A systematic approach to Diophantine equations: open problems

Bogdan Grechuk

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Abstract

This paper collects polynomial Diophantine equations that are amazingly simple to write down but are apparently difficult to solve.

1 Background and definitions

This document collects the current smallest open Diophantine equations in the sense described in [1]. The plan is to regularly update this document to keep this list up-to-date.

We start by recalling necessary definitions from [1]. A monomial in variables $x_1, \dots, x_n$ with integer coefficient is any expression of the form $M = ax_1^{k_1} \dots x_n^{k_n}$, where a is a non-zero integer and $k_1, \dots, k_n$ are non-negative integers. Integer $d = \sum_{i=1}^n k_i$ is called the degree of monomial M. Two monomials $M = ax_1^{k_1} \dots x_n^{k_n}$ and $M' = a' x_1^{k'_1} \dots x_n^{k'_n}$ are called similar if $k_i = k'_i$, $i = 1, \dots, n$.

A polynomial $P(x_1, \dots, x_n)$ with integer coefficients is the sum of any (finite) number of monomials. A polynomial Diophantine equation is an equation of the form

$$P(x_1, \dots, x_n) = 0. \tag{1}$$

We will assume that at least one of the monomials of P is non-constant, so that equation (1) has at least one variable. Without loss of generality we may also assume that P is in reduced form, that is, none of the monomials of P are similar. If polynomial P consists of k non-similar monomials with integer coefficients $a_1, \dots, a_k$ and degrees $d_1, \dots, d_k$, respectively, then the size of equation (1) is defined as

$$H(P) = \sum_{i=1}^k |a_i| 2^{d_i}. \tag{2}$$

It follows easily from the definition that, up to renaming the variables, there are only finitely many polynomial Diophantine equations of any given size. Hence, it is possible to order the equations by their size H, and solve them in this order.

The aim of this document is to list the current smallest open equations. We will list only one equation from an equivalence class, where the equations of the form (1) are called equivalent if they can be reduced to each other by a sequence of the following operations: (i) multiplications by a non-zero constant; (ii) substitutions in the form $x_i \to -x_i$; and (iii) renaming and permuting the variables.

We will consider unrestricted equations as well as equations in various categories. For example, we may restrict the number of variables in equation (1), its degree (the degree of equation (1) is the maximal degree of a monomial of P), or the number of monomials of P. We will also consider the following categories.

- homogeneous equations: that is, equations in which all monomials have the same degree.
- symmetric equations: that is, ones that do not change after permutation of variables. In other words, if a symmetric equation contains a monomial $ax_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $(\beta_1, \dots, \beta_n)$ is any permutation of $(\alpha_1, \dots, \alpha_n)$, then the equation must also contain a monomial $ax_1^{\beta_1} \dots x_n^{\beta_n}$.
- cyclic equations: that is, ones invariant under cyclic shift of variables $(x_1, \dots, x_n) \to (x_2, \dots, x_n, x_1)$. In other words, if a cyclic equation contains a monomial $ax_1^{\alpha_1} \dots x_n^{\alpha_n}$, it must contain all monomials of the form

$$ax_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad ax_2^{\alpha_1} \dots x_n^{\alpha_{n-1}} x_1^{\alpha_n}, \quad ax_3^{\alpha_1} \dots x_1^{\alpha_{n-1}} x_2^{\alpha_n}, \ \dots, \ ax_n^{\alpha_1} x_1^{\alpha_2} \dots x_{n-1}^{\alpha_n}.$$

H	Equation	H	Equation	H	Equation
13	$x^2+y^2+zt+1=0$	13	$xyz+t^2+1=0$	13	$x^2y+z^2+1=0$
13	$x^2+y^2+zt-1=0$	13	$xyz+t^2-1=0$	13	$x^2y+z^2-1=0$
13	$x^3+yz+1=0$	13	$x^2y+zt+1=0$	13	$x_1x_2x_3+x_4x_5+1=0$

Table 1: Equations of size $H=13$ for which the existence of polynomial parametrization is unknown.

- equations with independent monomials: that is, equations in which no two monomials share a variable.

One may ask various questions about equation (1). For example, how to describe all its integer (or rational) solutions, whether its solution set is finite, or whether it is non-empty. These questions will be discussed in the next sections.

2 Polynomial parametrization of all integer solutions

One of the basic questions one may ask about (1) is whether all its integer solutions can be parametrized.

