COMPACTLY SUPPORTED A¹-EULER CHARACTERISTICS OF SYMMETRIC POWERS OF CELLULAR VARIETIES

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ABSTRACT. The compactly supported \mathbb{A}^1 -Euler characteristic, introduced by Hoyois and later refined by Levine and others, is an anologue in motivic homotopy theory of the classical Euler characteristic of complex topological manifolds. It is an invariant on the Grothendieck ring of varieties $K_0(Var_k)$ taking values in the Grothendieck-Witt ring GW(k) of the base field k. The former ring has a natural power structure induced by symmetric powers of varieties. In a recent preprint, Pajwani and Pál construct a power structure on GW(k) and show that the compactly supported \mathbb{A}^1 -Euler characteristic respects these two power structures for 0-dimensional varieties, or equivalently étale k-algebras. In this paper, we define the class Sym_k of symmetrisable varieties to be those varieties for which the compactly supported \mathbb{A}^1 -Euler characteristic respects the power structures and study the algebraic properties of $K_0(Sym_k)$. We show that it includes all cellular varieties, and even linear varieties as introduced by Totaro. Moreover, we show that it includes non-linear varieties such as elliptic curves. As an application of our main result, we compute the compactly supported \mathbb{A}^1 -Euler characteristics of symmetric powers of Grassmannians and certain del Pezzo surfaces.

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1. INTRODUCTION

The compactly supported \mathbb{A}^1 -Euler characteristic χ^{mot} was first introduced in work of Hoyois [Hoy14], later refined by Levine [Lev20] for smooth projective schemes and extended to general varieties over a field in characteristic zero by Arcila-Maya, Bethea, Opie, Wickelgren and Zakharevich [AMBO⁺22], and to general varieties in characteristic not equal to 2 by Levine, Pepin-Lehalleur and Srinivas in [LPS24]. It is an algebro-geometric invariant that refines both the real and complex Euler characteristic of topological manifolds, as well as some additional arithmetic data. As opposed to the classical Euler characteristic, which takes values in \mathbb{Z} , the compactly supported \mathbb{A}^1 -Euler characteristic takes values in the Grothendieck-Witt ring GW(k) of the base field k, so it contains "quadratic" information. However, unlike the classical Euler characteristic, it can be difficult to compute $\chi^{\text{mot}}(X)$ even when X is a smooth projective variety. Papers such as [LPS24] and [Vie23] use the motivic Gauss-Bonnet Theorem of Levine-Raksit [LR20] to compute the compactly supported \mathbb{A}^1 -Euler characteristic of hypersurfaces in \mathbb{P}^n and complete intersections of hypersurfaces of the same degree in \mathbb{P}^n . Also, Brazelton, McKean and Pauli computed the compactly supported \mathbb{A}^1 -Euler characteristics of Grassmannians in [BMP23], using \mathbb{A}^1 -degrees. While this invariant can be difficult to work with, it has found use in enumerative geometry since it is analogous to the classical Euler characteristic of a manifold. We may use this invariant to obtain enumerative geometry counts which take values in GW(k), and papers such as [PP22] by Pajwani and Pál and [BBG24] by Blomme, Brugallé and Garay, use the compactly supported \mathbb{A}^1 -Euler characteristic to obtain arithmetic refinements of results in complex enumerative geometry, the first over a general base field and the second over the real numbers.

This paper is concerned with the compactly supported \mathbb{A}^1 -Euler characteristic of symmetric powers of varieties. These geometric objects are closely related to Hilbert schemes of points via the birational Hilbert-Chow morphism. They are of particular interest to people studying enumerative geometry, appearing for example in the Göttsche formula for Euler characteristics of Hilbert schemes of surfaces ([Göt90, Theorem 0.1], [PP22, Corollary 8.5]). Since these varieties are almost always singular if $\dim(X) \ge 2$, we cannot directly apply the motivic Gauss-Bonnet theorem of [LR20] to them, and as such their compactly supported \mathbb{A}^1 -Euler characteristics seem difficult to compute directly. Therefore, we instead use the power structures defined in [PP23] for this computation. We give a formula for the compactly supported \mathbb{A}^1 -Euler characteristic of symmetric powers of a large class of varieties that we call K_0 -étale linear, see Definition 3.1. Informally, K_0 -étale linear varieties are varieties whose class in $K_0(Var_k)$ decomposes into a sum with terms $[\mathbb{A}_{L}^{n}]$, where L/k is a finite separable extension (see Definition 3.1). These form a large class of varieties containing many widely studied varieties, such as cellular varieties (Lemma 3.3), del Pezzo surfaces of degree ≥ 5 (Theorem 5.7), certain tori (Example 3.2) and others. Our main result can be stated as follows:

Theorem 1.1 (Theorem 4.10). Let X be a K₀-étale linear variety over a base field k of characteristic 0 (see Definition 3.1), and for $n \in \mathbb{Z}_{\geq 0}$, write $X^{(n)} :=$ Symⁿ(X). Then $\chi^{\text{mot}}(X^{(n)}) = a_n(\chi^{\text{mot}}(X))$ for every n, where a_n denotes the function defining the power structure on GW(k) as in Definition 2.8.

The power of the above theorem lies in the fact that it is much easier to work with the power structure on GW(k) than it is to decompose the symmetric powers of K₀-étale linear varieties in general.

We define a variety X to be symmetrisable if χ^{mot} respects the power structure as in our main result, i.e. $\chi^{\text{mot}}(X^{(n)}) = a_n(\chi^{\text{mot}}(X))$ for all n, see Definition 4.1. Corollaries 6.4 and 6.6 show that the class of symmetrisable varieties contains curves of genus 1 and that these are not K₀-étale linear.

Theorem 1.2 (Corollary 6.4 and Corollary 6.6). Let C be a curve of genus 1. Then $\chi^{\text{mot}}(C^{(n)}) = a_n(\chi^{\text{mot}}(C)), \text{ but } [C] \notin K_0(\text{ÉtLin}_k)$

Additionally, we show in Theorem 4.17 that a variety over k must itself be symmetrisable if it becomes symmetrisable after base change to a finite extension L/k of odd degree. We use this to show that even dimensional Severi–Brauer varieties are symmetrisable in Corollary 4.19 even though they may not be K₀-étale linear. While we only show that curves of genus ≤ 1 are symmetrisable, in forthcoming work, Lukas Bröring and the third author [BV24] show that all curves are symmetrisable using different techniques.

We apply our main result in Theorem 5.8 to compute $\chi^{\text{mot}}(X^{(3)})$ for X a cubic surface; a computation which we believe would be difficult to do without using the power structure. Similarly, we use it to compute a generating series for χ^{mot} of the symmetric powers of a Grassmannian.

Theorem 1.3 (Corollary 5.4). There is a generating series for the compactly supported \mathbb{A}^1 -Euler characteristic of the symmetric power of a Grassmannian:

$$\sum_{t=0}^{\infty} \chi^{mot} (\operatorname{Gr}(d,r)^{(n)}) t^n = (1-t)^{-e(d,r)} (1-(\langle -1\rangle t))^{-o(d,r)} \in \operatorname{GW}(k)[[t]],$$

where e(d,r) is the d-th entry in the r-th row of Losanitsch's triangle, and $o(d,r) = \binom{r}{d} - e(d,r)$.

The above result enriches the generating series of the classical Euler characteristic of symmetric powers of Grassmannians, as the rank map $GW(k) \to \mathbb{Z}$ sends the form $\langle -1 \rangle$ to 1 and the sum e(d, r) + o(d, r) is the binomial coefficient $\binom{r}{d}$.

In Section 2, we recall notions required for our paper. We first restate the definition of the compactly supported \mathbb{A}^1 -Euler characteristic in Definition 2.3. To compute the compactly supported \mathbb{A}^1 -Euler characteristics of symmetric powers of varieties, we use the notion of a power structure on a ring, see Definition 2.6. We recall the existence natural power structures on both $K_0(\operatorname{Var}_k)$ and $\operatorname{GW}(k)$ following [GZLMH06] and [PP23]. We introduce the notion of a K_0 -étale linear variety in Section 3 (Definition 3.1), and prove some of their basic properties. Section 4 is concerned with proving the main theorem of this paper, using Göttsche's lemma for symmetric powers [Göt01, Lemma 4.4]. Section 5 then uses the main result to compute the Euler characteristics of Grassmannians and a sizeable class of del Pezzo surfaces. Finally in Section 6, we turn our attention to varieties which do not become K_0 -étale linear over any field, but are nonetheless symmetrisable.

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Notation. Fix k to be a field of characteristic 0. For a variety X over k, i.e. a reduced separated scheme of finite type over k, and $n \in \mathbb{Z}_{\geq 0}$, let $X^{(n)}$ be the n^{th} symmetric power of X, which is the quotient of X^n by the action of the symmetric group on n letters permuting the co-ordinates.

2. Compactly supported $\mathbb{A}^1\text{-}\mathrm{Euler}$ Characteristics and Symmetric Powers

In this section we recall results concerning compactly supported \mathbb{A}^1 -Euler characteristics of varieties, as well as the notion of a power structure on a ring.

2.1. The compactly supported \mathbb{A}^1 -Euler characteristic.

Definition 2.1. Let Var_k be the category of varieties over k. The Grothendieck ring of varieties $\operatorname{K}_0(\operatorname{Var}_k)$ is the free abelian group generated by isomorphism classes [X] of varieties $X \in \operatorname{Var}_k$ modulo the relation $[X] = [Z] + [X \setminus Z]$ for every closed immersion $Z \to X$ in Var_k , together with the multiplication given on generators by $[X][Y] = [X \times_k Y]$. Note that $1 = [\operatorname{Spec} k]$ and $0 = [\emptyset]$ in $\operatorname{K}_0(\operatorname{Var}_k)$. Denote the subring of $\operatorname{K}_0(\operatorname{Var}_k)$ which is generated by dimension 0 varieties by $\operatorname{K}_0(\operatorname{\acute{Et}}_k)$.

