TOWARDS A GENERAL THEORY OF DEPENDENT SUMS

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Abstract. We introduce dependent adders. A dependent adder A has for every $x \in A$ a way of adding together x many elements of A. We provide examples from many disparate branches of mathematics. Examples include the field with one element $\mathbb{F}_1$, the real numbers with integrals as sums, the category of categories with oplax colimits as sums. We also consider modules over dependent adders and provide examples.

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1. Introduction

This paper defines *dependent adders* and constructs examples for them. Before precisely defining them, we define them roughly and enumerate the examples we are interested in. Defined roughly, a *dependent adder* consists of a set A, an element $1 \in A$, for every $x \in A$ a set A^x, called the *set of x-indexed families* in A, and a function $\sum_i^x : A^x \to A$ called the *x-dependent sum function*, satisfying nice properties, like that $\sum_i^x 1 = x$ and that every sum of the form $\sum_j^{\sum_i^x f(i)} g(j)$ can be rewritten as a sum of the form $\sum_i^x \sum_j^{f(i)} g(\phi(i,j))$ for some well-behaved function $\phi(i,j)$ depending only on f and not on g. Examples of dependent adders include:

(1) The set of natural numbers $\mathbb{N}$. For each $x \in \mathbb{N}$, an x-indexed family of natural numbers is a function $f : \{1, \dots, x\} \to \mathbb{N}$, and its sum is the usual sum
$$\sum_i^x f(i) := \sum_{i=1}^x f(i)$$

(2) The set of non-negative real numbers $\mathbb{R}_{\geq 0}$. For each $x \in \mathbb{R}_{\geq 0}$, we define an x-indexed family of real numbers to be a continuous function $f : [0, x] \to \mathbb{R}_{\geq 0}$. The sum of such a function is defined to be the integral
$$\sum_i^x f(i) := \int_0^x f(t)dt$$
This integral always exists, is non-negative and finite.

(3) The category of small categories Cat. Define an I-indexed family of categories to be an isomorphism class of functors $F : I^{op} \to Cat$. The sum of F is defined to be the lax 2-colimit of F, also known as the Grothendieck construction
$$\sum_i^I F(i) := \underset{i \in I^{op}}{\mathrm{colim}}^{lax} F(i) = \int F$$
Alternatively, one can define an I-indexed family of categories to be an isomorphism class of functors $F : I \to Cat$ and define its sum to be the oplax colimit of F. We need to take isomorphism

classes of functors instead of just taking functors, because otherwise Cat would only satisfy our axioms up to isomorphisms, instead of satisfying them strictly.

(4) The p-adic integers $\mathbb{Z}_p$. For every $x \in \mathbb{Z}_p$ we define an x-indexed family to be a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$ we can find a sequence of natural numbers $x_n \in \mathbb{N}$ such that $x_n \to x$ in $\mathbb{Z}_p$. We define

$$\sum_i^x f(i) := \lim_{n\to\infty} \sum_{i=1}^{x_n} f(i)$$

It is an easy exercise in p-adic analysis to show that this limit always exists and is independent of the choice of sequence x_n. (See Lemma 3.2 for a proof of this claim).

(5) The set of all small cardinals $Card$, where a κ-indexed family is a function $\kappa \to Card$, and where the sums are the usual infinite sums of cardinals.

(6) The set of all small ordinal numbers Ord. For each ordinal α an α-indexed family is a function of sets $f : \alpha \to Ord$. The sum of f is defined via transfinite recusion on α:

$$\sum_x^0 f(x) := 0 \quad \sum_x^{\alpha+1} f(x) := (\sum_x^{\alpha} f(x)) + f(\alpha) \quad \sum_x^{\sup_{\beta<\alpha}\beta} f(x) := \sup_{\beta<\alpha} \sum_x^{\beta} f(x)$$

(7) The two element set $\mathbb{F}_1 := \{0, 1\}$, which is also sometimes called "the field with one element"[5]. We define a 0-indexed family to be a function from the empty set $\varnothing \to \mathbb{F}_1$, i.e. there is exactly one 0-indexed family, which we call the empty family and denote $\varnothing$. We define a 1-indexed family to be a function $f : \{*\} \to \mathbb{F}_1$, i.e. there are exactly two 1-indexed families 0 and 1 corresponding to the elements $\mathbb{F}_1$. We define

$$\sum_i^0 \varnothing; = 0 \quad \sum_i^1 0 := 0 \quad \sum_i^1 1 := 1$$

(8) The real unit interval $[0, 1]$. Just as for the real numbers, for each $x \in [0, 1]$ we define an x-indexed family to be a continuous function

$f : [0, x] \to [0, 1]$, and define the sum to be the integral

$$\sum_i^x f(i) := \int_0^x f(t)dt$$

This integral is again in $[0, 1]$, because for every t we have $f(t) \leq 1$, so

$$\int_0^x f(t)dt \leq \int_0^x 1dt = t \leq 1$$

(9) The set of integers $\mathbb{Z}$. For each $x \in \mathbb{Z}$, an x-indexed family of integers is a function $f : \mathbb{Z} \to \mathbb{Z}$. For $x \geq 0$ the sum of f is the usual sum. For $x < 0$ we define

$$\sum_i^x f(i) := \sum_{i=1}^{-x} - f(1 - i)$$

(10) The set of all (possibly negative) real numbers $\mathbb{R}$ with integrals as sums, where an x-indexed family is a continuous function $\mathbb{R} \to \mathbb{R}$.

(11) The interval $[-1, 1]$ with integrals as sums, where an x-indexed family is a continuous function $[-1, 1] \to [-1, 1]$.

(12) The complex numbers $\mathbb{C}$. For every $z \in \mathbb{C}$ we define a z-indexed family to be an entire holomorphic function $f : \mathbb{C} \to \mathbb{C}$. We can choose a smooth path $\gamma : [0, 1] \to \mathbb{C}$ that leads from 0 to z, i.e. $\gamma(0) = 0$ and $\gamma(1) = z$. We define

$$\sum_i^z f(i) := \int_\gamma f(t)dt = \int_0^1 f(\gamma(t))\gamma'(t)dt$$

By Cauchy's integral theorem this is independent of the choice of γ.

(13) Every commutative $\mathbb{Q}$-algebra R. For every $r \in R$ we define an r-indexed family to be a polynomial function $p : R \to R$, i.e. an element of the polynomial ring $p \in R[X]$. For every $d \in \mathbb{N}$ we define the d-th Faulhaber polynomial $F_d \in R[X]$ by

$$F_d := \frac{1}{d+1} \sum_{n=0}^{d} \binom{d+1}{n} B_n X^{d-n+1}$$

where B_n is the Bernoulli number with convention $B_1 = +\frac{1}{2}$. For $p \in R[X]$ we can write p in the form $p = \sum_{i=0}^{n} p_i X^i$ and define for every $x \in R$ the x-dependent sum of p by

$$\sum_i^x p(i) := \sum_{i=0}^n p_i \cdot F_i(x)$$

Then $\sum_i^x p(i)$ is a polynomial function in x and for every $m \in \mathbb{N}$ the m-dependent sum $\sum_i^m p(i)$ is equal to the usual sum $\sum_{i=1}^m p(i)$, because of Faulhaber's formula $F_d(m) = \sum_{i=1}^m i^d$. [7]

This paper tries to capture what all of these examples have in common in a unified framework. In Section 2 we define dependent adders, and very briefly investigate their most rudimentary properties. However we do not develop this theory very far in this paper. The majority of this paper is of a zoological nature and consists of definitions, examples, and proofs that the examples fit the definition. In Section 3 we show that all the examples we listed above are in fact dependent adders. In Section 4 and 5 we consider right and left modules over dependent adders. Given a dependent adder A, a right A-module M, is a place where one can form sums of the form $\sum_i^x f(i) \in M$ where $x \in A$, $f(i) \in M$. $\mathbb{F}_1$-modules are exactly the pointed sets. $\mathbb{N}$-modules are exactly the monoids. Every Banach space with its Bochner integrals is an $\mathbb{R}$-module. Every cocomplete category is a Cat-module. A left A-module M is a place where one can form sums of the form $\sum_i^m f(i) \in M$ where $m \in M$ and $f(i) \in A$. We show that the category of topological spaces an open maps Top_{open} is a left module over the category of $Sets$, if one defines for $X \in Top_{open}$ an X-indexed family of sets to be a presheaf $\mathscr{F}$ on X, and defines the sum of $\mathscr{F}$ to be the étalé space of $\mathscr{F}$.
Our investigations have mostly been inspired by $\mathbb{F}_1$-geometry [5] and Arakelov theory [2], and then derailed into something more general. The "field with one element" $\mathbb{F}_1$ and the "complete local ring at infinity" $[-1, 1]$ might not have binary sums $x + y$, but they have all dependent sums $\sum_i^x f(i)$.

2. Dependent Adders

Let $\mathscr{C}$ be a category with finite limits. We denote the terminal object of $\mathscr{C}$ by $*$.

Definition 2.1. A *dependent adder* in $\mathscr{C}$ consists of:

(1) An object $A \in \mathscr{C}$.

(2) A morphism $p : F \to A$ in $\mathscr{C}$. We informally think of it as being an A-indexed family of objects of $\mathscr{C}$. For any morphism $x : U \to A$ we write $[\![x]\!]$ for the pullback of the diagram $U \xrightarrow{x} A \xleftarrow{p} F$ and write p_x for the canonical map $[\![x]\!] \to U$.

We informally think of a morphism $x : U \to A$ to be a generalized element of A, and a morphism $[\![x]\!] \to A$ to be an x-indexed family of elements of A.

(3) A morphism $1_F : * \to F$, called the *unit*. We also define $1_A : * \to A$ by $1_A := p \circ 1_F$.

(4) For every $x : U \to A$ in $\mathscr{C}$ a function of sets

$$\sum^{x} : \mathrm{Hom}_{\mathscr{C}}([\![x]\!], A) \to \mathrm{Hom}_{\mathscr{C}}(U, A)$$

called the *x-dependent sum function*. This map is supposed to be natural in $x \in \mathrm{Ob}(\mathscr{C}/A)$ in the sense that for every triangle of the form

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "x"'] & & U' \arrow[dl, "y"] \\ & A & \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}([\![x]\!], A) \arrow[r, "\sum^{x}"] & \mathrm{Hom}_{\mathscr{C}}(U, A) \\ \mathrm{Hom}_{\mathscr{C}}([\![y]\!], A) \arrow[u, "(f \times_A id_F)^*"] \arrow[r, "\sum^{y}"] & \mathrm{Hom}_{\mathscr{C}}(U', A) \arrow[u, "f^*"'] \end{tikzcd}$$

(5) For every $x : U \to A$ and every $f : [\![x]\!] \to A$ a morphism

$$f^{\flat} : [\![f]\!] \to [\![\sum^{x} f]\!]$$

called the *flattening map of* f. This is supposed to be a morphism in $\mathscr{C}/U$ in the sense that the following diagram commutes:

$$\begin{tikzcd} \llbracket f \rrbracket \arrow[r, "f^\flat"] \arrow[d, "p_f"'] & \llbracket \sum^{x} f \rrbracket \arrow[d, "p_{(\sum^{x} f)}"] \\ \llbracket x \rrbracket \arrow[r, "p_x"] & U \end{tikzcd}$$

We now demand that the following axioms are satisfied:

(1) Right Unit axiom: For every $x : U \to A$ let $const_{1_A} : \llbracket x \rrbracket \to A$ be the composite function $\llbracket x \rrbracket \to * \xrightarrow{1_A} A$. We demand that

$$\sum^{x} const_{1_A} = x$$

(2) Left Unit axiom: For every $x : U \to A$ and $f : \llbracket x \rrbracket \to A$, let $const_{1_A} : \llbracket x \rrbracket \to A$ be the constant 1_A function. We demand that

$$\sum^{const_{1_A}} f \circ const_{1_A}^\flat = f$$

(3) Sum Associativity Axiom: For every $x : U \to A$, $f : \llbracket x \rrbracket \to A$ and $g : \llbracket \sum^{x} f \rrbracket \to A$ we demand that

$$\sum^{\sum^{x} f} g = \sum^{x} \sum^{f} g \circ f^\flat$$

(4) Flatten Associativity Axiom: For every $x : U \to A$, $f : \llbracket x \rrbracket \to A$, $g : \llbracket \sum^{x} f \rrbracket \to A$ the following diagram commutes

$$\begin{tikzcd} \llbracket g \circ f^\flat \rrbracket \arrow[r, "(g \circ f^\flat)^\flat"] \arrow[d, "f^\flat \times_A id_F"'] & \llbracket \sum^{f} g \circ f^\flat \rrbracket \arrow[d, "(\sum^{f} g \circ f^\flat)^\flat"] \\ \llbracket g \rrbracket \arrow[r, "g^\flat"] & \llbracket \sum^{x} \sum^{f} g \circ f^\flat \rrbracket \end{tikzcd}$$

Definition 2.2. A dependent adder A in $\mathscr{C}$ is called *commutative* if it satisfies the following Fubini axiom: For every $x : U \to A$, $y : U \to A$ and $f : \llbracket x \circ p_y \rrbracket \to A$, we have a canonical isomorphism $switch : \llbracket x \circ p_y \rrbracket \to \llbracket y \circ p_x \rrbracket$ and demand that

$$\sum^{y} \sum^{x \circ p_y} f = \sum^{x} \sum^{y \circ p_x} f \circ switch$$

All the examples in the introduction, except the ordinals Ord are commutative.

Definition 2.3. A dependent adder A in $\mathscr{C}$ has a *zero object* if there exists a morphism $0_A : * \to A$ such that

(1) For all $x : U \to A$ let $const_{U,0_A}$ be the composite $U \to * \overset{0_A}{\to} A$ and let $const_{[\![x]\!],0_A}$ be the composite $[\![x]\!] \to * \overset{0_A}{\to} A$. We demand that
$$\sum^{x} const_{[\![x]\!],0_A} = const_{U,0_A}$$
(2) For all $f : [\![0_A]\!] \to A$
$$\sum^{0_A} f = 0_A$$

All the examples in the introduction have a zero object.

2.1. **Categorical Structure.** Let A be a dependent adder in a category with pullbacks $\mathscr{C}$.

Given $x : U \to A$, $f : [\![x]\!] \to A$ and $g : [\![\sum^{x} f]\!] \to A$ we define the *composition of f and g* to be
$$f \boxtimes g := \sum^{f} (g \circ f^{\flat})$$
Then the Left Unit Axiom states $const_{1_A} \boxtimes f = f$.

