THE LOCAL-TO-GLOBAL PRINCIPLE VIA TOPOLOGICAL PROPERTIES OF THE BALMER-FAVI SUPPORT

NICOLA BELLUMAT

Abstract. Following the theory of stratification of tensor triangulated categories via Balmer-Favi support inaugurated by Barthel, Heard and Sanders, we prove the local versions of the well-known statements that the Balmer spectrum being noetherian or profinite scattered implies the local-to-global principle.

That is, given an object t of a tensor triangulated category $\mathcal{T}$ we show that if the Balmer-Favi support $\mathrm{Supp}(t)$ is a noetherian space, then the local-to-global principle holds for t. In the case where the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ is profinite, if the support $\mathrm{Supp}(t)$ is scattered then the local-to-global principle holds for the object t.

We conclude with an application of the last result to the examination of the support of injective superdecomposable modules in the derived category of an absolutely flat ring which is not semi-artinian.

1. Introduction

Our starting point is the following question, raised in [2]:

Question ([2, Question 21.8]). *Let t be an object of a rigidly-compactly generated tensor triangulated category $\mathcal{T}$. Suppose that the Balmer-Favi support* $\mathrm{Supp}(t)$ *is a noetherian subspace of the Balmer spectrum* $\mathrm{Spc}(\mathcal{T}^c)$*. Then, does the local-to-global principle hold for t?*

This problem was motivated by [1, Thm. 3.21], which states that in the case the whole Balmer spectrum is noetherian then the local-to-global principle holds. This result is nothing else that the adaptation to the setting of stratification via Balmer-Favi support presented in [1] of the analogous finding by Benson, Iyengar and Krause demonstrated in [3, Thm. 3.6].

The answer is positive and the claim will be proved in Theorem 4.7 below. The core of the proof consists in adapting the argument of [10] to our situation, where the filtration given by the Krull dimension is present only on $\mathrm{Supp}(t)$ but not necessarily the whole Balmer spectrum. We will need some preliminary results on the topology of spectral spaces, to ensure that the filtration on $\mathrm{Supp}(t)$ is still enough to provide the finite localizations and the associated colocalizations which are necessary to decompose t.

Inspired by this success and [10, Thm. 5.6], we show the following variant for tensor triangulated categories with profinite Balmer spectra:

Theorem. *Suppose* $\mathrm{Spc}(\mathcal{T}^c)$ *is profinite. If* $\mathrm{Supp}(t)$ *is a scattered space, then the local-to-global principle holds for t.*

Even in this case, the proof relies on using a filtration on $\mathrm{Supp}(t)$, namely the Cantor-Bendixon rank, which behaves well with respect to the spectral topology on $\mathrm{Spc}(\mathcal{T}^c)$ to decompose t.

Finally, we use this Theorem to deduce that in $\mathrm{D}(R)$, the derived category of a non semi-artinian absolutely flat ring, the support of an injective superdecomposable module must be contained in the maximal perfect subset of $\mathrm{Spc}(\mathrm{D}(R)^c)$, where this Balmer spectrum is homeomorphic to the Zariski spectrum $\mathrm{Spec}(R)$.

We construct one of such modules so that its support coincides with the maximal perfect subset. The construction depends on the correspondence between the Loewy series of R and the Cantor-Bendixon derivatives of the Zariski spectrum $\mathrm{Spec}(R)$.

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2. Setup and notation

We will mostly follow the notation and definitions of [2]. We will explicitly state when we introduce different notation.

From now on, $(\mathcal{T}, \otimes, \mathbb{1})$ denotes a fixed rigid-compactly generated tensor triangulated category. We make three assumptions on this tensor triangulated category: first, we assume its Balmer spectrum is weakly noetherian ([1, Def. 2.3]) so that we can define the Balmer-Favi support.

Second, we assume it satisfies the detection property ([1, Def. 2.15]), i.e. if there is an object $t \in \mathcal{T}$ such that $\mathrm{Supp}(t) = \emptyset$ then $t = 0$.

Third, we assume the tensor triangulated category $\mathcal{T}$ admits a model, i.e. it is the homotopy category of a monoidal model category or of a monoidal infinity category, or it is the underlying category of a strong stable monoidal derivator. This allows us to have a well-behaved theory of homotopy colimits. See [10, Rmk. 2.19] for more details.

Since we will be using only the notion of Balmer-Favi support, from now on we will drop the names Balmer-Favi and just call it support.

We introduce our own notation regarding the Rickard idempotents:

Definition 2.1. Let $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a Thomason subset, so we have the corresponding thick tensor ideal of compact objects $\mathcal{T}^c_Y$ by the classification theorem due to Balmer. Then we can invoke [6, Thm. 3.3.5] to obtain from $\mathcal{T}^c_Y$ an exact triangle

$$\Gamma_Y \mathbb{1} \to \mathbb{1} \to L_{Y^c}\mathbb{1} \to \Sigma\Gamma_Y \mathbb{1}$$

where $\Gamma_Y \mathbb{1} \in \mathrm{Loc}\langle \mathcal{T}^c_Y \rangle$ and $L_{Y^c}\mathbb{1} \in \mathrm{Loc}\langle \mathcal{T}^c_Y \rangle^{\perp}$ are the idempotents associated respectively to the acyclicization and localization with respect to $\mathrm{Loc}\langle \mathcal{T}^c_Y \rangle$.

In the notation of [2], we have the equalities

$$e_Y = \Gamma_Y \mathbb{1} \qquad\qquad f_Y = L_{Y^c}\mathbb{1}.$$

Since $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian, for any Balmer prime $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^c)$ there exist two Thomason subsets $Y_1, Y_2 \subseteq \mathrm{Spc}(\mathcal{T}^c)$ such that $\{\mathfrak{p}\} = Y_1 \cap Y_2$. We denote by $g(\mathfrak{p}) = \Gamma_{Y_1}\mathbb{1} \otimes L_{Y_2^c}\mathbb{1}$ the corresponding idempotent.

We make this choice so that the subscript of an idempotent coincides with its support and to emphasize that left and right Rickard idempotents correspond to finite colocalizations and localizations rispectively.

Remark 2.2. It is proved in [1, Lemma 2.13] that for any $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ Thomason subset and $t \in \mathcal{T}$ it holds

$$\mathrm{Supp}(\Gamma_Y t) = \mathrm{Supp}(t) \cap Y \qquad\qquad \mathrm{Supp}(L_{Y^c} t) = \mathrm{Supp}(t) \cap Y^c.$$

Using these equalities and the detection property, we can see that if $\mathrm{Supp}(t) \cap Y^c = \emptyset$ then $t \cong t \otimes \Gamma_Y \mathbb{1}$ and dually the equality $Y \cap \mathrm{Supp}(t) = \emptyset$ implies $t \cong t \otimes L_{Y^c}\mathbb{1}$.

Definition 2.3. We adopt the following abbreviations, for the sake of brevity. We set $X = \mathrm{Spc}(\mathcal{T}^c)$ to be the name of the Balmer spectrum of the tensor triangulated category in exam. If t is the object of $\mathcal{T}$ we are studying we denote by $S = \mathrm{Supp}(t)$ its support.

Moreover, given a generic spectral space W and a subset $V \subseteq W$ we denote by $\overline{V}^{\vee}$ the closure operator with respect to the dual topology $W^{\vee}$. That is, $\overline{V}^{\vee}$ is the smallest complement of a Thomason subset of W containing V.

3. Noetherian support: special cases

Our first objective is to show the claim of [2, Question 21.8] is true. That is, we show that if an object $t \in \mathcal{T}$ has noetherian support then the local-to-global principle holds for t. Before attempting the proof in full generality, we deal with two special cases: namely when S is a Thomason subset of X or the complement of a Thomason subset. This allows us to reduce to proving the claim for the left and right Rickard idempotents respectively.