Definition 2.2. The *Grothendieck-Witt ring of* k, denoted by GW(k), is the Grothendieck group completion of isometry classes of non-degenerate symmetric bilinear forms on finite dimensional k-vector spaces.

By [Lam05, §2, Theorem 4.1], GW(k) is generated by elements $\langle a \rangle$ for $a \in k^{\times}$, which are the classes of one-dimensional forms $(x, y) \mapsto axy$, subject to the relations

- (1) $\langle a \rangle = \langle ab^2 \rangle$ for $b \in k^{\times}$,
- (2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $b \in k^{\times}$,
- (3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$, and
- (4) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $b, a + b \in k^{\times}$.

Define $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle$, which we call the *hyperbolic form*. There is a canonical homomorphism rank : $\mathrm{GW}(k) \to \mathbb{Z}$, given by sending $\langle a \rangle \mapsto 1$ for all $a \in k^{\times}$. Note that for all $q \in \mathrm{GW}(k)$, $q \cdot \mathbb{H} = \mathrm{rank}(q)\mathbb{H}$.

To define χ^{mot} , we follow [LR20, Corollary 8.7] and [AMBO⁺22, Definition 1.4, Theorem 1.13]. For X a smooth projective scheme over k of dimension n, define a quadratic form $\chi^{Hdg}(X) \in \text{GW}(k)$ as follows.

• If n is odd, we set $\chi^{Hdg}(X) = m \cdot H$ where

$$m = \sum_{i+j < n} (-1)^{i+j} \dim_k(H^i(X, \Omega^j_{X/k})) - \sum_{i < j, i+j = n} \dim_k(H^i(X, \Omega^j_{X/k})).$$

• If n = 2p is even, we set $\chi^{Hdg}(X) = m \cdot H + Q$ where Q corresponds to the symmetric bilinear form given by

$$H^p(X, \Omega^p_{X/k}) \otimes H^p(X, \Omega^p_{X/k}) \xrightarrow{\cup} H^n(X, \Omega^n_{X/k}) \xrightarrow{\operatorname{Trace}} k$$

and

$$m = \sum_{i+j < n} (-1)^{i+j} \dim_k(H^i(X, \Omega^j_{X/k})) + \sum_{i < j, i+j = n} \dim_k(H^i(X, \Omega^j_{X/k})).$$

By [AMBO⁺22, Theorem 1.13], there exists a unique ring homomorphism, the compactly supported \mathbb{A}^1 -Euler characteristic

$$\chi_k^{\mathrm{mot}} : \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{GW}(k),$$

such that if X is a smooth projective connected variety, $\chi_k^{\text{mot}}([X]) = \chi^{Hdg}(X)$.

Definition 2.3. For a variety X over k, the compactly supported \mathbb{A}^1 -Euler characteristic $\chi_k^{\text{mot}}(X) \in \text{GW}(k)$ is the image of $[X] \in K_0(\text{Var}_k)$ under the above map.

Remark 2.4. When the base field is clear, we will drop the subscript k and simply write $\chi^{\text{mot}}(X)$ to mean $\chi^{\text{mot}}_k([X])$.

In [AMBO⁺22], this invariant is denoted by $\chi_c^{\mathbb{A}^1}$.

Remark 2.5. For X smooth and projective, the motivic Gauss–Bonnet Theorem ([LR20, Theorem 1.3]) implies that $\chi^{\text{mot}}(X)$ is the quadratic Euler characteristic of X. This is an invariant coming from motivic homotopy theory which was first studied by Hoyois in [Hoy14]. One obtains this invariant by applying the categorical Euler characteristic construction as defined by Dold-Puppe [DP80] to the stable motivic homotopy category $\mathbf{SH}(k)$ introduced by Morel-Voevodsky, see [Lev20, Section 2] for details. The quadratic Euler characteristic above is the motivation for the definition of the compactly supported \mathbb{A}^1 -Euler characteristic in [AMBO+22]. For this paper, we define χ_k^{mot} in terms of Hodge cohomology for ease of use, however this invariant should be thought of as one coming from motivic homotopy theory.

2.2. **Power structures.** In this section, we give a brief introduction to the power structures studied by Gusein-Zade, Luengo and Melle-Hernández in [GZLMH06] and by Pajwani and Pál in [PP23]. Informally, a power structure on a ring R is a way to make sense of the expression $f(t)^r$ for $r \in R$ and $f(t) \in 1 + tR[[t]]$, see e.g. [PP23, Definition 2.1] for a precise definition. By [GZLMH06, Proposition 1], under some finiteness assumptions it suffices to define $(1-t)^{-r}$ for $r \in R$ satisfying some conditions which we specify now.

Definition 2.6. Let R be a ring. A *finitely determined power structure* on R is a collection of functions $a_i : R \to R$ for $i \in \mathbb{Z}_{>0}$ such that:

- (1) $a_i(0) = 0$ and $a_i(1) = 1$.
- (2) $a_0(r) = 1, a_1(r) = r$ for all $r \in R$. (3) $a_n(r+s) = \sum_{i=0}^n a_i(r)a_{n-i}(s)$ for all $r, s \in R$.

For the purposes of this paper, all power structures will be finitely determined. Suppose R and S are rings with power structures on them given by functions a_i and b_i respectively, and let $f: R \to S$ be a ring homomorphism. Then we say that f respects the power structures if $f(a_i(r)) = b_i(f(r))$ for all i > 0 and $r \in \mathbb{R}$.

Remark 2.7. Gusein-Zade, Luengo and Melle-Hernández [GZLMH06, Page 3] proved that there is a canonical power structure on $K_0(Var_k)$, given on the level of quasiprojective varieties by functions S_n such that $S_n([X]) = [X^{(n)}]$. Their paper works over base field \mathbb{C} , however the construction works over a general base field of characteristic zero.

This paper is concerned with the following power structure on GW(k) from [PP23, Corollary 3.25].

Definition 2.8. For every $n \ge 0$, define functions $a_n : \mathrm{GW}(k) \to \mathrm{GW}(k)$ such that for $\alpha \in k^{\times}$

$$a_n(\langle \alpha \rangle) = \langle \alpha^n \rangle + \frac{n(n-1)}{2} t_\alpha,$$

where $t_{\alpha} = \langle 2 \rangle + \langle \alpha \rangle - \langle 1 \rangle - \langle 2\alpha \rangle$. Note that t_{α} is 2-torsion in GW(k). These functions uniquely define a power structure on GW(k) by [PP23, Corollary 3.25].

In fact it is shown that if there is a power structure b_n on GW(k) such that $\chi^{\text{mot}}(X^{(n)}) = b_n(\chi^{\text{mot}}(X))$ for X = Spec(L) where L/k is a quadratic étale algebra, then $b_n = a_n$ for all n. The power structure on GW(k) given by these a_n functions is therefore of interest for computing the compactly supported \mathbb{A}^1 -Euler characteristic of symmetric powers of varieties, since it predicts that $\chi^{\text{mot}}(X^{(n)}) = a_n(\chi^{\text{mot}}(X))$ for every variety X/k. It is currently an open question whether this is true for all varieties, however [PP23, Corollary 4.30] shows that this is true whenever X is dimension 0. We extend this result to K_0 -étale linear varieties in Theorem 4.10.

Remark 2.9. We have that $t_{\alpha} = 0$ if and only if $[\alpha] \cup [2] = 0 \in H^2_{\text{Gal}}(k, \mathbb{Z}/2\mathbb{Z})$, so in particular, $t_1 = t_{-1} = 0$. Corollary 3.28 of [PP23] therefore tells us if $-\cup [2]$ is the zero map, then $a_n(\langle \alpha \rangle) = \langle \alpha^n \rangle$ for all n. In particular, the power structure defined by the a_n functions will agree with the non factorial symmetric power structure on GW(k) as defined by McGarraghy in [McG05, Definition 4.1].

We will use the following results about this power structure later.

Lemma 2.10. Let $m \in \mathbb{Z}$ and let n be odd. Then $a_n(m\mathbb{H})$ is hyperbolic.

Proof. Since $m\mathbb{H} = m\langle 1 \rangle + m\langle -1 \rangle$ and $t_1 = t_{-1} = 0$, we see $a_n(m\mathbb{H}) = S_n(m\mathbb{H})$, where S_n is the non factorial symmetric power structure on $\mathrm{GW}(k)$ as in [McG05, Definition 4.1] so when $m \geq 0$ this follows by [McG05, Corollary 4.13]. For m < 0,

$$0 = a_n(m\mathbb{H} + (-m)\mathbb{H}) = \sum_{i=0}^n a_i(m\mathbb{H})a_{n-i}((-m)\mathbb{H}),$$

so the result follows by induction using that $q\mathbb{H} = \operatorname{rank}(q) \cdot \mathbb{H}$ for any $q \in \mathrm{GW}(k)$. \Box

Lemma 2.11. Let $q \in GW(k)$, and let n be a positive integer. Then

$$a_n(\langle -1 \rangle \cdot q) = \langle (-1)^n \rangle \cdot a_n(q)$$

Proof. The result is true for $q = \langle \alpha \rangle$ by a simple computation, and so the result holds by the additive formulae for the a_n functions.

3. K₀-étale linear varieties

In this section, we define K_0 -étale linear varieties and show that varieties of this class generalise cellular varieties in the sense of [Lev20], and linear varieties in the sense of Joshua's paper [Jos01].

Definition 3.1. Let $K_0(\text{ÉtLin}_k)$ be the subring of $K_0(\text{Var}_k)$ generated by $[\mathbb{A}_k^1]$ and classes of the form [Spec L] where L is a finite étale algebra over k. We say a variety X is K_0 -étale linear if the class $[X] \in K_0(\text{Var}_k)$ lies in $K_0(\text{ÉtLin}_k)$.

Since $[\mathbb{A}^1]^n = [\mathbb{A}^n]$, we see that \mathbb{A}^n is K₀-étale linear. More generally X is K₀-étale linear if and only if we can write

$$[X] = \sum_{i=0}^{n} m_i[\mathbb{A}^i] \cdot [\operatorname{Spec}(L_i)]$$

where L_i/k is a finite étale algebra over k.

