Composing Codensity Bisimulations

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Abstract

Proving compositionality of behavioral equivalence on state-based systems with respect to algebraic operations is a classical and widely studied problem. We study a categorical formulation of this problem, where operations on state-based systems modeled as coalgebras can be elegantly captured through distributive laws between functors. To prove compositionality, it then suffices to show that this distributive law lifts from sets to relations, giving an explanation of how behavioral equivalence on smaller systems can be combined to obtain behavioral equivalence on the composed system.

In this paper, we refine this approach by focusing on so-called codensity lifting of functors, which gives a very generic presentation of various notions of (bi)similarity as well as quantitative notions such as behavioral metrics on probabilistic systems. The key idea is to use codensity liftings both at the level of algebras and coalgebras, using a new generalization of the codensity lifting. The problem of lifting distributive laws then reduces to the abstract problem of constructing distributive laws between codensity liftings, for which we propose a simplified sufficient condition. Our sufficient condition instantiates to concrete proof methods for compositionality of algebraic operations on various types of state-based systems. We instantiate our results to prove compositionality of qualitative and quantitative properties of deterministic automata. We also explore the limits of our approach by including an example of probabilistic systems, where it is unclear whether the sufficient condition holds, and instead we use our setting to give a direct proof of compositionality.

In addition, we propose a composition of Komorida et al.'s codensity games for bisimilarities. A novel feature of this composition is that it can also compose game invariants, which are subsets of winning positions. Under our sufficient condition of the liftability of distributive laws, composed games give an alternative proof of the preservation of bisimilarities under the composition.

1 Introduction

Bisimilarity and its many variants are fundamental notions of behavioral equivalence on state-based systems. A classical question is whether a given notion of equivalence is *compositional* w.r.t. algebraic operations on these systems, such as parallel composition of concurrent systems, product constructions on automata, or other language constructs. The problem can become particularly challenging when moving from bisimilarity to quantitative notions such as behavioral metrics on probabilistic systems, where it is already non-trivial to even define what the right notion of compositionality is (see, e.g. [26, 13]).

To formulate and address the problem of compositionality of algebraic operations on state-based systems at a high level of generality, a natural formalism is that of *coalgebra* [19], which is parametric in the type of system, as modeled by a "behavior" endofunctor F. A key idea, originating in the seminal work of Turi and Plotkin [31], is that composition operations and in fact whole languages specifying state-based systems arise as *distributive laws* between F and a functor (or monad) T which models the syntax of a bigger programming language. In fact, Turi and Plotkin's *abstract GSOS laws* are forms of such distributive laws which guarantee compositionality of strong bisimilarity, generalizing the analogous result for specifications in the *GSOS format* for labeled transition systems [4]. Distributive laws here precisely capture algebra-coalgebra interaction needed for compositionality.

The results of Turi and Plotkin specifically apply to strong bisimilarity, and not directly to other notions such as similarity or behavioral metrics. To prove compositionality in these cases, the key observation is that the task is to define the composition operation at hand not only on state-based systems but also on relations (or, e.g., pseudometrics). This idea has been formalized in the theory of coalgebras by requiring a *lifting* of the distributive law that models the composition operator to a category of relations. The lifting of the behavior functor F then specifies the notion of bisimilarity at hand, and the lifting of the syntax functor T the way that relations on components can be combined into bisimulations on the composite system. It has been shown in [6] at a high level of generality that the existence of such liftings, referred to in this paper as *liftability*, ensures compositionality in this way. The generality there is offered by the use of liftings of the behavior functor in fibrations which goes back to [17], and which allows us to study not only relations but also, for instance, coinductively defined metrics or unary predicates (e.g., [23, 29, 16, 5]). But the main challenge here is to *prove liftability*; this is what we address in the current paper (Question 3.11.2).

We focus on the *codensity lifting* of behavior functors [29], which allows to model a wide variety of coinductive relations and predicates including (bi)similarity but also, for instance, behavioral metrics; these are commonly referred to as *codensity bisimilarity*. In fact, the codensity lifting directly generalizes the celebrated Kantorovich distance between probability distributions used to define, for instance, metrics between probabilistic systems [8]. The codensity lifting is parametric in a collection of modal operators, which makes it on the one hand very flexible and on the other hand much more structured than arbitrary liftings. Moreover, the codensity lifting allows to characterize *codensity games* [23], a generalization of classical bisimilarity games [30, 10] from transition systems to coalgebras, and from strong bisimilarity to a wide variety of coinductively defined relations, metrics and even topologies.

In this paper we study *compositionality of codensity bisimulations* with respect to composition operators on the underlying state-based systems, modeled as coalgebras. We model these composition operators as distributive laws between an n-ary product functor and the behavior functor F, referred to as *one-step composition operators.* We mainly tackle two problems (Question 3.11) in this paper: 1) How do we lift the n-ary product functor in order to capture non-trivial combination of relations, pseudometrics etc? 2) When is a one-step composition operator liftable?

A key idea for the first problem is to use codensity liftings not only to lift the behavior functor F and get our desired notion of equivalence, but to use another codensity lifting of the product functor, to explain syntactically how relations (or pseudometrics, etc) should be combined from components to the composite system. This combination can be a simple product between relations, but in many cases, such as for behavioral metrics, it needs to be more sophisticated; the flexibility offered by the codensity lifting helps to define the appropriate constructions on relations.

The second problem about liftability then becomes that of proving the existence of a *distributive law between codensity liftings.* To this end, we exhibit a sufficient condition that ensure the existence of this distributive laws, defined in terms of two properties: 1) commutation between the underlying modalities that define codensity liftings, and 2) an *approximation* property, which resembles the one used to prove expressiveness of modal logics in [24]. Underlying our approach is a combination of a new generalization of the codensity lifting beyond endofunctors, to allow to lift product functors, and the adjunction-based decomposition of codensity liftings proposed in [3]. The adjunction-based approach precisely allows us to arrive at our sufficient condition.

We instantiate our approach to pseudometrics on deterministic automata. We also revisit the compositionality of parallel composition w.r.t. behavioral metrics studied in [13]. In this case it is not clear whether our sufficient condition holds; but the framework nevertheless helps to prove liftability and thereby compositionality.

We further study a composition of codensity games. A key observation is that our *composite codensity games* consist of positions that are tuples of positions of codensity games for component systems. This design of the composite game directly leads to the compositionality of game invariants, which characterize winning positions. Assuming our sufficient condition of the liftability of distributive laws, we present an alternative proof of the preservation of bisimilarities along compositions (the inequality in Corollary 3.10).

In summary, the contributions of this paper are

- a generalization of the codensity lifting beyond endofunctors (Section 4), which can be used in a special case to lift products in various ways (Section 5);

- a sufficient condition for proving the existence of a distributive law between codensity liftings (Section 6), with several detailed examples (Section 7);

- a composition of codensity games, which also composes game invariants, and an alternative proof of the preservation of codensity bisimilarities under our sufficient condition (Section 8);

After the preliminaries (Section 2), we start the paper with a more detailed overview of our approach (Section 3).

$\triangleright$ *Related work* There is a wide range of results in the process algebra on compositionality and rule formats to prove it, which are usually focused on specific models (see, e.g., [27] for an overview). A full account is beyond the scope of this paper, which focuses instead on generality in the type of models and the type of coinductive predicates. Concerning general frameworks for compositionality in the theory of coalgebras, we have already mentioned Turi and Plotkin's abstract GSOS format; the main innovation in the current paper is that we go beyond bisimilarity by employing the codensity lifting.

In [6], it is shown that liftings of distributive laws to fibrations yield so-called *compatibility*, a property that ensures soundness of up-to techniques, and which implies compositionality. On the one hand, if one uses the so-called canonical relation lifting then all distributive laws lift, but this only concerns strong bisimilarity; on the other hand, in [6] examples beyond bisimilarity are studied but liftability there is proven on an ad-hoc basis. In the current paper we identify the codensity lifting as a sweet spot between the (restricted) canonical relation lifting and abstract, unrestricted lifting of functors, and focus instead on conditions that allow to prove liftability.

Many bisimilarity notions are known to be characterized by winning positions of certain safety games, including bisimilarity on Kripke frames [30], probabilistic bisimulation [11, 10], and bisimulation metric [25]. There are several coalgebraic frameworks [25, 12] that captures such a relationship between games and bisimilarities. In this paper, we focus on codensity games [23] that are naturally obtained by codensity liftings and coalgebras.

Codensity lifting is first introduced to lift monads across fibrations [21]. Its instances include the Kantorivich metric on Giry monad. Later, in [28], it is extended to lift endofunctors across fibrations, and is shown to be a generalization of Baldan et al.'s Kantorovich lifting [1].

2 Preliminaries

For a mathematical entity x equipped with the notion of product $(\times)$, by x^N we mean the $(\times)$-product of N-many copies of x. For instance, when $\mathbb{C}$ is a category, $\mathbb{C}^N$ denotes the product category of N-many copies of $\mathbb{C}$. By abuse of notation, we use the same letter N for the set $\{1, \dots, N\}$ and write $i \in N$ for $i \in \{1, \dots, N\}$.

For two natural transformations $\alpha : F \to G, \beta : G \to H$, their vertical composition is denoted by $\beta \bullet \alpha$, which is the componentwise composition of β and α. For a functor $p : \mathbb{E} \to \mathbb{B}$ and an object $X \in \mathbb{B}$, the *fiber category* over

X, denoted by $\mathbb{E}_X$, is the category whose objects are $P \in \mathbb{E}$ such that $pP = X$, and whose morphisms are f such that $pf = id_X$.

2.1 $\mathbf{CLat}_\sqcap$-fibration

A $\mathbf{CLat}_\sqcap$-*fibration* is a posetal fibration $p : \mathbb{E} \to \mathbb{B}$ such that each fiber $\mathbb{E}_X$ is a complete lattice and each pullback $f^* : \mathbb{E}_Y \to \mathbb{E}_X$ preserves all meets. The order relation of a fiber $\mathbb{E}_X$ is denoted by $\sqsubseteq$, the meet and the join by $\bigsqcap, \bigsqcup$, and the empty meet and join by $\top, \bot$. We remark that $\mathbf{CLat}_\sqcap$-fibrations are a special case of *topological functors* [18], where each fiber is a small partial order.

Examples of $\mathbf{CLat}_\sqcap$-fibrations are forgetful functors from the following categories into **Set**: 1) **Pre**, the category of preorders and monotone functions, 2) **EqRel**, the category of equivalence relations and relation-respecting functions, 3) **Top**, the category of topological spaces and continuous functions, 4) **PMet**, the category of pseudometric spaces and non-expansive functions, 5) **ERel**, the category of sets with endorelations, and relation-respecting functions. The forgetful functors from these categories are denoted by $U_{\mathbf{Pre}}$ etc.

A $\mathbf{CLat}_\sqcap$-fibration $p : \mathbb{E} \to \mathbb{B}$ is faithful, i.e., for any $\mathbb{E}$-object P, Q and $\mathbb{B}$-morphism $f : pP \to pQ$, there is at most one $\mathbb{E}$-morphism $\dot{f}$ such that $p\dot{f} = f$. When such an $\mathbb{E}$-morphism exists, we write $f : P \dot{\to} Q$. For instance, when p is the forgetful functor from **Top**, the notation $f : (X, O_X) \dot{\to} (Y, O_Y)$ for a function $f : X \to Y$ is equivalent to the statement: "f is continuous with respect to topologies O_X, O_Y". From the fiberdness, $f : P \dot{\to} Q$ is equivalent to the inequality $P \sqsubseteq f^*Q$.

2.2 Liftings

We extensively use the concept of *lifting* of various categorical structures in this paper. Let $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{C}$ be functors. A *lifting* of a functor $F : \mathbb{B} \to \mathbb{C}$ along p, q (or simply p when $p = q$) is a functor $\dot{F} : \mathbb{E} \to \mathbb{F}$ such that $q \circ \dot{F} = F \circ p$. Similarly, for liftings $\dot{F}, \dot{G}$ of $F, G : \mathbb{B} \to \mathbb{C}$ along p, q and a natural transformation $\alpha : F \Rightarrow G$, a *lifting* of α with respect to $\dot{F}, \dot{G}$ is a natural transformation $\dot{\alpha} : \dot{F} \Rightarrow \dot{G}$ such that $q \circ \dot{\alpha} = \alpha \circ p$.

To extend the concept of lifting to other categorical structures, it is convenient to set-up the *arrow 2-category* $\mathbf{CAT}^{\to}$ given by the following data: a 0-cell is a functor $p : \mathbb{E} \to \mathbb{B}$, a 1-cell from $p : \mathbb{E} \to \mathbb{B}$ to $q : \mathbb{F} \to \mathbb{C}$ is a pair $(F : \mathbb{B} \to \mathbb{C}, \dot{F} : \mathbb{E} \to \mathbb{F})$ of functors such that $\dot{F}$ is a lifting of F along p, q, and a 2-cell from $(F, \dot{F}) : p \to q$ to $(G, \dot{G})$ is a pair of natural transformations $(\alpha, \dot{\alpha})$ such that $\dot{\alpha}$ is a lifting of α with respect to $\dot{F}, \dot{G}$. This 2-category $\mathbf{CAT}^{\to}$ has 2-products: the terminal 0-cell is id_1, and the binary product of p_1, p_2 is the product functor $p_1 \times p_2$ (we only use finite ones in this paper). The evident forgetful 2-functor is denoted by $\mathrm{cod} : \mathbf{CAT}^{\to} \to \mathbf{CAT}$. Any categorical structure expressible within this 2-category corresponds to a pair of the categorical structure and its lifting along functors (and vice versa). For instance, a monad on $p : \mathbb{E} \to \mathbb{B}$ in $\mathbf{CAT}^{\to}$ is a pair of a monad on $\mathbb{B}$ together with its lifting along p (as a monad).

For the theory of coalgebraic bisimulation in Section 3.3, we often focus on the full-sub 2-category $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ of $\mathbf{CAT}^{\rightarrow}$ obtained by restricting 0-cells to $\mathbf{CLat}_\sqcap$-fibrations. We say that a 1-cell $(F, \dot{F}) : p \to q$ in $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ is *fibered* if $\dot{F}$ preserves Cartesian morphisms, or equivalently $(Ff)^*(\dot{F}P) = \dot{F}(f^*P)$ holds for any f, P.

The action of this 2-functor on hom-categories is denoted by the functor $\mathrm{cod}_{p,q} : \mathbf{CAT}^{\rightarrow}(p, q) \to \mathbf{CAT}(\mathrm{cod}(p), \mathrm{cod}(q))$. This is a $\mathbf{CLat}_\sqcap$-fibration, hence faithful. Thus, for a natural transformation $\alpha : F \Rightarrow G$ and liftings $\dot{F}, \dot{G}$ of F, G respectively, there is at most one lifting $\dot{\alpha} : \dot{F} \Rightarrow \dot{G}$ of α. When it exists, we say that α is *liftable* with respect to $\dot{F}, \dot{G}$.

$$
\begin{aligned}
&\mathbf{CAT}^{\mathbf{CLat}_\sqcap}(p, q)((F, \dot{F}), (G, \dot{G})) \\
&\cong \{\alpha : F \Rightarrow G \mid \forall P \in \mathrm{dom}(p) \ . \ \alpha_{pP} : \dot{F}P \dot{\to} \dot{G}P\}.
\end{aligned}
$$

If $\alpha_{pP} : \dot{F}P \dot{\to} \dot{G}P$ holds for P in a subclass C of objects in $\mathbb{E}$, we say that α is liftable *on* C.

3 Overview

The theory of *coalgebras* provides a categorical framework for expressing various transition systems in a unified manner. Let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$ be an endofunctor (called *behavior functor*). An F*-coalgebra* is a pair of an object X (called *carrier*) and a morphism $c : X \to F(X)$. A morphism from $c : X \to F(X)$ to $d : Y \to F(Y)$ is a morphism $h : X \to Y$ such that $Fh \circ c = d \circ h$.

Example 3.1. We write $\mathcal{P}$ for the covariant powerset functor on **Set**. A $\mathcal{P}$-coalgebra $c : X \to \mathcal{P}(X)$ bijectively corresponds to a binary relation on X, that is, a *Kripke frame*.

Example 3.2. Fix an alphabet Σ. We define the endofunctor F_{da} on **Set** by $F_{\mathrm{da}} \triangleq 2 \times (-)^\Sigma$. An F_{da}-coalgebra $c : X \to F_{\mathrm{da}}(X)$ bijectively corresponds to a deterministic automata over alphabet Σ.

The category of F-coalgebras and morphisms between them is denoted by $\mathbf{Coalg}(F)$, and the evident forgetful functor to $\mathbb{C}$ is denoted by U_F. We note that $(\mathbf{Coalg}(F))^N \cong \mathbf{Coalg}(F^N)$. When $(F, \dot{F}) : p \to p$ is a 1-cell in $\mathbf{CAT}^{\rightarrow}$, the application of p to $\dot{F}$-coalgebras extends to a functor $\mathbf{Coalg}(p) : \mathbf{Coalg}(\dot{F}) \to \mathbf{Coalg}(F)$.

3.1 Coalgebraic Bisimulation

One of the important concepts in state transition systems is the identification of two states that behave in the same way. A classical notion of behavioral equivalence is *bisimilarity*, which has been extensively studied in process algebra, coalgebra theory and modal logic. Recently, the concept of bisimulation, initially formulated as a binary relation, is expressed using other spatial structures, such as *pseudometrics* [2] and *topologies* [28]. A uniform treatment of these spatial representations of bisimulation have been developed using the framework of $\mathbf{CLat}_\sqcap$-fibrations [28]. We employ the formulation of bisimulation as *coalgebras* in suitable fibrations. This formulation goes back to the seminal work by Hermida and Jacobs [17], and the key ingredient is the *lifting* of behavior functors along fibrations. We recall their theory here.

