MEAN DIMENSION EXPLOSION OF INDUCED HOMEOMORPHISMS

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Abstract. Given X a compact metric space and $T : X \to X$ a continuous map, the induced hyperspace map $T_{\mathcal{K}}$ acts on the hyperspace $\mathcal{K}(X)$ of closed and nonempty subsets of X, and on the continuum hyperspace $\mathcal{C}(X) \subset \mathcal{K}(X)$ of connected sets. This work studies the mean dimension explosion phenomenon: when the base system T has zero topological entropy, but the mean dimension of the induced map $T_{\mathcal{K}}$ is infinite. In particular, this phenomenon is attained for Morse-Smale diffeomorphisms. Furthermore, for a circle homeomorphism H, the mean dimension explosion does not occur if, and only if, H is conjugated to a rotation. Finally, if the topological entropy of T is positive, then the metric mean dimension of $T_{\mathcal{K}}$ is infinite.

1. Introduction

The study of dynamical systems is mostly concerned with the complexity of the orbits of a map $T : X \to X$, where X is a compact metric space and T is continuous. There are numerous tools to evaluate such complexity, and the mean dimension is one of them. Proposed by M. Gromov in [Gro99], the mean dimension is a topological invariant of dynamical systems, and it is useful to classify maps acting on an infinite-dimensional topological space because the mean dimension of (X, T) is zero when the dimension of the phase space X is finite.

E. Lindenstrauss and B. Weiss later studied the mean dimension in [LW00], where they defined the metric mean dimension. It is a simpler tool to handle compared to the original mean dimension, as many different results have been presented in recent literature: [Hay17; LT18; CRV20; ARA24].

This work is interested in applying the mean dimension and metric mean dimension theory to the induced hyperspace map $T_{\mathcal{K}}$ acting on the hyperspace $\mathcal{K}(X)$ of all closed subsets of X and on the continuum hyperspace $\mathcal{C}(X)$ of all closed connected subsets of X because these hyperspaces are often infinite-dimensional topological spaces. Therefore, studying the mean dimension of the induced hyperspace map offers an alternative to classifying the complexity of well-known finite-dimensional dynamical systems. Moreover, we are also interested in how the dynamics of the base system T influence the complexity of the induced map $T_{\mathcal{K}}$, and vice versa.

In particular, we prove that a wide class of zero topological entropy dynamical systems acting on a finite-dimensional phase space have infinite mean dimension on the hyperspace, the phenomenon of *explosion*.

The study of a relation between the base system T and the induced map $T_{\mathcal{K}}$ began with W. Bauer and K. Sigmund in [BS75], where, in particular, they proved

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that *if* $T_{\mathcal{K}} : \mathcal{K}(X) \to \mathcal{K}(X)$ *is topologically transitive, then so is* $T : X \to X$. Not long ago, M. Lampart and P. Raith established in [LR10] a sufficient condition on (X, T) so that the topological entropy $h(T_{\mathcal{K}})$ is infinite, where $T_{\mathcal{K}}$ acts on $\mathcal{K}(X)$. So, it is a valid question if the metric mean dimension of a determined induced map is finite or infinite. Surprisingly, a sufficient condition on T is presented here so that the mean dimension of $T_{\mathcal{K}}$, denoted as $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}})$, is infinite.

Theorem A. *Let X be a continuum and $T : X \to X$ a homeomorphism. If the nonwandering set $\Omega(T)$ is a strict subset of X, then* $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}}) = \infty$.

The identity map Id on X satisfies $\Omega(\mathrm{Id}) = X$ and $\mathrm{mdim}(\mathcal{K}(X), \mathrm{Id}_{\mathcal{K}}) = 0$, hence the nonwandering set condition in Theorem A proves to be the most suitable in this context.

Moreover, a famous class of zero topological entropy systems satisfies the hypothesis of Theorem A, the Morse-Smale diffeomorphisms. Therefore, the mean dimension explosion occurs for the induced hyperspace map of these maps. In [BS75, Proposition 6], the authors proved that *if the topological entropy $h(T)$ is positive, then $h(T_{\mathcal{K}})$ is infinite.* This fact leads us to the following question:

Question. *Let X be a continuum and $T : X \to X$ a continuous map. If $h(T) > 0$, then we also have* $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}}) = \infty$*?*

The condition to be a continuum, that is, a connected and compact metric space, is necessary because D. Burguet and R. Shi gave in [BS22] an example of a dynamical system (X, T), where X is zero-dimensional, such that the topological entropy $h(T)$ is positive, but $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}})$ is zero.

The continuum hyperspace $\mathcal{C}(X)$ is an infinite-dimensional topological space when X is a continuum with no free arcs, then it is valid to ask if there is an analog of Theorem A to the continuum hyperspace. In particular, A. Arbieto and J. Bohorquez proved in [AB23, Theorem B] that *the topological entropy of the induced continuum map $F_{\mathcal{K}} : \mathcal{C}(N^m) \to \mathcal{C}(N^m)$ is either zero or infinite* where $F : N^m \to N^m$ is a Morse-Smale diffeomorphism acting on a connect and compact boundaryless m-dimensional manifold N^m. In this article it is proved the mean dimension version:

Theorem B. *Let $F : N^m \to N^m$ be a Morse-Smale diffeomorphism. Hence, the following dichotomy holds:*

- *if $m = 1$, then* $\mathrm{mdim}(\mathcal{C}(N^m), F_{\mathcal{K}}) = 0$*;*
- *if $m > 1$, then* $\mathrm{mdim}(\mathcal{C}(N^m), F_{\mathcal{K}}) = \infty$.

The hypothesis of Theorem B is weaker than the hypothesis of Theorem A because the nonwandering set of a Morse-Smale diffeomorphism is finite. As a consequence, Theorem B states that the mean dimension explosion also happens for the induced continuum map, when the dimension of the phase space is greater than one.

Another kind of dichotomy holds for the mean dimension as well. It is a stronger version of the following fact: *if H is a homeomorphism on S^1, then the topological entropy of $H_{\mathcal{K}} : \mathcal{K}(S^1) \to \mathcal{K}(S^1)$ is either zero or infinite.* It is proved in [LR10, Theorem 5] and based on this result we have the following statement:

Theorem C. *Given $H : S^1 \to S^1$ a homeomorphism, then $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}})$ is either 0 or ∞.*

The ideas used in the proof of Theorem C are similar to those presented in [LR10], where the presence of periodic orbits plays a crucial role. Furthermore, its proof allows us to classify orientation-preserving homeomorphisms on the circle according to the mean dimension of its induced hyperspace map. Therefore, the mean dimension explosion is completely understood for circle homeomorphisms.

Corollary D. *Let $H : S^1 \to S^1$ be an orientation-preserving homeomorphism, then $\operatorname{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = 0$ if, and only if, H is conjugated to a rotation.*

It is important to state that the mean dimension explosion phenomenon do not happen for the induced system of Borel probability measures $\mathcal{M}(X)$ equipped with the push-forward map $T_* : \mu \mapsto \mu \circ T^{-1}$, where $T : X \to X$ is a continuous map because D. Burguet and R. Shi proved in [BS22] that $h(T) > 0$ if, and only if, $\operatorname{mdim}(\mathcal{M}(X), T_*) = \infty$.