Example 3.2. We give examples of some K_0 -étale linear varieties.

- (1) Since there exists an open embedding $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with complement $\operatorname{Spec}(k)$, we have that $[\mathbb{P}^1] = [\mathbb{A}^1] + [\operatorname{Spec}(k)] \in \mathrm{K}_0(\operatorname{Var}_k)$, therefore \mathbb{P}^1 is K_0 -étale linear. More generally, \mathbb{P}^n is K_0 -étale linear since $[\mathbb{P}^n] = \sum_{i=0}^n [\mathbb{A}^i]$.
- (2) Let G be a one dimensional torus, defined by the vanishing set of the equation $x^2 - \alpha y^2 = 1 \subseteq \mathbb{A}^2$. Then G admits a compactification isomorphic to \mathbb{P}^1_k , with compliment Spec(L), where $L = k[x]/(x^2 - \alpha)$. Therefore, $[G] = [\mathbb{P}^1] - [\operatorname{Spec}(L)]$, and since L/k is a finite étale algebra, we see that G is K₀-étale linear. Similarly, any torus which is a product of 1-dimensional tori is K_0 -étale linear.
- (3) Consider $C = \{xy = 0\} \subseteq \mathbb{A}^2$. Note that $C \setminus \{(0,0)\} \cong \mathbb{G}_{m,k} \amalg \mathbb{G}_{m,k}$, so $[C] = 2[\mathbb{G}_{m,k}] + [\operatorname{Spec}(k)], \text{ so is } K_0\text{-\'etale linear.}$ (4) Let C denote the curve $y^2 z = x^3 \subseteq \mathbb{P}^2_{[x:y:z]}$. We see that

$$(C \setminus \{[0:0:1]\}) \cong \mathbb{A}^1$$

via the isomorphism $[x:y:z] \mapsto \frac{x}{y}$. Therefore in $K_0(\operatorname{Var}_k)$ we may write $[C] = [\mathbb{A}^1] + [\operatorname{Spec}(k)]$, so C is K₀-étale linear. In particular, K₀-étale linear varieties do not need to have smooth irreducible components.

Lemma 3.3. Let X/k be a cellular variety in the sense of [Lev20, Page 2189]. Then X is K_0 -étale linear.

Proof. As in [Lev20, Page 2189], X is cellular implies that there exists a filtration

 $\emptyset = X_0 \subseteq X_1 \subseteq X_2 \ldots \subseteq X_n = X$

such that $X_{i+1} \setminus X_i$ is a disjoint union of copies of \mathbb{A}_k^i . Let m_i denote the number of disjoint copies of \mathbb{A}_k^i in $X_{i+1} \setminus X_i$. In $\mathrm{K}_0(\mathrm{Var}_k)$, we may write

$$[X] = \sum_{i=0}^{n} [X_{i+1} \setminus X_i] = \sum_{i=0}^{n} m_i [\mathbb{A}^i],$$

and the claim is now clear.

Remark 3.4. The curve $\{xy = 0\} \subseteq \mathbb{A}^2$ is not cellular, even though its irreducible components are cellular and their intersection is cellular, so the class of K_0 -étale linear varieties is strictly larger than the class of cellular varieties.

Remark 3.5. In [MS23, Section 2], Morel and Sawant use a more general definition of cellular varieties by relaxing the condition on the stratification so that we only require that $X_{i+1} \setminus X_i$ is a disjoint union of cohomologically trivial varieties. Using this definition, \mathbb{A}^1 -contractible varieties are cellular, e.g. Hoyois, Krishna and Østvær have proven that Koras–Russell threefolds are \mathbb{A}^1 -contractible [HKØ16] so these would be cellular. However, due to the obtuse structure of $K_0(Var_k)$, it is unclear to the authors whether such varieties are K₀-étale linear.

Definition 3.6. Following Section 3 of Totaro's paper [Tot14] and [Jos01, Section 2], we say a variety X over k is 0-linear if it is isomorphic to \mathbb{A}_k^m for some $m \in \mathbb{N}$. A variety X over k is *n*-linear for $n \ge 1$ if there exists an open embedding $U \to V$ with complement Z, such that $X \in \{U, V, Z\}$ and the other two are (n-1)-linear. A variety X over k is *linear* if it is n-linear for some $n \in \mathbb{N}$. Let $\text{Lin}_k \subseteq \text{Var}_k$ be the full subcategory of Var_k of linear varieties over k.

The class of linear varieties includes any variety which admits a stratification into linear varieties. In particular, it includes all projective spaces, Grassmannians, flag varieties, and blowups of projective spaces in linear subvarieties.

Lemma 3.7. Let X/k be a linear variety. Then X is K_0 -étale linear.

Proof. The result is trivial if X is 0-linear, so suppose all m linear varieties are K_0 -étale linear for m < n for an induction. Let X be an (n + 1)-linear variety. By assumption, there exists an open embedding $U \hookrightarrow V$ with compliment Z such that X is isomorphic to either U, V or Z and the other two are n-linear. In particular, since [U] + [V] = [Z], and $K_0(\text{ÉtLin}_k)$ is a subring of $K_0(\text{Var}_k)$, if any two of [U], [V] and [Z] are in $K_0(\text{ÉtLin}_k)$, the third is also, so X is K_0 -étale linear.

Remark 3.8. The class of K_0 -étale linear varieties is strictly bigger than the class of linear varieties. For example, when L/k is a quadratic field extension, Spec(L) does not admit a stratification as in Definition 3.6.

When k is algebraically closed, that the class of K_0 -étale linear varieties are those whose class in $K_0(\operatorname{Var}_k)$ lie in the subring generated by \mathbb{A}^1 . Clearly all linear varieties lie in this subring. It is unclear if all K_0 -étale linear varieties over an algebraically closed field are linear. That is, there may exist varieties X/k that do not admit stratifications as above, but nevertheless the class [X] lies in $K_0(\operatorname{\acute{e}tLin}_k)$.

The class of K_0 -étale linear varieties is closed under natural geometric constructions: clearly they are closed under products and scissor relations, but we also have the following.

Lemma 3.9. Let X be K_0 -étale linear, and let $p : E \to X$ be a Zariski locally trivial fibre bundle whose fibre F is K_0 -étale linear. Then E is K_0 -étale linear.

Proof. Since $E \to X$ is Zariski locally trivialisable, we see [E] = [X][F] for example by Remark 4.1 of [Göt01]. Since $K_0(\text{ÉtLin}_k)$ is a ring and $[X], [F] \in K_0(\text{ÉtLin}_k)$ by assumption, the result follows.

Lemma 3.10. Let X be a smooth variety, and let Z be a smooth closed subvariety of X such that Z is K_0 -étale linear. Then the blow up, $Bl_Z(X)$ is K_0 -étale linear if and only if X is K_0 -étale linear.

Proof. Let $Y := \operatorname{Bl}_Z(X)$ and let E denote the exceptional divisor of the blow up. Note that $[Z] \in \operatorname{K}_0(\operatorname{\acute{EtLin}}_k)$. As $Z \to X$ is a regular closed immersion, E is given by the projectivization of the conormal bundle $\mathcal{N}_{Z/X}$ and is therefore a projective bundle over Z. It follows from Lemma 3.9 that $[E] \in \operatorname{K}_0(\operatorname{\acute{EtLin}}_k)$. By Bittner's theorem [Bit04, Theorem 3.1], we see that

$$[X] - [Z] = [X \setminus Z] = [Y \setminus E] = [Y] - [E] \in \mathcal{K}_0(\operatorname{Var}_k),$$

and therefore $[Y] \in K_0(\text{ÉtLin}_k)$ if and only if $[X] \in K_0(\text{ÉtLin}_k)$, since $K_0(\text{ÉtLin}_k)$ forms a subring of $K_0(\text{Var}_k)$.

4. Symmetrisable varieties

In this section, we prove the main result of this paper, namely that K_0 -étale linear varieties are *symmetrisable*, see Definition 4.1. We also show that $K_0(Sym_k)$ is a $K_0(\acute{EtLin}_k)$ -submodule of $K_0(Var_k)$.

4.1. Properties of symmetrisable varieties.

Definition 4.1. A variety X is symmetrisable if $\chi^{\text{mot}}(X^{(m)}) = a_m(\chi^{\text{mot}}(X))$ for all m. Let $\text{Sym}_k \subset \text{Var}_k$ be the full subcategory consisting of symmetrisable varieties.

Informally, symmetrisable varieties are varieties that are compatible with our power structures on $K_0(Var_k)$ and GW(k) under the morphism χ^{mot} . Let $K_0(Sym_k)$ be the subset of $K_0(Var_k)$ consisting of elements s such that

$$\chi^{\mathrm{mot}}(S_m(s)) = a_m(\chi^{\mathrm{mot}}(s))$$

for all $m \in \mathbb{Z}_{\geq 0}$, where S_m is the power structure on $K_0(\operatorname{Var}_k)$ induced by symmetric powers as in Definition 2.7. It is an abelian subgroup of $K_0(\operatorname{Var}_k)$ by [PP23, Lemma 2.9]. Note that a variety X is symmetrisable if and only if $[X] \in K_0(\operatorname{Sym}_k)$, and we later see in Corollary 4.11 that $K_0(\operatorname{Sym}_k)$ is the sub-abelian group of $K_0(\operatorname{Var}_k)$ generated by symmetrisable varieties, which justifies this notation.

This paper studies the structure of $K_0(Sym_k) \subseteq K_0(Var_k)$, and can therefore be thought of as a geometric extension of the purely arithmetic results of [PP23]. Indeed, [PP23, Corollary 4.30] shows that Sym_k contains all zero-dimensional varieties, so $K_0(\acute{E}t_k) \subseteq K_0(Sym_k)$. A slight modification of the arguments in [PP23] gives us the following.

Theorem 4.2. Let X be symmetrisable. Then $\chi_k^{\text{mot}}((X_L)^{(n)}) = a_n(\chi_k^{\text{mot}}(X_L))$ for all n and for all finite extensions L/k.