The Sum Associativity Axiom states that $\sum^{\sum^{x} f} g = \sum^{x} f \boxtimes g$.

The Flatten Associativity Axiom states that the following diagram commutes:

$$\begin{tikzcd}
[\![g \circ f^{\flat}]\!] \arrow[r, "(g \circ f^{\flat})^{\flat}"] \arrow[d, "f^{\flat} \underset{A}{\times} id_F"'] & [\![f \boxtimes g]\!] \arrow[d, "(f \boxtimes g)^{\flat}"] \\
[\![g]\!] \arrow[r, "g^{\flat}"] & [\![\sum^{x} f \boxtimes g]\!]
\end{tikzcd}$$

The $\boxtimes$ composition is associative in the following sense:

Lemma 2.4. For every $x : U \to A$, $f : [\![x]\!] \to A$, $g : [\![\sum^{x} f]\!] \to A$ and $h : [\![\sum^{x} f \boxtimes g]\!] \to A$ we have that
$$f \boxtimes (g \boxtimes h) = (f \boxtimes g) \boxtimes h$$

Proof. By definition of $\boxtimes$ we have

$$f \boxtimes (g \boxtimes h) = \sum^{f} (g \boxtimes h) \circ f^{\flat} = \sum^{f} (\sum^{g} h \circ g^{\flat}) \circ f^{\flat}$$

By naturality of dependent sums we have

$$\sum^{f} (\sum^{g} h \circ g^{\flat}) \circ f^{\flat} = \sum^{f} \sum^{g \circ f^{\flat}} h \circ g^{\flat} \circ (f^{\flat} \underset{A}{\times} id_F)$$

By the Flatten Associativity Axiom we have

$$\sum^{f} \sum^{g \circ f^{\flat}} h \circ g^{\flat} \circ (f^{\flat} \underset{A}{\times} id_F) = \sum^{f} \sum^{g \circ f^{\flat}} h \circ (f \boxtimes g)^{\flat} \circ (g \circ f^{\flat})^{\flat}$$

By the Sum Associativity Axiom we have

$$\sum^{f} \sum^{g \circ f^{\flat}} h \circ (f \boxtimes g)^{\flat} \circ (g \circ f^{\flat})^{\flat} = \sum^{\sum^{f} g \circ f^{\flat}} h \circ (f \boxtimes g)^{\flat} = (f \boxtimes g) \boxtimes h$$

□

We can associate to every dependent adder A in $\mathscr{C}$ a associated category $Fib(A)$ internal to the presheaf category $\mathrm{PSh}(\mathscr{C})$. We call this the *category of fibrations of* A, because in the case $A = Cat$ its global sections are equivalent to the category of small categories and Grothendieck fibrations.
A category in $\mathrm{PSh}(\mathscr{C})$ is the same thing as a presheaf of categories $Fib(A) : \mathscr{C}^{op} \to Cat$.
For $U \in \mathscr{C}$ define a category $Fib(A)(U)$ as follows: The objects of $Fib(A)(U)$ are morphisms $x : U \to A$ in $\mathscr{C}$. Given two objects $x : U \to A$ and $y : U \to A$ we define $\mathrm{Hom}_{Fib(A)(U)}(x, y) := \{f \in \mathrm{Hom}_{\mathscr{C}}([\![y]\!], A) | \sum^{y} f = x\}$. We call elements of this set A-fibrations. So an A-fibration $x \to y$ is a morphism $f : [\![y]\!] \to A$ whose sum is x.
For any object $x : U \to A$ we define id_x to be the map $const_{1_A} : [\![x]\!] \to A$ that is the composite $[\![x]\!] \to * \xrightarrow{1_A} A$.
For two fibrations $g : x \to y$ and $f : y \to z$ we define their composition by:

$$f \circ g := f \boxtimes g = \sum^{f} (g \circ f^{\flat})$$

This makes sense because $y = \sum^{z} f$, so g is a function $[\![\sum^{z} f]\!] \to A$ and then the formula is well-defined.

The Sum Associativity Axiom of A implies that the domain of $f \circ g$ is x. The left and right unit axioms of the dependent adder A imply the left and right unit axioms of the category $Fib(A)(U)$. The associativity of the composition of $Fib(A)(U)$ follows from Lemma 2.4.

One can now easily check that a morphism $U \to V$ in $\mathscr{C}$ induces a functor $Fib(A)(V) \to Fib(A)(U)$, so that $Fib(A)$ becomes a presheaf of categories $Fib(A) : \mathscr{C}^{op} \to Cat$. Equivalently this is also a category internal to $\mathrm{PSh}(\mathscr{C})$.

2.2. **Binary products.** Dependent sums imply binary products. For example in the ordinal numbers Ord we have the formula

$$\alpha \cdot \beta = \sum_{i=1}^{\beta} \alpha$$

and this same formula makes sense for arbitrary dependent adders.

Given a dependent adder A in $\mathscr{C}$, and two parallel maps $x, y : U \to A$ in $\mathscr{C}$ we have a canonical map $p_y : [\![y]\!] \to U$ and define $x \cdot y : U \to A$ by

$$x \cdot y := \sum^{y} x \circ p_y$$

With this we get a multiplication map

$$\mu_U : \mathrm{Hom}_{\mathscr{C}}(U, A) \times \mathrm{Hom}_{\mathscr{C}}(U, A) \to \mathrm{Hom}_{\mathscr{C}}(U, A)$$

which is natural in U. By the Yoneda lemma we obtain a map $\mu : A \times A \to A$ in $\mathscr{C}$. This map explicitly looks as follows: let $\pi_2 : A \times A \to A$ be the second projection. The first projection π_2 has a dependent sum function $\sum^{\pi_2} : \mathrm{Hom}_{\mathscr{C}}([\![\pi_2]\!], A) \to \mathrm{Hom}_{\mathscr{C}}(A \times A, A)$. Let $\pi_1 : [\![\pi_2]\!] \cong F \times A \to A$ be the first projection. The multiplication function $\mu : A \times A \to A$ is given by

$$\mu = \sum^{\pi_2} \pi_1$$

For all the examples from the introduction, this recovers the usual binary multiplication on the respective sets.

Lemma 2.5. *A is a monoid object in $\mathscr{C}$ with the multiplication function $\mu : A \times A \to A$ and the unit $1_A : * \to A$.*

If A is a commutative dependent adder, then it is a commutative monoid object.

Proof. By the Yoneda lemma it suffices to show for every $U \in \mathscr{C}$ that $\mathrm{Hom}_{\mathscr{C}}(U, A)$ is a monoid, respectively a commutative monoid.

The right unit axiom of the dependent adder A implies the left unit axiom of the monoid $\mathrm{Hom}_{\mathscr{C}}(U, A)$, because for $x : U \to A$ we have

$$const_{1_A} \cdot x = \sum^{x} const_{1_A} = x$$

The Sum Associativity Axiom of the dependent adder implies the associativity of the monoid in the following way: For $x, y, z : U \to A$ we have a commutative diagram.

$$\begin{tikzcd}
\llbracket y \circ p_z \rrbracket \arrow[rr, "(y \circ p_z)^\flat"] \arrow[d, "(p_z \underset{A}{\times} id_F)"'] \arrow[dr, "p_{y \circ p_z}"] & & \llbracket y \cdot z \rrbracket \arrow[d, "p_{y \cdot z}"] \\
\llbracket y \rrbracket \arrow[rr, bend right, "p_y"'] & \llbracket z \rrbracket \arrow[r, "p_z"] & U
\end{tikzcd}$$

Here the upper right part commutes, because flattening maps commute over U. The lower part commutes because both maps are the projection $U \underset{A}{\times} F \underset{A}{\times} F \to U$. Using this diagram and the Sum Associativity Axiom we get

$$x \cdot (y \cdot z) = \sum^{\sum^{z} y \circ p_z} x \circ p_{y \cdot z} = \sum^{z} \sum^{y \circ p_z} x \circ p_{y \cdot z} \circ (y \circ p_z)^\flat = \sum^{z} \sum^{y \circ p_z} x \circ p_y \circ (p_z \underset{A}{\times} id_F) =$$

$$= \sum^{z} (\sum^{y} x \circ p_y) \circ p_z = (x \cdot y) \cdot z$$

For the right unit axiom of the monoid we need to work a bit harder, because we can only directly apply the left unit axiom of the dependent adder A if U is of the form $\llbracket w \rrbracket$ for some $w \in \mathscr{C}/A$. Thankfully we can use the naturality of the dependent sums and the unit $1_F : * \to F$ to translate the problem into such a situation.

Let $c := const_1 : U \to A$ be the constant 1_A map. Consider the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[dr, dotted, "s"] \arrow[ddr, bend right, "id_U"'] & * \arrow[dr, "1_F"] & \\
& \llbracket c \rrbracket \arrow[r] \arrow[d, "p_c"] & F \arrow[d, "p"'] \\
& U \arrow[r, "c"] & A
\end{tikzcd}$$

The lower right square is a pullback. The outer diragram commutes. So we get $s : U \to [\![c]\!]$ with $p_c \circ s = id_U$. Let $\tilde{c} := c \circ p_c$ be the constant 1_A function $[\![c]\!] \to A$. We have a commutative triangle

$$\begin{tikzcd}
U \arrow[rr, "s"] \arrow[dr, "\tilde{c}\circ s"'] & & {[\![c]\!]} \arrow[dl, "\tilde{c}"] \\
& A &
\end{tikzcd}$$

so s is a morphism $\tilde{c} \circ s \to \tilde{c}$ in $\mathscr{C}/A$. The naturality of dependent sums now implies that for every $z : [\![\tilde{c}]\!] \to A$ we have

$$(\sum^{\tilde{c}} z) \circ s = \sum^{\tilde{c}\circ s} (z \circ (s \underset{A}{\times} id_F))$$

This in particular implies that for every $y : [\![c]\!] \to A$ we have

$$(y \cdot \tilde{c}) \circ s = (y \circ s) \cdot (\tilde{c} \circ s)$$

Now take $x : U \to A$. We want to show that $x \cdot c = x$. Let $y := x \circ p_c$. We have $y \cdot \tilde{c} = \sum^{\tilde{c}} const_y = \sum^{\tilde{c}} const_y \circ \tilde{c}^\flat = y$ by the left unit axiom of the dependent adder A. Then $x = y \circ s = (y \cdot \tilde{c}) \circ s = (y \circ s) \cdot (\tilde{c} \circ s) = x \cdot c$. So $\mathrm{Hom}_{\mathscr{C}}(U, A)$ is a monoid.

If the dependent adder is commutative, then the Fubini axiom and the left unit axiom of the monoid imply the commutativity of the monoid.

$$x \cdot y = (1_A \cdot x) \cdot y = \sum^{y} \sum^{x \circ p_y} const_{1_A} = \sum^{x} \sum^{y \circ p_x} const_{1_A} = y \cdot x$$

With the Yoneda lemma it follows that A is a monoid, respectively a commutative monoid, in $\mathscr{C}$. $\square$

In every dependent adder A we always have the right distributive law

$$y \cdot \sum_i^x f(i) = \sum_i^x y \cdot f(i)$$

but not necessarily the left distributive law. For example in Ord we have $(1 + 1) \cdot \omega \neq (1 \cdot \omega) + (1 \cdot \omega)$.

In all examples of dependent adders in the introduction, the binary multiplication here recovers the usual binary multiplication on these sets. For example in Cat the product $I \cdot J$ is the cartesian product of categories $I \times J$.

3. Examples

3.1. **Natural Numbers.** Let $\mathscr{C} = Set$ be the category of small sets. Let $A = \mathbb{N}$ be the set of natural numbers. For $n \in \mathbb{N}$ let $[n] := \{1, \dots, n\}$ be a set with n elements. Let $F := \coprod_{n \in \mathbb{N}} [n] = \{(n,i) | n, i \in \mathbb{N}, 1 \le i \le n\}$. Let $p : F \to A$ be the map sending (n,i) to n. The fiber of p over $n \in \mathbb{N}$ is the set $[n]$. Let $1_F : * \to F$ be the point $(1,1) \in F$. For any function $x : U \to \mathbb{N}$ can write $[\![x]\!] = U \times_A F$ as a coproduct:

$$[\![x]\!] = \coprod_{u \in U} [x(u)]$$

We define for every $x : U \to \mathbb{N}$ a natural function

$$\sum^{x} : \mathrm{Hom}_{Set}([\![x]\!], \mathbb{N}) \to \mathrm{Hom}_{Set}(U, \mathbb{N})$$

by sending $f : [\![x]\!] \to \mathbb{N}$ to the function $\sum^{x} f : U \to \mathbb{N}$ defined by

$$(\sum^{x} f)(u) := \sum_{i=1}^{x(u)} f(u,i)$$

In the case $U = *$ this is the map $\mathrm{Hom}_{Set}([x], \mathbb{N}) \to \mathbb{N}$ that sends f to $\sum_{i=1}^{x} f(i)$.

Next, for every $x : U \to \mathbb{N}$ and $f : [\![x]\!] \to \mathbb{N}$ we define the flattening map

$$f^{\flat} : \coprod_{u \in U} \coprod_{i \in [x(u)]} [f(u,i)] \to \coprod_{u \in U} [\sum_{i=1}^{x(u)} f(u,i)]$$

by

$$f^{\flat}(u,i,j) := (u, j + \sum_{k=1}^{i-1} f(u,k))$$

The Right Unit axiom $\sum_{i=1}^{x(u)} 1 = x(u)$ is obvious.

The Left Unit axiom follows from the following calculation:

$$(const_1 \boxtimes f)(u,i) = \sum_{j=1}^{1} f(u, j + \sum_{k=1}^{i-1} 1) = f(u, 1 + i - 1) = f(u,i)$$

The Sum Associativity Axiom

$$\sum_{j=1}^{\sum_{i=1}^{x(u)} f(u,i)} g(u,j) = \sum_{i=1}^{x(u)} \sum_{j=1}^{f(u,i)} g(u, j + \sum_{k=1}^{i-1} f(u,k))$$

is easily verified by induction over n.

