1}$ and $b_{2}<^{2} a_{2}$. Also let $a_{1}^{\prime}<^{1} a_{1}, c:=\left\langle a_{1}^{\prime}, a_{2}\right\rangle$ and $s_{1}, s_{2} \in \omega$ such that $c \cdot b=\left\langle s_{1}, s_{2}\right\rangle$. We have $c \wedge b=\left\langle a_{1}^{\prime}, b_{2}\right\rangle$. Since $c<a$ and $c \not \leq b$, the calculations of the previous paragraph apply. From $c \cdot b<b$ and $c \cdot b \not \leq a$ we have

$$
\begin{equation*}
s_{2} \leq^{2} b_{2} \quad b_{1} \geq^{1} s_{1}>^{1} a_{1}, \tag{3}
\end{equation*}
$$

which, together with $c \wedge b<c \cdot b$ imply $s_{2}=b_{2}$. We conclude that $c \cdot b \geq$ $\left\langle a_{1}, b_{2}\right\rangle=a \wedge b$, and hence

$$
\left\langle a_{1}^{\prime}, a_{2}\right\rangle=c=(c \cdot b) \cdot a \geq(a \wedge b) \cdot a=a \wedge b=\left\langle a_{1}, b_{2}\right\rangle,
$$

which contradicts the fact that $a_{1}^{\prime}<^{1} a_{1}$.
The hypotheses on $C_{i}$ are necessary: The square of the 2-element chain admits exactly two RRB operations.

Corollary 2.23. The only RRB structure admissible for the cartesian square of the naturals with the reverse order is given by the infimum.

### 2.5 Foliated trees

Some of our more general tools for proving associativity of posets involve trees. In this paper, a tree is a poset $(T, \leq)$, with top element 1 , such that for every $x \in T, x \uparrow:=\{y \in T: x \leq y\}$ is linearly ordered; a forest is a disjoint union of trees.

Proposition 2.24. Let $T$ be a forest and $x, y \in T$. If there is $a z$ such that $z \leq x, y$, then $x$ and $y$ are comparable.

We will say that a tree has finite branches if every chain in the tree is finite. Moreover, given a natural number $n$, we will say that a tree with finite branches has height $n$ if every chain has at most $n$ elements and there is at least one chain with $n$ elements. Finally, we call a tree $T$ foliated if for every $x \in T$, there is a minimal element below $x$. Note that a foliated tree might still have branches without a minimal element.

Theorem 2.25. The following are equivalent (in $Z F$ ):

1. Every foliated tree is associative.
2. Every tree with finite branches is associative.
3. Every tree with height 3 is associative.

## 4. The Axiom of Choice.

Proof. $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are trivial. Let us prove $3 \Rightarrow 4$. Let $\mathcal{F}$ be a non empty family of mutually disjoint non empty sets. We now define a tree order over $T:=\{\mathcal{F}\} \cup \mathcal{F} \cup \bigcup \mathcal{F}: x<y$ if and only if $x \neq y$, and $y=\mathcal{F}$ or $x \in y$ (we are considering $\mathcal{F} \cap \bigcup \mathcal{F}=\varnothing$. If this was not the case, the order obtained would not be that of a tree with height 3 . This can be fixed by considering $\{\mathcal{F}\} \times\{2\} \cup \mathcal{F} \times\{1\} \cup \bigcup \mathcal{F} \times\{0\}$ as the universe for $T$ and defining the order as: $x^{\prime}=\langle x, n\rangle<y^{\prime}=\langle y, m\rangle$ if and only if $y^{\prime}=\langle\mathcal{F}, 2\rangle$ or $x \in y$ and $\left.n<m\right)$. Note that this is a tree with height 3 and therefore it is associative. Fix an admissible RRB structure for $T$ and a minimal $m \in T$. Then $\{m \cdot B: B \in \mathcal{F}\}$ is transversal for $\mathcal{F}$ as for $B \in \mathcal{F}, m \cdot B$ is a minimal element below $B$ by Corollary 2.6.

Let's now see that $4 \Rightarrow 1$. To this end, we fix a foliated tree $T$ and we invoke the Axiom of Choice to define an admissible RRB structure for $T$.

Let $M:=\{x \in T: x$ is minimal $\}$. Define a well order over $M$ of type $\kappa$ for $\kappa$ a suitable ordinal. We can now think of $M$ as $M=\left\{x_{\alpha}: \alpha<\kappa\right\}$. We now proceed to decompose $T$ into disjoint convex chains $\left\{C_{\alpha}: \alpha<\kappa\right\}$. We define the chains $C_{\alpha}$ recursively. First, we take $C_{0}:=x_{0} \uparrow$. Now for $\alpha<\kappa$, we define $C_{\alpha}=x_{\alpha} \uparrow \backslash \bigcup_{\beta<\alpha} C_{\beta}$.
Claim 3. $\bigcup_{\alpha<\kappa} C_{\alpha}=T$.
Proof. $M \subset \bigcup_{\alpha<\kappa} C_{\alpha}$ is trivial Let $y \in T \backslash M$ and $M_{y}:=M \cap y \downarrow$, which is non-empty as the tree is foliated. Now, let $\alpha_{y}:=\min \left\{\beta<\kappa: x_{\beta} \in M_{y}\right\}$. By the construction of $\left\{C_{\alpha}: \alpha<\kappa\right\}$ we have that for every $\beta<\alpha_{y}, y \notin C_{\beta}$. Also $y \in x_{\alpha_{y}} \uparrow$ by hypothesis. Then $y \in x_{\alpha_{y}} \uparrow \backslash\left\{C_{\beta}: \beta<\alpha_{y}\right\}=C_{\alpha_{y}}$.

Let's now define a function $F$ in the following manner: For $y \in M$, $F(y)=y$, and for $y \in T \backslash M, F(y)=x_{\alpha_{y}}$ with $\alpha_{y}$ defined as in the proof of the last claim.

Claim 4. Let $y, z$ such $F(y) \leq z \leq y$, then $F(z)=F(y)$.
Proof. If $z \in M$, then we must have $F(z)=z=F(y)$ and the result holds trivially. Let's check that the claim holds for $z \in T \backslash M$. Note that since $M_{z} \subset M_{y}$, we get $\alpha_{y} \leq \alpha_{z}$. As we also have $x_{\alpha_{y}} \in M_{z}$ by hypothesis, we obtain that $\alpha_{z} \leq \alpha_{y}$. Then $F(z)=x_{\alpha_{z}}=x_{\alpha_{y}}=F(y)$.

Let's now define an RRB structure for $T$ :

$$
x \cdot y= \begin{cases}\min \{x, y\} & \text { if they are comparable } \\ F(y) & \text { otherwise }\end{cases}
$$

Let $x, y, z$ in $T$. Then

$$
(x \cdot y) \cdot z= \begin{cases}\min \{x, y, z\} & (1) \\ F(z) & (2) \vee(4) \\ F(y) & (3)\end{cases}
$$

Where

1. $x$ and $y$ comparable and $z$ and $\min \{x, y\}$ comparable; or equivalently $x, y, z$ mutually comparable (By Proposition 2.24).
2. $x$ and $y$ comparable and $z$ and $\min \{x, y\}$ incomparable. We consider the following subcases:
(a) $x \leq y, z$ and $x$ incomparable.
(b) $y<x, z$ and $y$ incomparable.
3. $y$ incomparable with $x$ and $F(y) \leq z$. We consider the following subcases:
(a) $F(y) \leq z \leq y, x$ and $y$ incomparable.
(b) $F(y) \leq y<z, x$ and $y$ incomparable.

This classification is exhaustive because by $2.24 z$ and $y$ must be comparable.
4. $y$ incomparable with $x$ and $F(y)$ incomparable with $z$. We consider the following subcases:
(a) $z<y$.
(b) $z$ and $y$ incomparable.

There are no more subcases because $F(y)$ is minimal below $y$ and $F(y)$ is incomparable with $z$.

Let's now check the value of $x \cdot(y \cdot z)$.

1. $x \cdot(y \cdot z)=\min \{x, y, z\}=(x \cdot y) \cdot z$.
2. (a) It cannot be $y \leq z$ (as this would imply $x \leq z$ ), then either $z$ and $y$ are incomparable or $z<y$. In the first case $x \cdot(y \cdot z)=x \cdot F(z)=$ $F(z)=x \cdot z=(x \cdot y) \cdot z$. In the second case $x \cdot(y \cdot z)=x \cdot z=(x \cdot y) \cdot z$.
(b) $y$ and $z$ are incomparable. Then $x \cdot(y \cdot z)=x \cdot F(z)=F(z)=$ $y \cdot z=(x \cdot y) \cdot z$.
3. (a) In this case we know by 2.24 that $x$ and $z$ must be incomparable as $x$ and $y$ are so: $x \leq z$ would imply $x \leq y$, as $z \leq x$, together with 2.24 would imply that $x$ and $y$ are comparable. We also have that $F(y)=F(z)$ by Claim 4. Then $x \cdot(y \cdot z)=x \cdot z=$ $F(z)=F(y)=F(y) \cdot z=(x \cdot y) \cdot z$.
(b) $x \cdot(y \cdot z)=x \cdot y=F(y)=F(y) \cdot z=(x \cdot y) \cdot z$.
4. (a) In this case $x, z$ are incomparable. Then $x \cdot(y \cdot z)=x \cdot z=F(z)=$ $(x \cdot y) \cdot z$.
(b) $x \cdot(y \cdot z)=x \cdot F(z)=F(z)=F(y) \cdot z=(x \cdot y) \cdot z$.

Example 2.26. We now present an application of Theorem 2.25. Let $T$ be the tree in (a) of Figure 8. In (b) of Figure 8 we can see a decomposition of $T$ into disjoint convex chains. For the RRB operation induced by this decomposition, we have that $x \cdot y=b$ and $y \cdot x=a$.


Figure 8: An application of Theorem 2.25.

Remark 2.27. If a tree has a minimal element but is not foliated, by Corollary 2.6 it cannot be associative.

Remark 2.28. The previous theorem still holds if we chose $T$ to be a forest of foliated trees instead of just a foliated tree. This result follows from the fact that if we have a forest of foliated trees $T$, then we can consider the poset $T^{\prime}:=T+\{1\}$, where $1 \notin T . T^{\prime}$ is a foliated tree and is therefore associative by the previous theorem. Finally, since $T$ is a decreasing subset of $T^{\prime}$, it is associative; in fact, it is a subalgebra with respect to the operation we defined in the previous theorem.

The last theorem in this section will show that certain homomorphic preimages of foliated trees are associative.

Theorem 2.29. Let $P$ be a poset. Suppose there exist a forest $T$ consisting of foliated trees and a surjective homomorphism $f: P \rightarrow T$ such that:

1. $f(x)<f(y)$ implies $x<y$.
2. For all $a \in f(P), f^{-1}(a)$ is an associative subposet of $P$ and in addition, if a is minimal in $T$, then $f^{-1}(a)$ has a minimum element.