Proof. This follows by an identical argument to [PP23, Subsection 4.3], which we sketch here for convenience. By [PP23, Corollary 4.24], X is symmetrisable implies that $X \times_k \operatorname{Spec}(K)$ is also symmetrisable for K/k a quadratic étale algebra. By repeatedly applying this result, for any multiquadratic étale algebra A/k we have $X \times_k \operatorname{Spec}(A) \in \operatorname{Sym}_k$. As in [PP23, Lemma 4.27], for any positive integer n the assignments $A \mapsto a_n(\chi^{\mathrm{mot}}(X \times_k A))$ and $A \mapsto \chi^{\mathrm{mot}}((X \times_k A)^{(n)})$ both define Witt-valued invariants, and the above shows they take the same values whenever A is a multiquadratic étale algebra, so the result [GMS03, Theorem 29.1] of Garibaldi, Merkurjev and Serre gives the result.

Corollary 4.3. The group $K_0(Sym_k)$ is a $K_0(\acute{Et}_k)$ -submodule of $K_0(Var_k)$.

Proof. Note that $K_0(Sym_k)$ is a $K_0(\acute{E}t_k)$ -module if and only if for all symmetrisable varieties X and finite separable field extensions L/k, $[Spec(L)][X] \in K_0(Sym_k)$, which is precisely the above theorem. \Box

Remark 4.4. It is an open question to determine whether $K_0(Sym_k) = K_0(Var_k)$. For some base fields k, it is true that $K_0(Sym_k) = K_0(Var_k)$. When $k = \mathbb{C}$, there is a canonical isomorphism $GW(\mathbb{C}) \cong \mathbb{Z}$ and [Lev20, Remark 1.3 (1)] allows us to compute $\chi^{mot}(X) = e_c(X(\mathbb{C}))$, where e_c denotes the compactly supported Euler characteristic of the topological space $X(\mathbb{C})$. We may then apply MacDonald's Theorem ([Mac62b]) to obtain $\chi^{mot}(X^{(n)}) = a_n(\chi^{mot}(X))$, so when $k = \mathbb{C}$, we have $K_0(Sym_{\mathbb{C}}) = K_0(Var_{\mathbb{C}})$, and we may argue as in [PP22, Theorem 2.10] to show the same is true when k is algebraically closed.

Similarly, when $k = \mathbb{R}$, [Lev20, Remark 1.3(2)] gives

$$\operatorname{sign}(\chi^{\operatorname{mot}}(X)) = e_c(X(\mathbb{R})).$$

We may then apply MacDonald's theorem ([Mac62b]) and [McG05, Proposition 4.14] to see $K_0(Sym_{\mathbb{R}}) = K_0(Var_{\mathbb{R}})$. Moreover, we may argue as in [PP22, Theorem

2.16] to obtain the same result whenever k is a real closed field. If we let J denote the kernel of the rank, signature and discriminant morphisms out of GW(k), then [PP22, Lemma 8.4] guarantees that for char(k) = 0, these power structures are compatible modulo the ideal J. In particular, $K_0(Sym_k) = K_0(Var_k)$ for all fields such that J = 0, which are precisely fields k such that the 2-primary virtual cohomological dimension $vcd_2(k) \leq 1$ by the n = 2, l = 2 case of the Milnor Conjecture, see the result [Mer81, Theorem 2.2] of Merkurjev. For this class of fields, the map

$$-\cup [2]: H^1(k, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(k, \mathbb{Z}/2\mathbb{Z})$$

is zero. In particular, in every known example where $K_0(Var_k) = K_0(Sym_k)$, the power structure on GW(k) from Definition 2.6 agrees with the non factorial symmetric power structure of [McG05]. It is unknown whether the non-vanishing of this map provides an obstruction to the compatibility of these power structures, and if so, whether this obstruction would be the only one to this compatibility.

4.2. Göttsche's lemma for symmetric powers. We give a detailed account of [Göt01, Lemma 4.4], as it will play a crucial role in the proof of Corollary 4.12, which is one of the main results of the paper. For $n \in \mathbb{N}$, let $X^{[n]}$ be the Hilbert scheme of n-points on X and let $\omega_n : X^{[n]} \to X^{(n)}$ be the Hilbert-Chow morphism, given by sending a subscheme Z to its support with multiplicities. There are stratifications of both $X^{[n]}$ and $X^{(s)}$ by partitions $\alpha \in P(n)$, where $\alpha = (n_1, \ldots, n_r)$, and also

$$n = \sum_{i=1}^{n} a_i i$$

where a_i is the number of *i*'s in the partition (n_1, \ldots, n_r) . We call $|\alpha| = r$ the *length* of the partition. A partition $\alpha \in P(n)$ defines the locally closed strata

$$X_{\alpha}^{(n)} = \{\xi = \sum_{i=1}^{r} n_i x_i \mid x_i \in X \text{ distinct}\}$$
$$X_{\alpha}^{[n]} = \omega_n^{-1} (X_{\alpha}^{(n)})_{\text{red}}$$

of $X^{(n)}$ and $X^{[n]}$, respectively. Let $X^{|\alpha|}_* = X^r_* \subset X^r$ be the subscheme of points $(x_1, \ldots, x_r) \in X^r$ for which the x_i are disjoint. There is a natural map $X^{|\alpha|}_* \to X^{(n)}_\alpha$ sending a point (x_1, \ldots, x_r) to

$$\sum_{i=1}^r n_i x_i \in X_{\alpha}^{(n)}.$$

Remark 4.5. By the fundamental theorem of symmetric polynomials and the definition of the symmetric power of an affine variety, we have

$$(\mathbb{A}^1)^{(n)} = \operatorname{Spec}(k[x_1, \dots, x_n]^{S_n}) = \operatorname{Spec}(k[e_1, \dots, e_n]) = \mathbb{A}^n.$$

Here, the e_i are the elementary symmetric polynomials in the variables x_i .

The following appears as [Göt01, Lemma 4.4]. For the convenience of the reader, we present the proof here, with some additional details.

Lemma 4.6. Let X be a variety and $n \in \mathbb{N}$. Let $p : (X \times \mathbb{A}^1)^{(n)} \to X^{(n)}$ be the projection map. Then $p : p^{-1}(X^{(n)}_{\alpha}) \to X^{(n)}_{\alpha}$ is a Zariski locally trivial vector bundle of rank n for all $\alpha \in P(n)$.

Proof. Let $n \in \mathbb{N}$ and $\alpha \in P(n)$. The key observation is that the square

is cartesian. Indeed, a point of the cartesian product $p^{-1}(X^{(n)}_{\alpha}) \times_{X^{(n)}_{\alpha}} X^{|\alpha|}_*$ is a pair

$$\left(\sum_{i=1}^{n} (x_i, t_i), (y_1, \dots, y_r)\right)$$

such that

$$\sum_{i=1}^n x_i = \sum_{j=1}^r n_j y_j,$$

and it follows that the choice of tuple $(t_1, \ldots, t_n) \in \prod_{i=1}^n ((\mathbb{A}^1)^{(i)})^{a_i}$ is free. By Remark 4.5, $\prod_{i=1}^n ((\mathbb{A}^1)^{(i)})^{a_i} \cong \mathbb{A}^n$. As ϕ is surjective and étale by construction, we have that $p: p^{-1}(X_{\alpha}^{(n)}) \to X_{\alpha}^{(n)}$ is an étale locally trivial vector bundle, trivialized by the étale cover ϕ . By Hilbert's Theorem 90 [Ser58, Théorème 2], this map is étale locally trivial if and only if it is Zariski locally trivial. This implies that p is also Zariski locally trivial, as required.

The following appears as the second half of [Göt01, Lemma 4.4], which again we present with additional details for convenience of the reader.

Corollary 4.7. Let X be a variety over k and let $l, n \in \mathbb{N}$. Then in $K_0(Var_k)$, we have $[(X \times \mathbb{A}^l)^{(n)}] = [X^{(n)} \times \mathbb{A}^{nl}]$.

Proof. Let $n \in \mathbb{N}$ and $\alpha \in P(n)$. Let $p : (X \times \mathbb{A}^l)^{(n)} \to X^{(n)}$ be the projection map. The proof is by induction on l. First, let l = 1. Then, by Lemma 4.6,

$$(X \times \mathbb{A}^{l})^{(n)}] = \sum_{\alpha \in P(n)} [p^{-1}(X_{\alpha}^{(n)})]$$
$$= \sum_{\alpha \in P(n)} [X_{\alpha}^{(n)} \times \mathbb{A}^{n}]$$
$$= [X^{(n)} \times \mathbb{A}^{n}]$$

as required. Now suppose the statement holds for all $l' \leq l$ for some $l \in \mathbb{N}$. By consecutive applications of the induction hypothesis for l' = 1 and l' = l,

$$[(X \times \mathbb{A}^{l+1})^{(n)}] = [(X \times \mathbb{A}^{l} \times \mathbb{A}^{1})^{(n)}]$$
$$= [(X \times \mathbb{A}^{l})^{(n)} \times \mathbb{A}^{n}]$$
$$= [X^{(n)} \times \mathbb{A}^{nl} \times \mathbb{A}^{n}]$$
$$= [X^{(n)} \times \mathbb{A}^{n(l+1)}],$$

as was to be shown, and the proof is done.

Using Göttsche's results above, we may quickly deduce the following.

Corollary 4.8. There is an equality $[(\mathbb{A}^m)^{(n)}] = [\mathbb{A}^{mn}]$ in $\mathrm{K}_0(\mathrm{Var}_k)$. *Proof.* Immediate, by taking $X = \mathrm{Spec}(k)$ in the above Corollary. \Box

Corollary 4.9. If X is a K_0 -étale linear variety, then $X^{(n)}$ is K_0 -étale linear.