We start by showing the claim if $S = Y$ is a Thomason subset.

By Remark 2.2 we have $t \cong t \otimes \Gamma_Y \mathbb{1}$. Therefore, it is enough to show the claim for $\Gamma_Y \mathbb{1}$, i.e. we have to prove

$$\Gamma_Y \mathbb{1} \in \mathrm{Locid}\langle g(\mathfrak{p}) : \mathfrak{p} \in Y \rangle.$$

Lemma 3.1. *Let $\mathfrak{p} \in Y$, where $Y \subseteq X$ is Thomason closed and noetherian. Then $\overline{\{\mathfrak{p}\}}$ is a Thomason subset of X.*

Proof. Since $\overline{\{\mathfrak{p}\}}$ is by definition a closed subset of the Balmer spectrum, we have to show that its complement is quasi-compact.

Set $K = \overline{\{\mathfrak{p}\}}$. Suppose $X \setminus K \subseteq \bigcup_{i \in I} U_i$ where U_i are open subsets of X. If we take the intersection with Y we obtain

$$(X \setminus K) \cap Y = Y \setminus K \subseteq \bigcup_i U_i \cap Y$$

where the equality comes from $K \subseteq Y$. Since Y is noetherian, its open subset $Y \setminus K$ is quasi-compact. Thus we can find a finite subcover $Y \setminus K \subseteq \bigcup_{j \in J} U_j \cap Y$ for some finite indexing subset $J \subseteq I$.

We have that $X \setminus Y$ is open and quasi-compact by the Thomason property. Hence

$$X \setminus Y \subseteq X \setminus K \subseteq \bigcup_i U_i$$

and we can refine to a finite subcover for some indexing subset $L \subseteq I$.

It follows

$$X \setminus K = X \setminus Y \cup Y \setminus K \subseteq \bigcup_{l \in L} U_l \cup \bigcup_{j \in J} U_j = \bigcup_{i \in J \cup L} U_i.$$

Since we found a finite subcover we proved $X \setminus K$ is quasi-compact. □

Corollary 3.2. *Let $Y \subseteq X$ be a noetherian Thomason subset. Then $Z \subseteq Y$ is a Thomason subset of X if and only if it is downward closed with respect to the ordering given by the inclusion of Balmer primes.*

Proof. That Thomason implies downward closed is trivial. For the other direction write

$$Z = \bigcup_{\mathfrak{p} \in Z} \overline{\{\mathfrak{p}\}}.$$

We observe that if Y can be written as $Y = \bigcup_{i \in I} Y_i$ with Y_i Thomason closed, then $\mathfrak{p} \in Y_i$ implies $\overline{\{\mathfrak{p}\}} \subseteq Y_i$ and by Lemma 3.1 we deduce $\overline{\{\mathfrak{p}\}}$ is Thomason closed. It follows Z is Thomason. □

Proposition 3.3. *Let $Y \subseteq X$ be a noetherian Thomason subset. Then the local-to-global principle holds for $\Gamma_Y \mathbb{1}$.*

Proof. With the above preliminaries we can repeat almost verbatim the proof of [1, Thm. 3.21]. This time, we have to show that

$$Y' = \{\mathfrak{p} \in Y : \Gamma_{\overline{\{\mathfrak{p}\}}} \mathbb{1} \in \operatorname{Locid}\langle g(\mathfrak{q}) : \mathfrak{q} \in Y \rangle\}$$

coincides with Y.

Suppose by absurd $Y' \neq Y$, by the noetherianity of Y there exists a minimal $\mathfrak{q} \in Y \setminus Y'$. Set $Z = \overline{\{\mathfrak{q}\}} \setminus \{\mathfrak{q}\}$. This is a downward closed subset of Y, hence it is a Thomason subset of the Balmer spectrum (Corollary 3.2) and we can form the corresponding idempotent $\Gamma_Z \mathbb{1}$.

As in [1, (3.22)] we have an exact triangle

$$\Gamma_Z \mathbb{1} \to \Gamma_{\overline{\{\mathfrak{q}\}}} \mathbb{1} \to g(\mathfrak{q}) \to \Sigma \Gamma_Z \mathbb{1}.$$

Even $Y' \subseteq Y$ is downward closed, hence Thomason. We have

$$\operatorname{Locid}\langle \Gamma_{Y'} \mathbb{1} \rangle = \operatorname{Locid}\langle \Gamma_{\overline{\{\mathfrak{p}\}}} \mathbb{1} : \mathfrak{p} \in Y' \rangle \subseteq \operatorname{Locid}\langle g(\mathfrak{q}) : \mathfrak{q} \in Y \rangle.$$

We have $Z \subseteq Y'$ by minimality of $\mathfrak{q}$, hence $\Gamma_Z \mathbb{1} \in \operatorname{Locid}\langle \Gamma_{Y'} \mathbb{1} \rangle$. Then the above exact triangle implies $\Gamma_{\overline{\{\mathfrak{q}\}}} \mathbb{1} \in \operatorname{Locid}\langle g(\mathfrak{p}) : \mathfrak{p} \in Y \rangle$. We deduce that even $\mathfrak{q} \in Y'$: a contradiction. □

Now suppose S is the complement of a Thomason subset of X. By Remark 2.2 we have $t \cong L_S \mathbb{1} \otimes t$, hence it suffices to prove the claim for $L_S \mathbb{1}$.

Remark 3.4. Let $Z \subseteq X$ be the complement of a Thomason subset. We can form the localization adjunction $L : \mathcal{T} \rightleftarrows L_Z\mathcal{T} : j$. We observe that L is a geometric morphism in the sense of [2, Terminology 13.1]: in this case $f^* = L$ and $f_* = j$.

In [1, Rmk. 1.23] it is explained that $\mathrm{Spc}(L)$ induces the identification of $\mathrm{Spc}(L_Z\mathcal{T}^c)$ with the subspace $Z \subseteq X$.

For $\mathfrak{p} \in Z$ we denote by $\kappa(\mathfrak{p})$ the idempotent of [1, Def. 2.7] formed in the category $L_Z\mathcal{T}$, to distinguish it from $g(\mathfrak{p})$ the idempotent associated to the same Balmer prime in the starting category $\mathcal{T}$. It is immediate from [2, Rmk. 13.7] that $\kappa(\mathfrak{p}) \cong L(g(\mathfrak{p}))$.

Suppose that for an object $u \in L_Z\mathcal{T}$ we showed

$$u \in \mathrm{Locid}\langle \kappa(\mathfrak{p}) \otimes u : \mathfrak{p} \in A\rangle$$

for an arbitrary subset $A \subseteq Z$. Then we can use [2, (13.4)] to deduce

$$ju \in \mathrm{Locid}\langle g(\mathfrak{p}) \otimes ju : \mathfrak{p} \in A\rangle.$$

Proposition 3.5. *Suppose $S \subseteq X$ is the complement of a Thomason subset. Then $L_S\mathbb{1}$ satisfies the local-to-global principle.*

Proof. We can consider $L_S\mathbb{1}$ as an object of the localized category $L_S\mathcal{T}$. Since $\mathrm{Spc}(L_S\mathcal{T}^c) \cong S$ is noetherian we can use [1, Thm. 3.21] to show

$$L_S\mathbb{1} \in \mathrm{Locid}\langle L_S g(\mathfrak{p}) : \mathfrak{p} \in S\rangle.$$

Now Remark 3.4 allows us to conclude

$$L_S\mathbb{1} \in \mathrm{Locid}\langle g(\mathfrak{p}) : \mathfrak{p} \in S\rangle.$$

□

4. Noetherian support: the general case

We now deal with the general case. Our strategy will be to construct a filtration of S by defining on it a Krull dimension function. Using this filtration we can prove the local-to-global principle by arguing by transfinite induction.