Definition 1. *Let $\mathbb{Z}^n$ be the set of vectors $(x_1,\dots,x_n)$ with integer coordinates x_i. We say that a subset $S\subseteq\mathbb{Z}^n$ is a polynomial family if there exist an integer $k\geq 0$ and polynomials $P_1,\dots,P_n$ in k variables $u_1,\dots,u_k$ with integer coefficients such that $(x_1,\dots,x_n)\in S$ if and only if there exist integers $u_1,\dots,u_k$ such that*

$$x_i=P_i(u_1,\dots,u_k),\quad i=1,\dots,n.$$

Problem 1. *Given a polynomial Diophantine equation (1), determine whether the set of its integer solutions is a finite unions of polynomial families.*

The smallest equations for which Problem 1 is open are listed in Table 1.

3 Describing all integer/rational solutions

In addition to polynomial parametrization, there are many other ways to describe all integer solutions of (1). For example, one may use

- rational expressions with a parameter in the denominator that are ensured to produce integer solutions by imposing the restriction that this parameter must be a divisor of a numerator, or
- recurrence relations, or
- present a set of "initial" solutions and a set of "transformations" mapping solutions to solutions, such that all solutions to (1) can be obtained from the initial solutions by applying a sequence of transformations.

Given a polynomial Diophantine equation (1), one may ask whether its set of integer solutions can be described in any "reasonable" way. Because we do not rigorously define the word "reasonable", this problem is a bit informal. Nevertheless, the smallest equations for which we currently do not know a "satisfiable" answer are the equations

$$y^2+z^2=x^3+1 \tag{3}$$

and

$$y^2+z^2=x^3-1. \tag{4}$$

of size $H=17$.

If we ask the same question about rational solutions, then the smallest open are the same equations (3) and (4), as well as the equation

$$y^2-x^2y+z^2+1=0 \tag{5}$$

H	Equation	H	Equation	H	Equation
28	$x^4 - y^3 - x + y = 0$	30	$x^4 + y^3 + x + 2y = 0$	31	$x^4 + y^3 + x - 2y - 1 = 0$
28	$x^4 - y^3 + x - y = 0$	30	$x^4 + y^3 + 2x - y = 0$	31	$x^4 + y^3 + x + 2y + 1 = 0$
29	$x^4 + y^3 + xy + 1 = 0$	30	$x^4 + y^3 + 2x + y = 0$	31	$x^4 + y^3 + 2x - y + 1 = 0$
29	$x^4 + y^3 + xy - 1 = 0$	30	$x^4 + y^3 + y^2 + x = 0$	31	$x^4 + y^3 + 2x + y + 1 = 0$
30	$x^4 - y^3 + xy + x = 0$	30	$x^4 + y^3 + xy - y = 0$	31	$x^4 + y^3 + 2x + y - 1 = 0$
30	$x^4 + y^3 - y^2 + x = 0$	30	$x^4 + y^3 + xy + y = 0$	31	$x^4 + y^3 + xy - y - 1 = 0$
30	$x^4 + y^3 + x - 2y = 0$	30	$x^4 + y^3 + xy + x = 0$	31	$x^4 + y^3 + xy - 3 = 0$
30	$x^4 + y^3 + x - y - 2 = 0$	31	$x^4 - y^3 + xy + x + 1 = 0$	31	$x^4 + y^3 + xy + y - 1 = 0$
30	$x^4 + y^3 + x + y + 2 = 0$	31	$x^4 - y^3 + xy + x - 1 = 0$	31	$x^4 + y^3 + xy + x + 1 = 0$
30	$x^4 + y^3 + x + y - 2 = 0$	31	$x^4 + y^3 + x - 2y + 1 = 0$	31	$x^4 + y^3 + xy + x - 1 = 0$

Table 2: Open equations in 2 variables of size $H \leq 31$.

of the same size $H = 17$. Solving these equations in rationals is equivalent to solving homogeneous equations

$$y^2t + z^2t - x^3 - t^3 = 0, \tag{6}$$

$$y^2t + z^2t - x^3 + t^3 = 0 \tag{7}$$

and

$$y^2t - x^2y + z^2t + t^3 = 0 \tag{8}$$

of size $H = 32$ in integers.

We next return to the problem of describing all integer solutions, and consider equations in various categories. The smallest open non-symmetric cyclic equation is

$$x^2y + y^2z + z^2x = 1 \tag{9}$$

of size $H = 25$. For this equation, we even do not know whether its integer solution set is finite or infinite. The smallest open symmetric equation is either

$$x^3 + y^3 + z^3 = 1 \tag{10}$$

of size $H = 25$ or

$$x^3 + y^3 + z^3 = 2 \tag{11}$$

of size $H = 26$, depending on whether you are satisfied with the description of integer solutions to (10) presented in [1]. Equation (11) is open without any doubts, because we do not even know whether it has infinitely many integer solutions that are not permutations of $(x, y, z) = (1 - 6t^3, -6t^2, 1 + 6t^3)$, $t \in \mathbb{Z}$.