In the latter part of this work, a study for the metric mean dimension of induced continuous maps is developed. If d is a metric on X, the upper metric mean dimension is denoted by $\overline{\operatorname{mdim}}(X, d, T)$. The standard metric for both hyperspaces is the Hausdorff metric d_H. One may wonder if, at least, W. Bauer and K. Sigmund result on the topological entropy [BS75, Proposition 6] is still valid for the metric mean dimension. The answer is affirmative and given by:

Theorem E. *Let (X, d) be a compact metric space and $T : X \to X$ a continuous map. If $h(T) > 0$, then $\overline{\operatorname{mdim}}(\mathcal{K}(X), d_H, T_{\mathcal{K}}) = \infty$.*

This result promises a new path to study zero entropy continuous maps. Indeed, X. Huang and X. Wang gave in [HW22] an example of a compact metric space (E, d) and a continuous map $\sigma : E \to E$, such that $\overline{\operatorname{mdim}}_M(\mathcal{K}(E), d_H, \sigma_{\mathcal{K}}) = 1$, where E is a subset of $[0, 1]^{\mathbb{N}}$ and σ is the shift transformation. Furthermore, Theorem E is a stronger answer to a problem set in [HW22]. As a consequence, unlike Theorem C, there is no dichotomy for the metric mean dimension of induced continuous maps. This example prompts the following question:

Question. *Given $\alpha > 0$, there exists a compact metric space (X, d) and a continuous map $T : X \to X$ such that $\overline{\operatorname{mdim}}(\mathcal{K}(X), d_H, T_{\mathcal{K}}) = \alpha$?*

Note that the converse of Theorem E is not true by the result of Theorem C, because all circle homeomorphisms have zero topological entropy.

If the phase space is now a topological metrizable manifold, then there is a similar result of Theorem E to the induced continuum map.

Theorem F. *Consider $G : N^m \to N^m$ a continuous map, where $m \geq 2$, such that $h(G) > 0$, then $\overline{\operatorname{mdim}}(\mathcal{C}(N^m), d_H, G_{\mathcal{K}}) = \infty$.*

Observe that Theorem F gives us a hint that if the phase space is sufficiently rich, that is, a manifold, then the complexity of the induced dynamics on the induced hyperspace explodes. Indeed, for differentiable dynamics, a wide set of induced diffeomorphisms has infinite metric mean dimension.

Corollary G. *Given N^m a compact and connected smooth manifold of dimension $m \geq 2$, there exists a residual set $\mathcal{R} \subset \mathit{Diff}^1(N^m)$ such that, for all $F \in \mathcal{R}$,*

$$\overline{\operatorname{mdim}}(\mathcal{C}(N^m), d_H, F_{\mathcal{K}}) = \infty. \tag{1.1}$$

It seems that the only way for an induced continuum hyperspace map to attain zero metric mean dimension is that it is conjugated to an isometry (see Proposition

4.5). So, if the answer to the following question is true, then the theory of the metric mean dimension of induced maps is almost done.

Question. *Given N^m a topological metrizable manifold and $G : N^m \to N^m$ a continuous map, then $\overline{mdim}(\mathcal{C}(N^m), d_H, G_{\mathcal{K}})$ is either zero or infinite? Moreover, the metric mean dimension is zero if, and only if, G is conjugated to an isometry?*

1.1. **Reading guide.** This work is organized as follows. In Section 2 there are given fundamental definitions to understand the theory discussed in this paper. Section 3 is dedicated to studying induced hyperspace maps of homeomorphisms and their mean dimension. Precisely, in §3.1 is presented the proof of Theorem A. In §3.2 is given the proof of Theorem B. The mean dimension explosion is completely understood for circle homeomorphisms and §3.3 includes the proof of Theorem C and Corollary D. Finally, the study of the metric mean dimension for the induced (continuum) hyperspace maps of continuous maps is given in Section 4, where the proof of Theorems E and F and its corollaries are presented.

2. Basic Definitions

From now on, (X, d) is a compact metric space, and $T : X \to X$ is a continuous function, or simply, a map. We start with some fundamental definitions. The *ω-limit set* of a point $x \in X$ is defined as the set of points $y \in X$ such that there is a subsequence $n_i \to \infty$ of positive integers such that $T^{n_i}(x) \to y$, and denoted by $\omega(x)$. It is well known that $\omega(x)$ is nonempty, compact, and invariant. Analogously, the *α-limit set* of a point $x \in X$ is the set of points $y \in X$ such that $T^{-n_i}(x) \to y$, if T is invertible. Denoted by $\alpha(x)$, it is nonempty, compact, and invariant.

The long-run behavior of all orbits of a dynamical system is contained in the set $L(T) = \overline{\bigcup_{x\in X} \omega(x) \cup \alpha(x)}$, called the *limit set* of T. It is also nonempty, compact, and invariant. It is not hard to see that these properties are also valid for the set

$$L(A) := \overline{\bigcup_{x\in A} \omega(x) \cup \alpha(x)},$$

where $A \subset X$ is nonempty, the *A-limit set of T*. Note that $L(A) \subseteq L(T)$. When X is connected, $\gamma \subset X$ is an *arc* if it is homeomorphic to a closed interval in $\mathbb{R}$.

An important set to study the dynamics of a system $T : X \to X$ is the *nonwandering set* of T, denoted as $\Omega(T)$. Given T a homeomorphism, a point $x \in \Omega(T)$ if, for any neighborhood U of x in X, there is a non-zero integer n such that $T^n(U) \cap U \neq \varnothing$. The nonwandering set is also nonempty, compact, and invariant.

2.1. **Hyperspaces and induced maps.** A *hyperspace* is a designated collection of subsets of X. In this work, we will study the following hyperspaces:

- $\mathcal{K}(X) = \{A \subset X; A \text{ is closed and nonempty}\}$;
- $\mathcal{C}(X) = \{A \in \mathcal{K}(X); A \text{ is connected}\}$, if X is a *continuum*, that is, a connected compact metric space.

In this text, we designate as *hyperspace* and *continuum hyperspace*, respectively. Always assume that a continuum is not a singleton. On both hyperspaces, we consider the Hausdorff metric d_H defined as

$$d_H(A, B) = \inf\{\varepsilon > 0; A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\},$$

where $A_\varepsilon = \bigcup_{x\in A}\{y \in X; d(x, y) < \varepsilon\}$ is the generalized ball of radius ε around A.

It is known that $(\mathcal{K}(X), d_H)$ is a compact metric space [IJ99, Theorem 3.5] and that $(\mathcal{C}(X), d_H)$ is a connected and compact metric space when X is connected [IJ99, Theorem 11.3]. Moreover, if X is a continuum with no free arcs[1], then both hyperspaces are infinite-dimensional topological spaces homeomorphic to the Hilbert cube $[0,1]^{\mathbb{Z}}$ (cf. [CS74]).

Given (Y, ρ) a compact metric space and $F : X \to Y$ a map, the *induced hyperspace map* $F_{\mathcal{K}} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is given by $F_{\mathcal{K}}(A) := F(A)$. Note that $F(A)$ is closed in Y, because F is continuous, hence $F_{\mathcal{K}}$ is well defined. It is also well known that $F_{\mathcal{K}}$ is continuous, and a homeomorphism, only if F is a homeomorphism. The *induced continuum map* is intuitively defined as the map $F_{\mathcal{K}}$ restricted to $\mathcal{C}(X)$, that is, $F_{\mathcal{K}} : \mathcal{C}(X) \to \mathcal{C}(Y)$. An alternate notation is $F_{\mathcal{K}}|_{\mathcal{C}(X)} : \mathcal{C}(X) \to \mathcal{C}(Y)$.

Remark 2.1. If $R : X \to Y$ is an isometry, then $R_{\mathcal{K}} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is also an isometry. Indeed, since R is an isometry, then, given $x_1, x_2 \in X$, $\rho(Rx_1, Rx_2) = d(x_1, x_2)$. Given $\varepsilon > 0$, consider two sets $A, B \subset X$ such that $d_H(A, B) = \varepsilon$. Note that $R(A) \subset R(B)_\varepsilon$ because $A \subset B_\varepsilon$ and R is an isometry. If $\varepsilon_0 \in (0, \varepsilon)$, then $R(A) \not\subset R(B)_{\varepsilon_0}$ because there is $a \in A$ and $b \in B$ such that $d(a, b) \geq \varepsilon_0$. Hence, $d(Ra, Rb) \geq \varepsilon_0$. Therefore, since there is no change in the distance between two sets, $\rho_H(R_{\mathcal{K}}(A), R_{\mathcal{K}}(B)) = d_H(A, B)$.