For the Flatten Associativity Axiom we need to show that the following diagram commutes

$$\begin{array}{ccc}
\coprod_{u\in U}\coprod_{i\in[x(u)]}\coprod_{j\in[f(u,i)]}[g(f^\flat(u,i,j))] & \xrightarrow{(g\circ f^\flat)^\flat} & \coprod_{u\in U}\coprod_{i\in[x(u)]}[\sum_{j=1}^{f(u,i)} g(f^\flat(u,i,j))] \\
\Big\downarrow{\scriptstyle f^\flat \times_A id_F} & & \Big\downarrow{\scriptstyle (f\boxtimes g)^\flat} \\
\coprod_{u\in U}\coprod_{k\in[\sum_{i=1}^{x(u)} f(u,i)]}[g(u,k)] & \xrightarrow{g^\flat} & \coprod_{u\in U}[\sum_{i=1}^{x(u)}\sum_{j=1}^{f(u,i)} g(f^\flat(u,i,j))]
\end{array}$$

The commutativity follows by the following calculation:

$$g^\flat((f^\flat \underset{A}{\times} id_F)(u,i,j,k)) = g^\flat(f^\flat(u,i,j),k) = g^\flat(u, j + \sum_{a=1}^{i-1} f(u,a), k) =$$

$$= (u, k + \sum_{b=1}^{j-1+\sum_{a=1}^{i-1} f(u,a)} g(u,b)) =$$

$$= (u, k + (\sum_{b=1}^{j-1} g(u, b + \sum_{a=1}^{i-1} f(u,a))) + \sum_{a=1}^{i-1}\sum_{b=1}^{f(u,a)} g(u, b + \sum_{c=1}^{a-1} f(u,c))) =$$

$$= (u, k + (\sum_{b=1}^{j-1} g(f^\flat(u,i,b))) + \sum_{a=1}^{i-1} (f \boxtimes g)(u,a)) =$$

$$= (f \boxtimes g)^\flat(u, i, k + \sum_{b=1}^{j-1} g(f^\flat(i,b))) = (f\boxtimes g)^\flat((g\circ f^\flat)^\flat(u,i,j,k))$$

which means the Flatten Associativity Axiom is satisfied and $\mathbb{N}$ is a dependent adder. The dependent adder is commutative because the Fubini axiom $\sum_{i=1}^{x(u)}\sum_{j=1}^{y(u)} f(u,i,j) = \sum_{j=1}^{y(u)}\sum_{i=1}^{x(u)} f(u,i,j)$ is obviously satisfied.

In the category of sets $\mathscr{C} = Set$ the entire u variable is kind of unnecessary, and it is enough to define dependent sums, flattening maps and prove the axioms for $U = *$. The u variable is however important in some other categories $\mathscr{C}$.

Let $FinSet$ be the full subcategory of Set on sets of the form $[n]$ for $n \in \mathbb{N}$. The presheaves of categories $Fib(\mathbb{N}) : Set^{op} \to Cat$ is representable, and it is represented by the category $FinSet$. By this we mean that for every $S \in Set$ we have a strict isomorphism of categories

$$Fib(\mathbb{N})(S) \cong FinSet^S$$

which is natural in S.
More precisely there is a bijection

$$\Phi : \mathrm{Hom}_{FinSet}([n],[m]) \to \mathrm{Hom}_{Fib(\mathbb{N})(*)}(n,m)$$

sending $f : [n] \to [m]$ to the function $\Phi(f) : [m] \to \mathbb{N}$ that sends $x \in [m]$ to the cardinality of the fiber of f over x.

$$\Phi(f)(x) := |f^{-1}(\{x\})|$$

It preserves composition in the sense that $\Phi(g \circ f) = \Phi(g) \boxtimes \Phi(f)$.

3.2. **Non-negative real numbers.** Let $\mathscr{C} = Top_{mtr}$ be the category of metrizable topological spaces and continuous maps between them. The category $\mathscr{C}$ has pullbacks which agree with the usual pullbacks of topological spaces.
Let $A := \mathbb{R}_{\geq 0}$ be the space of non-negative real numbers with euclidean topology.
Let $F := \{(x,y) \in \mathbb{R}^2 | 0 \leq x, 0 \leq y \leq x\}$ be equipped with the subspace topology from $\mathbb{R}^2$. Define $p : F \to \mathbb{R}_{\geq 0}$, $p(x,y) := x$. The fiber of p over x is the interval $[0,x]$.
We let $1_F : * \to F$ be the point $(1,1) \in F$.
For $x : U \to \mathbb{R}_{\geq 0}$ there is a homeomorphism

$$[\![x]\!] \cong \{(u,a) \in U \times \mathbb{R}_{\geq 0} | 0 \leq a \leq x(u)\}$$

and from now on we redefine $[\![x]\!]$ to mean the topological space $\{(u,a) \in U \times \mathbb{R}_{\geq 0} | 0 \leq a \leq x(u)\}$.
For $x : U \to \mathbb{R}_{\geq 0}$ define $\sum^x : \mathrm{Hom}_{Top_{mtr}}([\![x]\!], \mathbb{R}_{\geq 0}) \to \mathrm{Hom}_{Top_{mtr}}(U, \mathbb{R}_{\geq 0})$ on $f : [\![x]\!] \to \mathbb{R}_{\geq 0}$ by

$$(\sum^x f)(u) := \int_0^{x(u)} f(u,t)dt$$

This integral always exists and is finite, because f is continuous and for every $u \in U$ the interval $[0, x(u)]$ is compact. The map is easily seen to be natural in x, but we have to show that $\sum^{x} f$ is actually a continuous function. For this we will need to use the fact that U is metrizable.

Lemma 3.1. The function $g := \sum^{x} f : U \to \mathbb{R}_{\geq 0}$ is continuous.

Proof. Choose a metric d on U, and put on $[\![x]\!] = U \underset{\mathbb{R}_{\geq 0}}{\times} F$ the metric that is the sum of the metric d and the Manhattan distance on F. Take $u \in U$ and $\epsilon_1 > 0$. We want to show that the set $V := g^{-1}(B_{\epsilon_1}(g(u)))$ is open in U. Since U is metrizable, U is also a compactly generated topological space. So to show that V is open, it suffices to show for every compact subset $K \subseteq U$ that $K \cap V$ is open in K. So take an arbitrary compact subset $K \subseteq U$, and some $v \in K \cap V$. We need to show there exists $\delta > 0$ such that $K \cap B_\delta(v) \subseteq K \cap V$. Let $\epsilon_2 := \epsilon_1 - |g(v) - g(u)|$. We have $\epsilon_2 > 0$ and $B_{\epsilon_2}(g(v)) \subseteq B_{\epsilon_1}(g(u))$. Choose $\theta_1 > 0$ with $\theta_1 \cdot x(v) \leq \frac{\epsilon_2}{2}$. The set $K \underset{\mathbb{R}_{\geq 0}}{\times} F$ is a compact subset of $[\![x]\!]$. Therefore the restricted map $f : K \underset{\mathbb{R}_{\geq 0}}{\times} F \to \mathbb{R}_{\geq 0}$ is uniformly continuous. So there exists a $\delta_1 > 0$ such that for all $x, y \in K \underset{\mathbb{R}_{\geq 0}}{\times} F$, if $d(x,y) < \delta_1$ then $d(f(x), f(y)) < \theta_1$. Let M be the maximum of $f : K \underset{\mathbb{R}_{\geq 0}}{\times} F \to \mathbb{R}_{\geq 0}$. Choose $\theta_2 > 0$ such that $\theta_2 \cdot M \leq \frac{\epsilon_2}{2}$ and $\theta_2 \leq \delta_1$. Since $x : U \to \mathbb{R}_{\geq 0}$ is continuous, there exists $\delta_2 > 0$ such that $x(B_{\delta_2}(v)) \subseteq B_{\theta_2}(x(v))$. Let δ be the minimum of δ_1 and δ_2. Take $w \in K \cap B_\delta(v)$. We claim that $w \in V$. This means that $g(w) \in B_{\epsilon_1}(g(u))$. To show that it suffices to show $g(w) \in B_{\epsilon_2}(v)$. We have

$$g(w) = \int_0^{x(w)} f(w,t)dt = \int_{x(v)}^{x(w)} f(w,t)dt + \int_0^{x(v)} f(w,t)dt$$

Let's look at the first term of this sum:

$$\left|\int_{x(v)}^{x(w)} f(w,t)dt\right| \leq |x(w) - x(v)| \cdot M \leq \theta_2 \cdot M \leq \frac{\epsilon_2}{2}$$

With this we get

$$|g(w) - g(v)| \leq \frac{\epsilon_2}{2} + \left|\int_0^{x(v)} f(w,t) - f(v,t)dt\right| \leq$$

$$\leq \frac{\epsilon_2}{2} + x(v) \cdot \theta_1 \leq \epsilon_2$$

so $g = \sum^{x} f$ is continuous. $\square$

Given $x : U \to \mathbb{R}_{\geq 0}$ and $f : [\![x]\!] \to \mathbb{R}_{\geq 0}$ the flattening map is defined by

$$f^\flat : \coprod_{u \in U} \coprod_{a \in [0, x(u)]} [0, f(u,a)] \to \coprod_{u \in U} [0, \int_0^{x(u)} f(u,t)dt]$$

$$f^\flat(u, a, b) := (u, \int_0^a f(u,t)dt)$$

Here the $\coprod$ coproducts that we have written domain and codomain of $f^\flat$ are not literally coproducts of topological spaces, but they are coproducts of sets equipped with the subspace topology from $U \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, respectively $U \times \mathbb{R}_{\geq 0}$. Also note that $f^\flat(u, a, b)$ does not depend on b at all, so if we have continuous functions $x : U \to \mathbb{R}_{\geq 0}$, $f : [\![x]\!] \to \mathbb{R}_{\geq 0}$ and $g : [\![\sum^{x} f]\!] \to \mathbb{R}_{\geq 0}$, then

$$(f \boxtimes g)(u, a) = \int_0^{f(u,a)} g(f^\flat(u, a, t))dt = f(u, a) \cdot g(\int_0^a f(u, s)ds)$$

We now verify that $\mathbb{R}_{\geq 0}$ satisfies the axioms of a commutative dependent adder.

The Right Unit axiom follows from the fact that

$$\int_0^x 1dt = x$$

The Left Unit axiom follows from

$$(const_{1_A} \boxtimes f)(u, a) = 1 \cdot f(u, \int_0^a 1) = f(u, a)$$

For the Sum Associativity Axiom we need to show that for every continuous function $f : [\![x]\!] \to \mathbb{R}_{\geq 0}$ and $g : [\![\sum^{x} f]\!] \to \mathbb{R}_{\geq 0}$ we have

$$\int_0^{\int_0^x f(u,t)dt} g(u, s)ds = \int_0^x f(u, t) \cdot g(u, \int_0^t f(u, z)dz)dt$$

This follows through "integration by substitution" with an antiderivative of f. Let $h : [\![x]\!] \to \mathbb{R}_{\geq 0}$, $h(u, t) := \int_0^t f(u, s)ds$. Then h is continuously differentiable in the t variable with $\frac{d}{dt}h(u,t) = f(u,t)$, and we get

$$\int_0^{\int_0^x f(u,t)dt} g(u,s)ds = \int_{h(u,0)}^{h(u,x)} g(u,s)ds = \int_0^x f(u,t)\cdot g(u,h(u,t))dt$$

The Flatten Associativity Axiom follows from

$$g^\flat((f^\flat \underset{A}{\times} id_F)(u,a,b,c)) = g^\flat(u, \int_0^a f(u,t)dt, c) = \int_0^{\int_0^a f(u,t)dt} g(u,t)dt =$$

$$= \int_0^a (f \boxtimes g)(u,t)dt = (f\boxtimes g)^\flat(u,a,\int_0^b g(f^\flat(i,t))dt) = (f\boxtimes g)^\flat((g\circ f^\flat)^\flat(u,a,b,c))$$

so $\mathbb{R}_{\geq 0}$ is a dependent adder.

It is a commutative dependent adder because of Fubini's theorem

$$\int_0^x\int_0^y f(u,s,t)dsdt = \int_0^y\int_0^x f(u,s,t)dtds$$

3.3. **P-Adic integers.** Like in Section 3.2, we take $\mathscr{C} = Top_{mtr}$ the category of metrizable spaces.

Let q be a prime number. Let $A := \mathbb{Z}_q$. Let $F := \mathbb{Z}_q \times \mathbb{Z}_q$. Let $p : F \to A$ be the second projection $\mathbb{Z}_q \times \mathbb{Z}_q \to \mathbb{Z}_q$. The fiber of p over $x \in \mathbb{Z}_q$ is $\mathbb{Z}_q$. Let $1_F : * \to F$ be the point $(1,1) \in \mathbb{Z}_q \times \mathbb{Z}_q$. For $x : U \to \mathbb{Z}_q$ there is a homeomorphism $[\![x]\!] \cong U \times \mathbb{Z}_q$, so by $[\![x]\!]$ we will from now on simply mean $U \times \mathbb{Z}_q$.

To define the dependent sums we use the following lemma

Lemma 3.2. If $f : \mathbb{Z}_q \to \mathbb{Z}_q$ is a continuous function, and $g : \mathbb{N} \to \mathbb{Z}_q$ is the function $g(n) := \sum_{i=1}^{n} f(i)$ then g is continuous with respect to the q-adic topology on $\mathbb{N}$.

Proof. Take $N \in \mathbb{N}$. We need to show that there exists $M \in \mathbb{N}$ such that for all $x, y \in \mathbb{N}$ with $x - y$ divisible by q^M we have that $g(x) - g(y)$ is divisble by q^N.