Proof. For $[X] = [\mathbb{A}^n]$, this is immediate by the above. For $[X] = [\operatorname{Spec}(L) \times \mathbb{A}^n]$ where L/k is a finite étale algebra, this follows by Corollary 4.7, and for general $[X] \in \operatorname{K}_0(\operatorname{\acute{EtLin}}_k)$ this follows since we can write $[X] = \sum_i m_i [\operatorname{Spec}(L_i)] \mathbb{A}^i$ then apply the formulae for the functions defining power structures from Definition 2.6.

Theorem 4.10. Let X be a K_0 -étale linear variety over k. Then X is symmetrisable.

Proof. Recall that $\chi^{\text{mot}}(\mathbb{A}^m) = \langle (-1)^m \rangle$. Then by Corollary 4.8 and Lemma 2.11 we have

 $\chi^{\mathrm{mot}}((\mathbb{A}^m)^{(n)}) = \chi^{\mathrm{mot}}(\mathbb{A}^{mn}) = \langle (-1)^{mn} \rangle = a_n(\langle (-1)^m \rangle),$

where we use that $\chi^{\text{mot}}(\mathbb{A}^1) = \langle -1 \rangle$. Therefore, \mathbb{A}^d is symmetrisable. Corollary 4.3 then tells us that $[\mathbb{A}^d] \cdot [\text{Spec}(L)]$ is symmetrisable for any L/k a finite separable field extension. Since $K_0(\text{Sym}_k)$ is a finite abelian subgroup of $K_0(\text{Var}_k)$, any variety X such that $[X] = \sum_{i=0}^{n} m_i [\mathbb{A}^i] [\text{Spec}(L_i)]$ is also symmetrisable, which are all K_0 -étale linear varieties by Definition 3.1.

The above theorem allows us to justify the notation $K_0(Sym_k)$.

Corollary 4.11. The abelian group $K_0(Sym_k)$ is the abelian subgroup of $K_0(Var_k)$ generated by classes of symmetrisable varieties.

Proof. Let $s \in K_0(Sym_k)$. Since $K_0(Var_k)$ is generated as an abelian group by smooth projective varieties by Bittner's Theorem [Bit04, Theorem 3.1], we may write s = [X] - [Y] where both X and Y are smooth projective varieties.

Since Y is projective, it admits a closed embedding $Y \hookrightarrow \mathbb{P}^m$ for some m. Let $U := \mathbb{P}^m \setminus Y$. Then $s + [\mathbb{P}^m] = [X] + [\mathbb{P}^m] - [Y] = [X] + [U] = [X \amalg U]$. Since \mathbb{P}^m is K_0 -étale linear, it is symmetrisable by the above theorem. Therefore, since $K_0(\operatorname{Sym}_k)$ is closed under addition, we see that $s + [\mathbb{P}^m]$ lies in $K_0(\operatorname{Sym}_k)$. In particular, $X \amalg U$ is a symmetrisable variety, since $[X \amalg U] = s + [\mathbb{P}^m] \in K_0(\operatorname{Sym}_k)$. Rewriting $s = [X \amalg U] - [\mathbb{P}^m]$ shows that s can be written as a linear combination of the classes of symmetrisable varieties, as required.

Corollary 4.12. The subset $K_0(Sym_k)$ is a $K_0(ÉtLin_k)$ -submodule of $K_0(Var_k)$.

Proof. Let X be a symmetrisable variety. Note that $\chi^{\text{mot}}(X \times \mathbb{A}^l) = \langle (-1)^l \rangle \chi^{\text{mot}}(X)$ by multiplicativity of χ^{mot} . Corollary 4.7 gives us

$$\chi^{\mathrm{mot}}((X \times \mathbb{A}^l)^{(n)}) = \chi^{\mathrm{mot}}(X^{(n)} \times \mathbb{A}^{ln}) = \langle (-1)^{ln} \rangle a_n(\chi^{\mathrm{mot}}(X)).$$

Applying Lemma 2.11 gives

$$\langle (-1)^{ln} \rangle \cdot a_n(\chi^{\mathrm{mot}}(X)) = a_n(\langle (-1)^l \rangle \chi^{\mathrm{mot}}(X)) = a_n(\chi^{\mathrm{mot}}(\mathbb{A}^l \times X)),$$

and so $\mathbb{A}^l \times X$ is also symmetrisable. Combining this with Corollary 4.3 and [PP23, Lemma 2.12] tells us that $[X] \cdot \sum_{i=0}^n m_i[\mathbb{A}^i] \cdot [\operatorname{Spec}(L_i)] \in \mathrm{K}_0(\operatorname{Sym}_k)$ for integers m_i and finite étale algebras L_i . Since any element of $\mathrm{K}_0(\operatorname{\acute{EtLin}}_k)$ can be written as $\sum_{i=0}^n m_i[\mathbb{A}^i] \cdot [\operatorname{Spec}(L_i)]$, this means $\mathrm{K}_0(\operatorname{Sym}_k)$ is a submodule over $\mathrm{K}_0(\operatorname{\acute{EtLin}}_k)$. \Box

Corollary 4.13. Let Z be a smooth symmetrisable variety, let X be a smooth variety and let $Z \hookrightarrow X$ be a closed immersion. Then $Bl_Z(X)$ is symmetrisable if and only if X is symmetrisable.

Proof. The proof is identical to Lemma 3.10, replacing K_0 -étale linear varieties by symmetrisable varieties and using that $K_0(Sym_k)$ is a $K_0(\acute{EtLin}_k)$ submodule. \Box

Remark 4.14. It is a reasonable question to ask whether Theorem 4.10 holds over a field which is not of characteristic zero. By the discussion in [LPS24, Section 5.1], one can extend the quadratic Euler characteristic of [Lev20] on smooth projective schemes to a motivic measure $K_0(\operatorname{Var}_k) \to \operatorname{GW}(k)$ if k has odd positive characteristic. Also, the power structure of [PP23] on $\operatorname{GW}(k)$ is valid in odd positive characteristic. However, symmetric powers only define a power structure on $K_0(\operatorname{Var}_k)$ after inverting radicial surjective morphisms, which is also a necessity for Göttsche's lemma to hold in positive characteristic. Therefore the only obstruction to the above theorem holding in positive characteristic is to show that if X and Y are varieties over a field k of positive characteristic and $f: X \to Y$ is a radicial surjective morphism, one has that $\chi^{\operatorname{mot}}(X) = \chi^{\operatorname{mot}}(Y)$.

Remark 4.15. We see in Corollary 6.4 and Corollary 6.6 that curves of genus 1 are symmetrisable but not K_0 -étale linear. Therefore $K_0(Sym_k)$ is strictly larger than $K_0(\text{ÉtLin}_k)$, so Corollary 4.12 is always stronger than Theorem 4.10.

Remark 4.16. Theorem 4.10 can be rephrased in terms of Kapranov ζ -functions. Let X be a variety over k. We have a power series over $K_0(Var_k)$, the Kapranov ζ -function of X:

$$\zeta_{Kap}(t) := \sum_{n=0}^{\infty} [X^{(n)}] t^n \in \mathcal{K}_0(\operatorname{Var}_k)[[t]].$$

If $[X] \in K_0(Sym_k)$, then we may apply $\chi^{mot}(X)$ to obtain a power series in GW(k):

$$\zeta_{\chi(X)}(t) := \sum_{n=0}^{\infty} a_n(\chi^{\text{mot}}(X))t^n \in \text{GW}(k)[[t]].$$

In particular, for K_0 -étale linear varieties, we obtain a quadratically enriched ζ -function from the Kapranov ζ -function.

In [BHS⁺23], Bilu, Ho, Srinivasan, Vogt and Wickelgren study quadratically enriched ζ -functions related to the Hasse–Weil ζ -function used in the Weil conjectures. When working over finite fields, we may use the point counting measure on the Kapranov ζ -function to obtain the Hasse–Weil ζ -function used in the Weil conjectures. However, as discussed in Section 9 of [BHS⁺23], the link between their quadratically enriched ζ -functions and the Kapranov ζ -function is unclear.

While the link between the above power series $\zeta_{\chi(X)}(t)$ and the Kapranov ζ -function is clear, it is unclear whether the series $\zeta_{\chi(X)}(t)$ above has an connection to the ζ -functions of [BHS⁺23]. Even if we were to resolve the issues from Remark 4.14, it is also unclear whether the $\zeta_{\chi(X)}(t)$ would relate to the Weil conjectures.

4.3. Odd Galois twists.

Theorem 4.17. Let X, Y be varieties such that there exists a finite separable field extension L/k with [L:k] odd and $X_L \cong Y_L$. Then $\chi^{mot}(X) = \chi^{mot}(Y)$.

Proof. Note that $\chi_L^{mot}(X_L)$ is the image of $\chi_k^{mot}(X)$ under the base change map $\mathrm{GW}(k) \to \mathrm{GW}(L)$ by [PP22, Lemma 4.2] and similarly for Y. Therefore, since $X_L \cong Y_L$, we see $\chi_k^{mot}(X) - \chi_k^{mot}(Y)$ lies in the kernel of the base change map

 $K_0(Var_k) \to K_0(Var_L)$. Composing again with the map given by forgetting the base $K_0(Var_L) \to K_0(Var_k)$, this means

$$\chi_k^{\mathrm{mot}}(X \times \operatorname{Spec}(L)) - \chi_k^{\mathrm{mot}}(Y \times \operatorname{Spec}(L)) = 0 \in \operatorname{GW}(k).$$

By multiplicativity of χ^{mot} , we have

 $\chi^{\text{mot}}(\text{Spec}(L)) \cdot \left(\chi_k^{mot}(X) - \chi_k^{mot}(Y)\right) = 0.$

Note $\chi^{\text{mot}}(\text{Spec}(L)) = [\text{Tr}_{L/k}]$ by [Hoy14, Proposition 5.2]. Since [L:k] is odd, $[\text{Tr}_{L/k}] = [L:k]\langle 1 \rangle$ by a result of Bayer-Fluckiger and Lenstra [BFL90, Main Theorem, pages 356 and 359], so this implies that $\chi^{\text{mot}}_k(X) - \chi^{\text{mot}}_k(Y)$ is torsion of order dividing [L:k]. Since [L:k] is odd and all torsion in GW(k) is 2-primary order by e.g. [Pfi66, Satz 10], this gives us the result.