2.2. **Mean dimension.** Consider $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite open cover of X. The *order* $\mathrm{ord}(\mathcal{U})$ is the maximum integer $n \geq 0$ such that there is pairwisely distinct $i_0, ..., i_n \in I$ satisfying $U_{i_0} \cap ... \cap U_{i_n} \neq \varnothing$. A *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ of X such that for every $V_j \in \mathcal{V}$ there is $U_i \in \mathcal{U}$ with $V_j \subset U_i$. The *topological dimension* of X is given by the supremum of the *degree* $\mathcal{D}(\mathcal{U})$, that is the minimum order $\mathrm{ord}(\mathcal{V})$ over all refinements $\mathcal{V}$ of $\mathcal{U}$. (See details in Definition 1.6.7, [Eng78]).

For two open covers $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$, the *joint* is given by $\mathcal{U} \vee \mathcal{V} = \{U_i \cap V_j \ i \in I, j \in J\}$. Since $T : X \to X$ is continuous, observe that $T^{-1}\mathcal{U} = \{T^{-1}U_i; i \in I\}$ is also an open cover of X. Therefore, the *(topological) mean dimension* of (X, T) is given by

$$\mathrm{mdim}(X, T) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U})}{n},$$

where $\mathcal{U}$ runs over all finite open covers of X. A detailed explanation of the mean dimension is given in [LW00].

Some useful basic properties of mean dimension are:

1. the mean dimension is a topological invariant and takes values in $[0, \infty]$;
2. if the topological entropy of the dynamical system is finite, then its mean dimension is zero;
3. if Y is a closed T-invariant subset of X, then $\mathrm{mdim}(Y, T) \leq \mathrm{mdim}(X, T)$;
4. if X is finite-dimensional, then $\mathrm{mdim}(X, T) = 0$;
5. for any dynamical system (X, T), $\mathrm{mdim}(X, T^n) = n \cdot \mathrm{mdim}(X, T)$;
6. if the phase space is the Hilbert cube $[0,1]^{\mathbb{Z}}$, and $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$ is the shift transformation, then $\mathrm{mdim}([0,1]^{\mathbb{Z}}, \sigma) = 1$. More generally, $\mathrm{mdim}(([0,1]^d)^{\mathbb{Z}}, \sigma) = d$.

[1]A continuum X has no free arcs if given $\gamma \subset X$ homeomorphic to the unit interval $[0,1]$ the interior of γ is not open in X.

2.3. **Metric mean dimension.** Given $k \in \mathbb{N}$, define the dynamical distance d_k as

$$d_k(x,y) = \max\{d\big(T^i(x), T^i(y)\big), 0 \le i \le k-1\},$$

where T^0 is the identity, it is well known that d_k is still a distance function on X and generates the same topology as d. For $x \in X$ and $\varepsilon > 0$, we call $B_{(k,\varepsilon)}(x) = \{y \in X; d_k(x,y) < \varepsilon\}$ the (k,ε)-*dynamical ball.*

Call $A \subset X$ a (k,ε)-*separated set* if to any distinct points $x, y \in A$, $d_k(x,y) \ge \varepsilon$. Denote by $\mathrm{Sep}(T,k,\varepsilon)$ the maximal cardinality of a (k,ε)-separated set, that is finite by the compactness of X. We say that $E \subset X$ is a (k,ε)-*spanning set* for X if for any $x \in X$, there exists $y \in E$ such that $d_k(x,y) < \varepsilon$. Let $\mathrm{Span}(T,k,\varepsilon)$ be the minimum cardinality of any (k,ε)-spanning set. To clarify further notations, define

$$\mathrm{h}(T,\varepsilon) = \limsup_{k\to\infty} \frac{\log \mathrm{Sep}(T,k,\varepsilon)}{k} \text{ and } \widetilde{h}(T,\varepsilon) = \limsup_{k\to\infty} \frac{\log \mathrm{Span}(T,k,\varepsilon)}{k}.$$

It is well known that $h(T,\varepsilon) \ge \widetilde{h}(T,\varepsilon)$ (cf. [KH95]). Note that if $\varepsilon_1 < \varepsilon_2$, then $\widetilde{h}(T,\varepsilon_1) \ge \widetilde{h}(T,\varepsilon_2)$. The *topological entropy* $h(T)$ is the limit of both $h(T,\varepsilon)$ and $\widetilde{h}(T,\varepsilon)$ as $\varepsilon \to 0$, that is the same value. We are interested in $\widetilde{h}(T,\varepsilon)$ growth when the topological entropy is infinite. This motivates the definition of upper and lower metric mean dimension, given in [LW00].

The *lower metric mean dimension* and the *upper metric mean dimension* of (X,d,T) are defined by

$$\underline{\mathrm{mdim}}(X,d,T) = \liminf_{\varepsilon\to 0} \frac{h(T,\varepsilon)}{-\log\varepsilon} \quad \text{and} \quad \overline{\mathrm{mdim}}(X,d,T) = \limsup_{\varepsilon\to 0} \frac{h(T,\varepsilon)}{-\log\varepsilon}, \tag{2.1}$$

respectively. There is no problem with replacing $h(T,\varepsilon)$ by $\widetilde{h}(T,\varepsilon)$ in the above limit.

An important property that relates the mean dimension, the metric mean dimension, and the topological entropy is that

$$\mathrm{mdim}(X,T) \le \underline{\mathrm{mdim}}(X,d,T) \le \overline{\mathrm{mdim}}(X,d,T) \le h(T), \tag{2.2}$$

whenever d is a metric compatible with the topology on X [LW00, Theorem 4.2]. Indeed, note that if $h(T)$ is finite, then both topological and metric mean dimension are zero.

2.4. **Morse-Smale diffeomorphisms.** For N^m a m-dimensional compact and connected smooth manifold without boundary, set $\mathrm{Diff}^r(N^m)$ as the set of C^r diffeomorphisms endowed with the C^r topology, for $r \ge 1$. A periodic point $x \in N^m$ of period $k \ge 1$ for $F \in \mathrm{Diff}^r(N^m)$ is *hyperbolic* if the derivative $(DF^k)_x$ has its spectrum disjoint from the unit circle in $\mathbb{C}$. It is well known that in this case, we have the existence of stable and unstable manifolds of x, denoted by $W^s(x)$ and $W^u(x)$.

Definition 2.2. $F \in \mathrm{Diff}^r(N^m)$ is *Morse-Smale* if it satisfies the following conditions:

(1) the set of nonwandering points, $\Omega(F)$, have only a finite number of hyperbolic periodic points;
(2) the stable and unstable manifolds of the periodic points are all transversal to each other.

The Morse-Smale diffeomorphisms have the following properties:

- every Morse-Smale diffeomorphism has an *attractor* periodic point and a *repeller* periodic point, that is, the stable (unstable) manifold of the attractor (repeller) periodic point is an immersed submanifold of dimension m;
- if p is an attractor periodic point, then there is q a repeller periodic point such that the stable manifold of p intersects transversely the unstable manifold of q and $W^s(p) \cap W^u(q) \neq \varnothing$;
- The topological entropy of a Morse-Smale diffeomorphism is always zero.

Further information on Morse-Smale diffeomorphisms can be found at [AB23].

3. Mean dimension of induced homeomorphisms

This section is devoted to the study of the topological mean dimension of induced hyperspace maps for homeomorphisms defined in a compact and connected phase space.

