WEAKLY O-MINIMAL TYPES

SLAVKO MOCONJA AND PREDRAG TANOVIĆ

ABSTRACT. We introduce and study weak o-minimality in the context of complete types in an arbitrary first-order theory. A type $p \in S(A)$ is weakly o-minimal with respect to <, a relatively A-definable linear order on $p(\mathfrak{C})$, if every relatively definable subset has finitely many convex components; we prove that in that case the latter holds for all orders. Notably, we prove: (i) a monotonicity theorem for relatively definable functions on the locus of a weakly o-minimal type; (ii) weakly o-minimal types are dp-minimal, and the weak and forking non-orthogonality are equivalence relations on weakly o-minimal types. For a weakly o-minimal pair $\mathbf{p} = (p, <)$, we introduce the notions of the left- and right- \mathbf{p} -genericity of $a \models p$ over B; the latter is denoted by $B \triangleleft^{\mathbf{p}} a$. We prove that $\triangleleft^{\mathbf{p}}$ behaves particularly well on realizations of p: the $\triangleleft^{\mathbf{p}}$ -incomparability and $x \downarrow_{\operatorname{dom}(p)} y$ are the same equivalence relation and the quotient order is dense linear. We show that this naturally generalizes to the set of realizations of weakly o-minimal types from a fixed \measuredangle^w -class.

In seminal papers [13, 14] by Pillay and Steinhorn and [8] by Knight, Pillay and Steinhorn, the notions of o-minimal structures and theories were introduced, and a substantial theory of definable sets within the o-minimal framework was developed. In particular, they proved the Monotonicity Theorem and the Cell Decomposition Theorem, which are fundamental tools used in the analysis of definable sets in o-minimal structures. Extensive research has been conducted on o-minimal structures in the decades that followed, and applications in various areas outside logic, even outside mathematics, have been discovered. One direction of the research included generalization of the concept of o-minimality; this involved relaxing the o-minimality assumption and developing a theory of definable sets that resembles, as closely as possible, that for o-minimal structures.

Let $\mathcal{M} = (M, <, ...)$ be an infinite linearly ordered first-order structure. \mathcal{M} is *o-minimal* if every definable¹ subset of M is a finite union of points and open intervals (with endpoints in $M \cup \{\pm \infty\}$). There are several generalizations of o-minimality; some of them are:

- (Dickmann [3]) M is *weakly o-minimal* if every definable subset of M is a finite union of convex sets;
- (Belegradek, Stolboushkin and Taitslin [2]) M is quasi-o-minimal if every definable subset of M is a Boolean combination of Ø-definable sets, points, and open intervals;
- (Kudaĭbergenov [9]) \mathcal{M} is *weakly quasi-o-minimal* if every definable subset of M is a Boolean combination of \emptyset -definable sets and convex sets.

All of the above definitions refer to the distinguished linear order <; in fact, they depend on < in the sense that a structure can be (weakly, quasi- or weakly quasi-) o-minimal with respect to <, but it might not be the case for some other \emptyset -definable linear order. A complete first-order theory T is o-minimal (weakly o-minimal, quasi-o-minimal, weakly quasi-o-minimal) with respect to a distinguished \emptyset -definable linear order if all models (equivalently, some \aleph_0 -saturated model) of T are such. It is well known that the o-minimality of the structure \mathcal{M} (with respect to <) is always carried over to the theory $\mathrm{Th}(\mathcal{M})$ ([15]); however, this fails for weak, quasi- and weak quasi-o-minimality. Let us also mention that the weak quasi-o-minimality of the theory does not depend on the choice of order ([12, Theorem 1]); in general, this fails for o-minimal, weakly o-minimal and quasi-o-minimal theories.

The Monotonicity Theorem for an o-minimal structure (M, <, ...), [14, Theorem 4.2] states that for every definable function $f: M \to M$ there is a finite definable partition of M into points and open intervals such that f is either constant of strictly monotone on each member of the partition. This fails

¹Throughout the paper "definable" means "definable with parameters".

The authors are supported by the Science Fund of the Republic of Serbia, grant 7750027–SMART.

if the structure is not o-minimal; an explanation for this and a more detailed discussion of analogues of monotonicity and cell decomposition outside the o-minimal context can be found in Goodrick's article [5]. Among the weak analogues is a "local" version of monotonicity proved in the weakly o-minimal context by Macpherson, Marker, and Steinhorn in [10]. Roughly speaking, it states that for every definable function $f: M \to \overline{M}$ there is a finite convex partition of M such that f is locally constant or locally strictly monotone on each of the convex parts of the partition.

In this paper, we start a systematic study of weak o-minimality transferred to the locus of a complete type in an arbitrary complete (possibly multi-sorted) first-order theory T; we introduce weakly o-minimal types. The motivation for introducing this notion comes from our recent work on complete 1-types in weakly quasi-o-minimal theories ([12]) (which are all weakly o-minimal in our sense) and on (stationarily ordered, or) so-types ([11]); the latter include weakly o-minimal types. Here, we will adapt to the context of weakly o-minimal types and reprove several results from these two papers, as well as establish novel results. For example, in [12] we proved Theorem 1(i) for complete 1-types in a weakly quasi-o-minimal theory; however, Theorems 1(ii), 2, and 3 are novel. The results presented in this paper will be utilized in our forthcoming papers on Vaught's conjecture for weakly quasi-o-minimal theories.

Let \mathfrak{C} be a monster model of T, let A be a small subset of \mathfrak{C} , and let $p \in S(A)$. We will consider orders $(p(\mathfrak{C}), <)$, where < is a relatively A-definable linear order on $p(\mathfrak{C})$ and say that (p, <) is a weakly o-minimal pair over A if every relatively definable subset of $p(\mathfrak{C})$ is the union of a finite number of convex sets; a type p is weakly o-minimal if there exists such a pair.

We will prove that weakly o-minimal types have several favorable model-theoretic properties, both geometric and general. For example, we will prove that the weak o-minimality of a type does not depend on the choice of order and that a theory T is weakly quasi-o-minimal if and only if every complete type $p \in S_1(T)$ is weakly o-minimal. Among the geometric properties of weakly o-minimal types, the most interesting one is a version of the Monotonicity Theorem formulated in Theorem 1 below which, roughly speaking, says that every relatively definable function on the locus of a weakly o-minimal type is "weakly monotone"; this is explained as follows: Given a linear order (D, <) and an equivalence relation with convex classes E, we can define another linear order $<_E$ by reversing the order < within each class, but leaving the classes originally ordered:

$$x <_E y \Leftrightarrow (E(x, y) \land y < x) \lor (\neg E(x, y) \land x < y).$$

For an \subseteq -increasing sequence of convex equivalence relations $\vec{E} = (E_1, \ldots, E_n)$ we can iterate this construction and define $\langle_{\vec{E}} := (\ldots, (\langle_{E_1})_{E_2} \ldots)_{E_n}$. If (D', \langle') is another linear order and $f : D \to D'$ is an increasing function, then we say that f is (\langle, \langle') -increasing; similarly for (\langle, \langle') -decreasing and (\langle, \langle') -monotone. If f is $(\langle_{\vec{E}}, \langle')$ -monotone for some sequence of convex equivalence relations \vec{E} , then we may think of f as a weakly (\langle, \langle') -monotone function.

Theorem 1. (Weak monotonicity). Suppose that $(p, <_p)$ is a weakly o-minimal pair over A, (D, <) an A-definable linear order, and $f : p(\mathfrak{C}) \to D$ a relatively A-definable non-constant function. Then:

- (i) There exists a strictly increasing sequence of relatively A-definable convex equivalence relations $\vec{E} = (E_0, \ldots, E_n)$ on $p(\mathfrak{C})$ such that f is $((<_p)_{\vec{E}}, <)$ -increasing.
- (ii) There exists an increasing sequence of A-definable convex equivalence relations $\vec{F} = (F_0, \ldots, F_n)$ on (D, <) such that f is $(<_p, <_{\vec{F}})$ -increasing.

Under the assumptions of Theorem 1, as a consequence of the $((<_p)_{\vec{E}}, <)$ -monotonicity of f, we obtain the following:

- **Theorem 2.** (i) (Local monotonicity). There exists a convex relatively A-definable equivalence E on $p(\mathfrak{C})$, such that $E \neq \mathrm{id}_{p(\mathfrak{C})}$ and the restriction of f to each E-class is constant or strictly $(<_p, <)$ -monotone.
- (ii) (Upper monotonicity). There exists a convex relatively A-definable equivalence E on $p(\mathfrak{C})$, such that $E \neq p(\mathfrak{C})^2$ and one of the following two conditions holds for all x_1, x_2 realizing p: $[x_1]_E <_p [x_2]_E \Rightarrow f(x_1) < f(x_2)$ or $[x_1]_E <_p [x_2]_E \Rightarrow f(x_1) > f(x_2)$.

3

Assume for a moment that the underlying theory is weakly quasi-o-minimal and $f: \mathfrak{C} \to D$. Then all complete 1-types are weakly o-minimal, and Theorem 1 applies to each of them. Using a routine compactness argument, one obtains a finite definable partition of \mathfrak{C} such that the restriction of f to each member of the partition is weakly monotone; that is exactly the content of our aforementioned result [12, Theorem 2]. Similarly, we can derive an interesting definable form of local monotonicity (and a less interesting form of upper monotonicity, which we omit). In the following theorem, we state specific versions adjusted to the context of weakly o-minimal theories; note that part (ii) is a version of [10, Theorem 3.3].

Theorem 3. Suppose that $\text{Th}(\mathfrak{C}, <, ...)$ is weakly o-minimal, (D, \lhd) is an A-definable linear order, and $f : \mathfrak{C} \to D$ is an A-definable function.

- (i) There exists a finite convex A-definable partition \mathfrak{C} of \mathfrak{C} and an increasing sequence of A-definable convex equivalence relations \vec{E} on \mathfrak{C} such that f is $(<_{\vec{E}}, \lhd)$ -increasing on each member of the partition.
- (ii) There exists a finite convex A-definable partition \mathfrak{C} of \mathfrak{C} and a convex A-definable equivalence relation E with at most finitely many singleton classes on \mathfrak{C} such that $E = \bigcup_{C \in \mathfrak{C}} E_{\uparrow C}$ and for all $C \in \mathfrak{C}$ the restriction $f_{\uparrow \uparrow a \downarrow_E}$ is constant or strictly $(<, \lhd)$ -monotone uniformly for all $a \in C$.

In [11], we investigated the weak and forking non-orthogonality $(\measuredangle^w \text{ and } \pounds^f)$ of so-types over the same domain, say A, and proved that they are equivalence relations; we also proved that forking, viewed as a binary relation $x \not \perp_A y$ on the set of realizations of so-types over A, is an equivalence relation. As weakly o-minimal types are so-types, the same conclusions hold for them. We will reprove all of these results in Section 4 below. The main novelty here is the notion of (left and) right **p**-generic elements over a parameter set B, where $\mathbf{p} = (p, <)$ is an so-pair over A; this makes the proofs presented here intuitively clearer and considerably shorter than those in [11]. An element $a \models p$ is right (left) **p**-generic over B if the locus of $\operatorname{tp}(a/AB)$ is a final (initial) part of $(p(\mathfrak{C}), <)$. We show that left and right **p**-generic over B; $B \triangleleft^{\mathbf{p}} a$ denotes that a is right **p**-generic over B. We will prove that $\triangleleft^{\mathbf{p}}$, viewed as a binary relation on $p(\mathfrak{C})$, particularly well behaves; for example, $a \models p$ is left **p**-generic over $b \models p$ if and only if b is right **p**-generic over a (that is, $a \triangleleft^{\mathbf{p}} b$).

Theorem 4. Let $\mathbf{p} = (p, <)$ be an so-pair over A. Assume that p is non-algebraic.

(i) $x \not \perp_A y$ defines a convex equivalence relation, denoted by \mathcal{D}_p , on $(p(\mathfrak{C}), <)$.

(ii) $(p(\mathfrak{C}), \triangleleft^{\mathbf{p}})$ is a strict partial order in which the $\triangleleft^{\mathbf{p}}$ -incomparability agrees with the relation \mathcal{D}_p . (iii) $\triangleleft^{\mathbf{p}}$ and < agree on $p(\mathfrak{C})/\mathcal{D}_p$; $(p(\mathfrak{C})/\mathcal{D}_p, \triangleleft^{\mathbf{p}})$ is a dense linear order.

Two so-pairs over A, $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$, are weakly non-orthogonal ($\mathbf{p} \not\perp^w \mathbf{q}$) if $p \not\perp^w q$; they are directly non-orthogonal, denoted by $\delta_A(\mathbf{p}, \mathbf{q})$, if $p \not\perp^w q$ and for all $a \models p$ and $b \models q$ the following holds: a is left **p**-generic over b if and only if b is right **q**-generic over a. We will prove that $\not\perp^w$ and δ_A are equivalence relations on the set of all so-pairs over A, and that δ_A refines $\not\perp^w$ and splits each $\not\perp^w$ -class (which consists of non-algebraic types) into two classes; note that the pairs $\mathbf{r} = (r, <)$ and $\mathbf{r}^* = (r, >)$ are in the same $\not\perp^w$ -class, but in distinct δ_A -classes.

Let \mathcal{F} be a δ_A -class and let $\mathcal{F}(\mathfrak{C})$ be the set of realizations of all types from \mathcal{F} . We will say that $a \in \mathcal{F}(\mathfrak{C})$ is right \mathcal{F} -generic over $b \in \mathcal{F}(\mathfrak{C})$, denoted by $b \triangleleft^{\mathcal{F}} a$, if $b \triangleleft^{\mathbf{p}} a$ holds for some (equivalently all) $\mathbf{p} = (\operatorname{tp}(a/A), <) \in \mathcal{F}$. Let $\mathcal{D}_{\mathcal{F}} := \{(x, y) \in \mathcal{F}(\mathfrak{C})^2 \mid x \not\downarrow_A y\}$. The following generalizes Theorem 4.

Theorem 5. Let \mathcal{F} be a δ_A -class of non-algebraic so-pairs (or weakly o-minimal pairs) over A.

- (i) $(\mathfrak{F}(\mathfrak{C}), \triangleleft^{\mathfrak{F}})$ is a strict partial order.
- (ii) $\mathcal{D}_{\mathcal{F}}$ and $\triangleleft^{\mathcal{F}}$ -incomparability are the same equivalence relation on $\mathcal{F}(\mathfrak{C})$.
- (iii) $(\mathfrak{F}(\mathfrak{C})/\mathfrak{D}_{\mathfrak{F}}, \triangleleft^{\mathfrak{F}})$ is a dense linear order.

The paper is organized as follows. In Section 1, the terminology from model theory and terminology concerning linear orders is given. The basic properties of orders $<_{\vec{E}}$ are outlined. Relatively definable sets are studied in some detail as they play an important role further in the paper. We also recall the

definition and very basic properties of the so-types from [11]. Weakly o-minimal types are introduced in Section 2, and few basic facts about them are proved. In Section 3, we characterize all relatively definable linear orders on the locus of a weakly o-minimal type and deduce several corollaries, including Theorems 1–3. Section 4 deals with so-types. We study forking independence of realizations of sotypes, and prove Theorems 4, 5 and the related aforementioned results on non-orthogonality. In Section 5, we prove that weakly o-minimal types are dp-minimal. We also provide some analysis of indiscernible sequences of realizations of weakly o-minimal types; in particular, we show that they have a bit stronger property than the distality (as defined by Simon in [17]).

1. Preliminaries

We use standard concepts and notation from model theory. We work in \mathfrak{C} , a large, saturated (monster) model of a complete, first-order (possibly multi-sorted) theory T in a first-order language L. Both singletons and tuples of elements from \mathfrak{C} are denoted by a, b, c, \ldots ; |a| denotes the length of the tuple a. The letters A, B, A', B', \ldots are reserved for small subsets (of cardinality $< |\mathfrak{C}|$) of the monster, while C, D, C', D', \ldots are used to denote arbitrary sets of tuples. By an L_C -formula $\phi(x)$ we mean a formula whose parameters are from C; by a formula we mean an $L_{\mathfrak{C}}$ -formula. The set of all the realizations of $\phi(x)$ in \mathfrak{C} is usually denoted by $\phi(\mathfrak{C})$, but sometimes, when we want to emphasize |x| = n, we also denote it by $\phi(\mathfrak{C}^n)$. Sets of this form are said to be C-definable; a set is definable if it is C-definable for some parameter set C. A partial type p(x) is any small finitely consistent set of $L_{\mathfrak{C}}$ -formulae that is closed under conjunctions; $p(\mathfrak{C})$ (or $p(\mathfrak{C}^{|x|})$) denotes the set of all realizations of p(x). A subset $D \subset \mathfrak{C}^n$ is type-definable (over A) if $D = p(\mathfrak{C})$ for some partial type p(x) (over A).

The space of all complete n-types over the parameters C is denoted by $S_n(C)$; the basic clopen subsets are of the form $[\phi] = \{p \in S_n(C) \mid \phi(x) \in p\}$, where $\phi(x)$ is an L_C -formula and |x| = n. $S(C) := \bigcup_{n \in \mathbb{N}} S_n(C)$. In particular, $S(\mathfrak{C})$ is the set of all global finitary types. A global type $\mathfrak{p}(x)$ is A-invariant if $(\phi(x, b_1) \leftrightarrow \phi(x, b_2)) \in \mathfrak{p}$ for all L_A -formulae $\phi(x, y)$ and all tuples b_1, b_2 of length |y|that satisfy $b_1 \equiv b_2(A)$; the type \mathfrak{p} is invariant if it is A-invariant for some small set of parameters A. For an A-invariant global type \mathfrak{p} and a linear order (I, <), a sequence of tuples $(a_i \mid i \in I)$ is a Morley sequence in \mathfrak{p} over A if $a_i \models \mathfrak{p}_{\uparrow A a_{< i}}$ holds for all $i \in I$. Note that we allow Morley sequences to have an arbitrary (even finite) order-type. Dividing and forking have the usual meaning, and by $a \bigcup_A B$ we denote that $\operatorname{tp}(a/AB)$ does not fork over A. The types $p, q \in S(A)$ are weakly orthogonal, denoted by $p \perp^w q$, if $p(x) \cup q(y)$ determines a complete type over A; they are forking orthogonal, denoted by $p \perp^f q$, if $a \bigcup_A b$ holds for all $a \models p$ and $b \models q$.

1.1. Linear orders. Notation related to linear orders is mainly standard. Let (X, <) be a linear order, and let $D \subseteq X$.

- D is convex if $a, b \in D$ and a < c < b imply $c \in D$.
- D is an *initial part* if $a \in D$ and b < a imply $b \in D$; D is a *left-eventual part* if it contains a nonempty initial part. The final parts and right-eventual parts are defined dually.
- A subset $C \subseteq D$ is a *convex component* of D if C is a maximal, convex subset of D.
- The set of all convex components forms a partition of D; therefore, the meaning of D has a finite number of convex components is clear.
- For nonempty $Y, Y' \subseteq X$ we write Y < Y' if y < y' holds for all $y \in Y$ and $y' \in Y'$; we write x < Y and Y < x instead of $\{x\} < Y$ and $Y < \{x\}$.
- $x \in X$ is an upper (lower) bound of D if D < x (x < D).
- *D* is *upper (lower) bounded* if an upper (lower) bound exists; *D* is *bounded* if it is upper and lower bounded. Note that the set of all upper (lower) bounds of *D* is a final (initial) part.
- $\sup D_1 \leq \sup D_2$ (where $D_1, D_2 \subseteq X$) denotes that any upper bound of D_2 is an upper bounds of D_1 too; $\sup D_1 < \sup D_2$, $\inf D_1 \leq \inf D_2$ and $\inf D_1 < \inf D_2$ have analogous meanings.
- An equivalence relation $E \subseteq X \times X$ is *convex* if all *E*-classes $[x]_E$ ($x \in X$) are convex subsets of X; in that case, the quotient set X/E is naturally linearly ordered by <.

Note that all of the above definitions should be read as with respect to (X, <). Further in the text, whenever the meaning of the order is not clear from the context, we emphasize it in some way; for example, we say that D is convex in (X, <) or that D is a <-convex subset of X, etc.

If $(X, <_X)$ and $(Y, <_Y)$ are linear orders and $f: X \to Y$, we say that f is $(<_X, <_Y)$ -increasing if $x <_X x'$ implies $f(x) \leq_Y f(x')$ for all $x, x' \in X$. In that case, the kernel relation $\operatorname{Ker}(f)$, defined by f(x) = f(x'), is a convex equivalence relation on (X, \leq_X) , and the mapping defined by $[x]_{\operatorname{Ker}(f)} \mapsto f(x)$ is an order isomorphism between $(X/\operatorname{Ker}(f), <_X)$ and $(f(X), <_Y)$. Also, f is strictly $(<_X, <_Y)$ -increasing if $x <_X x'$ implies $f(x) <_Y f(x')$ for all $x, x' \in X$.