Let $p : \mathbb{E} \to \mathbb{B}$ be a $\mathbf{CLat}_\sqcap$-fibration and $c : X \to F(X)$ be an F-coalgebra. Hermida and Jacobs' theory first chooses a *lifting* $\dot{F}$ of F along p; the main body of the definition of bisimulation is pack into it. Then we formulate a $\dot{F}$-*bisimulation* on a coalgebra $c : X \to F(X)$ to be a $\dot{F}$-coalgebra $\dot{c} : P \to \dot{F}(P)$ such that $p(\dot{c}) = c$ (hence $p(P) = X$). The morphism $\dot{c}$ witnesses that P is respected by the underlying transition system $p(c)$. Since there exists at most one such arrow $\dot{c}$ for a given P, a $\dot{F}$-bisimulation bijectively corresponds to $P \in \mathbb{E}_X$ satisfying $P \sqsubseteq c^* \circ \dot{F}(P)$. We thus define the set of bisimulations on c by

$$\mathbf{Bisim}(\dot{F}, c) \triangleq \{P \in \mathbb{E}_X \mid P \sqsubseteq c^* \circ \dot{F}(P)\},$$

and impose a partial order on it by restricting the one on $\mathbb{E}_X$. Since $\mathbb{E}_X$ is a complete lattice, it contains the *greatest* postfixpoint $\nu(c^* \circ \dot{F})$ corresponding to the *bisimilarity.*

Example 3.3. We express the definition of bisimulation for $\mathcal{P}$-coalgebras. We take the $\mathbf{CLat}_\sqcap$-fibration $U_{\mathbf{EqRel}}$ and take the following lifting $\dot{\mathcal{P}}$ of $\mathcal{P}$ along $U_{\mathbf{EqRel}}$:

$$\begin{aligned}\dot{\mathcal{P}}(X, P) \triangleq (\mathcal{P}(X), \{(U, V) \mid\ & \forall x \in U \ .\ \exists y \in V \ .\ (x, y) \in P \wedge \\ & \forall x \in V \ .\ \exists y \in U \ .\ (x, y) \in P\}).\end{aligned}$$

Then P is a $\dot{\mathcal{P}}$-bisimulation on $c : X \to \mathcal{P}(X)$ if and only if it satisfies the standard bisimulation condition:

$$\begin{aligned}\forall (x, y) \in P \ .\ & \forall x' \in c(x) \ .\ \exists y' \in c(y) \ .\ (x', y') \in P \wedge \\ & \forall x' \in c(y) \ .\ \exists y' \in c(x) \ .\ (x', y') \in P.\end{aligned}$$

Example 3.4. We represent language-equivalent states in deterministic automata by the $\mathbf{CLat}_\sqcap$-fibration $U_{\mathbf{EqRel}}$ and the following lifting $\dot{F}_{\mathrm{da}}$ of F_{da} along p:

$$\dot{F}_{\mathrm{da}}(P) \triangleq \{((t_1, \rho_1), (t_2, \rho_2)) \mid t_1 = t_2 \wedge \forall a \in \Sigma \ .\ (\rho_1(a), \rho_2(a)) \in P\}.$$

Let P be a $\dot{F}_{\mathrm{da}}$-bisimulation on $c : X \to F_{\mathrm{da}}(X)$. Then any pair of states $(x, y) \in P$ are language equivalent, that is, the set of words accepted from x coincides with that of y. Moreover, $\dot{F}_{\mathrm{da}}$-bisimilarity coincides with the set of all language-equivalent state pairs.

3.2 Composing F-Coalgebras

Let $F : \mathbb{C} \to \mathbb{C}$ be an endofunctor over a category $\mathbb{C}$. We model an N-ary composition operation for F-coalgebras by a pair of

1. a functor $T : \mathbb{C}^N \to \mathbb{C}$ (called *structure functor*) describing how the operation transforms coalgebra carriers, and
2. a natural transformation $\lambda \colon T \circ F^N \Rightarrow F \circ T$, describing the composition of a transition system by merging one-step transition of its arguments.

The second component induces the lifting $T_\lambda : (\mathbf{Coalg}(F))^N \to \mathbf{Coalg}(F)$ of T along $(U_F)^N, U_F$ given by

$$T_\lambda \triangleq \lambda \circ T(c_1, \cdots, c_N).$$

We call the pair (T, λ) an *N-ary one-step composition operation* for F-coalgebras.

Example 3.5. We introduce a binary one-step composition operation $(\times, \lambda)$ on $\mathcal{P}$-coalgebras. It takes the binary product of coalgebra carriers. The distributive law $\lambda^{\mathcal{P}}_{X,Y} : \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X \times Y)$ is given by $\lambda^{\mathcal{P}}_{X,Y}(A, B) \triangleq A \times B$.

Example 3.6. We introduce a binary one-step composition operation $(\times, \lambda^{\mathrm{da}})$ on F_{da}-coalgebras. It takes the binary product of coalgebra carriers. The distributive law $\lambda^{\mathrm{da}}_{X,Y} \colon F_{\mathrm{da}}X \times F_{\mathrm{da}}Y \to F_{\mathrm{da}}(X \times Y)$ is given by

$$\lambda^{\mathrm{da}}_{X,Y}((t_1, \rho_1), (t_2, \rho_2)) \triangleq (t_1 \wedge t_2,\ \lambda a\ .\ (\rho_1(a), \rho_2(a))).$$

3.3 Composing Coalgebraic Bismulations

The central theme of this paper is to deepen the understanding of the interaction between composition operations on coalgebras and the concept of bisimulation. The question we address is:

Question 3.7. Let (T, λ) be an N-ary one-step composition operation for F-coalgebras. How can we compose bisimulations $P_1, \cdots, P_N$ on F-coalgebras $c_1, \cdots, c_N$ into a bisimulation on the composed F-coalgebra $T_\lambda(c_1, \cdots, c_N)$?

In other words, the problem is about extending the composition operation on F-coalgebras to bisimulations. Using Hermida and Jacobs' coalgebraic formulation of bisimulation, we rephrase Question 3.7 as follows:

Question 3.8. Let $(F, \dot{F}) : p \to p$ be a 1-cell in $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$. How can we lift T_λ along $(\mathbf{Coalg}(p))^N, \mathbf{Coalg}(p)$?

$$\begin{tikzcd}
(\mathbf{Coalg}(\dot{F}))^N \arrow[r, dotted, "?"] \arrow[d, "(\mathbf{Coalg}(p))^N"'] & \mathbf{Coalg}(\dot{F}) \arrow[d, "\mathbf{Coalg}(p)"] \\
(\mathbf{Coalg}(F))^N \arrow[r, "T_\lambda"] & \mathbf{Coalg}(F)
\end{tikzcd} \tag{1}$$

Thanks to the coalgebraic formulation of bisimulation, we notice that a lifting of the distributive law with respect to liftings of relevant functors yields the operation (1) on $\dot{F}$-bisimulations that is compatible with the operation on F-coalgebras. We formally state this principle that guides this research as follows. This itself is not new, and a limited version is seen as [6, Proposition 6.3].

Theorem 3.9 (composition of bisimulations)**.** *Let* $(F, \dot{F}) : p \to p$ *and* $(T, \dot{T}) : p^N \to p$ *be 1-cells in* $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ *and* $(\lambda, \dot{\lambda}) : (T, \dot{T}) \circ (F^N, \dot{F}^N) \to (F, \dot{F}) \circ (T, \dot{T})$ *be a 2-cell. Then* $(T_\lambda, \dot{T}_{\dot{\lambda}}) : (\mathbf{Coalg}(p))^N \to \mathbf{Coalg}(p)$ *is a 1-cell in* $\mathbf{CAT}^{\to}$. □

A restricted version of this theorem is also available. Let C be a subclass of objects in $\mathbb{E}$. If λ is liftable on C^N, then we obtain a 1-cell $(T_\lambda, \dot{T}_{\dot{\lambda}}) : (\mathbf{Coalg}(p)|_C)^N \to \mathbf{Coalg}(p)$; here $\mathbf{Coalg}(p)|_C$ is the restriction of $\mathbf{Coalg}(p)$ to the category of $\dot{F}$-coalgebras over objects in C. This restriction will be used after Proposition 7.3.

Corollary 3.10 (preservation of bisimilarities)**.** *In the setting of Theorem 3.9, for any F-coalgebra $c_i : X_i \to F(X_i)$ $(i = 1 \cdots N)$,*

$$\dot{T}(\nu(c_1^* \circ \dot{F}), \cdots, \nu(c_N^* \circ \dot{F})) \sqsubseteq \nu(T_\lambda(c_1, \cdots, c_N)^* \circ \dot{F}). \qquad \square$$

The preservation holds because $\nu(c^* \circ \dot{F})$ is the domain of the terminal object of $\mathbf{Coalg}(\dot{F})_c$ for each F-coalgebra c, and $\dot{T}_{\dot{\lambda}}$ maps coalgebras in $\mathbf{Coalg}(\dot{F})_{c_i}$ $(i \in N)$ to a coalgebra in $\mathbf{Coalg}(\dot{F})_{T_\lambda(c_1, \cdots, c_N)}$.

This theorem merely says that when all ingredients $F, \cdots, \dot{\lambda}$ are available, the lifting $\dot{T}_{\dot{\lambda}}$ is available. In practice, the data $F, \dot{F}, T, \lambda$ are given by hand when designing transition systems and bisimulations on them, while a problematic part is defining a lifting of the structure $\dot{T}$, and make the distributive law λ *liftable* with respect to $\dot{T} \circ (\dot{F})^N$ and $\dot{F} \circ \dot{T}$.

Regarding lifting the behavior functor F, recently a systematic method called the *codensity lifting* has been introduced [29]. This is a generalization of Kantorovich lifting [2], a classical notion of distance between distributions, to general $\mathbf{CLat}_\sqcap$-fibrations. The advantage of the codensity lifting is its flexibility; it has some parameters and by varying it we obtain various liftings. Yet, it provides more structure than assuming arbitrary liftings of F and T. Another important point is that *codensity bisimulations*, where $\dot{F}$ is a codensity lifting, have an interesting game-theoretic characterization. This flexibility and relevance to bisimulation games is attractive, so we address the problem of having the codensity liftings and the liftable λ. This involves two specific questions. One is about extending the codensity lifting technique to lift not only behavior functors, but also structure functors T.

Question 3.11. Let (T, λ) be an N-ary one-step composition operation for F-coalgebras.

1. How do we non-trivially lift structure functor (such as N-ary product functors) in order to capture composition operators at the level of relations?
2. When is λ liftable as a distributive law between codensity liftings of F and T?

To answer to these questions, we employ Beohar et al.'s recent decomposition result of codensity liftings [3]. Their decomposition separates fibration-specific parts in the codensity lifting from the central part that does the actual lifting task. Upon their decomposition, in Section 5, we first answer to (1) by generalizing the codensity lifting to arbitrary functors. Then in Section 6, we answer to (2) by providing a sufficient condition to lift λ to a distributive law between codensity liftings. The proof of the sufficiency takes advantage of the decomposition and a 2-categorical nature of the central part in the decomposition.

4 Generalizing the Codensity Lifting

We recall the multiple parameter codensity lifting in [29]. It lifts an endofunctor $F : \mathbb{B} \to \mathbb{B}$ along a $\mathbf{CLat}_\sqcap$-fibration $p : \mathbb{E} \to \mathbb{B}$ using three parameters $\Omega, \boldsymbol{\Omega}, \tau$. We pack $(p, \Omega, \boldsymbol{\Omega})$ into the following data.

Definition 4.1. Let A be a discrete category. A $\mathbf{CLat}_\sqcap$*-fibration with truth values* is a tuple $(p : \mathbb{E} \to \mathbb{B}, \Omega : A \to \mathbb{B}, \boldsymbol{\Omega} : A \to \mathbb{E})$ of functors such that $\Omega = p \circ \boldsymbol{\Omega}$ and p is a $\mathbf{CLat}_\sqcap$-fibration. The notation for a $\mathbf{CLat}_\sqcap$-fibration with truth values is $(\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B})$.

Let $(\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B})$ be a $\mathbf{CLat}_\sqcap$-fibration with truth values. The codensity lifting of a functor $F : \mathbb{B} \to \mathbb{B}$ takes as argument a natural transformation $\tau : F \circ \Omega \to \Omega$. The components $\tau_a : F \circ \Omega(a) \to \Omega(a)$ are referred to in coalgebraic modal logic as *modalities*; τ is just an indexed collection of these, and below we will often refer to τ itself as a modality. The *codensity lifting* of $F : \mathbb{B} \to \mathbb{B}$ with τ along p is given by

$$[F, \tau]^{\boldsymbol{\Omega}} X \triangleq \textstyle\bigsqcap_{a \in A, k \in \mathbb{E}(X, \boldsymbol{\Omega}(a))} (\tau_a \circ F(pk))^* (\boldsymbol{\Omega}(a)).$$

It is an endofunctor over $\mathbb{E}$, and is a lifting of F along p. When $A = 1$, it is called the *single-parameter* codensity lifting. $\dot{F}$-bisimulations whose lifting part $\dot{F}$ is a codensity lifting are called *codensity bisimulation*.

Example 4.2. We write 2 for the two-point set $\{\mathrm{false}, \mathrm{true}\}$ and Eq_2 for the object $(2, \{(x, x) \mid x \in 2\})$ in $\mathbf{EqRel}$. We identify it as a functor of type $1 \to \mathbf{EqRel}$, and form a $\mathbf{CLat}_\sqcap$-fibration with truth values $(2, \mathrm{Eq}_2) : Id_1 \to U_{\mathbf{EqRel}}$. The codensity lifting of $\mathcal{P}$ with the modality $\diamond : \mathcal{P}(2) \to 2$ given by $\diamond(U) = true \iff true \in U$ coincides with the lifting $\dot{\mathcal{P}}$ in Example 3.3 [23]. Therefore the standard bisimulations on $\mathcal{P}$-coalgebras are expressible as codensity bisimulations.

Example 4.3. We adopt the following $\mathbf{CLat}_\sqcap$-fibration with truth values $(\Omega, \boldsymbol{\Omega}) : Id_{\Sigma \uplus \{\epsilon\}} \to U_{\mathbf{EqRel}}$ where Ω and $\boldsymbol{\Omega}$ are functors constantly returning 2 and Eq_2, respectively. We define the modality for F_{da} by

$$\tau_a(t, \rho) = \begin{cases} t & (a = \epsilon) \\ \rho(a) & (a \in \Sigma). \end{cases}$$

Then the codensity lifting $[F_{\mathrm{da}}, \tau]^{\boldsymbol{\Omega}}$ maps $(X, P) \in \mathbf{EqRel}$ to the set $F_{\mathrm{da}}(X) = 2 \times X^\Sigma$ paired with the following equivalence relation:

$$\{((t_1, \rho_1), (t_2, \rho_2)) \mid t_1 = t_2 \wedge \forall a \in \Sigma \ . \ (\rho_1(a), \rho_2(a)) \in P\}.$$

The $[F_{\mathrm{da}}, \tau]^{\boldsymbol{\Omega}}$-bisimilarity on c identifies the states that accepts the same language.

To organize the discussion, we package all the ingredients into the following data. It specifies both abstract transition systems and a notion of bisimulation on them.

Definition 4.4. *Codensity bisimulation data* consists of a $\mathbf{CLat}_\sqcap$-fibration with truth values $(\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B})$, an endofunctor $F : \mathbb{B} \to \mathbb{B}$, and a natural transformation $\tau : F \circ \Omega \to \Omega$.

4.1 Decomposition of Codensity Lifting

Let $((\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be codensity bisimulation data. Recently, Beohar et al. [3] introduced a decomposition of the codensity lifting as a "sandwich" of a monotone function within Galois connections. We adopt their decomposition as it is useful for analyzing the interaction between codensity liftings. In this paper, we describe their decomposition in fibered category theory. The decomposition is given as

$$[F, \tau]^{\boldsymbol{\Omega}} = R^{p,\boldsymbol{\Omega}} \circ \mathrm{Sp}^A(F, \tau) \circ L^{p,\boldsymbol{\Omega}}, \quad (L^{p,\boldsymbol{\Omega}} \dashv R^{p,\boldsymbol{\Omega}}) \tag{2}$$

where $\mathrm{Sp}^A(F, \tau)$ is an endofunctor over a suitable category that we introduce below. The corresponding equation in Beohar et al. is in [3, Section 4.4]. Their presentation is based on indexed lattices, and left and right adjoints are swapped.[1]

The description of the components of (2) is in order. First, we observe that the product $[A, s^{op}] \colon [A, \mathbf{Pred}^{op}] \to [A, \mathbf{Set}^{op}]$ of A-fold copies of the opposite of the subobject fibration $U_{\mathbf{Pred}} : \mathbf{Pred} \to \mathbf{Set}$ is a $\mathbf{CLat}_\sqcap$-fibration. Now the adjunction $L^{p,\boldsymbol{\Omega}} \dashv R^{p,\boldsymbol{\Omega}}$ in (2) is given between $\mathbb{E}$ and the category

[1] Our choice of left and right adjoint is consistent with the codensity lifting of *monads* (using Eilenberg-Moore algebra) [22].

obtained by the change-of-base of the fibration $[A, U^{op}_{\mathbf{Pred}}]$ along the functor $H(b) \triangleq (\mathbb{B}(b, \Omega(-)))^{op}$:

$$\begin{tikzcd}
\mathbb{E} \arrow[r, shift left, "L^{p,\mathbf{\Omega}}"] \arrow[r, phantom, "\perp"] \arrow[drr, bend right, "p"'] & \mathrm{Sp}^A(\mathbb{B}, \Omega) \arrow[l, shift left, "R^{p,\mathbf{\Omega}}"] \arrow[r, "\overline{H}"] \arrow[d, "r^A_{\mathbb{B},\Omega}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[A, \mathbf{Pred}^{op}]} \arrow[d, "{[A, U^{op}_{\mathbf{Pred}}]}"] \\
& \mathbb{B} \arrow[r, "H"'] & {[A, \mathbf{Set}^{op}]}
\end{tikzcd}$$

Concretely, an object of $\mathrm{Sp}^A(\mathbb{B}, \Omega)$ is a pair of an object $X \in \mathbb{B}$ and a mapping $S : A \to \mathbf{Set}$ such that for any $a \in A$, $S(a) \subseteq \mathbb{B}(X, \Omega(a))$. A morphism from (X, S) to (Y, T) is a morphism $f : X \to Y$ such that for any $a \in A$ and $k \in T(a)$, $k \circ f \in S(a)$. The evident projection functor $r^A_{\mathbb{B},\Omega} : \mathrm{Sp}^A(\mathbb{B}, \Omega) \to \mathbb{B}$ is a $\mathbf{CLat}_\sqcap$-fibration, because s is so.