We will need later the following result

Proposition 4.1. *Let $K \subseteq W$ be a quasi-compact subset of a generic spectral space, then*

$$\overline{K}^{\vee} = \mathrm{Gen}(K) = \{w \in W : \exists k \in K \ k \in \overline{\{w\}}\}.$$

Proof. This is an immediate consequence of [4, Thm. 4.1.5]. □

We will need an adaptation of the notion of Krull dimension in the sense of [10, Def. 4.1] which can be defined on noetherian spectral spaces.

Definition 4.2. Let K be a noetherian T_0 space. We denote by Ord the class of all ordinals. We define a function

$$\dim_K : K \to \mathrm{Ord}$$

by transfinite recursion.

Given $k \in K$ we set $\dim_K(k) = 0$ if and only if $\{k\}$ is closed in K. We denote by $K_{\leq 0}$ the subset of these points and by $K_{>0}$ its complement.

Let $\alpha+1$ be the successor of the ordinal number α and suppose we already defined $K_{\leq\alpha}$, the subset of K given by the points of dimension lesser or equal to α. We define $K_{>\alpha}$ as the complement of $K_{\leq\alpha}$, then we say that a point $k \in K$ has dimension $\alpha+1$ if and only if $k \in K_{>\alpha}$ and $\{k\}$ is closed in $K_{>\alpha}$.

If λ is a limit ordinal we do not assign dimension λ to any element of K. Consequently, we define $K_{\leq\lambda} = \bigcup_{\alpha<\lambda} K_{\leq\alpha}$.

We name this function the *(Krull) dimension function* on K. We call the *(Krull) dimension* of K the least ordinal α such that $K = K_{\leq\alpha}$.

If the ambient space is clear from the context, we drop it from the subscript and write the dimension function just dim.

Remark 4.3. While Definition 4.2 is exactly the same of the Krull dimension of [10, Def. 4.1], we cannot immediately deduce that it is well defined from the results of [10] because we are not assuming the base space to be spectral. We observe that by [4, Cor. 8.1.7] a noetherian T_0 space is spectral if and only if it is sober.

Lemma 4.4. *Let K be a noetherian T_0 space. The dimension function on K of Definition 4.2 is well defined.*

Proof. We define the following relation on the elements of K: we set $x \leq y$ if and only if $x \in \overline{\{y\}}$. The T_0 assumption implies that this relation is a partial ordering.

Since K is noetherian we have that it has minimal elements with respect to this order: otherwise we have a descending sequence of distinct elements $k_1 > k_2 > k_3 > \dots$, but this would imply that the sequence of closed subsets

$$\overline{\{k_1\}} \supsetneq \overline{\{k_2\}} \supsetneq \overline{\{k_3\}} \supsetneq \dots$$

does not stabilize. This is a contradiction with the noetherianity of K.

By definition, these minimal points have dimension 0 and they form the subset $K_{\leq 0}$.

Now let α be an ordinal and assume that $K_{\leq\alpha}$ has been defined. We set $K_{>\alpha} = K \setminus K_{\leq\alpha}$. Since it is a subset of K even this is noetherian and T_0. Using the argument of before, it admits minimal elements with respect to the partial order: these elements by definition are the elements of dimension $\alpha+1$ of K. We set $K_{\leq\alpha+1}$ as the union of $K_{\leq\alpha}$ and the set of this elements.

If we defined $K_{\leq\alpha}$ for any $\alpha < \lambda$ with λ a limit ordinal, we set $K_{\leq\lambda} = \bigcup_{\alpha<\lambda} K_{\leq\alpha}$.

Now we have to show that any element of K has a dimension assigned. Suppose we have $x \in K$ with $\dim(x)$ not defined. Then we can define the following subset $\{k \in K : k \leq x,\ \dim(k) \text{ not defined}\}$. As before, noetherianity implies this subspace has a minimal element. Therefore, without loss of generality we can assume $\dim(k)$ is defined for all $k < x$.

If there are no elements of K such that $k < x$ it follows x is a minimal point, hence it has dimension 0. If instead there are elements different from x contained in the closure of $\{x\}$ we can verify that $\dim(x) = \sup_{k<x} \dim(k) + 1$. Therefore, the function $\dim\colon K \to \mathrm{Ord}$ is well defined.

We conclude by noticing that, once established each point of K has a dimension assigned, the existence of a dimension for the whole K follows from the fact that K is a set and the class of ordinals is well-ordered. Thus there exists a minimal ordinal δ such that $K_{\leq\delta} = K$. □

Remark 4.5. Since $S \subseteq X$ is a noetherian subset of a spectral space, it satisfies the assumptions of Lemma 4.4 hence the Krull dimension function is defined on it.

By its very definition, the Krull dimension of Definition 4.2 satisfies the properties (i), (ii) and (iii) of [10, Def. 3.1]. However, since we are not considering a noetherian space which is necessarily spectral, this dimension function will not be spectral in the sense of [10, Def. 3.1]. That is, we are not guaranteed that $S_{\leq\alpha}$ will be a Thomason subspace of S. More importantly, even if S were a spectral subspace of X (i.e. a proconstructible subspace, see [10, Prop. 2.5]) we cannot guarantee that $S_{\leq\alpha}$ are Thomason subsets of the ambient space X.

Nevertheless, we will remedy that by constructing appropriate Thomason subsets using Proposition 4.1.

Example 4.6. If K is a noetherian spectral space then its dimension is a successor ordinal. By [4, Thm. 8.1.11 (ix)] the space K has finitely many irreducible components. Since K is sober it follows that each one of such components is the closure of a point, which is clearly maximal with respect to the specialization ordering.

By the properties defining spectral dimensions ([10, Def. 3.1]) it follows that the dimension of K coincides with the maximum of the dimensions of these maximal points. But by definition the dimension of one point is a successor ordinal, thus their maximum must also be a successor ordinal.

If instead K were not spectral, it could happen its dimension is a limit ordinal. We provide an explicit example. Consider the spectral space $(\omega+1)^\vee$, the dual of the linearly ordered set $\omega+1$ endowed with the coarse lower topology ([4, A.8 (ii)]). This has a basis of open quasi-compact subsets given by $\{y \in \omega+1 : y > x\}$ for x ranging among all the elements of $\omega+1$. It follows easily that $y \in \overline{\{x\}}$ if and only if $y \leq x$ and the space $(\omega+1)^\vee$ is noetherian.

We can consider its subspace $2\mathbb{N}$ given by the ordinal numbers which are even natural numbers. This is a noetherian subspace of a spectral space and it is easy to see that $\dim_{2\mathbb{N}}(2n) = n$. It follows that the dimension of $2\mathbb{N}$ is ω.

Importantly, we observe that $(2\mathbb{N})_{\leq i}$ is a Thomason subset of $(\omega+1)^\vee$ only for the starting case $i = 0$.

Theorem 4.7. *Let* $\mathrm{Supp}(t) \subseteq \mathrm{Spc}(\mathcal{T}^c)$ *be a noetherian subset, then the local-to-global principle holds for* $t \in \mathcal{T}$*.*

Proof. We first observe that since S is noetherian, each of its subsets is quasi-compact. Thus by Proposition 4.1 we have $\overline{S}^\vee = \mathrm{Gen}(S)$ and similarly for any ordinal α we have $\overline{S_{>\alpha}}^\vee = \mathrm{Gen}(S_{>\alpha})$.

We prove by induction that for every ordinal α the following inclusion holds

$$\Gamma_{\mathrm{Gen}(S_{>\alpha})^c} t \in \mathrm{Locid}\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in S_{\leq\alpha}\rangle.$$

Taking α to be the dimension of S allows us to conclude.