In the rest of the subsection, we recall the construction of orders $<_{\vec{E}}$ from [12] and state their basic properties.

Definition 1.1. Let (X, <) be a linear order and E a convex equivalence relation on X. Define:

$$x <_E y$$
 iff $(E(x, y) \land y < x) \lor (\neg E(x, y) \land x < y).$

It is easy to see that $(X, <_E)$ is a linear order; the order $<_E$ reverses the order < within each E-class, but the classes in the quotient order remain originally ordered. In particular, E is a $<_E$ -convex equivalence relation, and $(X/E, <) = (X/E, <_E)$ holds. Furthermore, let E' be another convex equivalence relation on (X, <) that is \subseteq -comparable (either finer or coarser) to E. It is easy to see that E' is a $<_E$ -convex equivalence, so the order $(<_E)_{E'}$ is well defined. Similarly, the order $(<_{E'})_E$ is well-defined; it is not hard to see that $(<_E)_{E'} = (<_{E'})_E$ holds.

Definition 1.2. Let (X, <) be a linear order and $\vec{E} = (E_1, \ldots, E_n)$ a sequence of convex equivalence relations on X, such that any two of them are \subseteq -comparable. Define:

$$<_{\vec{E}} := (\ldots (<_{E_1})_{E_2} \ldots)_{E_n}.$$

- **Remark 1.3.** (a) If the order (X, <) and the sequence \vec{E} in the previous definition are definable, then the resulting order $<_{\vec{E}}$ is definable with the same parameters. Similarly, if X is type-definable over A and < and \vec{E} are relatively A-definable, then $<_{\vec{E}}$ is relatively A-definable.
- (b) We have already remarked that $\langle (E_1, E_2) = \langle (E_2, E_1) \rangle$ holds for any pair of \subseteq -comparable convex equivalences. By induction, if \vec{E} is a sequence of convex equivalences such that any two of them are \subseteq -comparable, it is easy to prove that the order $\langle_{\vec{E}}\rangle$ does not depend on the order of elements of \vec{E} : $\langle_{\vec{E}} = \langle_{\pi(\vec{E})}\rangle$ holds for any permutation $\pi(\vec{E})$ of \vec{E} .
- (c) It is very easy to see that $(\langle E \rangle_E = \langle a | ways holds and is only slightly harder to verify <math>(\langle E \rangle_E) = \langle E | E \rangle_E = \langle$
- (d) It is rather straightforward to verify that $\langle_{\vec{E}} \neq \langle_{\vec{E'}}\rangle$ holds for any pair of distinct strictly increasing sequences of convex equivalence relations \vec{E} and $\vec{E'}$ that do not contain the identity relation.
- (e) Let $D \subseteq X$ be <-convex. Then D properly intersects at most two E_1 -classes (the endpoints of D/E_1 in the quotient order X/E_1). Thus, D has at most three $\langle E_1$ -convex components. Each of these components has at most three $(\langle E_1 \rangle)_{E_2}$ -components, etc. Thus, D can have a maximum of $3^n <_{\vec{E}}$ -convex components. Taking into account (c), it follows that a subset $D \subseteq X$ has finitely many <-convex components if and only if D has finitely many $<_{\vec{E}}$ -convex components.

1.2. Relative definability.

Definition 1.4. Let p(x) be a partial type over A. A set $X \subseteq p(\mathfrak{C})$ is relatively *B*-definable within $p(\mathfrak{C})$ if $X = D \cap p(\mathfrak{C})$ holds for some *B*-definable set $D \subseteq \mathfrak{C}^{|x|}$. In that case, any formula $\phi(x)$ that defines D is called a *relative definition of* X within $p(\mathfrak{C})$, and we also say that X is relatively defined by $\phi(x)$ within $p(\mathfrak{C})$.

Clearly, the family of relatively *B*-definable subsets of a type-definable set is closed for finite Boolean combinations. Also, if the subset $P \subseteq \mathfrak{C}^n$ is type-definable over *A*, then so is any finite power of *P*. Therefore, the relative definability of the relations on *P* is well defined. For example, if a relation

 $R \subseteq P^2$ is relatively defined by a formula $\phi(x, y)$ and if (P, R) is a linear order, then we say that ϕ relatively defines a linear order on P. Similarly, if $P \subseteq \mathfrak{C}^n$ and $Q \subseteq \mathfrak{C}^m$ are type-definable sets, then so is the set $P \times Q$ and the relative definability of (graphs of) functions $f: P \to Q$ is well defined.

Several interesting properties of relatively definable relations on $p(\mathfrak{C})$ can be transferred to a definable neighborhood $\theta(\mathfrak{C}) \supseteq p(\mathfrak{C})$ (where $\theta(x) \in p$). For example, assume that $\phi(x, y)$ relatively defines a pre-order on $p(\mathfrak{C})$. Let $\psi(x, y, z) := \phi(x, x) \land (\phi(x, y) \land \phi(y, z) \to \phi(x, z))$. Then $p(x) \cup p(y) \cup p(z) \vdash \psi(x, y, z)$, so by compactness there exists a $\theta(x) \in p(x)$ such that $\{\theta(x), \theta(y), \theta(z)\} \vdash \psi(x, y, z)$. Therefore, $\phi(x, y)$ relatively defines a pre-order on $\theta(\mathfrak{C})$ and $(p(\mathfrak{C}), \phi(p(\mathfrak{C})))$ is a suborder.

The key point in the above argument is that the theory of pre-orders is universally axiomatizable, so that the property " $\phi(x, y)$ relatively defines a pre-order on $p(\mathfrak{C})$ " can be expressed by a sentence saying that " $\psi(x, y, z)$ holds for all x, y, z realizing p". Formally, this is expressed by the following $L_{\infty,\omega}$ -sentence: $(\forall x, y, z) \left(\bigwedge_{\theta \in p} (\theta(x) \land \theta(y) \land \theta(z)) \rightarrow \psi(x, y, z) \right)$, which will be informally denoted by $(\forall x, y, z \models p) \psi(x, y, z)$. More generally, we will consider $L_{\infty,\omega}$ -sentences denoted informally by $(\forall x_1 \models p_1) \ldots (\forall x_n \models p_n) \psi(x_1, \ldots, x_n)$, where $p_1(x_1), \ldots, p_n(x_n)$ are partial types and $\psi(x_1, \ldots, x_n)$ an $L_{\mathfrak{C}}$ formula, and call them tp-universal sentences; the properties of relations (and their defining formulae) expressed by these sentences are called tp-universal properties. For example, " $\phi(x, y)$ relatively defines a pre-order on $p(\mathfrak{C})$ " and " \leq is a relatively definable pre-order on $p(\mathfrak{C})$ " are tp-universal properties. The following is a version of the compactness that will be applied further in the text when dealing with tp-universal properties.

Fact 1.5. Suppose that $p_1(x_1), \ldots, p_n(x_n)$ are partial types and $\phi(x_1, \ldots, x_n)$ is an $L_{\mathfrak{C}}$ -formula such that $\mathfrak{C} \models (\forall x_1 \models p_1) \ldots (\forall x_n \models p_n) \phi(x_1, \ldots, x_n)$. Then there are formulae $\theta_i(x_i) \in p_i$ for all $1 \leq i \leq n$ such that:

$$\mathfrak{C} \models (\forall x_1 \dots x_n) \left(\bigwedge_{1 \le i \le n} \theta'_i(x_i) \to \phi(x_1, \dots, x_n) \right)$$

for all formulae $\theta'_i(x_i)$ such that $\theta'_i(\mathfrak{C}) \subseteq \theta_i(\mathfrak{C})$ $(1 \leq i \leq n)$.

Remark 1.6. A finite conjunction of tp-universal sentences is equivalent to a tp-universal sentence. For example, $(\forall x \models p)(\forall y \models q)\phi(x, y) \land (\forall x \models r)(\forall y \models p)\psi(x, y)$ is equivalent to $(\forall x, t \models p)(\forall y \models q)(\forall z \models r)(\phi(x, y) \land \psi(z, t)).$

One typical application of Fact 1.5 is the following. Let $\mathcal{P} = (p(\mathfrak{C}); R_1, \ldots, R_n)$, where p(x) is a partial type over A and where each R_i is a relatively A-definable finitary relation on $p(\mathfrak{C})$; let ϕ_i relatively define R_i . Suppose that some interesting property of \mathcal{P} can be expressed by a tp-universal sentence $(\forall x_1, \ldots, x_m \models p) \ \psi(x_1, \ldots, x_m)$, where the formula $\psi \in L_A$ is built from ϕ_1, \ldots, ϕ_n (viewed as atomic); then Fact 1.5 produces A-definable superstructures $(\theta(\mathfrak{C}); \phi_1(\theta(\mathfrak{C})), \ldots, \phi_n(\theta(\mathfrak{C})))$ of \mathcal{P} with the same property; we will call them *definable extensions* of \mathcal{P} . In all future applications, we will fix the sequence of all relevant formulas before applying Fact 1.5.

- **Example 1.7.** (a) Let $\mathcal{P} = (p(\mathfrak{C}); <)$ be a relatively A-definable linear order; we will always assume that < is defined by the formula x < y. Clearly, that \mathcal{P} is a linear order is expressible by a tp-universal sentence (built from x < y), so by Fact 1.5 there is $\theta(x) \in p$ such that x < y defines a linear order, also denoted by <, on $\theta(\mathfrak{C}); (\theta(\mathfrak{C}); <)$ is a definable extension of \mathcal{P} .
- (b) Consider 𝒫 = (p(𝔅); <, E), where < is a linear order and E is a convex equivalence relation; we will always implicitly assume that E is relatively defined by E(x, y). Each of the following properties: "x < y defines a linear order on p(𝔅)", "E(x, y) defines an equivalence relation on p(𝔅)" and "E-classes are <-convex subsets of p(𝔅)" is a tp-universal property (the latter is expressed by (∀x, y, z ⊨ p)(E(x, y) ∧ x < z < y → E(x, z))). By Remark 1.6, the conjunction of these properties is also tp-universal, so by Fact 1.5, there exists an A-definable extension (θ(𝔅); <, Ê) of 𝔅 such that < is a linear order and Ê is a convex equivalence relation on θ(𝔅).</p>
- (c) Let $\mathcal{P} = (p(\mathfrak{C}); \langle E_1, \ldots, E_n)$ be a relatively A-definable linear order with a \subseteq -increasing sequence (E_1, \ldots, E_n) of relatively A-definable <-convex equivalence relations (each E_i defined by an L_A -formula $E_i(x, x')$). Again, we can describe this by a tp-universal sentence, so an A-definable

extension $(\theta(\mathfrak{C}), <, \hat{E}_1, \ldots, \hat{E}_n)$ of \mathcal{P} can be found such that $(\hat{E}_1, \ldots, \hat{E}_n)$ is a \subseteq -increasing sequence of <-convex equivalence relations on $\theta(\mathfrak{C})$.

Let $f: p(\mathfrak{C}) \to q(\mathfrak{C})$ be a relatively definable function; we will always implicitly assume that (the graph of) f is relatively defined by f(x, y). In general, this is not expressible by a tp-universal sentence built from f(x, y), although one part of the conclusion of 1.5 always holds: f has a definable extension relatively defined by f(x, y).

Fact 1.8. Let p(x) and q(y) be partial types over A and $f : p(\mathfrak{C}) \to q(\mathfrak{C})$ a relatively A-definable function. Then:

- (i) There are $\theta_p(x) \in p$ and $\theta_q(y) \in q$ and a definable extension of $f, \hat{f} : \theta_p(\mathfrak{C}) \to \theta_q(\mathfrak{C})$, relatively defined by f(x, y).
- (ii) The image $f(p(\mathfrak{C}))$ is an A-type-definable subset of $q(\mathfrak{C})$.
- (iii) The kernel relation of f, Ker f, defined by f(x) = f(x'), is a relatively A-definable equivalence relation on $p(\mathfrak{C})$.
- (iv) The inverse image $f^{-1}(D)$ of a relatively A-definable subset $D \subseteq q(\mathfrak{C})$ is a relatively A-definable subset of $p(\mathfrak{C})$.

Proof. (i) The sentence $(\forall x, x' \models p)(\forall y \models q)(f(x, y) \land f(x', y) \rightarrow x = x')$ expresses that f(x, y) relatively defines a partial function. By Fact 1.5 there is a formula $\theta_q(y) \in q$ such that f(x, y) relatively defines a partial function from $p(\mathfrak{C})$ to $\theta_q(\mathfrak{C})$; clearly, this function is total. Now, "f(x, y) relatively defines a function $p(\mathfrak{C}) \rightarrow \theta_q(\mathfrak{C})$ " is expressed by $(\forall x \models p)(\exists_1 y)(\theta_q(y) \land f(x, y))$. By Fact 1.5 there is a $\theta_p(x) \in p$ such that f(x, y) relatively defines a function $\hat{f} : \theta_p(\mathfrak{C}) \rightarrow \theta_q(\mathfrak{C})$. This proves (i).

Fix $\hat{f}: \theta_p(\mathfrak{C}) \to \theta_q(\mathfrak{C})$, an A-definable extension of f, given by (i). It is easy to see that: (ii) the type $\{(\exists x)(\hat{f}(x) = y \land \psi(x)) \mid \psi(x) \in p(x)\}$ defines $f(p(\mathfrak{C}))$; (iii) $\hat{f}(x) = \hat{f}(x')$ relatively defines Ker f. (iv) Suppose that D is relatively defined by $\phi(y)$ within $q(\mathfrak{C})$. Then the set $f^{-1}(D)$ is relatively

defined by $(\exists y)(\hat{f}(x) = y \land \theta_p(x) \land \theta_q(y) \land \phi(y))$.

Further in the paper, when dealing with tp-universal properties that involve a relatively definable function, say $f: p(\mathfrak{C}) \to q(\mathfrak{C})$, we will proceed similarly as in the proof of part (i) of the previous fact: First, we collect all relevant properties of relatively definable relations on $q(\mathfrak{C})$ and apply Fact 1.5 to find an appropriate $\theta_q(y) \in q$ such that, in addition, f(x, y) relatively defines a function from $p(\mathfrak{C})$ into $\theta_q(\mathfrak{C})$. Then we proceed with this and other properties involving p(x). Here is an example.

Example 1.9. Suppose that p(x) and q(y) are partial types over A, $<_p$ and $<_q$ are relatively A-definable orders on $p(\mathfrak{C})$ and $q(\mathfrak{C})$, respectively, and $f: p(\mathfrak{C}) \to q(\mathfrak{C})$ is a relatively A-definable strictly $(<_p, <_q)$ -increasing function. We find $\theta_p(x) \in p$ and $\theta_q(y) \in q$, such that the A-definable structure determined by $\theta_p(x)$, $\theta_q(y)$, $x <_p x'$, $y <_q y'$ and f(x, y) has the properties listed above, as follows. First, apply Fact 1.5 to:

 $-y <_q y'$ relatively defines a linear order on $q(\mathfrak{C})$, and

-f(x,y) relatively defines a partial function from $p(\mathfrak{C})$ into $q(\mathfrak{C})$.

Let $\theta_q(y) \in q$ be such that $y <_q y'$ defines a linear order on $\theta_q(\mathfrak{C})$ and f(x, y) relatively defines a function from $p(\mathfrak{C})$ into $\theta_q(\mathfrak{C})$. The desired formula $\theta_p(x) \in p$ is obtained by applying Fact 1.5 to:

 $-x <_p x'$ relatively defines a linear order on $p(\mathfrak{C})$, and

-f(x,y) relatively defines a strictly $(<_p,<_q)$ -increasing function from $p(\mathfrak{C})$ into $\theta_q(\mathfrak{C})$.

1.3. Stationarily ordered types. In this subsection, we recall basic facts about stationarily ordered types from [11].

Definition 1.10. A complete type $p \in S(A)$ is stationarily ordered (or so-type for short), if there exists a relatively A-definable linear order < on $p(\mathfrak{C})$ such that for every relatively definable set $D \subseteq p(\mathfrak{C})$, (exactly) one of the sets D and $p(\mathfrak{C}) \setminus D$ is left-eventual, and (exactly) one of them is right-eventual in $(p(\mathfrak{C}), <)$. In that case we say that (p, <) is an so-pair over A.

Definition 1.11. For an so-pair $\mathbf{p} = (p, <)$, define the left (\mathbf{p}_l) and the right (\mathbf{p}_r) globalization of \mathbf{p} :

 $\mathbf{p}_{l}(x) := \{ \phi(x) \in L_{\mathfrak{C}} \mid \phi(\mathfrak{C}) \cap p(\mathfrak{C}) \text{ is left-eventual in } (p(\mathfrak{C}), <) \} \text{ and } \mathbf{p}_{r}(x) := \{ \phi(x) \in L_{\mathfrak{C}} \mid \phi(\mathfrak{C}) \cap p(\mathfrak{C}) \text{ is right-eventual in } (p(\mathfrak{C}), <) \}.$

We collect the basic properties of the defined globalizations. These were proved in [11, Remark 3.5, Lemma 3.6]. However, to maintain the completeness of the presentation, we provide a proof here.

Fact 1.12. Let $\mathbf{p} = (p, <)$ be an so-pair over A.

- (i) Both \mathbf{p}_l and \mathbf{p}_r are complete global types that extend p.
- (ii) Both \mathbf{p}_l and \mathbf{p}_r are A-invariant types, so, in particular, they are nonforking extensions of p. Moreover, \mathbf{p}_l and \mathbf{p}_r are the only A-invariant globalizations of p.
- (iii) For all $B \supseteq A$ the locus $\mathbf{p}_{r \upharpoonright B}(\mathfrak{C})$ is a final part of $(p(\mathfrak{C}), <)$, and the locus $\mathbf{p}_{l \upharpoonright B}(\mathfrak{C})$ is an initial part of $(p(\mathfrak{C}), <)$.
- (iv) For $a, b \models p, a \models \mathbf{p}_{l \upharpoonright Ab}$ iff $b \models \mathbf{p}_{r \upharpoonright Aa}$. In other words, (a, b) is a Morley sequence in \mathbf{p}_r over A iff (b, a) is a Morley sequence in \mathbf{p}_l over A.
- (v) For any relatively A-definable linear order \triangleleft on $p(\mathfrak{C}), (p, \triangleleft)$ is also an so-pair over A.

Proof. (i) is easy. For (ii), note that the property being a left-eventual (right-eventual) subset of $p(\mathfrak{C})$ is invariant under automorphisms from $\operatorname{Aut}_A(\mathfrak{C})$, so both \mathbf{p}_l and \mathbf{p}_r are A-invariant. To see that they are the only two A-invariant extensions, suppose that \mathfrak{p} is an A-invariant global extension of p. Note that by A-invariance, either $(a < x) \in \mathfrak{p}$ for all $a \models p$ or $(x < a) \in \mathfrak{p}$ for all $a \models p$. In the first case, it is easy to see that any $\phi(x) \in \mathfrak{p}$ has an arbitrarily large realization in $p(\mathfrak{C})$, so $\phi(\mathfrak{C}) \cap p(\mathfrak{C})$ is a right-eventual subset of $(p(\mathfrak{C}), <)$, that is, $\phi(x) \in \mathbf{p}_r$; $\mathfrak{p} = \mathbf{p}_r$ follows. Similarly, $(x < a) \in \mathfrak{p}$ implies $\mathfrak{p} = \mathbf{p}_l$.

(iii) Clearly, the locus $\mathbf{p}_{r\uparrow B}(\mathfrak{C})$ is right-eventual in $(p(\mathfrak{C}), <)$, so $\{x \in p(\mathfrak{C}) \mid a < x\} \subseteq \mathbf{p}_{r\uparrow B}(\mathfrak{C})$ holds for some and hence for all $a \models \mathbf{p}_{r\uparrow B}$. Thus, $\mathbf{p}_{r\uparrow B}(\mathfrak{C})$ is final in $(p(\mathfrak{C}), <)$. Similarly, $\mathbf{p}_{l\uparrow B}(\mathfrak{C})$ is an initial part of $(p(\mathfrak{C}), <)$.

(iv) Suppose $b \models \mathbf{p}_{r \uparrow Aa}$ and let $c \models p$ be such that $b \models \mathbf{p}_{l \uparrow Ac}$; in particular, we have a < b < c. By (iii), c > b and $b \models \mathbf{p}_{r \uparrow Aa}$ imply $c \models \mathbf{p}_{r \uparrow Aa}$, while a < b and $b \models \mathbf{p}_{l \uparrow Ac}$ imply $a \models \mathbf{p}_{l \uparrow Ac}$. Thus, $ab \equiv ac \equiv bc$ (A). Now $b \models \mathbf{p}_{l \uparrow Ac}$ implies $a \models \mathbf{p}_{l \uparrow Ab}$ as \mathbf{p}_r is A-invariant. The other implication is similar.