Beohar et al. showed that the following assignments form a left adjoint functor [3, Theorem 7]:

$$L^{p,\mathbf{\Omega}}(P) \triangleq (pP, \lambda a \,.\, p(\mathbb{E}(P, \mathbf{\Omega}(a)))), \qquad L^{p,\mathbf{\Omega}}(f) \triangleq f,$$

and the object part of its right adjoint satisfies

$$R^{p,\mathbf{\Omega}}(X, S) = \textstyle\bigsqcap_{a \in A, k \in S(a)} k^*(\mathbf{\Omega}(a)).$$

The middle part $\mathrm{Sp}^A(\mathbb{B}, \Omega)$ in (2) is defined as a lifting of an endofunctor $F : \mathbb{B} \to \mathbb{B}$ along $r^A_{\mathbb{B},\Omega}$ using a modality $\tau : F \circ \Omega \to \Omega$. It is an extension of the predicate lifting $(P : X \to \Omega) \mapsto (\tau \circ F(P) : F(X) \to \Omega)$, which is commonly used in the theory of coalgebras, to sets of morphisms into Ω.

$$\mathrm{Sp}^A(F, \tau)(X, S) \triangleq (FX, \lambda a \,.\, \{\tau_a \circ Fk \mid k \in S(a)\}) \tag{3}$$
$$\mathrm{Sp}^A(F, \tau)(f) \triangleq Ff. \tag{4}$$

By composing the above concrete characterizations of $L^{p,\mathbf{\Omega}}(P)$, $R^{p,\mathbf{\Omega}}(X, S)$ and $\mathrm{Sp}^A(F, \tau)$, it is immediate that the codensity lifting satisfies the equation (2).

4.2 Generalizing Codensity Liftings

From Beohar et al.'s decomposition, we notice that the actual lifting job of F is done by $\mathrm{Sp}^A(F, \tau)$. We speculate that an extension of Sp to natural transformations would also help lifting distributive laws with respect to codensity liftings. We therefore exhibit a 2-functorial nature of the Sp construction. Its domain is the *lax coslice 2-category* $A//\mathbf{CAT}$ under a discrete category A, which is defined as follows:

- A 0-cell is a pair of a category $\mathbb{C}$ with a functor $\Omega : A \to \mathbb{C}$.
- A 1-cell from $(\mathbb{C}, \Omega)$ to $(\mathbb{D}, \Pi)$ is a pair of a functor $F : \mathbb{C} \to \mathbb{D}$ and a natural transformation $\tau : F \circ \Omega \to \Pi$.

- A 2-cell from $(F, \tau) : (\mathbb{C}, \Omega) \to (\mathbb{D}, \Pi)$ to (G, υ) is a natural transformation $\alpha : F \to G$ such that $\upsilon \bullet (\alpha \circ \Omega) = \tau$.

Observe that an endo-1-cell on $(\mathbb{C}, \Omega)$ bijectively corresponds to a pair of an endofunctor F on $\mathbb{C}$ and a modality $\tau : F \circ \Omega \to \Omega$.

Theorem 4.5. *The following assignments* $\mathbf{Sp}^A$ *form a 2-functor of type* $A//\mathbf{CAT} \to \mathbf{CAT}^{\mathbf{CLat}_\sqcap}$.

$$\mathbf{Sp}^A(\mathbb{C}, \Omega) = r^A_{\mathbb{C},\Omega} \qquad \mathbf{Sp}^A(F, \tau) = (F, \mathrm{Sp}^A(F, \tau)) \qquad \mathbf{Sp}^A(\alpha) = \alpha$$

Here, the functor $\mathrm{Sp}^A(F, \tau)$ *is the extension of* (3) *to arbitrary 1-cell* $(F, \tau) : (\mathbb{C}, \Omega) \to (\mathbb{D}, \Pi)$ *by the same formulas* (3),(4). *Moreover,* $\mathrm{Sp}^A(F, \tau)$ *preserves Cartesian morphisms and all meets.* □

This construction suggests that a categorical structure in $A//\mathbf{CAT}$ is transferred to $\mathbf{CLat}_\sqcap$-fibrations equipped with the same categorical structure and its lifting. We employ this property to transfer distributive laws in $A//\mathbf{CAT}$ to $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ in Theorem 6.2.

As the construction of $\mathrm{Sp}^A(F, \tau)$ is extended to arbitrary 1-cells $(\mathbb{C}, \Omega) \to (\mathbb{D}, \Pi)$, it is natural to put *different* adjoints in the decomposition (2). This leads us to the following generalization:

Definition 4.6. Let $(\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B})$ and $(\Pi, \mathbf{\Pi}) : Id_A \to (q : \mathbb{F} \to \mathbb{C})$ be $\mathbf{CLat}_\sqcap$-fibrations with truth values. The *codensity lifting* of $F : \mathbb{B} \to \mathbb{C}$ along p, q with a natural transformation $\tau : F \circ \Omega \to \Pi$, denoted by $[F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}}$, is defined by

$$\begin{aligned} [F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}} &\triangleq R^{q,\mathbf{\Pi}} \circ \mathrm{Sp}^A(F, \tau) \circ L^{p,\mathbf{\Omega}} \\ &= \textstyle\bigsqcap_{a \in A, k \in \mathbb{E}(-,\mathbf{\Omega}(a))} (\tau_a \circ F(pk))^*(\mathbf{\Pi}(a)). \end{aligned} \tag{5}$$

We show some properties of this generalized codensity lifting. The following shows that it is the largest lifting that makes τ liftable with respect to $\dot{F} \circ \mathbf{\Omega}$ and $\mathbf{\Pi}$. Here, two liftings $\dot{F}, \ddot{F}$ of F are compared by: $\dot{F} \sqsubseteq \ddot{F}$ if $\dot{F}(P) \sqsubseteq \ddot{F}(P)$ for any P.

Theorem 4.7. *For any* $(F, \dot{F}) \in \mathbf{CAT}^{\mathbf{CLat}_\sqcap}(p, q)$, $\dot{F} \sqsubseteq [F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}}$ *if and only if* $\tau : \dot{F} \circ \mathbf{\Omega} \dot{\to} \mathbf{\Pi}$ *in the fibration* $\mathrm{cod}_{\mathrm{Id}_A, q}$ *(§2.2).* □

This is the generalization of the universal property possessed by codensity liftings [29, Theorem 5.14].

When the indexing set A is one-point 1, the codensity lifting $[F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}}$ is determined by the object $[F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}}\mathbf{\Omega}$.

Proposition 4.8. *Assume* $A = 1$ *in the setting of Definition 4.6. Then* $[F, \tau]^{\mathbf{\Omega},\mathbf{\Pi}} = R^{q,[F,\tau]^{\mathbf{\Omega},\mathbf{\Pi}}\mathbf{\Omega}} \circ \mathrm{Sp}^1(F, \mathrm{id}) \circ L^{p,\mathbf{\Omega}}$. □

Proposition 4.9. *Assume* $A = 1$ *in the setting of Definition 4.6. Let* $(F, \dot{F}) : p \to q$ *be a fibered one-cell. If* $R^{p,\mathbf{\Omega}} \circ L^{p,\mathbf{\Omega}} = \mathrm{id}$ *and* $\dot{F}$ *preserves fibered meets, the following statements hold.*

1. $\dot{F} = R^{q,\dot{F}\mathbf{\Omega}} \circ \mathrm{Sp}^1(F, \mathrm{id}) \circ L^{p,\mathbf{\Omega}}$.

2. $[F,\tau]^{\mathbf{\Omega},\mathbf{\Pi}} = \dot{F}$ *if and only if* $[F,\tau]^{\mathbf{\Omega},\mathbf{\Pi}}\mathbf{\Omega} = \dot{F}\mathbf{\Omega}$. □

The assumption $R^{p,\mathbf{\Omega}} \circ L^{p,\mathbf{\Omega}} = \mathrm{id}$ is mild enough for all examples of $\mathbf{CLat}_\sqcap$-fibrations in this paper. The first statement means $\dot{F}$ has the same property as $[F,\tau]^{\mathbf{\Omega},\mathbf{\Pi}}$ written in Proposition 4.8, and it immediately yields the second statement. This proposition will be used to get a criterion for checking $[F,\tau]^{\mathbf{\Omega},\mathbf{\Pi}} = \dot{\times} : \mathbb{E}^2 \to \mathbb{E}$ (Corollary 5.1).

5 Codensity Liftings of structure Functors

We use the generalized codensity lifting to lift the structure functor. Let $(\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B})$ be a $\mathbf{CLat}_\sqcap$-fibration with truth values. Its N-fold tupling $(\langle\Omega\rangle_N, \langle\mathbf{\Omega}\rangle_N) : Id_A \to (p^N : \mathbb{E}^N \to \mathbb{B}^N)$ is again a $\mathbf{CLat}_\sqcap$-fibration with truth values; here $\langle\Omega\rangle_N$ denotes the tupling $\langle\Omega, \cdots, \Omega\rangle : A \to \mathbb{B}^N$ of N-fold copies of Ω (and the same for $\langle\mathbf{\Omega}\rangle_N$). We call the codensity lifting of a structure functor $T : \mathbb{B}^N \to \mathbb{B}$ with $\sigma : T \circ \langle\Omega\rangle_N \to \Omega$ along p^N, p the *N-codensity lifting*. Concretely,

$$\begin{aligned}
&[T,\sigma]^{\langle\mathbf{\Omega}\rangle_N,\mathbf{\Omega}}(P_1, \cdots, P_N) \\
&= \bigsqcap_{\substack{a \in A, \\ k_i \in \mathbb{E}(P_i, \mathbf{\Omega}(a))}} (\sigma_a \circ T(pk_1, \cdots, pk_n))^* \mathbf{\Omega}(a).
\end{aligned}$$

We overload the notation for the N-codensity lifting on that for the codensity lifting: when N can be read-off from T, we simply write $[T,\sigma]^{\mathbf{\Omega}}$ to mean $[T,\sigma]^{\langle\mathbf{\Omega}\rangle_N,\mathbf{\Omega}}$.

5.1 Product Functors by Codensity Liftings

One of the most fundamental operators for composing processes and state transition systems is *parallel composition*. Typically it generates a new transition system whose states are pairs $x||y$ of component system's states. This suggests that the carrier of the composed system is the *binary product* of the carrier of its components. We thus study the case where the structure functor T is $\times : \mathbb{B}^2 \to \mathbb{B}$.

In this section we illustrate some liftings of the binary product functor, and show that they can be expressed by the single-parameter 2-codensity lifting. It is easy to apply results here to the multiple-parameter codensity lifting because it is the intersection of single-parameter one.

5.1.1 Product functor on the total category

Suppose that the base category $\mathbb{B}$ has a binary product functor $\times : \mathbb{B}^2 \to \mathbb{B}$. Since $p : \mathbb{E} \to \mathbb{B}$ is a $\mathbf{CLat}_\sqcap$-fibration, the functor $\dot{\times} : \mathbb{E}^2 \to \mathbb{E}$ defined by

$$P \mathbin{\dot{\times}} Q \triangleq \pi^* P \sqcap \pi'^* Q \quad (\pi, \pi' \text{ are 1st and 2nd projections})$$

is a fibered lifting of $\times$ along p^2, p. It also preserves all meets. In fact, it is the binary product functor on $\mathbb{E}$ [20, Proposition 9.2.1].

When can $\dot{\times}$ be expressed by the codensity lifting $[\times, \sigma]^{\mathbf{\Omega}}$? Since Proposition 4.9 is applicable to $\dot{\times}$, we obtain the following:

Corollary 5.1. *Let $(\Omega, \mathbf{\Omega}) : Id_1 \to (p : \mathbb{E} \to \mathbb{B})$ be a $\mathbf{CLat}_\sqcap$-fibration with truth values such that $\mathbb{B}$ has binary products, and $R^{p,\mathbf{\Omega}} \circ L^{p,\mathbf{\Omega}} = \mathrm{id}$. Then for any modality $\sigma : \Omega \times \Omega \to \Omega$, we have $\dot{\times} = [\times, \sigma]^{\mathbf{\Omega}}$ if and only if $[\times, \sigma]^{\mathbf{\Omega}}(\mathbf{\Omega}, \mathbf{\Omega}) = \mathbf{\Omega} \dot{\times} \mathbf{\Omega}$.* $\square$

Therefore one can check $\dot{\times} = [\times, \sigma]^{\mathbf{\Omega}}$ only by checking the equality $[\times, \sigma]^{\mathbf{\Omega}}(\mathbf{\Omega}, \mathbf{\Omega}) = \mathbf{\Omega} \dot{\times} \mathbf{\Omega}$.

Example 5.2. Upon the same $\mathbf{CLat}_\sqcap$-fibration with truth values $(2, \mathrm{Eq}_2) : Id_1 \to U_{\mathbf{EqRel}}$ in Example 4.2, let $\wedge \colon 2 \times 2 \to 2$ be the logical conjunction. Then the binary product $\dot{\times}$ on $\mathbf{EqRel}$ is the 2-codensity lifting $[\times, \wedge]^{\mathrm{Eq}_2}$.

5.1.2 Metric lifting of the binary product by a modality

Here we focus on the $\mathbf{CLat}_\sqcap$-fibration with truth values $(\mathbb{I}, (\mathbb{I}, d_\mathbb{I})) : Id_1 \to U_{\mathbf{PMet}}$ where $\mathbb{I}$ is the interval $[0,1]$ and $(\mathbb{I}, d_\mathbb{I}) \in \mathbf{PMet}$ is the Euclidean distance over the interval on it. We will use it for quantitative bisimulations in §7.1 and §7.3.

We introduce a metric lifting of the binary product functor using a binary function on the interval (which we also call a modality). We show that it is equal to the codensity lifting given by the modality.

Proposition 5.3. *Let $\sigma \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ be a function. For each pseudometric spaces $(X, d_X), (Y, d_Y)$, we define the function $\times^\sigma(d_X, d_Y) \colon (X \times Y)^2 \to \mathbb{I}$ by*

$$\times^\sigma(d_X, d_Y)((x,y),(x',y')) \coloneqq \sigma(d_X(x,x'), d_Y(y,y')).$$

Then this is a lifting of the product functor $\times \colon \mathbb{B}^2 \to \mathbb{B}$ along $U_{\mathbf{PMet}}$ if and only if σ satisfies the following condition:

$$\begin{cases} \sigma \text{ is monotone}, \\ \sigma(0,0) = 0, \\ \sigma(a,b) - \sigma(c,d) \leq \sigma(|a-c|, |b-d|). \end{cases} \tag{6}$$

Moreover, if σ satisfies (6) then $\times^\sigma = [\times, \sigma]^{d_\mathbb{I}}$. $\square$

For example, the modalities $\sigma_\oplus(a,b) \coloneqq 1 - (1-a)(1-b)$, $\sigma_{\mathrm{av}}(a,b) \coloneqq (a+b)/2$, and $\sigma_\vee(a,b) \coloneqq \max(a,b)$ satisfy (6), see Appendix A.2. The modalities $\sigma_\otimes(a,b) \coloneqq a \cdot b$ and $\sigma_\wedge(a,b) \coloneqq \min(a,b)$ do not satisfy the last condition of (6). However, their codensity liftings are equal to those of $\sigma_\oplus$ and $\sigma_\vee$, respectively. This follows from the following proposition.

Proposition 5.4. *Let $f \colon \Omega \to \Omega$ be an isomorphism in $\mathbb{B}$ satisfying $f^*\mathbf{\Omega} = \mathbf{\Omega}$. Then $[T, \sigma]^{\mathbf{\Omega}} = [T, \sigma \circ T(f^N)]^{\mathbf{\Omega}} = [T, f \circ \sigma]^{\mathbf{\Omega}}$.* $\square$

Putting $f(a) \coloneqq 1 - a$, the proposition above implies $[\times, \sigma_\otimes]^{d_\mathbb{I}} = [\times, \sigma_\oplus]^{d_\mathbb{I}}$ and $[\times, \sigma_\wedge]^{d_\mathbb{I}} = [\times, \sigma_\vee]^{d_\mathbb{I}}$ because $\sigma_\otimes = f \circ \sigma_\oplus \circ (f \times f)$ and $\sigma_\wedge = f \circ \sigma_\vee \circ (f \times f)$ hold,

6 Codensity Lifting of One-Step Composition

As discussed at the end of §3, our focus is on *modalities that lifts one-step composition operations for coalgebras.* Technically, this is equivalent to the liftability of distributive law $\lambda\colon T \circ F^N \to F \circ T$ with respect to codensity liftings of F, T.

Definition 6.1. Let $((\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be codensity bisimulation data, and (T, λ) be an N-ary one-step composition on F-coalgebras. We say that *a modality* $\sigma\colon T \circ \langle\Omega\rangle_N \to \Omega$ *lifts* λ *along* p if λ is liftable with respect to $[T, \sigma]^{\boldsymbol{\Omega}} \circ ([F, \tau]^{\boldsymbol{\Omega}})^N$ and $[F, \tau]^{\boldsymbol{\Omega}} \circ [T, \sigma]^{\boldsymbol{\Omega}}$.