Corollary 4.18. Let X, Y/k be varieties such that $X_L \cong Y_L$ for some L/k with [L:k] odd, and suppose that Y is symmetrisable. Then X is symmetrisable.

Proof. The isomorphism $X_L \cong Y_L$ induces isomorphisms $(X^{(n)})_L \cong (Y^{(n)})_L$ for all n. The result then follows from the above theorem and the assumption on Y. \Box

Corollary 4.19. Let X/k be a Severi-Brauer variety of even dimension n. Then X is symmetrisable.

Proof. Since X is split by an odd degree extension, there exists an odd degree separable field extension L/k, such that $X_L \cong \mathbb{P}_L^n$. As \mathbb{P}^n is K₀-étale linear, it is symmetrisable by Theorem 4.10, so the result follows by the above.

Corollary 4.20. Let X/k be a symmetrisable variety and let L/k be an odd dimensional extension. Then the Weil restriction $\operatorname{Res}_{L/k}(X_L)$ is symmetrisable.

Proof. Note that $\operatorname{Res}_{L/k}(X_L)_L \cong \prod_{i=1}^{[L:k]} X_L$. The result follows instantly.

5. Computations of symmetric powers of symmetrisable varieties

In this section, we first show that some natural classes of varieties are symmetrisable, and then compute the compactly supported \mathbb{A}^1 -Euler characteristics of symmetric powers of varieties using the power structure on $\mathrm{GW}(k)$.

5.1. **Grassmannians.** It is well known that Grassmannians are linear varieties in the sense of Definition 3.6, see for example [Jos01, Example 2.2], and are therefore symmetrisable by Theorem 4.10. In this subsection, we compute the Euler characteristic of symmetric powers of Grassmannians. Because linear varieties have Euler characteristics consisting of sums of the classes $\langle 1 \rangle$ and $\langle -1 \rangle$ in GW(k), it is useful to know how the power structure behaves with respect to these classes.

Lemma 5.1. For all $m, n \in \mathbb{N}$, we have $a_n(m\langle (-1)^i \rangle) = \binom{m+n-1}{n} \langle (-1)^{in} \rangle$.

Proof. By Lemma 2.11, $a_n(m\langle (-1)^i \rangle) = \langle (-1)^{in} \rangle a_n(m\langle 1 \rangle)$, so without loss of generality assume i = 0. The proof is by double induction on m and n. Note that the identity holds whenever m = 1 or n = 1 by Definitions 2.6 and 2.8.

Now fix $n, m \in \mathbb{N}$, and assume that the identity holds for all $M, N \in \mathbb{N}$ such that $M \leq m$ and $N \leq n$. Then

$$a_n((m+1)\langle 1\rangle) = \sum_{i=0}^n a_i(m\langle 1\rangle)a_{n-i}(\langle 1\rangle) = \sum_{i=0}^n \binom{m+i-1}{i} = \binom{m+n}{n},$$

where the last equality follows from the hockey-stick identity for binomial coefficients, so the identity also holds for n and m + 1. Moreover,

$$a_{n+1}(m\langle 1\rangle) = a_{n+1}((m-1)\langle 1\rangle) + \sum_{i=0}^{n} a_i((m-1)\langle 1\rangle)a_{n+1-i}(\langle 1\rangle)$$
$$= a_{n+1}((m-1)\langle 1\rangle) + \langle 1\rangle \binom{m-1+n}{n}$$
$$= \langle 1\rangle \sum_{i=0}^{m-1} \binom{n+i}{n}$$
$$= \langle 1\rangle \binom{m+n}{n+1},$$

where the last equality follows from the hockey-stick identity again. Hence the identity also holds for n + 1 and m, which completes the proof.

Brazelton, McKean and Pauli [BMP23, Theorem 8.4] computed the \mathbb{A}^1 -Euler characteristic of Grassmannians over a field k which admits a real embedding $k \to \mathbb{R}$ by using a theorem of Bachmann and Wickelgren [BW23, Theorem 5.11]. We give a purely combinatorial proof which works over any field of characteristic zero, so without needing the condition that the field admits a real embedding. We will use Losanitsch's triangle, OEIS-sequence A034851, which is a summand of Pascal's triangle. It is a well-known combinatorial object constructed for example by Cigler [Cig17, Section 3]. The *d*-th entry in the *r*-th row is denoted by e(d, r) and we define $o(d, r) = {r \choose d} - e(d, r)$. The numbers e(d, r) and o(d, r) satisfy the following recurrence relations:

(1) if d is even, then

$$e(d-1, r-1) + e(d, r-1) = e(d, r)$$

 $o(d-1, r-1) + o(d, r-1) = o(d, r)$; and

(2) if d is odd, then

$$e(d-1, r-1) + o(d, r-1) = e(d, r)$$

$$o(d-1, r-1) + e(d, r-1) = o(d, r).$$

Closed formulae for the entries of Losanitsch's triangle and its complement in Pascal's triangle are given by

$$\begin{split} e(d,r) &= \frac{1}{2} \left(\binom{r}{d} + \mathbf{1}_A(r,d) \binom{\lfloor r/2 \rfloor}{\lfloor d/2 \rfloor} \right) \\ o(d,r) &= \frac{1}{2} \left(\binom{r}{d} - \mathbf{1}_A(r,d) \binom{\lfloor r/2 \rfloor}{\lfloor d/2 \rfloor} \right), \end{split}$$

where $A \subseteq \mathbb{N} \times \mathbb{N}$ is the subset of pairs (r, d) with either r odd or r and d both even, and $\mathbf{1}_A$ is its indicator function. The closed formula is proved by induction from the recurrence relations.

We now compute χ^{mot} of a Grassmannian.

Theorem 5.2. The compactly supported \mathbb{A}^1 -Euler characteristic of the Grassmannian $\operatorname{Gr}(d, r)$ is

(1)
$$\chi^{\text{mot}}(\operatorname{Gr}(d,r)) = e(d,r)\langle 1 \rangle + o(d,r)\langle -1 \rangle,$$

with e(d, r) and o(d, r) as above.

Proof. The closed immersion $\operatorname{Gr}(d-1, r-1) \to \operatorname{Gr}(d, r)$, yields the recursive formula

$$\chi^{\mathrm{mot}}(\mathrm{Gr}(d,r)) = \chi^{\mathrm{mot}}(\mathrm{Gr}(d-1,r-1)) + \langle (-1)^d \rangle \chi^{\mathrm{mot}}(\mathrm{Gr}(d,r-1)),$$

If d is even, then

$$\begin{split} \chi^{\text{mot}}(\text{Gr}(d,r)) &= \chi^{\text{mot}}(\text{Gr}(d-1,r-1)) + \chi^{\text{mot}}(\text{Gr}(d,r-1)), \\ &= (e(d-1,r-1) + e(d,r-1))\langle 1 \rangle \\ &+ (o(d-1,r-1) + o(d,r-1))\langle -1 \rangle. \end{split}$$

If d is odd, then

$$\begin{split} \chi^{\text{mot}}(\text{Gr}(d,r)) &= \chi^{\text{mot}}(\text{Gr}(d-1,r-1)) + \langle -1 \rangle \chi^{\text{mot}}(\text{Gr}(d,r-1)), \\ &= (e(d-1,r-1) + o(d,r-1)) \langle 1 \rangle \\ &+ (o(d-1,r-1) + e(d,r-1)) \langle -1 \rangle. \end{split}$$

The result follows from the recurrence relations for e(d, r) and o(d, r) above. \Box

Theorem 5.3. The compactly supported \mathbb{A}^1 -Euler characteristic of the n-th symmetric power of the Grassmannian $\operatorname{Gr}(d, r)$ is given by

$$\chi^{\text{mot}}(\text{Gr}(d,r)^{(n)}) = \sum_{i=0}^{n} \binom{e(d,r)+i-1}{i} \binom{o(d,r)+n-i-1}{n-i} \langle (-1)^{n-i} \rangle$$

Proof. Since Grassmannians are linear, they are K_0 -étale linear, so apply Theorem 4.10 to Theorem 5.2 to obtain

$$\begin{split} \chi^{\mathrm{mot}}(\mathrm{Gr}(d,r)^{(n)}) &= a_n(\chi^{\mathrm{mot}}(\mathrm{Gr}(d,r))) \\ &= a_n\left(e(d,r)\langle 1\rangle + o(d,r)\langle -1\rangle\right) \\ &= \sum_{i=0}^n a_{n-i}\left(e(d,r)\langle 1\rangle\right) a_i\left(o(d,r)\langle -1\rangle\right) \\ &= \sum_{i=0}^n \binom{e(d,r)+n-i-1}{n-i}\binom{o(d,r)+i-1}{i}\langle (-1)^i\rangle, \end{split}$$

where we use Lemma 5.1 in order to go to the last line.

Corollary 5.4. The generating series for the compactly supported \mathbb{A}^1 -Euler characteristic of symmetric powers of $\operatorname{Gr}(d, r)$ is given by

$$\sum_{t=0}^{\infty} \chi^{mot}(\operatorname{Gr}(d,r)^{(n)})t^n = (1-t)^{-e(d,r)}(1-\langle -1\rangle t)^{-o(d,r)} \in \operatorname{GW}(k)[[t]]$$

Proof. This follows immediately from Theorem 5.3 by taking the Taylor series expansion of the terms $(1-t)^{-e(d,r)}$ and $(1-\langle -1\rangle t)^{-o(d,r)}$.

5.2. del Pezzo surfaces. In this subsection, we use the techniques from Section 4 to show a large class of del Pezzo surfaces are symmetrisable. These are a class of surfaces of arithmetic interest, famous for the fact that over the complex numbers they contain a finite number of lines lying on them. This number is well known from enumerative geometry. For example, smooth projective cubic surfaces are del Pezzo surfaces of degree 3, which contain exactly 27 lines over the complex numbers. Quadratically enriched counts of the number of lines on del Pezzo surfaces were

achieved by Kass-Wickelgren [KW21, Theorem 2] in degree 3, by Darwin [Dar22, Theorem 1.2] in degree 4, and these were generalised to the degree \geq 3 case in [KLSW23b] and [KLSW23a] by Kass, Levine, Solomon and Wickelgren.