Then, $P$ is an associative poset.
Proof. We begin by establishing two claims which will be necessary for this proof:

Claim 5. If $f(z) \neq f(y)=f(x)$ and $z \leq y$, then $z \leq x$
Proof. $z \leq y \Longrightarrow f(z) \leq f(y)$, as $f(z) \neq f(y), f(z)<f(y)=f(x)$, then $z<x$ by 1 .

Claim 6. For all $x, y \in P$, if the set $\{x, y\}$ has a lower bound, then either $x$ and $y$ are comparable, or $f(x)=f(y)$.

Proof. Let $z$ be such a lower bound. That is $z \leq x$ and $z \leq y$. Then $f(z) \leq f(x)$ and $f(z) \leq f(y)$. As the codomain of $f$ is a forest, $f(x)$ and $f(y)$ are comparable by 2.24. If $f(x) \neq f(y)$, then either $f(x)<f(y)$ or $f(y)<f(x)$. In both of these cases $x$ and $y$ are comparable.

Let's take an element 1 not belonging to $T$. Let $\tilde{F}: T+\{1\} \rightarrow T+\{1\}$ a function defined like the one in 2.25. That is, a function such that $\tilde{F}(a)$ is minimal below $a$ for ever $a$ in $T$, and also, for every minimal $b \in T, \tilde{F}^{-1}(b)$ is a convex chain containing $b$.

Let's define $F: P \rightarrow P$ as

$$
F(x)=\min \left\{f^{-1}(\tilde{F}(f(x)))\right\}
$$

Note that if $f(x)=f(y)$ then $F(x)=F(y)$.
Claim 7. If $F(y) \leq z \leq y$ then $F(y)=F(z)$.

Proof. If $f(y)=f(z)$, the result holds trivially. Otherwise, we have that $f(z)<\tilde{F}(y)$. Note that $f(F(y))=\tilde{F}(f(y)) \leq f(z)<f(y)$ and therefore by $4, \tilde{F}(f(z))=\tilde{F}(f(y))$. This tells us, by definition of $F$, that $F(z)=$ $F(y)$.

Let's take for every $a \in T$, an admissible RRB $\tilde{\sim}_{a}$ for $f^{-1}(a)$. We now define a binary operation $\cdot$ over $P$ in the following manner:

$$
x \cdot y= \begin{cases}\min \{x, y\} & x, y \text { comparable }  \tag{4}\\ x \tilde{\sim}_{a} y & f(x)=f(y)=a \\ F(y) & x, y \text { incomparable and } f(x) \neq f(y)\end{cases}
$$

Note that for all $x, y \in P$ we have $x \cdot F(y)=F(y)$ : if they are comparable, as $F(y)$ is minimal in $P$ we must have $F(y) \leq x$. If $f(x)=f(F(y))=a$, then they are comparable (as $F(y)=\min f^{-1}(f(F(y)))$ and we can apply the previous reasoning. Otherwise, we have that $x \cdot F(y)=F(F(y)))=F(y)$ by definition of $F$ and $\tilde{F}$.

To lighten the notation we will omit the reference to $a$ in $\tilde{r}_{a}$, writing just : instead. The associative law $(x \cdot y) \cdot z=x \cdot(y \cdot z)$ for $x, y, z \in$ $P$ is proved by the following case distinction. We first consider whether $f(x)$ is equal to $f(y)$. Next we consider whether $f(y)=f(z)$. Thirdly, we consider the comparability condition between $x$ and $y$. And lastly, we consider the comparability condition between $y$ and $z$. We will showcase some representative cases.

Consider that $f(x)=f(y) \neq f(z) x \leq y$ and $y, z$ incomparable. Then $x \cdot(y \cdot z)=x \cdot F(z)=F(z)=x \cdot z=(x \cdot y) \cdot z$ because $x$ and $z$ are incomparable. This is because, if $x \leq y$ and $y$ is incomparable with $z$, it cannot be the case that $z \leq x$. It also cannot be $x \leq z$ because that would imply that $y, z$ has a lower bound, which contradicts 6 . Now consider that $f(x) \neq f(y)=f(z) x, y$ incomparable, $y \leq z$. Then $(x \cdot y) \cdot z=F(y) \cdot z=$ $F(y)=F(y \tilde{\sim} z)=x \cdot(y \tilde{\sim} z)=x \cdot(y \cdot z)$ because $F(y) \leq z, F(y)=F(y \tilde{\sim})$, and because $x$ is incomparable with $y \approx z$ (since it cannot be $x \leq y \approx z$ by 5, and it cannot be $y \approx z \leq x$ because that, together with 6 , would imply $z \leq x$ and that would mean $y \leq x$ ).

Example 2.30. We now present an example of an application of Theorem 2.29. The theorem tells us that the poset depicted in (a) of Figure 9 is associative.

## 3 An adjunction

Until now, our focus has been on a specific class of ordered algebraic structures. We've explored the relationship between right-regular bands and associative posets, whose respective categories can be canonically linked by

(a)

(b)

Figure 9: An application of Theorem 2.29.
a forgetful functor. Now we present a more general result: The existence of left adjoints for a broad family of forgetful functors.

We fix a first-order language $L$ with at least an $n$-ary relation symbol $R$.
Definition 3.1. Let $\mathcal{K}$ be a class of $L$-structures. We will say that $R$ is definable by conjunctions of identities over $\mathcal{K}$ if there exist $L$-terms $t_{1}\left(x_{1}, \ldots, x_{n}\right)$, $\ldots, t_{m}\left(x_{1}, \ldots, x_{n}\right), s_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, s_{m}\left(x_{1}, \ldots, x_{n}\right)$ such that
$\mathcal{K} \vDash \forall x_{1}, \ldots, x_{n} R\left(x_{1}, \ldots, x_{n}\right) \Longleftrightarrow \bigwedge_{1 \leq i \leq m} t_{i}\left(x_{1}, \ldots, x_{n}\right)=s_{i}\left(x_{1}, \ldots, x_{n}\right)$.
Definition 3.2. Let $\mathcal{K}$ be a class of $L$-structures. We denote the set of function symbols of $L$ by $L_{\mathrm{alg}}$, and denote the reducts of $\mathcal{K}$ to $L_{\mathrm{alg}}$ and to $\{R\}$ by $\mathcal{K}_{\text {alg }}$ and $\mathcal{K}_{R}$, respectively.

From now on, $\mathcal{K}$ will denote a class of $L$-structures such that $R$ is definable by conjunctions of identities over $\mathcal{K}$. Throughout this section we will study the interplay between the categories $\mathrm{K}_{\text {alg }}$ and $\mathrm{K}_{R}$, which consist of the classes $\mathcal{K}_{\text {alg }}$ and $\mathcal{K}_{R}$ and their homomorphisms, respectively.

There is a canonical forgetful functor $U: \mathrm{K}_{\text {alg }} \rightarrow \mathrm{K}_{R}$ which assigns to every $\mathbf{A} \in \mathcal{K}_{\text {alg }}$ the structure $\mathbf{A}_{R}:=\left(A, R^{\mathbf{A}}\right)$ where

$$
\left\langle a_{1}, \ldots, a_{n}\right\rangle \in R^{\mathbf{A}} \Longleftrightarrow \mathbf{A} \vDash \bigwedge_{1 \leq i \leq m} t_{i}\left(a_{1}, \ldots, a_{n}\right)=s_{i}\left(a_{1}, \ldots, a_{n}\right) ;
$$

and to every $\mathrm{K}_{\text {alg-morphism }} f: \mathbf{A} \rightarrow \mathbf{B}$, the $\mathrm{K}_{R}$-morphism $U f: \mathbf{A}_{R} \rightarrow \mathbf{B}_{R}$ given by $U f(a)=f(a)$ (that is, $U$ preserves morphism as maps).

We will show:
Theorem 3.3. If $\mathcal{K}_{\text {alg }}$ is a variety, the forgetful functor $U: \mathrm{K}_{\text {alg }} \rightarrow \mathrm{K}_{R}$ has a left adjoint $F$.

We will start by defining the functor $F$ on objects.
For each set $X$, let's consider the $\mathcal{K}_{\text {alg }}$-free algebra over $X$, denoted by $\mathbf{F}_{\mathcal{K}}(X)$. We construe $\mathbf{F}_{\mathcal{K}}(X)$ as the quotient of the term algebra $T(X)$,
by the least congruence $\delta_{X}$ such that the quotient belongs to $\mathcal{K}_{\text {alg }}$. For $t \in$ $T(X)$, we will denote by $\llbracket t \rrbracket$ the equivalence class of $t$ under that congruence. In particular, $\{\llbracket x \rrbracket: x \in X\}$ is the set of free generators of $\mathbf{F}_{\mathcal{K}}(X)$.

We know that $\mathbf{F}_{\mathcal{K}}(X)$ has the following universal property: For every $\mathbf{B} \in \mathcal{K}_{\text {alg }}$ and any function $\alpha: X \rightarrow B$, there exists a homomorphism $\beta: \mathbf{F}_{\mathcal{K}}(X) \rightarrow \mathbf{B}$ such that $\alpha(x)=\beta(\llbracket x \rrbracket)$ for every $x \in X$. We will use the following particular application:

Proposition 3.4. Let $X$ and $Y$ be sets and $f: X \rightarrow Y$ a function. Then the map given by $\tilde{f}(\llbracket t(\bar{x}) \rrbracket)=\llbracket t(f(\bar{x})) \rrbracket$ is a morphism $\tilde{f}: \mathbf{F}_{\mathcal{K}}(X) \rightarrow \mathbf{F}_{\mathcal{K}}(Y)$.

Now, for $\mathbf{X} \in \mathcal{K}_{R}$, consider the congruence $\theta_{\mathbf{X}}$ on $\mathbf{F}_{\mathcal{K}}(X)$ generated by

$$
\left\{\left\langle\llbracket t_{i}\left(x_{1}, \ldots, x_{n}\right) \rrbracket, \llbracket s_{i}\left(x_{1}, \ldots x_{n}\right) \rrbracket\right\rangle \mid 1 \leq i \leq m,\left\langle x_{1}, \ldots, x_{n}\right\rangle \in R^{\mathbf{X}}\right\}
$$

and denote the quotient $\mathbf{F}_{\mathcal{K}}(X) / \theta_{\mathbf{X}}$ by $F \mathbf{X}$.
We will also need Grätzer's version of Mal'cev's characterization of compact congruences.