Recall that the liftability of α may be restricted to a class C of objects in $\mathbb{E}$. This restriction only appears in §7.3.

Below we provide a sufficient condition for the modality σ to lift a distributive law. Our approach employs the lifting of the distributive law λ by the 2-functor Sp introduced in Theorem 4.5, and the decomposed definition of the generalized codensity lifting (5).

First, we restrict σ so that λ becomes a 2-cell $A/\!/\mathbf{CAT}$ of type

$$(T, \sigma) \circ (F^N, \tau^N) \Rightarrow (F, \tau) \circ (T, \sigma).$$

Its image by the 2-functor Sp yields a lifting $\dot{\lambda}$ of λ, whose type is

$$\mathrm{Sp}^A(T, \sigma) \circ \mathrm{Sp}^A(F^N, \tau^N) \Rightarrow \mathrm{Sp}^A(F, \tau) \circ \mathrm{Sp}^A(T, \sigma).$$

This enables us to *interchange* the center of two codensity liftings that meets in the middle of their composition (below we put $R \triangleq R^{p,\boldsymbol{\Omega}}, L \triangleq L^{p,\boldsymbol{\Omega}}, R_N \triangleq R^{p^N,\langle\boldsymbol{\Omega}\rangle_N}, L_N \triangleq L^{p^N,\langle\boldsymbol{\Omega}\rangle_N}$):

$$\begin{aligned}
&[T, \sigma]^{\boldsymbol{\Omega}} \circ ([F, \tau]^{\boldsymbol{\Omega}})^N \\
&= [T, \sigma]^{\boldsymbol{\Omega}} \circ [F^N, \tau^N]^{\langle\boldsymbol{\Omega}\rangle_N} \quad (([F, \tau]^{\boldsymbol{\Omega}})^N = [F^N, \tau^N]^{\langle\boldsymbol{\Omega}\rangle_N}) \\
&= R \circ \mathrm{Sp}^A(T, \sigma) \circ L_N \circ R_N \circ \mathrm{Sp}^A(F^N, \tau^N) \circ L_N \\
&\Rightarrow R \circ \mathrm{Sp}^A(T, \sigma) \circ \mathrm{Sp}^A(F^N, \tau^N) \circ L_N \quad (\text{by } L_N \dashv R_N) \\
&\Rightarrow R \circ \mathrm{Sp}^A(F, \tau) \circ \mathrm{Sp}^A(T, \sigma) \circ L_N.
\end{aligned} \tag{7}$$

Overall, the above natural transformation is again a lifting of λ, because the last natural transformation is a lifting of id.

Next, we impose the last line (7) to be equal to the composition $[F, \tau]^{\boldsymbol{\Omega}} \circ [T, \sigma]^{\boldsymbol{\Omega}}$ of codensity liftings. Overall, we obtain a desired lifting of λ with respect to $[T, \sigma]^{\boldsymbol{\Omega}} \circ ([F, \tau]^{\boldsymbol{\Omega}})^N$ and $[F, \tau]^{\boldsymbol{\Omega}} \circ [T, \sigma]^{\boldsymbol{\Omega}}$.

Theorem 6.2 (sufficient condition)**.** *Let* $((\Omega, \boldsymbol{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ *be codensity bisimulation data and* (T, λ) *be an* N*-ary composition operation on* F*-coalgebras. Then a modality* σ *lifts* λ *along* p *if the following conditions hold.*

1. σ *makes* λ *a 2-cell of type* $(T, \sigma) \circ (F^N, \tau^N) \Rightarrow (F, \tau) \circ (T, \sigma)$ *in* $A/\!/\mathbf{CAT}$*, and*

2. $R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau) \circ \mathrm{Sp}^A(T,\sigma) \circ L^{p^N,\langle\mathbf{\Omega}\rangle_N} = [F,\tau]^{\mathbf{\Omega}} \circ [T,\sigma]^{\mathbf{\Omega}}$. $\square$

The first condition is equivalent to $\sigma_a \circ T((\tau_a)_{i\in N}) = \tau_a \circ F\sigma_a \circ \lambda$ for each $a \in A$, and it induces $\mathrm{Sp}^A(T,\sigma) \circ \mathrm{Sp}^A(F^N,\tau^N) = \lambda^*\mathrm{Sp}^A(F,\tau) \circ \mathrm{Sp}^A(T,\sigma)$.

We investigate the second condition. First, it can be expressed as "$\mathrm{Sp}^A(T,\sigma) \circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}(P)$ is approximating to $[F,\tau]^{\mathbf{\Omega}}$ for each $P \in \mathbb{E}^N$" by the concept of *approximating families* introduced by Komorida et al. [24] for expressivity of coalgebraic modal logics. Note that they consider a restricted setting such that $\mathbf{\Omega}$ is constant on A.

Definition 6.3. An object $S \in \mathrm{Sp}^A(\mathbb{B},\Omega)$ is *approximating* to the codensity lifting $[F,\tau]^{\mathbf{\Omega}}$ if

$$R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)(S) \sqsubseteq R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau) \circ L^{p,\mathbf{\Omega}} \circ R^{p,\mathbf{\Omega}}(S).$$

That is, for each $a \in A$ and $k \in \mathbb{E}(\bigsqcap_{a'\in A,k'\in S_{a'}} k'^*\mathbf{\Omega}(a'), \mathbf{\Omega}(a))$, $\bigsqcap_{a'\in A,k'\in S_{a'}} (\tau_{a'} \circ Fk')^*\mathbf{\Omega}(a') \sqsubseteq (\tau_a \circ Fpk)^*\mathbf{\Omega}(a)$.

It is equivalent to $R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)(S) = R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau) \circ L^{p,\mathbf{\Omega}} \circ R^{p,\mathbf{\Omega}}(S)$ by the counit of $L^{p,\mathbf{\Omega}} \dashv R^{p,\mathbf{\Omega}}$. Thus (2) in Theorem 6.2 can be written as "$\mathrm{Sp}^A(T,\sigma) \circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}(P)$ is approximating to $[F,\tau]^{\mathbf{\Omega}}$ for each $P \in \mathbb{E}^N$".

Next we show a result for the approximating condition that is applicable when we can express $k \in \mathbb{E}(\bigsqcap_{a'\in A,k'\in S_{a'}} k'^*\mathbf{\Omega}(a'), \mathbf{\Omega}(a))$ by a subset of S_a.

Proposition 6.4. *Assume that each hom-poset $(\mathbb{B}(X,\Omega), \leq_X)$ is a complete lattice, and we write $\bigvee$ for the join. Then $(X,S) \in \mathrm{Sp}^A(\mathbb{B},\Omega)$ is approximating to $[F,\tau]^{\mathbf{\Omega}}$ if the following conditions hold: for each $a \in A$,*

1. *For each $S' \subseteq \mathbb{C}(X',\Omega(a))$, $\tau_a \circ F\bigvee_{f\in S'} f = \bigvee_{f\in S'} \tau_a \circ Ff$.*
2. *For each $S' \subseteq \mathbb{C}(X',\Omega(a))$, $\bigsqcap_{f\in S'} f^*\mathbf{\Omega} \sqsubseteq (\bigvee_{f\in S'} f)^*\mathbf{\Omega}(a)$.*
3. *For each $k \in \mathbb{E}(\bigsqcap_{a'\in A,k'\in S_{a'}} k'^*\mathbf{\Omega}(a'), \mathbf{\Omega}(a))$, there exists $S'_k \subseteq S_a$ s.t. $pk = \bigvee_{k'\in S'_k} k'$.*

Additionally, if these conditions hold, $(X,S_a) \in \mathrm{Sp}^1(\mathbb{B},\Omega(a))$ is approximating to $[F,\tau_a]^{\mathbf{\Omega}(a)}$ for each $a \in A$. $\square$

Example 6.5 (Bisimulation for $\mathcal{P}$-coalgebras)**.** This example aims for showing the composition of the standard bisimulations.

$\triangleright$ *Codensity bisimulation data* We take the $\mathbf{CLat}_\sqcap$-fibration with truth values $(2,\mathrm{Eq}_2) : Id_1 \to U_{\mathbf{EqRel}}$ and the covariant powerset functor $\mathcal{P}$ (Example 3.1) with the modality $\diamond\colon \mathcal{P}2 \to 2$ (Example 4.2) to form codensity bisimulation data. We have $[\mathcal{P},\diamond]^{\mathrm{Eq}_2} = \dot{\mathcal{P}}$ (Example 3.3).

$\triangleright$ *Binary one-step composition operation for $\mathcal{P}$-coalgebras* We adopt $(\times,\lambda^{\mathcal{P}})$ in Example 3.5; recall that the distributive law $\lambda^{\mathcal{P}}$ is given by $\lambda^{\mathcal{P}}_{X,Y}(A,B) = A\times B$.

$\triangleright$ *Modality for $\times$* We adopt the modality $\wedge : 2\times 2 \to 2$ in Example 5.2. The codensity lifting $[\times,\wedge]^{\mathrm{Eq}_2}$ coincides with the binary product of predicates as in Example 5.2.

Proposition 6.6. *The modality* $\wedge$ *lifts* $\lambda^{\mathcal{P}}$ *along* $U_{\mathbf{EqRel}}$. $\square$

This proposition, together with Corollary 3.10, implies that for two $\mathcal{P}$-coalgebras $c_1 : X_1 \to \mathcal{P}(X_1)$ and $c_2 : X_2 \to \mathcal{P}(X_2)$, if x_i and x_i' are $[\mathcal{P}, \diamond]^{\mathrm{Eq}_2}$-bisimilar in c_i ($i \in \{1, 2\}$), the pairs (x_1, x_2) and (x_1', x_2') are $[\mathcal{P}, \diamond]^{\mathrm{Eq}_2}$-bisimilar in the composed $\mathcal{P}$-coalgebra $\lambda^{\mathcal{P}} \circ (c_1 \times c_2)$.

The proof of Proposition 6.6 is the following: $\wedge$ satisfies two conditions in Theorem 6.2: 1) For each $A, B \subseteq 2$, $\diamond A \wedge \diamond B = \diamond\{a \wedge b \mid a \in A, b \in B\}$. 2) As discussed before Definition 6.3, it's equivalent to approximating objects $\{\wedge \circ (k_P \times k_Q) \mid k_P : P \to \mathrm{Eq}_2, k_Q : Q \to \mathrm{Eq}_2\}$ for each $P, Q \in \mathbf{EqRel}$. We can prove it by Proposition 6.4 with the ordered object $(2, \leq)$ where $\leq$ is the order satisfying false $\leq$ true. One easily verifies the first and second condition in Proposition 6.4. For the third condition, for each $a \in A$ and $k \colon [\times, \wedge]^{\mathrm{Eq}_2}(P, Q) \to \mathrm{Eq}_2$, we define $S_k' \coloneqq \{\wedge \circ (\chi_{[x]_P} \times \chi_{[y]_Q}) \mid k(x, y) = 1\}$ where $\chi_{[x]_P}, \chi_{[y]_Q}$ are characteristic functions. Then $pk = \bigvee_{k' \in S_k'} k'$ holds.

Example 6.7. We next capture language equivalence of deterministic automata using codensity bisimilarity.

$\triangleright$ *Codensity bisimulation data* They are set-up in Example 4.3. We package the $\mathbf{CLat}_{\sqcap}$-fibration with truth values $(\Omega, \mathbf{\Omega}) : Id_{\Sigma \uplus \{\epsilon\}} \to U_{\mathbf{EqRel}}$, the behavior functor F_{da} and the modality τ there into codensity bisimulation data.

$\triangleright$ *Binary one-step composition operation for* F_{da}*-coalgebras* We adopt the one $(\times, \lambda^{\mathrm{da}})$ in Example 3.6.

$\triangleright$ *Modality for* $\times$ We adopt the modality σ given by $\sigma_a \triangleq \wedge$ (the logical conjunction; here $a \in \Sigma \uplus \{\epsilon\}$). The binary codensity lifting $[\times, \wedge]^{\mathrm{Eq}_2}$ is equal to that in Example 5.2, so that $[\times, \wedge]^{\mathrm{Eq}_2} = \dot{\times}$.

Proposition 6.8. *The modality* σ *lifts* λ^{da} *along* $U_{\mathbf{EqRel}}$. $\square$

As a consequence, given deterministic automata $c_1 : X_1 \to F_{\mathrm{da}}(X_1)$ and $c_2 : X_2 \to F_{\mathrm{da}}(X_2)$, if $x_i, x_i' \in X_i$ are language equivalent in c_i ($i \in \{1, 2\}$), the pairs (x_1, x_2) and (x_1', x_2') are also language equivalent in the product automaton $\lambda^{\mathrm{da}} \circ (c_1 \times c_2)$.

The proof of Proposition 6.8 is as follows. The modality $\wedge$ satisfies two conditions in Theorem 6.2: 1) For each $(t, \rho), (t', \rho') \in 2 \times 2^{\Sigma}$, both $\wedge \circ \tau_a$ and $\tau_\epsilon \circ (2 \times \wedge^{\Sigma}) \circ \lambda^{\mathrm{da}}$ map $((t, \rho), (t', \rho'))$ to $t \wedge t$ if $a = \epsilon$ and $\rho(a) \wedge \rho'(a)$ otherwise. 2) We can prove it by Proposition 6.4. The second and third conditions are the same as Example. 6.5. The first condition in Proposition 6.4 holds because for each $S' \subseteq \mathbf{Set}(X', 2)$ and $(t, \rho) \in 2 \times X'^{\Sigma}$, both $\tau_a \circ 2 \times (\bigvee_{f \in S'} f)^{\Sigma}$ and $\bigvee_{f \in S'} \tau_a \circ 2 \times f^{\Sigma}$ map (t, ρ) to t if $a = \epsilon$ and $\bigvee_{f \in S'} f(\rho(a))$ otherwise.

As a side note, when we prove the sufficient condition by Proposition 6.4, we get the liftability of λ with respect to the composition of not only $[T, \sigma]^{\mathbf{\Omega}}$ and $[F, \tau]^{\mathbf{\Omega}}$ but also $[T, \sigma]^{\mathbf{\Omega}}$ and $[F, \tau_a]^{\mathbf{\Omega}(a)}$ for each $a \in A$. The latter is stronger than the former. It enables us to compose bisimulations with respect to each $a \in A$.

Lemma 6.9. *Let α be a natural transformation $F \Rightarrow G$. Then α is a 2-cell $(F, \tau) \Rightarrow (G, \upsilon)$ in $A//\mathbf{CAT}$ if and only if α is a 2-cell $(F, \tau_a) \Rightarrow (G, \upsilon_a)$ in $1//\mathbf{CAT}$ for each $a \in A$.* $\square$

Proposition 6.10. *If a distributive law $\lambda\colon T \circ F^N \Rightarrow F \circ T$ is liftable w.r.t. $[T, \sigma]^{\mathbf{\Omega}} \circ ([F, \tau_a]^{\mathbf{\Omega}(a)})^N$ and $[F, \tau_a]^{\mathbf{\Omega}(a)} \circ [T, \sigma]^{\mathbf{\Omega}}$ for each $a \in A$, then σ lifts λ.* $\square$

6.1 Transferring Liftable Modalities

We next show that the *transfer* of liftable modalities from one $\mathbf{CLat}_\sqcap$-fibration to another. We first see the interaction between L/R and the Sp.

Lemma 6.11. *Let $(F, \dot{F}) \in \mathbf{CAT}^{\mathbf{CLat}_\sqcap}(p, q)$.*

1. *$L^{q, \dot{F}\mathbf{\Omega}} \circ \dot{F} \sqsubseteq \mathrm{Sp}^A(F, id) \circ L^{p,\mathbf{\Omega}}$. They are equal when $\dot{F}$ is full.*
2. *$\dot{F} \circ R^{p,\mathbf{\Omega}} \sqsubseteq R^{q,\dot{F}\mathbf{\Omega}} \circ \mathrm{Sp}^A(F, id)$. They are equal when $\dot{F}$ is fibered and preserving fibered meets.* $\square$

When $F = Id$, these properties are working behind Komorida et al.'s argument of transferring codensity bisimilarities [23]. Let $(Id, G) : p \to q$ be a 1-cell in $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ such that G is full, fibered and preserving fiberwise meets; such a 1-cell is called a *transfer situation* in [23]. Then the 1-cell commutes with codensity liftings, that is, $G \circ [F, \tau]^{\mathbf{\Omega}} = [F, \tau]^{G \circ \mathbf{\Omega}} \circ G$. From this commutativity, together with the adjoint lifting theorem of Hermida and Jacobs [17, Theorem 2.14] (see also [29, 23, 32] for relevant results), the preservation of the codensity bisimilarity is obtained:

$$G(\nu(c^* \circ [F, \tau]^{\mathbf{\Omega}})) = \nu(c^* \circ [F, \tau]^{G \circ \mathbf{\Omega}}) \quad (c : X \to F(X)).$$

Such G not only transfers the codensity bisimilarities, but also the liftability property of modalities.

Proposition 6.12. *Let $((\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be codensity bisimulation data and (T, λ) be an N-ary composition operation for F-coalgebras. Let $(Id, G) : p \to (q : \mathbb{F} \to \mathbb{B})$ be a 1-cell in $\mathbf{CAT}^{\mathbf{CLat}_\sqcap}$ such that G is full, fibered and preserving fibered meets. If a modality σ lifts λ along p, then in the $\mathbf{CLat}_\sqcap$-fibration with truth values $((\Omega, G \circ \mathbf{\Omega}) : Id_A \to (q : \mathbb{F} \to \mathbb{B}))$, σ lifts λ along q .*

The proof uses the equality $G \circ [T, \sigma]^{\mathbf{\Omega}} = [T, \sigma]^{\langle \mathbf{\Omega} \rangle_N} \circ G^N$ derivable from Lemma 6.11.