(v) Let $x \triangleleft y$ be an L_A -formula that defines \triangleleft . Clearly, for $a \models p$, either $x \triangleleft a$ or $a \triangleleft x$ belongs to \mathbf{p}_l . Without loss of generality, suppose that the former is the case. By the A-invariance of \mathbf{p}_l , $(x \triangleleft a) \in \mathbf{p}_l$ for all $a \models p$. By (iv) and the A-invariance of \mathbf{p}_r , $(a \triangleleft x) \in \mathbf{p}_r$ for all $a \models p$. It suffices to prove that $\mathbf{p}_{l \upharpoonright B}(\mathfrak{C})$ is an initial part of $(p(\mathfrak{C}), \triangleleft)$ for all $B \supseteq A$; indeed, this implies that every formula from \mathbf{p}_l is left-eventual in $(p(\mathfrak{C}), \triangleleft)$, and an analogous argument shows that every formula from \mathbf{p}_r is right-eventual in $(p(\mathfrak{C}), \triangleleft)$, so we conclude that (p, \triangleleft) is an so-pair over A.

So, let $B \supseteq A$, $a \models \mathbf{p}_{l \upharpoonright B}$ and $b \models p$, $b \lhd a$. Since $(a \lhd x) \in \mathbf{p}_r$, $b \models \mathbf{p}_{r \upharpoonright Aa}$ follows. By saturation find $c \models \mathbf{p}_{l \upharpoonright B}$ such that $a \models \mathbf{p}_{l \upharpoonright Bc}$; in particular, $a \models \mathbf{p}_{l \upharpoonright Ac}$, so $c \models \mathbf{p}_{r \upharpoonright Aa}$ by (iv). Since $b \models \mathbf{p}_{r \upharpoonright Aa}$, $c \models \mathbf{p}_{r \upharpoonright Aa}$ and $\mathbf{p}_{r \upharpoonright Aa}(\mathfrak{C})$ is a final part of $(p(\mathfrak{C}), <)$ by (iii), we conclude b < c. Since $c \models \mathbf{p}_{l \upharpoonright B}$ and $\mathbf{p}_{l \upharpoonright B}(\mathfrak{C})$ is an initial part of $(p(\mathfrak{C}), <)$ by (iii), b < c implies $b \models \mathbf{p}_{l \upharpoonright B}$, and we are done.

2. Weakly o-minimal types

The notion of a weakly o-minimal type, as defined below, was first observed by Belegradek, Peterzil and Wagner in [1, p.1130], where they remark that every complete 1-type in a quasi-o-minimal theory is weakly o-minimal. In this section, we introduce weakly o-minimal orders and types and prove a few basic facts. For example, we prove that relatively definable equivalence relations on the locus of a weakly o-minimal type are convex and pairwise \subseteq -comparable. We also show that the weak o-minimality of a type (over A) is preserved under relatively A-definable mappings.

Definition 2.1. Let *P* be a type-definable set and < a relatively definable linear order on *P*. We will say that the order (P, <) is *weakly o-minimal* if every relatively definable subset of *P* has finitely many convex components.

Definition 2.2. A complete type $p(x) \in S(A)$ is weakly o-minimal if there exists a relatively Adefinable linear order < such that $(p(\mathfrak{C}), <)$ is a weakly o-minimal order. In that case, we say that (p, <) is a weakly o-minimal pair over A.

Remark 2.3. (a) If (P, <) is a weakly o-minimal order and $Q \subseteq P$ is a type-definable subset, then the suborder (Q, <) is also weakly o-minimal.

- (b) Every complete type that extends a weakly o-minimal type is also weakly o-minimal. In fact, if (p, <) is a weakly o-minimal pair over $A, B \supseteq A$ and the type $q \in S(B)$ extends p, then the pair (q, <) is weakly o-minimal over B because $(q(\mathfrak{C}), <)$ is a suborder of $(p(\mathfrak{C}), <)$.
- (c) If the theory T is weakly o-minimal with respect to <, then (P, <) is a weakly o-minimal order for every type-definable set $P \subset \mathfrak{C}$; in particular, the pair (p, <) is weakly o-minimal for every complete 1-type p. In fact, the latter holds even if T is weakly quasi-o-minimal with respect to <.
- (d) The weak o-minimality of a type is preserved by passing from T to T^{eq} . More precisely, if (p, <) is a weakly o-minimal pair over $A \subseteq \mathfrak{C}$, then p, viewed as a T^{eq} -type of a real sort, is also weakly o-minimal (as witnessed by <).
- (e) If (p(𝔅), <) is a weakly o-minimal order, then so is (p(𝔅), <_{*Ē*}) for any sequence *Ē* of pairwise ⊆-comparable, relatively definable, <-convex equivalence relations on p(𝔅); this is a consequence of Remark 1.3(e). Similarly, if (p, <) is a weakly o-minimal pair, then so is (p, <_{*Ē*}) for any sequence *Ē* of pairwise ⊆-comparable, relatively *A*-definable <-convex equivalence relations.</p>
- (f) It is easy to see that every weakly o-minimal type is an so-type. In fact, every weakly o-minimal pair over A, say $\mathbf{p} = (p, <)$, is an so-pair over A; therefore, \mathbf{p}_r and \mathbf{p}_l , the right and left globalizations of \mathbf{p} , are well defined.
- (g) The main advantage of weakly o-minimal types compared to so-types is that weak o-minimality transfers to complete extensions.

Weakly o-minimal types have the following important property (later proved in Corollary 3.4): Every weakly o-minimal type $p \in S(A)$ is weakly o-minimal with respect to *any* relatively A-definable order on $p(\mathfrak{C})$; that is, the order $(p(\mathfrak{C}), \lhd)$ is weakly o-minimal for any relatively A-definable order \lhd on $p(\mathfrak{C})$. However, this does not hold for all relatively \mathfrak{C} -definable orders, as illustrated in the following example.

Example 2.4. Consider the structure $\mathcal{M} = (\mathbb{R}, <, S)$ where S(x) = x + 1. \mathcal{M} is o-minimal as a reduct of the ordered group of reals, and the theory $T = \text{Th}(\mathcal{M})$ eliminates quantifiers. Since any translation $x \mapsto x + r$ is an automorphism of \mathcal{M} , there is a unique complete type $p \in S_1(\emptyset)$; (p, <) is a weakly o-minimal pair over \emptyset . The only \emptyset -definable linear orders on \mathbb{R} are: < and its reverse >; this follows by elimination of quantifiers. Hence p is weakly o-minimal with respect to all \emptyset -definable orders. Let \lhd be defined by:

- for $x \in [0,1)$ and $y \in [1,2)$: $x \triangleleft y$ iff $x+1 \leq y$, and $y \triangleleft x$ iff x+1 > y;
- for all other pairs (x, y) define $x \triangleleft y$ iff $x \lt y$.

It is not hard to see that \lhd is a {0}-definable linear order on \mathbb{R} . The order (\mathbb{R}, \lhd) is not weakly o-minimal, since the formula $x < \frac{1}{2}$ alternates on the sequence

$$\cdots \lhd \frac{1}{5} \lhd 1 + \frac{1}{5} \lhd \frac{1}{4} \lhd 1 + \frac{1}{4} \lhd \frac{1}{3} \lhd 1 + \frac{1}{3}.$$

Therefore, the type $p \in S_1(T)$ is weakly o-minimal and $(p(\mathfrak{C}), \prec)$ is a weakly o-minimal order for all \emptyset -definable orders on $p(\mathfrak{C})$, but the order $(p(\mathfrak{C}), \lhd)$ is not weakly o-minimal.

Recall that a function $f: X \to Y$ is $(<_X, <_Y)$ -increasing, where $<_X$ is a linear order on X and $<_Y$ a linear order on Y, if $x <_X x'$ implies that $f(x) \leq_Y f(x')$ holds for all $x, x' \in X$.

Lemma 2.5. (i) Suppose that (P, <) is a weakly o-minimal order, D is a definable set, and $f : P \to D$ is a relatively definable function with a convex kernel. For $y, y' \in f(P)$ define: $y <_f y'$ iff $f^{-1}(\{y\}) < f^{-1}(\{y'\})$. Then $(f(P), <_f)$ is a weakly o-minimal order, and $f : P \to f(P)$ is $(<, <_f)$ -increasing.

(ii) The weak o-minimality of types is preserved under interdefinability, that is, if $p, q \in S(A)$ are interdefinable, then p is weakly o-minimal iff q is such.

Proof. (i) Let P be type-defined by p(x) and let $D = \psi(\mathfrak{C})$. Suppose that x < y and f(x, y) relatively define < and f respectively. The kernel relation, Ker f, is relatively defined by $(\exists y)(\psi(y) \land f(x, y) \land f(x', y))$ on P. We have the following:

- x < x' defines a linear order on $p(\mathfrak{C})$ and f(x, y) defines a function $p(\mathfrak{C}) \to \psi(\mathfrak{C})$;

 $- (\exists y)(\psi(y) \land f(x,y) \land f(x',y)) \text{ defines a convex equivalence relation on } (p(\mathfrak{C}),<)).$

This is expressible by a tp-universal sentence, so by Fact 1.5 there exist a definable extension $(\theta(\mathfrak{C}), <)$ of $(p(\mathfrak{C}), <)$ and a definable extension $\hat{f} : \theta(\mathfrak{C}) \to \psi(\mathfrak{C})$ of f such that Ker \hat{f} is a <-convex equivalence relation on $\theta(\mathfrak{C})$. By the convexity of Ker \hat{f} , it is easy to see that $\hat{f}^{-1}(\{y\}) < \hat{f}^{-1}(\{y'\})$ defines a linear order on $\hat{f}(\theta(\mathfrak{C}))$ that agrees with $<_f$ on f(P). By Fact 1.8, the set f(P) is type-definable, so $<_f$ is relatively definable on f(P). It is easy to see that the function $f : P \to f(P)$ is $(<, <_f)$ -increasing.

Now we prove that $(f(P), <_f)$ is a weakly o-minimal order. Let D' be a relatively definable subset of f(P); we need to show that D' has finitely many convex components in $(f(P), <_f)$. By Fact 1.8, the inverse image $f^{-1}(D')$ is a relatively definable subset of P; the weak o-minimality of (P, <) implies that $f^{-1}(D')$ has finitely many <-convex components, say n. Since $f: P \to f(P)$ is $(<,<_f)$ -increasing and surjective, it follows that the set $D' = f(f^{-1}(D'))$ has exactly n convex components in $(f(P),<_f)$. Thus, $(f(P),<_f)$ is weakly o-minimal.

(ii) follows easily from (i).

Lemma 2.6. Let (p, <) be a weakly o-minimal pair over A and let $B \supseteq A$.

- (i) If D is a relatively B-definable subset of $p(\mathfrak{C})$, then every convex component of D, as well as each of the sets $\{x \in p(\mathfrak{C}) \mid x < D\}$ and $\{x \in p(\mathfrak{C}) \mid D < x\}$, is relatively B-definable.
- (ii) If $q \in S(B)$ is an extension of p, then q(x) is determined by the subtype of all L_B -formulae that relatively define a convex subset. In particular, $q(\mathfrak{C})$ is a convex subset of $p(\mathfrak{C})$ and (q, <) a weakly o-minimal pair over B.

Proof. (i) Let $D = \phi(p(\mathfrak{C}))$. Denote by \mathcal{C}_0 the finite convex partition of $p(\mathfrak{C})$ consisting of all the convex components of the sets D and $D^c = \neg \phi(p(\mathfrak{C}))$. Let $n = |\mathcal{C}_0|$. Then all the sets mentioned in the conclusion of the lemma are members of \mathcal{C}_0 ; we will prove that every member of \mathcal{C}_0 is relatively B-definable. Note that "x < y defines a linear order on $p(\mathfrak{C})$ " and " $\phi(x)$ determines a convex partition of $p(\mathfrak{C})$ with at most n elements" are tp-universal properties; the latter is expressed by:

$$\models (\forall x_0, \dots, x_n \models p) \left(x_0 < x_1 < \dots < x_n \rightarrow \bigvee_{i < n} (\phi(x_i) \leftrightarrow \phi(x_{i+1})) \right)$$

By Fact 1.5 there is a formula $\theta(x) \in p$ such that x < y defines a linear order on $\theta(\mathfrak{C})$ and $\phi(x)$ induces a convex partition \mathcal{C}_1 of $(\theta(\mathfrak{C}), <)$ with $\leq n$ members. Then each component from \mathcal{C}_0 is the intersection of the corresponding component from \mathcal{C}_1 with $p(\mathfrak{C})$. Since the members of \mathcal{C}_1 are *B*-definable, the desired conclusion follows.

(ii) Fix $q \in S(B)$ that extends p. For each $\phi(x) \in q$ let $D_{\phi} = \phi(\mathfrak{C}) \cap p(\mathfrak{C})$, then $q(\mathfrak{C}) = \bigcap_{\phi(x) \in q} D_{\phi}$. For each $\phi \in q$ the set D_{ϕ} has finitely many convex components on $p(\mathfrak{C})$ and, by part (i), each of them is relatively *B*-definable. Notice that the L_B -formulae that relatively define the components can be chosen pairwise inconsistent, in which case exactly one of them, say $\theta_{\phi}(x)$, belongs to q(x); denote by C_{ϕ} the component relatively defined by $\theta_{\phi}(x)$. Then $q(\mathfrak{C}) = \bigcap_{\phi(x) \in q} C_{\phi}$ and $\{\theta_{\phi}(x) \mid \phi \in q\} \vdash q(x)$. Finally, since each C_{ϕ} is a convex subset of $p(\mathfrak{C})$, so is the intersection $\bigcap_{\phi(x) \in q} C_{\phi} = q(\mathfrak{C})$.

Corollary 2.7. Let $\mathbf{p} = (p, <)$ be a weakly o-minimal pair over A and let $B \supseteq A$. Then the set $S_p(B) = \{q \in S_n(B) \mid p \subseteq q\}$ is linearly ordered by <; $\mathbf{p}_{l \uparrow B} = \min S_p(B)$ and $\mathbf{p}_{r \uparrow B} = \max S_p(B)$.

Proof. By Lemma 2.6(ii), $\{q(\mathfrak{C}) \mid q \in S_p(B)\}$ is a convex partition of $p(\mathfrak{C})$, so it is naturally linearly ordered by <. The second assertion is valid by Fact 1.12(iii).

In Lemma 2.6(ii), we proved that every complete type extending a weakly o-minimal p has a convex locus in $(p(\mathfrak{C}), <)$ (for any < witnessing the weak o-minimality of p. Now we show that this actually characterizes weakly o-minimal types.

Lemma 2.8. Let $p \in S(A)$ and < be a relatively A-definable linear order on $p(\mathfrak{C})$. If for all $B \supseteq A$ and all $q \in S(B)$ that extend p the locus $q(\mathfrak{C})$ is convex in $(p(\mathfrak{C}), <)$, then (p, <) is a weakly o-minimal pair.

Proof. Denote $\lambda = 2^{\aleph_0 + |T| + |A|}$. By way of contradiction, suppose that formula $\phi(x, b)$ relatively defines a subset of $p(\mathfrak{C})$ that has infinitely many convex components in $(p(\mathfrak{C}), <)$. Let $y = (x_\alpha)_{\alpha < \lambda^+}$. By compactness, the set $\pi(y) = \{x_\alpha < x_\beta \mid \alpha < \beta < \lambda^+\} \cup \{\neg(\phi(x_\alpha, b) \leftrightarrow \phi(x_{\alpha+1}, b)) \mid \alpha < \lambda^+\}$ is satisfiable; let $a = (a_\alpha)_{\alpha < \lambda^+}$ realize $\pi(y)$. Since there are at most λ extensions of p over Ab, there are $\alpha < \beta < \lambda^+$ such that $a_\alpha \equiv a_\beta$ (Ab). Note that $a_\alpha \not\equiv a_{\alpha+1}$ (Ab), so $\beta > \alpha + 1$. Since also $a_\alpha < a_{\alpha+1} < a_\beta$, this contradicts the assumption that the locus of $\operatorname{tp}(a_\alpha/Ab)$ is convex within $(p(\mathfrak{C}), <)$.

Definition 2.9. For a complete type $p \in S(A)$, define \mathcal{E}_p as the set of all relatively A-definable equivalence relations on $p(\mathfrak{C})$; $\mathbf{1}_p \in \mathcal{E}_p$ is the complete relation $p(\mathfrak{C})^2$.

Proposition 2.10. Let (p, <) be a weakly o-minimal pair over A. Then:

- (i) Every relation from \mathcal{E}_p is <-convex;
- (ii) $(\mathcal{E}_p, \subseteq)$ is a linear order.

Proof. (i) Let $E \in \mathcal{E}_p$. By way of contradiction, suppose that E is not convex and choose $a_1 < b_1 < a_2$ realizing p such that $\models E(a_1, a_2) \land \neg E(a_1, b_1)$. Let $f \in \operatorname{Aut}_A(\mathfrak{C})$ map a_1 to a_2 . Define $f(a_n) = a_{n+1}$ and $f(b_n) = b_{n+1}$ for $n \ge 1$. Then $\models E(a_n, a_{n+1}) \land \neg E(a_n, b_n)$ holds for all $n \ge 1$. Therefore, members of the sequence $a_1 < b_1 < a_2 < b_2 < a_3 < \ldots$ alternately satisfy the formula $E(a_1, x)$; that contradicts weak o-minimality of (p, <).

(ii) Let $E_1, E_2 \in \mathcal{E}_p$ and $a \models p$. It suffices to prove $[a]_{E_1} \subseteq [a]_{E_2}$ or $[a]_{E_2} \subseteq [a]_{E_1}$. Suppose, for the sake of contradiction, that $b \in [a]_{E_1} \setminus [a]_{E_2}$ and $c \in [a]_{E_2} \setminus [a]_{E_1}$. By (a), both $[a]_{E_1}$ and $[a]_{E_2}$ are convex, so either $b < [a]_{E_2}$ and $[a]_{E_1} < c$, or $c < [a]_{E_1}$ and $[a]_{E_2} < b$. Without loss, suppose that the former holds. Take $f \in \operatorname{Aut}_A(\mathfrak{C})$ such that f(b) = a, and let f(a) = a'; clearly, a < a' as b < a, and $a' \in [a]_{E_1}$ as $\models E_1(b, a)$. Thus, a' < c as $[a]_{E_1} < c$. Since a < a' < c, $a, c \in [a]_{E_2}$, and since $[a]_{E_2}$ is convex, we obtain $\models E_2(a, a')$. Thus, $\models E_2(f^{-1}(a), f^{-1}(a'))$ holds, that is, $\models E_2(b, a)$; a contradiction.

Corollary 2.11. If (p, <) is a weakly o-minimal pair over A, then Ker $f \in \mathcal{E}_p$ is a convex equivalence relation for any relatively A-definable function f from $p(\mathfrak{C})$ into an A-definable set.

Proof. By Fact 1.8 the kernel Ker f is relatively A-definable, so it is convex by Proposition 2.10(i).

Proposition 2.12. Let (p, <) be a weakly o-minimal pair over A and let $q \in S(A)$. Suppose that $f: p(\mathfrak{C}) \to q(\mathfrak{C})$ is a relatively A-definable function. Then:

- (i) $(q, <_f)$ is a weakly o-minimal pair over A, where $<_f$ is defined by $y <_f y'$ iff $f^{-1}(\{y\}) < f^{-1}(\{y'\})$; in particular, q is a weakly o-minimal type.
- (ii) $f: (\mathcal{E}_p, \subseteq) \to (\mathcal{E}_q, \subseteq)$ is an order-epimorphism whose restriction $\{E \in \mathcal{E}_p \mid \text{Ker } f \subseteq E\} \to \mathcal{E}_q$ is an order-isomorphism;
- (iii) For all $E \in \mathcal{E}_p$, $[x]_E \mapsto [f(x)]_{f(E)}$ defines a function $f_E : p(\mathfrak{C})/E \to q(\mathfrak{C})/f(E)$.

Proof. (i) By Corollary 2.11, the kernel relation, Ker f, is convex on $(p(\mathfrak{C}), <)$. Hence, the weakly ominimal order $(p(\mathfrak{C}), <)$ and the function f satisfy the assumptions of Lemma 2.5(i); we conclude that $(f(p(\mathfrak{C})), <_f)$ is a weakly o-minimal order. As $f(p(\mathfrak{C})) = q(\mathfrak{C})$, the pair $(q, <_f)$ is weakly o-minimal over A.