Lemma 3.5. Let A be any algebra and let $c, d \in A, \bar{a}, \bar{b} \in A^{j}$. Then $\langle c, d\rangle \in$ $\mathrm{Cg}^{A}(\bar{a}, \bar{b})$ if and only if there exist $(j+m)$-ary terms $r_{1}(\bar{x}, \bar{u}), \ldots, r_{k}(\bar{x}, \bar{u})$, with $k$ odd, and $\bar{z} \in A^{m}$ such that:

$$
\begin{aligned}
c & =r_{1}(\bar{a}, \bar{z}) & & \\
r_{l}(\bar{b}, \bar{z}) & =r_{l+1}(\bar{b}, \bar{z}) & & \text { l odd }, \\
r_{l}(\bar{a}, \bar{z}) & =r_{l+1}(\bar{a}, \bar{z}) & & \text { l even }, \\
r_{k}(\bar{b}, \bar{z}) & =d . & &
\end{aligned}
$$

Lemma 3.6. Let $\mathbf{X}, \mathbf{Y} \in \mathcal{K}_{R}$ and $f: \mathbf{X} \rightarrow \mathbf{Y}$ an homomorphism. The function $F f: F \mathbf{X} \rightarrow F \mathbf{Y}$ given by $F f\left(\llbracket t(\bar{x}) \rrbracket / \theta_{\mathbf{X}}\right)=\llbracket t(f(\bar{x})) \rrbracket / \theta_{\mathbf{Y}}$ is well defined and is a morphism.
Proof. Let $f$ be as in the assumptions, and let $\tilde{f}: \mathbf{F}_{\mathcal{K}}(X) \rightarrow \mathbf{F}_{\mathcal{K}}(Y)$ be the morphism provided by Proposition 3.4 for $f$. In particular, we have $\tilde{f}(\llbracket x \rrbracket)=\llbracket f(x) \rrbracket$ for every $x \in X$. Now, take $c, d \in \mathbf{F}_{\mathcal{K}}(X)$ (where, $c=\llbracket t(\bar{x}) \rrbracket$ and $d=\llbracket s(\bar{x}) \rrbracket$ for some $s, t \in T(X))$ such that $\langle c, d\rangle \in \theta_{\mathbf{X}}$. If we can show that $\langle\tilde{f}(c), \tilde{f}(d)\rangle \in \theta_{\mathbf{Y}}$, we can then define $F f\left(\llbracket t(\bar{x}) \rrbracket / \theta_{\mathbf{X}}\right):=\tilde{f}(\llbracket t(\bar{x}) \rrbracket) / \theta_{\mathbf{Y}}$, and by how we have chosen $\tilde{f}$, this function will satisfy what we want. Since $\langle c, d\rangle \in \theta_{\mathbf{X}}$ is witnessed by finitely many tuples from the generators of $\theta_{\mathbf{X}}$, namely $\left\langle\llbracket t_{i_{p}}\left(x_{1}^{p}, \ldots, x_{n}^{p}\right) \rrbracket, \llbracket s_{i_{p}}\left(x_{1}^{p}, \ldots, x_{n}^{p}\right) \rrbracket\right\rangle$ for some appropriate $x_{l}^{p} \in X$ (with $l=1, \ldots, n, p=1, \ldots, j$ ), Lemma 3.5 gives us $(j+m)$-ary terms $r_{1}(\bar{x}, \bar{u}), \ldots, r_{k}(\bar{x}, \bar{u})$, with $k$ odd, and $\bar{z} \in \mathbf{F}_{\mathcal{K}}(X)^{m}$ such that:

$$
\begin{align*}
c & =r_{1}(\llbracket \bar{W} \rrbracket, \bar{z}), & &  \tag{5}\\
r_{l}(\llbracket \bar{V} \rrbracket, \bar{z}) & =r_{l+1}(\llbracket \bar{V} \rrbracket, \bar{z}) & & \text { for odd } l,  \tag{6}\\
r_{l}(\llbracket \bar{W} \rrbracket, \bar{z}) & =r_{l+1}(\llbracket \bar{W} \rrbracket, \bar{z}) & & \text { for even } l,  \tag{7}\\
r_{k}(\llbracket \bar{V} \rrbracket, \bar{z}) & =d & & \tag{8}
\end{align*}
$$

where

$$
\begin{aligned}
\overline{\mathrm{x}} & :=x_{1}^{1}, \ldots, x_{n}^{1}, \ldots, x_{1}^{j}, \ldots, x_{n}^{j} \\
\bar{W} & =\bar{W}(\overline{\mathrm{x}}):=t_{i_{1}}\left(x_{1}^{1}, \ldots, x_{n}^{1}\right), \ldots, t_{i_{j}}\left(x_{1}^{j}, \ldots, x_{n}^{j}\right) \\
\bar{V} & =\bar{V}(\overline{\mathrm{x}}):=s_{i_{1}}\left(x_{1}^{1}, \ldots, x_{n}^{1}\right), \ldots, s_{i_{j}}\left(x_{1}^{j}, \ldots, x_{n}^{j}\right),
\end{aligned}
$$

with the following abuses of notation: We take equivalence classes term-wise,

$$
\llbracket \bar{W} \rrbracket=\llbracket t_{i_{1}}\left(x_{1}^{1}, \ldots, x_{n}^{1}\right) \rrbracket, \ldots, \llbracket t_{i_{j}}\left(x_{1}^{j}, \ldots, x_{n}^{j}\right) \rrbracket,
$$

and apply functions in the same way:

$$
\begin{aligned}
h(\bar{W}) & =h\left(t_{i_{1}}\left(x_{1}^{1}, \ldots, x_{n}^{1}\right)\right), \ldots, h\left(t_{i_{j}}\left(x_{1}^{j}, \ldots, x_{n}^{j}\right)\right) \\
\bar{W}(h(\overline{\mathrm{x}})) & =t_{i_{1}}\left(h\left(x_{1}^{1}\right), \ldots, h\left(x_{n}^{1}\right)\right), \ldots, t_{i_{j}}\left(h\left(x_{1}^{j}\right), \ldots, h\left(x_{n}^{j}\right)\right)
\end{aligned}
$$

Using the fact that $\left\langle f\left(x_{1}^{i}\right), \ldots, f\left(x_{n}^{i}\right)\right\rangle \in R^{\mathbf{Y}}$ for every $i$, we will show that the terms obtained witness the fact that $\langle\tilde{f}(c), \tilde{f}(d)\rangle \in \theta_{\mathbf{Y}}$ by transitivity.

$$
\begin{aligned}
\tilde{f}(c) & =\tilde{f}\left(r_{1}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket, \bar{z})\right) & & \text { by }(5) \\
& =r_{1}(\tilde{f}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{1}(\llbracket \bar{W}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) . & & \text { by definition of } \tilde{f}
\end{aligned}
$$

Observe now that $\left\langle r_{1}(\llbracket \bar{W}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})), r_{1}(\llbracket \bar{V}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z}))\right\rangle \in \theta_{\mathbf{Y}}$, where the second term appears for $l=1$ below. In general, for odd $l$,

$$
\begin{aligned}
r_{l}(\llbracket \bar{V}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) & =r_{l}(\bar{V}(\llbracket f(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \delta_{Y} \text { is a congruence } \\
& =r_{l}(\bar{V}(\tilde{f}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \text { by definition of } \tilde{f} \\
& =r_{l}(\tilde{f}(\bar{V}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{l}(\tilde{f}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \delta_{X} \text { is a congruence } \\
& =\tilde{f}\left(r_{l}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket, \bar{z})\right) & & \tilde{f} \text { is a morphism } \\
& =\tilde{f}\left(r_{l+1}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket, \bar{z})\right) & & \text { by }(6) \\
& =r_{l+1}(\tilde{f}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{l+1}(\tilde{f}(\bar{V}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \delta_{X} \text { is a congruence } \\
& =r_{l+1}(\bar{V}(\tilde{f}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{l+1}(\llbracket \bar{V}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) & & \text { by definition of } \tilde{f} .
\end{aligned}
$$

In general, $\left\langle r_{l+1}(\llbracket \bar{W}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})), r_{l+1}(\llbracket \bar{V}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z}))\right\rangle \in \theta_{\mathbf{Y}}$, for $l \leq k-1$. Similarly, for even $l$,

$$
\begin{aligned}
r_{l}(\llbracket \bar{W}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) & =r_{l}(\tilde{f}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \text { by definition of } \tilde{f} \\
& =\tilde{f}\left(r_{l}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket), \bar{z}\right) & & \tilde{f} \text { is a morphism } \\
& =\tilde{f}\left(r_{l+1}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket, \bar{z})\right) & & \text { by }(7) \\
& =r_{l+1}(\tilde{f}(\llbracket \bar{W}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{l+1}(\llbracket \bar{W}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) & & \text { by definition of } \tilde{f} .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
r_{k}(\llbracket \bar{V}(f(\overline{\mathrm{x}})) \rrbracket, \tilde{f}(\bar{z})) & =r_{k}(\bar{V}(\llbracket f(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \delta_{Y} \text { is a congruence } \\
& =r_{k}(\bar{V}(\tilde{f}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \text { by definition of } \tilde{f} \\
& =r_{k}(\tilde{f}(\bar{V}(\llbracket \overline{\mathrm{x}} \rrbracket)), \tilde{f}(\bar{z})) & & \tilde{f} \text { is a morphism } \\
& =r_{k}(\tilde{f}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket), \tilde{f}(\bar{z})) & & \delta_{X} \text { is a congruence } \\
& =\tilde{f}\left(r_{k}(\llbracket \bar{V}(\overline{\mathrm{x}}) \rrbracket, \bar{z})\right) & & \tilde{f} \text { is a morphism } \\
& =\tilde{f}(d) & & \text { by }(8) .
\end{aligned}
$$

Therefore the function $F f$ is well defined. Now take $q$ an $n$-ary operation symbol in $L_{\text {alg }}$. We have that

$$
\begin{aligned}
F f\left(q^{\left.\left.F \mathbf{X}_{\left(\llbracket u_{1} \rrbracket / \theta_{\mathbf{X}}\right.}, \ldots, \llbracket u_{n} \rrbracket / \theta_{\mathbf{X}}\right)\right)}\right. & =\tilde{f}\left(q^{\mathbf{F}_{\mathcal{K}}(X)}\left(\llbracket u_{1} \rrbracket, \ldots, \llbracket u_{n} \rrbracket\right) / \theta_{\mathbf{X}}\right) \\
& =q^{\mathbf{F}_{\mathcal{K}}(Y)}\left(\tilde{f}\left(\llbracket u_{1} \rrbracket\right), \ldots, \tilde{f}\left(\llbracket u_{n} \rrbracket\right)\right) / \theta_{\mathbf{Y}} \\
& =q^{F \mathbf{Y}}\left(\tilde{f}\left(\llbracket u_{1} \rrbracket\right) / \theta_{\mathbf{Y}}, \ldots, \tilde{f}\left(\llbracket u_{n} \rrbracket\right) / \theta_{\mathbf{Y}}\right) \\
& =q^{F \mathbf{Y}}\left(F f\left(\llbracket u_{1} \rrbracket / \theta_{\mathbf{X}}\right), \ldots, F f\left(\llbracket u_{n} \rrbracket / \theta_{\mathbf{X}}\right)\right)
\end{aligned}
$$

so $F f$ is indeed a morphism.
For a structure $\mathbf{X} \in \mathrm{K}_{R}$, we define an inclusion $\eta_{\mathbf{X}}: \mathbf{X} \rightarrow U F \mathbf{X}$ by $\eta_{\mathbf{X}}(a):=\llbracket a \rrbracket / \theta \mathbf{X}$. The previous lemma allows us to prove the following:

Lemma 3.7. For every $\mathbf{X}$ and $\mathbf{Y}$ in $\mathrm{K}_{R}$, and $\mathrm{K}_{R}$-morphism $h: \mathbf{Y} \rightarrow \mathbf{X}$ :

$$
\begin{equation*}
U F h \circ \eta_{\mathbf{Y}}=\eta_{\mathbf{X}} \circ h \tag{9}
\end{equation*}
$$

Proof. Let $y \in Y$. We have: $\left(\eta_{\mathbf{X}} \circ h\right)(y)=\eta_{\mathbf{X}}(h(y))=\llbracket h(y) \rrbracket / \theta_{\mathbf{X}}$. Notice that, by the definition of $F h$, this element is equal to $F h\left(\llbracket y \rrbracket / \theta_{\mathbf{Y}}\right)=(F h \circ$ $\left.\eta_{\mathbf{Y}}\right)(y)$. And since $U$ preserves morphisms as functions, we have that this is equal to $\left(U F h \circ \eta_{\mathbf{Y}}\right)(y)$.