Example 6.13. Consider the $\mathbf{CLat}_\sqcap$-fibration $U_{\mathbf{ERel}}\colon \mathbf{ERel} \to \mathbf{Set}$ and the inclusion functor $i\colon \mathbf{EqRel} \hookrightarrow \mathbf{ERel}$. Then $(U_{\mathbf{EqRel}}, U_{\mathbf{ERel}}, i)$ is a transfer situation. In Example 6.5, we showed that $\sigma_\wedge$ lifts $\lambda^{\mathcal{P}}$ along $U_{\mathbf{EqRel}}$. Proposition 6.12 induces that $\sigma_\wedge$ lifts $\lambda^{\mathcal{P}}$ also along $U_{\mathbf{ERel}}$.

7 Examples of Lifting Distributive laws via Modalities

7.1 Bisimilarity Pseudometric for Deterministic Automata

We next visit *bisimilarity pseudometric* for deterministic automata [2, 5], which is a quantitative extension of language equivalence. We fix a weight $w \in \mathbb{I}$. We regard the interval $\mathbb{I}$ as a quantitative extension of binary truth values via the cast function $i : 2 \to \mathbb{I}$ defined by $i(\mathrm{true}) = 0$ and $i(\mathrm{false}) = 1$.

$\triangleright$ *Codensity bisimulation data* We use the same coalgebraic formulation of deterministic automata as F_{da}-coalgebras as Example 6.7. The difference from that section is that we employ the category of pseudometric spaces for modeling bisimilarity pseudometric. We adopt the $\mathbf{CLat}_{\sqcap}$-fibration with truth values $(\Omega, \mathbf{\Omega}) : Id_{\Sigma \uplus \{\epsilon\}} \to U_{\mathbf{PMet}}$ where $\mathbf{\Omega}$ constantly returns the Euclidean space $(\mathbb{I}, d_{\mathbb{I}})$ over the $[0, 1]$-interval. We pair it with the behavior functor F_{da} for deterministic automata and the following modality $\tau_a \colon F_{\mathrm{da}}(\mathbb{I}) \to \mathbb{I}$ to obtain codensity bisimulation data:

$$\tau_a(t, \rho) = \begin{cases} t & a = \epsilon \\ w \cdot \rho(a) & a \in \Sigma. \end{cases}$$

The codensity lifting $[F_{\mathrm{da}}, \tau]^{\mathbf{\Omega}}$ maps $(X, d) \in \mathbf{PMet}$ to the space over $F_{\mathrm{da}}X$ with the following pseudometric:

$$((t_1, \rho_1), (t_2, \rho_2)) \mapsto max\{|t_1 - t_2|, w \cdot \max_{a \in \Sigma} d(\rho_1(a), \rho_2(a))\}.$$

The $[F_{\mathrm{da}}, \tau]^{\mathbf{\Omega}}$-bisimilarity on a F_{da}-coalgebra $c : X \to F_{\mathrm{da}}(X)$ is a quantitative extension of the language equivalence on X. It maps $(x, x') \in X^2$ to 0 if the languages of x, x' are the same, and w^n otherwise where n is the minimum length of a word that is accepted from one and not from the other. This notion corresponds to language equivalence if $w = 1$.

$\triangleright$ *Binary one-step composition operation* We reuse $(\times, \lambda^{\mathrm{da}})$ in Example 3.6.

$\triangleright$ *Modality for* $\times$ We adopt $\sigma_\wedge \colon \mathbb{I}^2 \to \mathbb{I}$ given in §5.1.2. As we saw in §5.1.2, the binary codensity lifting $[\times, \sigma_\wedge]^{d_{\mathbb{I}}}$ maps $(X, d_1), (X, d_2) \in \mathbf{PMet}$ to the space on $X \times Y$ with the following pseudometric:

$$((x, y), (x', y')) \mapsto \max(d_1(x, x'), d_2(y, y')).$$

Proposition 7.1. *The modality $\sigma_\wedge$ lifts λ^{da} along $U_{\mathbf{PMet}}$.* □

This liftability yields a bound of bisimilarity pseudometrics of composite automata: given deterministic automata $c_1 \colon X_1 \to F_{\mathrm{da}}(X_1)$ and $c_2 \colon X_2 \to F_{\mathrm{da}}(X_2)$, the distance between (x, y) and (x', y') in the product automaton $\lambda^{\mathrm{da}} \circ (c_1 \times c_2)$ is bounded by $\max(v_1, v_2)$ where v_1 is the distance between x, x' in c_1 and v_2 is the distance between y, y' in c_2.

The proof is to show that two conditions in Theorem 6.2. See Appendix A.3.1 for its proof.

7.2 Similarity Pseudometric for Deterministic Automata

Let us consider similarity for the previous subsection. We get a similarity framework just by relaxing the symmetry condition of **PMet** and taking an asymmetric truth-value domain instead of the Euclidean distance $d_{\mathbb{I}}$. For omitted proofs, see Appendix A.3.2.

$\triangleright$ *Codensity bisimulation data* Let **LMet** be the category of Lawvere metric spaces (meaning asymmetric pseudometric spaces) and non-expansive maps. The forgetful functor $U_{\mathbf{LMet}} \colon \mathbf{LMet} \to \mathbf{Set}$ is a $\mathbf{CLat}_{\sqcap}$-fibration like $U_{\mathbf{PMet}}$. We adopt the $\mathbf{CLat}_{\sqcap}$-fibration with truth values $(\Omega, \mathbf{\Omega}) : Id_A \to U_{\mathbf{LMet}}$ such that $\mathbf{\Omega}$ constantly returns the space over the interval with the Lawvere metric $d^{\mathrm{as}}_{\mathbb{I}}(x, y) \coloneqq \max(0, y{-}x)$. We pair it with the behavior functor F_{da} for deterministic automata and the same modality τ in §7.1 to form codensity bisimulation data.

The codensity lifting $[F_{\mathrm{da}}, \tau]^{\mathbf{\Omega}}$ maps $(X, d) \in \mathbf{LMet}$ to the space on $F_{\mathrm{da}}(X)$ with the following Lawvere metric:

$$((t_1, \rho_1), (t_2, \rho_2)) \mapsto max\{d^{\mathrm{as}}_{\mathbb{I}}(i(t_1), i(t_2)), w \cdot \max_{a\in\Sigma} d(\rho_1(a), \rho_2(a))\};$$

recall that $i : 2 \to \mathbb{I}$ is the cast function (§7.1). Note that $d^{\mathrm{as}}_{\mathbb{I}}$ maps $(\mathrm{true}, \mathrm{false})$ to 1 and the others to 0. In §7.1 we used the euclidean distance $d_{\mathbb{I}}$ instead of $d^{\mathrm{as}}_{\mathbb{I}}$, which also maps $(\mathrm{false}, \mathrm{true})$ to 1. It makes a difference in the bisimilarity as below.

The $[F_{\mathrm{da}}, \tau]^{\mathbf{\Omega}}$-bisimilarity on a F_{da}-coalgebra $c : X \to F_{\mathrm{da}}(X)$ is a similarity pseudometric on X for the deterministic automaton c. It maps $(x, x') \in X^2$ to 0 if the language of x is included in that of x', and w^n otherwise where n is the minimum length of a word that is accepted from x and not from x'. A bisimialrity pseudometric in §7.1 detects words that are accepted from x' and not from x while a similarity pseudometric doesn't.

$\triangleright$ *Binary one-step composition operation* We reuse $(\times, \lambda^{\mathrm{da}})$ in Example 3.6.

$\triangleright$ *Modality for* $\times$ We adopt $\sigma_\wedge \colon \mathbb{I}^2 \to \mathbb{I}$ given in §5.1.2. The binary codensity lifting $[\times, \sigma_\wedge]^{d^{\mathrm{as}}_{\mathbb{I}}}$ maps (X, d_1) and (Y, d_2) to the Lawvere metric on $X \times Y$

$$((x, y), (x', y')) \mapsto \max(d_1(x, x'), d_2(y, y')).$$

Proposition 7.2. *The modality $\sigma_\wedge$ lifts λ^{da} along $U_{\mathbf{LMet}}$.* $\square$

This liftability yields a bound of similarity pseudometrics of composite automata: given deterministic automata $c_1 \colon X_1 \to F_{\mathrm{da}}(X_1)$ and $c_2 \colon X_2 \to F_{\mathrm{da}}(X_2)$, the distance between (x, y) and (x', y') in the product automaton $\lambda^{\mathrm{da}} \circ (c_1 \times c_2)$ is bounded by $\max(v_1, v_2)$ where v_1 is the distance between x, x' in c_1 and v_2 is the distance between y, y' in c_2.

The proof is to show that two conditions in Theorem 6.2. See Appendix A.3.2 for its proof.

7.3 Bisimulation Metric for Markov Decision Processes

Bisimulation metric is a quantitative notion of behavioral equivalences for probabilistic systems [9, 7, 34]. We shall see liftability of a distributive law for composing bisimulation metrics [9].

$\triangleright$ *Codensity bisimulation data* We first set-up coalgebraic formulation of Markov decision processes (MDPs) and bisimulation metric. Let us write $\mathcal{D}\colon \mathbf{Set} \to \mathbf{Set}$ for the probability distribution functor which maps X to the set of distributions on X. Then MDPs are naturally modeled by $\mathcal{P} \circ \mathcal{D}$-coalgebras $c : X \to \mathcal{P}\mathcal{D}(X)$. Hereafter we omit the composition between $\mathcal{P}$ and $\mathcal{D}$.

For bisimulation metrics, let us provide two codensity bisimulation data for the Kantorovich and Hausdorff lifting.

The first codensity bisimulation data is $((\mathbb{I}, d_{\mathbb{I}}) : Id_1 \to U_{\mathbf{PMet}}, \mathcal{D}, e)$ where e is the expectation function $e\colon \mathcal{D}\mathbb{I} \to \mathbb{I}$. The codensity lifting $[\mathcal{D}, e]^{d_{\mathbb{I}}}\colon \mathbf{PMet} \to \mathbf{PMet}$ maps $d \in \mathbf{PMet}_X$ to the Kantorovich pseudometric $\mathcal{K}(d)$ on $\mathcal{D}X$:

$$(\mu_1, \mu_2) \mapsto \sup_{k \in \mathbf{PMet}((X,d),(\mathbb{I},d_{\mathbb{I}}))} |\Sigma_{x\in X} k(x) \cdot (\mu_1(x) - \mu_2(x))|.$$

The second codensity bisimulation data is $((\mathbb{I}, d_{\mathbb{I}}) : Id_1 \to U_{\mathbf{PMet}}, \mathcal{P}, \inf)$. The codensity lifting $[\mathcal{P}, \inf]^{d_{\mathbb{I}}}\colon \mathbf{PMet} \to \mathbf{PMet}$ mapping $d \in \mathbf{PMet}_X$ to the Hausdorff $\mathcal{H}(d)$ pseudometric on $\mathcal{P}X$:

$$\begin{aligned}(A_1, A_2) \mapsto & \sup_{k \in \mathbf{PMet}((X,d),(\mathbb{I},d_{\mathbb{I}}))} |\inf_{a_1 \in A_1} k(a_1) - \inf_{a_2 \in A_2} k(a_2)| \\ &= \max(\sup_{a_1\in A_1} \inf_{a_2 \in A_2} d(a_1, a_2), \sup_{a_2 \in A_2} \inf_{a_1 \in A_1} d(a_1, a_2))\end{aligned}$$

Notice that $\sup \emptyset = 0$ and $\inf \emptyset = 1$, hence $\mathcal{H}(d)(\emptyset, A_2) = 1$ for $A_2 \neq \emptyset$. The equality is proved in [23, Appendix 3].

Then a $[\mathcal{P}, \inf]^{d_{\mathbb{I}}} \circ [\mathcal{D}, e]^{d_{\mathbb{I}}}$-bisimulation on c is called a *bisimulation metric*, and the bisimilarity is called a *bisimilarity metric* [7]. It maps a pair of states (x, y) to a distance that measures quantitative analogue to behavioral equivalences.

$\triangleright$ *Binary one-step composition operation* We have seen the binary one-step composition $(\times, \lambda^{\mathcal{P}})$ for $\mathcal{P}$ in Example 6.5. We next introduce the one for $\mathcal{D}$ by

$$\lambda^{\mathcal{D}}(\mu, \mu') \coloneqq (x, y) \mapsto \mu(x) \cdot \mu'(y).$$

This induces the binary one-step composition $(\times, \lambda^{\mathcal{P}\mathcal{D}})$ for $\mathcal{P}\mathcal{D}$-coalgebras by

$$\lambda^{\mathcal{P}\mathcal{D}} \triangleq (\mathcal{P} \circ \lambda^{\mathcal{D}}) \bullet (\lambda^{\mathcal{P}} \circ \mathcal{D}^2).$$

$\triangleright$ *Modality for* $\times$ From the discussion above, it is sufficient to find a modality σ for $\times$ that lifts both distributive laws $\lambda^{\mathcal{P}}$ and $\lambda^{\mathcal{D}}$. Then both distributive laws $\lambda^{\mathcal{P}}$ and $\lambda^{\mathcal{D}}$ become liftable, hence so is $\lambda^{\mathcal{P}\mathcal{D}}$. Among several modalities introduced in §5.1.2, only $\sigma_{\oplus}$ lifts both binary operations. Therefore:

Proposition 7.3. *The distributive law $\lambda^{\mathcal{PD}}$ is liftable on countable pseudometric spaces w.r.t. $[\times, \sigma_\oplus]^{d_\mathbb{I}} \circ ([\mathcal{P}, \inf]^{d_\mathbb{I}} \circ [\mathcal{D}, e]^{d_\mathbb{I}})^N$ and $[\mathcal{P}, \inf]^{d_\mathbb{I}} \circ [\mathcal{D}, e]^{d_\mathbb{I}} \circ [\times, \sigma_\oplus]^{d_\mathbb{I}}$.* □

As a consequence of the proposition above, given two MDPs $c_i : X_i \to \mathcal{PD}(X_i)$ over at most countable states X_i $(i = 1, 2)$, if d_i is a bisimulation metric for c_i, then the mapping

$$d((x, y), (x', y')) \triangleq \sigma_\oplus(d_1(x, x'), d_2(y, y'))$$

becomes a bisimulation metric for the composite MDP $\lambda^{\mathcal{PD}} \circ (c_1 \times c_2)$. Moreover, the above proposition implies preservation of *bisimilarity metrics.* If d_i is the bisimilarity metric for c_i, then the bisimilarity metric for the composite MDP $\lambda^{\mathcal{PD}} \circ (c_1 \times c_2)$ is bounded by d. This bound itself was shown in [13]. We give a proof in terms of composition of distributive laws.

We prove Proposition 7.3 by checking if $\sigma_\oplus$ lifts both binary operations $(\times, \lambda^{\mathcal{P}})$ and $(\times, \lambda^{\mathcal{D}})$. Other σ written in §5.1.2 do not satisfy all conditions in the lemma.

Lemma 7.4. *Let $\sigma \colon \mathbb{I}^2 \to \mathbb{I}$ be a modality satisfying* (6)*, so that $[\times, \sigma]^{d_\mathbb{I}} = \times^\sigma$.*

1. *The modality σ lifts $\lambda^{\mathcal{P}}$ if σ is concave, σ preserves infimums, and $\sigma(1, x) = \sigma(x, 1) = 1$ for any $x \in \mathbb{I}$.*
2. *The modality σ lifts $\lambda^{\mathcal{D}}$ on countable pseudometric spaces if σ is concave.*

□

The second statement comes from the following lemma, which is a direct adaptation of [13, Theorem 2.20] in the context of pseudometric spaces. It can be proved in almost the same way as the original theorem; here we rely on the Kantorovich-Rubinstein duality theorem on at most countable sets [35].

Lemma 7.5. *Let X, Y be at most countable sets, $(X, d_1), (Y, d_2) \in$ **PMet**, and $\sigma : \mathbb{I}^2 \to \mathbb{I}$ be a concave function satisfying* (6)*. Then for each $\mu_1, \mu_1' \in \mathcal{D}X$ and $\mu_2, \mu_2' \in \mathcal{D}Y$,*

$$\mathcal{K}(d)(\lambda^{\mathcal{D}}_{X,Y}(\mu_1, \mu_2), \lambda^{\mathcal{D}}_{X,Y}(\mu_1', \mu_2')) \leq \sigma(\mathcal{K}(d_1)(\mu_1, \mu_1'), \mathcal{K}(d_2)(\mu_2, \mu_2')) \qquad (8)$$

where d is the pseudometric on $X \times Y$ given by $d((x, y), (x', y')) \coloneqq \sigma(d_1(x, x'), d_2(y, y'))$. □

We didn't apply Theorem 6.2 to this case because as of now, it remains unsolved that the modality $\sigma_\oplus$ satisfies the second condition in Theorem 6.2 while it satisfies the first one.

8 Composing Codensity Games

We introduce a composition of codensity games, and show that it preserves game invariants. As a consequence, we provide an alternative proof of the preservation of bisimilarities (the inequality shown in Corollary 3.10) under the sufficient condition in Theorem 6.2. Equivalently, the preservation of bisimilarities is restated as follows:

$$\forall P_1, \cdots, P_N \ . \ \bigwedge_{i \in N} \big(P_i \sqsubseteq \nu(c_i^* \circ [F,\tau]^{\mathbf{\Omega}})\big) \\ \implies [T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i}) \sqsubseteq \nu(T_\lambda(c_1, \cdots, c_N)^* \circ [F,\tau]^{\mathbf{\Omega}}).$$

We call P_i a *witness* of bisimilarity, and call the above formula the *compositionality of witnesses*.