(ii) Here, the main task is to prove that f determines a function from \mathcal{E}_p to \mathcal{E}_q . Fix $E \in \mathcal{E}_p$ and we will prove $f(E) \in \mathcal{E}_q$. As Ker $f \in \mathcal{E}_p$, by Proposition 2.10(ii), we have two possibilities: $E \subseteq \text{Ker } f$ and Ker $f \subseteq E$. Clearly, $E \subseteq \text{Ker } f$ implies $f(E) = \text{id}_{q(\mathfrak{C})} \in \mathcal{E}_q$ and we are done, so from now on assume Ker $f \subseteq E$. Suppose that f is relatively defined by the L_A -formula f(x, y). By Fact 1.5 there is an A-definable set $D \supseteq q(\mathfrak{C})$, defined by $\psi(y)$ say, such that f(x, y) relatively defines a function $p(\mathfrak{C}) \to D$. Then the formula $(\exists y)(\psi(y) \land f(x, y) \land f(x', y))$ relatively defines the kernel Ker f, so we have the following properties:

- (1) $(p(\mathfrak{C}), <, E)$ is a linear order with a convex equivalence relation;
- (2) f(x, y) relatively defines a function $p(\mathfrak{C}) \to D$;
- (3) $(\exists y)(\psi(y) \land f(x,y) \land f(x',y))$ relatively defines a convex equivalence on $(p(\mathfrak{C}), <)$;
- $(4) \models (\forall x, x' \models p)((\exists y)(\psi(y) \land f(x, y) \land f(x', y)) \to E(x, x')).$

Here, (3) says that Ker f is convex on $(p(\mathfrak{C}), <)$ and (4) says Ker $f \subseteq E$. Clearly, (1)–(4) are tpuniversal properties, so by Fact 1.5 there exists an A-definable set $D_p \supseteq p(\mathfrak{C})$ such that letting \hat{E} and \hat{f} be as usual, we have: $(D_p, <, \hat{E})$ is a linear order with a convex equivalence relation; $\hat{f} : D_p \to D$; Ker \hat{f} is a convex equivalence on $(D_p, <)$ and Ker $\hat{f} \subseteq \hat{E}$. Furthermore, by replacing D with $\hat{f}(D_p)$ the kernel relation does not change, so we can also assume that \hat{f} is surjective. Note that Ker $\hat{f} \subseteq \hat{E}$ implies that $\hat{f}(\hat{E})$ is an equivalence relation on D_q ; clearly, $\hat{f}(\hat{E})$ is A-definable. Then $\hat{f}(\hat{E})_{\uparrow q(\mathfrak{C})} = f(E)$ is a relatively A-definable equivalence relation on $q(\mathfrak{C})$, so $f(E) \in \mathcal{E}_q$. Therefore, f maps \mathcal{E}_p to \mathcal{E}_q .

To show that f is surjective, let $F \in \mathcal{E}_q$ be relatively defined by F(y, y'). Then $E' = \{(x, x') \in p(\mathfrak{C})^2 \mid \models F(f(x), f(x'))\}$ is an equivalence relation on $p(\mathfrak{C})$ that is relatively defined by $(\exists y, y' \in D)(f(x, y) \land f(x', y') \land F(y, y'))$, so $E' \in \mathcal{E}_p$. As f(E') = F is clearly true, the function $f : \mathcal{E}_p \to \mathcal{E}_q$ is surjective; it follows that f is an order-epimorphism.

Finally, note that Ker $f \subseteq E_1 \subset E_2$ implies $f(E_1) \subset f(E_2)$, so $f : \{E \in \mathcal{E}_p \mid \text{Ker } f \subseteq E\} \to \mathcal{E}_q$ is an order-isomorphism. This completes the proof of (ii). (iii) follows from (ii)

Remark 2.13. A consequence of the previous proposition is that, roughly speaking, the quotient (p/E, <) of a weakly o-minimal pair (p, <) over A by a relatively A-definable (convex) equivalence relation E is also a weakly o-minimal pair. In general, p/E is not a \mathfrak{C}^{eq} -type, so formally it cannot be a weakly o-minimal type. So, instead of p/E we will work with $p/\hat{E} \in S^{eq}(A)$, where \hat{E} is a definable (convex) extension of E. The pair $(p/\hat{E}, <)$ is weakly o-minimal by the previous proposition, and p/E is interdefinable with p/\hat{E} in the sense that for each $a \models p$, hyperimaginary $[a]_E$ and imaginary $[a]_{\hat{E}} \in \mathfrak{C}^{eq}$ are interdefinable (meaning $\operatorname{Aut}_{A[a]_E}(\mathfrak{C}) = \operatorname{Aut}_{A[a]_{\hat{E}}}(\mathfrak{C})$).

In the next lemma, we will state some basic properties of the $(<_{\vec{E}}, \lhd)$ -increasing functions that will be used in the proof of the monotonicity theorems.

Lemma 2.14. Suppose that (p, <) and (q, \lhd) are weakly o-minimal pairs over A, $\vec{E} = (E_1, \ldots, E_n) \in (\mathcal{E}_p \setminus \{\mathbf{1}_p\})^n$ an \subseteq -increasing sequence, and $f : p(\mathfrak{C}) \to q(\mathfrak{C})$ a non-constant, relatively A-definable function such that $\operatorname{Ker}(f) \subseteq E_1$. Then:

(i) f is $(<_{\vec{E}}, \lhd)$ -increasing iff it is $(<, \lhd_{f(\vec{E})})$ -increasing; (ii) If f is $(<_{\vec{E}}, \lhd)$ -increasing, then: (a) $f_{E_n} : p(\mathfrak{C})/E_n \to q(\mathfrak{C})/f(E_n)$ is strictly $(<, \lhd)$ -increasing; (b) $f_{E_k} : p(\mathfrak{C})/E_k \to q(\mathfrak{C})/f(E_k)$ is strictly $(<_{(E_{k+1},...,E_n)}, \lhd)$ -increasing for all $k \leq n$.

Proof. Since Ker $f \subseteq E_1$ holds, we can apply Proposition 2.12(ii) and conclude that $(f(E_1), \ldots, f(E_n))$ is an increasing sequence of convex equivalences on $q(\mathfrak{C})$. In particular:

(*)
$$(a,b) \in E_k \iff (f(a),f(b)) \in f(E_k)$$
 holds for all $k \le n$.

(i) We will prove the case n = 1; the general case follows from this one and Remark 1.3(b) by easy induction. To prove (\Rightarrow) , suppose that f is $(<_{E_1} \lhd)$ -increasing, and let a < b be realizations of p. We consider two cases. If $(a, b) \in E_1$, then $b <_{E_1} a$, so $f(b) \triangleleft f(a)$ as f is $(<_{E_1}, \lhd)$ -increasing. Since also $(f(a), f(b)) \in f(E_1)$ by (*), we conclude $f(a) \triangleleft_{f(E_1)} f(b)$. On the other hand, if $(a, b) \notin E_1$, then $a <_{E_1} b$, so $f(a) \triangleleft f(b)$ as f is $(<_{E_1}, \lhd)$ -increasing. Since also $(f(a), f(b)) \notin f(E_1)$ by (*), we conclude $f(a) \triangleleft_{f(E_1)} f(b)$. Therefore, for all $a, b \models p, a < b$ implies $f(a) \triangleleft_{f(E_1)} f(b)$, so f is $(<, \triangleleft_{f(E_1)})$ -increasing. This completes the proof of (\Rightarrow) . To prove (\Leftarrow) , note that f being $(<, \lhd_{f(E_1)})$ -increasing is

the same as being $((<_{E_1})_{E_1}, \lhd_{f(E_1)})$ -increasing as $(<_{E_1})_{E_1} = <$. So, by (\Rightarrow) , f is $(<_{E_1}, (\lhd_{f(E_1)})_{f(E_1)})$ -increasing, that is, it is $(<_{E_1}, \lhd)$ -increasing as $(\lhd_{f(E_1)})_{f(E_1)} = <$.

(ii) Suppose that f is $(\langle \vec{E}, \triangleleft \rangle)$ -increasing. To prove part (a), assume that $a, b \models p$ and $[a]_{E_n} < [b]_{E_n}$. Since E_n is maximal in \vec{E} , the orders \langle and $\langle_{\vec{E}}$ agree on the E_n -classes, so $f([a]_{E_n}) \triangleleft f([b]_{E_n})$ follows as f is $(\langle \vec{E}, \triangleleft \rangle)$ -increasing. Furthermore, by (*), $[a]_{E_n} < [b]_{E_n}$ implies $(f(a), f(b)) \notin f(E_n)$, so the classes $[f(a)]_{f(E_n)}$ and $[f(b)]_{f(E_n)}$ are distinct; $[f(a)]_{f(E_n)} \triangleleft [f(b)]_{f(E_n)}$ follows. Therefore, f_{E_n} is strictly (\langle, \triangleleft) -increasing, which proves part (a). To prove (b), consider $(p(\mathfrak{C}), \langle_{(E_{k+1},\ldots,E_n)})$ instead of $(p(\mathfrak{C}), \langle\rangle)$. Then f is $((\langle_{(E_{k+1},\ldots,E_n)})_{(E_1,\ldots,E_k)}, \triangleleft)$ -increasing by Remark 1.3(b), so f_{E_k} is strictly $(\langle_{(E_{k+1},\ldots,E_n)}, \triangleleft)$ -increasing by part (a). \Box

3. Relatively definable orders and monotonicity theorems

In this section, we characterize relatively definable linear orders on the locus of a weakly o-minimal type and, as corollaries, obtain monotonicity theorems. The most technically demanding part is Theorem 3.3, where we prove that any pair of relatively definable orders on the locus of a weakly o-minimal type p satisfies $\triangleleft = \langle_{\vec{E}}$ for some increasing sequence of relations $\vec{E} \in (\mathcal{E}_p)^{<\omega}$. This result has already been proved in [12, Proposition 5.2] in the context of weakly quasi-o-minimal theories. Although the proof there *ad verbum* goes through in the context of weakly o-minimal types, we present a simpler argument here.

Lemma 3.1. Let (p, <) be a weakly o-minimal pair over A and $\phi(x, y)$ an L_A -formula. Let $C(a) = \phi(\mathfrak{C}, a)$. Suppose that for some (any) $a \models p$, C(a) is an initial part of $\{x \in p(\mathfrak{C}) \mid a < x\}$. Then there do not exist $a_1 < b_0 < a_0$ realizing p with $C(b_0) < C(a_0)$ and $C(b_0) \cup C(a_0) \subseteq C(a_1)$.

Proof. By way of contradiction, assume that $a_1 < b_0 < a_0$ are realizations of p such that $C(b_0) < C(a_0)$ and $C(b_0) \cup C(a_0) \subseteq C(a_1)$; in particular, $a_1 < C(b_0) < a_0$. Let $f \in \operatorname{Aut}_A(\mathfrak{C})$ map a_0 to a_1 , and for $n \ge 0$ define $a_{n+1} = f(a_n)$ and $b_{n+1} = f(b_n)$; by induction we have $a_{n+1} < b_n < a_n$, $C(a_n) \subseteq C(a_{n+1})$ and $a_{n+1} < C(b_n) < a_n$. Since $a_0 \in C(a_0) \subseteq C(a_1) \subseteq C(a_n) \subseteq \ldots$, we conclude $\models \phi(a_0, a_n)$ as $\phi(x, a_n)$ relatively defines $C(a_n)$. Also, $a_{n+1} < C(b_n) < a_n$ implies $a_0 > C(b_0) > C(b_1) > \ldots$, so $a_0 \notin C(b_n)$, that is, $\models \neg \phi(a_0, b_n)$. Therefore, the members of the sequence $\cdots < a_2 < b_1 < a_1 < b_0 < a_0$ alternately satisfy the formula $\phi(a_0, x)$, contradicting the fact that (p, <) is a weakly o-minimal pair. \Box

Definition 3.2. Let $\mathbf{p} = (p, <)$ be an so-pair over A and let \lhd be a relatively A-definable linear order on $p(\mathfrak{C})$. We say that the order \lhd has the same orientation as < if for some (all) $a \models p$ the formula $a \lhd x$ relatively defines a right-eventual part of $(p(\mathfrak{C}), <)$.

Let (p, <) be an so-pair and let \lhd be a relatively definable order on $p(\mathfrak{C})$. By the definition of so-pairs, for some (any) $a \models p$, some final part of $(p(\mathfrak{C}), <)$ is contained in the set defined by $a \lhd x$ or by $x \lhd a$. Therefore, \lhd or its reverse \lhd^* has the same orientation as <.

Theorem 3.3. Suppose that $\mathbf{p} = (p, <)$ is a weakly o-minimal pair over A and \lhd a relatively A-definable linear order on $p(\mathfrak{C})$. Then there exists a unique strictly increasing sequence of equivalence relations $\vec{E} = (E_0, E_1, ..., E_n) \in \mathcal{E}_p^{n+1}$ such that $E_0 = \mathrm{id}_{p(\mathfrak{C})}$ and $\lhd = <_{\vec{E}}$. Moreover, < and \lhd have the same orientation if and only if $E_n \neq \mathbf{1}_p$.

Proof. For $a \models p$, the subsets of $(a, +\infty)_{\mathbf{p}} = \{x \in p(\mathfrak{C}) \mid a < x\}$ relatively defined by the formulae $a \lhd x$ and $x \lhd a$ have finitely many convex components in $p(\mathfrak{C}, <)$. These components are relatively Aa-definable by Lemma 2.6(i), form a <-convex partition $C_0(a) < C_1(a) < \cdots < C_n(a)$ of $(a, +\infty)_{\mathbf{p}}$ and alternately satisfy $x \lhd a$ and $a \lhd x$; note that we obtain the same partition if we work with the reverse \lhd^* instead of \lhd . We proceed by induction on n.

If n = 0, then for any $a, b \models p$, a < b implies $b \in C_0(a)$, so $a \lhd b$ or $b \lhd a$ is valid, depending on whether $C_0(a)$ is determined by $a \lhd x$ or $x \lhd a$. Thus, $\triangleleft \Box \lhd \neg \neg \triangleleft \bullet^*$, that is, either $\triangleleft = \lhd \circ \neg \triangleleft \bullet^*$ holds by linearity. So taking $\vec{E} = (\mathrm{id}_{p(\mathfrak{C})})$ or $\vec{E} = (\mathrm{id}_{p(\mathfrak{C})}, \mathbf{1}_p)$ completes the proof.

Assume that $n \ge 1$. For k = 1, ..., n let E_k be the equivalence relation in $p(\mathfrak{C})$ given by $\sup C_{k-1}(x) = \sup C_{k-1}(y)$ (the sets $C_{k-1}(x)$ and $C_{k-1}(y)$ have the same set of strict upper bounds

in $(p(\mathfrak{C}), <)$). We will prove that $E_k \in \mathcal{E}_p$ and that they are \triangleleft -convex. Moreover, it will turn out that $(\mathrm{id}_{p(\mathfrak{C})}, E_1, \ldots, E_n)$ or $(\mathrm{id}_{p(\mathfrak{C})}, E_1, \ldots, E_n, \mathbf{1}_p)$ is the desired sequence.

For a while, we will assume that the elements of $C_n(a)$ satisfy $a \triangleleft x$, that is, that the orders < and \lhd have the same orientation, so the elements of $C_{n-1}(a)$ satisfy $x \triangleleft a$. Define $C(a) = \bigcup_{i < n} C_i(a)$; then inf C(a) = a, and $x \in C(y)$ is a relatively A-definable relation on $p(\mathfrak{C})$. Also note that max C(a) does not exist, for if $a' = \max C(a)$ and $a'' = \max C(a')$, then $a' \in C_{n-1}(a)$ and $a'' \in C_n(a) \cap C_{n-1}(a')$, so $a' \lhd a, a \lhd a''$ and $a'' \lhd a'$; this is impossible. Finally, since $C_{n-1}(a)$ is a final part of C(a), we have $E_n(x, y)$ iff $\sup C(x) = \sup C(y)$.

Claim 1. $b \in C_{n-1}(a)$ implies $C(b) \subseteq C(a)$.

Proof. Assume $b \in C_{n-1}(a)$; then a < b and $b \lhd a$. The set C(a) is convex and contains $b = \min C(b)$ so, to prove $C(b) \subseteq C(a)$, it suffices to show that some final part of C(b) is contained in C(a); we will prove $C_{n-1}(b) \subseteq C(a)$. For any $x \in C_{n-1}(b)$ we have b < x and $x \lhd b$. Combining with a < b and $b \lhd a$ we derive a < x and $x \lhd a$, so $x \in (a, +\infty)_{\mathbf{p}}$ and $x \notin C_n(a)$, and thus $x \in C(a)$. This proves $C_{n-1}(b) \subseteq C(a)$ and completes the proof of the claim.

Claim 2. If $C(a) \cap C(a_1) \neq 0$, then $E_n(a, a_1)$ holds.

Proof. Suppose not. Without loss, let $\sup C(a) < \sup C(a_1)$. Since $C_{n-1}(a_1)$ is a final part of $C(a_1)$, there exists a $a_0 \in C_{n-1}(a_1)$ such that $C(a) < a_0$; clearly, $C(a) < C(a_0)$. By Claim 1, $a_0 \in C_{n-1}(a_1)$ implies $C(a_0) \subseteq C(a_1)$. Additionally, note that $\sup C(a) < \sup C(a_1)$ implies that the nonempty set $C(a) \cap C(a_1)$ is a final part of C(a). Since $C_{n-1}(a)$ is also a final part of C(a), we conclude $C_{n-1}(a) \cap C(a_1) \neq \emptyset$. Choose $b_0 \in C_{n-1}(a) \cap C(a_1)$; $b_0 \in C(a_1)$ in particular implies $a_1 < b_0$. By Claim 1, $b_0 \in C_{n-1}(a)$ implies $C(b_0) \subseteq C(a)$, which together with $C(a) < a_0$ implies $C(b_0) < C(a_0)$ and $b_0 < a_0$. Now, $C(b_0) < C(a_0)$, $b_0 \in C(a_1)$ and $C(a_0) \subseteq C(a_1)$ imply $C(b_0) \subseteq C(a_1)$. Therefore, we have $a_1 < b_0 < a_0$ such that $C(b_0) < C(a_0)$ and $C(a_0) \cup C(b_0) \subseteq C(a_1)$; this is impossible by Lemma 3.1.

Claim 3. $E_n(x,y)$ iff $x \in C(y) \lor y \in C(x) \lor x = y$, and $E_n \in \mathcal{E}_p$ is a \triangleleft -convex equivalence relation.

Proof. By Claim 2 we see that $E_n(x, y)$ if and only if $C(x) \cap C(y) \neq 0$. Since $\max C(x)$ does not exist, $C(x) \cap C(y) \neq 0$ is easily seen to be equivalent to $x \in C(y) \lor y \in C(x) \lor x = y$, and since $x \in C(y)$ is a relatively A-definable relation, the relation E_n is also relatively A-definable. It remains to show that E_n is \lhd -convex, so assume that $x \lhd z \lhd y$ and $E_n(x, y)$ hold. If $C(z) \cap (C(x) \cup C(y)) \neq 0$, then by Claim 2 we have $z \in [x]_{E_n} = [y]_{E_n}$ and we are done. So suppose $C(z) \cap (C(x) \cup C(y)) = 0$. Since these are convex sets and $C(x) \cap C(y) \neq 0$, we see that $C(z) < C(x) \cup C(y)$ or $C(x) \cup C(y) < C(z)$ hold. To rule out the first option, note that C(z) < C(x) implies $x \in C_n(z)$, which contradicts $x \lhd z$. The second option is impossible, as C(y) < C(z) implies $z \in C_n(y)$, which contradicts $z \lhd y$.

As a consequence of Claim 3 we see that orders < and \lhd agree on p/E_n : $[x]_{E_n} < [y]_{E_n}$ iff $[x]_{E_n} \lhd [y]_{E_n}$. Now, consider the convex decomposition of $(a, +\infty)_p$ with respect to the formulae $x \lhd_{E_n} a$ and $a \lhd_{E_n} x$. Since the order \lhd_{E_n} reverses the order \lhd within each E_n -class and maintains the order of E_n -classes, it follows that the corresponding decomposition is $C_0(a) < \cdots < C_{n-2}(a) < C_{n-1}(a) \cup C_n(a)$ and that the elements of the final component $C_{n-1}(a) \cup C_n(a)$ satisfy the formula $a \lhd_{E_n} x$. By the induction hypothesis, relations (E_1, \ldots, E_{n-1}) are relatively A-definable and \lhd_{E_n} -convex; note that each of them refines E_n , so they are all \lhd -convex. By the induction hypothesis and Remark 1.3(b,c) we also have $<= (\lhd_{E_n})_{(\mathrm{id}_p(\mathfrak{C}), E_1, \ldots, E_{n-1})} = \lhd_{(\mathrm{id}_p(\mathfrak{C}), E_1, \ldots, E_{n-1})}$, which is equivalent to $\lhd =<(\mathrm{id}_{p(\mathfrak{C}), E_1, \ldots, E_n)}$.

So far, assuming that the elements of $C_n(a)$ satisfy $a \triangleleft x$, that is, the orders < and \triangleleft have the same orientation, we proved $\triangleleft = <_{(\mathrm{id}_{p(\mathfrak{C})}, E_1, \dots, E_n)}$. Now, if the elements of $C_n(a)$ satisfy $x \triangleleft a$, then the orders < and \triangleleft^* have the same orientation, so $<= \lhd_{(\mathrm{id}_{p(\mathfrak{C})}, E_1, \dots, E_n, \mathbf{1}_p)}$ easily follows. To finalize the proof of the theorem, it remains to notice that the uniqueness of \vec{E} follows from Remark 1.3(d). \Box

3.1. Monotonicity theorems and other corollaries of Theorem 3.3.

Corollary 3.4. If (p, <) is a weakly o-minimal pair over A, then so is the pair (p, \lhd) for every relatively A-definable linear order \lhd on $p(\mathfrak{C})$.