We begin with the starting case $\alpha = 0$. By Remark 2.2 it follows $t \cong t \otimes L_{\mathrm{Gen}(S)}\mathbb{1}$, hence we can consider $\Gamma_{\mathrm{Gen}(S_{>0})^c} L_{\mathrm{Gen}(S)} t$. This can be interpreted as an object of the the localized category $L_{\mathrm{Gen}(S)}\mathcal{T}$ whose Balmer spectrum coincides with $\mathrm{Gen}(S)$. Observe that the minimal points of $\mathrm{Gen}(S)$ are given by the elements of $S_{\leq 0}$. Since these points are closed and contained in a noetherian subset it follows they are visible (Corollary 3.2), thus $S_{\leq 0}$ is a Thomason subset of $\mathrm{Gen}(S)$. Hence we have $\Gamma_{\mathrm{Gen}(S_{>0})^c} L_{\mathrm{Gen}(S)} t \cong \Gamma_{S_{\leq 0}} t$, where $\Gamma_{S_{\leq 0}}$ refers to the colocalization in the category $L_{\mathrm{Gen}(S)}\mathcal{T}$. By the case when S is Thomason (Proposition 3.3), we deduce $\Gamma_{S_{\leq 0}} t \in \mathrm{Locid}\langle L_{\mathrm{Gen}(S)} g(\mathfrak{p}) \otimes t : \mathfrak{p} \in S_{\leq 0}\rangle$. Now Remark 3.4 implies

$$\Gamma_{\mathrm{Gen}(S_{>0})^c} t \in \mathrm{Locid}\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in S_{\leq 0}\rangle.$$

We now argue how to prove the claim for $\alpha + 1$, assuming it has been showed for α. We invoke the exact triangle

$$\Gamma_{\mathrm{Gen}(S_{>\alpha})^c} t \to \Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t \to L_{\mathrm{Gen}(S_{>\alpha})}\Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t \to \Sigma\Gamma_{\mathrm{Gen}(S_{>\alpha})^c} t.$$

We consider $L_{\mathrm{Gen}(S_{>\alpha})}\Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t$ as an object of the localization $L_{\mathrm{Gen}(S_{>\alpha})}\mathcal{T}$. We observe that the minimal elements of $\mathrm{Gen}(S_{>\alpha})$ coincide with the points of $S_{\leq\alpha+1} \cap S_{>\alpha}$. Again these points are closed and contained in a noetherian subset, hence they are visible by Corollary 3.2 and we deduce $S_{\leq\alpha+1} \cap S_{>\alpha}$ is a Thomason subset of $\mathrm{Gen}(S_{>\alpha})$. It follows we have $L_{\mathrm{Gen}(S_{>\alpha})}\Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t \cong \Gamma_{S_{\leq\alpha+1}\cap S_{>\alpha}} t$, where the last functor refers to the colocalization in the category $L_{\mathrm{Gen}(S_{>\alpha})}\mathcal{T}$. By the case with Thomason support (Proposition 3.3) we have

$$\Gamma_{S_{\leq\alpha+1}\cap S_{>\alpha}} t \in \mathrm{Locid}\langle L_{\mathrm{Gen}(S_{>\alpha})} g(\mathfrak{p}) \otimes t : \mathfrak{p} \in S_{\leq\alpha+1} \cap S_{>\alpha}\rangle$$

which implies by Remark 3.4

$$L_{\mathrm{Gen}(S_{>\alpha})}\Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t \in \mathrm{Locid}\langle g(\mathfrak{p}) \otimes t : \mathfrak{p} \in S_{\leq\alpha+1} \cap S_{>\alpha}\rangle.$$

Recalling the previous exact triangle, this inclusion together with the one coming from the induction assumption imply

$$\Gamma_{\mathrm{Gen}(S_{>\alpha+1})^c} t \in \mathrm{Locid}\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in S_{\leq\alpha+1}\rangle.$$

We now prove the claim for a limit ordinal λ, assuming it has been proved for all ordinals $\alpha < \lambda$.

Using an adaptation of [10, Lemma 3.8], exchanging the subspaces $X_{\leq\alpha}$ in the reference for the subspaces $\mathrm{Gen}(S_{>\alpha})^c$ employed here, we have

$$\Gamma_{\mathrm{Gen}(S_{>\lambda})^c} t \cong \underset{\alpha<\lambda}{\underrightarrow{\mathrm{holim}}}\, \Gamma_{\mathrm{Gen}(S_{>\alpha})^c} t$$

and the inductive assumption easily implies

$$\underset{\alpha<\lambda}{\underrightarrow{\mathrm{holim}}}\, \Gamma_{\mathrm{Gen}(S_{>\alpha})^c} t \in \mathrm{Locid}\left\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in \bigcup_{\alpha<\lambda} S_{\leq\alpha}\right\rangle$$

but since there are no points of S with dimension exactly λ it follows that this last localizing ideal coincides with $\mathrm{Locid}\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in S_{\leq\lambda}\rangle$. $\square$

5. Scattered support in a profinite Balmer spectrum

The nature of [2, Question 21.8] was to weaken the assumption guaranteeing the local-to-global principle to just the support of the examined object. We apply this reasoning to [10, Thm. 5.6] and prove that if the Balmer spectrum X is profinite and the support S is scattered then the local-to-global principle holds for t.

Remark 5.1. By [4, Thm. 1.3.4] and [4, Prop. 1.3.20] for a spectral space the following conditions are equivalent:

(i) it is a profinite space (also said Boolean or Stone space);
(ii) it is Hausdorff;
(iii) it is T_1;
(iv) it has the constructible topology.

Remark 5.2. We will adopt the notation of [4, §4.3] regarding Cantor-Bendixon derivatives and the definitions of scattered and perfect spaces.

We warn the reader that in this convention a perfect space is a space with no isolated points. A different choice, often found in the literature, is to call a space with no isolated points by *dense-in-itself* and to use perfect to denote the closed dense-in-itself subset of a space, which coincides with the maximal dense-in-itself subset of the space.

We start by providing a crucial lemma regarding the topology of the Balmer spectrum. Since X has the constructible topology its open subsets coincide with the Thomason subsets. As in [10, Def. 5.1], for an ordinal α and a scattered set C we set $C_{\leq\alpha}$ to be the subset of C given by the points with Cantor-Bendixon rank lesser or equal to α.

Lemma 5.3. *Let W be a T_1 space and $C \subseteq W$ a scattered subspace. Then for any ordinal α we have that*

$$\overline{C} \setminus C_{\leq\alpha}$$

is a closed subset of W. Alternatively, $C_{\leq\alpha}$ is an open subset of $\overline{C}$.

As a consequence, $C_{\leq\alpha+1} \setminus C_{\leq\alpha}$ is an open discrete subset of $\overline{C} \setminus C_{\leq\alpha}$.

Proof. We work by transfinite induction on α. We first show that $C_{\leq 0}$ is open in $\overline{C}$. By definition, for any $w \in C_{\leq 0}$ there is an open subset U of $\overline{C}$ such that $U \cap C = \{w\}$. Since W is a T_1 space, the singleton $\{w\}$ is closed, hence $U \cap \{w\}^c$ is an open subset contained in $\overline{C} \setminus C$. But because C is dense in $\overline{C}$, the set difference $\overline{C} \setminus C$ has empty interior. We deduce $U = \{w\}$, consequently $C_{\leq 0}$ is an open discrete subset of $\overline{C}$.