Definition 8.1 (witness)**.** Let $((\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be codensity bisimulation data and $c\colon X \to F(X)$ be an F-coalgebra. An object $P \in \mathbb{E}_X$ is a *witness* of the codensity bisimilarity on c if $P \sqsubseteq \nu\big(c^* \circ [F,\tau]^{\mathbf{\Omega}}\big)$.

8.1 Compositionality of Invariants

We briefly recall definitions and results on codensity games. We reserve the variable i for N-indices, and write $\overrightarrow{x_i}$ for the sequence of mathematical entities x_i (such as morphisms and objects) indexed by $i \in N$.

Definition 8.2. A *safety game* is a game $\mathcal{G} = (Q_{\mathbf{D}}, Q_{\mathbf{S}}, E)$ played by two players $\mathbf{D}$ (Duplicator) and $\mathbf{S}$ (Spoiler) where $Q_{\mathbf{D}}, Q_{\mathbf{S}}$ are sets of positions of $\mathbf{D}, \mathbf{S}$ respectively and $E \subseteq (Q_{\mathbf{D}} \times Q_{\mathbf{S}} \cup Q_{\mathbf{S}} \times Q_{\mathbf{D}})$ is a set of possible moves. A play of $\mathcal{G}$ is a finite or infinite sequence of positions $q_0, q_1, \ldots$ such that $(q_i, q_{i+1}) \in E$ for each i. The player $\mathbf{S}$ wins a play if the sequence is finite and the last position is in $Q_{\mathbf{D}}$, and the player $\mathbf{D}$ wins the game otherwise.

A *strategy* of $\mathbf{D}$ is a partial function $s\colon Q^* \times Q_{\mathbf{D}} \rightharpoonup Q_{\mathbf{S}}$ where Q^* is the set of finite sequences of $Q_{\mathbf{D}} \uplus Q_{\mathbf{S}}$. A strategy s of $\mathbf{D}$ is *winning* from q if $\mathbf{D}$ wins any play $q_0, q_1, \ldots$ such that $q_0 = q$ and $q_{i+1} = s(q_0, q_1, \ldots, q_i)$ for each $q_i \in Q_{\mathbf{D}}$. A position $q \in Q_{\mathbf{S}}$ is *winning* (for $\mathbf{D}$) if there exists a winning strategy s of $\mathbf{D}$ from q.

Winning positions on a safety games are characterizes by *invariants*, which also induce winning strategies of $\mathbf{D}$.

Definition 8.3. A set $\mathcal{V} \subseteq Q_{\mathbf{S}}$ is an *invariant* for $\mathbf{D}$ if for each $q \in \mathcal{V}$ and $q' \in Q_{\mathbf{D}}$, $(q, q') \in E$ implies that there is $q'' \in \mathcal{V}$ such that $(q', q'') \in E$.

Proposition 8.4. *A position $q \in Q_{\mathbf{S}}$ is winning if and only if there is an invariant $\mathcal{V}$ for $\mathbf{D}$ such that $q \in \mathcal{V}$.* □

Codensity games are safety games that are induced by codensity liftings and coalgebras.

Definition 8.5 (codensity games [23])**.** Let $((\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be codensity bisimulation data and $c\colon X \to F(X)$ be a F-coalgebra. The *codensity game* $\mathcal{G}_c$ is the safety game $(Q_{\mathbf{D}}, Q_{\mathbf{S}}, E)$ played by two players $\mathbf{D}, \mathbf{S}$ where

$Q_{\mathbf{D}} \coloneqq \{(a,k) \mid a \in A, k \in \mathbb{B}(X, \Omega(a))\}$, $Q_{\mathbf{S}} \coloneqq \mathrm{Obj}(\mathbb{E}_X)$, and $E \coloneqq \to_{\mathbf{S}} \cup \to_{\mathbf{D}}$ is given by

$$P \to_{\mathbf{S}} (a,k) \text{ if } \tau_a \circ Fk \circ c \colon P \dot{\nrightarrow} \mathbf{\Omega}(a),$$
$$(a,k) \to_{\mathbf{D}} P' \text{ if } k \colon P' \dot{\nrightarrow} \mathbf{\Omega}(a).$$

We show characterizations of invariants and winning positions of codensity games, which are required in the latter part.

Proposition 8.6 ([23]). *1. A set $\mathcal{V} \subseteq \mathrm{Obj}(\mathbb{E}_X)$ is an invariant for $\mathbf{D}$ in the codensity game $\mathcal{G}_c$ if and only if $\bigsqcup_{P \in \mathcal{V}} P$ is a $[F,\tau]^{\mathbf{\Omega}}$-bisimulation on c.*

2. A position $P \in \mathbb{E}_X$ is winning in $\mathcal{G}_c$ if and only if P is a witness of the codensity bisimilarity on c. □

Now we move to a codensity game of a composite coalgebra. Given F-coalgebras $c_i \colon X_i \to F(X_i)$ $(i \in N)$, the codensity game $\mathcal{G}_{T_\lambda(\overrightarrow{c_i})}$ of the composite coalgebra $T_\lambda(\overrightarrow{c_i})$ is given by

$$(P \in \mathbb{E}_{T(\overrightarrow{X_i})}) \to_{\mathbf{S}} (a,k) \text{ if } \tau_a \circ Fk \circ (\lambda \circ T(\overrightarrow{c_i})) \colon P \dot{\nrightarrow} \mathbf{\Omega}(a),$$
$$(a \in A, k \in \mathbb{B}(T(\overrightarrow{X_i}), \Omega(a))) \to_{\mathbf{D}} P' \text{ if } k \colon P' \dot{\nrightarrow} \mathbf{\Omega}(a).$$

In general, the codensity game $\mathcal{G}_{T_\lambda(\overrightarrow{c_i})}$ may not be composed from each game of $\mathcal{G}_{c_i}$: in particular, positions and moves may not be composed by products, and therefore invariants may not be.

Instead, we define a composite codensity game (Definition 8.7), where invariants are compositionally preserved:

Definition 8.7 (composite codensity games)**.** Let $((\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbb{E} \to \mathbb{B}), F, \tau)$ be a codensity bisimulation data and $c_i \colon X_i \to F(X_i)$ $(i \in N)$ be F-coalgebras. The *composite codensity game* $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ is the safety game $(Q_{\mathbf{D}}, Q_{\mathbf{S}}, E)$ by two players $\mathbf{D}, \mathbf{S}$ where $Q_{\mathbf{D}} \coloneqq \{(a, \overrightarrow{k_i}) \mid a \in A, k_i \in \mathbb{C}(X_i, \Omega(a))\}$, $Q_{\mathbf{S}} \coloneqq \prod_i \mathrm{Obj}(\mathbb{E}_{X_i})$, and $E \coloneqq \to_{\mathbf{S}} \cup \to_{\mathbf{D}}$ is given by

$$\overrightarrow{P_i} \to_{\mathbf{S}} (a, \overrightarrow{k_i}) \text{ if } \sigma_a \circ T(\overrightarrow{\tau_a \circ Fk_i \circ c_i}) \colon [T,\sigma]^{\mathbf{\Omega}} \overrightarrow{P_i} \dot{\nrightarrow} \mathbf{\Omega}(a),$$
$$(a, \overrightarrow{k_i}) \to_{\mathbf{D}} \overrightarrow{P'_i} \text{ if there is } i \in N \text{ s.t. } k_i \colon P'_i \dot{\nrightarrow} \mathbf{\Omega}(a).$$

The following proposition shows that we can construct an invariant by composing invariants of component games $\mathcal{G}_{c_i}$.

Proposition 8.8. *If each $\mathcal{V}_i \subseteq \mathrm{Obj}(\mathbb{E}_{X_i})$ is an invariant of $\mathcal{G}_{c_i}$ for $\mathbf{D}$ then $\prod_{i \in N} \mathcal{V}_i$ is an invariant of $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ for $\mathbf{D}$.* □

8.2 Preservation of Bisimilarities

Under the sufficient condition in Theorem 6.2, invariants of composite codensity games $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ also characterizes codensity bisimilarities on composite coalgebras

$T_\lambda(\overrightarrow{c_i})$. We introduce a join-closure operation for invariants: for a set $\mathcal{V} \subseteq \prod_{i\in N} \mathrm{Obj}(\mathbb{E}_{X_i})$, we write $\overline{\mathcal{V}}$ for $\mathcal{V} \cup \left\{ \left(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\right)_{i\in N} \right\}$. We remark that if a set $\mathcal{V}$ is an invariant of a codensity game $\mathcal{G}_c$ for $\mathbf{D}$ then so is $\overline{\mathcal{V}}$.

Theorem 8.9. *Assume that the sufficient condition in Theorem 6.2 holds and let $\mathcal{V} \subseteq \prod_{i\in N} \mathrm{Obj}(\mathbb{E}_{X_i})$ be a set.*

1. *The set $\mathcal{V}$ is an invariant for $\mathbf{D}$ in $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ if*
   $\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} [T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})$ *is a $[F,\tau]^{\mathbf{\Omega}}$-bisimulation on $T_\lambda(\overrightarrow{c_i})$.*

2. *The set $\overline{\mathcal{V}}$ is an invariant for $\mathbf{D}$ in $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ if and only if*
   $\bigsqcup_{\overrightarrow{P_i}\in\overline{\mathcal{V}}} [T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})$ *is a $[F,\tau]^{\mathbf{\Omega}}$-bisimulation on $T_\lambda(\overrightarrow{c_i})$.* □

Corollary 8.10. *Assume that the sufficient condition in Theorem 6.2 holds. If $\overrightarrow{P_i}$ is a winning position for $\mathbf{D}$ with an invariant $\overline{\mathcal{V}}$ then $[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})$ is a witness on $T_\lambda(\overrightarrow{c_i})$.* □

Finally, through a composite codensity game, we prove the compositionality of witnesses (and equivalently the preservation of bisimilarities). If each P_i is a witness on the codensity bisimilarity on c_i, P_i is a winning position for $\mathbf{D}$ in $\mathcal{G}_{c_i}$ by Proposition 8.6. Then Proposition 8.4 ensures the existence of invariants $\overline{\mathcal{V}}_i$ of $\mathcal{G}_{c_i}$ such that $P_i \in \overline{\mathcal{V}}_i$. Proposition 8.8 and $\prod_{i\in N} \overline{\mathcal{V}}_i = \overline{\prod_{i\in N} \mathcal{V}_i}$ imply that $\overrightarrow{P_i}$ is a winning position with the invariant $\overline{\prod_{i\in N} \mathcal{V}_i}$. Therefore, Corollary 8.10 concludes that $[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})$ is a witness of the codensity bisimilarity on $T_\lambda(\overrightarrow{c_i})$.

9 Conclusion

In this paper, we have presented generalized codensity liftings in a 2-categorical framework. This development has facilitated the derivation of liftings of structure functors, especially binary product functors, and enhanced our understanding and manipulation of codensity liftings.

By integrating the 2-categorical framework of generalized codensity liftings and structure functors through codensity liftings, we were able to identify a sufficient condition to lift a distributive law between functors into one between codensity liftings. Additionally, we have investigated our sufficient condition from the perspective of compositional reasoning of codensity games.

For future work, we will continue to explore and expand the scope of our approach, particularly considering different structure functors or different notions of compatibility as they appear in the study of up-to techniques.

Recently, higher-order abstract GSOS frameworks [14, 33, 15] are actively studied: they extend behavior endofunctors to bifunctors and use certain dinatural transformations for distributive laws. Seeking a higher-order extension of our framework, that is, studying a liftability of distributive laws along codensity liftings in a higher-order setting is an interesting future direction.

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A Omitted Proofs

A.1 Omitted Proofs for Section 5

Proposition A.1 (Proof for Example. 5.2)**.** *In Example. 5.2, the* $\mathbf{CLat}_\sqcap$*-fibrations with truth values* $(2,(2,\mathrm{Eq}_2)) : Id_1 \to U_{\mathbf{EqRel}}$ *and* $(2,(2,\{\emptyset,\{\mathrm{true}\},2\})) : Id_1 \to U_{\mathbf{Top}}$ *satisfy* $R^{p,\mathbf{\Omega}} L^{p,\mathbf{\Omega}} = \mathrm{id}$.

Proof. $R^{p,\mathbf{\Omega}} L^{p,\mathbf{\Omega}} = \mathrm{id}$ is equivalent to $\bigsqcap_{k\colon \mathbf{\Omega}\to\mathbf{\Omega}} k^*\mathbf{\Omega} = \mathbf{\Omega}$.

- $((2,(2,\mathrm{Eq}_2)) : Id_1 \to U_{\mathbf{EqRel}})$: $\bigsqcap_{k\colon \mathbf{\Omega}\to\mathbf{\Omega}} k^*\mathbf{\Omega} = \{(x,y) \mid \forall k\colon \mathbf{\Omega} \to \mathbf{\Omega}.\ kx = ky\} = \mathbf{\Omega}$ holds. The last equality is because (x,y) in l.h.s. satisfies $x = y$ by letting $k \coloneqq \mathrm{id}$, and (x,y) in r.h.s. satisfies $kx = ky$ for each $k\colon \mathbf{\Omega} \to \mathbf{\Omega}$.

- $((2,(2,\{\emptyset,\{\mathrm{true}\},2\})) : Id_1 \to U_{\mathbf{Top}})$: $\bigsqcap_{k\colon \mathbf{\Omega}\to\mathbf{\Omega}} k^*\mathbf{\Omega}$ is the topology generated by $\bigcup_{k\colon \mathbf{\Omega}\to\mathbf{\Omega}} \{k^{-1}A \mid A \in \mathbf{\Omega}\}$. $\mathbf{\Omega}$ is included in the topology by $k \coloneqq \mathrm{id}\colon \mathbf{\Omega} \to \mathbf{\Omega}$, and the topology is included in $\mathbf{\Omega}$ because $k^{-1}A \in \mathbf{\Omega}$ for each $k\colon \mathbf{\Omega} \to \mathbf{\Omega}$ and $A \in \mathbf{\Omega}$.

$\square$

Proof of Proposition 5.3.

1. $(\Rightarrow)$ Suppose $\times^\sigma\colon \mathbf{PMet}^2 \to \mathbf{PMet}$ is a lifting of the product functor $\times$. Consider arbitrary $a,b,c,d \in [0,1]$. For $x,x',x'' \in \{a,b,c,d\}$ we write $P_x \in \mathbf{PMet}_2$ for the pseudometric satisfying $P_x(0,1) = x$, and $P_{x,x',x''} \in \mathbf{PMet}_3$ (when $x + x' \le x''$) for the one mapping $(0,1)$ to x, $(1,2)$ to x', $(0,2)$ to $x + x'$ where $2 = \{0,1\}$ and $3 = \{0,1,2\}$. The modality σ is monotone because $a \le c$ and $b \le d$ imply $\sigma(a,b) = \sigma(P_a(0,1),P_b(0,1)) \le \sigma(P_c(0,1),P_d(0,1)) = \sigma(c,d)$ by monotonicity of the functor $\times^\sigma$. $\sigma(0,0) = \times^\sigma(P_a,P_a)((0,0),(0,0)) = 0$ because $\times^\sigma(P_a,P_a)$ is a pseudometric. $\sigma(a,b) = \times^\sigma(P_{c,|a-c|,a}, P_{d,|b-d|,b})((0,0),(2,2)) \le \times^\sigma(P_{c,|a-c|,a}, P_{d,|b-d|,b})((0,0),(1,1)) + \times^\sigma(P_{c,|a-c|,a}, P_{d,|b-d|,b})((1,1),(2,2)) = \sigma(|a-c|,|b-d|) + \sigma(c,d)$. Thus we have $\sigma(a,b) - \sigma(c,d) \le \sigma(|a-c|,|b-d|)$.

   $(\Leftarrow)$ Suppose σ satisfies (6). Then $\times^\sigma(P,Q)$ is a pseudometric for each $P,Q \in \mathbf{PMet}$: For arbitrary $x,x',x'' \in pP$, $y,y',y'' \in pQ$,

   - $\times^\sigma(P,Q)((x,y),(x,y)) = \sigma(P(x,x),Q(y,y)) = \sigma(0,0) = 0$,
   - $\times^\sigma(P,Q)((x,y),(x',y')) = \sigma(P(x,x'),Q(y,y')) = \sigma(P(x',x),Q(y',y)) = \times^\sigma(P,Q)((x',y'),(x,y))$,
   - $\times^\sigma(P,Q)((x,y),(x'',y'')) = \sigma(P(x,x''),Q(y,y'')) \le \sigma(P(x,x')+P(x',x''),Q(y,y')+Q(y',y'')) \le \sigma(P(x,x'),Q(y,y'))+\sigma(P(x',x''),Q(y',y'')) = \times^\sigma(P,Q)((x,y),(x',y'))+\times^\sigma(P,Q)((x',y'),(x'',y''))$.

   For each $f\colon P \to P'$ and $g\colon Q \to Q'$ in $\mathbf{PMet}$, $f \times g\colon\ \times^\sigma (P,Q) \dot{\to} \times^\sigma (P',Q')$ by monotonicity of σ.

2. The third condition of (6) is equivalent to $|\sigma(a,b)-\sigma(c,d)| \le \sigma(|a-c|,|b-d|)$ for each $a,b,c,d \in [0,1]$, and it corresponds to $\sigma\colon\ \times^\sigma (d_{[0,1]}) \dot{\to} d_{[0,1]}$.