Proof. Suppose that (p, <) is a weakly o-minimal pair over A and \lhd is a relatively A-definable linear order on $p(\mathfrak{C})$. By Theorem 3.3 there is a sequence $\vec{E} \in \mathcal{E}_p^{<\omega}$ such that $\lhd =<_{\vec{E}}$. By Remark 2.3(e), (p, \lhd) is a weakly o-minimal pair over A.

Corollary 3.5. Suppose that $\mathbf{p} = (p, <)$ is a weakly o-minimal pair over A and \leq a relatively A-definable total pre-order on $p(\mathfrak{C})$. Then there exists a unique strictly increasing sequence of equivalence relations $\vec{E} = (E_0, \ldots, E_n) \in \mathcal{E}_p^{n+1}$ such that E_0 is defined by $x \leq y \land y \leq x$ and for all $x, y \in p(\mathfrak{C})$:

$$x \leq y$$
 if and only if $E_0(x,y) \lor x <_{\vec{E}} y$.

Proof. Note that $E_0 \in \mathcal{E}_p$, so it is convex by Proposition 2.10(i). By Fact 1.5 there is an A-definable set $D \supseteq p(\mathfrak{C})$ such that: x < y defines a linear order on D, $x \leq y$ defines a total pre-order on D, and $x \leq y \land y \leq x$ defines a convex equivalence relation, \hat{E}_0 , on (D, <). The canonical projection $\pi : p(\mathfrak{C}) \to p/\hat{E}_0(\mathfrak{C})$ is relatively A-definable, so by Proposition 2.12(i) the pair $(p/\hat{E}_0, <)$ is weakly o-minimal. Define: $[x]_{\hat{E}_0} \lhd [y]_{\hat{E}_0}$ iff $\neg \hat{E}_0(x, y) \land x \leq y$; it is easy to see that \lhd is a relatively A-definable linear order on $p/\hat{E}_0(\mathfrak{C})$, so we can apply Theorem 3.3. Let $\vec{E}' = (\mathrm{id}_{p/\hat{E}_0(\mathfrak{C})}, E'_1, \ldots, E'_n) \in \mathcal{E}_{p/\hat{E}_0}^{n+1}$ be a strictly increasing sequence such that $\lhd =<_{\vec{E}'}$. By Proposition 2.12(ii) there is a strictly increasing sequence $\vec{E} = (E_0, E_1, \ldots, E_n) \in \mathcal{E}_p^{n+1}$ such that $\pi(\vec{E}) = \vec{E'}$. For all $x, y \in p(\mathfrak{C})$ satisfying $\neg \hat{E}_0(x, y)$ we have:

$$x \leq y \Leftrightarrow [x]_{\hat{E}_0} \lhd [y]_{\hat{E}_0} \Leftrightarrow [x]_{\hat{E}_0} <_{\vec{E}'} [y]_{\hat{E}_0} \Leftrightarrow x <_{\vec{E}} y,$$

where the last equivalence easily holds by induction on n.

We will now prove Theorem 1.

Theorem 3.6 (Weak monotonicity). Suppose that $\mathbf{p} = (p, <_p)$ is a weakly o-minimal pair over A, (D, <) is an A-definable linear order, and $f : p(\mathfrak{C}) \to D$ is a relatively A-definable non-constant function.

- (i) There exists a unique strictly increasing sequence of equivalence relations $\vec{E} = (E_0, \ldots, E_n) \in \mathcal{E}_p^{n+1}$ such that $E_0 = \text{Ker } f$ and f is $((<_p)_{\vec{E}}, <)$ -increasing.
- (ii) There exists an increasing sequence of A-definable convex equivalence relations $\vec{F} = (F_0, \ldots, F_n)$ on (D, <) such that f is $(<_p, <_{\vec{F}})$ -increasing.

Proof. (i) The conclusion follows easily by Corollary 3.5, after noting that $f(x) \leq f(y)$ relatively defines a total pre-order on $p(\mathfrak{C})$.

(ii) Let q = f(p) and let \vec{E} satisfy the conclusion of (i). Then $f : p(\mathfrak{C}) \to q(\mathfrak{C})$ is $((<_p)_{\vec{E}}, <)$ increasing. By Proposition 2.12(ii), $f(\vec{E}) = (f(E_0), \ldots, f(E_n))$ is an increasing sequence of convex,
relatively A-definable equivalence relations on $q(\mathfrak{C})$, and f is $(<_p, <_{f(\bar{E})})$ -increasing by Lemma 2.14(i).
Now, the sequence \vec{F} satisfying the conclusion in (ii) can be found by routine compactness.

In the following two theorems we deduce Theorem 2.

Theorem 3.7 (Local monotonicity). Suppose that $\mathbf{p} = (p, <_p)$ is a weakly o-minimal pair over A, (D, <) is a A-definable linear order, and $f : p(\mathfrak{C}) \to D$ is a relatively A-definable non-constant function. Then there exists $E \in \mathcal{E}_p \setminus \{ \mathrm{id}_{p(\mathfrak{C})} \}$ such that the restriction of f to each E-class is either constant or strictly $(<_p, <)$ -monotone.

Proof. If Ker $f \neq \operatorname{id}_{p(\mathfrak{C})}$ then $E = \operatorname{Ker} f$ satisfies the conclusion of the theorem, as f is constant on each E-class. If f is strictly $(<_p, <)$ -monotone, then $E = \mathbf{1}_p$ satisfies the conclusion. The remaining case is where $\operatorname{Ker} f = \operatorname{id}_{p(\mathfrak{C})}$ holds and f is not strictly $(<_p, <)$ -monotone. Let $\vec{E} = (E_0, \ldots, E_n) \in \mathcal{E}_p^{n+1}$ be given by Theorem 3.6(i). In particular, $E_0 = \operatorname{Ker} f = \operatorname{id}_{p(\mathfrak{C})}$, and f not strictly $(<_p, <)$ -monotone implies $n \geq 1$. Then we easily see that $E = E_1$ satisfies the conclusion of the theorem.

Theorem 3.8 (Upper monotonicity). Suppose that $\mathbf{p} = (p, <_p)$ is a weakly o-minimal pair over A, (D, <) is a A-definable linear order and $f: p(\mathfrak{C}) \to D$ is a relatively A-definable non-constant function.

(i) There exists a $E \in \mathcal{E}_p \setminus \{\mathbf{1}_p\}$ such that one of the following two conditions holds for all x_1, x_2 realizing p:

 $[x_1]_E <_p [x_2]_E \Rightarrow f(x_1) < f(x_2)$ or $[x_1]_E <_p [x_2]_E \Rightarrow f(x_1) > f(x_2)$.

(ii) If q = f(p), then there exists a $E \in \mathcal{E}_p \setminus \{\mathbf{1}_p\}$ such that the function $f_E : p(\mathfrak{C})/E \to q(\mathfrak{C})/f(E)$, defined by $f_E([x]_E) = [f(x)]_{f(E)}$, is strictly $(<_p, <)$ -monotone.

Proof. (i) Let $\vec{E} = (E_0, \ldots, E_n) \in \mathcal{E}_p^{n+1}$ be an increasing sequence given by Theorem 3.6(i). If $E_n \neq \mathbf{1}_p$ then $<_p$ agrees with $(<_p)_{(E_0,\ldots,E_n)}$ on $p(\mathfrak{C})/E_n$, so for $E = E_n$ the first option of (i) holds. Otherwise, $>_p$ agrees with $(<_p)_{(E_0,\ldots,E_{n-1})}$ on $p(\mathfrak{C})/E_{n-1}$, so for $E = E_{n-1}$ the second option of (i) holds.

(ii) Let $E \in \mathcal{E}_p \setminus \{\mathbf{1}_p\}$ satisfy the conclusion of (i). Then the function f_E is well defined by Proposition 2.12(iii), and strictly $(<_p, <)$ -monotone by (i).

Now we turn to the context of weakly o-minimal theories and prove Theorem 3.

Theorem 3.9. Suppose that $\text{Th}(\mathfrak{C}, <, \ldots)$ is weakly o-minimal, (D, \lhd) is an A-definable linear order and $f : \mathfrak{C} \to D$ is an A-definable function. Then:

- (i) There exists a finite convex A-definable partition \mathfrak{C} of \mathfrak{C} and an increasing sequence of A-definable convex equivalence relations \vec{E} on \mathfrak{C} such that f is $(<_{\vec{E}}, \lhd)$ -increasing on each member of \mathfrak{C} .
- (ii) There exists a finite convex A-definable partition $\mathfrak C$ of $\mathfrak C$ and a convex A-definable equivalence relation E on \mathfrak{C} with finitely many finite classes, such that $E = \bigcup_{C \in \mathfrak{C}} E_{\uparrow C}$ and the restriction $f_{\upharpoonright [a]_E}$ is constant or strictly $(<, \lhd)$ -monotone uniformly for all $a \in C$.

Proof. (i) First, note that the pair (p, <) is weakly o-minimal for every $p \in S_1(A)$. For each $p \in S_1(A)$ we will find a formula $\theta_p \in p$ and a sequence $\vec{E}_p \in \mathcal{E}_p^{<\omega}$ such that

 $\theta_p(\mathfrak{C})$ is a <-convex subset of \mathfrak{C} and $f_{\restriction \theta_p(\mathfrak{C})}$ is $(<_{\vec{E}_n}, \lhd)$ -increasing, (1)

in the following way. If f is constant on $p(\mathfrak{C})$, then by compactness there is a $\theta_p(x) \in p$ such that f is constant on $\theta_p(\mathfrak{C})$; by the weak o-minimality of the theory we may suppose that $\theta_p(\mathfrak{C})$ is convex. Set $\vec{E}_p = \mathrm{id}_{p(\mathfrak{C})}$ and note that $f_{\restriction \theta_p(\mathfrak{C})}$ is $(<_{\vec{E}_p}, \lhd)$ -increasing, so condition (1) is satisfied in this case. The other case is where f is non-constant on $p(\mathfrak{C})$. Then by Theorem 3.6(i) there is an increasing sequence $E'_p \in \mathcal{E}_p^{<\omega}$ such that $f_{\restriction p(\mathfrak{C})}$ is $(<_{\vec{E'_n}}, \lhd)$ -increasing. Note that the following are tp-universal properties:

- $\begin{array}{l} \ \vec{E}'_p \ \text{is an increasing sequence of } <-\text{convex equivalence relations on } p(\mathfrak{C}); \\ \ f: p(\mathfrak{C}) \to D \ \text{is a} \ (<_{\vec{E}'_p}, \lhd)\text{-increasing function.} \end{array}$

By Fact 1.5 there is a $\theta_p(x) \in p$ and an increasing sequence of A-definable <-convex equivalence relations $\vec{E_p}$ on $\theta_p(\mathfrak{C})$ such that $\vec{E_p}_{\restriction p(\mathfrak{C})} = \vec{E'_p}$ and $f_{\restriction \theta_p(\mathfrak{C})}$ is $(<_{\vec{E_p}}, \lhd)$ -increasing; again, we can assume that $\theta_p(\mathfrak{C})$ is a <-convex subset of \mathfrak{C} , so condition (1) is satisfied.

Since $\{[\theta_p(x)] \mid p \in S_1(A)\}$ is an open cover of $S_1(A)$, by compactness, we can find a finite subcover $\{[\theta_{p_1}(x)], \ldots, [\theta_{p_n}(x)]\}$. By a simple modification, we can assume that $\theta_{p_i}(x)$'s are mutually contradictory. It remains to construct the sequence \vec{E} . First, note that we may assume that all \vec{E}_{p_i} 's are of the same length. Indeed, if m is the maximal length, then every shorter sequence \vec{E}_{p_i} can be expanded by adding an appropriate number of $\mathrm{id}_{\theta_{p_i}(\mathfrak{C})}$ at the beginning; this does not change $\langle_{\vec{E}_{p_i}}$. So, let $\vec{E}_{p_i} = (E_{p_i,1}, \dots, E_{p_i,m})$. Now, define $\vec{E} = (E_1, \dots, E_m)$ in the obvious way: set E_j equal to $E_{p_i,j}$ on the part $\theta_{p_i}(\mathfrak{C})$, leaving the elements of different parts unrelated. Clearly, $\langle_{\vec{E}}$ equals $\langle_{\vec{E}_{p_i}}$ on $\theta_{p_i}(\mathfrak{C})$, so the conclusion follows.

(ii) We need the following observation: if F is a convex definable equivalence relation on \mathfrak{C} , then the set $\{a \in \mathfrak{C} \mid [a]_F$ is infinite} is definable, which follows from the fact that weak o-minimality of T guarantees that F has only finitely many classes with finitely many but more than one element.

For each $p \in S_1(A)$ we find a $\theta_p(x) \in p$ and an A-definable convex equivalence relation E_p on $\theta_p(\mathfrak{C})$, such that $\theta_p(\mathfrak{C})$ is convex and:

- (1) If $f_{\uparrow p(\mathfrak{C})}$ is constant, then $f_{\uparrow \theta_p(\mathfrak{C})}$ is also constant and $E_p = \theta_p(\mathfrak{C})^2$;
- (2) If $f_{\uparrow p(\mathfrak{C})}$ is non-constant, then each E_p -class is infinite and $f_{\uparrow [a]_{E_p}}$ is constant/strictly $(<, \lhd)$ increasing/strictly $(<, \lhd)$ -decreasing uniformly for all $a \in \theta_p(\mathfrak{C})$.

(1) is fulfilled as in the proof of (i). For (2), assume that f is non-constant. By Theorem 3.7, there exists a $E_p \in \mathcal{E}_p \setminus \{\mathrm{id}_{p(\mathfrak{C})}\}\$ such that $f_{\uparrow[a]_{E_p}}$ is constant/strictly $(<,\lhd)$ -increasing/strictly $(<,\lhd)$ -decreasing on $[a]_{E_p}$ uniformly for all $a \in p(\mathfrak{C})$; for simplicity, assume that $f_{\uparrow[a]_{E_p}}$ is strictly $(<,\lhd)$ -increasing. Note that $E_p \neq \mathrm{id}_{p(\mathfrak{C})}$ implies that each E_p -class is infinite. As in the proof of part (i), we find $\theta_p(x) \in p$ and a convex A-definable equivalence relation on $\theta_p(\mathfrak{C})$, also denoted by E_p , such that the restriction of fto each E_p -class is strictly $(<,\lhd)$ -increasing. By the above observation, we can further shrink $\theta_p(\mathfrak{C})$ so that condition (2) is satisfied.

As in the proof of (i), choose a finite subcover $\{[\theta_{p_1}(x)], \ldots, [\theta_{p_n}(x)]\}$ of $\{[\theta_p(x)] \mid p \in S_1(A)\}$. Let $\mathcal{C} = \{\theta_{p_i}(\mathfrak{C}) \mid i \leq n\}$. For each $C = \theta_{p_i}(\mathfrak{C}) \in \mathcal{C}$ denote $E_C := E_{p_i}$. Thus, we have a finite A-definable convex cover \mathcal{C} of \mathfrak{C} , and for each $C \in \mathcal{C}$ a convex A-definable definable equivalence relation E_C on C, such that at least one of the following two conditions is satisfied:

- (3) $f_{\uparrow C}$ is constant and $E_C = C^2$;
- (4) Each E_C -class is infinite and $f_{[a]_{E_C}}$ is constant/strictly $(<, \lhd)$ -increasing/strictly $(<, \lhd)$ -decreasing uniformly for all $a \in C$.

Refine \mathcal{C} in an obvious way to become a convex partition of \mathfrak{C} ; attach to each member of the partition the restriction of an appropriately chosen E_C . Note that after this modification, we have at most finitely many "new" finite E_C -classes (parts of previously infinite classes); each of those classes is Adefinable, so we can split each of them into single-element classes and form a new A-definable convex partition, each of whose members satisfies at least one of conditions (3) and (4). It is easy to see that the partition \mathcal{C} and the relation $E = \bigcup_{C \in \mathcal{C}} E_C$ satisfy the conclusion of (ii).

3.2. Weak quasi-o-minimality.

The following proposition describes the weak quasi-o-minimality of a theory as a "local" property of its complete 1-types. This extends [11, Theorem 1(ii)], in which we showed that the weak quasi-o-minimality of T does not depend on the particular choice of the linear order.

Proposition 3.10. A complete first-order theory T with infinite models is weakly quasi-o-minimal if and only if every type $p \in S_1(T)$ is weakly o-minimal.

Proof. It is easy to see that if T is weakly quasi-o-minimal with respect to <, then (p, <) is a weakly o-minimal pair over \emptyset for every $p \in S_1(T)$. For the converse, assume that every $p \in S_1(T)$ is weakly o-minimal. In particular, every $p \in S_1(T)$ is linearly ordered (there is a relatively \emptyset -definable linear order on $p(\mathfrak{C})$), so by routine compactness we may find an \emptyset -definable linear order on whole \mathfrak{C} . We prove that T is weakly quasi-o-minimal with respect to <.

Let $D \subseteq \mathfrak{C}$ be definable. By Corollary 3.4, (p, <) is a weakly o-minimal pair for every $p \in S_1(T)$, so $D \cap p(\mathfrak{C})$ has finitely many convex components, say n_p , in $(p(\mathfrak{C}), <)$. By compactness, as in the proof of Lemma 2.6(i), we find $\theta_p(x) \in p$ such that $D \cap \theta_p(\mathfrak{C})$ has n_p convex components in $(\theta_p(\mathfrak{C}), <)$, say $D_{p,i}, 1 \leq i \leq n_p$: $D \cap \theta_p(\mathfrak{C}) = \bigcup_{i=1}^{n_p} D_{p,i}$. Note that each of $D_{p,i}$ is definable. Setting $D_{p,i}^{conv}$ as the convex hull of $D_{p,i}$ in $(\mathfrak{C}, <)$, we have $D_{p,i} = D_{p,i}^{conv} \cap \theta_p(\mathfrak{C})$, so $D \cap \theta_p(\mathfrak{C}) = \bigcup_{i=1}^{n_p} D_{p,i}^{conv} \cap \theta_p(\mathfrak{C})$. Since $\{\theta_p(x) \mid p \in S_1(T)\}$ covers $S_1(T)$, by compactness we find a finite sub-cover $\{\theta_{p_j}(x) \mid 1 \leq j \leq m\}$. We have:

$$D = D \cap \mathfrak{C} = D \cap \bigcup_{j=1}^{m} \theta_{p_j}(\mathfrak{C}) = \bigcup_{j=1}^{m} D \cap \theta_{p_j}(\mathfrak{C}) = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_{p_j}} D_{p_j,i}^{conv} \cap \theta_{p_j}(\mathfrak{C}),$$

which is a Boolean combination of convex and \emptyset -definable sets, and we are done.

The proposition motivates the following definition.

Definition 3.11. A partial type $\pi(x)$ is weakly quasi-o-minimal over A if $\pi(x)$ is over A and every $p \in S(A)$ extending $\pi(x)$ is weakly o-minimal; in that case we say that the set $\pi(\mathfrak{C})$ is weakly quasi-o-minimal over A.

Note that if $\pi(x)$ is weakly quasi-o-minimal over A, then $\pi(x)$ is weakly quasi-o-minimal over any $B \supseteq A$ as complete extensions of weakly o-minimal types are weakly o-minimal by Lemma 2.6(ii).

Proposition 3.12. Let P be type-definable over A. Then P is weakly quasi-o-minimal over A if and only if there exists a relatively A-definable linear order < on P such that every relatively definable subset of P is a Boolean combination of <-convex and relatively A-definable sets. In that case, the latter is true for any relatively A-definable linear order < on P.

Proof. The proof of Proposition 3.10 with obvious modifications goes through.

Let (X, <) be a linear order and $\mathcal{C} = (C_1, \ldots, C_n)$ a partition of X. By $<_{\mathcal{C}}$ we denote the order obtained by keeping the original order within each component and defining $C_1 <_{\mathcal{C}} \cdots <_{\mathcal{C}} C_n$. Combining the arguments from previous proofs, the following theorem can be routinely derived.

Theorem 3.13. Suppose that a type-definable set P is weakly quasi-o-minimal over A, < is a relatively A-definable linear order on P, (D, \lhd) is an A-definable linear order, and $f : P \to D$ is a relatively A-definable function. Then there are A-definable extensions $(\hat{P}, <)$ of (P, <) and $\hat{f} : \hat{P} \to D$, an A-definable partition \mathbb{C} of \hat{P} and an increasing sequence of A-definable $<_{\mathbb{C}}$ -convex equivalence relations \vec{E} on \hat{P} such that \hat{f} is $((<_{\mathbb{C}})_{\vec{E}}, \lhd)$ -increasing on each member of \mathbb{C} .