For a structure $\mathbf{X} \in \mathrm{K}_{R}$, we define a congruence $D_{\mathbf{X}}$ on $T(X)$ as follows: $\langle t(\bar{x}), s(\bar{x})\rangle \in D_{\mathbf{X}}$ if and only if $t^{\mathbf{A}}(\bar{x})=s^{\mathbf{A}}(\bar{x})$ for every $\mathbf{A} \in \mathcal{K}_{\text {alg }}$ such that $U \mathbf{A}=\mathbf{X}$. It is clear that $\delta_{X} \subseteq D_{\mathbf{X}}$.

Lemma 3.8. Given a structure $\mathbf{X} \in \mathrm{K}_{R}$, and an algebra $\mathbf{A} \in \mathcal{K}_{F}$ such that $U \mathbf{A}=\mathbf{X}$ there exists a morphism $g: F \mathbf{X} \rightarrow \mathbf{A}$ such that $g\left(\eta_{\mathbf{X}}(x)\right)=x$ for every $x \in X$.

Proof. Note there is a morphism $f: T(X) / D_{\mathbf{X}} \rightarrow \mathbf{A}$ for which $f\left(x / D_{\mathbf{X}}\right)=$ $x$, given by $f\left(t(\bar{x}) / D_{\mathbf{X}}\right):=t^{\mathbf{A}}(\bar{x})$; this is well defined by definition of $D_{\mathbf{X}}$.

We have that $\theta_{\mathbf{X}} \subseteq D_{\mathbf{X}} / \delta_{X}$ : For every $\bar{x} \in X^{n}$ such that $\bar{x} \in R^{\mathbf{X}}$ we have that $\left\langle t_{i}(\bar{x}), s_{i}(\bar{x})\right\rangle \in D_{\mathbf{X}}$ for $1 \leq i \leq m$ and hence

$$
\left\{\left\langle\llbracket t_{i}(\bar{x}) \rrbracket, \llbracket s_{i}(\bar{x}) \rrbracket\right\rangle: 1 \leq i \leq m, \bar{x} \in R^{\mathbf{X}}\right\} \subseteq D_{\mathbf{X}} / \delta_{X}
$$

by definition of $D_{\mathbf{X}} / \delta_{X}$. Since $\theta_{\mathbf{X}}$ is generated by those pairs, we obtain the inclusion.

We now know that there is a projection morphism

$$
\pi:\left(T(X) / \delta_{X}\right) / \theta_{\mathbf{X}} \rightarrow\left(\left(T(X) / \delta_{X}\right) / \theta_{\mathbf{X}}\right) /\left(\left(D_{\mathbf{X}} / \delta_{X}\right) / \theta_{\mathbf{X}}\right)
$$

which satisfies

$$
\pi\left(\llbracket x \rrbracket / \theta_{\mathbf{X}}\right)=\left(\llbracket x \rrbracket / \theta_{\mathbf{X}}\right) /\left(\left(D_{\mathbf{X}} / \delta_{X}\right) / \theta_{\mathbf{X}}\right)
$$

By the Second Isomorphism Theorem, there are isomorphisms

$$
h_{1}:\left(\left(T(X) / \delta_{X}\right) / \theta_{\mathbf{X}}\right) /\left(\left(D_{\mathbf{X}} / \delta_{X}\right) / \theta_{\mathbf{X}}\right) \rightarrow\left(T(X) / \delta_{X}\right) /\left(D_{\mathbf{X}} / \delta_{X}\right)
$$

and

$$
h_{2}:\left(T(X) / \delta_{X}\right) /\left(D_{\mathbf{X}} / \delta_{X}\right) \rightarrow T(X) / D_{\mathbf{X}}
$$

which satisfy

$$
h_{1}\left(\left(\llbracket x \rrbracket / \theta_{\mathbf{X}}\right) /\left(\left(D_{\mathbf{X}} / \delta_{X}\right) / \theta_{\mathbf{X}}\right)\right)=\llbracket x \rrbracket /\left(D_{\mathbf{X}} / \delta_{X}\right)
$$

and

$$
h_{2}\left(\llbracket x \rrbracket /\left(D_{\mathbf{X}} / \delta_{X}\right)\right)=x / D_{\mathbf{X}}
$$

We can now take $g:=f \circ h_{2} \circ h_{1} \circ \pi: F \mathbf{X} \rightarrow A$ which is a morphism satisfying $g\left(\eta_{\mathbf{X}}(x)\right)=x$.

Definition 3.9. We define the map $F: \mathrm{K}_{R} \rightarrow \mathrm{~K}_{\text {alg }}$ which assigns to every structure $\mathbf{X}$ the algebra $F \mathbf{X}$ and to every morphism $f: \mathbf{X} \rightarrow \mathbf{Y}$, the morphism $F f: F \mathbf{X} \rightarrow F \mathbf{Y}$.

Lemma 3.10. $F: \mathrm{K}_{R} \rightarrow \mathrm{~K}_{\text {alg }}$ is a functor.
Proof. Given $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathrm{K}_{R}$, and $f: \mathbf{X} \rightarrow \mathbf{Y}, g: \mathbf{Y} \rightarrow \mathbf{Z} \mathrm{K}_{R}$-morphisms, we show that $F g \circ F f=F(g \circ f)$. Let $x \in F \mathbf{X}$ and assume $x=\llbracket t(\bar{x}) \rrbracket / \theta \mathbf{x}$ for some $t$ and $\bar{x}$. We have

$$
\left.\left.(F g \circ F f)(x)=F g\left(\llbracket t(f(\bar{x})) \rrbracket / \theta_{\mathbf{Y}}\right)=t(g f(\bar{x})) / R_{Z}\right) / \theta_{Z}\right)=F(g \circ f)(x) .
$$

Now let $\mathbf{X} \in \mathcal{K}_{R}$; we show that $F 1_{\mathbf{X}}=1_{F \mathbf{X}}$. Take $x=\llbracket t(\bar{x}) \rrbracket / \theta \mathbf{X} \in F \mathbf{X}$. We have that

$$
F 1_{\mathbf{X}}(x)=\llbracket t\left(1_{\mathbf{X}}(\bar{x})\right) \rrbracket / \theta \mathbf{X}=\llbracket t(\bar{x}) \rrbracket / \theta \mathbf{X}=x
$$

Consider now the map $\varphi_{\mathbf{X}, \mathbf{A}}: \mathrm{K}_{\text {alg }}(F \mathbf{X}, \mathbf{A}) \rightarrow \mathrm{K}_{R}(\mathbf{X}, U \mathbf{A})$ between homsets given by $\varphi_{\mathbf{X}, \mathbf{A}}(f)=U f \circ \eta_{\mathbf{X}}$.

Theorem 3.11. The triple $\langle F, U, \varphi\rangle$ is an adjunction from $\mathrm{K}_{R}$ to $\mathrm{K}_{\mathrm{alg}}$.

Proof. We first prove that $\varphi_{\mathbf{X}, \mathbf{A}}$ is injective. Let $g: F \mathbf{X} \rightarrow \mathbf{A}$ and $h: F \mathbf{X} \rightarrow$ $\mathbf{A}$ be two $\mathcal{K}_{F}$-morphisms such that $\varphi_{\mathbf{X}, \mathbf{A}}(h)=\varphi_{\mathbf{X}, \mathbf{A}}(g)$, i.e.,

$$
U h\left(\eta_{\mathbf{X}}(x)\right)=\left(U h \circ \eta_{\mathbf{X}}\right)(x)=\left(U g \circ \eta_{\mathbf{X}}\right)(x)=U g\left(\eta_{\mathbf{X}}(x)\right)
$$

for all $x \in \mathbf{X}$. From this we deduce that $g$ and $h$ coincide on all elements of $F \mathbf{X}$ of the form $\llbracket x \rrbracket / \theta_{\mathbf{X}}=\eta_{\mathbf{X}}(x)$ : We have that $h\left(\eta_{\mathbf{X}}(x)\right)$ equals $U h\left(\eta_{\mathbf{X}}(x)\right)$ since $U$ preserves morphisms as functions. By hypothesis, $U h\left(\eta_{\mathbf{X}}(x)\right)=$ $U g\left(\eta_{\mathbf{X}}(x)\right)$, the latter being equal to $g\left(\eta_{\mathbf{X}}(x)\right)$ for the same reason. Therefore, $h\left(\eta_{\mathbf{X}}(x)\right)=g\left(\eta_{\mathbf{X}}(x)\right)$ for all $x \in \mathbf{X}$. Now, let $y \in F \mathbf{X}$ be of the form $x=\llbracket t\left(x_{1}, \ldots, x_{k}\right) \rrbracket / \theta_{\mathbf{X}}$. Since $g$ and $h$ are morphisms, we have:

$$
\begin{aligned}
g(y) & =g\left(\llbracket t\left(x_{1}, \ldots, x_{k}\right) \rrbracket / \theta_{\mathbf{X}}\right) \\
& =t^{\mathbf{A}}\left(g\left(\llbracket x_{1} \rrbracket / \theta_{\mathbf{X}}\right), \ldots, g\left(\llbracket x_{k} \rrbracket / \theta_{\mathbf{X}}\right)\right) \\
& =t^{\mathbf{A}}\left(h\left(\llbracket x_{1} \rrbracket / \theta_{\mathbf{X}}\right), \ldots, h\left(\llbracket x_{k} \rrbracket / \theta_{\mathbf{X}}\right)\right) \\
& =h\left(\llbracket t\left(x_{1}, \ldots, x_{k}\right) \rrbracket / \theta_{\mathbf{X}}\right) \\
& =h(y)
\end{aligned}
$$

which means that $g=h$.
Next we show that $\varphi_{\mathbf{X}, \mathbf{A}}$ is surjective. Let $f: \mathbf{X} \rightarrow U \mathbf{A}$ be a $\mathrm{K}_{R^{-}}$ morphism. Take $F f: F \mathbf{X} \rightarrow F U \mathbf{A}$ the $\mathrm{K}_{\text {alg }}$-morphism given by Lemma 3.6, and compose it with the morphism $g: F U \mathbf{A} \rightarrow \mathbf{A}$ given by Lemma 3.8. Now we have $\varphi_{\mathbf{X}, \mathbf{A}}(g \circ F f)(x)=U(g \circ F f) \circ \eta_{\mathbf{X}}(x)$. Since $U$ preserves morphisms as functions, this element is equal to $g\left(F f\left(\eta_{\mathbf{X}}(x)\right)\right)$ which, by the definition of $F f$, is equal to $g\left(\eta_{U \mathbf{A}}(f(x))\right)$, and this last term is equal to $f(x)$ by construction of $g$. Therefore, $f=\varphi_{\mathbf{X}, \mathbf{A}}(g \circ F f)$.