Since $\mathbb{Z}_q$ is compact, f is uniformly continuous. So there exists a $K \in \mathbb{N}$, such that for all $x, y \in \mathbb{Z}_q$ if $x - y \in q^K\mathbb{Z}_q$ then $f(x) - f(y) \in q^N\mathbb{Z}_q$. Let

$M := K + N$. Take $x, y \in \mathbb{N}$ with $x - y = q^M \cdot z$ for some $z \in \mathbb{N}$. Without loss of generality assume $x \geq y$. Then we have

$$g(x)-g(y) = \sum_{i=y+1}^{x} f(i) = \sum_{i=1}^{x-y} f(i+y) = \sum_{i=1}^{q^M z} f(i+y) = \sum_{i=1}^{q^N}\sum_{j=1}^{q^K z} f(j+y+(i-1)q^K z)$$

Now for all i, j there exists $w_{i,j}$ such that $f(j + y + (i-1)q^K z) = f(j + y) + q^N w_{i,j}$, and then

$$\sum_{i=1}^{q^N}\sum_{j=1}^{q^K z} f(j + y + (i-1)q^K z) = q^N(\sum_{j=1}^{q^K z} f(j + y)) + q^N(\sum_{i=1}^{q^N}\sum_{j=1}^{q^K z} w_{i,j})$$

This is divisible by q^N so g is continuous. $\square$

In the above lemma, since $\mathbb{Z}_q$ is Cauchy-complete, the map $g : \mathbb{N} \to \mathbb{Z}_q$ induces a map from the Cauchy completion of $\mathbb{N}$ to $\mathbb{Z}_q$. Since the Cauchy completion of $\mathbb{N}$ with respect to the q-adic topology is $\mathbb{Z}_q$, we thus get a continuous map $\tilde{g} : \mathbb{Z}_q \to \mathbb{Z}_q$, $\tilde{g}(x) = \lim_{\substack{x_n \to x \\ x_n \in \mathbb{N}}} g(x)$. We write $\sum_{i=1}^{x} f(i)$ for $\tilde{g}(x)$.

$$\sum_{i=1}^{x} f(i) := \lim_{\substack{x_n \to x \\ x_n \in \mathbb{N}}} \sum_{i=1}^{x_n} f(i)$$

We can now define the dependent sum function of $\mathbb{Z}_q$. For any continuous function $x : U \to \mathbb{Z}_q$ define

$$\sum^{x} : \mathrm{Hom}_{Top_{mtr}}(U \times \mathbb{Z}_q, \mathbb{Z}_q) \to \mathrm{Hom}_{Top_{mtr}}(U, \mathbb{Z}_q)$$

by

$$(\sum^{x} f)(u) := \sum_{i=1}^{x(u)} f(u, i)$$

The continuity of this function can be proven using an argument very similar to the one from Lemma 3.1.

For any $f : U \times \mathbb{Z}_q \to \mathbb{Z}_q$ we define the flattening function of f by

$$f^\flat : U \times \mathbb{Z}_q \times \mathbb{Z}_q \to \mathbb{Z}_q, f^\flat(u, i, j) := j + \sum_{k=1}^{i-1} f(u, k)$$

The right and left unit axiom, both Associativity Axioms and Fubini axiom follow exactly like for the natural numbers $\mathbb{N}$ from Section 3.1.

3.4. **Small Categories.** We first explicitly construct oplax colimits of categories. We then show that Cat is a dependent adder with oplax colimits as dependent sums. We then show that Cat^{op} is a dependent adder with lax colimits / Grothendieck constructions as dependent sums.

3.4.1. *Explicit construction of oplax colimits.* For any functor $f : I \to Cat$ define its *oplax colimit* $\operatorname{colim}^{oplax}_{i\in I} f(i)$ to be the following category:

(1) The objects are pairs (i,j) with $i \in I$, $j \in f(i)$.
(2) The morphism $(i,j) \to (i',j')$ are pairs of morphisms (α,β) where $\alpha : i \to i'$ in I and $\beta : f(\alpha)(j) \to j'$ in $f(i')$.

The universal property of this particular oplax colimit construction is as follows: For any $i \in I$ we have a functor $\iota_i : f(i) \to \operatorname{colim}^{oplax}_{j\in I} f(j)$. For every $\alpha : i \to i'$ in I we have a natural transformation $\iota_\alpha : \iota_i \to \iota_{i'} \circ f(\alpha)$, such that $\iota_{id_i} = id_{\iota_i}$ and $\iota_{\beta\circ\alpha} = (\iota_\beta * f(\alpha)) \circ \iota_\alpha$.
Whenever we have another object X together with for every $i \in I$ a functor $\omega_i : f(i) \to X$, and for every $\alpha : i \to i'$ in I a natural transformation $\omega_\alpha : \omega_i \to \omega_{i'} \circ f(\alpha)$ such that $\omega_{id_i} = id_{\omega_i}$ and $\omega_{\beta\circ\alpha} = (\omega_\beta * f(\alpha)) \circ \omega_\alpha$ then there is a unique functor $\omega : \operatorname{colim}^{oplax}_{j\in I} f(j) \to X$, such that $\omega_i = \omega \circ \iota_i$ and $\omega_\alpha = \omega * \iota_\alpha$.
Usually the universal property of oplax colimits requires the functor ω to not be unique, but only be unique up to isomorphism. However the particular category we constructed above satisfies the stricter universal property where ω is unique up to equality. See [1, Section 2] for the usual theory of lax and oplax limits.

3.4.2. *Oplax colimits as sums.* We postulate two strongly inaccessible cardinals $\kappa_0 < \kappa_1$. A category is called *small* if it is smaller than κ_0. A category is called *moderately small* if it is smaller than κ_1.
The 1-category of small categories is denoted Cat. The 1-category of moderately small categories is denoted CAT.
For all moderately small categories I, J let $\operatorname{Hom}_{hCAT}(I,J)$ be a skeleton of the category of functors $\operatorname{Hom}_{CAT}(I,J)$. So for every functor $F : I \to J$ there exists exactly one functor $\operatorname{Ho}(F) : I \to J$ such that $F \cong \operatorname{Ho}(F)$ and $\operatorname{Ho}(F) \in \operatorname{Hom}_{hCAT}(I,J)$.

Let $hCAT$ be the category whose objects are moderately small categories, and whose morphism sets are given by $\mathrm{Hom}_{hCAT}(I, J)$. Given two functors $\mathrm{Ho}(F) : I \to J$ and $\mathrm{Ho}(G) : J \to K$ their composition is defined by taking sk of their composition in CAT:

$$\mathrm{Ho}(F) \underset{hCAT}{\circ} \mathrm{Ho}(G) := \mathrm{Ho}(\mathrm{Ho}(F) \underset{CAT}{\circ} \mathrm{Ho}(G))$$

With this composition $hCAT$ is a category.

Then $hCAT$ is in fact equivalent to the homotopy category of CAT, if we put on CAT the canonical model structure constructed in [8], in which weak equivalences are categorical equivalences and fibrations are Joyal isofibrations. We just prefer to present the morphisms of $hCAT$ as particular chosen functors instead of presenting them as isomorphism classes of functors.

The category $hCAT$ has finite limits. In general pullbacks in $hCAT$ do not need to coincide with pullbacks in CAT. However if we take a pullback of a Grothendieck fibration or opfibration in CAT, then it will also be a pullback in $hCAT$, because Grothendieck fibrations and opfibrations are Joyal isofibrations, and in any right proper model category a strict pullback of a fibration is a homotopy pullback.

Let $\mathscr{C} := hCAT$ be the homotopy category of moderately small categories. Let $A := Cat$ be the 1-category of small categories.

Consider the inclusion functor $\iota : A \to CAT$, $\iota(I) := I$. This functor corresponds to a Grothendieck op-fibration $p^{\heartsuit} : F \to A$. We define $p := \mathrm{Ho}(p^{\heartsuit})$. So we now have a morphism $p : F \to A$ in $\mathscr{C}$. The fiber of p over some $I \in A$ is isomorphic to $I \in CAT$. Define $1_F^{\heartsuit} : 1 \to F$ as the map that picks out the terminal category $1 \in A$ and the unique object in that category $* \in 1$. Define the unit 1_F by $1_F := \mathrm{Ho}(1_F^{\heartsuit})$.

Since pullbacks of Grothendieck op-fibrations are Grothendieck op-fibrations, we have for any functor $x : U \to A$ in $hCAT$ an isomorphism

$$[\![x]\!] \cong \operatorname*{colim}_{u \in U}^{oplax} x(u)$$

For every $x : U \to A$ we define the dependent sum function

$$\sum^{x} : \mathrm{Hom}_{\mathscr{C}}([\![x]\!], A) \to \mathrm{Hom}_{\mathscr{C}}(U, A)$$

on a functor $f : [\![x]\!] \to A$ as follows:

We define a functor $\sum^{x}(f)^{\heartsuit} : U \to A$ that is defined on objects by

$$\sum^{x}(f)^{\heartsuit}(u) := \operatorname*{colim}_{i \in x(u)}^{oplax} f(u,i)$$

Given an arrow $\alpha : u \to v$ in U we need to define an arrow $\sum^{x}(f)^{\heartsuit}(\alpha) : \sum^{x}(f)^{\heartsuit}(u) \to \sum^{x}(f)^{\heartsuit}(v)$ in A, by using the universal property of the oplax colimit in the codomain. For all $i \in x(u)$ we then have an arrow $(\alpha, id_{x(\alpha)(i)}) : (u,i) \to (v, x(\alpha)(i))$ in $\operatorname*{colim}_{u \in U}^{oplax} x(u) \cong [\![x]\!]$. We then get a composite functor $f(u,i) \overset{f(\alpha, id_{x(\alpha)(i)})}{\to} f(v, x(\alpha)(i)) \to \operatorname*{colim}_{k \in x(v)}^{oplax} f(v,k)$. For every morphism $\beta : i \to j$ in $x(u)$ we have a diagram

$$\begin{tikzcd}
f(u,i) \arrow[r, "f(\alpha, id_{x(\alpha)(i)})"] \arrow[d, "f(id_u, \beta)"'] & f(v, x(\alpha)(i)) \arrow[d, "f(id_v, x(\alpha)(\beta))"'] \arrow[dr] \arrow[d, phantom, shift left=40, "\Downarrow"] & \\
f(u,j) \arrow[r, "f(\alpha, id_{x(\alpha)(j)})"'] & f(v, x(\alpha)(j)) \arrow[r] & \operatorname*{colim}_{k \in x(v)}^{oplax} f(v,k)
\end{tikzcd}$$

where the left square commutes, and in the right triangle there is a canonical natural transformation.
By the universal property of the oplax colimit $\sum^{x}(f)^{\heartsuit}(u) = \operatorname*{colim}_{i \in x(u)}^{oplax} f(u,i)$ we get a morphism $\sum^{x}(f)^{\heartsuit}(\alpha) : \sum^{x}(f)^{\heartsuit}(u) \to \sum^{x}(f)^{\heartsuit}(v)$. Then $\sum^{x}(f)^{\heartsuit} : U \to A$ is a functor. We define $\sum^{x}(f) := \mathrm{Ho}(\sum^{x}(f)^{\heartsuit})$.
Next we need to define the flattening map

$$f^{\flat} : [\![f]\!] \to [\![\sum^{x}(f)]\!]$$

We have canonical isomorphisms

$$[\![f]\!] \cong \operatorname*{colim}_{(u,i) \in [\![x]\!]}^{oplax} f(u,i) \cong \operatorname*{colim}_{(u,i) \in \operatorname*{colim}_{u \in U}^{oplax} x(u)}^{oplax} f(u,i)$$

$$[\![\sum^{x} f]\!] \cong \operatorname*{colim}_{u \in U}^{oplax} \operatorname*{colim}_{i \in x(u)}^{oplax} f(u,i)$$

and using the universal property of oplax colimits one can construct a natural isomorphism of categories

$$\operatorname*{colim}_{(u,i) \in \operatorname*{colim}_{u \in U}^{oplax} x(u)}^{oplax} f(u,i) \xrightarrow{\sim} \operatorname*{colim}_{u \in U}^{oplax} \operatorname*{colim}_{i \in x(u)}^{oplax} f(u,i)$$

So there is a canonical isomorphism $f^{\flat,\heartsuit} : [\![f]\!] \to [\![\sum^{x} f]\!]$ and we define $f^{\flat} := \mathrm{Ho}(f^{\flat,\heartsuit})$.

We now verify that A satisfies all the axioms of a dependent adder.

For any category I we have a natural isomorphism

$$\operatorname*{colim}_{i\in I}^{oplax} 1 \cong I$$

and this implies that for any map $x : U \to Cat$ in $hCAT$ we have a strict equality $\sum^{x} const_{1_A} = x$ of morphisms in $hCAT$, so it satisfies the Right Unit Axiom.

The Left Unit Axiom follows from the fact that if I is a small category and $F_I : 1 \to Cat$ is the functor sending the unique object $*$ from the terminal category 1 to $I \in Cat$, then I is the oplax colimit of F_I.

$$\operatorname*{colim}_{i\in 1}^{oplax} F_I(i) \cong F_I(*) = I$$

The Sum Associativity Axiom follows from the fact that for any category I, functor $F : I \to Cat$ and $G : \operatorname*{colim}_{i\in I}^{oplax} F(i) \to Cat$ there is a natural isomoprhism

$$\operatorname*{colim}_{(i,j)\in \operatorname*{colim}_{i\in I}^{oplax} F(i)}^{oplax} G(i,j) \to \operatorname*{colim}_{i\in I}^{oplax} \operatorname*{colim}_{j\in F(i)}^{oplax} G(i,j)$$

For the Flatten Associativity Axiom one needs to verify for every $x : U \to Cat$, $f : \operatorname*{colim}_{u\in U}^{oplax} x(u) \to Cat$ and $g : \operatorname*{colim}_{u\in U}^{oplax} \operatorname*{colim}_{i\in x(u)}^{oplax} f(u,i) \to Cat$ that the following diagram commutes up to natural isomorphism

$$\begin{tikzcd}
\operatorname*{colim}_{(u,i,j)\in \operatorname*{colim}_{(u,i)\in \operatorname*{colim}_{u\in U}^{oplax} x(u)}^{oplax} f(u,i)}^{oplax} g(u,i,j) \arrow[r] \arrow[d] & \operatorname*{colim}_{(u,i)\in \operatorname*{colim}_{u\in U}^{oplax} x(u)}^{oplax} \operatorname*{colim}_{j\in f(u,i)}^{oplax} g(u,i,j) \arrow[d] \\
\operatorname*{colim}_{(u,i,j)\in \operatorname*{colim}_{u\in U}^{oplax} \operatorname*{colim}_{i\in x(u)}^{oplax} f(u,i)}^{oplax} g(u,i,j) \arrow[r] & \operatorname*{colim}_{u\in U}^{oplax} \operatorname*{colim}_{i\in x(u)}^{oplax} \operatorname*{colim}_{j\in f(u,i)}^{oplax} g(u,i,j)
\end{tikzcd}$$

where all maps are constructed canonically using the universal properties of the oplax colimits. If this diagram commutes up to isomorphism in CAT then the corresponding diagram in $hCAT$ commutes strictly, and then Cat satisfies the Flatten Associativity Axiom.

Cat also satisfies the Fubini axiom because oplax colimits commute, so it is a commutative dependent adder.