4. Non-orthogonality and orientation

In this section, we study forking independence in the context of so-types. We introduce the notion of **p**-genericity, and prove Theorems 4 and 5. We also prove that the following binary relations are equivalences: forking on the set of realizations of all so-types over a fixed domain A, weak and forking non-orthogonality of so-types over A, and direct non-orthogonality of so-pairs over A; As we remarked before, this was proven in [11], but we find the current presentation substantially simpler and intuitive. Almost all the results of this section are obtained in the context of so-types, exemptions are 4.10 and 4.21 which rely on the preservation of weak 0-minimality in extensions.

Lemma 4.1. Let $\mathbf{p} = (p, <)$ be an so-pair over A and let $\phi(x, b)$ be any formula. Then the type $p(x) \cup \{\phi(x, b)\}$ forks over A if and only if $\phi(x, b)$ relatively defines a bounded subset of $(p(\mathfrak{C}), <)$.

Proof. For one direction of the equivalence, assume that $p(x) \cup \{\phi(x,b)\}$ forks over A. By Fact 1.12(ii), \mathbf{p}_l and \mathbf{p}_r are nonforking extensions of p, so $\phi(x,b) \notin \mathbf{p}_l(x) \cup \mathbf{p}_r(x)$. Then $\phi(x,b) \notin \mathbf{p}_r(x)$ implies that $p(\mathfrak{C}) \cap \phi(\mathfrak{C}, b)$ is upper bounded, while $\phi(x,b) \notin \mathbf{p}_l(x)$ implies that $p(\mathfrak{C}) \cap \phi(\mathfrak{C}, b)$ is lower bounded. Therefore, $p(\mathfrak{C}) \cap \phi(\mathfrak{C}, b)$ is bounded in $(p(\mathfrak{C}), <)$. For the other direction, assume that $p(\mathfrak{C}) \cap \phi(\mathfrak{C}, b)$ is bounded. Let $b_0 = b$, and let $a_0, a_1 \models p$ be such that $a_0 < p(\mathfrak{C}) \cap \phi(\mathfrak{C}, b_0) < a_1$. By compactness there is $\theta(x) \in p$ such that $a_0 < \theta(\mathfrak{C}) \cap \phi(\mathfrak{C}, b_0) < a_1$. Let $f \in \operatorname{Aut}_A(\mathfrak{C})$ be such that $f(a_0) = a_1$. Define $a_{n+1} := f(a_n)$ and $b_{n+1} := f(b_n)$; each $b_n \models \operatorname{tp}(b/A)$. By induction, we see that $a_n < \theta(\mathfrak{C}) \cap \phi(\mathfrak{C}, b_n) < a_{n+1}$. So $\{\theta(\mathfrak{C}) \cap \phi(\mathfrak{C}, b_n) \mid n < \omega\}$ is 2-inconsistent, which shows that $p(x) \cup \{\phi(x, b)\}$ divides over A.

Immediately from the lemma, we have the following corollary.

Corollary 4.2. Let $\mathbf{p} = (p, <)$ be an so-pair over A and let $B \supseteq A$.

- (i) The type p has exactly two global nonforking extensions: \mathbf{p}_r and \mathbf{p}_l .
- (ii) The only nonforking extensions of p in S(B) are $\mathbf{p}_{r \upharpoonright B}$ and $\mathbf{p}_{l \upharpoonright B}$.
- (iii) The following holds for all $q \in S(B)$ that extend p: q forks over A if and only if the locus $q(\mathfrak{C})$ is bounded in $(p(\mathfrak{C}), <)$.

Notice that condition " $p(x) \cup \{\phi(x)\}$ forks over A" from Lemma 4.1 does not refer to any particular relatively definable order < on the locus of the so-type $p \in S(A)$, so the equivalent condition, $\phi(p(\mathfrak{C}))$ is bounded in $(p(\mathfrak{C}), <)$, holds for all relatively A-definable orders on $p(\mathfrak{C})$.

Definition 4.3. Let $p(x) \in S(A)$ be an so-type. We will say that $\phi(x)$ is a *p*-bounded formula if $p(x) \cup \{\phi(x)\}$ forks over A; *p*-bounded subsets of $p(\mathfrak{C})$ are those that are relatively defined by *p*-bounded formulas.

Notice that any formula that relatively defines a *p*-bounded set is *p*-bounded, too.

Definition 4.4. Let $\mathbf{p} = (p, <)$ be an so-pair over A and B a small set. Define:

$$\mathcal{L}_{\mathbf{p}}(B) := (\mathbf{p}_{l \upharpoonright AB})(\mathfrak{C}); \quad \mathcal{R}_{\mathbf{p}}(B) := (\mathbf{p}_{r \upharpoonright AB})(\mathfrak{C}); \quad \mathcal{D}_{p}(B) := \{a \in p(\mathfrak{C}) \mid a \not\perp_{A} B\}.$$

Recall that $p, q \in S(A)$ are forking orthogonal, $p \perp^f q$, if $a \, \bigcup_A b$ holds for all $a \models p$ and $b \models q$.

Lemma 4.5. Let $\mathbf{p} = (p, <)$ be an so-pair over A.

- (i) $\mathcal{L}_{\mathbf{p}}(B)$ is an initial and $\mathcal{R}_{\mathbf{p}}(B)$ is a final part of $(p(\mathfrak{C}), <)$.
- (ii) $a \, {\bf p}_{\bf a} B$ if and only if $a \in \mathcal{L}_{\bf p}(B) \cup \mathfrak{R}_{\bf p}(B)$.
- (iii) $\mathcal{D}_p(\tilde{B}) = p(\mathfrak{C}) \smallsetminus (\mathcal{L}_p(B) \cup \mathfrak{R}_p(B))$ is a convex (possibly empty), p-bounded subset of $(p(\mathfrak{C}), <)$; $\mathcal{D}_p(B)$ is the union of all p-bounded, relatively AB-definable subsets of $p(\mathfrak{C})$.
- (iv) There are three possible cases: 1° $p \not\preceq^{f} \operatorname{tp}(B/A)$. Then $\mathcal{L}_{\mathbf{p}}(B) < \mathcal{D}_{p}(B) < \mathcal{R}_{\mathbf{p}}(B)$ is a convex partition of $p(\mathfrak{C})$; 2° $p \perp^{f} \operatorname{tp}(B/A)$ and $p \not\preceq^{w} \operatorname{tp}(B/A)$. Then $\mathcal{D}_{p}(B) = \emptyset$ and $\mathcal{L}_{\mathbf{p}}(B) < \mathcal{R}_{\mathbf{p}}(B)$ is a convex partition of $p(\mathfrak{C})$.
 - $\mathcal{Z} \ p \perp^w \operatorname{tp}(B/A). \ Then \ \mathcal{L}_{\mathbf{p}}(B) = \mathfrak{R}_{\mathbf{p}}(B) = p(\mathfrak{C}) \ and \ \mathfrak{D}_p(B) = \varnothing.$

Proof. (i) is Fact 1.12(iii), (ii)–(iv) follow by (i) and Corollary 4.2.

By part (i) of the previous lemma, the set $\mathcal{R}_{\mathbf{p}}(B)$ is a final part of $(p(\mathfrak{C}), <)$, so its elements can be thought of as being realizations of p which are "as far to the <-right (from the point of view) of B as possible". This is formalized in the next definition.

Definition 4.6. Let $\mathbf{p} = (p, <)$ be an so-pair over A, B be a small set of parameters, and $a \in p(\mathfrak{C})$. We say that a is right \mathbf{p} -generic over B, denoted by $B \triangleleft^{\mathbf{p}} a$, if $a \in \mathcal{R}_{\mathbf{p}}(B)$; similarly, a is left \mathbf{p} -generic over B if $a \in \mathcal{L}_{\mathbf{p}}(B)$.

Remark 4.7. Let $\mathbf{p} = (p, <)$ be an so-pair over A.

- (a) There exist left- and right \mathbf{p} -generic elements over any small set B.
- (b) By Fact 1.12(iv), for $a, b \models p, a \in \mathcal{L}_{\mathbf{p}}(b)$ iff $b \in \mathcal{R}_{\mathbf{p}}(a)$, that is, a is left **p**-generic over b iff b is right **p**-generic over a.
- (c) Lemma 4.5(i), $B \triangleleft^{\mathbf{p}} a < a'$ implies $B \triangleleft^{\mathbf{p}} a'$.
- (d) By Lemma 4.5(ii), for all $a \models p$, $a \downarrow_A B$ holds if and only if a is left or right **p**-generic over B.
- (e) If $a, b \models p$ then a is left (resp. right) \mathbf{p} -generic over b if and only if b is left (resp. right) \mathbf{p}^* -generic over a, where $\mathbf{p}^* = (p, >)$ is the reverse of \mathbf{p} .
- (f) By Lemma 4.5(iv), $p \not\preceq^{f} \operatorname{tp}(B/A)$ and $B \lhd^{\mathbf{p}} a$ imply $\mathcal{D}_{p}(B) < a$.
- (g) Note that we did not choose a special symbol to denote the left **p**-genericity. However, "a is left **p**-generic over B" can be expressed by $B \triangleleft^{\mathbf{p}^*} a$.

Lemma 4.8. Let $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ be so-pairs over A. Assume $B \triangleleft^{\mathbf{p}} a$. Then there exists $b \models q$ such that $B \triangleleft^{\mathbf{q}} b \triangleleft^{\mathbf{p}} a$.

Proof. Choose $b' \models q$ satisfying $B \lhd^{\mathbf{q}} b'$, and then $a' \models p$ satisfying $Bb' \lhd^{\mathbf{p}} a'$; in particular, $B \lhd^{\mathbf{p}} a'$ and $b' \lhd^{\mathbf{p}} a'$ hold. Then $B \lhd^{\mathbf{p}} a'$ and $B \lhd^{\mathbf{p}} a$ imply $\operatorname{tp}(a/AB) = \operatorname{tp}(a'/AB) = \operatorname{p}_{r \upharpoonright AB}$. Let $f \in \operatorname{Aut}_{AB}(\mathfrak{C})$ map a' to a. Put b = f(b'). Then $B \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a'$ implies the desired conclusion $B \lhd^{\mathbf{q}} b \lhd^{\mathbf{p}} a$. \Box

Lemma 4.9. $a \downarrow_{A} b \Leftrightarrow b \downarrow_{A} a$ holds for all realizations of so-types over A.

Proof. Suppose that $p = \operatorname{tp}(a/A)$ and $q = \operatorname{tp}(b/A)$ are so-types and $b \not \perp_A a$; in particular, $q \not \perp^f p$. Set $a_0 := a$ and $b_0 := b$ and note $b_0 \in \mathcal{D}_q(a_0)$. Choose orders $<_p$ and $<_q$ such that $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ are so-pairs over A. By Lemma 4.5(iv), $\mathcal{D}_q(a_0)$ is a nonempty bounded subset of $q(\mathfrak{C})$, and by taking $d_0 \in \mathcal{L}_{\mathbf{q}}(a_0)$ and $d_1 \in \mathcal{R}_{\mathbf{q}}(a_0)$ we have $d_0 <_q \mathcal{D}_q(a_0) <_q d_1$. Let $f \in \operatorname{Aut}_A(\mathfrak{C})$ be such that $f(d_0) = d_1$; set $a_{n+1} := f(a_n)$ and $b_{n+1} := f(b_n)$ for n = 0, 1. Then $\mathcal{D}_q(a_0) <_q \mathcal{D}_q(a_1) <_q \mathcal{D}_q(a_2)$ and $b_i \in \mathcal{D}_q(a_i)$ for i = 0, 1, 2. Note that the sequence (a_0, a_1, a_2) is $<_p$ -monotone, so by possibly reversing the order, we may assume that it is $<_p$ -increasing.

Since $\mathcal{D}_q(a_0) <_q b_1 \in \mathcal{D}_q(a_1)$ we have $a_0 \neq a_1$ (Ab₁), and since $\mathcal{D}_q(a_1) \ni b_1 <_q \mathcal{D}_q(a_2)$ we have $a_1 \neq a_2$ (Ab₁). Consider the locus P of tp(a_1/Ab_1). Since $a_0 <_p a_1$, $a_1 \in P$, and $a_0 \notin P$, P is not an initial part of $(p(\mathfrak{C}), <_p)$, so a_1 is not left **p**-generic over b_1 . Similarly, $a_1 <_p a_2$, $a_1 \in P$ and $a_2 \notin P$ imply that P is not a final part of $(p(\mathfrak{C}), <_p)$, so a_1 is also not right **p** generic over b_1 . Thus, by Lemma 4.5(ii), $a_1 \not \downarrow_A b_1$. Applying f^{-1} we get $a \not \downarrow_A b$, and we are done.

Lemma 4.10. Let $p \in S(A)$ be an so-type and let $B \supseteq A$.

- (i) If $q = \operatorname{tp}(a/B)$ is a nonforking extension of p, then $\mathcal{D}_p(a) \subseteq q(\mathfrak{C})$.
- (ii) If q in (i) is also an so-type over B (which is the case if p is weakly o-minimal, for example), then $\mathcal{D}_p(a) \subseteq \mathcal{D}_q(a)$.
- (iii) For all $a, a' \models p$: $a \bigcup_A B$ and $a' \not \sqcup_A a$ imply $a \equiv a'$ (B).

Proof. (i) Let $\mathbf{p} = (p, <)$ be an so-pair over A. Since q is a nonforking extension of p, by Corollary 4.2(ii) we have $q = \mathbf{p}_{l \uparrow B}$ or $q = \mathbf{p}_{r \uparrow B}$; by reversing the order < if necessary, we can assume $q = \mathbf{p}_{r \uparrow B}$; hence, $B \lhd^{\mathbf{p}} a$. Next, we show that $\mathcal{D}_p(a)$ is lower bounded in $q(\mathfrak{C})$. By Lemma 4.8 there exists $a' \models p$ such that $B \lhd^{\mathbf{p}} a' \lhd^{\mathbf{p}} a$. By Remark 4.7(b), $a' \lhd^{\mathbf{p}} a$ implies $a' \in \mathcal{L}_{\mathbf{p}}(a)$, so $a' < \mathcal{D}_p(a)$ holds by Lemma 4.5(iv). $B \lhd^{\mathbf{p}} a'$ implies $a' \models q, a' \in q(\mathfrak{C})$ is a lower bound of $\mathcal{D}_p(a)$. As $q(\mathfrak{C})$ is a final part of $(p(\mathfrak{C}), <)$ we conclude $\mathcal{D}_p(a) \subseteq q(\mathfrak{C})$.

(ii) From the proof of (i), we have that $\mathcal{D}_p(a)$ is lower bounded in $q(\mathfrak{C})$; $\mathcal{D}_p(a)$ is also upper bounded by any a'' which satisfies $a \triangleleft^{\mathbf{q}} a''$. So, for each $b \in \mathcal{D}_p(a)$, the locus of $\operatorname{tp}(b/Ba)$ is bounded in $(q(\mathfrak{C}), <)$, and hence $b \in \mathcal{D}_q(a)$ by Corollary 4.2(iii).

(iii) Suppose $a \, {\downarrow}_A B$ and let $q = \operatorname{tp}(a/B)$. By (i), we have $\mathcal{D}_p(a) \subseteq q(\mathfrak{C})$, so every element $a' \in \mathcal{D}_p(a)$ realizes q, that is, $a \equiv a'$ (B).

Now, we can prove Theorem 4.

Theorem 4.11. Let $\mathbf{p} = (p, <)$ be an so-pair over A. Assume that p is non-algebraic.

- (i) $x \not \perp_A y$ defines a convex equivalence relation, denoted by \mathcal{D}_p , on $(p(\mathfrak{C}), <)$.
- (ii) $(p(\mathfrak{C}), \triangleleft^{\mathbf{p}})$ is a strict partial order in which $\triangleleft^{\mathbf{p}}$ -incomparability agrees with the relation \mathcal{D}_p .
- (iii) $\triangleleft^{\mathbf{p}}$ and < agree on $p(\mathfrak{C})/\mathfrak{D}_p$; $(p(\mathfrak{C})/\mathfrak{D}_p, \triangleleft^{\mathbf{p}})$ is a dense linear order.

Proof. (i) The reflexivity is clear and the symmetry follows from Lemma 4.9. For transitivity, assume $a, b, c \models p, a \not\perp_A b$, and $b \not\perp_A c$. If $a \perp_A c$ were true, then, by Lemma 4.10(iii), $a \perp_A c$ and $a \not\perp_A b$ would imply $a \equiv b$ (Ac), which contradicts $b \not\perp_A c$. Therefore, $a \not\perp_A c$.

(ii) Clearly, $\triangleleft^{\mathbf{p}}$ is antireflexive and the transitivity follows by Remark 4.14(c). To prove the other claim, let $a, b \models p$. By Remark 4.7(d), $a \not \perp_A b$ holds if and only if a is neither left nor right **p**-generic over b. By Remark 4.7(b), a is left **p**-generic over b if and only if b is right **p**-generic over a. Therefore, $a \not \perp_A b$ holds if and only if a is not right **p**-generic over b and b is not right **p**-generic over a, that is, if a and b are $\triangleleft^{\mathbf{p}}$ -incomparable.

(iii) By (i), the quotient $p(\mathfrak{C})/\mathfrak{D}_p$ is linearly ordered by <. If $[x]_{\mathfrak{D}_p} \triangleleft^{\mathbf{p}} [y]_{\mathfrak{D}_p}$ then $x \triangleleft^{\mathbf{p}} y$ and thus x < y. Conversely, if $[x]_{\mathfrak{D}_p} < [y]_{\mathfrak{D}_p}$ then $x \bigsqcup_A y$, so by (ii) either $x \triangleleft^{\mathbf{p}} y$ or $y \triangleleft^{\mathbf{p}} x$ holds; the latter is ruled out by x < y. Therefore, $\triangleleft^{\mathbf{p}}$ and < agree on $p(\mathfrak{C})/\mathfrak{D}_p$. So $\triangleleft^{\mathbf{p}}$ is a linear order on $p(\mathfrak{C})/\mathfrak{D}_p$. To prove the density, assume $[a]_{\mathfrak{D}_p} \triangleleft^{\mathbf{p}} [b]_{\mathfrak{D}_p}$. Then $a \triangleleft^{\mathbf{p}} b$, so by Lemma 4.8 there exists $c \models p$ such that $a \triangleleft^{\mathbf{p}} c \triangleleft^{\mathbf{p}} b$; then $[a]_{\mathfrak{D}_p} \triangleleft^{\mathbf{p}} [c]_{\mathfrak{D}_p} .$

4.1. Orientation.

Let $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ be so pairs. We have chosen $a \triangleleft^{\mathbf{q}} b$ to describe that " $b \in q(\mathfrak{C})$ is as far to the $<_q$ -right of a as possible"; the underlying intuition would be justified if "b is far to the $<_q$ -right of a" and "a is far to the $<_p$ -left of b" would be equivalent. Note that option (I) in part (i) of the following lemma says just that, which motivates the definition of direct non-orthogonality of so-pairs.

Lemma 4.12. Suppose that $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ are so-pairs over A and $p \measuredangle^w q$. (i) Exactly one of the following two conditions holds: (I) For all $a \models p$ and $b \models q$: $a \in \mathcal{L}_{\mathbf{p}}(b) \Leftrightarrow b \in \mathfrak{R}_{\mathbf{q}}(a)$ and $a \in \mathfrak{R}_{\mathbf{p}}(b) \Leftrightarrow b \in \mathcal{L}_{\mathbf{q}}(a)$.

(II) For all
$$a \models p$$
 and $b \models q$: $a \in \mathcal{L}_{\mathbf{p}}(b) \Leftrightarrow b \in \mathcal{L}_{\mathbf{q}}(a)$ and $a \in \mathcal{R}_{\mathbf{p}}(b) \Leftrightarrow b \in \mathcal{R}_{\mathbf{q}}(a)$.

(ii) (I) is equivalent to: $a \triangleleft^{\mathbf{q}} b$ and $b \not\models^{\mathbf{p}} a$ hold for some $a \models p$ and $b \models q$.

(iii) (II) is equivalent to: $a \triangleleft^{\mathbf{q}} b$ and $b \triangleleft^{\mathbf{p}} a$ hold for some $a \models p$ and $b \models q$.

Proof. (i) Let $a \models p$ and $b \models q$. By Lemma 4.5(ii) we know that $a \downarrow_A b$ is equivalent to $a \in \mathcal{L}_{\mathbf{p}}(b) \cup \mathcal{R}_{\mathbf{p}}(b)$. By the independence symmetry, proved in Lemma 4.9, one more equivalent condition is $b \in \mathcal{L}_{\mathbf{q}}(a) \cup \mathcal{R}_{\mathbf{q}}(a)$. Therefore, for all $x \models q$ we have:

(*)
$$a \in \mathcal{L}_{\mathbf{p}}(x) \cup \mathcal{R}_{\mathbf{p}}(x)$$
 if and only if $x \in \mathcal{L}_{\mathbf{q}}(a) \cup \mathcal{R}_{\mathbf{q}}(a)$.