Now let's see that $\varphi$ satisfies the naturality conditions. That is, we want to show that for every pair of objects $\mathbf{X} \in \mathrm{K}_{R}$ and $\mathbf{A} \in \mathrm{K}_{\mathrm{alg}}$, every $\mathrm{K}_{\mathrm{alg}}$-morphism $f: F \mathbf{X} \rightarrow \mathbf{A}$, every $\mathrm{K}_{R}$-morphism $h: \mathbf{Y} \rightarrow \mathbf{X}$, and every $\mathrm{K}_{\mathrm{alg}}-$ morphism $k: \mathbf{A} \rightarrow \mathbf{B}$, it holds that $\varphi_{\mathbf{X}, \mathbf{B}}(k \circ f)=U k \circ \varphi_{\mathbf{X}, \mathbf{A}}(f)$ and $\varphi_{\mathbf{Y}, \mathbf{A}}(f \circ F h)=\varphi_{\mathbf{X}, \mathbf{A}} f \circ h$.

On the one hand, we have $\varphi_{\mathbf{X}, \mathbf{B}}(k \circ f)=U(k \circ f) \circ \eta_{\mathbf{X}}=U k \circ U f \circ \eta_{\mathbf{X}}=$ $U k \circ \varphi_{\mathbf{X}, \mathbf{A}}(f)$, as desired.

On the other hand, we have $\varphi_{\mathbf{Y}, \mathbf{A}}(f \circ F h)=U(f \circ F h) \circ \eta_{\mathbf{Y}}=U f \circ U F h \circ$ $\eta_{\mathbf{Y}}=U f \circ \eta_{\mathbf{X}} \circ h=\varphi_{\mathbf{X}, \mathbf{A}}(f) \circ h$, since $U F h \circ \eta_{\mathbf{Y}}=\eta_{\mathbf{X}} \circ h$ by Equation 9 from Lemma 3.7.

Therefore, the naturality conditions are satisfied. Thus, $\varphi$ is a natural bijection between $\mathrm{K}_{\mathrm{alg}}(F \mathbf{X}, \mathbf{A})$ and $\mathrm{K}_{R}(\mathbf{X}, U \mathbf{A})$, and hence the triple $\langle F, U, \varphi\rangle$ forms an adjunction.

Corollary 3.12. Let RRB and AP be the categories of $R R B$ s with semigroup homomorphisms and associative posets with non decreasing maps, respectively. Then the forgetful functor $U: \mathrm{RRB} \rightarrow \mathrm{AP}$ admits a left adjoint.

Example 3.13. In Figure 10 we can see an example of the partial order underlying $F \mathbf{P}$ for a simple poset $\mathbf{P}=(P, \leq)$. The product of two distinct maximal elements in $U F \mathbf{P}$ is one of the two elements in the second level.


Figure 10

## 4 Factor congruences of right-regular bands

In the following, we provide an application of the order-theoretic perspective on bands to the study of direct product decompositions of RRBs with a central element. This generalizes the results from [5].

One key idea from that paper (which studies join-semilattices) was that existing binary infima in finite direct products of join-semilattices must factorize and satisfy some distributive and absorption properties with respect to the join. Since the $(\leq, \cdot)$-posemigroups associated to RRBs correspond to meet-semilattices, all concepts here will be dual those in [5]; in particular, we will be speaking of partial binary suprema. We have the analogous
Lemma 4.1. For any pair of $R R B s C, D$ and elements $c, e \in C$ and $d, f \in$ $D$, the element $\langle c, d\rangle \vee\langle e, f\rangle$ exists in $C \times D$ if and only if $c \vee e$ and $d \vee f$ exist.

The main difference between this section and [5] is that, unlike $\vee$ nor $\wedge$, the RRB operation • is not a commutative in general. We do assume that at least one element commutes with every other:
Definition 4.2. Given an $\operatorname{RRB}(A, \cdot)$, an element $c \in A$ is said to be central if for all $x \in A$, we have $x \cdot c=c \cdot x$.

We will highlight modifications to the proofs in the sequel.

### 4.1 Decomposition into direct product of RRBs

For the remainder of this chapter, we will write formulas in the language $\{\cdot, \vee\}$; occurrences of $\leq$ can be reduced using (2). The formula " $x \vee y=z$ " will be interpreted as " $z$ is the supremum of $\{x, y\}$ ":

$$
x, y \leq z \&(\forall u: x, y \leq u \rightarrow z \leq u)
$$

unless otherwise specified; in general, every equation $t_{1}=t_{2}$ involving $\vee$ should be interpreted as "if one of the terms exists, so does the other and they are equal." It is easy to check that the laws of associativity hold for the partial operation $\vee$ in every poset.

From now on, let $A$ be an RRB and $c \in A$ be a fixed central element. Let $\varphi\left(c, x_{1}, x_{2}, x\right)$ be the conjunction of the following formulas:

```
\(\operatorname{comm} x \cdot x_{1}=x_{1} \cdot x\) and \(x \cdot x_{2}=x_{2} \cdot x\).
    dist \(x=\left(x \cdot x_{1}\right) \vee\left(x \cdot x_{2}\right)\).
    \(\mathbf{p}_{1} x_{1}=\left(x \cdot x_{1}\right) \vee\left(c \cdot x_{1}\right)\).
    \(\mathbf{p}_{2} x_{2}=\left(x \cdot x_{2}\right) \vee\left(c \cdot x_{2}\right)\).
\(\operatorname{prod} x_{1} \cdot x_{2}=x \cdot c\).
```

Note that comm is the only formula that does not appear in the formula $\varphi\left(c, x_{1}, x_{2}, x\right)$ defined in [5]. Note also that, due to the assumption that $c$ is central and comm, Lemma 2.3(2) guarantees that all products appearing in $\varphi$ except for the last one are actually infima. We will write " $x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle_{c}$ " to denote that $\varphi\left(c, x_{1}, x_{2}, x\right)$ holds.

Definition 4.3. Suppose that $I_{1}, I_{2}$ are subsemigroups of $A$. We will say that $A$ is the $c$-direct product of $I_{1}$ and $I_{2}$, and write $A=I_{1} \times_{c} I_{2}$, if and only if the following conditions hold:

Perm The elements of $I_{1}$ commute with those of $I_{2}$.
Mod1 For all $x, y \in A, x_{1} \in I_{1}$, and $x_{2} \in I_{2}$, if $x \cdot c \leq x_{1} \cdot x_{2}$, then

$$
\begin{aligned}
& \left(\left(x \cdot x_{1}\right) \vee\left(x \cdot x_{2}\right)\right) \cdot y=\left(x \cdot x_{1} \cdot y\right) \vee\left(x \cdot x_{2} \cdot y\right), \\
& y \cdot\left(\left(x \cdot x_{1}\right) \vee\left(x \cdot x_{2}\right)\right)=\left(y \cdot x \cdot x_{1}\right) \vee\left(y \cdot x \cdot x_{2}\right) .
\end{aligned}
$$

Mod2 For all $x, y \in A, x_{1} \in I_{1}$, and $x_{2} \in I_{2}$, if $x \geq x_{1} \cdot x_{2}$, then

$$
\begin{aligned}
& \left(\left(x \cdot x_{i}\right) \vee\left(c \cdot x_{i}\right)\right) \cdot y=\left(x \cdot x_{i} \cdot y\right) \vee\left(c \cdot x_{i} \cdot y\right) \\
& y \cdot\left(\left(x \cdot x_{i}\right) \vee\left(c \cdot x_{i}\right)\right)=\left(y \cdot x \cdot x_{i}\right) \vee\left(y \cdot c \cdot x_{i}\right)
\end{aligned}
$$

for $i=1,2$.
Abs For all $x_{1}, y_{1} \in I_{1}$, and $z_{2} \in I_{2}$, we have: $x_{1} \vee\left(y_{1} \cdot z_{2}\right)=$ $x_{1} \vee\left(y_{1} \cdot c\right)$ (and exchanging the roles of $I_{1}$ and $I_{2}$ ).
Exi $\forall x_{1} \in I_{1}, x_{2} \in I_{2} \exists x \in A: x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle_{c}$.
Onto $\forall x \in A \exists x_{1} \in I_{1}, x_{2} \in I_{2}: x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle_{c}$.
Note that these are "two-sided" versions of the conditions in [5], plus the commutativities Perm.

For the sake of brevity, we will omit the reference to $c$, writing $x=$ $\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle$ instead of $x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle_{c}$. Note that Exi implies:

Ori $\forall x_{1} \in I_{1}, x_{2} \in I_{2}\left(x_{1} \cdot x_{2} \leq c\right)$.

### 4.2 The representation lemma

Lemma 4.4. Suppose $A=I_{1} \times_{c} I_{2}$. Then $x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle$ defines an isomorphism $\left\langle x_{1}, x_{2}\right\rangle \stackrel{\varphi}{\mapsto} x$ between $I_{1} \times I_{2}$ and $A$.

Proof. The proof that the function $\left\langle x_{1}, x_{2}\right\rangle \stackrel{\varphi}{\mapsto} x$ is well-defined is very similar to that in [5], but it involves essential uses of Perm and comm. It is left as an interesting exercise for the reader.