Now suppose we dealt the case $C_{\leq\alpha}$. We work in the subspace $\overline{C} \setminus C_{\leq\alpha}$. We observe that $C \setminus C_{\leq\alpha}$ is dense in $\overline{C} \setminus C_{\leq\alpha}$. Indeed, their difference coincides with

$$(\overline{C} \setminus C_{\leq\alpha}) \setminus (C \setminus C_{\leq\alpha}) = \overline{C} \setminus C$$

which as empty interior since C is dense in $\overline{C}$. Arguing as in the previous case, we have that $C_{\leq\alpha+1} \setminus C_{\leq\alpha}$ is an open discrete subset of $\overline{C} \setminus C_{\leq\alpha}$. Thus

$$(\overline{C} \setminus C_{\leq\alpha}) \setminus (C_{\leq\alpha+1} \setminus C_{\leq\alpha}) = \overline{C} \setminus C_{\leq\alpha+1}$$

is a closed subset of $\overline{C}$, hence it is also closed in W.

If $\alpha = \lambda$ is a limit ordinal we have that $C_{\leq\lambda} = \bigcup_{\beta<\lambda} C_{\leq\beta}$ and consequently

$$\overline{C} \setminus C_{\leq\lambda} = \bigcap_{\beta<\lambda} \overline{C} \setminus C_{\leq\beta}.$$

Being this an intersection of closed subsets, it is a closed subset of W. □

Theorem 5.4. *Assume* $\mathrm{Spc}(\mathcal{T}^c)$ *is a profinite space. Let* $\mathrm{Supp}(t) \subseteq \mathrm{Spc}(\mathcal{T}^c)$ *be a scattered subset, then the local-to-global principle holds for $t \in \mathcal{T}$.*

Proof. Using Remark 3.4, it is enough to show the local-to-global principle holds for $L_{\overline{S}}t$ in the category $L_{\overline{S}}\mathcal{T}$.

We prove by transfinite induction that for any ordinal α it holds

$$\Gamma_{S_{\leq\alpha}} L_{\overline{S}} t \in \mathrm{Locid}\langle t \otimes g(\mathfrak{p}) : \mathfrak{p} \in S_{\leq\alpha} \rangle,$$

where $\Gamma_{S_{\le\alpha}}$ denotes the colocalization functor associated with $(L_{\overline{S}}\mathcal{T})^c_{S_{\le\alpha}}$. This exists since Lemma 5.3 implies that $S_{\le\alpha}$ is a Thomason subset of the profinite Balmer spectrum $\overline{S} \cong \mathrm{Spc}(L_{\overline{S}}\mathcal{T}^c)$.

If $\alpha = 0$, $S_{\le 0}$ is an open and discrete subset of $\overline{S}$. Hence, we can use the argument of [10, Lemma 3.13] to prove that we have a decomposition

$$\Gamma_{S_{\le 0}} L_{\overline{S}} t \cong \coprod_{\mathfrak{p}\in S_{\le 0}} \Gamma_{\{\mathfrak{p}\}} L_{\overline{S}} t \cong \coprod_{\mathfrak{p}\in S_{\le 0}} t \otimes g(\mathfrak{p})$$

and the claim immediately follows.

Now suppose the claim has been verified for an ordinal α, we show it must follow for the successor $\alpha+1$. We invoke the exact triangle

$$\Gamma_{S_{\le\alpha}} L_{\overline{S}} t \to \Gamma_{S_{\le\alpha+1}} L_{\overline{S}} t \to L_{\overline{S}\setminus S_{\le\alpha}} \Gamma_{S_{\le\alpha+1}} L_{\overline{S}} t \to \Sigma\Gamma_{S_{\le\alpha}} L_{\overline{S}} t.$$

If we prove the local-to-global principle holds for $L_{\overline{S}\setminus S_{\le\alpha}} \Gamma_{S_{\le\alpha+1}} L_{\overline{S}} t$ we are done. Again, we can use Remark 3.4 to reduce to showing the principle for this object as an object of the localizing subcategory $L_{\overline{S}\setminus S_{\le\alpha}}\mathcal{T}$. In this category, we have an isomorphism

$$L_{\overline{S}\setminus S_{\le\alpha}} \Gamma_{S_{\le\alpha+1}} L_{\overline{S}} t \cong \Gamma_{S_{\le\alpha+1}\setminus S_{\le\alpha}} L_{\overline{S}\setminus S_{\le\alpha}} t$$

where on the right hand side we are invoking the colocalization functor associated with $S_{\le\alpha+1}\setminus S_{\le\alpha}$ which is a Thomason subset of $\overline{S}\setminus S_{\le\alpha}$ by Lemma 5.3. To be more precise, $S_{\le\alpha+1}\setminus S_{\le\alpha}$ is an open discrete subset of the profinite Balmer spectrum $\overline{S}\setminus S_{\le\alpha} \cong \mathrm{Spc}(L_{\overline{S}\setminus S_{\le\alpha}}\mathcal{T}^c)$, hence we can argue as in the case $\alpha = 0$ to obtain a decomposition

$$\Gamma_{S_{\le\alpha+1}\setminus S_{\le\alpha}} L_{\overline{S}\setminus S_{\le\alpha}} t \cong \coprod_{\mathfrak{p}\in S_{\le\alpha+1}\setminus S_{\le\alpha}} t \otimes g(\mathfrak{p}).$$

Finally, we consider the case when α is a limit ordinal λ. We can adapt the argument of [10, Lemma 3.8], exchanging the subspaces $X_{\le\alpha}$ in the reference for the subspaces $S_{\le\beta}$ used here, to obtain the isomorphism

$$\Gamma_{S_{\le\lambda}} L_{\overline{S}} t \cong \underset{\beta<\lambda}{\underrightarrow{\mathrm{holim}}}\, \Gamma_{S_{\le\beta}} L_{\overline{S}} t.$$

Now the claim easily follows from the inductive assumption: the fact that $\Gamma_{S_{\le\beta}} L_{\overline{S}} t \in \mathrm{Locid}\langle t\otimes g(\mathfrak{p}) : \mathfrak{p}\in S_{\le\beta}\rangle$ for all ordinals $\beta < \lambda$ implies

$$\Gamma_{S_{\le\lambda}} L_{\overline{S}} t \in \mathrm{Locid}\left\langle t\otimes g(\mathfrak{p}) : \mathfrak{p} \in \bigcup_{\beta<\lambda} S_{\le\beta}\right\rangle = \mathrm{Locid}\langle t\otimes g(\mathfrak{p}) : \mathfrak{p}\in S_{\le\lambda}\rangle.$$

The claim taken for α equal to the Cantor-Bendixon rank of S is the local-to-global principle for $L_{\overline{S}} t$. ☐

Example 5.5. Suppose that $\mathcal{T}$ does not satisfy the local-to-global principle. By [2, Thm. 6.4], this is equivalent to the subcategory

$$\mathrm{Locid}\langle g(\mathfrak{p}) : \mathfrak{p}\in X\rangle^{\perp}$$

not being zero. If X is profinite, we claim that the support of any non zero object of such subcategory must be a perfect subspace of the Balmer spectrum.

Indeed, let t be one of such objects with $S = \mathrm{Supp}(t)$ and assume the Cantor-Bendixon derivative δS is a proper subset of S. Then we can repeat the argument of Theorem 5.4 to obtain the exact triangle

$$\coprod_{\mathfrak{p}\in S_{\le 0}} g(\mathfrak{p})\otimes t \to t \to L_{\overline{S}\setminus S_{\le 0}} t \to \Sigma \coprod_{\mathfrak{p}\in S_{\le 0}} g(\mathfrak{p})\otimes t.$$

By the definition of t, the first morphism must be trivial, hence the triangle splits giving us

$$L_{\overline{S}\setminus S_{\le 0}} t \cong t \oplus \coprod_{\mathfrak{p}\in S_{\le 0}} g(\mathfrak{p})\otimes t.$$

Observe that for any $\mathfrak{p}\in S_{\le 0}$, this prime belongs to the support of the object on the right hand side, while it does not belong to the support of the object on the left hand side. This is an absurd, hence we conclude $S = \delta S$.