Note that, by A-invariance of \mathbf{p}_l and \mathbf{p}_r , each of $a \in \mathcal{L}_{\mathbf{p}}(x)$ and $a \in \mathcal{R}_{\mathbf{p}}(x)$ determines a completion of q(x) in S(Aa); denote these completions by q_L and q_R , respectively, and note that they are different because of $p \not\perp^w q$. Since $x \in \mathcal{L}_{\mathbf{q}}(a)$ determines the type $\mathbf{q}_{l \uparrow Aa}(x)$ and $x \in \mathcal{R}_{\mathbf{q}}(a)$ determines $\mathbf{q}_{r \uparrow Aa}(x)$, the equivalence in (*) can be expressed by $\{q_L, q_R\} = \{\mathbf{q}_{l \uparrow Aa}, \mathbf{q}_{r \uparrow Aa}\}$. Here, we have two possibilities. The first is $q_R(x) = \mathbf{q}_{l \uparrow Aa}(x)$ and $q_L(x) = \mathbf{q}_{r \uparrow Aa}(x)$ (for all $a \models p$); this is equivalent to (I). The other possibility, $q_L(x) = \mathbf{q}_{l \uparrow Aa}(x)$ and $q_R(x) = \mathbf{p}_{r \uparrow Aa}(x)$, corresponds to (II).

(ii) and (iii) Let $a \models p$ and $b \models q$ satisfy $a \triangleleft^{\mathbf{q}} b$. Notice that $b \not\models^{\mathbf{p}} a$ is inconsistent with (II) and is therefore equivalent to (I). Similarly, $b \triangleleft^{\mathbf{p}} a$ is equivalent to (II).

Definition 4.13. Let $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ be so-pairs over A and $p \not\perp^w q$. The pairs \mathbf{p} and \mathbf{q} are directly non-orthogonal, denoted by $\delta_A(\mathbf{p}, \mathbf{q})$, if option (I) of Lemma 4.12 holds.

Remark 4.14. (a) Immediately from the definition, we find that δ is symmetric: $\delta_A(\mathbf{p}, \mathbf{q})$ iff $\delta_A(\mathbf{q}, \mathbf{p})$. (b) By Lemma 4.12, $p \not\perp^w q$ implies that exactly one of $\delta_A(\mathbf{p}, \mathbf{q})$ and $\delta_A(\mathbf{p}, \mathbf{q}^*)$ is true.

(c) So-pairs $\mathbf{p} = (p, <)$ and $\mathbf{p}' = (p, <')$ are directly non-orthogonal if and only if the orders < and <' have the same orientation (in the sense of Definition 3.2). In that case, $B \triangleleft^{\mathbf{p}} a$ iff $B \triangleleft^{\mathbf{p}'} a$ for all B and $a \models p$.

Lemma 4.15. Let $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ be so-pairs over A and $p \not\perp^w q$. Then $\delta_A(\mathbf{p}, \mathbf{q})$ is equivalent to each of the following conditions:

- (1) for all $a_1 <_p a_2 \models p$, $\Re_{\mathbf{q}}(a_1) \supseteq \Re_{\mathbf{q}}(a_2)$, i.e. $a_1 <_p a_2 \triangleleft^{\mathbf{q}} x$ implies $a_1 \triangleleft^{\mathbf{q}} x$;
- (2) for all $a_1, a_2 \models p, a_1 \triangleleft^{\mathbf{p}} a_2$ iff $\mathfrak{R}_{\mathbf{q}}(a_1) \supseteq \mathfrak{R}_{\mathbf{q}}(a_2)$;
- (3) There are no $a \models p$ and $b \models q$ such that $a \triangleleft^{\mathbf{q}} b$ and $b \triangleleft^{\mathbf{p}} a$.

Proof. $\delta_A(\mathbf{p}, \mathbf{q}) \Rightarrow (1)$ Assume $\delta(\mathbf{p}, \mathbf{q})$. Let $a_1 <_p a_2$ realize p. Suppose $b \in \mathcal{R}_{\mathbf{q}}(a_2)$. Then b is right \mathbf{q} -generic over a_2 , so direct non-orthogonality implies that a_2 is left \mathbf{p} -generic over b, that is, $a_2 \in \mathcal{L}_{\mathbf{p}}(b)$. Combining with $a_1 <_p a_2$ we derive $a_1 \in \mathcal{L}_{\mathbf{p}}(b)$, so direct nonorthogonality implies $b \in \mathcal{R}_{\mathbf{q}}(a_1)$.

 $(1)\Rightarrow(2)$ Assume (1) and let $a_1, a_2 \models p$. For (\Rightarrow) , assume $a_1 \triangleleft^{\mathbf{p}} a_2$. Choose $a' \models p$ satisfying $\mathcal{R}_{\mathbf{q}}(a_1) \supseteq \mathcal{R}_{\mathbf{q}}(a')$ and choose a'_2 satisfying $a'a_1 \triangleleft^{\mathbf{p}} a'_2$. In particular $a_1 \triangleleft^{\mathbf{p}} a'_2$, hence $a_1a_2 \equiv a_1a'_2$ (A), so we can find $a \models p$ such that $a_1a_2a \equiv a_1a'_2a'$ (A); $\mathcal{R}_{\mathbf{q}}(a_1) \supseteq \mathcal{R}_{\mathbf{q}}(a)$ and $aa_1 \triangleleft^{\mathbf{p}} a_2$ hold. Since $a \triangleleft^{\mathbf{p}} a_2$, by (1), $\mathcal{R}_{\mathbf{q}}(a) \supseteq \mathcal{R}_{\mathbf{q}}(a_2)$, which together with $\mathcal{R}_{\mathbf{q}}(a_1) \supseteq \mathcal{R}_{\mathbf{q}}(a)$ implies $\mathcal{R}_{\mathbf{q}}(a_1) \supseteq \mathcal{R}_{\mathbf{q}}(a_2)$.

For (\Leftarrow) , assume $\Re_{\mathbf{q}}(a_1) \supseteq \Re_{\mathbf{q}}(a_2)$; let $b \in \Re_{\mathbf{q}}(a_1) \smallsetminus \Re_{\mathbf{q}}(a_2)$ and note $a_1 \not\equiv a_2$ (Ab). By Lemma 4.9, $a_1 \bigcup_A b$ as $b \in \Re_{\mathbf{q}}(a_1)$, so $a_1 \not\equiv a_2$ (Ab) implies $a_1 \bigcup_A a_2$ by Lemma 4.10(iii), so $a_2 \bigcup_A a_1$ by Lemma 4.9. Thus either $a_2 \in \Re_{\mathbf{p}}(a_1)$, i.e. $a_1 \triangleleft^{\mathbf{p}} a_2$, or $a_2 \in \mathcal{L}_{\mathbf{p}}(a_1)$. The latter implies $a_1 \in \Re_{\mathbf{p}}(a_2)$ by Remark 4.7(b), i.e. $a_2 \triangleleft^{\mathbf{p}} a_1$, so $\Re_{\mathbf{q}}(a_2) \supseteq \Re_{\mathbf{q}}(a_1)$ by (\Rightarrow); a contradiction.

 $(2) \Rightarrow (3)$ Assume (2) and, towards a contradiction, suppose $a \lhd^{\mathbf{q}} b \lhd^{\mathbf{p}} a$. Let $a' \models p$ be such that $b \notin \mathcal{R}_{\mathbf{q}}(a')$. Since $b \in \mathcal{R}_{\mathbf{q}}(a)$ as $a \lhd^{\mathbf{q}} b$, and $\mathcal{R}_{\mathbf{q}}(a)$ and $\mathcal{R}_{\mathbf{q}}(a')$ are \subseteq -comparable as the final parts of $(p(\mathfrak{C}), <_p)$, we conclude $\mathcal{R}_{\mathbf{q}}(a) \supseteq \mathcal{R}_{\mathbf{q}}(a')$. By (2), $a \lhd^{\mathbf{p}} a'$ follows; in particular, $a <_p a'$. Since $a \in \mathcal{R}_{\mathbf{p}}(b)$ as $b \lhd^{\mathbf{p}} a$, $a <_p a'$ implies $a' \in \mathcal{R}_{\mathbf{p}}(b)$. Thus, $a \equiv a'$ (Ab), but this contradicts $b \in \mathcal{R}_{\mathbf{q}}(a) \smallsetminus \mathcal{R}_{\mathbf{q}}(a')$. (3) $\Rightarrow \delta_A(\mathbf{p}, \mathbf{q})$ follows from Lemma 4.12(ii).

In the following lemma, we prove a more general form of transitivity of \triangleleft .

Lemma 4.16. Suppose that $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ are directly non-orthogonal so-pairs over A. (i) For all a, b and $B: B \lhd^{\mathbf{p}} a$ and $a \lhd^{\mathbf{q}} b$ imply $B \lhd^{\mathbf{q}} b$. (ii) For all a, b and $B: B \lhd^{\{\mathbf{p},\mathbf{q}\}} a$ and $a \lhd^{\{\mathbf{p},\mathbf{q}\}} b$ imply $B \lhd^{\{\mathbf{p},\mathbf{q}\}} b$ (where $\lhd^{\{\mathbf{p},\mathbf{q}\}} = \lhd^{\mathbf{p}} \cup \lhd^{\mathbf{q}}$). (iii) $(p(\mathfrak{C}) \cup q(\mathfrak{C}), \lhd^{\mathbf{p},\mathbf{q}})$ is a strict partial order.

Proof. (i) Suppose not. Then $b \in \mathcal{R}_{\mathbf{q}}(a) \smallsetminus \mathcal{R}_{\mathbf{q}}(B)$. Since $\mathcal{R}_{\mathbf{q}}(B)$ and $\mathcal{R}_{\mathbf{q}}(a)$ are the final parts of $(q(\mathfrak{C}), <_q)$ we get $\mathcal{R}_{\mathbf{q}}(B) \subsetneq \mathcal{R}_{\mathbf{q}}(a)$. Let $b' \in \mathcal{R}_{\mathbf{q}}(B)$. Choose $a' \models p$ such that $b' \in \mathcal{L}_{\mathbf{q}}(a')$. Then $\mathcal{L}_{\mathbf{q}}(a') <_q \mathcal{R}_{\mathbf{q}}(a')$ and $b' \in \mathcal{L}_{\mathbf{q}}(a')$ together imply $b' <_q \mathcal{R}_{\mathbf{q}}(a')$, which combined with $b' \in \mathcal{R}_{\mathbf{q}}(B)$ gives $\mathcal{R}_{\mathbf{q}}(a') \subsetneq \mathcal{R}_{\mathbf{q}}(a)$. Therefore, $\mathcal{R}_{\mathbf{q}}(a') \subsetneq \mathcal{R}_{\mathbf{q}}(a)$ holds and, in particular, $a \not\equiv a'$ (AB). Now, by Lemma 4.15 $\mathcal{R}_{\mathbf{q}}(a) \supsetneq \mathcal{R}_{\mathbf{q}}(a')$ implies $a \lhd^{\mathbf{p}} a'$, which together with $B \lhd^{\mathbf{p}} a$ gives $a, a' \models \mathbf{p}_{r \upharpoonright AB}$ and, in particular, $a \equiv a'$ (AB); a contradiction.

- (ii) Clearly, $\delta_A(\mathbf{x}, \mathbf{y})$ holds for all $\mathbf{x}, \mathbf{y} \in \{\mathbf{p}, \mathbf{q}\}$, so by part (i), $B \triangleleft^{\mathbf{x}} a \triangleleft^{\mathbf{y}} b$ implies $B \triangleleft^{\mathbf{y}} b$.
- (iii) Irreflexivity is clear and transitivity follows from part (ii).

Theorem 4.17. Let \mathcal{P} denote the set of all so-pairs over A or the set of all weakly o-minimal pairs over A. Denote by \mathcal{T} the set of all types corresponding to an element of \mathcal{P} and by $\mathcal{T}(\mathfrak{C})$ the set of all realizations of types of \mathcal{T} .

- (i) \measuredangle^w is an equivalence relation on both \Im and \Im . (Here, we say $(p, <_p) \measuredangle^w (q, <_q)$ iff $p \measuredangle^w q$.)
- (ii) δ_A is an equivalence relation on \mathfrak{P} ; δ_A refines \measuredangle^w by splitting each class consisting of non-algebraic types into two classes, with each of them consisting of the reverses of the other class.
- (iii) $x \not\perp_A y$ is an equivalence relation on $\mathfrak{T}(\mathfrak{C})$.
- (iv) \measuredangle^f is an equivalence relation on Υ .

Proof. Clearly, both δ_A and \measuredangle^w are reflexive and symmetric.

(i) To prove the transitivity of \angle^w , assume that $p, q, r \in S(A)$ are so-types such that $p \angle^w q$ and $q \angle^w r$. Choose relatively A-definable orders $<_p, <_q$ and $<_r$ such that $\mathbf{p} = (p, <_p)$, $\mathbf{q} = (q, <_q)$, and $\mathbf{r} = (r, <_r)$ are so-pairs over A. By Remark 4.14(b), after reversing the order $<_q$ if needed, we can assume $\delta_A(\mathbf{p}, \mathbf{q})$, and then, after reversing $<_r$ if needed, we can assume $\delta_A(\mathbf{q}, \mathbf{r})$. Let $a \models p$. Choose b_1, b_2, c_1, c_2 that satisfy $c_1 \lhd^{\mathbf{q}} b_1 \lhd^{\mathbf{p}} a$ and $a \lhd^{\mathbf{q}} b_2 \lhd^{\mathbf{r}} c_2$. We claim $\operatorname{tp}(c_1 a/A) \neq \operatorname{tp}(c_2 a/A)$. Otherwise, there would be $b' \models q$ such that $c_2 \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a$. Then $a \lhd^{\mathbf{q}} b_2 \lhd^{\mathbf{r}} c_2 \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a$. Since $\delta_A(\mathbf{p}, \mathbf{q})$ and $\delta_A(\mathbf{q}, \mathbf{r})$ hold, we can successively apply Lemma 4.16(i):

$$a \lhd^{\mathbf{q}} b_2 \lhd^{\mathbf{r}} c_2 \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a \quad \Rightarrow \quad a \lhd^{\mathbf{r}} c_2 \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a \quad \Rightarrow \quad a \lhd^{\mathbf{q}} b' \lhd^{\mathbf{p}} a \quad \Rightarrow \quad a \lhd^{\mathbf{p}} a; \text{ a contradiction.}$$

This proves the claim: $\operatorname{tp}(c_1a/A) \neq \operatorname{tp}(c_2a/A)$, so $p \not\perp^w r$ holds; $\not\perp^w$ is an equivalence relation.

(ii) To prove the transitivity of δ_A , assume that $\mathbf{p} = (p, <_p)$, $\mathbf{q} = (q, <_q)$ and $\mathbf{r} = (r, <_r)$ are so-pairs, $\delta_A(\mathbf{p}, \mathbf{q})$ and $\delta_A(\mathbf{q}, \mathbf{r})$. By part (i), $p \not\perp^w q$ and $q \not\perp^w r$ imply $p \not\perp^w r$. We will verify condition (1) from Lemma 4.15: Assuming $a_1 <_p a_2 \lhd^\mathbf{r} c$ we need to prove $a_1 \lhd^\mathbf{r} c$. By Lemma 4.8 there exists a $b \models q$ that satisfies $a_2 \lhd^\mathbf{q} b \lhd^\mathbf{r} c$. Then $a_1 <_p a_2 \lhd^\mathbf{q} b$, by condition (1) from Lemma 4.15, implies $a_1 \lhd^\mathbf{q} b$. By Lemma 4.16(i), $a_1 \lhd^\mathbf{q} b \lhd^\mathbf{r} c$ implies $a_1 \lhd^\mathbf{r} c$. Therefore, δ_A is an equivalence relation. Finally, each (non-algebraic) $\not\perp^w$ -class is split into two δ_A -classes by Remark 4.14(b).

(iii) By Lemma 4.9, $\not\perp$ is symmetric and the reflexivity is clear. To prove transitivity, assume $a \not\perp_A b$ and $a \downarrow_A c$, and we prove $b \downarrow_A c$. Let $p = \operatorname{tp}(a/A), q = \operatorname{tp}(b/A)$ and $r = \operatorname{tp}(c/A)$; these are so-types, and note that $a \not\perp_A b$ implies $p \not\perp^w q$. Thus, if $p \perp^w r$, then $q \perp^w r$ holds by (i); $b \downarrow_A c$ follows, and we are done. So suppose $p \not\perp^w r$; By (i), $q \not\perp^w r$ holds also. Choose orders $<_p, <_q$ and $<_r$ such that the corresponding so-pairs \mathbf{p}, \mathbf{q} and \mathbf{r} are in the same δ_A -class. Then $a \downarrow_A c$ implies that a is left- or right \mathbf{p} -generic over c; without loss (by reversing all three orders if necessary) assume $a \lhd^r c$. By Lemma 4.8 there exists a $b' \models q$ such that $a \lhd^q b' \lhd^r c$. By Remark 4.7(f), $a \lhd^q b'$ and $q \not\perp^f p$ (as $b \not\perp_A a$) imply $\mathcal{D}_q(a) <_q b'$, which together with $b \in \mathcal{D}_q(a)$ yields $b <_q b'$. Furthermore, $b' \lhd^r c$ by direct non-orthogonality implies $b' \in \mathcal{L}_q(c)$. Now, $b <_q b'$ and $b' \in \mathcal{L}_q(c)$ imply $b \in \mathcal{L}_q(c)$; in particular, $b \downarrow_A c$, as desired.

(iv) Follows easily from (iii).

Definition 4.18. Let \mathcal{F} be a δ_A -class of so-pairs over A. Define:

- $\mathfrak{F}(\mathfrak{C})$ is the set of all realizations of types from \mathfrak{F} ;
- $\mathcal{D}_{\mathcal{F}} = \{(a, b) \in \mathcal{F}(\mathfrak{C}) \times \mathcal{F}(\mathfrak{C}) \mid a \not \sqcup_{a} b\};$

• $b \in \mathcal{F}(\mathfrak{C})$ is right \mathcal{F} -generic over $a \in \mathcal{F}(\mathfrak{C})$, denoted by $a \triangleleft^{\mathcal{F}} b$, if $a \triangleleft^{\mathbf{q}} b$ holds for some (equivalently, any, by Remark 4.14(c)) pair $\mathbf{q} = (q, <_q) \in \mathcal{F}$ such that $b \in q(\mathfrak{C})$.

Remark 4.19. By Remark 4.7(d), $a, b \in \mathcal{F}(\mathfrak{C})$ are $\triangleleft^{\mathcal{F}}$ -comparable if and only if $a \bigsqcup_{A} b$.

Now, we prove Theorem 5.

Theorem 4.20. Let \mathcal{F} be a δ_A -class of non-algebraic so-pairs (or weakly o-minimal pairs) over A.

- (i) $(\mathfrak{F}(\mathfrak{C}), \triangleleft^{\mathfrak{F}})$ is a strict partial order.
- (ii) $\mathcal{D}_{\mathfrak{F}}$ and $\triangleleft^{\mathfrak{F}}$ -incomparability are the same equivalence relation on $\mathfrak{F}(\mathfrak{C})$.
- (iii) $(\mathfrak{F}(\mathfrak{C})/\mathfrak{D}_{\mathfrak{F}}, \triangleleft^{\mathfrak{F}})$ is a dense linear order.

Proof. (i) The antireflexivity is clear. To prove the transitivity, assume that $a \triangleleft^{\mathcal{F}} b \triangleleft^{\mathcal{F}} c$. Choose $\mathbf{p}, \mathbf{q} \in \mathcal{F}$ such that $a \triangleleft^{\mathbf{p}} b \triangleleft^{\mathbf{q}} c$. By Lemma 4.16(i), we have $a \triangleleft^{\mathbf{q}} c$, proving transitivity.

(ii) By Remark 4.19, $\mathcal{D}_{\mathcal{F}}$ and $\triangleleft^{\mathcal{F}}$ -incomparability agree on $\mathcal{F}(\mathfrak{C})$; $\mathcal{D}_{\mathcal{F}}$ is an equivalence relation by Theorem 4.17(iii).

(iii) First we claim: $a \triangleleft^{\mathcal{F}} b$ iff $[a]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} [b]_{\mathcal{D}_{\mathcal{F}}}$. We prove only the left-to-right implication; the other one is immediate. Assume $a \triangleleft^{\mathcal{F}} b$. Let $a' \in [a]_{\mathcal{D}_{\mathcal{F}}}$ be arbitrary. Then, by (ii), $a' \triangleleft^{\mathcal{F}} b$ or $b \triangleleft^{\mathcal{F}} a'$ holds. The latter is impossible: $a \triangleleft^{\mathcal{F}} b \triangleleft^{\mathcal{F}} a'$, by (i), implies $a \triangleleft^{\mathcal{F}} a'$, which contradicts $a' \in [a]_{\mathcal{D}_{\mathcal{F}}}$. Therefore, $a' \triangleleft^{\mathcal{F}} b$ is valid for all $a' \in [a]_{\mathcal{D}_{\mathcal{F}}}$; we conclude $[a]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} b$. Similarly, we obtain $[a]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} [b]_{\mathcal{D}_{\mathcal{F}}}$. This proves the claim.