The function defined by $\varphi$ is surjective by Onto; to see it is injective, let $x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle$ and $x=\left\langle\left\langle y_{1}, y_{2}\right\rangle\right\rangle$. We have:

$$
\begin{aligned}
x_{1} & =x_{1} \vee\left(x_{1} \cdot x_{2}\right) & & \\
& =x_{1} \vee(x \cdot c) & & \text { by prod } \\
& =x_{1} \vee\left(y_{1} \cdot y_{2}\right) & & \text { by prod again } \\
& =x_{1} \vee\left(y_{1} \cdot c\right) & & \text { by Abs }
\end{aligned}
$$

and then $x_{1} \geq y_{1} \cdot c$. Similarly, $y_{1} \geq x_{1} \cdot c$ and in conclusion, by Lemma 2.3(1),

$$
\begin{equation*}
x_{1} \cdot c=y_{1} \cdot c . \tag{10}
\end{equation*}
$$

On the other hand,

$$
\begin{aligned}
x \cdot y_{1} & =y_{1} \cdot x & & \text { by comm } \\
& =y_{1} \cdot\left(\left(x \cdot x_{1}\right) \vee\left(x \cdot x_{2}\right)\right) & & \text { by dist } \\
& =\left(y_{1} \cdot x \cdot x_{1}\right) \vee\left(y_{1} \cdot x \cdot x_{2}\right) & & \text { by Mod1 } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee\left(x \cdot y_{1} \cdot x_{2}\right) & & \text { by comm } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee\left(x \cdot y_{1} \cdot x_{2} \cdot c\right) & & \text { by Ori } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee(x \cdot c) & & c \text { central and } x \cdot c \leq x_{1}, y_{1}, x_{2}, y_{2} \\
& =x \cdot y_{1} \cdot x_{1}, & & \text { by the same argument. }
\end{aligned}
$$

and also

$$
\begin{aligned}
x \cdot x_{1} & =\left(\left(x \cdot y_{1}\right) \vee\left(x \cdot y_{2}\right)\right) \cdot x_{1} & & \text { by dist } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee\left(x \cdot y_{2} \cdot x_{1}\right) & & \text { by Mod1 } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee\left(x \cdot x_{1} \cdot y_{2}\right) & & \text { by Perm } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee\left(x \cdot x_{1} \cdot y_{2} \cdot c\right) & & \text { by Ori } \\
& =\left(x \cdot y_{1} \cdot x_{1}\right) \vee(x \cdot c) & & c \text { central and } x \cdot c \leq x_{1}, y_{1}, x_{2}, y_{2} \\
& =x \cdot y_{1} \cdot x_{1} . & & \text { by the same argument. }
\end{aligned}
$$

Thus, we obtain $x \cdot x_{1}=x \cdot y_{1}$. (Here, we must use the fact that $x \cdot y_{1}=y_{1} \cdot x$ because otherwise, it is not possible to prove that $x \cdot y_{1}=x \cdot x_{1}$ ). Combining this with (10) and using $\mathbf{p}_{\mathbf{1}}$, we have

$$
x_{1}=\left(x \cdot x_{1}\right) \vee\left(c \cdot x_{1}\right)=\left(x \cdot y_{1}\right) \vee\left(c \cdot y_{1}\right)=y_{1} .
$$

By the same reasoning, $x_{2}=y_{2}$. The preceding part requires a more extensive development than the proof in [5].

Next we prove that $\varphi$ preserves •. Suppose $x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle$ and $z=$ $\left\langle\left\langle z_{1}, z_{2}\right\rangle\right\rangle$; since each $I_{1}, I_{2}$ is a subsemigroup, we know that $x_{j} \cdot z_{j} \in I_{j}$ for $j=1,2$. We want to show $x \cdot z=\left\langle\left\langle x_{1} \cdot z_{1}, x_{2} \cdot z_{2}\right\rangle\right\rangle$. The property prod is immediate (and therefore, we can apply Mod1 and Mod2 to $x \cdot z$ ). Now we prove dist:

$$
\begin{aligned}
x \cdot z & =\left(\left(x \cdot x_{1}\right) \vee\left(x \cdot x_{2}\right)\right) \cdot z & & \text { by dist for } x \\
& =\left(x \cdot x_{1} \cdot z\right) \vee\left(x \cdot x_{2} \cdot z\right) . & & \text { by Mod1 }
\end{aligned}
$$

This last term is equal to

$$
\begin{equation*}
\left(x \cdot x_{1} \cdot z \cdot z_{1}\right) \vee\left(x \cdot x_{1} \cdot z \cdot z_{2}\right) \vee\left(x \cdot x_{2} \cdot z \cdot z_{1}\right) \vee\left(x \cdot x_{2} \cdot z \cdot z_{2}\right), \tag{11}
\end{equation*}
$$

by dist for $z$ and Mod1. Note that, for any $y$,

$$
\begin{aligned}
\left(y \cdot z \cdot z_{1}\right) & =y \cdot z \cdot\left(z \cdot z_{1} \vee c \cdot z_{1}\right) & & \text { by } \mathbf{p}_{1} \text { for } z_{1} \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(y \cdot z \cdot c \cdot z_{1}\right) & & \text { by Mod1 for } z \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(y \cdot z_{1} \cdot z_{2} \cdot z_{1}\right) & & \text { by prod for } z \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(y \cdot z_{2} \cdot z_{1}\right) & & \text { by Lemma } 2.1 \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(z_{2} \cdot y \cdot z_{1}\right) & & \text { by Perm } \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(z_{1} \cdot z_{2} \cdot y \cdot z_{1}\right) & & \text { by Lemma } 2.1 \text { again } \\
& =\left(y \cdot z \cdot z_{1}\right) \vee\left(z \cdot c \cdot y \cdot z_{1}\right) & & \text { by prod for } z \\
& =\left(z \cdot y \cdot z \cdot z_{1}\right) \vee\left(z \cdot y \cdot c \cdot z_{1}\right) & & \text { by Lemma } 2.1 \text { and because } c \text { is central } \\
& =z \cdot y \cdot\left(z \cdot z_{1} \vee c \cdot z_{1}\right) & & \text { by Mod1 for } z \\
& =z \cdot y \cdot z_{1} . & &
\end{aligned}
$$

This development is necessary in order to permute $x_{1}$ and $x_{2}$ with $z$ in the first and fourth term of (11) respectively, as we cannot guarantee that • commutes over these elements. This equality will be used multiple times in this proof, always for the same reason. Similarly, $z \cdot y \cdot z_{2}=y \cdot z \cdot z_{2}$.

We can now rewrite the term (11) as follows:

$$
\left(x \cdot z \cdot x_{1} \cdot z_{1}\right) \vee\left(x \cdot x_{1} \cdot z \cdot z_{2}\right) \vee\left(x \cdot x_{2} \cdot z \cdot z_{1}\right) \vee\left(x \cdot z \cdot x_{2} \cdot z_{2}\right)
$$

Note that

$$
\begin{aligned}
x \cdot x_{1} \cdot z \cdot z_{2} & =x \cdot x_{1} \cdot z_{2} \cdot z & & \text { by comm } \\
& =x \cdot x_{1} \cdot z_{2} \cdot c \cdot z & & \text { by Ori } \\
& =x \cdot c \cdot x_{1} \cdot z_{2} \cdot z_{1} \cdot z_{2} & & \text { by Lema } 2.1 \text { and } c \text { is central } \\
& =x_{1} \cdot x_{2} \cdot x_{1} \cdot z_{2} \cdot z_{1} \cdot z_{2} & & \text { by prod } \\
& =x_{2} \cdot x_{1} \cdot z_{1} \cdot z_{2} & & \\
& =x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2} . & & \text { by Perm }
\end{aligned}
$$

Similarly, $x \cdot x_{2} \cdot z \cdot z_{1}=x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2}$
Let's see that $\left(x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2}\right) \leq\left(x \cdot x_{1} \cdot z \cdot z_{1}\right)$ :

$$
\begin{array}{rlrl}
\left(x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2}\right) \cdot\left(x \cdot z \cdot x_{1} \cdot z_{1}\right) & =x_{2} \cdot z_{2} \cdot x \cdot x_{1} \cdot z \cdot z_{1} & & \\
& =x_{1} \cdot x_{2} \cdot z_{2} \cdot x \cdot z \cdot z_{1} & & \text { by comm and Perm } \\
& =x \cdot c \cdot z_{2} \cdot x \cdot z \cdot z_{1} & & \text { by prod for } x \\
& =z_{2} \cdot x \cdot z \cdot c \cdot z_{1} & & \\
& =z_{2} \cdot x \cdot c \cdot z \cdot c \cdot z_{1} & \\
& =z_{2} \cdot x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2} \cdot z_{1} & & \text { by prod for } z \\
& =x_{1} \cdot x_{2} \cdot z_{2} \cdot z_{1} & & \\
& =x_{1} \cdot x_{2} \cdot z_{1} \cdot z_{2} . &
\end{array}
$$

Therefore, we can eliminate the two middle terms in Equation (11) and obtain dist for $x \cdot z$ :

$$
(x \cdot z)=\left((x \cdot z) \cdot\left(x_{1} \cdot z_{1}\right)\right) \vee\left((x \cdot z) \cdot\left(x_{2} \cdot z_{2}\right)\right)
$$

Here, the method for eliminating the two terms also differs from what was done in [5]. We can obtain $\mathbf{p}_{\mathbf{1}}$ and $\mathbf{p}_{\mathbf{2}}$ in a similar way. We prove $\mathbf{p}_{\mathbf{1}}$ :

$$
\begin{aligned}
x_{1} \cdot z_{1} & =\left(\left(x \cdot x_{1}\right) \vee\left(c \cdot x_{1}\right)\right) \cdot z_{1} & & \text { by } \mathbf{p}_{1} \text { for } x \\
& =\left(x \cdot x_{1} \cdot z_{1}\right) \vee\left(c \cdot x_{1} \cdot z_{1}\right) . & & \text { by Mod2 }
\end{aligned}
$$

By $\mathbf{p}_{1}$ for $z$ followed by Mod2 on each term of the supremum, the last term is equal to

$$
\left(x \cdot x_{1} \cdot z \cdot z_{1}\right) \vee\left(x \cdot x_{1} \cdot c \cdot z_{1}\right) \vee\left(c \cdot x_{1} \cdot z \cdot z_{1}\right) \vee\left(x_{1} \cdot c \cdot z_{1}\right)
$$

Due to the observed fact (that $z \cdot x_{1} \cdot z_{1}=x_{1} \cdot z \cdot z_{1}$ ) and the fact that $c$ is central, we can rewrite this term as

$$
\begin{equation*}
\left(x \cdot z \cdot x_{1} \cdot z_{1}\right) \vee\left(x \cdot c \cdot x_{1} \cdot z_{1}\right) \vee\left(x_{1} \cdot z \cdot c \cdot z_{1}\right) \vee\left(c \cdot x_{1} \cdot z_{1}\right) \tag{12}
\end{equation*}
$$