3.4.3. *Lax colimits as sums.* One can also make the category of small categories Cat into a dependent adder with lax colimits instead of oplax colimits. For a functor $F : I \to CAT$ we can define the lax limit $\operatorname*{colim}^{lax}_{i\in I} F(i)$ of F as the opposite of the oplax colimit of the composite $I \xrightarrow{F} CAT \xrightarrow{op} CAT$.

$$\operatorname*{colim}^{lax}_{i\in I} F(i) := (\operatorname*{colim}^{oplax}_{i\in I} F(i)^{op})^{op}$$

This is also known as the Grothendieck construction $\int F$.
We have a functor $\iota : Cat \to CAT$ defined by $\iota(I) := I^{op}$. It corresponds to a Grothendieck op-fibration $p : F \to Cat$, whose fiber over some $I \in Cat$ is isomorphic to I^{op}.
Given a functor $F : I^{op} \to Cat$ we define the sum of F to be its lax colimit / Grothendieck construction.

$$\sum^{I} F := \operatorname*{colim}^{lax}_{i\in I^{op}} F(i) = \int F$$

We then have natural isomorphisms

$$\operatorname*{colim}^{lax}_{i\in I^{op}} 1 \cong I$$

$$\operatorname*{colim}^{lax}_{(i,j)\in(\operatorname*{colim}^{lax}_{i\in I^{op}} F(i))^{op}} G(i,j) \cong \operatorname*{colim}^{lax}_{i\in I^{op}} \operatorname*{colim}^{lax}_{j\in F(i)^{op}} G(i,j)$$

which make Cat into a dependent adder in $hCAT$.

3.5. **Commutative $\mathbb{Q}$-algebras.** Let R be a commutative $\mathbb{Q}$-algebra. Let $\mathscr{C} := Sch/R$ be the category of schemes over Spec(R). Let $A := \mathbb{A}^1_R$ be the affine line. We define $F := \mathbb{A}^2_R$ and let $p : F \to \mathbb{A}^1_R$ be the projection to the second variable. For every $x : U \to \mathbb{A}^1_R$ we have $[\![x]\!] \cong \mathbb{A}^1_U$. Let $S := \Gamma(U, \mathscr{O}_U)$ be the ring of global sections of U. Then morphisms $x : U \to \mathbb{A}^1_R$ correspond to elements $x \in S$, and morphisms $p : \mathbb{A}^1_U \to \mathbb{A}^1_R$ corresponds to elements $p \in S[X]$. So for every $x \in S$ and every $p \in S[X]$ we need to define a sum $\sum^{x} p \in S$. For this we will use Faulhaber's formula [7]. For any $d \in \mathbb{N}$ define the d-th Faulhaber polynomial $F_d \in S[X]$ by

$$F_d := \frac{1}{d+1} \sum_{n=0}^{d} \binom{d+1}{n} B_n X^{d-n+1}$$

where $\binom{d+1}{n}$ is the binomial coefficient and B_n is the Bernoulli number with the convention $B_1 = +\frac{1}{2}$. Faulhaber's formula states that for any $d, n \in \mathbb{N}$

$$\sum_{k=1}^{n} k^d = F_d(n)$$

For any polynomial $p \in S[X]$ we can write p as a sum of monomials $p = \sum_{i=0}^{n} p_i \cdot X^i$ and then define $\sum(p) \in S[X]$ by

$$\sum(p) := \sum_{i=0}^{n} p_i \cdot F_i$$

Then $\sum(p)$ is a polynomial satisfying $\sum(p)(m) = \sum_{i=1}^{m} p(i)$ for all $m \in \mathbb{N}$.

We define for all $x \in S$ and $p \in S[X]$ that $\sum^{x}(p) := \sum(p)(x)$.
With this we have defined the dependent sum function of the dependent adder $\mathbb{A}^1_R$.
Next, for every $p \in S[X]$ we need to define a flattening function $p^\flat : \mathbb{A}^2_U \to \mathbb{A}^1_U$ over U. Such a morphism corresponds to an element in $S[X, Y]$. We define $p^\flat \in S[X, Y]$ by $p^\flat := Y + \sum^{X-1}(p)$.
We now need to verify the axioms of a dependent adder. The Left Unit Axiom is true because $\sum^{x} 1 = F_0(x) = x$. The Right Unit Axiom is true because $\sum^{const_1}(p \cdot const_1^\flat) = p(1 + \sum^{X-1} 1) = p(X) = p$.
For the Sum Associativity Axiom we need to show for all R-algebras S and all $x \in S$, $f, g \in S[X]$, that $\sum^{\sum^{x}(f)}(g) = \sum^{x}(\sum^{f} g \circ f^\flat)$ in S. To show it we will use the following lemma:

Lemma 3.3. Let B be a commutative $\mathbb{Q}$-algebra. If for two polynomials $p, q \in B[X]$ we have $p(n) = q(n)$ for all $n \in \mathbb{N}$, then $p = q$ in $B[X]$.

Proof. We claim that for all $f \in B[X]$, if $f(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ in $B[X]$. Once we have shown this claim the lemma follows with $f := p - q$. We show this claim by induction over the degree d of f.
Suppose that for all polynomials g of degree d, if $g(n) = 0$ for all $n \in \mathbb{N}$, then $g = 0$.
Let $f \in B[X]$ be a polynomial of degree $d + 1$ such that $f(n) = 0$ for all $n \in \mathbb{N}$. Write $f = \sum_{i=0}^{d+1} f_i X^i$. Then $0 = f(0) = f_0$. So $f = X \cdot (\sum_{i=1}^{d+1} f_i X^{i-1})$. Let $g := \sum_{i=1}^{d+1} f_i X^{i-1}$. For all $n \in \mathbb{N}$ with $n \neq 0$ we have $g(n) = \frac{f(n)}{n} = 0$. So $g(X + 1) = \sum_{i=1}^{d+1} f_i (X + 1)^{i-1}$ is a polynomial of degree d that vanishes on all of $\mathbb{N}$. By inductive hypothesis $g(X + 1) = 0$ in $B[X]$. This then implies $g = 0$ and then $f = 0$. $\square$

For $p \in S[T]$ and $n \in \mathbb{N}$ we know that $\sum^{n+1}(p) = p(n+1) + \sum^{n}(p)$. With Lemma 3.3 it follows that for all $x \in S$ we have $\sum^{x+1}(p) = p(x+1) + \sum^{x}(p)$. We can then show inductively for all $n \in \mathbb{N}$, $x \in S$ and $p \in S[T]$ that $\sum^{x+n}(p) = \sum^{x}(p) + \sum^{n}(p(T+x))$. By Lemma 3.3 it follows that for all $x, y \in S$ and $p \in S[T]$ that $\sum^{x+y}(p) = \sum^{x}(p) + \sum^{y}(p(T+x))$. We can then inductively show for all $n \in \mathbb{N}$, and $f, g \in S[T]$ that

$$\sum^{\sum\limits^{n}(f)}(g) = \sum_{j=1}^{\sum\limits_{i=1}^{n} f(i)} g(j) = \sum_{i=1}^{n}\sum_{j=1}^{f(i)} g(j + \sum_{k=1}^{i-1} f(k)) = \sum^{n}(\sum^{f}(g \circ f^{\flat}))$$

From Lemma 3.3 we then get the Sum Associativity Axiom for all $x \in S$ and all $f, g \in S[T]$.

As soon as one has the Sum Associativity Axiom, one can prove the Flatten Associativity Axiom exactly like we did for the natural numbers in Section 3.1.

We also claim that $\mathbb{A}^1_R$ satisfies the Fubini axiom. This can be shown as follows. Firstly we have for all $n \in \mathbb{N}$ and $p, q \in S[T]$ that $\sum^{n}(p+q) = \sum^{n}(p) + \sum^{n}(q)$. So Lemma 3.3 implies for all $p, q \in S[T]$ and $x \in S$ that $\sum^{x}(p+q) = \sum^{x}(p) + \sum^{x}(q)$. One can then show inductively that for all $n \in \mathbb{N}$, $x \in S$ and $p \in S[T_1][T_2]$ that $\sum^{x}(\sum^{n}(p)) = \sum^{n}(\sum^{x}(switch(p))$ where $switch : S[T_1][T_2] \to S[T_1][T_2]$ is isomorphism exchanging T_1 and T_2, and then the Fubini axiom follows from Lemma 3.3.

3.6. **All the other examples.**

(1) The set of small ordinals Ord is a dependent adder in the category of moderately small sets SET. We take the function $p : F \to Ord$ whose fiber $p^{-1}(\{\alpha\})$ at some $\alpha \in Ord$ is the underlying set of α. For every α we define the α-dependent sum function

$$\sum^{\alpha} : \mathrm{Hom}_{SET}(\alpha, Ord) \to Ord$$

by transfinite induction over α. $\sum^{0}$ is defined by $\sum^{0} f := 0$. $\sum^{\alpha+1}$ is defined by $\sum^{\alpha+1} f := (\sum^{\alpha}(f|_\alpha)) + f(\alpha)$. Here $f|_\alpha$ means the restriction of f to $\alpha \subseteq \alpha + 1$. Also, it is important here to put the $f(\alpha)$ on the right of the sum, and not on the left. If $\alpha = \bigcup\limits_{\beta<\alpha} \beta$ is a limit ordinal, then we define $\sum^{\alpha} f := \sup_{\beta<\alpha} \sum^{\beta} f$. These α-dependent sum functions then assemble together for every $x : U \to Ord$ into an x-dependent sum function

$$\sum^{x} : \mathrm{Hom}_{SET}([\![x]\!], Ord) \to \mathrm{Hom}_{SET}(U, Ord)$$

Note that the indexing conventions here are slightly shifted compared to those from Section 3.1, because here $\sum^{\alpha} f$ refers to the sum of $f(i)$ for all $0 \le i < \alpha$, while in Section 3.1 the sum $\sum^{n} f$ referred to the sum of $f(i)$ for all $1 \le i \le n$.

For $f : [\![x]\!] \to Ord$ the flattening function $f^{\flat} : \coprod_{u\in U} \coprod_{\alpha\in x(u)} f(\alpha) \to \coprod_{u\in U} \sum^{x(u)}(f)$ is defined by $f^{\flat}(u, i, j) := (u, (\sum^{i}(f)) + j)$. It is important that j is here on the right side of the sum.

One can then easily verify all the axioms of dependent adders by using a lot of transfinite induction.

(2) The "set of all small cardinals" $Card$ is a dependent adder in the category of moderately small sets SET. It works exactly like the ordinal numbers Ord.

(3) The "field with one element" $\mathbb{F}_1 = \{0, 1\}$ works exactly like the natural numbers $\mathbb{N}$. See Section 3.1.

(4) The unit interval $[0, 1]$ works exactly like the real numbers $\mathbb{R}_{\ge 0}$. See Section 3.2.

(5) The integers $\mathbb{Z}$ are a dependent adder in Set. We define $p : F \to \mathbb{Z}$ to be the function whose fibers are $\mathbb{Z}$ everywhere. So p is the second product projection $p : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Given some $x : U \to \mathbb{Z}$ and a function $f : U \times \mathbb{Z} \to \mathbb{Z}$ we define its sum by

$$\sum^{x}(f)(u) := \begin{cases} \sum_{i=1}^{x(u)} f(u, i) & , x(u) \ge 0 \\ \sum_{i=1}^{-x(u)} -f(u, 1-i) & , x(u) < 0 \end{cases}$$

The flattening function $f^{\flat} : U \times \mathbb{Z} \times \mathbb{Z} \to U \times \mathbb{Z}$ is given by $f^{\flat}(u, i, j) := (u, j + \sum_{k=1}^{i-1} f(u, k))$. This makes $\mathbb{Z}$ into a dependent adder.

Just as a remark: The reason we don't take for p the function whose fiber over n is $\{1, \ldots, |n|\}$, is because then the usual flattening map $f^\flat : \coprod_{u \in U} \coprod_{i \in \{1,\dots,|x(u)|\}} \{1,\dots,|f(i)|\} \to \coprod_{u \in U} \{1,\dots,\sum_{i=1}^{x(u)} f(u,i)\}$ would be impossible to define, because if f takes on some negative values, then $j + \sum_{k=1}^{i-1} f(u,k)$ might not lie in $\{1,\dots,\sum_{i=1}^{x(u)} f(u,i)\}$. If a dependent adder contains negative numbers, one has to add some redundant information in the fibers of p.

(6) The reals $\mathbb{R}$ work exactly like the non-negative reals $\mathbb{R}_{\geq 0}$ from Section 3.2, except like for the integers $\mathbb{Z}$ we let $p : F \to \mathbb{R}$ be the second product projection $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

(7) The "complete local ring at infinity" $[-1,1]$ works exactly like $\mathbb{R}$ above.

(8) For the complex numbers $\mathbb{C}$ we choose as background category $\mathscr{C}$ the category of complex analytic spaces from [3]. This category has pullbacks by [3, Corollary 0.32]. In this category a morphism $\mathbb{C} \to \mathbb{C}$ is an entire holomorphic function $\mathbb{C} \to \mathbb{C}$. Making $\mathbb{C}$ into a dependent adder in this category works very similar to the real numbers $\mathbb{R}$.

Finally a class of degenerate examples not mentioned in the introduction: If $\mathscr{C}$ is a category with pullbacks and M is a monoid in $\mathscr{C}$, then $A := M$ can be made into a dependent adder in $\mathscr{C}$. We define $F := M$, $p := id_M : F \to M$. Then for every $x : U \to M$ we have $[\![x]\!] \cong U$.We define a natural function $\mathrm{Hom}_{\mathscr{C}}(U,M) \to \mathrm{Hom}_{\mathscr{C}}(U,M)$ by sending $y : U \to M$ to the multiplication $y \cdot x : U \to M$. For every $f : [\![x]\!] = U \to M$ we define $f^\flat := id_U : [\![f]\!] = U \to U = [\![\sum^x f]\!]$. With this M becomes a dependent adder.

4. Right Modules over Dependent Adders

Let $\mathscr{C}$ be a category with pullbacks and A a dependent adder in $\mathscr{C}$.

Definition 4.1. A *dependent right A-module* consists of

(1) An object M in $\mathscr{C}$

(2) For every $x : U \to A$ a dependent sum function

$$\sum^x : \mathrm{Hom}_{\mathscr{C}}([\![x]\!], M) \to \mathrm{Hom}_{\mathscr{C}}(U, M)$$

natural in $x \in \mathscr{C}/A$.

For $x : U \to A$, $f : [\![x]\!] \to A$ and $g : [\![\sum^x f]\!] \to M$ we define $f \boxtimes g := \sum^f g \circ f^\flat$.

satisfying the following axioms

(1) Unit axiom: For $x : U \to A$ and $f : [\![x]\!] \to M$

$$\sum^{const_1} f \circ const_1^\flat = f$$

(2) Sum associativity axiom: For $x : U \to A$, $f : [\![x]\!] \to A$ and $g : [\![\sum^x f]\!] \to M$

$$\sum^{\sum^x f} g = \sum^x \sum^f g \circ f^\flat$$

While we are at the present moment still hesitant to provide a definition of morphisms of dependent adders, we do not hesitate to define morphisms of dependent right modules.