6. Injective superdecomposable modules over non semi-artinian absolutely flat rings

In this last section, we apply Theorem 5.4 to the study of the support of injective superdecomposable modules over non semi-artinian absolutely flat rings. We refer the reader to [9] for the importance these objects have in the theory of tensor triangular geometry: they provide examples of objects not satisfying the local-to-global principle.

From now on, all rings which we consider will be unital and commutative.

Let R be an absolutely flat ring which is not semi-artinian. In [9, Thm. 4.7] it is proved that any injective superdecomposable R-module is right orthogonal to the idempotents $g(\mathfrak{p})$. Therefore, Example 5.5 implies that its support must be a perfect subset of $\mathrm{Spc}(\mathrm{D}(R)^c)$.

We will present an injective superdecomposable R-module whose support coincides with the maximal perfect subset of $\mathrm{Spc}(\mathrm{D}(R)^c)$.

Before starting, we recall some basic facts which we will use through this section:

(a) as consequence of [11, Thm. 4.1], there is a homeomorphism between the Zariski spectrum $\mathrm{Spec}(A)$ of the ring A and the Balmer spectrum $\mathrm{Spc}(\mathrm{D}(A)^c)$ given by

$$\begin{aligned} \mathrm{Spec}(A) &\cong \mathrm{Spc}(\mathrm{D}(A)^c) \\ P &\mapsto \mathfrak{p} = \{M \in \mathrm{D}(A)^c : M_P \simeq 0\} \end{aligned}$$

where M_P denotes the localization at the prime ideal P.

In the case of a general ring, one should be careful because this map is inclusion reversing. However, in the case of absolutely flat rings there are no proper inclusion of prime ideals. Under this identification we denote $X = \mathrm{Spec}(R)$ for the absolutely flat, non semi-artinian ring R we are considering.

(b) For an absolutely flat ring A, given a generic element $a \in A$ it is possible to concoct an idempotent element $a' \in A$ such that $(a) = (a')$. Thus, all the principal ideals of A are generated by idempotents. See [7, Lemma 2].

(c) Given an ideal $I \subseteq A$ of an absolutely flat ring, the quotient A/I is also absolutely flat. This guarantees that all the rings we will construct later will remain absolutely flat.

Lemma 6.1. *Let A be an absolutely flat ring. Then there is a bijection*

$$\{\text{non-zero minimal ideals of } A\} \leftrightarrow \{\text{isolated points of } \mathrm{Spec}(A)\}.$$

Proof. Suppose we start with a simple ideal I, this can be written as (a) for an idempotent element $a \in A$. Thus $1-a$ is the orthogonal idempotent, we set $J = (1-a)$ and claim this is a maximal ideal. By construction we have $A = I + J$, now we show $I \cap J = 0$. An element of $I \cap J$ can be written as $x = \lambda a = \mu(1-a)$, but using the fact that a is idempotent we have $x = ax = \mu a(1-a) = 0$. Hence we proved $A = I \oplus J$.

We now show J is maximal. Suppose we have $J \subsetneq M$ for another ideal M, then taking the intersection of M with A we get $M = IM \oplus JM$. Since $J \subsetneq M$ it must follow $IM \neq 0$, thus there exists $x \in M$ such that $ax \neq 0$. By minimality of I it follows $(a) = (ax)$ hence $IM = I$. We have both a and $1-a$ belong to M, thus $M = A$. This shows J must be a maximal ideal.

By construction, we have $J \in U(a)$. If we prove $U(a) = \{J\}$ this proves the prime is isolated in the Zariski spectrum. Let M be a maximal ideal different from J. My taking the intersection of M with the decomposition $A = I \oplus J$ we get $M = IM \oplus JM$, if $IM = 0$ we deduce $M \subseteq J$ contradicting the maximality of M. Hence $IM \neq 0$, this mean there exists $x \in M$ with $ax \neq 0$. By minimality of $I = (a)$, we deduce $(a) = (ax) \subseteq M$, thus $M \in V(a)$. Therefore the only prime ideal in $U(a)$ is J.

Suppose now we start from P an isolated prime ideal. The Zariski spectrum has a basis given by $U(x)$ for $x \in A$. Since $U(x) \cap U(y) = U(xy)$, we have that P is isolated iff $\{P\} = U(a)$ for some $a \in A$. Since $U(a)$ depends only on the ideal generated by a and A is absolutely flat, we can assume a is idempotent. Observe that $R = (a) \oplus (1-a)$ as before. Moreover, $a(1-a) = 0 \in P$ and P being prime implies $1-a \in P$. Thus $(1-a) \subseteq P$.

We now prove $(a) \cap P = 0$. First observe that for any maximal ideal M different from P it holds $(a) \subseteq M$. Indeed, if $U(a) = \{P\}$ then $V(a) = \{M \in \mathrm{Spec}(A) : M \neq P\}$. Therefore, we have

$$(a) \subseteq \bigcap_{M \neq P} M$$

and it follows

$$(a) \cap P \subseteq \bigcap_{M \in \mathrm{Spec}(A)} M.$$

But the intersection of all prime ideals coincides with the nilradical of A, this ring is reduced being absolutely flat hence the nilradical is 0. We conclude $(a) \cap P = 0$.

Thus $(1-a) = P$, since $\forall p \in P$ we have $p = pa + p(1-a) = p(1-a)$. From the maximality of P it follows (a) is minimal. Indeed, suppose $I \subsetneq (a)$ is not zero, then $P + I$ is an ideal of A which is strictly bigger than P, thus $I + P = A$. It follows $a \in I + P$, i.e. $a = ax + p$ with $ax \in I$ and $p \in P$. But $a = a^2 = a^2x + ap = ax \in I$ and we conclude $I = (a)$. Thus (a) is non-zero minimal. $\square$

We give the following notation for the socle filtration of modules over a ring. It is not the standard one, but it will be useful for our purpose.

Definition 6.2. Let B be an arbitrary ring and let M be a B-module. We set sM to be the socle of M. Recall that the socle is defined to be the sum of all simple submodules of the module in question. We set $s_0M = 0$ and $s_1M = sM$. Now we define recursively $s_\alpha M$, a submodule of M, for every ordinal α.

Suppose we defined $s_\alpha M$, then we set $s_{\alpha+1}M$ to be the submodule of M such that $s_{\alpha+1}M/s_\alpha M$ is the socle of $M/s_\alpha M$. I.e. $s_{\alpha+1}M/s_\alpha M = s(M/s_\alpha M)$.

If λ is a limit ordinal, then we set $s_\lambda M = \bigcup_{\alpha<\lambda} s_\alpha M$.

This sequence of submodules is called *Loewy series* of M. The *Loewy length* of M is defined to be the minimal ordinal δ such that $s_\delta M = M$, if such an ordinal exists.

The ring B being semi-artinian is equivalent to the existence of the Loewy length for B, considered as B-module.

Lemma 6.3. *Let A be an absolutely flat ring. Then we have*

$$\mathrm{Spec}(A/sA) \cong \{P \in \mathrm{Spec}(A) : sA \subseteq P\} \cong \delta\mathrm{Spec}(A).$$

Proof. We have to show that $sA \subseteq P$ iff P is not isolated in $\mathrm{Spec}(A)$.

Suppose we have P an isolated prime. Then in Lemma 6.1 we saw there is a decomposition $A = P \oplus I$ with I a simple module. Since $I \subseteq sA$ it follows $sA \nsubseteq P$.

Now suppose the prime P does not contain sA. By definition, $sA = \sum_{I \text{ simple}} I$ hence there must exit a simple module I such that $I \nsubseteq P$. In Lemma 6.1 we saw that $I = (a)$ with $a \in A$ idempotent and there is a decomposition $A = (a) \oplus (1-a)$. Again $P = (1-a)$ and $\{P\} = U(a)$, thus P is isolated. $\square$

This result allows us to produce a correspondence between Loewy series and Cantor-Bendixon derivatives.