Definition 5.5. A *del Pezzo surface* is a smooth projective variety of dimension 2 whose anticanonical bundle is ample. The *degree* of a del Pezzo surface is the self intersection number of the anticanonical class. An *exceptional curve* on a del Pezzo surface is a curve with self intersection number -1.

Theorem 5.6 (Demazure, Théorème 1 of [Dem80]). Let \overline{k} be an algebraically closed field and let X/\overline{k} be a del Pezzo surface of degree d. Then $1 \le d \le 9$ and either:

- (1) $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and d = 8.
- (2) X is isomorphic to the blow up of \mathbb{P}^2 at 9-d points in general position.

While del Pezzo surfaces can be arithmetically very complicated, once we base change to the algebraic closure of our field, they are linear, so we would expect large classes of del Pezzo surfaces to be symmetrisable.

Theorem 5.7. Let X/k be a del Pezzo surface of degree ≥ 5 such that $X(k) \neq \emptyset$. Then X is symmetrisable.

Proof. This proceeds by checking on a case by case basis that the conditions we have already established for X to be symmetrisable hold in our cases. We appeal to [Poo17, Section 9.4], and we sketch the main results here.

Suppose X is a del Pezzo surface of degree 9. Then X is an even dimensional Severi Brauer variety, and since $X(k) \neq \emptyset$, we see $X \cong \mathbb{P}^2$, so X is K₀-étale linear and so symmetrisable by Theorem 4.10.

Suppose X is degree 8. By [Poo17, Proposition 9.4.12], we see X is either $\operatorname{Res}_{L/k}(C)$ where L/k is a quadratic étale algebra and C is a conic, or \mathbb{P}^2 blown up at a point. In the latter case the result holds by Corollary 4.13. Suppose X is the Weil restriction of a conic. Then $X(k) \neq \emptyset$ implies that $C(L) \neq \emptyset$ so $C_L \cong \mathbb{P}^1_L$. Therefore $\operatorname{Res}_{L/k}(C) = \mathbb{P}^1 \times \mathbb{P}^1$, which is symmetrisable by Theorem 4.10.

Suppose X is degree 7. Then Proposition 9.4.17 of [Poo17] tells us that X is isomorphic to the blow up of \mathbb{P}^2 at a closed subscheme isomorphic to a finite étale algebra of degree 2, so the result holds by Corollary 4.13.

Suppose X has degree 5. Then by [Poo17, Proposition 9.4.20], if there exists a k-point of X lying on none of the exceptional curves in X, we may obtain X through blowing up and and blowing down \mathbb{P}^2 . By [Poo17, Theorem 9.4.29], this is always the case in characteristic 0. Therefore, X is symmetrisable.

Suppose X is degree 6, and let $x \in X(k)$. Using [Poo17, Proposition 9.4.20], if x lies on exactly one exceptional curve E_i , then we may blow down X to obtain a del Pezzo surface of degree 7. Therefore, X is symmetrisable by the degree 7 case and Corollary 4.13. If x lies on an intersection of exceptional curves, then we may blow down two other exceptional curves to express X obtain a del Pezzo surface of degree 8, so X is symmetrisable by Corollary 4.13 and the degree 8 case. Finally, if x does not lie on any exceptional curves, we blow up X at x to obtain a del Pezzo surface of degree 5, so X is symmetrisable by the degree 5 case and Corollary 4.13.

If X is a del Pezzo surface of degree 5 or 7, the condition that $X(k) \neq \emptyset$ is immediate. If X is a del Pezzo surface of degree 9, then it is a Severi–Brauer surface so it is symmetrisable by Corollary 4.19. We also see del Pezzo surfaces of degree 6 which are given by blow ups of Severi–Brauer surfaces at a point are also symmetrisable by Corollary 4.13, so the above also holds for certain del Pezzo surfaces with no k-point. We can also use our results to show the following.

Theorem 5.8. Let X/k be a diagonal cubic surface. Then X is symmetrisable.

Proof. We first claim that the diagonal cubic surface Y defined by the equation

$$Y : x^{3} + y^{3} + z^{3} = t^{3} \subseteq \mathbb{P}^{3}_{[x:y:z:t]}$$

is K₀-étale linear. Note that Y contains 2 skew lines defined over k: namely, the lines $L_1 : \{x = t, y = -z\}$ and $L_2 : \{x = -t, y = z\}$. Therefore we may blow Y down at these 2 lines to obtain a del Pezzo surface of degree 5. This del Pezzo surface will have a k-point, since the skew lines are defined over k, so is symmetrisable by Theorem 5.7, and therefore Y is symmetrisable by Corollary 4.13.

For the general case, a diagonal cubic surface X is defined by an equation

$$X: a_1 x^3 + a_2 y^3 + a_3 z^3 = t^3 \subseteq \mathbb{P}^3_{[x:y:z:t]},$$

for some $a_1, a_2, a_3 \in k^{\times}$. Therefore, X and Y become isomorphic over the field $L = k(\sqrt[3]{a_1}, \sqrt[3]{a_2}, \sqrt[3]{a_3})$. Note that $[L:k] = 3^i$ where $i \in \{0, 1, 2, 3\}$. In particular, we can apply Theorem 4.17 to obtain the result.

To demonstrate the computational utility of Theorem 4.10, we give an explicit computation of the compactly supported \mathbb{A}^1 -Euler characteristic of the third symmetric power of a class of cubic surfaces. Let $\alpha, \beta, \gamma \in k^{\times}$ be non squares. Let Ybe a closed embedding of $\operatorname{Spec}(k(\sqrt{\alpha})) \amalg \operatorname{Spec}(k(\sqrt{\beta})) \amalg \operatorname{Spec}(k(\sqrt{\gamma}))$ into \mathbb{P}^2 such that the six points of $Y_{\overline{k}}(\overline{k})$ lie in general position in $\mathbb{P}^2_{\overline{k}}(\overline{k})$. Let $X := \operatorname{Bl}_Y(\mathbb{P}^2)$, so X/k is a smooth cubic surface which is symmetrisable by Corollary 4.13.

Proposition 5.9. We see that

$$\chi^{\mathrm{mot}}(X) = 2\langle 1 \rangle + 4\langle -1 \rangle + \langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle.$$

Proof. By the blow up formula for χ^{mot} , we see that

$$\chi^{\mathrm{mot}}(X) = \chi^{\mathrm{mot}}(\mathbb{P}^2) + \chi^{\mathrm{mot}}(E) - \chi^{\mathrm{mot}}(Y),$$

where E is the exceptional divisor of the blow up. The exceptional divisor is a \mathbb{P}^1 bundle over Y, so $\chi^{\mathrm{mot}}(E) = \chi^{\mathrm{mot}}(\mathbb{P}^1) \cdot \chi^{\mathrm{mot}}(Y)$. Note that $\chi^{\mathrm{mot}}(\mathbb{P}^1) = \mathbb{H}$, so $\chi^{\mathrm{mot}}(E) - \chi^{\mathrm{mot}}(Y) = \langle -1 \rangle \cdot \chi^{\mathrm{mot}}(Y)$. By [Hoy14, Proposition 5.2], we see that if L/k is a finite field extension, then $\chi^{\mathrm{mot}}(\mathrm{Spec}(L)) = \mathrm{Tr}_{L/k}$, where $[\mathrm{Tr}_{L/k}]$ is the trace form on L. If $L = k(\sqrt{\alpha})$, then computing the trace form in the basis $1, \sqrt{\alpha}$ gives $\chi^{\mathrm{mot}}(\mathrm{Spec}(L)) = \langle 2 \rangle + \langle 2\alpha \rangle = \langle 1 \rangle + \langle \alpha \rangle$. Additivity of χ^{mot} implies

$$\chi^{\mathrm{mot}}(Y) = 3\langle 1 \rangle + \langle \alpha \rangle + \langle \beta \rangle + \langle \gamma \rangle.$$

Finally, $\chi^{\text{mot}}(\mathbb{P}^2) = 2\langle 1 \rangle + \langle -1 \rangle$. Put together, this gives

$$\chi^{\text{mot}}(X) = 2\langle 1 \rangle + 4\langle -1 \rangle + \langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle.$$

Corollary 5.10. For X/k our cubic surface as above, the compactly supported \mathbb{A}^1 -Euler characteristic of its third symmetric power is given by

$$\chi^{\text{mot}}(X^{(3)}) = 60\mathbb{H} + 14\langle -1 \rangle + 8(\langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle) + 2(\langle -\alpha\beta \rangle + \langle -\alpha\gamma \rangle + \langle -\beta\gamma \rangle) + \langle -\alpha\beta\gamma \rangle + t_{\alpha\beta} + t_{\beta\gamma} + t_{\alpha\gamma}.$$

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Proof. Since X is symmetrisable, we see that $\chi^{\text{mot}}(X^{(3)}) = a_3(\chi^{\text{mot}}(X))$. We may compute $a_3(\chi^{\text{mot}}(X))$, using the additive formula for the a_n s, to obtain

$$\chi^{\text{mot}}(X^{(3)}) = a_3(2\langle 1 \rangle + 4\langle -1 \rangle) + a_2(2\langle 1 \rangle + 4\langle -1 \rangle) \cdot (\langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle) + (2\langle 1 \rangle + 4\langle -1 \rangle) \cdot a_2(\langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle) + a_3(\langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle).$$

Write $\phi = \langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle$ to ease notation. Standard computations utilising [PP23, Lemmas 3.18 and 3.19] give us

$$a_{3}(2\langle 1 \rangle + 4\langle -1 \rangle) = 24\mathbb{H} + 8\langle -1 \rangle$$

$$a_{2}(2\langle 1 \rangle + 4\langle -1 \rangle)\phi = 24\mathbb{H} + 5\phi$$

$$(2\langle 1 \rangle + 4\langle -1 \rangle) \cdot a_{2}(\phi) = 12\mathbb{H} + 6\langle -1 \rangle + 2(\langle -\alpha\beta \rangle + \langle -\beta\gamma \rangle + \langle -\alpha\gamma \rangle)$$

$$a_{3}(\phi) = 3\phi + \langle -\alpha\beta\gamma \rangle + t_{\alpha\beta} + t_{\beta\gamma} + t_{\alpha\gamma},$$

and putting this together gives

$$\chi^{\text{mot}}(X^{(3)}) = 60\mathbb{H} + 14\langle -1 \rangle + 8\left(\langle -\alpha \rangle + \langle -\beta \rangle + \langle -\gamma \rangle\right) \\ + 2\left(\langle -\alpha\beta \rangle + \langle -\alpha\gamma \rangle + \langle -\beta\gamma \rangle\right) + \langle -\alpha\beta\gamma \rangle + t_{\alpha\beta} + t_{\beta\gamma} + t_{\alpha\gamma}.$$

6. Curves of genus ≤ 1

In this section we show that curves of geometric genus ≤ 1 are symmetrisable, but curves of geometric genus > 0 are not K₀-étale linear. Therefore, curves of genus 1 give examples to show the inclusion K₀(ÉtLin_k) \subseteq K₀(Sym_k) is strict.