3.1. **Proof of Theorem A.** The following lemma is fundamental to understanding a sufficient condition on why the topological mean dimension of an induced map is unbounded. Precisely, if there is a wandering arc for a homeomorphism, then the mean dimension of its induced map is infinite.

Lemma 3.1. *Let X be a compact and connected topological space and $T : X \to X$ be a homeomorphism. If there is an arc γ such that $T^{n_1}(\gamma) \cap T^{n_2}(\gamma) = \varnothing$, for $n_1 \neq n_2 \in \mathbb{Z}$, then mdim$(\mathcal{K}(X), T_{\mathcal{K}}) = \infty$.*

Proof. Given $\varphi : [0, 1] \to \gamma$ a homeomorphism and $k \in \mathbb{N}$, choose k disjoint closed intervals J_i in $[0, 1]$. For each J_i, $1 \leq i \leq k$, consider a homeomorphism $\psi_i : [0, 1] \to J_i$. If $x = (x_1, ..., x_k)$ is a point in $[0, 1]^k$, then $\varphi(\{\psi_i(x_i), 1 \leq i \leq k\}) \in \mathcal{K}(X)$. Set $\psi(x) = \{\psi_i(x_i), 1 \leq i \leq k\}$, hence $\varphi \circ \psi(x)$ is an element of $\mathcal{K}(X)$. Denote $\gamma_i = \varphi(J_i)$ a subset of γ.

Define a map $\Phi : ([0, 1]^k)^{\mathbb{Z}} \to \mathcal{K}(X)$, for $\xi = (..., \xi_{-1}, \xi_0, \xi_1, ...) \in ([0, 1]^k)^{\mathbb{Z}}$, as

$$\Phi(\xi) = \Phi(..., \xi_{-1}, \xi_0, \xi_1, ...) = \overline{\bigcup_{n\in\mathbb{Z}} T^n \circ \varphi \circ \psi(\xi_n)}.$$

Notice that $T^n \circ \varphi \circ \psi(\xi_n) \subset T^n(\gamma)$, for each $n \in \mathbb{Z}$, hence

$$\overline{\bigcup_{n\in\mathbb{Z}} T^n \circ \varphi \circ \psi(\xi_n)} = \bigcup_{n\in\mathbb{Z}} \{T^n \circ \varphi \circ \psi_i(\xi_{n,i}), 1 \leq i \leq k\} \cup \bigcup_{i=1}^{k} L(\gamma_i),$$

where $\xi_{n,i} \in [0, 1]$, and $T^{n_1} \circ \varphi \circ \psi(\xi_{n_1}) \neq T^{n_2} \circ \varphi \circ \psi(\xi_{n_2})$, because $T^{n_1}(\gamma)$ and $T^{n_2}(\gamma)$ are disjoint. Given that $T_{\mathcal{K}}(L(\gamma_i)) = L(\gamma_i)$, for each $1 \leq i \leq k$, we also have

$$T_{\mathcal{K}} \circ \Phi(\xi) = \overline{\bigcup_{n\in\mathbb{Z}} T^{n+1} \circ \varphi \circ \psi(\xi_n)} = \overline{\bigcup_{n\in\mathbb{Z}} T^n \circ \varphi \circ \psi(\xi_{n-1})} = \Phi \circ \sigma^{-1}(\xi), \tag{3.1}$$

where $\sigma : ([0, 1]^k)^{\mathbb{Z}} \to ([0, 1]^k)^{\mathbb{Z}}$ is the shift transformation. Lindenstrauss & Weiss proved in [LW00] that $\mathrm{mdim}(([0, 1]^k)^{\mathbb{Z}}, \sigma) = k$. Since the mean dimension is a topological invariant, the rest of this proof is dedicated to proving that Φ is injective and continuous. Note that equation (3.1) also shows that $\Phi\big(([0, 1]^k)^{\mathbb{Z}}\big)$ is invariant by $T_{\mathcal{K}}$.

It is not hard to prove that Φ is injective because if $\xi \neq \eta \in ([0, 1]^k)^{\mathbb{Z}}$, then there is $\ell \in \mathbb{Z}$ such that $\xi_\ell \neq \eta_\ell$. That is, there is $i \in \{1, ..., k\}$ such that $\xi_{\ell,i} \neq \eta_{\ell,i}$. Then, $\psi_i(\xi_{\ell,i}) \neq \psi_i(\eta_{\ell,i})$. Therefore, by construction, $\Phi(\xi) \neq \Phi(\eta)$.

To prove continuity, consider $\xi \in ([0,1]^k)^{\mathbb{Z}}$ and $(\xi^{(j)})_{j\in\mathbb{N}}$ a convergent sequence to ξ in the product topology of $([0,1]^k)^{\mathbb{Z}}$. Particularly, for each $n \in \mathbb{Z}$, $\xi_n^{(j)}$ converges to ξ_n in $[0,1]^k$. Then, for all $i \in \{1,...,k\}$, $T^n \circ \varphi \circ \psi_i(\xi_{n,i}^{(j)}) \to T^n \circ \varphi \circ \psi_i(\xi_{n,i})$ as $j \to \infty$. Since such convergence is valid for all $n \in \mathbb{Z}$, then $\Phi(\xi^{(j)}) \to \Phi(\xi)$.

Finally, Φ is a homeomorphism of $([0,1]^k)^{\mathbb{Z}}$ onto it is image. Hence

$$\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}}) \geq \mathrm{mdim}(\Phi(([0,1]^k)^{\mathbb{Z}}), T_{\mathcal{K}}) = k.$$

Since $k \in \mathbb{N}$ is arbitrary, then $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}}) = \infty$. □

It is not hard to see that Theorem A is an easy consequence of the prior lemma.

Proof of Theorem A. Remember that the nonwandering set is compact, hence $X \setminus \Omega(T)$ is an open set. Given $x \in X \setminus \Omega(T)$, there is U_x a neighborhood of x such that $T^n(U_x) \cap U_x = \varnothing$, where n is an non-zero integer, because T is a homeomorphism. As a consequence, $T^{n_1}(U_x) \cap T^{n_2}(U_x) = \varnothing$, for $n_1 \neq n_2$ in $\mathbb{Z}$. Let γ_x be an arc contained in U_x, then it satisfy the hypothesis of Lemma 3.1, hence $\mathrm{mdim}(\mathcal{K}(X), T_{\mathcal{K}}) = \infty$. □

For each $x \in X \setminus \Omega(T)$, it is not hard to prove that $\lim_{n\to\infty} f^n(x) \in \Omega(T)$, in a general way, it is well known that $L(T) \subseteq \Omega(T)$. Whenever the nonwandering set is finite, the topological entropy of T is zero [LR10, Proposition 1 & 2]. Therefore, Theorem A proves that for homeomorphisms with finite nonwandering set occurs the explosion phenomenon. The most common examples of these systems are the Morse-Smale diffeomorphisms, defined in §2.4. The following result is trivial if one recalls inequality (2.2).

Corollary 3.2. *Let X be a continuum and $T : X \to X$ a homeomorphism. If the nonwandering set $\Omega(T)$ is a strict subset of X, then $\underline{\mathrm{mdim}}(\mathcal{K}(X), \rho, T_{\mathcal{K}}) = \infty$, when ρ is any metric compatible with the topology of $\mathcal{K}(X)$.*

3.2. **Proof of Theorem B.** The core of its proof relies on the same idea of Theorem A: We connect the wandering arcs through its points to form a connected set. Each connected set will be an element of the continuum hyperspace.

If N^m is a one-dimensional connect and compact boundaryless manifold, that is, a manifold homeomorphic to the circle S^1, then the topological entropy $h(F_{\mathcal{K}})$ restricted to $\mathcal{C}(N^m)$ is zero for any homeomorphism F [LR10, Theorem 1 & 4]. In this case, $\overline{\mathrm{mdim}}_M(\mathcal{C}(N^m), d_H, F_{\mathcal{K}}) = 0$. Therefore, $\mathrm{mdim}(\mathcal{C}(N^m), F_{\mathcal{K}}) = 0$.