Therefore it induces $\times^\sigma(P,Q) \sqsubseteq [\times,\sigma]^{d_{[0,1]}}(P,Q)$ for each $P,Q \in \mathbf{PMet}$ by Proposition 4.7. For each $P,Q \in \mathbf{PMet}$, $\times^\sigma(P,Q) \sqsupseteq [\times,\sigma]^{d_{[0,1]}}(P,Q)$ because for each $x,x' \in pP$ and $y,y' \in pQ$, letting $k_1(t) \coloneqq P(x,t)$ and $k_2(t) \coloneqq Q(y,t)$,

$$\begin{aligned}\times^\sigma(P,Q)((x,y),(x',y')) &= \sigma(P(x,x'),Q(y,y'))\\ &= \sigma(k_1(x'),k_2(y'))\\ &= |\sigma(k_1(x),k_2(y)) - \sigma(k_1(x'),k_2(y'))|\\ &= (\sigma\circ k_1\times k_2)^* d_{[0,1]}((x,y),(x',y'))\\ &\sqsupseteq [\times,\sigma]^{d_{[0,1]}}(P,Q)((x,y),(x',y')).\end{aligned}$$

$\square$

Proposition A.2. *The modalities $\sigma_\oplus,\sigma_\vee$ defined by $\sigma_\oplus(a,b) \coloneqq 1-(1-a)(1-b)$ and $\sigma_\vee(a,b) \coloneqq \max(a,b)$ satisfy* (6).

Proof. For each $\sigma \in \{\sigma_\oplus,\sigma_\vee\}$, It is easy to show that σ is monotone and $\sigma(0,0) = 0$. Here we only show $\sigma(a,b) - \sigma(c,d) \le \sigma(|a-c|,|b-d|)$ for each $a,b,c,d \in [0,1]$.

- ($\sigma = \sigma_\oplus$): Let $v_1 \coloneqq a$, $\epsilon_1 \coloneqq a-c$, $v_2 \coloneqq b$, and $\epsilon_2 \coloneqq b-d$. Then $\max(0,\epsilon_i) \le v_i \le \min(1,1+\epsilon_i)$ $(i=1,2)$ and (lhs) $= \sigma_\oplus(v_1,v_2) - \sigma_\oplus(v_1-\epsilon_1,v_2-\epsilon_2) = \epsilon_1(1-v_2)+\epsilon_2(1-v_1)+\epsilon_1\epsilon_2$.

  If $\epsilon_1,\epsilon_2 \ge 0$, (lhs) $\le \epsilon_1(1-\epsilon_2)+\epsilon_2(1-\epsilon_1)+\epsilon_1\epsilon_2 = \sigma_\oplus(\epsilon_1,\epsilon_2) \le$ (rhs).

  If $\epsilon_1 \ge 0$ and $\epsilon_2 \le 0$, (lhs) $\le \epsilon_1(1-\epsilon_2)+\epsilon_2(1-1)+\epsilon_1\epsilon_2 = \epsilon_1 \le$ (rhs).

  If $\epsilon_1 \le 0$ and $\epsilon_2 \ge 0$, (lhs) $\le \epsilon_1(1-1)+\epsilon_2(1-\epsilon_1)+\epsilon_1\epsilon_2 = \epsilon_2 \le$ (rhs).

  If $\epsilon_1 \le 0$ and $\epsilon_2 \le 0$, (lhs) $\le \epsilon_1(1-(1+\epsilon_2))+\epsilon_2(1-(1+\epsilon_1))+\epsilon_1\epsilon_2 = -\epsilon_1\epsilon_2 \le 0 \le$ (rhs).

- ($\sigma = \sigma_\vee$): Since $\sigma_\vee(x,y) = \sigma_\vee(y,x)$ for each $x,y \in [0,1]$, we can assume $a \le b$ without loss of generality.

  If $c \le d$, (lhs) $= b-d \le |b-d| \le$ (rhs).

  If $c > d$, (lhs) $= b-c$. If $b \le c$, (lhs) $\le 0 \le$ (rhs). Otherwise ($b \ge c$), $b-c \le b-d$ because $d \le c \le b$. Therefore (lhs) $= b-c \le |b-d| \le$ (rhs).

$\square$

Proof of Proposition 5.4. For each $(P_i)_{i\in N} \in \mathbb{E}^N$ and $a \in A$, $\{\sigma\circ T((k_i)_{i\in N}) \mid k_i\colon P_i \dot\to \mathbf{\Omega}(a)\} = \{\sigma\circ T((f\circ k_i)_{i\in N}) \mid k_i\colon P_i\dot\to\mathbf{\Omega}(a)\}$ because for each $(k_i\colon P_i\dot\to\mathbf{\Omega}(a))_{i\in N}$, $(f\circ k_i)_{i\in N}, \left(f^{-1}\circ k_i\right)_{i\in N} : P_i\dot\to\mathbf{\Omega}(a)$ by $f^*\mathbf{\Omega} = \mathbf{\Omega} = (f^{-1})^*\mathbf{\Omega}$. It induces $\mathrm{Sp}^A(T,\sigma)\circ L^{p,\langle\mathbf{\Omega}\rangle_N} = \mathrm{Sp}^A(T,\sigma\circ T(f^N))\circ L^{p,\langle\mathbf{\Omega}\rangle_N}$. Therefore, $[T,\sigma]^{\mathbf{\Omega}} = R^{p,\langle\mathbf{\Omega}\rangle_N}\circ \mathrm{Sp}^A(T,\sigma)\circ L^{p,\langle\mathbf{\Omega}\rangle_N} = R^{p,\langle\mathbf{\Omega}\rangle_N}\circ\mathrm{Sp}^A(T,\sigma\circ T(f^N))\circ L^{p,\langle\mathbf{\Omega}\rangle_N} = [T,\sigma\circ T(f^N)]^{\mathbf{\Omega}}$.

Similarly, for each $S \in \mathrm{Sp}^A(\mathbb{B}^N,\mathbf{\Omega})$ and $a \in A$, $\bigsqcap_{a\in A,(k_i)_{i\in N}\in S}(\sigma\circ T((k_i)_{i\in N}))^*\mathbf{\Omega}(a) = \bigsqcap_{a\in A,(k_i)_{i\in N}\in S}(\sigma\circ T((k_i)_{i\in N}))^* f^*\mathbf{\Omega}(a) = \bigsqcap_{a\in A,(k_i)_{i\in N}\in S}(f\circ\sigma\circ T((k_i)_{i\in N}))^*\mathbf{\Omega}(a)$. It induces $R^{p,\langle\mathbf{\Omega}\rangle_N}\circ\mathrm{Sp}^A(T,\sigma) = R^{p,\langle\mathbf{\Omega}\rangle_N}\circ\mathrm{Sp}^A(T,f\circ\sigma)$. Therefore, $[T,\sigma]^{\mathbf{\Omega}} = R^{p,\langle\mathbf{\Omega}\rangle_N}\circ\mathrm{Sp}^A(T,\sigma)\circ L^{p,\langle\mathbf{\Omega}\rangle_N} = R^{p,\langle\mathbf{\Omega}\rangle_N}\circ\mathrm{Sp}^A(T,f\circ\sigma)\circ L^{p,\langle\mathbf{\Omega}\rangle_N} = [T,f\circ\sigma]^{\mathbf{\Omega}}$. $\square$

A.2 Omitted Proofs for Section 6

Lemma A.3. *In the setting of Proposition 6.4, if the three conditions in the proposition hold, then for each $a \in A$,*

$$R^{p,\mathbf{\Omega}(a)} \circ \mathrm{Sp}^1(F,\tau_a)(S_a) \sqsubseteq R^{p,\mathbf{\Omega}(a)} \circ \mathrm{Sp}^1(F,\tau_a) \circ L^{p,\mathbf{\Omega}(a)} \circ R^{p,\mathbf{\Omega}}(S).$$

Proof. For each $a \in A$ and $k \colon \prod_{a' \in A, k' \in S_{a'}} k'^{*}\mathbf{\Omega}(a') \to \mathbf{\Omega}(a)$,

$$\begin{aligned}
\prod_{k' \in S_a} (\tau_a \circ Fk')^{*}\mathbf{\Omega}(a) &\sqsubseteq \prod_{k' \in S'_k} (\tau_a \circ Fk')^{*}\mathbf{\Omega}(a) && \text{since } S'_k \subseteq S_a \\
&\sqsubseteq (\bigvee_{k' \in S'_k} \tau_a \circ Fk')^{*}\mathbf{\Omega}(a) && \text{by the second condition} \\
&= (\tau_a \circ F(\bigvee_{k' \in S'_k} k'))^{*}\mathbf{\Omega}(a) && \text{by the first condition} \\
&= (\tau_a \circ Fpk)^{*}\mathbf{\Omega}(a). && \text{by the third condition}
\end{aligned}$$

□

Proof of Proposition 6.4. Because $R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)(S') = \prod_{a \in A} R^{p,\mathbf{\Omega}(a)} \circ \mathrm{Sp}^1(F,\tau_a)(S'_a)$ for each $S' \in \mathrm{Sp}^A(\mathbb{B},\Omega)$, S is approximating to $[F,\tau]^{\mathbf{\Omega}}$ is equivalent to $R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)(S) \sqsubseteq R^{p,\mathbf{\Omega}(a)} \circ \mathrm{Sp}^1(F,\tau_a) \circ L^{p,\mathbf{\Omega}(a)} \circ R^{p,\mathbf{\Omega}}(S)$ for each $a \in A$. Lemma A.3 and $R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)(S) \sqsubseteq R^{p,\mathbf{\Omega}(a)} \circ \mathrm{Sp}^1(F,\tau_a)(S_a)$ for each $a \in A$ conclude the proof. □

Proof of Proposition 6.12. Suppose that λ is liftable with respect to $[T,\sigma]^{\mathbf{\Omega}}$, $[F,\tau]^{\mathbf{\Omega}}$, i.e. there is a natural transformation $\dot{\lambda} \colon [T,\sigma]^{\mathbf{\Omega}}([F,\tau]^{\mathbf{\Omega}})^N \Rightarrow [F,\tau]^{\mathbf{\Omega}}[T,\sigma]^{\mathbf{\Omega}}$ above λ. Then we have the natural transformation $G \circ \dot{\lambda}$ above λ. The natural transformation $G \circ \dot{\lambda}$ is from $[T,\sigma]^{G\mathbf{\Omega}} \circ ([F,\tau]^{G\mathbf{\Omega}})^N \circ G^N$ to $[F,\tau]^{G\mathbf{\Omega}} \circ [T,\sigma]^{G\mathbf{\Omega}} \circ G^N$ because $G \circ [T,\sigma]^{\mathbf{\Omega}} \circ ([F,\tau]^{\mathbf{\Omega}})^N = [T,\sigma]^{G\mathbf{\Omega}} \circ ([F,\tau]^{G\mathbf{\Omega}})^N \circ G^N$ and $G \circ [F,\tau]^{\mathbf{\Omega}} \circ [T,\sigma]^{\mathbf{\Omega}} = [F,\tau]^{G\mathbf{\Omega}} \circ [T,\sigma]^{G\mathbf{\Omega}} \circ G^N$ hold by the discussion of §6.1.

Because G is full, it ensures the existence of a natural transformation from $[T,\sigma]^{G\mathbf{\Omega}}([F,\tau]^{G\mathbf{\Omega}})^N$ to $[F,\tau]^{G\mathbf{\Omega}}[T,\sigma]^{G\mathbf{\Omega}}$ above λ. □

Proof of Proposition 6.10. $[T,\sigma]^{\mathbf{\Omega}} \circ ([F,\tau]^{\mathbf{\Omega}})^N \Rightarrow \prod_{a \in A} [T,\sigma]^{\mathbf{\Omega}} \circ ([F,\tau_a]^{\mathbf{\Omega}(a)})^N \Rightarrow \prod_{a \in A} [F,\tau_a]^{\mathbf{\Omega}(a)} \circ [T,\sigma]^{\mathbf{\Omega}} = [F,\tau]^{\mathbf{\Omega}} \circ [T,\sigma]^{\mathbf{\Omega}}$. The first natural transformation is above id and the second one is above λ. □

A.3 Omitted Proofs for Section 7

A.3.1 Bisimilarity Pseudometric

Proposition A.4. *In the setting of §7.1, the tuple $((\Omega,\mathbf{\Omega}) : Id_A \to U_{\mathbf{PMet}}, (\times, \lambda^{\mathrm{da}}), \sigma_\wedge)$ satisfies the conditions in Theorem 6.2.*

Proof. 1) For each $(t,\rho),(t',\rho') \in 2 \times [0,1]^\Sigma$, $(\sigma_\wedge \circ \tau_\epsilon \times \tau_\epsilon)((t,\rho),(t',\rho')) = t \wedge t' = (\tau_\epsilon \circ (2 \times \sigma_\wedge^\Sigma) \circ \lambda)((t,\rho),(t',\rho'))$ and $(\sigma_\wedge \circ \tau_a \times \tau_a)((t,\rho),(t',\rho')) = w \cdot (\rho(a) \wedge \rho'(a)) = (\tau_a \circ (2 \times \sigma_\wedge^\Sigma) \circ \lambda)((t,\rho),(t',\rho'))$ $(a \in \Sigma)$.

2) We aim to show that for each $(d_1,d_2) \in \mathbf{PMet}_X \times \mathbf{PMet}_Y$, $a \in A$, $k\colon [\times,\sigma_\wedge]^{d_{[0,1]}}(d_1,d_2) \to d_{[0,1]}$, $(t_1,f_1),(t_2,f_2) \in (2 \times (X \times Y)^\Sigma)$,

$$\begin{aligned}&|(\tau_a \circ (2 \times k^\Sigma))(t_1,f_1) - (\tau_a \circ (2 \times k^\Sigma))(t_2,f_2)|\\ \le &\sup_{\substack{a' \in A,\\ k_1\colon d_1 \to d_{[0,1]},\\ k_2\colon d_2 \to d_{[0,1]}}} |(\tau_{a'} \circ (2 \times (\sigma_\wedge \circ k_1 \times k_2)^\Sigma))(t_1,f_1) - (\tau_{a'} \circ (2 \times (\sigma_\wedge \circ k_1 \times k_2)^\Sigma))(t_2,f_2)|.\end{aligned}$$

If $a = \epsilon$, (lhs) $= |t_1 - t_2| =$ (rhs). If $a \in \Sigma$, noting that $|k(x_1,y_1) - k(x_2,y_2)| \le \max(d_1(x_1,x_2), d_2(y_1,y_2))$ for each $x_1,x_2 \in X$ and $y_1,y_2 \in Y$, (lhs) $= w \cdot |kf_1(a) - kf_2(a)| \le w \cdot \max(d_1(x_1,x_2), d_2(y_1,y_2)) = |(\tau_a \circ (2 \times (\sigma_\wedge \circ k_1 \times k_2)^\Sigma))(t_1,f_1) - (\tau_a \circ (2 \times (\sigma_\wedge \circ k_1 \times k_2)^\Sigma))(t_2,f_2)| \le$ (rhs) where $f_1(a) = (x_1,y_1)$, $f_2(a) = (x_2,y_2)$, $k_1(t) \coloneqq 1 - d_1(x_1,t)$, $k_2(t) \coloneqq 1 - d_2(x_2,t)$.

□

A.3.2 Similarity Pseudometric

Lemma A.5. $\min(a,b) - \min(c,d) \le \max(a-c, b-d)$ *for each* $a,b,c,d \in [0,1]$.

Proof. If $a \le b$ and $c \le d$, (lhs) $= a - c \le$ (rhs). If $a \ge b$ and $c \ge d$, (lhs) $= b - d \le$ (rhs). If $a \le b$ and $c \ge d$, (lhs) $= a - d \le b - d \le$(rhs). If $a \ge b$ and $c \le d$, (lhs) $= b - c \le a - c \le$(rhs). □

Proposition A.6. $d_{\mathbb{I}}^{\mathrm{as}}$ *is a Lawvere metric.*

Proof. For each $a,b,c \in [0,1]$,

- $d_{\mathbb{I}}^{\mathrm{as}}(a,a) = 0$ is easy.
- $d_{\mathbb{I}}^{\mathrm{as}}(a,c) \le d_{\mathbb{I}}^{\mathrm{as}}(a,b) + d_{\mathbb{I}}^{\mathrm{as}}(b,c)$ is because: If $a \le b$ and $b \le c$, (lhs) $= c - a =$ (rhs). If $a \ge b$ and $b \ge c$, (lhs) $= 0 =$ (rhs). If $a \le b$ and $b \ge c$, $c - a \le b - a =$ (rhs) and $0 \le b - a =$ (rhs). If $a \ge b$ and $b \le c$, $c - a \le c - b =$ (rhs) and $0 \le c - b =$ (rhs).

□

Proposition A.7. *1. The codensity lifting* $[F,\tau]^{d_{\mathbb{I}}^{\mathrm{as}}}$ *maps* $(X,d) \in \mathbf{LMet}$ *to the pseudometric on* FX *given by* $((t_1,\rho_1),(t_2,\rho_2)) \mapsto max\{d_{\mathbb{I}}^{\mathrm{as}}(t_1,t_2), w \cdot \max_{a\in\Sigma} d(\rho_1(a),\rho_2(a))\}$.

2. The binary codensity lifting $[\times,\sigma_\wedge]^{d_{\mathbb{I}}^{\mathrm{as}}}$ *maps* $(d_1,d_2) \in \mathbf{LMet}_X \times \mathbf{LMet}_Y$ *to the Lawvere metric on* $X \times Y$ *given by* $((x,y),(x',y')) \mapsto \max(d_1(x,x'), d_2(y,y'))$.

Proof. 1. Fix arbitrary $d \in \mathbf{LMet}_X$ and $(t_1,\rho_1),(t_2,\rho_2) \in 2 \times X^\Sigma$. It is easy to see that $[F,\tau_\epsilon]^{d_{\mathbb{I}}^{\mathrm{as}}}(d)((t_1,\rho_1),(t_2,\rho_2)) = d_{\mathbb{I}}^{\mathrm{as}}(t_1,t_2)$. Let us prove that $[F,\tau_a]^{d_{\mathbb{I}}^{\mathrm{as}}}(d)((t_1,\rho_1),(t_2,\rho_2)) = w \cdot d(\rho_1(a),\rho_2(a))$ for each $a \in \Sigma$.