6.1. Curves of genus ≤ 1 are symmetrisable.

Proposition 6.1. Let C be a smooth projective curve, and let n be odd. Then $\chi^{\text{mot}}(C^{(n)}) = a_n(\chi^{\text{mot}}(C)).$

Proof. Note that $C^{(n)}$ is smooth projective of odd dimension, so $\chi^{\text{mot}}(C^{(n)})$ is hyperbolic by [Lev20, Corollary 3.2]. Similarly, since $\chi^{\text{mot}}(C)$ is hyperbolic and n is odd, $a_n(\chi^{\text{mot}}(C))$ is hyperbolic by Lemma 2.10, so we only need to check that these have the same rank. Since the rank is invariant under base change of fields, we may argue as in the proof of [PP22, Theorem 2.10] to reduce the result to the case where $k = \mathbb{C}$, where the result holds by a result of MacDonald [Mac62a, (4.4)].

Corollary 6.2. Smooth projective curves of genus 0 are symmetrisable.

Proof. Let C be a smooth projective curve of genus 0. By the above proposition, we only need to show $\chi^{\text{mot}}(C^{(n)}) = a_n(\chi^{\text{mot}}(C))$ when n is even. If $C = \mathbb{P}^1$, then C is linear, so the result follows by Theorem 4.10. Assume therefore that C is a conic with $C(k) = \emptyset$, so there exists a quadratic extension L/k with $C(L) \neq \emptyset$. In particular, we see there is a closed embedding $\text{Spec}(L) \hookrightarrow C$. We then see that $\text{Spec}(L)^{(n)}$ is a closed subvariety of $C^{(n)}$. Moreover, since $C^{(n)}$ is a k-form of \mathbb{P}^1 , we see that $C^{(n)}$ will be given by a k-form of $(\mathbb{P}^1)^{(n)} = \mathbb{P}^n$, so is also a Severi–Brauer variety. Note that $\text{Spec}(L)^{(n)}$ is given by the disjoint union of Spec(k) and $\frac{n}{2}$ copies of Spec(L), so in particular, Spec(k) embeds as a closed subvariety into $C^{(n)}$, so $C^{(n)}(k) \neq \emptyset$. Since $C^{(n)}$ is a Severi–Brauer variety however, we see $C^{(n)} = \mathbb{P}^n$ again by Châtelet's Theorem ([Poo17, Proposition 4.5.10]). The result is now clear. \Box

Proposition 6.3. Let C be a smooth projective curve of genus g > 0. Suppose that n > 2g - 2. Then $\chi^{\text{mot}}(C^{(n)}) = 0$.

Proof. Following the proof of [Mus11, Theorem 7.33], we may define a morphism $C^{(n)} \to \operatorname{Pic}^n(C)$ which is defined on points over the algebraic closure \bar{k} of k as follows. A point of $C^{(n)}$ over \bar{k} is a divisor D on C of degree n and we send D to $\mathcal{O}_C(D)$. If n > 2g - 2, this map makes $C^{(n)}$ into a Zariski locally trivial bundle of degree n - g over $\operatorname{Pic}^n(C)$ whose fibre B^{n-g} is a Severi–Brauer variety of dimension n - g. We now find that

$$\chi^{\mathrm{mot}}(C^{(n)}) = \chi^{\mathrm{mot}}(B^{n-g})\chi^{\mathrm{mot}}(\mathrm{Pic}^n(C)).$$

We note that $\chi^{\text{mot}}(\text{Pic}^n(C)) = 0$ by Theorem 5.48 of [PP22]. This gives the desired result.

Corollary 6.4. Let C be a curve of geometric genus ≤ 1 . Then C is symmetrisable.

Proof. We first reduce to the case where C is smooth and projective. Let \overline{C} be a compactification of C, and let \tilde{C} be a normalisation of \overline{C} . Then \tilde{C} and C are birational and dimension 1, so the rational map $\tilde{C} \dashrightarrow C$ allows us to realise $[\tilde{C}] - [A] = [C] - [A']$, where A, A' are dimension 0 varieties. Since $K_0(Sym_k)$ is an sub-abelian group of $K_0(Var_k)$ and dimension 0 varieties are symmetrisable by [PP23, Corollary 4.30], [C] is an element of $K_0(Sym_k)$ if and only if $[\tilde{C}]$ is. Therefore without loss of generality, assume C is smooth and projective.

If g(C) = 0, we appeal to Corollary 6.2. If g(C) = 1, Proposition 6.3 guarantees that $\chi^{\text{mot}}(C^{(n)}) = a_n(\chi^{\text{mot}}(C))$ for all n > 0. Therefore, for all n, we have that $\chi^{\text{mot}}(C^{(n)}) = a_n(\chi^{\text{mot}}(C))$, so $[C] \in K_0(\text{Sym}_k)$.

6.2. Curves of genus > 0 are not K₀-étale linear.

Theorem 6.5. Let X/k be a geometrically connected variety which is not geometrically stably rational. Then X is not K_0 -étale linear.

Proof. Let \overline{k} be an algebraic closure of k. Suppose that $[X] \in K_0(\text{ÉtLin}_k)$. Then clearly after base changing to \overline{k} , we have $X_{\overline{k}} \in K_0(\text{ÉtLin}_{\overline{k}})$. Since \overline{k} is algebraically closed, all varieties of dimension 0 are disjoint unions of $\text{Spec}(\overline{k})$, and therefore $K_0(\text{ÉtLin}_{\overline{k}})$ is simply the subring of $K_0(\text{Var}_k)$ generated by \mathbb{A}^1 . This question therefore reduces to the case where k is algebraically closed.

Let I denote the ideal of $K_0(\operatorname{Var}_k)$ generated by \mathbb{A}^1 . Then by a result of Larsen and Lunts [LL03, Proposition 2.7], we have an isomorphism $K_0(\operatorname{Var}_k)/I \cong \mathbb{Z}[SB]$, where $\mathbb{Z}[SB]$ is the ring whose underlying additive group is the free abelian group generated by stable-birational classes of varieties. By [LL03, Proposition 2.7], a connected smooth projective variety X/k is stably rational if and only if the class [X] in $K_0(\operatorname{Var}_k)$ is congruent to 1 (mod I). By assumption, X is not stably rational and is connected, so $[X] \neq 1 \in \mathbb{Z}[SB]$, which implies that $[X] \neq n \in \mathbb{Z}[SB]$ for some $n \in \mathbb{Z}$, since X is connected. This is precisely saying that there is no element $Y \in I$ such that $Y + n[\operatorname{Spec}(k)] = [X]$ for any $n \in \mathbb{Z}$. Since every element of $K_0(\operatorname{\acute{EtLin}_{\overline{k}})$ can be written in this manner, [X] is not in $K_0(\operatorname{\acute{EtLin}_{\overline{k}})$. \Box **Corollary 6.6.** Let X be a geometrically connected curve of geometric genus > 0. Then X is not K_0 -étale linear.

Proof. As above, we may reduce to the case where k is algebraically closed and X is connected, and as in Corollary 6.4, we may reduce to the case where X is smooth and projective. Suppose X is stably rational, so it is unirational. Since X is a curve, this would imply X is rational by a result of Luroth ([Lür75]). By the Riemann–Hurwitz formula g(X) = 0. For X not genus 0, it is not geometrically stably rational, so we can apply the above theorem.

As mentioned in Remark 4.15, the above shows that the inclusion $K_0(\text{ÉtLin}_k) \subseteq K_0(\text{Sym}_k)$ is always strict. In [BV24], it is shown that all curves are symmetrisable. Since we know that $K_0(\text{Sym}_k)$ is a module over $K_0(\text{ÉtLin}_k)$, we can combine the above result with the result from [BV24] to obtain large classes of symmetrisable varieties which are not K_0 -étale linear.

Remark 6.7. For other invariants in motivic homotopy theory, we may obtain obstructions to nice behaviour for varieties which are not \mathbb{A}^1 -connected. For example, [KLSW23b, Definition 2.30] introduces the notion of a global \mathbb{A}^1 -degree of a morphism $f : X \to Y$, which descends to an element of $\mathrm{GW}(k)$ only when Yis \mathbb{A}^1 -connected. Therefore, one might expect \mathbb{A}^1 -connectedness to be a necessary condition for symmetrisability. However, when k is a number field, there exist smooth projective curves of genus ≥ 1 that are not \mathbb{A}^1 -connected. It is therefore not true that over a number field $\mathrm{K}_0(\mathrm{Sym}_k)$ is given by the Grothendieck group of \mathbb{A}^1 -connected varieties.

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