Note that $\left(x \cdot c \cdot x_{1} \cdot z_{1}\right)=x \cdot\left(c \cdot x_{1} \cdot z_{1}\right) \leq\left(c \cdot x_{1} \cdot z_{1}\right)$, so we can eliminate the second term of the supremum. To eliminate the third term, we can rewrite it as:

$$
\begin{aligned}
\left(x_{1} \cdot z \cdot c \cdot z_{1}\right) & =\left(x_{1} \cdot z_{1} \cdot z_{2} \cdot z_{1}\right) & & \text { By prod for } z \\
& =\left(x_{1} \cdot z_{2} \cdot z_{1}\right) & & \\
& =\left(x_{1} \cdot z \cdot c\right) & & \text { by prod for } z \\
& =\left(c \cdot x_{1} \cdot z\right) . & & c \text { is central }
\end{aligned}
$$

Let's see that $\left(c \cdot x_{1} \cdot z\right) \leq\left(c \cdot x_{1} \cdot z_{1}\right)$ :

$$
\begin{aligned}
\left(c \cdot x_{1} \cdot z\right) \cdot\left(c \cdot x_{1} \cdot z_{1}\right) & =z \cdot c \cdot x_{1} \cdot z_{1} & & \\
& =z_{1} \cdot z_{2} \cdot x_{1} \cdot z_{1} & & \text { by prod for } z \\
& =x_{1} \cdot z_{1} \cdot z_{2} & & \text { by Perm } \\
& =x_{1} \cdot z \cdot c & & \text { by prod for } z \\
& =c \cdot x_{1} \cdot z . & & c \text { is central }
\end{aligned}
$$

Then Equation (15) becomes:

$$
\left(x \cdot z \cdot x_{1} \cdot z_{1}\right) \vee\left(c \cdot x_{1} \cdot z_{1}\right)
$$

and this is equal to

$$
\left((x \cdot z) \cdot\left(x_{1} \cdot z_{1}\right)\right) \vee\left(c \cdot\left(x_{1} \cdot z_{1}\right)\right)
$$

### 4.3 The factorization theorem

We begin by introducing some notation. Recall that ker $f$ (cfr. Claim 1) is always a congruence when $f: A \rightarrow B$ is a homomorphism. We say that two congruences $\theta, \delta \in \operatorname{Con}(A)$ are complementary factor congruences if $\theta \cap \delta=\operatorname{Id}_{A}$ and their relational compositions $\theta \circ \delta$ and $\delta \circ \theta$ equal $A \times$ $A$. Complementary factor congruences are in 1-1 correspondence to direct product decompositions.

As $\langle\langle\cdot, \cdot\rangle\rangle$ defines an isomorphism, there exist canonical projections $\pi_{j}$ : $A \rightarrow I_{j}$ with $j=1,2$ such that:

$$
\begin{equation*}
\forall x \in A, x_{1} \in I_{1}, x_{2} \in I_{2}: x=\left\langle\left\langle x_{1}, x_{2}\right\rangle\right\rangle \Longleftrightarrow \pi_{1}(x)=x_{1} \text { and } \pi_{2}(x)=x_{2} . \tag{13}
\end{equation*}
$$

Let's define, for any congruence $\theta$ on $A$, the set $I_{\theta}:=\{a \in A: a \theta c\}$.
Theorem 4.5. Let $A$ be an $R R B$ and $c \in A$ be a central element. The mappings

$$
\begin{array}{ccc}
\langle\theta, \delta\rangle & \stackrel{\text { I }}{\longleftrightarrow} & \left\langle I_{\theta}, I_{\delta}\right\rangle \\
\left\langle\operatorname{ker} \pi_{2}, \operatorname{ker} \pi_{1}\right\rangle & \stackrel{\mathrm{K}}{\longleftrightarrow} & \left\langle I_{1}, I_{2}\right\rangle
\end{array}
$$

are mutually inverse maps defined between pairs of complementary factor congruences of $A$ and the set of pairs of subsemigroups $I_{1}, I_{2}$ of $A$ such that $A=I_{1} \times{ }_{c} I_{2}$.

Proof. The only part of this proof that differs in a nontrivial way from [5] is the verification that Mod1 and Mod2 hold

The mapping $a \mapsto\langle a / \theta, a / \delta\rangle$ is an isomorphism between $A$ and $A / \theta \times$ $A / \delta$. Under this isomorphism, $I_{\theta}$ corresponds to $\left\{\left\langle c^{\prime}, a^{\prime \prime}\right\rangle: a^{\prime \prime} \in A / \delta\right\}$ and $I_{\delta}$ corresponds to $\left\{\left\langle a^{\prime}, c^{\prime \prime}\right\rangle: a^{\prime} \in A / \theta\right\}$, where $c^{\prime}=c / \theta$ and $c^{\prime \prime}=c / \delta$. From now on, we will identify $I_{\theta}$ and $I_{\delta}$ with their respective isomorphic images
and verify the axioms for $I_{\theta} \times{ }_{c} I_{\delta}=A$ in $A / \theta \times A / \delta$. Note that since $c$ is central, $c^{\prime}$ and $c^{\prime \prime}$ are also central.

To verify Mod1, suppose $x=\left\langle x^{\prime}, x^{\prime \prime}\right\rangle, y=\left\langle y^{\prime}, y^{\prime \prime}\right\rangle, x_{1}=\left\langle c^{\prime}, x_{1}^{\prime \prime}\right\rangle \in I_{\theta}$, and $x_{2}=\left\langle x_{2}^{\prime}, c^{\prime \prime}\right\rangle \in I_{\delta}$. Notice that $x \cdot c \leq x_{1} \cdot x_{2}$ implies

$$
\begin{equation*}
x^{\prime} \cdot c^{\prime} \leq c^{\prime} \cdot x_{2}^{\prime} \quad x^{\prime \prime} \cdot c^{\prime \prime} \leq x_{1}^{\prime \prime} \cdot c^{\prime \prime} \tag{14}
\end{equation*}
$$

From the first inequality in (14), we obtain

$$
\begin{array}{rlr}
\left(x^{\prime} \cdot c^{\prime}\right) \cdot\left(x^{\prime} \cdot x_{2}^{\prime}\right) & =x^{\prime} \cdot x_{2}^{\prime} \cdot c^{\prime} & c^{\prime} \text { is central } \\
& =x^{\prime} \cdot c^{\prime} \cdot x_{2} \cdot c^{\prime} \\
& =\left(x^{\prime} \cdot c^{\prime}\right) \cdot\left(x_{2}^{\prime} \cdot c^{\prime}\right) \\
& =x^{\prime} \cdot c^{\prime}
\end{array}
$$

In other words, $x^{\prime} \cdot c^{\prime} \leq x^{\prime} \cdot x_{2}^{\prime}$.
Similarly, from the second inequality in (14), we obtain $x^{\prime \prime} \cdot c^{\prime \prime} \leq x^{\prime \prime} \cdot x_{1}^{\prime \prime}$.
That is:

$$
\begin{equation*}
x^{\prime} \cdot c^{\prime} \leq x^{\prime} \cdot x_{2}^{\prime} \quad x^{\prime \prime} \cdot c^{\prime \prime} \leq x^{\prime \prime} \cdot x_{1}^{\prime \prime} \tag{15}
\end{equation*}
$$

and therefore, we have $x^{\prime} \cdot c^{\prime} \cdot y^{\prime} \leq x^{\prime} \cdot x_{2}^{\prime} \cdot y^{\prime}$ and $x^{\prime \prime} \cdot c^{\prime \prime} \cdot y^{\prime \prime} \leq x^{\prime \prime} \cdot x_{1}^{\prime \prime} \cdot y^{\prime \prime}$. Applying Lemma 4.1, we obtain:

$$
\begin{aligned}
\left(x \cdot x_{1} \vee x \cdot x_{2}\right) \cdot y & =\left\langle\left(x^{\prime} \cdot c^{\prime} \vee x^{\prime} \cdot x_{2}^{\prime}\right) \cdot y^{\prime},\left(x^{\prime \prime} \cdot x_{1}^{\prime \prime} \vee x^{\prime \prime} \cdot c^{\prime \prime}\right) \cdot y^{\prime \prime}\right\rangle \\
& =\left\langle x^{\prime} \cdot x_{2}^{\prime} \cdot y^{\prime}, x^{\prime \prime} \cdot x_{1}^{\prime \prime} \cdot y^{\prime \prime}\right\rangle \\
& =\left\langle x^{\prime} \cdot c^{\prime} \cdot y^{\prime} \vee x^{\prime} \cdot x_{2}^{\prime} \cdot y^{\prime}, x^{\prime \prime} \cdot x_{1}^{\prime \prime} \cdot y^{\prime \prime} \vee x^{\prime \prime} \cdot c^{\prime \prime} \cdot y^{\prime \prime}\right\rangle \\
& =x \cdot x_{1} \cdot y \vee x \cdot x_{2} \cdot y
\end{aligned}
$$

Now, to distribute to the left, the equations (14) also imply

$$
\begin{aligned}
y^{\prime} \cdot x^{\prime} \cdot c^{\prime} & =y^{\prime} \cdot x^{\prime} \cdot c^{\prime} \cdot x_{2}^{\prime} \cdot c \\
& =y^{\prime} \cdot x^{\prime} \cdot x_{2}^{\prime} \cdot c \\
& =c \cdot y^{\prime} \cdot x^{\prime} \cdot x_{2}^{\prime}
\end{aligned}
$$

This gives us $y^{\prime} \cdot x^{\prime} \cdot c^{\prime} \leq y^{\prime} \cdot x^{\prime} \cdot x_{2}^{\prime}$ and, analogously, $y^{\prime \prime} \cdot x^{\prime \prime} \cdot c^{\prime} \leq y^{\prime \prime} \cdot x^{\prime \prime} \cdot x_{1}^{\prime \prime}$. Then, together with (15), we have

$$
\begin{aligned}
y \cdot\left(x \cdot x_{1} \vee x \cdot x_{2}\right) & =\left\langle y^{\prime} \cdot\left(x^{\prime} \cdot c^{\prime} \vee x^{\prime} \cdot x_{2}^{\prime}\right), y^{\prime \prime} \cdot\left(x^{\prime \prime} \cdot x_{1}^{\prime \prime} \vee x^{\prime \prime} \cdot c^{\prime \prime}\right)\right\rangle \\
& =\left\langle y^{\prime} \cdot x^{\prime} \cdot x_{2}^{\prime}, y^{\prime \prime} \cdot x^{\prime \prime} \cdot x_{1}^{\prime \prime}\right\rangle \\
& =\left\langle y^{\prime} \cdot x^{\prime} \cdot c^{\prime} \vee y^{\prime} \cdot x^{\prime} \cdot x_{2}^{\prime}, y^{\prime \prime} \cdot x^{\prime \prime} \cdot x_{1}^{\prime \prime} \vee y^{\prime \prime} \cdot x^{\prime \prime} \cdot c^{\prime \prime}\right\rangle \\
& =y \cdot x \cdot x_{1} \vee y \cdot x \cdot x_{2}
\end{aligned}
$$

We leave Mod2 to the reader.