Proposition 6.4. *Let A be an absolutely flat ring. Then for any ordinal α we have*

$$\mathrm{Spec}(A/s_\alpha A) \cong \delta^\alpha \mathrm{Spec}(A).$$

Proof. The argument is a basic transfinite induction, using Lemma 6.3.

If $\alpha = 0$ the claim is trivial. Suppose we showed the claim for a general ordinal α. Then we have

$$\begin{aligned}\delta^{\alpha+1}\mathrm{Spec}(A) &= \delta\delta^\alpha\mathrm{Spec}(A) = \delta\mathrm{Spec}(A/s_\alpha A) = \\ &= \mathrm{Spec}\Big((A/s_\alpha A)/(s_{\alpha+1}A/s_\alpha A)\Big) \cong \mathrm{Spec}(A/s_{\alpha+1}A)\end{aligned}$$

where the second equality comes from the induction assumption, the third relies on Lemma 6.3 and the definition of $s_{\alpha+1}A$.

Now suppose λ is a limit ordinal. By definition $\delta^\lambda\mathrm{Spec}(A) = \bigcap_{\alpha<\lambda} \delta^\alpha\mathrm{Spec}(A)$, using the inductive assumption we deduce $\delta^\lambda\mathrm{Spec}(A) = \bigcap_{\alpha<\lambda}\mathrm{Spec}(A/s_\alpha A)$. Observe that the filtration

$$0 = s_0A \subseteq s_1A \subseteq \cdots \subseteq s_\alpha A \subseteq s_{\alpha+1}A \subseteq \cdots \subseteq s_\lambda A \subseteq \ldots$$

induces the sequence of embeddings of spectral spaces

$$\mathrm{Spec}(A) \hookleftarrow \mathrm{Spec}(A/s_1A) \hookleftarrow \cdots \hookleftarrow \mathrm{Spec}(A/s_\alpha A) \hookleftarrow \mathrm{Spec}(A/s_{\alpha+1}A) \hookleftarrow \cdots \hookleftarrow \mathrm{Spec}(A/s_\lambda A) \hookleftarrow \ldots$$

and recall we have the identification $\mathrm{Spec}(A/s_\alpha A) \cong \{P \in \mathrm{Spec}(A) : s_\alpha A \subseteq P\}$.

We conclude

$$\bigcap_{\alpha<\lambda} \operatorname{Spec}(A/s_\alpha A) = \{P \in \operatorname{Spec}(A) : s_\alpha A \subseteq P\ \forall \alpha < \lambda\} = \left\{P \in \operatorname{Spec}(A) : \bigcup_{\alpha<\lambda} s_\alpha A \subseteq P\right\} =$$
$$= \{P \in \operatorname{Spec}(A) : s_\lambda A \subseteq P\} = \operatorname{Spec}(A/s_\lambda A).$$

□

The immediate consequence of this result is the following:

Corollary 6.5. *Let A be an absolutely flat ring. Then it is semi-artinian if and only if* $\operatorname{Spec}(A)$ *is scattered.*

To conclude the correspondence between the Loewy series of an absolutely flat ring A and the topology of its Zariski spectrum, we provide an explicit description of $s_\alpha A$ in terms of the prime ideals filtered by the Cantor-Bendixon rank.

Lemma 6.6. *Let A be an absolutely flat ring. Then we have*

$$sA = \bigcap_{P \in \delta\operatorname{Spec}(A)} P.$$

Proof. We first prove that $Q \in \operatorname{Spec}(A)$ is not isolated if and only if $\bigcap_{P\in\delta\operatorname{Spec}(A)} P \subseteq Q$.

For the non-trivial implication we need to show Q isolated implies $\bigcap_{P\in\delta\operatorname{Spec}(A)} P \nsubseteq Q$.

Given $Q \in \operatorname{Spec}(A)$ an isolated prime, we set $I \in A$ to be non-zero minimal ideal of A associated to Q by the correspondence of Lemma 6.1. Observe that for a prime ideal $P \neq Q$ we have $I \cap P$ is either 0 or I, by the minimality of I. If this intersection were 0, it would follow that $P \subseteq Q$, which is an absurd since P is maximal. We conclude $P \cap I = I$, i.e. $I \subseteq P$.

Varying P among all elements of $\delta\operatorname{Spec}(A)$ we deduce $I \subseteq \bigcap_{P\in\delta\operatorname{Spec}(A)} P$. If it held $\bigcap_{P\in\delta\operatorname{Spec}(A)} P \subseteq Q$ we would get $I \subseteq Q$, an absurd since $A = Q \oplus I$ by construction.

This equivalence together with Lemma 6.3 implies that the subsets of $\operatorname{Spec}(A)$ given by $V\left(\bigcap_{P\in\delta\operatorname{Spec}(A)} P\right)$ and $V(sA)$ coincide. This means that the radicals of the ideals $\bigcap_{P\in\delta\operatorname{Spec}(A)} P$ and sA coincide, but since A is absolutely flat all ideals are radical. This proves the claim. □

Proposition 6.7. *Let A be an absolutely flat ring. Then for any ordinal α it holds*

$$s_\alpha A = \bigcap_{P\in\delta^\alpha\operatorname{Spec}(A)} P.$$

Proof. We argue by transfinite induction on α.

If $\alpha = 0$ we have $s_0 A = 0$ by definition, while $\bigcap_{P\in\delta^0\operatorname{Spec}A} P = \bigcap_{P\in\operatorname{Spec}A} P$. But the intersection of all maximal ideals of a ring coincides with the nilradical, in our case A is reduced hence its nilradical is zero.

The case $\alpha = 1$ is Lemma 6.6.

Now suppose the claim has been proved up to an ordinal α, then we show the claim is true for $\alpha + 1$.

Applying Lemma 6.6 to the ring $A/s_\alpha A$ we obtain

$$s_{\alpha+1}A/s_\alpha A = s(A/s_\alpha R) = \bigcap_{Q\in\delta\operatorname{Spec}(A/s_\alpha A)} Q.$$

Under the identification of Proposition 6.4 this set can be described as

$$\bigcap_{Q\in\delta\operatorname{Spec}(A/s_\alpha A)} Q = \bigcap_{P\in\delta^{\alpha+1}\operatorname{Spec}(A)} P/s_\alpha A.$$

Observe it holds the inclusion

$$\left(\bigcap_{P\in\delta^{\alpha+1}\operatorname{Spec}(A)} P\right)/s_\alpha A \subseteq \bigcap_{P\in\delta^{\alpha+1}\operatorname{Spec}(A)} P/s_\alpha A = s_{\alpha+1}A/s_\alpha A.$$

However, a priori the first inclusion could be proper, hence we can only deduce $\bigcap_{P\in\delta^{\alpha+1}\operatorname{Spec}(A)} P \subseteq s_{\alpha+1}A$.

Observe that the equality

$$s_{\alpha+1}A/s_\alpha A = \bigcap_{P\in\delta^{\alpha+1}\operatorname{Spec}(A)} P/s_\alpha A$$

implies $s_{\alpha+1}A/s_\alpha A \subseteq P/s_\alpha A$ for all $P \in \delta^{\alpha+1}\mathrm{Spec}(A)$, thus $s_{\alpha+1}A \subseteq P$. This gives us the inverse inclusion $s_{\alpha+1}A \subseteq \bigcap_{P\in\delta^{\alpha+1}\mathrm{Spec}(A)} P$ and we conclude these two ideals coincide.