Definition 4.2. Given two dependent right A-modules M and N, an *A-linear map* $M \to N$ consists of a morphism $\phi : M \to N$ in $\mathscr{C}$, such that for every $x : U \to A$ in $\mathscr{C}$ and $f : [\![x]\!] \to M$ we have

$$\phi \circ (\sum^x (f)) = \sum^x (\phi \circ f)$$

We now provide a few examples of right modules over dependent adders, and the linear maps between them.

4.1. **$\mathbb{F}_1$-modules and pointed sets.** An $\mathbb{F}_1$-dependent right module M is a pointed set. The basepoint is given by the 0-dependent sum of the unique function $\varnothing \to M$.

An $\mathbb{F}_1$-linear map is a basepoint-preserving function.

4.2. **$\mathbb{N}$-modules and monoids.**

Theorem 4.1. A dependent right $\mathbb{N}$-module M is the same thing as a monoid.

Proof. Given any monoid M we obtain an $\mathbb{N}$-dependent module in the following way: We write the binary operation of M by $+$. For any $n \in \mathbb{N}$ and function $f : \{1, \dots, n\} \to M$ we write $\sum_{i=1}^{n} f(i)$ for the sum $f(1) + f(2) + \cdots + f(n)$. For every $x : U \to \mathbb{N}$ in Set we define $\sum^x : \mathrm{Hom}_{Set}([\![x]\!], M) \to \mathrm{Hom}_{Set}(U, M)$ by

$$\sum^x (f)(u) := \sum_{i=1}^{x(u)} f(u, i)$$

With these dependent sums M is a dependent right $\mathbb{N}$-module.

Conversely, if M is a dependent right $\mathbb{N}$-module, then we can make M into a monoid. In the following we will use list notation for functions $f : \{1, \ldots, n\} \to S$ going from $\{1, \ldots, n\}$ into some set S. Given $n \in \mathbb{N}$ and n elements $s_1, \ldots, s_n \in S$ we write $\langle s_1, \ldots, s_n \rangle$ for the function $\{1, \ldots, n\} \to S$ sending i to s_i

So for example $\langle 0 \rangle$ is the function $0 : \{1\} \to \mathbb{N}$ sending 1 to 0. We can form the set $[\![\langle 0 \rangle]\!] = \varnothing$, and have a dependent sum function $\sum\limits^{\langle 0 \rangle} : \mathrm{Hom}_{Set}(\varnothing, M) \to \mathrm{Hom}_{Set}(\{1\}, M)$. We have a unique function $\langle \rangle : \varnothing \to M$. and define the unit $0_M \in M$ by defining

$$\langle 0_M \rangle := \sum^{\langle 0 \rangle} \langle \rangle$$

This a priori just defines a function $\langle 0_M \rangle : \{1\} \to M$, but by 0_M we of course just mean the unique element of M in the range of that function.
For two elements $a, b \in M$ we define $a + b \in M$ by

$$\langle a + b \rangle := \sum^{\langle 2 \rangle} \langle a, b \rangle$$

We now need to show that this satisfies the unit axioms and associativity axiom of a monoid. Take some element $a \in M$ and show $0_M + a = a$. In $\mathbb{N}$ we have $\langle 1 \rangle = \langle 0 + 1 \rangle = \sum\limits^{\langle 2 \rangle} \langle 0, 1 \rangle$. So we get

$$\langle a \rangle = \sum^{\langle 1 \rangle} \langle a \rangle = \sum^{\sum^{\langle 2 \rangle} \langle 0,1 \rangle} \langle a \rangle = \sum^{\langle 2 \rangle} \sum^{\langle 0,1 \rangle} \langle a \rangle \circ \langle 0, 1 \rangle^\flat$$

Let $\iota_1 : \{1\} \to \{1, 2\}$ be the function sending 1 to 1, and let $\iota_2 : \{1\} \to \{1, 2\}$ be the function sending 1 to 2. The naturality of dependent sums implies that $(\sum\limits^{\langle 0,1 \rangle} \langle a \rangle \circ \langle 0, 1 \rangle^\flat) \circ \iota_1 = \sum\limits^{\langle 0,1 \rangle \circ \iota_1} \langle a \rangle \circ \langle 0, 1 \rangle^\flat \circ (\iota_1 \underset{A}{\times} id_F) = \sum\limits^{\langle 0 \rangle} \langle \rangle = \langle 0_M \rangle$ and $(\sum\limits^{\langle 0,1 \rangle} \langle a \rangle \circ \langle 0, 1 \rangle^\flat) \circ \iota_2 = \sum\limits^{\langle 0,1 \rangle \circ \iota_2} \langle a \rangle \circ \langle 0, 1 \rangle^\flat \circ (\iota_2 \underset{A}{\times} id_F) = \sum\limits^{\langle 1 \rangle} \langle a \rangle = \langle a \rangle$ so in total we get

$$\langle a \rangle = \sum^{\langle 2 \rangle} \sum^{\langle 0,1 \rangle} \langle a \rangle \circ \langle 0, 1 \rangle^\flat = \sum^{\langle 2 \rangle} \langle 0_M, a \rangle = \langle 0_M + a \rangle$$

so M satisfies the left unit axiom. By a similar argument it also satisfies the right unit axiom $a + 0_M = a$. For associativity we calculate

$$\sum^{\langle 3 \rangle} \langle a, b, c \rangle = \sum^{\sum^{\langle 2 \rangle} \langle 2,1 \rangle} \langle a, b, c \rangle = \sum^{\langle 2 \rangle} \sum^{\langle 2,1 \rangle} \langle a, b, c \rangle \circ \langle 2, 1 \rangle^\flat = \langle (a + b) + c \rangle$$

$$\sum^{\langle 3\rangle}\langle a,b,c\rangle = \sum^{\sum^{\langle 2\rangle}\langle 1,2\rangle}\langle a,b,c\rangle = \sum^{\langle 2\rangle}\sum^{\langle 1,2\rangle}\langle a,b,c\rangle \circ \langle 1,2\rangle^{\flat} = \langle a+(b+c)\rangle$$

so $+$ is associative and M is a monoid. $\square$

Theorem 4.2. A $\mathbb{N}$-linear map between dependent $\mathbb{N}$-modules is the same thing as a monoid homomorphism.

Proof. Given a $\mathbb{N}$-linear map $\phi : M \to N$ we have for every $a, b \in M$ that

$$\phi(0_M) = \phi(\sum^{\langle 0\rangle}\langle\rangle) = \sum^{\langle 0\rangle}\langle\rangle = 0_N$$

and

$$\phi(a+b) = \phi(\sum^{\langle 2\rangle}\langle a,b\rangle) = \sum^{\langle 2\rangle}\langle \phi(a),\phi(b)\rangle = \phi(a)+\phi(b)$$

so ϕ is a monoid homomorphism. Conversely, if $\phi : M \to N$ is a monoid homomorphism, then we can show by induction that $\phi(\sum_{i=1}^{x} f(i)) = \sum_{i=1}^{x}\phi(f(i))$ and then ϕ is a $\mathbb{N}$-linear map. $\square$

4.3. **$\mathbb{R}$-modules and Banach spaces.** Let $\mathbb{R}$ be the dependent adder of all (possibly negative) real numbers.

Lemma 4.3. Giving a dependent right $\mathbb{R}$-module is equivalent to the following data: We have a metrizable space M, and for every $x \in \mathbb{R}$ and continuous function $f : \mathbb{R} \to M$ an element $\int_0^x f(t)dt$ in M, such that

(1) (Continuity): For every metrizable space U, continuous function $x : U \to \mathbb{R}$ and continuous function $f : U \times \mathbb{R} \to M$ the function $\int_0^{x(u)} f(u,t)dt$ is continuous in u.
(2) (Unit): For every $m \in M$, if $const_m : \mathbb{R} \to M$ denotes the constant m function then we have
$$\int_0^1 const_m(t)dt = m$$
(3) (Substitution): For every $x \in \mathbb{R}$, continuous function $f : \mathbb{R} \to M$ and continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ we have that
$$\int_0^{h(x)} f(t)dt = \int_0^x f(h(t)) \cdot h'(t)dt$$

Proof. If the Continuity Axiom is satisfied, then the function sending $x : U \to \mathbb{R}$ and $f : \mathbb{R} \to M$ to $\int\limits_0^{x(u)} f(u,t)dt$ serves as a natural dependent sum function $\sum\limits^{x} f$ for the dependent right $\mathbb{R}$-module M. Also every function $\sum\limits^{x} : \mathrm{Hom}_{Top_{mtr}}([\![x]\!], M) \to \mathrm{Hom}_{Top_{mtr}}(U, M)$ that is natural in $x : U \to \mathbb{R}$ is necessarily of the above form, because for any $u \in U$ we have a map $\langle u \rangle : 1 \to U$, and then naturality tells us that $(\sum\limits^{x} f) \circ \langle u \rangle = \sum\limits^{x \circ \langle u \rangle} f \circ (\langle u \rangle \underset{A}{\times} id_F)$, and this implies that the dependent sum functions for all maps of the form $U \to \mathbb{R}$ are determined by dependent sum functions for maps of the form $1 \to \mathbb{R}$

The Unit Axiom we stated above is easily seen to be equivalent to the unit axiom of dependent right $\mathbb{R}$-modules. The Substitution Axiom stated above is equivalent to the Sum Associativity Axiom of dependent right $\mathbb{R}$-modules, because of the Fundamental Theorem of Calculus. □

So in summary, a dependent right $\mathbb{R}$-module is a space in which one can continuously form integrals of functions, and where one has a "integration by substitution" rule.

If one wants to study dependent right $\mathbb{R}$-modules, it is probably a good idea to only look at modules that have a zero element $0_M \in M$ satisfying $\int\limits_0^0 f(t)dt = 0_M$ for all $f : \mathbb{R} \to M$. Without such a zero element, dependent right $\mathbb{R}$-modules can be empty or disconnected, while with a zero element they are always path-connected.

For example, every Banach space is a dependent right $\mathbb{R}$-module, with Bochner integrals as dependent sums.

Theorem 4.3. Let X be a Banach space. For every continuous function $x : U \to \mathbb{R}$ and continuous function $f : U \times \mathbb{R} \to X$ define $\sum\limits^{x}(f) : U \to X$ by using the Bochner integral

$$\sum^{x}(f)(u) := \int\limits_0^{x(u)} f(u,t)dt = \begin{cases} \int\limits_{[0,x(u)]} f(u,t)dt & , x(u) \geq 0 \\ -\int\limits_{[x(u),0]} f(u,t)dt & , x(u) < 0 \end{cases}$$

Then X is a right $\mathbb{R}$-module.

Proof. Take $x : U \to \mathbb{R}$, $f : [\![x]\!] \to X$ and $u \in U$. Assume without loss of generality $x(u) \geq 0$. We need to show that the integral $\int\limits_{[0,x(u)]} f(u,t)dt$

exists. Since $f(u,-) : [0, x(u)] \to X$ is a continuous function, it is Bochner-measurable. The composite function $[0, x(u)] \overset{f(u,-)}{\to} X \overset{||.||_X}{\to} \mathbb{R}$ is Lebesgue-integrable, because it is continuous and $[0, x(u)]$ is compact. Therefore by Bochner's criterium, the function $f(u,-) : [0, x(u)] \to X$ is Bochner-integrable, and the integral $\int\limits_{[0,x(u)]} f(u,t)dt$ exists in X.
The Continuity Axiom can be shown exactly like in Lemma 3.1.
The Unit Axiom $\int\limits_{[0,1]} xdt = x$ is obvious, because the constant x function is a simple function.
Let us now prove the Substitution Axiom. For any differentiable function $f : [0, x] \to [0, y]$ between real intervals and every differentiable function $g : [0, y] \to X$ going into a Banach space X, we have the chain rule $(g \circ f)'(t) = g'(f(t)) \cdot f'(t)$.
Also for any continuously differentiable function $f : [0, x] \to X$ going into a Banach space X we have a Fundamental Theorem of Calculus, stating $f(x) - f(0) = \int\limits_0^x f'(t)dt$. This is proven in [4, Proposition A.2.3].
By combining the chain rule with the Fundamental Theorem of Calculus we obtain the substitution rule. □

Theorem 4.4. Let X, Y be Banach spaces. Then a continuous linear operator $T : X \to Y$ is the same thing as an $\mathbb{R}$-linear map $T : X \to Y$ in the dependent module sense between the corresponding dependent $\mathbb{R}$-modules.

Proof. If $T : X \to Y$ is a continuous linear operator between Banach spaces, then for every $x \in \mathbb{R}$ and continuous function $f : \mathbb{R} \to X$ we have that

$$T \int\limits_0^x f(t)dt = \int\limits_0^x Tf(t)dt$$

because every Bochner integral is a limit of integrals of simple function, and continuous functions commute with limits, and linear operators commute with integrals of simple functions. This implies that T is a $\mathbb{R}$-linear map in the dependent module sense. Conversely, take an $\mathbb{R}$-linear map in the dependent module sense $T : X \to Y$. By definition T is a morphism in Top_{mtr}, so T is continuous. We need to show that for all $x, y \in X$ that $T(x + y) = T(x) + T(y)$. Define $\phi_{x,y} : [0, 1] \to X$, $\phi(t) := t \cdot y + (1 - t) \cdot x$.