The smallest open three-monomial equations are

$$(a) \quad x^2y^2 + xz^2 + z = 0, \qquad (b) \quad x^2y^2 + xz^2 + y = 0, \tag{12}$$

$$(a) \quad x^3y + yz^2 + z = 0, \qquad (b) \quad x + x^2y^2 + z^3 = 0 \tag{13}$$

and

$$x + x^3y + yz^2 = 0. \tag{14}$$

of size $H = 26$. Equations (12) (a) and (12) (b) are equivalent, and so are the equations (13) (a) and (13) (b), see [1]. Whether equations (12) are open or not is debatable, because [1] contains some description of their solution sets, and everyone may decide for themselves whether to accept this as a "reasonable" description. Equations (13) and (14) are open without any doubts, because they have some obvious solutions, see [1], and it is open whether they have any other solutions.

H	Equation
325	$x^6+x^5y+x^3y^3+xy^5+y^6+x+y+1=0$
325	$x^6-x^5y-x^3y^3-xy^5+y^6+x+y-1=0$
325	$x^6-x^4y^2+x^3y^3-x^2y^4+y^6+x+y+1=0$
325	$x^6-x^4y^2-x^3y^3-x^2y^4+y^6+x+y-1=0$
325	$x^6+3x^3y^3+y^6+x+y+1=0$
325	$x^6-3x^3y^3+y^6+x+y-1=0$

Table 3: The smallest open symmetric two-variable equations

H	Equation	H	Equation	H	Equation
40	$x^4+2y^3+z^3=0$	56	$x^4+3y^3+2z^3=0$	56	$x^5+y^4-2z^2=0$
48	$x^4+2y^3+2z^3=0$	56	$x^4+4y^3+z^3=0$	56	$x^5+y^4+2z^2=0$
48	$x^4+3y^3+z^3=0$	56	$2x^4+2y^3+z^3=0$	60	$x^5+y^4-3z^2=0$
48	$2x^4+y^3+z^3=0$	56	$x^4-y^4+3z^3=0$	60	$x^5+y^4+3z^2=0$
48	$x^4-y^4+2z^3=0$	56	$2x^4-y^4+z^3=0$		
48	$x^4+y^4+2z^3=0$	56	$x^5+2y^3+z^3=0$		

Table 4: Open equations of the form (18) with $1/p+1/q+1/r<1$ of size $H\leq 60$.

The smallest open two-variable equations are listed in Table 2. For each of these equations it is known that the integer solution set is finite, and the open question is to list all integer solutions. All equations in Table 2 have genus 3. The smallest open equation of genus 2 is the equation

$$x^4-x^2+xy+y^3=0 \tag{15}$$

of size $H=32$.

We may also consider various intersections of the above categories. For example, the smallest open symmetric homogeneous equation is

$$x^3+y^3+z^3+t^3+s^3=0 \tag{16}$$

of size $H=40$. The smallest open cyclic homogeneous equations are (16) and the equation

$$x^2y+y^2z+z^2t+t^2s+s^2x=0 \tag{17}$$

of the same size. The smallest open symmetric two-variable equations are listed in Table 3.

Another famous problem is to list all primitive solutions to an equation of the form

$$ax^p+by^q+cz^r=0, \tag{18}$$

with $1/p+1/q+1/r<1$. An integer solution (x,y,z) to (18) is called primitive if $\gcd(x,y,z)=1$, and it is known that under condition $1/p+1/q+1/r<1$ the set of primitive solutions is finite. Table 4 presents the smallest equations of the form (18) for which the author of this document does not know the list of their primitive solutions.

4 Existence of arbitrary large integer solutions

The question of describing all integer solutions in a “reasonable” way is informal. This motivates to consider the following restricted but more formal question.