From now on, suppose that N^m is a compact manifold, for $m > 1$. Since F is a Morse-Smale diffeomorphism, then there is, up to an iterate of F, p and q hyperbolic fixed points such that $W^s(p) \cap W^u(q) \neq \varnothing$. Just as in the proof of Theorem A, consider $\gamma \subset W^s(p) \cap W^u(q)$ an arc such that $F^{n_1}(\gamma) \cap F^{n_2}(\gamma) = \varnothing$, for $n_1 \neq n_2$ in $\mathbb{Z}$.

By the Hartman-Grobman Theorem, consider V a neighborhood of p such that there is a homeomorphism $\psi : V \to \psi(V) \subset \mathbb{R}^m$ where $0 \in \psi(V)$ and $(DF)_p \circ \psi = \psi \circ F$. Note that we can shrink V so that $F(V) \subset V$ and $\psi(V)$ be a convex set. Let $K \in \mathbb{N}$ be such that $F^K(\gamma) \subset V$, hence $\psi \circ F^K(\gamma)$ is a curve in $\mathbb{R}^m$. Observe that $\psi \circ F^{K+1}(\gamma)$ is also a curve in $\psi(V)$.

Given $z \in \psi \circ F^K(\gamma)$ and $w \in \psi \circ F^{K+1}(\gamma)$, set $\beta(z,w,t)$ as the straight line connecting z to w, where $\beta(z,w,0) = z$ and $\beta(z,w,1) = w$. Therefore, $\beta(z,w,t)$ is, in particular, a continuous function on the first two coordinates. Define the curve $\kappa(x,y,t) = F^{-K} \circ \psi^{-1}(\beta(z,w,t))$ that connects a point x in γ to y in $F(\gamma)$, then $\Gamma : \gamma \times F(\gamma) \to \mathcal{C}(N^m)$ set as $\Gamma(x,y) = \{\kappa(x,y,t), t \in [0,1]\}$ is a continuous function on both coordinates.

Let $k \in \mathbb{N}$ and consider $\gamma_j \subset \gamma$ connected and closed sets, for $1 \leq j \leq k$. Since each γ_j is also a curve, there is $\tau_j : [0,1] \to \gamma_j$ a homeomorphism. Given $\xi \in ([0,1]^k)^{\mathbb{Z}}$, define $\Phi : ([0,1]^k)^{\mathbb{Z}} \to \mathcal{C}(N^m)$ as

$$\Phi(..., \xi_{-1}, \xi_0, \xi_1, ...) = \bigcup_{j=1}^{k} \overline{\bigcup_{i\in\mathbb{Z}} F^i(\Gamma(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j})))},$$

where $\xi_i = (\xi_{i,1}, ..., \xi_{i,k})$ and $\xi_{i,j} \in [0,1]$. The point $\Phi(\xi) \in \mathcal{K}(N^m)$ is indeed a connected set of N^m because, for all $i \in \mathbb{Z}$,

$$F^i(\Gamma(\tau_j(\xi_{i,j}), F\circ\tau_j(\xi_{i+1,j})))\cap F^{i+1}(\Gamma(\tau_j(\xi_{i+1,j}), F\circ\tau_j(\xi_{i+2,j}))) = \{F^{i+1}\circ\tau_j(\xi_{i+1,j})\},$$

and also $\lim_{i\to\infty} F^{i+1} \circ \tau_j(\xi_{i+1,j}) \to p$, and $\lim_{i\to-\infty} F^{i+1} \circ \tau_j(\xi_{i+1,j}) \to q$, since $F^{i+1} \circ \tau_j(\xi_{i+1,j}) \in F^{i+1}(\gamma)$. Therefore, $\Phi(\xi)$ is a finite union of connected sets with a common point in both p and q, hence a connected set.

To illustrate, for each $j \in \{1, ..., k\}$, the curve $\overline{\bigcup_{i\in\mathbb{Z}} F^i(\kappa(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j})))}$ connect q to p and pass through the sequence $(F^i \circ \tau_j(\xi_{i,j}))_{i\in\mathbb{N}}$ in N^m, as represented in Figure 1.

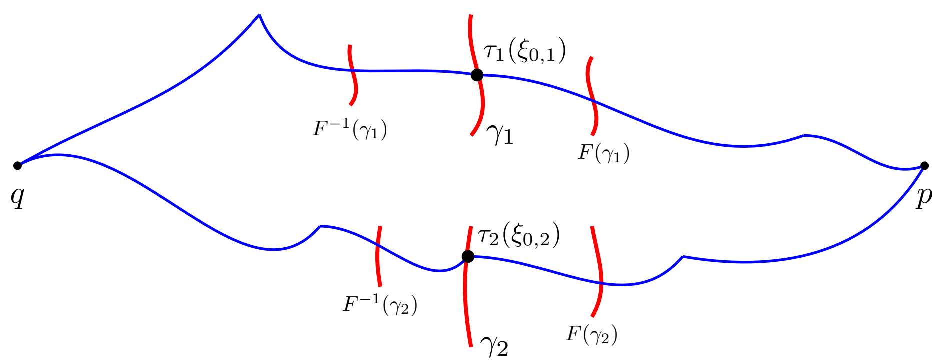

FIGURE 1. The connected set in blue is an example of $\Phi(\xi)$ for $k = 2$.

Observe that

$$F_{\mathcal{K}}\left(\overline{\bigcup_{i\in\mathbb{Z}} F^i(\kappa(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j})))}\right) = \overline{\bigcup_{i\in\mathbb{Z}} F^{i+1}(\kappa(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j})))}.$$

Thus, $F_{\mathcal{K}} \circ \Phi(\xi) = \Phi \circ \sigma^{-1}(\xi)$.

If Φ is a homeomorphism between $([0,1]^k)^{\mathbb{Z}}$ and its image, then, since the mean dimension is a topological invariant, $\text{mdim}(\mathcal{C}(N^m), F_{\mathcal{K}}) \geq \text{mdim}(\Phi(([0,1]^k)^{\mathbb{Z}}), F_{\mathcal{K}}) \geq k$, because $\text{mdim}(([0,1]^k)^{\mathbb{Z}}, \sigma) = k$. Therefore, $\text{mdim}(\mathcal{C}(N^m), F_{\mathcal{K}}) = \infty$, because $k \in \mathbb{N}$ is arbitrary. Thus, our task is reduced to prove that Φ is continuous and injective.

To prove that Φ is injective, consider $\xi \neq \eta$ in $([0,1]^k)^{\mathbb{Z}}$, hence, there is $i \in \mathbb{Z}$ such that $\xi_i \neq \eta_i$. Specifically, there is $j \in \{1, ..., k\}$ such that $\xi_{i,j} \neq \eta_{i,j}$. Therefore, $\Gamma(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j})) \neq \Gamma(\tau_j(\eta_{i,j}), F \circ \tau_j(\eta_{i+1,j}))$ as elements in $\mathcal{C}(N^m)$. Since F is, in particular, a homeomorphism, then $F^i(\Gamma(\tau_j(\xi_{i,j}), F \circ \tau_j(\xi_{i+1,j}))) \neq$

$F^i(\Gamma(\tau_j(\eta_{i,j}), F\circ\tau_j(\eta_{i+1,j})))$. Therefore, the curves $\overline{\bigcup_{i\in\mathbb{Z}} F^i(\Gamma(\tau_j(\xi_{i,j}), F\circ\tau_j(\xi_{i+1,j})))}$ and $\overline{\bigcup_{i\in\mathbb{Z}} F^i(\Gamma(\tau_j(\eta_{i,j}), F\circ\tau_j(\eta_{i+1,j})))}$ are different elements of $\mathcal{C}(N^m)$. Thus, $\Phi(\xi)\neq\Phi(\eta)$.