For every $\epsilon > 0$ with $\epsilon < 1$ define $\gamma_\epsilon : [0,2] \to X$ by

$$\gamma_\epsilon(t) := \begin{cases} x & , t < 1-\epsilon \\ y & , t > 1+\epsilon \\ \phi_{x,y}(\frac{t-1+\epsilon}{2\epsilon}) & , 1-\epsilon \le t \le 1+\epsilon \end{cases}$$

Then each γ_ϵ is continuous and

$$T(x+y) = T(\lim_{\epsilon\to 0}\int_0^2 \gamma_\epsilon(t)dt) = \lim_{\epsilon\to 0}\int_0^2 T(\gamma_\epsilon(t))dt = T(x) + T(y)$$

so T is a linear operator in the usual Banach space sense. □

4.4. *Cat*-**modules and cocomplete categories.** If M is a cocomplete category, then M is a Cat-module in the following way:
For every $x : U \to Cat$ we define a function

$$\sum^{x} : \mathrm{Hom}_{hCAT}([\![x]\!], M) \to \mathrm{Hom}_{hCAT}(U, M)$$

in the following way: For every $f : [\![x]\!] \to M$ and we define a functor $\sum^{x}(f)^{\heartsuit} : U \to M$ by sending $u \in U$ to

$$\sum^{x}(f)^{\heartsuit}(u) := \operatorname*{colim}_{i\in x(u)} f(u,i)$$

and sending a morphism $\alpha : u \to v$ in U to the canonical map

$$\operatorname*{colim}_{i\in x(u)} f(u,i) \to \operatorname*{colim}_{i\in x(v)} f(v,i)$$

in M. We define $\sum^{x}(f) := \mathrm{Ho}(\sum^{x}(f)^{\heartsuit})$. Then M satisfies the Unit Axiom, because if we have an object $m \in M$ and consider the diagram $f : 1 \to M$ sending the unique object of 1 to m, then the colimit of f is isomorphic to m again. M satisfies the Sum Associativity axiom because for every functor $F : I \to Cat$ and $G : \operatorname*{colim}^{oplax}_{i\in I} F(i) \to M$ we have a canonical isomorphism

$$\operatorname*{colim}_{(i,j)\in\operatorname*{colim}^{oplax}_{i\in I} F(i)} G(i,j) \cong \operatorname*{colim}_{i\in I}\operatorname*{colim}_{j\in F(i)} G(i,j)$$

in M. So M is a right Cat-module.
If M is a complete category, then M is also a Cat-module, because in any such category we have a canonical isomorphism

$$\lim_{(i,j)\in\operatorname*{colim}^{oplax}_{i\in I} F(i)} G(i,j) \cong \lim_{i\in I}\lim_{j\in F(i)} G(i,j)$$

Theorem 4.5. Let C, D be cocomplete categories, and regard C, D as Cat-modules with colimits as dependent sums. Then a functor $F : C \to D$ is a Cat-linear map if and only if F preserves small colimits, in the usual sense of sending small colimit cocones to small colimit cocones.

Proof. Let $F : C \to D$ be a Cat-linear map. Let $G : I \to C$ be a functor. It is immediately clear that $F(\operatorname{colim}_{i \in I} G(i)) \cong \operatorname{colim}_{i \in I} F(G(i))$. But for F to preserve colimits it does not just need to send colimit objects to colimit objects, but it needs to preserve colimit cocones. Let $U = (\cdot \to \cdot)$ be the category with two objects and one morphism between them. Let $x : U \to Cat$ be the constant functor sending everything to $I \in Cat$. Then $[\![x]\!] \cong I \times U$. To define a functor $g : I \times U \to Cat$ we need to define two functors $g_0, g_1 : I \to Cat$ and a natural transformation $\tau : g_0 \to g_1$. Let $g_0 := G$. Let g_1 be the constant functor sending every $i \in I$ to $\operatorname{colim}_{j \in I} G(j)$. Let $\tau : g_0 \to g_1$ be the colimiting cocone of G. So for every $i \in I$ the map $\tau_i : G(i) \to \operatorname{colim}_{j \in I} G(j)$ is the canonical colimit inclusion map.
Since F is a Cat-linear map we have

$$F \circ \sum^{x} g = \sum^{x} F \circ g$$

This means we have a commutative diagram

$$\begin{tikzcd} F(\operatorname{colim}_{i \in I} G(i)) \arrow[r, "\sim"] \arrow[d] & \operatorname{colim}_{i \in I} F(G(i)) \arrow[d] \\ F(\operatorname{colim}_{i \in I} g_1(i)) \arrow[r] & \operatorname{colim}_{i \in I} F(g_1(i)) \end{tikzcd}$$

Now g_1 is a constant functor, so $\operatorname{colim}_{i \in I} g_1(i) = \operatorname{colim}_{i \in I} G(i)$ and $\operatorname{colim}_{i \in I} F(g_1(i)) = F(\operatorname{colim}_{i \in I} G(i))$. The above diagram then implies that F preserves not just the colimit object but the whole colimiting cocone. $\square$

Similarly, if C, D are complete categories regarded as Cat-modules with limits as dependent sums, then a functor $F : C \to D$ is a Cat-linear map if F preserves small limits.

5. Left Modules over Dependent Adders

Definition 5.1. Given a category with pullbacks $\mathscr{C}$ and a dependent adder A, an *A-dependent left module* consists of

(1) An object $M \in \mathscr{C}$

(2) A morphism $p : F_M \to M$ in $\mathscr{C}$. For every $m : U \to M$ we write $[\![m]\!]_M$ for the pullback $U \underset{M}{\times} F_M$.

(3) For every $m : U \to M$ a function of sets

$$\sum^{m} : \mathrm{Hom}_{\mathscr{C}}([\![m]\!]_M, A) \to \mathrm{Hom}_{\mathscr{C}}(U, M)$$

natural in $m \in \mathscr{C}/M$.

(4) For every $m : U \to M$ and $f : [\![m]\!]_M \to A$ a flattening function

$$f^{M,\flat} : [\![f]\!] \to [\![\sum^{m} f]\!]_M$$

over U. For $m : U \to M$, $f : [\![m]\!]_M \to A$ and $g : [\![\sum^{m} f]\!]_M \to A$ we define $f \boxtimes g := \sum^{f} g \circ f^{M,\flat}$.

such that the following axioms are satisfied

(1) Unit Axiom: For $m : U \to M$ and $const_1 : [\![m]\!]_M \to A$ the constant 1_A function, we demand

$$\sum^{m} const_1 = m$$

(2) Sum Associativity Axiom: For $m : U \to M$, $f : [\![m]\!]_M \to A$ and $g : [\![\sum^{m} f]\!]_M \to A$ we demand that

$$\sum^{\sum^{m} f} g = \sum^{m} \sum^{f} g \circ f^{M,\flat}$$

(3) Flatten Associativity Axiom For $m : U \to M$, $f : [\![m]\!]_M \to A$ and $g : [\![\sum^{m} f]\!]_M \to A$ the following diagram commutes

$$\begin{tikzcd}
{[\![g \circ f^{M,\flat}]\!]} \arrow[r, "(g \circ f^{M,\flat})^{\flat}"] \arrow[d, "f^{M,\flat} \underset{A}{\times} id_F"'] & {[\![f \boxtimes g]\!]} \arrow[d, "(f \boxtimes g)^{M,\flat}"] \\
{[\![g]\!]} \arrow[r, "g^{M,\flat}"] & {[\![\sum^{x} f \boxtimes g]\!]}
\end{tikzcd}$$

The interval $[0, n]$ has both a right and a left $[0, 1]$-dependent module structure. The right module structure comes from the fact that for any $x \in [0, 1]$ and continuous map $f : [0, x] \to [0, n]$ we have $\int_0^x f(t)dt \in [0, n]$. The left module structure comes from the fact that for any $x \in [0, n]$ and continuous map $f : [0, x] \to [0, 1]$ we have $\int_0^x f(t)dt \in [0, n]$.

5.1. Top_{open} **as a left** Set**-module using étalé spaces of presheaves.** The category of topological spaces and open maps Top_{open} has, up to isomorphism, the structure of a left Set-dependent module. For any topological space $X \in Top_{open}$ we define an X-indexed family of sets to be a presheaf $\mathscr{F}$ on X, and define the sum of such a sheaf to be its étalé space $Et(\mathscr{F})$.
Let $\mathscr{C} = hCAT$ be the homotopy category of small categories from Section 3.4. Let $A := Set$ be the category of very small sets, Set is a dependent adder in $hCAT$, quite similarly to how Cat is a dependent adder in $hCAT$. For any $x : U \to Set$ we have $[\![x]\!] \cong \operatorname{colim}^{oplax}_{u\in U} x(u)$, where $x(u)$ is regarded as a discrete category. The dependent sums of Set are the coproducts: Given $f : [\![x]\!] \to Set$ we define $\sum^x f := \mathrm{Ho}(\lambda u. \coprod_{t\in x(u)} f(u,t))$. The flattening map of $f^\flat$ of f is given by taking Ho of the natural isomorphism

$$\operatorname*{colim}^{oplax}_{(u,t)\in \operatorname{colim}^{oplax}_{u\in U} x(u)} f(u,t) \to \operatorname*{colim}^{oplax}_{u\in U} \coprod_{t\in x(u)} f(u,t)$$

where one needs to note that $\coprod_{t\in x(u)} f(u,t)$ is in fact the same thing as $\operatorname{colim}^{oplax}_{t\in x(u)} f(u,t)$, because oplax colimits over discrete categories are just coproducts. With this Set is a dependent adder in $hCAT$.
Now let $M := Top_{open}$ the category of very small topological spaces with open maps.
We quickly recall the construction of the étalé space $Et(\mathscr{F})$ of a presheaf $\mathscr{F}$ on a topological space X. See [6, Section II.6] for a reference. For each $x \in X$ let $\mathscr{F}_x$ be the stalk of $\mathscr{F}$ at x. The underlying set of $Et(\mathscr{F})$ is the coproduct of all the stalks of $\mathscr{F}$.

$$Et(\mathscr{F}) = \coprod_{x\in X} \mathscr{F}_x$$

For every open subset $U \subseteq X$ and section $s \in \mathscr{F}(U)$, we have for every $x \in U$ an element $s_x \in \mathscr{F}_x$ and can define the set $\epsilon_{\mathscr{F}}(U,s) := \{(x,s_x) | x \in U\} \subseteq Et(\mathscr{F})$. We put on $Et(\mathscr{F})$ the topology generated by the sets $\epsilon_{\mathscr{F}}(U,s)$ for all open $U \subseteq X$ and $s \in \mathscr{F}(U)$. The notation $\epsilon_{\mathscr{F}}(U,s)$ will be used below a few times.
Étalé spaces are functorial: If $\mathscr{F},\mathscr{G}$ are presheaves on X and $\tau : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then for every $x \in X$ we get a map on stalks $\tau_x : \mathscr{F}_x \to \mathscr{G}_x$, and these assemble together into a map on étalé spaces $Et(\tau) : Et(\mathscr{F}) \to Et(\mathscr{G})$. This map is always an open map, because $Et(\tau)(\epsilon_{\mathscr{F}}(U,s)) = \epsilon_{\mathscr{G}}(U,\tau_U(s))$. With this construction $Et : \mathrm{Psh}(X) \to$

Top_{open} is a functor from the category of presheaves on X to the category of topological spaces with open maps.
For any open map $f : X \to Y$ in Top_{open} and every presheaf $\mathscr{F}$ on Y define a presheaf $f^*(\mathscr{F})$ on X by sending $U \in \mathrm{Ouv}(X)$ to $f^*(\mathscr{F})(U) := \mathscr{F}(f(U))$.

Lemma 5.2. For every open map $f : X \to Y$ in Top_{open} and every presheaf $\mathscr{F}$ on Y there is a natural open map

$$Et_f : Et(f^*(\mathscr{F})) \to Et(\mathscr{F})$$

in Top_{open}

Proof. For every $x \in X$ we have a map on stalks

$$f^*(\mathscr{F})_x = \operatorname*{colim}_{x\in U\in \mathrm{Ouv}(X)} \mathscr{F}(f(U)) \to \operatorname*{colim}_{x\in V\in \mathrm{Ouv}(Y)} \mathscr{F}(V) = \mathscr{F}_{f(x)}$$

because for every open neighborhodd U of x in X, $f(U)$ is an open neighborhood of $f(x)$ in Y. These maps assemble together into a map

$$Et_f : Et(f^*(\mathscr{F})) = \coprod_{x\in X} f^*(\mathscr{F})_x \to \coprod_{y\in Y} \mathscr{F}_y = Et(\mathscr{F})$$

which is open because $Et_f(\epsilon_{f^*(\mathscr{F})}(U,s)) = \epsilon_{\mathscr{F}}(f(U),s)$. □

Let us now explain how étalé spaces make Top_{open} into a left Set-module.
Consider the functor $\mathrm{Ouv} : Top_{open} \to CAT$, sending $X \in Top$ to the poset $\mathrm{Ouv}(X)^{op}$ of open subsets of X with reverse inclusions as morphisms, and sending an open map $f : X \to Y$ to the image functor $\mathrm{Ouv}(f) : \mathrm{Ouv}(X)^{op} \to \mathrm{Ouv}(Y)^{op}$. The functor Ouv gives rise to a Grothendieck opfibration $p^{\heartsuit} : \operatorname*{colim}^{oplax}_{X\in Top_{open}} \mathrm{Ouv}(X)^{op} \to Top_{open}$.
Define $F_M := \operatorname*{colim}^{oplax}_{X\in Top_{open}} \mathrm{Ouv}(X)^{op}$ and $p_M := \mathrm{Ho}(p^{\heartsuit})$.
For any $X \in Top_{open}$ the fiber of p_M over X is $\mathrm{Ouv}(X)^{op}$, and a morphism $\mathrm{Ouv}(X)^{op} \to Set$ in $hCAT$ is an isomorphism class of presheaves on X.
More generally, for any functor $x : U \to Top_{open}$ in $hCAT$ we have

$$[\![x]\!]_M \cong \operatorname*{colim}^{oplax}_{u\in U} \mathrm{Ouv}(x(u))^{op}$$

For any functor $x : U \to Top_{open}$ in $hCAT$ we need to define a map

$$\sum^{x} : \mathrm{Hom}_{hCAT}([\![x]\!]_M, Set) \to \mathrm{Hom}_{hCAT}(U, Top_{open})$$

For every $f : [\![x]\!]_M \to Set$ in $hCAT$ we define a functor $(\sum^{x} f)^{\heartsuit} : U \to Top_{open}$ the following way: For any object $u \in U$ we have an inclusion functor $\iota_u : \mathrm{Ouv}(x(u))^{op} \to \operatorname*{colim}^{oplax}_{v\in U} \mathrm{Ouv}(x(v))^{op} \cong [\![x]\!]_M$. Then $f \circ \iota_u$:

$\mathrm{Ouv}(x(u))^{op} \to Set$ is a presheaf on $x(u)$. We define $\sum^{x}(f)^{\heartsuit}(u)$ to be the étalé space of this presheaf.

$$\sum^{x}(f)^{\heartsuit}(u) := Et(f \circ \iota_u)$$

Given a morphism $\alpha : u \to v$ in U we get a natural transformation $\iota_u \Rightarrow \iota_v \circ \mathrm{Ouv}(x(\alpha))$, where $x(\alpha) : \mathrm{Ouv}(x(u))^{op} \to \mathrm{Ouv}(x(v))^{op}$ is the image functor associated to the open map $x(\alpha)$. This then induces a morphism of presheaves $\tau_\alpha : f \circ \iota_u \to f \circ \iota_v \circ \mathrm{Ouv}(x(\alpha))$, which then induces an open map on étalé spaces $Et(\tau_\alpha) : Et(f \circ \iota_u) \to Et(f \circ \iota_v \circ \mathrm{Ouv}(x(\alpha)))$. By Lemma 5.2 we have a canonical open map $Et(f \circ \iota_v \circ \mathrm{Ouv}(x(\alpha))) \to Et(f \circ \iota_v)$. So in total we obtain an open map

$$\sum^{x}(f)^{\heartsuit}(u) = Et(f \circ \iota_u) \to Et(f \circ \iota_v) = \sum^{x}(f)(v)$$

so $\sum^{x}(f)^{\heartsuit} : U \to Top_{open}$ is a functor.