($\geq$): Letting $k(t) := d(\rho_1(a), t)$, $k\colon d \to d_{\mathbb{I}}^{\mathrm{as}}$ and (lhs) $\geq d_{\mathbb{I}}^{\mathrm{as}}(w \cdot k\rho_1(a), w \cdot k\rho_2(a)) = w \cdot d(\rho_1(a), \rho_2(a)) =$ (rhs). ($\leq$): For each $k\colon d \to d_{\mathbb{I}}^{\mathrm{as}}$, $d_{\mathbb{I}}^{\mathrm{as}}(w \cdot k\rho_1(a), w \cdot k\rho_2(a)) = w \cdot d_{\mathbb{I}}^{\mathrm{as}}(k\rho_1(a), k\rho_2(a)) \leq w \cdot d(\rho_1(a), \rho_2(a))$. Therefore we have (lhs) $\leq$ (rhs).

2. Fix arbitrary $d_1, d_2 \in \mathbf{LMet}_X$ and $(x, y), (x', y') \in X \times Y$. Let us prove $[\times, \sigma_\wedge]^{d_{\mathbb{I}}^{\mathrm{as}}}(d_1, d_2)((x, y), (x', y')) = \max(d_1(x, x'), d_2(y, y'))$. ($\geq$): Letting $k_1(t) := 1 - d_1(t, x')$ and $k_2(t) := 1 - d_2(t, y')$, $k_1\colon d_1 \to d_{\mathbb{I}}^{\mathrm{as}}$ and $k_2\colon d_2 \to d_{\mathbb{I}}^{\mathrm{as}}$. Therefore we have

$$\begin{aligned} (lhs) &\geq d_{\mathbb{I}}^{\mathrm{as}}(\min(k_1 x, k_2 y), \min(k_1 x', k_2 y')) \\ &= \max(d_1(x, x'), d_2(y, y')) = (rhs). \end{aligned}$$

($\leq$): For each $k_1\colon d_1 \to d_{\mathbb{I}}^{\mathrm{as}}$ and $k_2\colon d_2 \to d_{\mathbb{I}}^{\mathrm{as}}$,

$$\begin{aligned} &d_{\mathbb{I}}^{\mathrm{as}}(\min(k_1 x, k_2 y), \min(k_1 x', k_2 y')) \\ &\leq \min(k_1 x', k_2 y') - \min(k_1 x, k_2 y) \\ &\leq \max(k_1 x' - k_1 x, k_2 y' - k_2 y) \\ &\leq \max(d_1(x, x'), d_2(y, y')) \end{aligned}$$

by Lemma A.5. Therefore we have (lhs) $\leq$ (rhs).

$\square$

Proposition A.8. *In the setting of §7.2, the tuple* $((\Omega, \mathbf{\Omega}) : Id_A \to (p : \mathbf{LMet} \to \mathbf{Set}), (\times, \lambda^{\mathrm{da}}), \sigma_\wedge)$ *satisfies the conditions in Theorem 6.2.*

Proof. We can prove it in the same way as Proposition A.4. $\square$

A.3.3 Bisimulation Metric

Proof of Lemma 7.4. 1) We can prove it by letting $d := [T, \sigma]^{\mathbf{\Omega}}(d_1, d_2)$ and $z := \sigma$ in Lemma 7.5.

2) Let $P \in \mathbf{PMet}_X$ and $Q \in \mathbf{PMet}_Y$. $[T, \sigma]^{\mathbf{\Omega}}([\mathcal{P}, \inf]^{\mathbf{\Omega}} P, [\mathcal{P}, \inf]^{\mathbf{\Omega}} Q) \sqsubseteq \lambda^{\mathcal{P}*}[\mathcal{P}, \inf]^{\mathbf{\Omega}}[T, \sigma]^{\mathbf{\Omega}}(P, Q)$ is equivalent to for any $A_1, A_2 \in \mathcal{P}X, B_1, B_2 \in \mathcal{P}Y$,

$$\sigma(([\mathcal{P}, \inf]^{\mathbf{\Omega}} P)(A_1, A_2), ([\mathcal{P}, \inf]^{\mathbf{\Omega}} Q)(B_1, B_2)) \geq ([\mathcal{P}, \inf]^{\mathbf{\Omega}}[T, \sigma]^{\mathbf{\Omega}}(P, Q))(\lambda^{\mathcal{P}}(A_1, B_1), \lambda^{\mathcal{P}}(A_2, B_2)). \tag{9}$$

If $\lambda^{\mathcal{P}}(A_1, B_1) = \lambda^{\mathcal{P}}(A_2, B_2) = \emptyset$, then (rhs) $= 0$ by the definition of $[\mathcal{P}, \inf]^{\mathbf{\Omega}}$ so the inequality (9) holds.

If $\lambda^{\mathcal{P}}(A_1, B_1) = \emptyset$ and $\lambda^{\mathcal{P}}(A_2, B_2) \neq \emptyset$, then ($A_1 = \emptyset$ or $B_1 = \emptyset$), $A_2 \neq \emptyset$, and $B_2 \neq \emptyset$ by definition of $\lambda^{\mathcal{P}}$. Thus at least one of $([\mathcal{P}, \inf]^{\mathbf{\Omega}} P)(A_1, A_2)$ and $([\mathcal{P}, \inf]^{\mathbf{\Omega}} Q)(B_1, B_2)$ is equal to 1 by the definition of $[\mathcal{P}, \inf]^{\mathbf{\Omega}}$. Because $\sigma(1, a) = \sigma(a, 1) = 1$ for any $a \in [0, 1]$, we have (lhs) $= 1$. It indicates the inequality (9) holds.

Assume that $\lambda^{\mathcal{P}}(A_1, B_1) \neq \emptyset$ and $\lambda^{\mathcal{P}}(A_2, B_2) \neq \emptyset$. The definition of $[\mathcal{P}, \inf]^{\mathbf{\Omega}}$ gives that

$$\begin{aligned}(\mathrm{rhs}) = \max\big(&\sup_{(a_1,b_1)\in\lambda^{\mathcal{P}}(A_1,B_1)} \inf_{(a_2,b_2)\in\lambda^{\mathcal{P}}(A_2,B_2)} [T,\sigma]^{\mathbf{\Omega}}(P,Q)((a_1,b_1),(a_2,b_2)),\\ &\sup_{(a_2,b_2)\in\lambda^{\mathcal{P}}(A_2,B_2)} \inf_{(a_1,b_1)\in\lambda^{\mathcal{P}}(A_1,B_1)} [T,\sigma]^{\mathbf{\Omega}}(P,Q)((a_1,b_1),(a_2,b_2))\big).\end{aligned}$$

Therefore it's enough to show that two inequalities

$$(lhs) \geq \sup_{(a_1,b_1)\in\lambda^{\mathcal{P}}(A_1,B_1)} \inf_{(a_2,b_2)\in\lambda^{\mathcal{P}}(A_2,B_2)} [T,\sigma]^{\mathbf{\Omega}}(P,Q)((a_1,b_1),(a_2,b_2))$$

and

$$(lhs) \geq \sup_{(a_2,b_2)\in\lambda^{\mathcal{P}}(A_2,B_2)} \inf_{(a_1,b_1)\in\lambda^{\mathcal{P}}(A_1,B_1)} [T,\sigma]^{\mathbf{\Omega}}(P,Q)((a_1,b_1),(a_2,b_2)).$$

Let us show the former. The other one can be shown similarly.

For any $(a_1, b_1) \in \lambda^{\mathcal{P}}(A_1, B_1)$,

$$\begin{aligned}(\mathrm{lhs}) &= \sigma(([\mathcal{P},\inf]^{\mathbf{\Omega}}P)(A_1,A_2),([\mathcal{P},\inf]^{\mathbf{\Omega}}Q)(B_1,B_2))\\ &\geq \sigma(\sup_{a_1'\in A_1}\inf_{a_2'\in A_2} P(a_1',a_2'), \sup_{b_1'\in B_1}\inf_{b_2'\in B_2} Q(b_1',b_2')) && \text{by the definition of } [\mathcal{P},\inf]^{\mathbf{\Omega}}\\ &\geq \sigma(\inf_{a_2'\in A_2} P(a_1,a_2'), \inf_{b_2'\in B_2} Q(b_1,b_2')) && \text{since } \sigma \text{ is monotone}\\ &= \inf_{a_2'\in A_2, b_2'\in B_2} \sigma(P(a_1,a_2'),Q(b_1,b_2')) && \text{since } \sigma \text{ preserves infimums}\\ &= \inf_{(a_2',b_2')\in\lambda^{\mathcal{P}}(A_2,B_2)} [T,\sigma]^{\mathbf{\Omega}}(P,Q)((a_1,b_1),(a_2',b_2')).\end{aligned}$$

□

A.4 Omitted Proofs for Section 8

Lemma A.9. *Assume that two conditions in Theorem 6.2 hold. A set* $\mathcal{V} \subseteq |\prod_{i\in N}\mathbb{E}_{X_i}|$ *is an invariant for* $\mathbf{D}$ *in the composite codensity game* $\mathcal{G}^{T,\sigma}_{\overrightarrow{c_i}}$ *if and only if* $\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}}[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i}) \sqsubseteq (T_\lambda(\overrightarrow{c_i}))^*[F,\tau]^{\mathbf{\Omega}}[T,\sigma]^{\mathbf{\Omega}}(\Big(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\Big)_{i\in N})$.

Proof.

$$\begin{aligned}\lambda^*[F,\tau]^{\mathbf{\Omega}}[T,\sigma]^{\mathbf{\Omega}} &= \lambda^* R^{p,\mathbf{\Omega}} \circ \mathrm{Sp}^A(F,\tau)\circ \mathrm{Sp}^A(T,\sigma)\circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}\\ &\text{(by the condition (2) in Theorem 6.2)}\\ &= R^{p,\mathbf{\Omega}}\circ\lambda^*\mathrm{Sp}^A(F,\tau)\circ\mathrm{Sp}^A(T,\sigma)\circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}\\ &\text{(since } R^{p,\mathbf{\Omega}} \text{ preserves cartesian morphisms)}\\ &= R^{p,\mathbf{\Omega}}\circ\mathrm{Sp}^A(T,\sigma)\circ\mathrm{Sp}^A(F^N,\tau^N)\circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}\\ &\text{(by the condition (1) in Theorem 6.2 and discussion after Theorem 6.2)}\\ &= R^{p,\mathbf{\Omega}}\circ\mathrm{Sp}^A(T\circ F^N,\sigma\bullet(T\circ\tau^N))\circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}.\end{aligned}$$

From this equation,

$$
\begin{aligned}
&\bigsqcup_{\langle P_i\rangle_{i\in N}\in\mathcal{V}} [T,\sigma]^{\mathbf{\Omega}}(\langle P_i\rangle_{i\in N}) \sqsubseteq (\lambda\circ T(\langle c_i\rangle_{i\in N}))^*[F,\tau]^{\mathbf{\Omega}}([T,\sigma]^{\mathbf{\Omega}}(\langle \bigsqcup_{\langle P_i\rangle_{i\in N}\in\mathcal{V}} P_i\rangle_{i\in N}))\\
&\iff \textstyle\bigsqcup_{\langle P_i\rangle_{i\in N}\in\mathcal{V}}[T,\sigma]^{\mathbf{\Omega}}(\langle P_i\rangle_{i\in N})\\
&\sqsubseteq (T(\langle c_i\rangle_{i\in N}))^* R^{p,\mathbf{\Omega}}\circ \mathrm{Sp}^A(T\circ F^N, \sigma\bullet(T\circ\tau^N))\circ L^{p^N,\langle\mathbf{\Omega}\rangle_N}(\langle \bigsqcup_{\langle P_i\rangle_{i\in N}\in\mathcal{V}} P_i\rangle_{i\in N})\\
&= \bigsqcap_{a\in A,\langle k_i\in\mathbb{E}(\bigsqcup_{\langle P_i\rangle_i\in\mathcal{V}} P_i,\mathbf{\Omega})\rangle_{i\in N}} \big(\sigma\circ T(\langle \tau_a\circ Fpk_i\circ c_i\rangle_{i\in N})\big)^*\mathbf{\Omega}\\
&\iff \forall\langle P_i\rangle_{i\in N}\in\mathcal{V}.\ \forall a\in A.\ \forall\langle k_i\colon \bigsqcup_{\langle P_i'\rangle_i\in\mathcal{V}} P_i'\dot{\to}\mathbf{\Omega}\rangle_{i\in N}.\ [T,\sigma]^{\mathbf{\Omega}}(\langle P_i\rangle_{i\in N})\sqsubseteq \big(\sigma\circ T(\langle\tau_a\circ Fk_i\circ c_i\rangle_{i\in N})\big)^*\mathbf{\Omega}\\
&\iff \forall\langle P_i\rangle_{i\in N}\in\mathcal{V}.\ \forall a\in A.\ \forall\langle k_i\in\mathbb{C}(X_i,\Omega)\rangle_{i\in N}.\\
&\qquad \big(\forall\langle P_i'\rangle_{i\in N}\in\mathcal{V}.\ \forall i\in N\ k_i\colon P_i'\dot{\to}\mathbf{\Omega}\Rightarrow [T,\sigma]^{\mathbf{\Omega}}(\langle P_i\rangle_{i\in N})\sqsubseteq \big(\sigma\circ T(\langle\tau_a\circ Fk_i\circ c_i\rangle_{i\in N})\big)^*\mathbf{\Omega}\big)\\
&\iff \forall\langle P_i\rangle_{i\in N}\in\mathcal{V}.\ \forall a\in A.\ \forall\langle k_i\in\mathbb{C}(X_i,\Omega)\rangle_{i\in N}.\\
&\qquad \big([T,\sigma]^{\mathbf{\Omega}}(\langle P_i\rangle_{i\in N})\not\sqsubseteq \big(\sigma\circ T(\langle\tau_a\circ Fk_i\circ c_i\rangle_{i\in N})\big)^*\mathbf{\Omega}\Rightarrow \exists\langle P_i'\rangle_{i\in N}\in\mathcal{V}.\ \exists i\in N\ k_i\colon P_i'\dot{\not\to}\mathbf{\Omega}\big).
\end{aligned}
$$

□

Proof of Theorem 8.9. (1) If $\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}}[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})$ is a $[F,\tau]^{\mathbf{\Omega}}$-bisimulation on $T_\lambda(\overrightarrow{c_i})$, then $\mathcal{V}$ satisfies the inequality in Lemma A.9 because $\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}}[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i}) \sqsubseteq (T_\lambda(\overrightarrow{c_i}))^*[F,\tau]^{\mathbf{\Omega}}\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}}[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i}) \sqsubseteq (T_\lambda(\overrightarrow{c_i}))^*[F,\tau]^{\mathbf{\Omega}}[T,\sigma]^{\mathbf{\Omega}}(\big(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\big)_{i\in N})$. Therefore $\mathcal{V}$ is an invariant by Lemma A.9.

(2) Since $\big(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\big)_{i\in N}\in\overline{\mathcal{V}}$ and $[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})\sqsubseteq[T,\sigma]^{\mathbf{\Omega}}(\big(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\big)_{i\in N})$ for each $\overrightarrow{P_i}\in\overline{\mathcal{V}}$, $\bigsqcup_{\overrightarrow{P_i}\in\overline{\mathcal{V}}}[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i}) = [T,\sigma]^{\mathbf{\Omega}}(\big(\bigsqcup_{\overrightarrow{P_i}\in\mathcal{V}} P_i\big)_{i\in N}) = [T,\sigma]^{\mathbf{\Omega}}(\big(\bigsqcup_{\overrightarrow{P_i}\in\overline{\mathcal{V}}} P_i\big)_{i\in N})$. Lemma A.9 concludes the proof. □

Proof of Proposition 8.8. Assume that each $\mathcal{V}_i\subseteq\mathbb{E}_{X_i}$ is an invariant of $\mathcal{G}_{c_i}$ for $\mathbf{D}$. For any $a\in A$ and $(k_i\in\mathbb{C}(X_i,\Omega))_{i\in N}$,

$$
\begin{aligned}
[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{(\tau_a\circ Fk_i\circ c_i)^*\mathbf{\Omega}(a)}) &= \bigsqcap_{a'\in A,\big(k_i'\colon(\tau_a\circ Fk_i\circ c_i)^*\mathbf{\Omega}\to\mathbf{\Omega}\big)_{i\in N}} \big(\sigma_{a'}\circ T(\overrightarrow{pk_i'})\big)^*\mathbf{\Omega}(a')\\
&\sqsubseteq \big(\sigma_a\circ T(\overrightarrow{\tau_a\circ Fk_i\circ c_i})\big)^*\mathbf{\Omega}(a)
\end{aligned}
\tag{10}
$$

For arbitrary $\overrightarrow{P_i}\in\prod_{i\in N}\mathcal{V}_i$, $a\in A$, and $(k_i\in\mathbb{C}(X_i,\Omega))_{i\in N}$, if $[T,\sigma]^{\mathbf{\Omega}}(\overrightarrow{P_i})\not\sqsubseteq \big(\sigma_a\circ T(\overrightarrow{\tau_a\circ Fk_i\circ c_i})\big)^*\mathbf{\Omega}(a)$, we have $\exists i\in N.\ P_i\not\sqsubseteq(\tau_a\circ Fk_i\circ c_i)^*\mathbf{\Omega}(a)$ by (10). Since $\mathcal{V}_i$ is an invariant for $\mathbf{D}$, $\exists P_i'\in\mathcal{V}_i.\ k_i\colon P_i'\dot{\not\to}\mathbf{\Omega}(a)$. Thus we have $\overrightarrow{P_i'}\in\mathcal{V}$ such that $\exists i\in N\ k_i\colon P_i'\dot{\not\to}\mathbf{\Omega}(a)$. □