To prove the continuity of Φ, consider $(\xi^{(n)})_{n\in\mathbb{N}}$ converging to ξ in $([0,1]^k)^{\mathbb{Z}}$ with the product topology, that is, for each $i\in\mathbb{Z}$ and $j\in\{1,...,k\}$, $\xi^{(n)}_{i,j}\to\xi_{i,j}$. The continuity of Γ and τ_j guarantees that $\Gamma(\tau_j(\xi^{(n)}_{i,j}), F\circ\tau_j(\xi^{(n)}_{i+1,j}))$ converges to $\Gamma(\tau_j(\xi_{i,j}), F\circ\tau_j(\xi_{i+1,j}))$ as $n\to\infty$ and for each $j\in\{1,...,k\}$. Then, $F^i(\Gamma(\tau_j(\xi^{(n)}_{i,j}), F\circ\tau_j(\xi^{(n)}_{i+1,j})))$ goes to $F^i(\Gamma(\tau_j(\xi_{i,j}), F\circ\tau_j(\xi_{i+1,j})))$ for all $i\in\mathbb{Z}$. Therefore, $\Phi(\xi^{(n)})\to\Phi(\xi)$.

3.3. **Explosion on the circle.** The explosion of the mean dimension for the induced hyperspace map is better understood for homeomorphisms defined in one-dimensional topological manifolds because they all have zero topological entropy. Precisely, there is a dichotomy for such homeomorphisms: the mean dimension of its induced hyperspace map is zero or infinite. The following lemma states this dichotomy for interval homeomorphisms and is crucial to prove the analog state for circle homeomorphisms.

Lemma 3.3. *Let $T:[0,1]\to[0,1]$ be a homeomorphism such that T^2 is not the identity, then mdim$(\mathcal{K}([0,1]),T_{\mathcal{K}})=\infty$.*

Proof. The first step is to prove that T^2 is increasing. Indeed, since T is a homeomorphism, T is a strictly increasing or decreasing function. If T is increasing, then it is obvious. If T is decreasing, then $T(a)<T(b)$, when $a>b$ in $[0,1]$. Therefore, $T^2(a)>T^2(b)$.

Since T^2 is not the identity, then $T^2(x)<x$ or $T^2(x)>x$, for some $x\in[0,1]$. Suppose, without loss of generality, that $T^2(x)<x$, hence $T^{2n_2}(x)<T^{2n_1}(x)$ for $n_2>n_1$ in $\mathbb{Z}$.

Notice that $(T^{2n}(x))_{n\geq 0}$ is a decreasing sequence, then $T^{2n}(x)$ converges to a fixed point of T^2. The similar happens to $(T^{2k}(x))_{k\leq 0}$, because $T^{2k}(x)$ is a increasing sequence. Remember that T^2 has at least two fixed points because T^2 is increasing.

The second step is to choose a closed interval J contained in the interval $(T^2(x),x)$, hence $T^{2n_2}(J)\cap T^{2n_1}(J)=\varnothing$ for $n_2>n_1$ in $\mathbb{Z}$, and $\lim_{n\to\pm\infty}T^{2n}(J)$ is a fixed point of T^2. Since J satisfies the condition of the Lemma 3.1, then $\mathrm{mdim}(\mathcal{K}([0,1]),T^2_{\mathcal{K}})=\mathrm{mdim}(\mathcal{K}([0,1]),T_{\mathcal{K}})=\infty$. □

Remark 3.4. If $T^2:[0,1]\to[0,1]$ is the identity, then $\mathrm{mdim}(\mathcal{K}([0,1]),T^2_{\mathcal{K}})=\mathrm{mdim}(\mathcal{K}([0,1]),T_{\mathcal{K}})=0$. This proves the dichotomy for homeomorphisms defined on the unit interval $[0,1]$.

To prove the explosion phenomenon on the circle S^1 we will recur to the consequences of the rotation number theory of circle homeomorphisms. A great exposition of this subject is given at [FG22]. Recall that two dynamical systems $T:X\to X$ and $F:Y\to Y$ are *topologically conjugated* or simply *conjugated* if there is a homeomorphism $\psi:X\to Y$ such that $\psi\circ F=T\circ\psi$.

Proof of Theorem C. Suppose that H is a circle homeomorphism with a periodic point, then there is $x\in S^1$ and $q\in\mathbb{N}$ such that $H^q(x)=x$. Every $z\in S^1$ can be written as $z = x \cdot e^{2\pi i t}$ for $t \in [0,1]$. The function $\psi : S^1 \to [0,1]$ defined as $\psi(z) = t$ conjugates H^q to a homeomorphism T on $[0,1]$ given by $T = \psi \circ H^q \circ \psi^{-1}$. If T^2 is the identity, then H^{2q} is the identity. Thus, $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = 0$. Otherwise, by Lemma 3.3, there is $J \subset [0,1]$ wandering interval for T^2, then $\psi^{-1}(J)$ is a wandering interval for H^{2q}. Therefore, by Theorem A, $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = \infty$.

The last step is to consider that H has no periodic points. If H is conjugated to an irrational rotation R_θ by the homeomorphism ϕ, then it is not hard to show that $H_{\mathcal{K}}$ is conjugated to $(R_\theta)_{\mathcal{K}}$ by the homeomorphism $\phi_{\mathcal{K}}$. Since $(R_\theta)_{\mathcal{K}}$ is an isometry, by Remark 2.1, and the topological entropy is preserved by conjugation, then the entropy of $H_{\mathcal{K}}$ is zero. Therefore, $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = 0$. Otherwise, the non-wandering set of H, $\Omega(H)$, is a Cantor set [FG22, Proposition 2.5]. In this case, by the proof of Theorem A, it is well known that there is a closed wandering interval $J \subset S^1 \setminus \Omega(H)$, that is, its images by H are pairwise disjoint, such that $|H^n(J)| \to 0$ and $H^n(J) \to \Omega(H)$ when $|n| \to \infty$. Then, $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = \infty$. □

Given $H : S^1 \to S^1$ an orientation-preserving homeomorphism, if H is conjugated to a rotation, then H is an isometry to some metric compatible with the usual topology on S^1. Therefore, the metric mean dimension of its induced hyperspace map is zero. Conversely, if $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = 0$, we have two cases:

- if H has no periodic points, then H is conjugated to an irrational rotation, by Theorem C;
- if H has a periodic point of period $q \in \mathbb{N}$, then all periodic orbits of H has period q [FG22, Proposition 2.4]. Suppose that H^q is not the identity, then $T := \psi \circ H^q \circ \psi^{-1}$, as in the proof of Theorem C, is not the identity. By the hypothesis, T^2 is the identity, hence H^{2q} is the identity. Contradiction with the period of its periodic orbits. Thus, H^q is the identity. In this case, it is well known that H is topologically conjugated to a rational rotation [CK94].

As a consequence, this discussion proves Corollary D.

Remark 3.5. The orientation-preserving condition on Corollary D is necessary because if H is a reflection, then $\mathrm{mdim}(\mathcal{K}(S^1), H_{\mathcal{K}}) = 0$ because H^2 is the identity. But, H has two fixed points and is not conjugated to the identity. Therefore, H is not conjugated to a rotation.

4. Metric mean dimension of induced continuous maps

For the systems that are yet not possible to calculate the mean dimension of its induced hyperspace map, i.e., continuous maps $T : X \to X$ such that $h(T) > 0$, the alternative is to calculate the metric mean dimension. It is easier to compute because it relies on the separation of points in the phase space by the acting dynamics.