Now define $\sum^{x}(f) := \mathrm{Ho}(\sum^{x}(f)^{\heartsuit})$. Then $\sum^{x}$ is a natural function

$$\sum^{x} : \mathrm{Hom}_{hCAT}([\![x]\!]_M, Set) \to \mathrm{Hom}_{hCAT}(U, Top_{open})$$

Next we need to define for any small category D, functor $x : D \to Top_{open}$ and functor $f : [\![x]\!]_M \to Set$ the flattening map

$$f^{M,\flat} : \operatorname*{colim}_{d \in D}^{oplax} \operatorname*{colim}_{U \in \mathrm{Ouv}(x(d))^{op}}^{oplax} f(d,U) \to \operatorname*{colim}_{d \in D}^{oplax} \mathrm{Ouv}(Et(f \circ \iota_d))^{op}$$

where the $f(d,U)$ are regarded as discrete categories. Just to make the notation a bit more intuitive, define for every $d \in D$ that $X_d := x(d)$ and $\mathscr{F}_d := f \circ \iota_d$. Then X_d is a topological space and $\mathscr{F}_d$ is a presheaf on X_d, and we have to define a map

$$f^{M,\flat} : \operatorname*{colim}_{d \in D}^{oplax} \operatorname*{colim}_{U \in \mathrm{Ouv}(X_d)^{op}}^{oplax} \mathscr{F}_d(U) \to \operatorname*{colim}_{d \in D}^{oplax} \mathrm{Ouv}(Et(\mathscr{F}_d))^{op}$$

We have for every $d \in D$ a map $\epsilon_{\mathscr{F}_d} : \operatorname*{colim}_{U \in \mathrm{Ouv}(X_d)^{op}}^{oplax} \mathscr{F}_d(U) \to \mathrm{Ouv}(Et(\mathscr{F}_d))^{op}$ sending an open subset $U \subseteq X_d$ and a section $s \in \mathscr{F}_d(U)$ to the open subset $\epsilon_{\mathscr{F}_d}(U,s) = \{(x, s_x) | x \in X_d\}$ of the étalé space $Et(\mathscr{F}_d)$. If we have a morphism $\alpha : (U,s) \to (V,t)$ in $\operatorname*{colim}_{U \in \mathrm{Ouv}(X_d)^{op}}^{oplax} \mathscr{F}_d(U)$ then we have an inclusion $U \supseteq V$ of open subsets and a morphism $s|_V \to t$ in the discrete category $\mathscr{F}_d(V)$. So we have in fact an identity $s|_V = t$. This implies an inclusion of subsets $\epsilon_{\mathscr{F}_d}(U,s) \supseteq \epsilon_{\mathscr{F}_d}(V,t)$ in $Et(\mathscr{F}_d)$. So $\epsilon_{\mathscr{F}_d} : \operatorname*{colim}_{U \in \mathrm{Ouv}(X_d)^{op}}^{oplax} \mathscr{F}_d(U) \to \mathrm{Ouv}(Et(\mathscr{F}_d))^{op}$ is in fact a functor. One can now check that this functor is natural in d. This naturality claim comes down to the assertion that for any morphism $\alpha : d \to e$ in D and $U \in \mathrm{Ouv}(X_d)^{op}$ and $s \in \mathscr{F}_d(U)$ we have

$$\sum^{x}(f)^{\heartsuit}(\alpha)(\epsilon_{\mathscr{F}_d}(U,s)) = \epsilon_{\mathscr{F}_e}(x(\alpha)(U), f(\alpha, id_U)(s))$$

and this assertion is in fact true. With this naturality, we can then use the universal property of the oplax colimit to get a functor $f^{M,\flat,\heartsuit}$ that satisfies $f^{M,\flat,\heartsuit} \circ \iota_d = \epsilon_{\mathscr{F}_d}$. We can then define $f^{M,\flat} := \mathrm{Ho}(f^{M,\flat,\heartsuit})$ and then have a flattening function for our left Set-module Top_{open}.

We now need to verify the axioms of a left Set-module.

Unit Axiom: Let X be a topological space, and $\mathscr{F}$ the constant 1 presheaf $\mathscr{F}(U) = 1$. Then for every $x \in X$ we also have $\mathscr{F}_x \cong 1$, and using this we obtain a homeomorphism $Et(\mathscr{F}) \cong X$ in Top_{open}. Since this homeomorphism is natural in X, this implies the Unit Axiom for the left Set-module Top_{open}.

Sum Associativity Axiom: Take a topological space X, a presheaf $\mathscr{F}$ on X and a presheaf $\mathscr{G}$ on $Et(\mathscr{F})$.

To show the Sum Associativity Axiom we need to show that there is a homeomorphism

$$\sum^{\sum^{X}\mathscr{F}}\mathscr{G} \cong \sum^{X}\sum^{\mathscr{F}}\mathscr{G} \circ \mathscr{F}^{M,\flat}$$

natural in X, $\mathscr{F}$ and $\mathscr{G}$.

The space $\sum^{\sum^{X}\mathscr{F}}\mathscr{G}$ is homeomorphic the étalé space $Et(\mathscr{G})$ of $\mathscr{G}$.

The map $\mathscr{F}^{M,\flat}$ is the functor $\epsilon_{\mathscr{F}} : \operatorname{colim}^{oplax}_{V \in \mathrm{Ouv}(X)^{op}} \mathscr{F}(V) \to \mathrm{Ouv}(Et(\mathscr{F}))^{op}$ sending (V, s) to $\epsilon_{\mathscr{F}}(V, s) = \{(x, s_x) | x \in V\}$.

Let $\mathscr{F} \boxtimes \mathscr{G} := \sum^{\mathscr{F}}\mathscr{G} \circ \mathscr{F}^{M,\flat}$. Then $\mathscr{F} \boxtimes \mathscr{G}$ is isomorphic to the presheaf on X that sends an open subset $V \subseteq X$ to $\coprod_{s \in \mathscr{F}(V)} \mathscr{G}(\epsilon_{\mathscr{F}}(V, s))$.

$$(\mathscr{F} \boxtimes \mathscr{G})(V) \cong \coprod_{s \in \mathscr{F}(V)} \mathscr{G}(\epsilon_{\mathscr{F}}(V, s))$$

To prove the Sum Associativity Axiom we now need to show that there is a natural homeomorphism

$$Et(\mathscr{G}) \cong Et(\mathscr{F} \boxtimes \mathscr{G})$$

Lemma 5.3. For every $x \in X$ there is a natural isomorphism

$$(\mathscr{F} \boxtimes \mathscr{G})_x \cong \coprod_{s \in \mathscr{F}_x} \mathscr{G}_{(x,s)}$$

where $\mathscr{G}_{(x,s)}$ means the stalk of $\mathscr{G}$ at $(x,s) \in \coprod_{x \in X} \mathscr{F}_x = Et(\mathscr{F})$.

Proof. We have $(\mathscr{F} \boxtimes \mathscr{G})_x \cong \operatorname*{colim}_{x \in U \in \mathrm{Ouv}(X)} \coprod_{s \in \mathscr{F}(U)} \mathscr{G}(\epsilon_{\mathscr{F}}(U,s))$. Now for every open neighborhood U of x and every $s \in \mathscr{F}(U)$ we have a map $\mathscr{F}(U) \to \mathscr{F}_x$, and we know that $\epsilon_{\mathscr{F}}(U,s)$ is an open neighborhood of $(x,s) \in Et(\mathscr{F})$ so we have a map $\mathscr{G}(\epsilon_{\mathscr{F}}(U,s)) \to \mathscr{G}_{(x,s)}$. These maps assemble together into a map $(\mathscr{F} \boxtimes \mathscr{G})_x \to \coprod_{s \in \mathscr{F}_x} \mathscr{G}_{(x,s)}$. We claim that this map is surjective: If we have $(s,t) \in \coprod_{s \in \mathscr{F}_x} \mathscr{G}_{(x,s)}$, then there exists an open neighborhood V of (x,s) in $Et(\mathscr{F})$ and a section $\tilde{t} \in \mathscr{G}(V)$ such that $t = \tilde{t}_{(x,s)}$. Since the topology of $Et(\mathscr{F})$ is generated by open sets of the form $\epsilon_{\mathscr{F}}(W, \tilde{s})$, we know there exists an open set $W \subseteq X$ and some $\tilde{s} \in \mathscr{F}(W)$ such that $(x,s) \in \epsilon_{\mathscr{F}}(W,\tilde{s}) \subseteq V$. Then $(\tilde{s}, t|_{\epsilon_{\mathscr{F}}(W,\tilde{s})})$ lies in $\coprod_{y \in \mathscr{F}(W)} \mathscr{G}(\epsilon_{\mathscr{F}}(W,y)) \cong (\mathscr{F} \boxtimes \mathscr{G})(W)$ and $(\tilde{s}, t|_{\epsilon_{\mathscr{F}}(W,\tilde{s})})_x = (s,t)$. One can similarly check that the above map is injective, and then it is an isomorphism of sets. □

With this lemma we get a natural bijective map $\Phi : Et(\mathscr{G}) \to Et(\mathscr{F} \boxtimes \mathscr{G})$ defined by

$$Et(\mathscr{G}) = \coprod_{y \in Et(\mathscr{F})} \mathscr{G}_y = \coprod_{(x,s) \in \coprod_{x \in X} \mathscr{F}_x} \mathscr{G}_{(x,s)} \cong \coprod_{x \in X} \coprod_{s \in \mathscr{F}_x} \mathscr{G}_{(x,s)} \cong \coprod_{x \in X} (\mathscr{F} \boxtimes \mathscr{G})_x = Et(\mathscr{F} \boxtimes \mathscr{G})$$

We now just need to show that this map is open and continuous.

Lemma 5.4. For all open $U \subseteq X$, $s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(\epsilon_{\mathscr{F}}(U,s))$ we have

$$\epsilon_{\mathscr{F} \boxtimes \mathscr{G}}(U,(s,t)) = \Phi(\epsilon_{\mathscr{G}}(\epsilon_{\mathscr{F}}(U,s),t))$$

where Φ is the natural bijective map $\Phi : Et(\mathscr{G}) \to Et(\mathscr{F} \boxtimes \mathscr{G})$.

Proof.

$$\Phi(\epsilon_{\mathscr{G}}(\epsilon_{\mathscr{F}}(U,s),t)) = \Phi(\{(y,t_y) | y \in \epsilon_{\mathscr{F}}(U,s)\}) =$$

$$= \Phi(\{(y,t_y) | y = (x,s_x), x \in U\}) = \{(x,(s,t)_x) | x \in U\} = \epsilon_{\mathscr{F} \boxtimes \mathscr{G}}(U,(s,t))$$

□

This lemma then shows that our map $Et(\mathscr{G}) \to Et(\mathscr{F} \boxtimes \mathscr{G})$ is a homeomorphism. This homeomorphism is natural in X , $\mathscr{F}$ and $\mathscr{G}$ and this then proves the Sum Associativity Axiom of Top_{open}.
Flatten Associativity Axiom: Take functors $x : D \to Top_{open}$, $f : [\![x]\!]_M \to Set$ and $g : [\![\sum^{x} f]\!] \to Set$.

Write $X_d := x(d)$, $\mathscr{F}_d := f(d,-)$, $\mathscr{G}_d := g(d,-)$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
\operatorname*{colim}^{oplax}_{d\in D} \operatorname*{colim}^{oplax}_{(U,s)\in \operatorname*{colim}^{oplax}_{U\in \mathrm{Ouv}(X_d)^{op}} \mathscr{F}_d(U)} \mathscr{G}_d(\epsilon_{\mathscr{F}_d}(U,s)) \arrow[r] \arrow[d] & \operatorname*{colim}^{oplax}_{d\in D} \operatorname*{colim}^{oplax}_{U\in \mathrm{Ouv}(X_d)^{op}} (\mathscr{F}_d \boxtimes \mathscr{G}_d)(U) \arrow[d] \\
\operatorname*{colim}^{oplax}_{d\in D} \operatorname*{colim}^{oplax}_{V\in \mathrm{Ouv}(Et(\mathscr{F}_d))^{op}} \mathscr{G}_d(V) \arrow[r] & \operatorname*{colim}^{oplax}_{d\in D} \mathrm{Ouv}(Et(\mathscr{G}_d))^{op}
\end{tikzcd}$$

Since $\mathrm{Ouv}(Et(\mathscr{G}_d))^{op}$ is a pre-order, any two morphisms with the same domain and codomain coincide in it. For this reason the commutativity of the above diagram can be "checked on objects", in the sense that it commutes if and only if for every object $d \in D$ the following diagram commutes

$$\begin{tikzcd}
\operatorname*{colim}^{oplax}_{(U,s)\in \operatorname*{colim}^{oplax}_{U\in \mathrm{Ouv}(X_d)^{op}} \mathscr{F}_d(U)} \mathscr{G}_d(\epsilon_{\mathscr{F}_d}(U,s)) \arrow[r] \arrow[d, "\epsilon_{\mathscr{F}_d} \times_A id_F"] & \operatorname*{colim}^{oplax}_{U\in \mathrm{Ouv}(X_d)^{op}} (\mathscr{F}_d \boxtimes \mathscr{G}_d)(U) \arrow[d, "\epsilon_{\mathscr{F}_d \boxtimes \mathscr{G}_d}"] \\
\operatorname*{colim}^{oplax}_{V\in \mathrm{Ouv}(Et(\mathscr{F}_d))^{op}} \mathscr{G}_d(V) \arrow[r, "\epsilon_{\mathscr{G}_d}"] & \mathrm{Ouv}(Et(\mathscr{G}_d))^{op}
\end{tikzcd}$$

And this follows from Lemma 5.4. So Top_{open} is a left Set-module.

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