As in [5], the characterization of direct representations assume a simpler forms when $c$ is an endpoint of the poset. For example,

Theorem 4.6. Let $A$ be an RRB with identity 1. Then $A=I_{1} \times I_{2}$ if and only if $I_{1}, I_{2} \leq A$ satisfy:

Perm The elements of $I_{1}$ commute with those of $I_{2}$.
Abs For all $x_{1}, y_{1} \in I_{1}$ and $z_{2} \in I_{2}$, we have: $x_{1} \vee\left(y_{1} \cdot z_{2}\right)=$ $x_{1} \vee y_{1}$ (and interchanging $I_{1}$ and $I_{2}$ ).
Onto $I_{1} \cdot I_{2}=A$.
Moreover, $I_{1}$ and $I_{2}$ are filters of $A$.

## 5 Conclusion

We have presented many examples which showcase the usefulness of an order theoretical point of view for studying bands. We proved that we can define, for certain posets, a band operation by invoking some of its structural properties, such as the existence of a special order-preserving function for a normal poset, or a decomposition into disjoint convex subchains with minimum for a foliated tree. More so, we showed that in the first case, this is the only way of defining a right-normal band operation over a normal poset. Here, natural definability questions arise. There appears to be no way of canonically assigning a right-normal band operation to every normal poset. We can then ask ourselves if being able to define this assignment is equivalent to the Axiom of Choice or some fragment of it.

We proved the equivalence of the associativity of foliated trees and the Axiom of Choice. This, together with the observations regarding rightnormal bands, shows that assigning a (right-regular or right-normal) band operation to an associative poset is highly non-canonical. We believe that a broader family of trees might be proven to be associative. It appears to be that for associative trees there must exist, perhaps as some sort of limit, some order type that "occurs densely". That is, a type order which appears in arbitrary low levels of the tree. Therefore, this leads us to believe that every tree admiting a decomposition into disjoint convex subchains with isomorphic initial segments might be associative.

The following is a list of problems/questions which remain open:
Question 5.1. Is every associative disjoint union of meet-semilattices a normal poset?

We know that every disjoint union of meet-semilattices with minimum is normal, so if the answer to this question is negative, at least one of the meet-semilattices in our counterexample must be infinite.

Question 5.2. Is there a family of posets whose "normality" is equivalent to the Axiom of Choice?

A result of this sort would be an analogue of Theorem 2.25
Question 5.3. Is every disjoint union of finite associative posets associative?

We believe the answer to this question to be negative. From Theorem 2.29 we know that every disjoint union of associative posets with minimum is associative, so if a counterexample exists, at leats one of the associative posets must not have a minimum.

Question 5.4. Can Theorem 2.25 be extended to prove associativity of a more general class of posets/trees?

Question 5.5. Is there a more suitable notion of morphism of associative posets which yields a more interesting category?

Acknowledgment : We thank Dr. Miguel Campercholi for suggesting that we study the existence of a left adjoint for the forgetful functor between the categories of RRB and associative poset.

## A Additional proofs

## A. 1 The crown is not associative

In the next lemmas, we assume that there is an admissible RRB structure for the crown poset. By minimality we obtain:

Lemma A.1. 1. $3 \cdot 2 \in\{4,5\}$.
2. $4 \cdot 1 \in\{3,5\}$.
3. $5 \cdot 0 \in\{3,4\}$.

So far, we have the situation pictured in Table 1.
Lemma A.2. 1. $5 \cdot 0=3$ if and only if $3 \cdot 2=5$.
2. $3 \cdot 2=4$ if and only if $4 \cdot 1=3$.
3. $5 \cdot 0=4$ if and only if $4 \cdot 1=5$.

Proof. For the first item, assume $5 \cdot 0=3$. If $3 \cdot 2 \neq 5$, by Lemma A.1, $3 \cdot 2=4$. We obtain, using Lemma 2.1(2),

$$
4=4 \cdot 0=3 \cdot 2 \cdot 0=5 \cdot \underline{0 \cdot 2 \cdot 0}=5 \cdot 2 \cdot 0=5 \cdot 0=3,
$$

| $\cdot$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $?$ | $?$ | 3 | 4 | 5 |
| 1 | $?$ | 1 | $?$ | 3 | 4 | 5 |
| 2 | $?$ | $?$ | 2 | 3 | 4 | 5 |
| 3 | 3 | 3 | $A$ | 3 | 4 | 5 |
| 4 | 4 | $B$ | 4 | 3 | 4 | 5 |
| 5 | $C$ | 5 | 5 | 3 | 4 | 5 |

Table 1: The partial crown product; $A \in\{4,5\}, B \in\{3,5\}, C \in\{3,4\}$.
a contradiction. Thus we have the direct implication. For the converse, the map given by the permutation (53)(20) is an isomorphism.

For the other items, there are isomorphisms that send the first equivalence to the other two.

Proposition A.3. The crown poset does not admit an RRB structure.
Proof. Assume it is related to a posemigroup. Then all previous lemmas apply.

$$
\begin{aligned}
4 \cdot 1=3 & \Longleftrightarrow 3 \cdot 2=4 & & \text { Lemma A.2 } \\
& \Longleftrightarrow 3 \cdot 2 \neq 5 & & \text { Lemma A.1 } \\
& \Longleftrightarrow 5 \cdot 0 \neq 3 & & \text { Lemma A.2 } \\
& \Longleftrightarrow 5 \cdot 0=4 & & \text { Lemma A.1 } \\
& \Longleftrightarrow 4 \cdot 1=5 & & \text { Lemma A.2. }
\end{aligned}
$$

This contradiction shows that Table 1 can't be completed to obtain an associative product.

## A. 2 Preimages of foliated trees

We present here the complete analysis of the case distinction required by the last paragraphs of the proof of Theorem 2.29.

Each case will be named by a 4 -tuple. Its first and second coordinates determine if $f(y)$ is equal to $f(x)$ and $f(z)$ respectively; if $f(x)=f(y)$, then the first number in the name will be 1 , otherwise it will be 2 . Similar remarks hold for the second coordinate. The third and fourth coordinate determine the comparability conditions between $x$ and $y$, and $y$ and $z$ respectively; if $x \leq y$ the third coordinate will be 1 , if $y<x$ the third coordinate will be 2 , and if $x$ and $y$ are incomparable, the third coordinate will be 3 . The fourth coordinate behaves analogously. If an asterisk is present in any coordinate, it means the reasoning applied in that case holds for all possible choices for that coordinate. As an example of these conventions, the name $[1.2 .3 .2]$
corresponds to the case in which $f(x)=f(y), f(y) \neq f(z), x$ and $y$ are incomparable, and $z<y$.

In the following calculations, we denote the product defined in (4) by juxtaposition.
[1.1.*.*] In this case:

$$
(x y) z=(x \tilde{\cdot} y) \tilde{\sim} z=x \tilde{\sim}(y \tilde{\cdot} z)=x(y z)
$$

[*.*.1.1]

$$
(x y) z=x z=x=x y=x(y z)
$$

[*.*.1.2]

$$
(x y) z=x z=x(y z)
$$

[*.*.2.1]

$$
(x y) z=y z=y=x y=x(y z)
$$

[*.*.2.2]

$$
(x y) z=y z=z=x z=x(y z)
$$

## [1.2.1.3]

$$
x(y z)=x F(z)=F(z)=x z=(x y) z
$$

[1.2.2.3]

$$
(x y) z=y z=F(z)=x F(z)=x(y z)
$$

As $x \cdot F(z)=F(z)$.

## [1.2.3.1]

$$
(x y) z=(x \tilde{\cdot} y) z=x \tilde{\sim} y=x y=x(y z)
$$

As $(x \sim y)<z$ by Claim 5.

## [1.2.3.2]

$$
(x y) z=(x \sim y) z=z=x z=x(y z)
$$

by Claim 5.

## [1.2.3.3]

$$
(x y) z=(x \tilde{\sim} y) z=F(z)=x F(z)=x(y z)
$$

because $(x \sim y)$ must be incomparable with $z$ by Claim 6 as $x \sim y \leq y$.
[2.1.1.3]

$$
(x y) z=x z=x=x(y \tilde{\wedge})=x(y z)
$$

by Claim 5.

## [2.1.2.3]

$$
(x y) z=y z=y \tilde{\cdot} z=x(y \tilde{\sim} z)=x(y z)
$$

by Claim 5 .

## [2.1.3.1]

$$
(x y) z=F(y) z=F(y)=x y=x(y z)
$$

as $F(y) \leq y \leq z$.

## [2.1.3.2]

$$
(x y) z=F(y) z=F(z) z=F(z)=x z=x(y z)
$$

because $F(y)=F(z)$ and the fact that $x$ and $z$ are incomparable .

$$
\begin{equation*}
(x y) z=F(y) z=F(y)=F(y \tilde{\sim} z)=x(y \tilde{\sim} z)=x(y z) \tag{2.1.3.3}
\end{equation*}
$$

as $F(y) \leq z, F(y)=F(y \tilde{\sim})$ and $x$ and $y \tilde{\sim} z$ are incomparable.

## [2.2.1.3]

$$
(x y) z=x z=F(z)=x F(z)=x(y z)
$$

as $x$ and $z$ are incomparable.

## [2.2.2.3]

$$
(x y) z=y z=F(z)=x F(z)=x(y z)
$$

[2.2.3.1]

$$
(x y) z=F(y) z=F(y)=x y=x(y z)
$$

[2.2.3.2]

$$
(x y) z=F(y) z=F(z)=x z=x(y z)
$$

Because if $F(y) \leq z$ then $F(y)=F(z)$. Otherwise they are incomparable.

## [2.2.3.3]

$$
(x y) z=F(y) z=F(z)=x F(z)=x(y z)
$$

because $F(y)$ and $z$ are incomparable.

## References

[1] K.S. Brown, Semigroups, rings, and Markov chains, J. Theor. Probab. 13: 871-938 (2000).
[2] J. Gerhard, The lattice of equational classes of idempotent semigroups, Journal of Algebra 15: 195-224 (1970).
[3] A. Mistlbacher, An order-theoretical characterization of varieties of bands, Canadian Mathematical Bulletin 34: 368-374 (1991).
[4] P. SÁnchez Terraf, Directly indecomposables in semidegenerate varieties of connected po-groupoids, Order 25: 377-386 (2008).
[5] P. Sánchez Terraf, Factor congruences in semilattices, Revista de la Unión Matemática Argentina 52: 1-10 (2011).


[^0]:    *Universidad Nacional de Córdoba. Facultad de Matemática, Astronomía, Física y Computación.

    Centro de Investigación y Estudios de Matemática (CIEM-FaMAF). Córdoba. Argentina.
    Supported by Secyt-UNC project 33620230100751 CB and Conicet PIP project 11220210100508 CO .
    ${ }^{\dagger}$ Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, UBA. Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, UNLPAM.