To simplify further notation, denote the Hausdorff distance d_H by D. Given $k \in \mathbb{N}$ and for $A, B \in \mathcal{K}(X)$, the dynamical Hausdorff distance for $T_{\mathcal{K}} : \mathcal{K}(X) \to \mathcal{K}(X)$ is set as

$$D_k(A,B) = \max\{D(T^i_{\mathcal{K}}(A), T^i_{\mathcal{K}}(B)), 0 \le i \le k-1\}$$

and, for $\varepsilon > 0$, the Hausdorff (k,ε)-dynamical ball around $A \in \mathcal{K}(X)$ is defined as $B^H_{(k,\varepsilon)}(A) = \{B \in \mathcal{K}(X); D_k(A,B) < \varepsilon\}$.

Observe that, given $x \in X$, $B^H_{(k,\varepsilon)}(\{x\}) = \{C \in \mathcal{K}(X); C \subset B_{(k,\varepsilon)}(x)\}$. Indeed, if $C \subset B_{(k,\varepsilon)}(x)$, then, for all $i \in \{0, ..., k-1\}$, $T^i_{\mathcal{K}}(C) \subset T^i_{\mathcal{K}}(\{x\})_\varepsilon$. Moreover, clearly, $T^i_{\mathcal{K}}(\{x\}) \subset T^i_{\mathcal{K}}(C)_\varepsilon$. On the other way, if $C \in B^H_{(k,\varepsilon)}(\{x\})$, then $D(T^i_{\mathcal{K}}(C), T^i_{\mathcal{K}}(\{x\})) < \varepsilon$ for all $i \in \{0, ..., k-1\}$. That is, $T^i(C) \subset B_\varepsilon(T^i(x))$. Therefore, $C \subset B_{(k,\varepsilon)}(x)$.

The following lemma compares the size of the separated and spanning sets for the base and the induced system, respectively, and states that the spanning set for the induced system is exponentially bigger than the separated set for the base system. It is the main tool to prove Theorem E.

Lemma 4.1. *Let (X, d) be a compact metric space and $T : X \to X$ a continuous map. For all $\varepsilon > 0$ and $k \in \mathbb{N}$,*

$$Span(T_{\mathcal{K}}, k, \varepsilon/2) \geq 2^{Sep(T,k,\varepsilon)} - 1. \tag{4.1}$$

Proof. Let $\{p_1, ..., p_N\}$ be a (k, ε)-separated set of maximal cardinality for $T : X \to X$. Observe that, for $i \neq j$, $B_{(k,\varepsilon/2)}(p_i) \cap B_{(k,\varepsilon/2)}(p_j) = \varnothing$; otherwise, $d_k(p_i, p_j) < \varepsilon$. For A_1, A_2 nonempty distinct subsets of $\{p_1, ..., p_N\}$,

$$B^H_{(k,\varepsilon/2)}(A_\ell) = \left\{ C \in \mathcal{K}(X); C \subset \bigcup_{x \in A_\ell} B_{(k,\varepsilon/2)}(x) \text{ and } C \cap B_{(k,\varepsilon/2)}(x) \neq \varnothing, \forall x \in A_\ell \right\}.$$

The above first condition guarantees that, for any $z \in C$ and any $s \in \{0, ..., k-1\}$, there is $w \in T^s_{\mathcal{K}}(A_\ell)$ such that $d(T^s(z), w) < \varepsilon/2$. Therefore, $T^s_{\mathcal{K}}(C) \subset T^s_{\mathcal{K}}(A_\ell)_{\varepsilon/2}$. The second condition states that, given $x \in A_\ell$ and $s \in \{0, ..., k-1\}$, there is $z \in C$ such that $d(T^s(z), T^s(x)) < \varepsilon/2$. Thus, $T^s_{\mathcal{K}}(A_\ell) \subset T^s_{\mathcal{K}}(C)_{\varepsilon/2}$. Conversely, if $C \in B^H_{(k,\varepsilon/2)}(A_\ell)$, then $D(T^s_{\mathcal{K}}(C), T^s_{\mathcal{K}}(A_\ell)) < \varepsilon/2$, for all $s \in \{0, ..., k-1\}$. That is, $T^s_{\mathcal{K}}(C) \subset T^s_{\mathcal{K}}(A_\ell)_{\varepsilon/2}$ and $T^s_{\mathcal{K}}(A_\ell) \subset T^s_{\mathcal{K}}(C)_{\varepsilon/2}$. It is not hard to see that these conditions imply this alternative definition of $B^H_{(k,\varepsilon/2)}(A_\ell)$.

Consider $p_j \in A_1$ such that $p_j \notin A_2$. Given $F \in B^H_{(k,\varepsilon/2)}(A_1)$, we have that $F \cap B_{(k,\varepsilon/2)}(p_j) \neq \varnothing$, that is, there is $y \in F$ such that $y \notin B_{(k,\varepsilon/2)}(x)$, for all $x \in A_2$, because $B_{(k,\varepsilon/2)}(p_j) \cap B_{(k,\varepsilon/2)}(x) = \varnothing$. Thus, $B^H_{(k,\varepsilon/2)}(A_1) \cap B^H_{(k,\varepsilon/2)}(A_2) = \varnothing$. Therefore, all nonempty subsets of $\{p_1, ..., p_N\}$ are elements of a $(k, \varepsilon/2)$-spanning set of minimum cardinality for $\mathcal{K}(X)$ and this proves inequality (4.1). □

The next discussion proves that, if the base system has a sufficient growth of separated orbits, that is, $h(T) > 0$, then, by the light of Lemma 4.1, its induced hyperspace map has superexponential growth of separated orbits.

Proof of Theorem E. It is a proof by contrapositive. Let $L \in \mathbb{R}$ be such that $\overline{\text{mdim}}(\mathcal{K}(X), D, T_{\mathcal{K}}) \leq L$. By definition, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{\widetilde{h}(T_{\mathcal{K}}, \varepsilon/2)}{-\log(\varepsilon/2)} \leq L.$$

That is,

$$(-\log \varepsilon + \log 2) \cdot L \geq \widetilde{h}(T_{\mathcal{K}}, \varepsilon/2) = \limsup_{k \to \infty} \frac{\log \text{Span}(T_{\mathcal{K}}, k, \varepsilon/2)}{k}.$$

Thus, there is $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$,

$$(-\log \varepsilon + \log 2) \cdot L \geq \frac{\log \text{Span}(T_{\mathcal{K}}, k, \varepsilon/2)}{k}.$$

Inequality (4.1) implies that

$$(-\log\varepsilon+\log 2)\cdot L\cdot k\geq \log \operatorname{Span}(T_{\mathcal{K}},k,\varepsilon/2)\geq \log(2^{\operatorname{Sep}(T,k,\varepsilon)}-1).$$

From now on we will suppose that $\operatorname{Sep}(T,k,\varepsilon)>1$. Otherwise, $h(T)$ is always zero. Then,

$$(-\log\varepsilon+\log 2)\cdot L\cdot k\geq \log(2^{\operatorname{Sep}(T,k,\varepsilon)}-1)\geq \log(2^{\operatorname{Sep}(T,k,\varepsilon)-1})=(\operatorname{Sep}(T,k,\varepsilon)-1)\cdot\log 2.$$

Again, apply the inequality $t-1\geq t/2$, when $t\geq 2$, to obtain

$$(-\log\varepsilon+\log 2)\cdot L\cdot k\geq \operatorname{Sep}(T,k,\varepsilon)\cdot\frac{\log 2}{2}.$$

Taking the log function on both sides and dividing by k, we get

$$\frac{\log(-\log\varepsilon+\log 2)}{k}+\frac{\log k}{k}+\frac{\log L}{k}\geq\frac{\log\operatorname{Sep}(T,k,\varepsilon)}{k}+\frac{\log(\frac{\log 2}{2})}{k}.$$

Passing the limit when $k\to\infty$, we finally obtain $0\geq h(T,\varepsilon)$, for all $\varepsilon\in(0,\varepsilon_0)$. Therefore, $h(T)=0$. □

Remark 4.2. A trivial consequence is that if $\overline{\operatorname{mdim}}(\mathcal{K}(X),D,T_{\mathcal{K}})$ is bounded, then the topological entropy $h(T)$ of the base system is zero.