From (i), (ii) and the claim, we immediately conclude that $(\mathcal{F}(\mathfrak{C})/\mathcal{D}_{\mathcal{F}}, \triangleleft^{\mathcal{F}})$ is a linear order. For density, assume $[a]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} [b]_{\mathcal{D}_{\mathcal{F}}}$. Then $a \triangleleft^{\mathcal{F}} b$, so by Lemma 4.8 there exists $c \equiv a$ (A) such that $a \triangleleft^{\mathcal{F}} c \triangleleft^{\mathcal{F}} b$; $[a]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} [c]_{\mathcal{D}_{\mathcal{F}}} \triangleleft^{\mathcal{F}} [b]_{\mathcal{D}_{\mathcal{F}}}$ follows by the claim. \Box

In the next proposition, we make a connection between direct non-orthogonality of weakly o-minimal pairs and direct non-orthogonality of their nonforking extensions.

Proposition 4.21. Let \mathcal{F} be a δ_A -class of weakly o-minimal pairs over A and let $B \supseteq A$. For each pair $\mathbf{p} = (p, <_p) \in \mathcal{F}$ set $p_B = \mathbf{p}_{r \upharpoonright B}$ and $\mathbf{p}_B = (p_B, <_p)$.

- (i) There exists a (unique) δ_B -class, \mathfrak{F}_B , which contains all the pairs $\{\mathbf{p}_B \mid \mathbf{p} \in \mathfrak{F}\}$. In particular, $p_B \not\perp^w q_B$ holds for all types from \mathfrak{F} .
- (ii) $\mathfrak{F}_B(\mathfrak{C})$ is a final part of $(\mathfrak{F}(\mathfrak{C}), \triangleleft^{\mathfrak{F}})$, that is, $B \triangleleft^{\mathfrak{F}} a \triangleleft^{\mathfrak{F}} b$ implies $B \triangleleft^{\mathfrak{F}} b$.
- (iii) The restriction of $\mathbb{D}_{\mathcal{F}_B}$ is a convex equivalence relation on $(\mathfrak{F}_B(\mathfrak{C}), \triangleleft^{\mathfrak{F}})$, that is $B \triangleleft^{\mathfrak{F}} a \triangleleft^{\mathfrak{F}} b \triangleleft^{\mathfrak{F}} c$ and $a \not \perp_B c$ imply $b \not \perp_B a$ and $b \not \perp_B c$.

Proof. (i) We need to prove that $\delta_B(\mathbf{p}_B, \mathbf{q}_B)$ is valid for all $\mathbf{p}, \mathbf{q} \in \mathcal{F}$. Fix $\mathbf{p}, \mathbf{q} \in \mathcal{F}$. First, we prove $p_B \not\perp^w q_B$. Choose $a, a' \models p$ and $b \models q$ that satisfy $B \triangleleft^\mathbf{p} a \triangleleft^\mathbf{q} b \triangleleft^\mathbf{p} a'$. Then, by Lemma 4.16(i), $a, a' \models p_B$ and $b \models q_B$. Also, $\delta_A(\mathbf{p}, \mathbf{q})$ and $a \triangleleft^\mathbf{q} b \triangleleft^\mathbf{p} a'$ imply that a is left \mathbf{p} -generic and a' is right \mathbf{p} -generic over b, so $a \not\equiv a'$ (Ab) holds, and, in particular, $ab \not\equiv a'b$ (B). Therefore, the types $\operatorname{tp}(ab/B)$ and $\operatorname{tp}(a'b/B)$ are distinct completions of $p_B(x) \cup q_B(y)$; $p_B \not\perp^w q_B$ follows. To prove $\delta_B(\mathbf{p}_B, \mathbf{q}_B)$, by Lemma 4.15 it suffices to show that $a \triangleleft^{\mathbf{q}_B} b \triangleleft^{\mathbf{p}_B} a$ is impossible for all $a \models p_B$ and $b \models q_B$. Otherwise, $a \triangleleft^{\mathbf{q}_B} b \triangleleft^{\mathbf{p}_B} a$ which contradicts $\delta_A(\mathbf{p}, \mathbf{q})$. This proves $\delta_B(\mathbf{p}_B, \mathbf{q}_B)$.

(ii) Follows from Lemma 4.16.

S. MOCONJA AND P. TANOVIĆ

5. DP-MINIMALITY

In this section, we prove that weakly o-minimal types are dp-minimal and that their indiscernible sequences enjoy some nice properties. For example, in Proposition 5.8 below, we prove that every Morley sequence I of realizations of a weakly o-minimal type p remains Morley after replacing every element $a \in I$ with an element $a' \in \mathcal{D}_p(a)$.

We will very briefly recall the notions of NIP and dp-minimality for (partial) types, without recalling the original definition of dp-minimal types, as we will not use it in this paper; a detailed exposition of NIP can be found in Simon's book [16]. Rather, we recall two characterizations of NIP types (see [7, Claim 2.1]), since we will use them later. We also recall an equivalent characterization of dp-minimality due to Kaplan, Onshuus, and Usvyatsov (see [6, Proposition 2.8]).

Definition 5.1. Let p(x) be a (partial) type over A. The type p(x) is:

- (a) *NIP* if there does *not* exist a formula $\varphi(x, y)$, an *A*-indiscernible sequence $(b_n \mid n < \omega)$ of tuples of length |y| and $a \models p$ such that $\models \varphi(a, b_n)$ iff *n* is even. Equivalently, there is no formula $\varphi(x, y)$, an *A*-indiscernible sequence $(a_n \mid n < \omega)$ in *p* and a tuple *b*, such that $\models \varphi(a_n, b)$ iff *n* is even;
- (b) dp-minimal if there does not exist a formula $\varphi(x, y)$, an A-indiscernible sequence I of tuples of length |y| and $a \models p$ such that $\varphi(a, y)$ has at least four alternations on I.

Fact 5.2. (i) Every dp-minimal type is NIP.

- (ii) Every weakly o-minimal type is NIP.
- (iii) If p(x) over A is NIP and $a_i \models p$ for i < n, then $tp(a_0, \ldots, a_{n-1}/A)$ is NIP.

Proof. (i) is obvious by the first characterization from the definition of NIP types. (ii) If (p, <) is a weakly o-minimal pair over A, then every A-indiscernible sequence of realizations of p is monotone with respect to <, so weak o-minimality implies that the second characterization of NIP types above is satisfied. (iii) follows from [6, Theorem 4.11].

We prove that a weakly o-minimal type is dp-minimal. In the proof, we will use the following simple, Helly-style fact.

Fact 5.3. Let (X, <) be a linear order and $\{S_n \mid n < \omega\}$ a family of non-empty subsets of X such that there is $N < \omega$ such that each S_n has at most N convex components. Suppose that $\{S_n \mid n < \omega\}$ has the 2-intersection property. Then there is an infinite $I \subseteq \omega$ such that $\{S_n \mid n \in I\}$ has the finite intersection property.

Proof. Without loss we may assume that each S_n has N convex components, and write $S_n = C_n^1 \cup \cdots \cup C_n^N$ with $C_n^1 < \cdots < C_n^N$ being convex. For n < m choose minimal i and minimal j such that $C_n^i \cap C_m^j \neq \emptyset$. By Ramsey's theorem, there is an infinite $I \subseteq \omega$ such that the chosen pairs (i, j) are the same for all n < m in I. If i = j then $\{C_n^i \mid n \in I\}$ is a 2-consistent family of convex sets, so it is k-consistent for all $k \ge 2$ by Helly's theorem, and hence $\{S_n \mid n \in I\}$ is k-consistent for all $k \ge 2$, too.

Consider the case i < j. If n < m in I then $C_m^i < C_n^i$ as $C_m^i < C_m^j$ and j is minimal such that $C_n^i \cap C_m^j \neq \emptyset$. Write $I = \{n_0, n_1, n_2, \ldots\}$ in increasing order. Then $C_{n_2}^i < C_{n_1}^i < C_{n_0}^i$. For each $l \ge 3$, the set $C_{n_l}^j$ meets both $C_{n_2}^i$ and $C_{n_0}^i$, so by convexity it completely contains $C_{n_1}^i$. Therefore, $\{S_{n_l} \mid l \ge 3\}$ contains $C_{n_1}^i$ at its intersection, so it is k-consistent for all $k \ge 2$. The proof in the case i > j is similar.

Proposition 5.4. Suppose that p(x) is a partial type over A whose completions over A are all weakly *o-minimal*. Then p(x) is dp-minimal.

Proof. Suppose not. Let $\varphi(x, y)$ be a formula, $I = (b_n \mid n < \omega)$ a A-indiscernible sequence, and c realizing p such that $\varphi(c, y)$ has at least four alternations on I. Let $i_0 < i_1 < i_2 < i_3$ be such that:

$$(*) \qquad \qquad \models \varphi(c, b_{i_0}) \land \neg \varphi(c, b_{i_1}) \land \varphi(c, b_{i_2}) \land \neg \varphi(c, b_{i_3}).$$

Put $q = \operatorname{tp}(c/A)$. By assumption, q is a weakly o-minimal type, so $(q, <_q)$ is a weakly o-minimal pair over A for some (any) relatively A-definable order $<_q$ on $q(\mathfrak{C})$. Consider the relatively definable subsets S_n of $q(\mathfrak{C})$ relatively defined by $\varphi(x, b_{2n}) \wedge \neg \varphi(x, b_{2n+1})$. By weak o-minimality of q and indiscernibility of I, for some $N < \omega$ each S_n has $N <_q$ -convex components. Furthermore, by the indiscernibility of I and (*) we see that $\{S_n \mid n < \omega\}$ is 2-consistent. Then by Fact 5.3 there is an infinite $J \subseteq \omega$ such that $\{S_n \mid n \in J\}$ has the finite intersection property; in fact, by the indiscernibility of I, we may conclude that $\{S_n \mid n < \omega\}$ has the finite intersection property. By saturation we find $a \in \bigcap_{n < \omega} S_n$, but this contradicts the fact that p is NIP (Fact 5.2(ii)).

Corollary 5.5. Every weakly o-minimal type is dp-minimal, and every weakly quasi-o-minimal theory is dp-minimal.

In the rest of this section, we focus on indiscernible sequences of realizations of a weakly o-minimal type. In part (ii) of the following lemma, we show that they have a bit stronger property than the distality as defined by Simon in [17] (see [17, Lemma 2.7, Corollary 2.9] and [4, Definition 4.21]).

Lemma 5.6. Let (p, <) be a weakly o-minimal pair over A, let $a_0, a_1 \models p$ and $B \supseteq A$. Suppose that I and J are sequences of realizations of p such that I + J is infinite and <-increasing. Then:

- (i) If I has no maximum, $I < a_0 < a_1$ and $I + a_1$ is B-indiscernible, then $I + a_0$ is B-indiscernible.
- (ii) If I + J is B-indiscernible, I has no maximum, J has no minimum, and $I < a_0 < J$, then $I + a_0 + J$ is B-indiscernible.
- (iii) If $I + a_0 + a_1 + J$ is B-indiscernible, $a \models p$ and $a_0 < a < a_1$, then at least one of the sequences $I + a_0 + a + J$ and $I + a + a_1 + J$ is B-indiscernible.

Proof. (i) Let I_0 be a finite subset of I. Since I has no maximum, there exists $a \in I$ such that $I_0 < a$. Then $a < a_0 < a_1$ and the sequence $I_0 + a + a_1$ is B-indiscernible, so $a \equiv a_1$ (BI_0) holds. Since p is weakly o-minimal, the locus of type $tp(a_1/BI_0)$ is a convex subset of $(p(\mathfrak{C}), <)$ by Lemma 2.6(ii), so $a < a_0 < a_1$ implies $a_0 \equiv a_1$ (BI_0). In particular, the sequence $I_0 + a_0$ is B-indiscernible. Since this holds for all finite $I_0 \subseteq I$, $I + a_0$ is B-indiscernible.

(ii) Let $I_0 \subseteq I$ and $J_0 \subseteq J$ be finite. Choose $a_i \in I$ and $b_j \in J$ that satisfy $I_0 < a_i$ and $b_j < J_0$. Then the sequence $I_0 + a_i + b_j + J_0$ is *B*-indiscernible, and $a_i < a_0 < b_j$. By Lemma 2.6(ii) the locus of $\operatorname{tp}(a_i/BI_0J_0) = \operatorname{tp}(b_j/BI_0J_0)$ is convex in $(p(\mathfrak{C}), <)$, so $a_0 \equiv a_i$ (BI_0J_0) , therefore the sequence $I_0 + a_0 + J_0$ is *B*-indiscernible. Since this holds for all finite $I_0 \subseteq I$ and $J_0 \subseteq J$, the sequence $I + a_0 + J$ is *B*-indiscernible.

(iii) Choose $a_{\frac{1}{2}} \models p$ such that $I + a_0 + a_{\frac{1}{2}} + a_1 + J$ is *B*-indiscernible; this is possible by compactness as I + J is infinite. Then $a_0 < a_{\frac{1}{2}} < a_1$, so *a* is in one of the intervals $(a_0, a_{\frac{1}{2}}]$ and $[a_{\frac{1}{2}}, a_1)$. First, assume $a \in (a_0, a_{\frac{1}{2}}]$. Notice that a_0 and $a_{\frac{1}{2}}$ realize the same type, say *q*, over a_1BIJ . By Lemma 2.6(ii), $q(\mathfrak{C})$ is a convex subset of $p(\mathfrak{C})$, so $a_0 < a \leq a_{\frac{1}{2}}$ implies $a \models q$. In particular, $a \equiv a_{\frac{1}{2}}(a_1BIJ)$ implies that $I + a + a_1 + J$ is *B*-indiscernible. Similarly, $a \in [a_{\frac{1}{2}}, a_1)$ implies the indiscernibility of $I + a_0 + a + J$.

Corollary 5.7. Let (p, <) be a weakly o-minimal pair over A and $B \supseteq A$. Suppose that I and J are increasing sequences of realizations of p such that I + J is infinite and B-indiscernible. Let $a \models p$ satisfy I < a < J. Then by removing at most one element except a from the sequence I + a + J the sequence remains B-indiscernible. More precisely, at least one of the following conditions holds:

(1) I + a + J is *B*-indiscernible;

(2) $a_0 = \max I$ exists and the sequence $(I - a_0) + a + J$ is B-indiscernible;

(3) $a_1 = \min J$ exists and the sequence $I + a + (J - a_1)$ is B-indiscernible.

Proof. If I has no maximum and J has no minimum, then I + a + J is indiscernible by Lemma 5.6(ii). If both $a_0 = \max I$ and $a_1 = \min J$ exist, then the sequence $(I - a_0) + a_0 + a_1 + (J - a_1)$ is B-indiscernible and $a_0 < a < a_1$, so by Lemma 5.6(iii) at least one of conditions (2) and (3) holds. If I has no maximum and $a_1 = \min J$ exists, then $I < a < a_1$ and $I + a_1$ is $B(J - a_1)$ -indiscernible, so condition (3) holds by Lemma 5.6(i); similarly, if $a_0 = \max I$ and J has no minimum, then condition (2) holds. **Proposition 5.8.** Let $\mathbf{p} = (p, <)$ be a weakly-o-minimal pair over A. Suppose that $I = (a_j \mid j \in J)$ is a (possibly finite) Morley sequence in \mathbf{p}_r over A.

- (i) Suppose that $a \models p$ is in the <-convex hull of I. Then we can remove at most one element from I and insert a so that the sequence remains Morley over A.
- (ii) If $I' = (a'_j | j \in J)$ is a sequence of realizations of p such that $a'_j \not \perp_A a_j$ for all $j \in J$, then I' is a Morley sequence in \mathbf{p}_r over A.

Proof. (i) By extending I if necessary, we may assume that I is infinite; also, we may assume $a \notin I$. Let $I^- = \{x \in I \mid x < a\}$ and $I^+ = \{x \in I \mid a < x\}$. Then $\{I^-, I^+\}$ is a partition of I and $I^- < a < I^+$ also holds; the assumptions of Corollary 5.7 are satisfied, so by removing at most one element from $I^- + a + I^+$ except a we get an A-indiscernible sequence. The obtained sequence is Morley over A as it is A-indiscernible and contains an infinite subsequence which is Morley over A.

(ii) It suffices to prove the statement for a finite I. So, assume that $I = (a_0, \ldots, a_n)$ is a Morley sequence in \mathbf{p}_r over A. Extend I to a longer Morley sequence $I_0 = (a_{-1}, a_0, \ldots, a_n, a_{n+1})$; we will show that the sequence $I'_0 = (a_{-1}, a'_0, \ldots, a'_n, a_{n+1})$ has the same type over A as I_0 . Note that $a_{k-1} < \mathcal{D}_p(a_k) < a_{k+1}$ holds for all $k = 0, 1, \ldots, n$, so $a_k \not \perp_A a'_k$ implies $a_{k-1} < a'_k < a_{k+1}$. By part (i), we can replace some element of I_0 , say a_i , by a'_k so that the sequence remains Morley. In particular, $a_{i-1} < a'_k < a_{i+1}$ is satisfied, so i = k. Therefore, replacing a_k by a'_k in I_0 does not change the type $\operatorname{tp}(I_0/A)$.

References

- Belegradek, Oleg, Ya'acov Peterzil, and Frank Wagner. "Quasi-o-minimal structures." The Journal of Symbolic Logic 65.3 (2000): 1115-1132.
- [2] Oleg V. Belegradek, A. P. Stolboushkin, and M.A.Taitslin. Generic queries over quasi-a-minimal domains, Proceedings of the Fourth International Symposium LFCS (S. Adian and A. Nerode, eds.). Lecture Notes in Computer Science, vol. 1234, Springer-Verlag, (1997): pp. 21-32.
- [3] Dickmann, Max A. "Elimination of quantifiers for ordered valuation rings." The Journal of Symbolic Logic 52.1 (1987): 116-128.
- [4] Estevan, Pedro Andrés, and Itay Kaplan. "Non-forking and preservation of NIP and dp-rank." Annals of Pure and Applied Logic 172.6 (2021): 102946.
- [5] Goodrick, John. "Definable sets in dp-minimal ordered abelian groups" (Model theoretic aspects of the notion of independence and dimension). (2022) http://hdl.handle.net/2433/277116
- [6] Kaplan, Itay, Alf Onshuus, and Alexander Usvyatsov. "Additivity of the dp-rank." Transactions of the American Mathematical Society 365.11 (2013): 5783-5804.
- [7] Kaplan, Itay, and Pierre Simon. "Witnessing dp-rank." Notre Dame Journal of Formal Logic 55.3 (2014): 419-429.
- [8] Knight, Julia F., Anand Pillay, and Charles Steinhorn. "Definable sets in ordered structures. II." Transactions of the American Mathematical Society 295.2 (1986): 593-605.
- [9] Kudaĭbergenov, K. Zh. "Weakly quasi-o-minimal models." Siberian Advances in Mathematics 20 (2010): 285-292.
- [10] Macpherson, Dugald, David Marker, and Charles Steinhorn. "Weakly o-minimal structures and real closed fields." Transactions of the American Mathematical Society 352.12 (2000): 5435-5483.
- [11] Moconja, Slavko, and Predrag Tanović. "Stationarily ordered types and the number of countable models." Annals of Pure and Applied Logic 171.3 (2020): 102765.
- [12] Moconja, Slavko, and Predrag Tanović. "Does weak quasi-o-minimality behave better than weak o-minimality?." Archive for Mathematical Logic 61.1 (2022): 81-103.
- [13] Pillay, Anand, and Charles Steinhorn. "Definable sets in ordered structures." Bulletin of the American Mathematical Society 11 (1984): 159-162.
- [14] Pillay, Anand, and Charles Steinhorn. "Definable sets in ordered structures. I." Transactions of the American Mathematical Society 295.2 (1986): 565-592.
- [15] Pillay, Anand, and Charles Steinhorn. "Definable Sets in Ordered Structures. III." Transactions of the American Mathematical Society, 309.2 (1988): 469–476.
- [16] Simon, Pierre. A guide to NIP theories. Cambridge University Press, 2015.
- [17] Simon, Pierre. "Distal and non-distal NIP theories." Annals of Pure and Applied Logic 164.3 (2013): 294-318.

(S. Moconja) UNIVERSITY OF BELGRADE, FACULTY OF MATHEMATICS, BELGRADE, SERBIA *Email address:* slavko@matf.bg.ac.rs

(P. Tanović) MATHEMATICAL INSTITUTE OF THE SERBIAN ACADEMY OF SCIENCES AND ARTS, BELGRADE, SERBIA Email address: tane@mi.sanu.ac.rs