Let λ be a limit ordinal and suppose we showed the claim for all $\alpha < \lambda$. By definition $s_\lambda A = \bigcup_{\alpha<\lambda} s_\alpha A$ and we have $s_\alpha A = \bigcap_{P\in\delta^\alpha\mathrm{Spec}(A)} P$. Observe that for $\beta < \gamma$ we have the inclusion $\bigcap_{P\in\delta^\beta\mathrm{Spec}(A)} P \subseteq \bigcap_{P\in\delta^\delta\mathrm{Spec}(A)} P$. Thus, it follows that

$$\bigcup_{\alpha<\lambda} s_\alpha A = \bigcup_{\alpha<\lambda}\left(\bigcap_{P\in\delta^\alpha\mathrm{Spec}(A)} P\right) = \varinjlim_{\alpha<\lambda} \bigcap_{P\in\delta^\alpha\mathrm{Spec}(A)} P = \bigcap_{P\in\bigcap_{\alpha<\lambda}\delta^\alpha\mathrm{Spec}(A)} P = \bigcap_{P\in\delta^\lambda\mathrm{Spec}(A)} P.$$

This concludes the induction argument. □

Corollary 6.8. *Let R be an absolutely flat ring which is not semi-artinian. Let σ be the minimal ordinal such that $s_\sigma R$ stabilizes. Hence the ring $R/\bigcap_{P\in\delta^\sigma X} P$ is an absolutely flat ring with trivial socle and its Zariski spectrum corresponds to the maximal perfect subspace of X.*

Proof. We have just to put together the results proved above. Proposition 6.7 states that $s_\sigma R = \bigcap_{P\in\delta^\sigma X} P$ and Proposition 6.4 tells us that $\mathrm{Spec}(R/s_\sigma R)$ coincides with $\delta^\sigma X$. Since $s_\sigma R = s_{\sigma+1}R$ we have $\delta^\sigma X = \delta^{\sigma+1}X$, hence this subspace coincides with the maximal perfect subset of X. Since the spectrum of $R/s_\sigma R$ has no isolated points Lemma 6.1 implies that this ring has no minimal non-zero ideals. □

Example 6.9. Let k be any field and set $R = \prod_{\mathbb{N}} k$ which is an absolutely flat ring. In this case $\mathrm{Spec}(R)$ is homeomorphic to $\beta\mathbb{N}$, the Stone-Cech compactification of the natural numbers. This space can be subdivided in its set of isolated points, corresponding to the principal ultrafilters on $\mathbb{N}$, and its maximal perfect subset, given by the non-principal ultrafilters.

For $i \in \mathbb{N}$, we set e_i to be the element of R with entry 1 at its i-th coordinate and 0 otherwise. Then $(1-e_i)$ is the isolated prime ideal corresponding to the principal ultrafilter induced by i. We have a decomposition $R = (1-e_i) \oplus (e_i)$ as in Lemma 6.1. It follows $s_1R = \oplus_i(e_i) = \sum_{\mathbb{N}} k$, coherently with the usual computations in the literature. The direct sum being the intersection of the prime ideals associated to the non-principal ultrafilters corresponds to the known fact that the cofinite filter coindices with the intersection of all non-principal ultrafilters.

In this situation, the ring invoked in Corollary 6.8 is $\prod_{\mathbb{N}} k/\sum_{\mathbb{N}} k$ which is known to have trivial socle.

We are finally ready to construct the injective superdecomposable module we wanted.

Theorem 6.10. *Let R be an absolutely flat ring which is not semi-artinian. Let σ be the minimal ordinal such that $s_\sigma R$ stabilizes. Let E denote the injective hull of $R/\bigcap_{P\in\delta^\sigma X} P$. Then E is an injective superdecomposable module and as an element of $\mathrm{D}(R)$ its support coincides with $\delta^\sigma X$, the maximal perfect subset of X.*

Proof. We first show E is superdecomposable. By [5, Lemma 5.14] this is equivalent to proving that $R/s_\sigma R$ does not admit uniform submodules, i.e. for any non-zero ideal $I \subseteq R/s_\sigma R$ there exist non-zero ideals $J_1, J_2 \subseteq I$ such that $J_1 \cap J_2 = 0$.

Let us consider a generic non-zero ideal I and take $i \in I \setminus \{0,1\}$. Since we are interested only in the associated principal ideal $(i) \subseteq I$ we can assume i is idempotent. Using the orthogonal idempotent $1-i$ we obtain the decomposition $R/s_\sigma R = (i) \oplus (1-i)$.

First consider the case $(1-i)\cap I = 0$, which would imply $I = (i)$. We showed in Corollary 6.8 that $R/s_\sigma R$ has no non-zero minimal ideals. Hence (i) must have a proper non-zero submodule, say $(xi) \subsetneq (i)$ where again we can assume the element x to be idempotent. The fact that $ix \neq i$ implies $i(1-x) \neq 0$. Therefore, (ix) and $(i(1-x))$ provide two non-zero submodules of I with trivial intersection.

Now consider the case $(1-i)\cap I \neq 0$, i.e. there exits some idempotent x such that $(x(1-i)) \subseteq J$ and $(x(1-i)) \neq 0$. If it held $xi = 0$, this would imply $0 \neq (x) \subseteq I$. In this situation the ideals (x) and (i) are non-zero submodules of I such that $(i)\cap(x) = 0$, just as we need. Thus, we assume $xi \neq 0$ and in this instance (xi) and $(x(1-i))$ are the non-zero submodules of I doing the job.

We finally show that $\mathrm{Supp}(E) = \delta^\sigma X$. We recall two important facts about the tensor-triangulated category $\mathrm{D}(R)$. First, since all R-modules are flat, the tensor product of modules is an exact functor, thus the derived tensor product on $\mathrm{D}(R)$ coincides with the non-derived tensor product of chains. Second, since the spectrum X is T_1 we have that the idempotent $g(P)$ associated to the prime P coincides with $L_P\mathbb{1} = R_P$, the localization at P. Furthermore, it is easy to prove that the localization morphism $R \to R_P$ is surjective and its kernel coincides with P, thus we deduce $g(P) \cong R/P$.

We have that the support of the ring $R/s_\sigma R$ coincides with $\delta^\sigma X$. Indeed, for any prime $Q \in X$ we can compute

$$R/s_\sigma R \otimes g(Q) = R/s_\sigma R \otimes R/Q = R/(s_\sigma R + Q).$$

Since $s_\sigma R = \bigcap_{P \in \delta^\sigma X} P$, if $Q \in \delta^\sigma X$ it follows $s_\sigma R + Q = Q$ and $R/s_\sigma R \otimes g(Q) \neq 0$. If instead $Q \notin \delta^\sigma X$, by Proposition 6.7 and Proposition 6.4 we have $\bigcap_{P \in \delta^\sigma X} P \not\subset Q$. Thus $Q + \bigcap_{P \in \delta^\sigma X} P$ is an ideal strictly containing the maximal ideal Q, hence it must coincide with the whole ring R and consequently $R/s_\sigma R \otimes g(Q) = 0$.

If we tensor the inclusion $R/s_\sigma R \hookrightarrow E$ with $g(P)$ we still have an inclusion since $g(P)$ is flat. Therefore, it follows that $\mathrm{Supp}(R/s_\sigma R) \subseteq \mathrm{Supp}(E)$. Example 5.5 and [9, Thm. 4.7] give us the inverse inclusion $\mathrm{Supp}(E) \subseteq \delta^\sigma X$ and we conclude. □

Remark 6.11. One could wonder if the ring $R/s_\sigma R$ is already injective, so that it is not necessary to form its injective hull. Unfortunately, this seems too optimistic: if we consider $R = \prod_{\mathbb{N}} k$ as in Example 6.9 then the main theorem of [8] implies that $\prod_{\mathbb{N}} k / \sum_{\mathbb{N}} k$ is not an injective R-module.

Since this is one of the simplest examples of non semi-artinian absolutely flat rings, this should indicate we should not expect $R/s_\sigma R$ to be injective in general.

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