For the following result, N^m is a m-dimensional compact and connected metrizable manifold without boundary. The distance function in N^m is set as d. Observe that, when $m\geq 2$, N^m has no free arcs, then $\mathcal{C}(N^m)$ is an infinite-dimensional topological space.

Let $\varepsilon>0$ and $k\in\mathbb{N}$, the continuum Hausdorff (k,ε)-dynamical ball around $\gamma\in\mathcal{C}(N^m)$, when γ is an arc, is given by

$$\mathbf{B}^H_{(k,\varepsilon)}(\gamma)=\{F\in\mathcal{C}(N^m);F\subset\bigcup_{x\in\gamma}B_{(k,\varepsilon)}(x)\text{ and }F\cap B_{(k,\varepsilon)}(x)\neq\varnothing,\forall x\in\gamma\}.$$

Proof of Theorem F. Let $\{p_1,...,p_M\}$ be a (k,ε)-separated set of maximal cardinality for $G:N^m\to N^m$, as in the proof of Lemma 4.1. Recall that, for $i\neq j$, $B_{(k,\varepsilon/2)}(p_i)\cap B_{(k,\varepsilon/2)}(p_j)=\varnothing$.

Suppose that $1\leq i<j\leq M$, then define by $[i,j]$ the arc with p_i and p_j as endpoints such that for all $\ell\in\{1,2,...,M\}$ different from i and j the intersection $[i,j]\cap B_{(k,\varepsilon/2)}(p_\ell)=\varnothing$. This is possible because N^m has topological dimension of at least two. Given $P\subset\{p_1,...,p_M\}$, define $\Gamma(P)=[i_1,i_2]\cup...\cup[i_{t-1},i_t]$, when $P=\{p_{i_1},p_{i_2},...,p_{i_t}\}$. If $P=\{p_i\}$, then $\Gamma(P):=P$. Note that $\Gamma(P)$ is also an arc when P has at least two points. Therefore, we have an injective function from the nonempty subsets of $\{p_1,...,p_M\}$ to $\mathcal{C}(N^m)$.

Consider $P\neq Q$ nonempty subsets of $\{p_1,...,p_M\}$, and suppose, without loss of generality, that there is $p_j\in P$ such that $p_j\notin Q$. Let $F\in\mathbf{B}^H_{(k,\varepsilon/4)}(\Gamma(P))$. If F is an element of $\mathbf{B}^H_{(k,\varepsilon/4)}(\Gamma(Q))$, then there is $y\in F$ and $x\in\Gamma(Q)$ such that $d_k(y,x)\leq\varepsilon/4$. Hence, $d_k(p_j,x)\leq d_k(p_j,y)+d_k(y,x)\leq\varepsilon/2$. Contradiction, because $\Gamma(Q)\cap B_{(k,\varepsilon/2)}(p_j)=\varnothing$, by construction.

Therefore, each $\Gamma(P)$, for nonempty $P\subset\{p_1,...,p_M\}$, is an element of the same $(k,\varepsilon/4)$-spanning set of minimum cardinality for $\mathcal{C}(N^m)$. This proves that $\operatorname{Span}(G_{\mathcal{K}}|_{\mathcal{C}(N^m)},k,\varepsilon/4)\geq 2^{\operatorname{Sep}(G,k,\varepsilon)}-1$. Finally, proceeding similarly as in the proof of Theorem E, if $\overline{\operatorname{mdim}}(\mathcal{C}(N^m),D,G_{\mathcal{K}})$ is bounded, then $h(G)=0$. □

Remark 4.3. If N^1 is the interval $[0,1]$ or the circle S^1, then both continuum hyperspaces are homeomorphic to a bi-dimensional manifold with boundary [IJ99, Example 5.1 & 5.2]. It is not hard to construct a continuous map $G : N^1 \to N^1$ such that $h(G) > 0$, but we always have $\overline{\mathrm{mdim}}(\mathcal{C}(N^1), d_H, G_{\mathcal{K}}) \leq 2$ [VV17, Remark 4]. Therefore, Theorem F is not valid for one-dimensional manifolds.

Denote by $C^0(N^m)$ the set of continuous maps of N^m to itself. It is a complete metric space when endowed with the distance $d_0(F,G) := \max_{x\in N^m} d(F(x), G(x))$. In the same way, the set of homeomorphisms of N^m, denoted by $\mathrm{Homeo}(N^m)$, is also a complete metric space with the metric $D_0(F,G) = d_0(F,G) + d_0(F^{-1}, G^{-1})$. K. Yano proved in [Yan80] that there are open and dense sets $\mathcal{E}_0 \subset C^0(N^m)$ and $\mathcal{E} \subset \mathrm{Homeo}(N^m)$, when $m \geq 2$, such that any element of $\mathcal{E}_0$ or $\mathcal{E}$ has positive topological entropy. Therefore, this leaves us with the following result.

Corollary 4.4. *Given N^m a compact and connected topological metrizable manifold of dimension m bigger than 1, there are open and dense sets $\mathcal{E}_0 \subset C^0(N^m)$ and $\mathcal{E} \subset Homeo(N^m)$ such that for any $G \in \mathcal{E}$ or $G \in \mathcal{E}_0$,*

$$\overline{mdim}(\mathcal{K}(N^m), D, G_{\mathcal{K}}) = \overline{mdim}(\mathcal{C}(N^m), D, G_{\mathcal{K}}) = \infty.$$

So, in the continuous world, it remains to classify maps with zero topological entropy according to the metric mean dimension of its induced map. In the differentiable world, consider N^m a compact and connected smooth manifold endowed with the Riemannian distance. It is well known that $\mathrm{Diff}^1(N^m)$ is a complete metric space with the Whitney C^1 topology. S. Covisier proved in [Cro10, Theorems A & B] that there is a residual set, that is, an enumerable intersection of open and dense sets, $\mathcal{R} \subset \mathrm{Diff}^1(N^m)$ such that, given $F \in \mathcal{R}$ either F is a Morse-Smale diffeomorphism or F has a horseshoe. In the case that F possesses a horseshoe, then the topological entropy of F is positive [BW95, Lemma 1.3]. Therefore, if $F \in \mathcal{R}$ is a Morse-Smale diffeomorphism, then, by Theorem B, equality (1.1) holds. If $F \in \mathcal{R}$ and is not a Morse-Smale diffeomorphism, then, by Theorem F, equality (1.1) also holds. Hence, Corollary G is proved.

Recall that Morse-Smale diffeomorphisms have zero topological entropy. Another class of zero entropy systems is those that conjugate to an isometry. Consider $T : X \to X$ a continuous injective map such that T is conjugated by ψ to an isometry R of (X,d). Since $R_{\mathcal{K}}$ is an isometry on $\mathcal{K}(X)$, by Remark 2.1, then the entropy of $T_{\mathcal{K}}$ is zero, because it is conjugated to $R_{\mathcal{K}}$ by $\psi_{\mathcal{K}}$. Therefore, by the inequality (2.2), the following statement is true.

Proposition 4.5. *Let (X,d) be a compact metric space and $T : X \to X$ a map that is topologically conjugated to an isometry. Then, $\overline{mdim}(\mathcal{K}(X), D, T_{\mathcal{K}}) = 0$.*

5. Final Remarks

The mean dimension explosion phenomenon hints that the induced hyperspace map has too many different orbits, hence its complexity is overwhelmed by the slight separation of points in the base system. Future research may be directed toward classifying zero topological entropy systems with finite (metric) mean dimension of their induced maps.

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