H	Equation	H	Equation	H	Equation
21	$2x^2 - xyz - y^2 - 1 = 0$	22	$y^2 + x^2y + z^2x - 2 = 0$	22	$z^2 - xy^2 - x^3 - 2 = 0$
21	$2x^2 - xyz + y^2 + 1 = 0$	22	$y^2 + x^2y + z^2x + 2 = 0$	22	$z^2 + y^2z + x^3 - 2 = 0$
21	$y^2 + x^2y + z^2x + 1 = 0$	22	$z^2 - xy^2 - x^3 + 2 = 0$	22	$y^2 + xyz + x^3 + 2 = 0$

Table 5: Equations with open Problem 2 of size $H \leq 22$

H	Equation	H	Equation	H	Equation
64	$x^4 + x^3y + xy^3 - z^4 = 0$	64	$x^4 - y^4 + x^2yz + yz^3 = 0$	64	$x^4 + xy^3 + z^4 + t^4 = 0$
64	$x^4 + x^2y^2 + y^3z - yz^3 = 0$	64	$x^4 + y^4 + x^2yz - yz^3 = 0$	64	$x^4 + y^4 + z^3t - zt^3 = 0$
64	$x^4 - x^2y^2 + y^3z + yz^3 = 0$	64	$x^3z + y^3x + y^2z^2 - z^3y = 0$		
64	$x^4 + x^2y^2 + y^3z + z^4 = 0$	64	$x^4 - y^3z + xyz^2 + xz^3 = 0$		

Table 6: Three-variable equations of size $H \leq 64$ with open Problem 3.

Problem 2. *Given a Diophantine equation* (1)*, determine whether for any $k \geq 0$ it has a solution such that*

$$\min(|x_1|, \dots, |x_n|) \geq k. \tag{19}$$

If yes, the problem is solved. If not, then describe all its integer solutions.

The second part of Problem 2 ("If not, then describe all its integer solutions.") is informal but trivial for all equations of small size. For such equations, Problem 2 reduces to the question whether (1) has integer solutions satisfying (19) for every k. This question is completely formal and rigorous. The smallest equations for which it is open are the ones listed in Table 5.

For homogeneous equations, this question reduces to the following problem.

Problem 3. *Given a homogeneous polynomial $P(x_1, \dots, x_n)$ with integer coefficients, determine whether equation* (1) *has an integer solution such that all variables are different from* 0.

The smallest homogeneous equations with open Problem 3 are listed in Table 6.

Problem 3 may also be studied for homogeneous equations of the form

$$ax^d + by^d + cz^d = 0, \qquad a,b,c,d \in \mathbb{Z}, \quad a,b,c \neq 0, \quad d > 0, \tag{20}$$

in which case it reduces to investigation which rational numbers are the sums/differences of two rational d-th powers. The smallest equation of the form (20) for which Problem 3 is open is the equation

$$4x^5 + 4y^5 + 11z^5 = 0 \tag{21}$$

of size $H = 608$. The next smallest is the equation

$$7x^4 - 7y^4 = 25z^4 \tag{22}$$

of size $H = 624$.

5 The finiteness problem

In general, Problem 2 is still non-rigorous because of the ambiguity of the term "describe" in its second part. We next consider the following more restricted but completely rigorous question.

Problem 4. *Given a Diophantine equation, either list all its integer solutions, or prove that there are infinitely many of them.*

H	Equation	H	Equation	H	Equation
22	$z^2 + y^2z + x^3 - 2 = 0$	23	$z^2 + y^2z + x^3 - 3 = 0$	24	$z^2 + y^2z - z + x^3 + 2 = 0$
22	$y^2 + xyz + x^3 + 2 = 0$	23	$z^2 + y^2z + x^3 + 3 = 0$	24	$y^2 + xyz + x^3 + 4 = 0$
23	$z^2 + y^2z + x^3 - x - 1 = 0$	23	$y^2 + xyz + x^3 - 3 = 0$	24	$y^2 + xyz + x^3 - x - 2 = 0$
23	$z^2 + y^2z + x^3 + x - 1 = 0$	24	$z^2 + y^2z + x^3 - x - 2 = 0$	24	$y^2 + xyz + y + x^3 + 2 = 0$
23	$z^2 + y^2z + x^3 + x + 1 = 0$	24	$z^2 + y^2z + x^3 - x + 2 = 0$		

Table 7: Equations of size $H \leq 24$ with open Problem 4.

The smallest equations for which Problem 4 is open are listed in Table 7.

The smallest cyclic equation with open Problem 4 is (9). The smallest open symmetric equation is

$$x^3 + y^3 + z^3 = 3 \tag{23}$$

of size $H = 27$. This is also the smallest open equation with independent monomials. The next-smallest one is the equation

$$x^4 + y^3 + z^2 + 1 = 0 \tag{24}$$

of size $H = 29$. The smallest open two-variable equations are the ones in Table 2. The smallest open three-monomial equation is

$$x^3y^2 = z^3 + 2 \tag{25}$$

of size $H = 42$. For homogeneous equations, Problem 4 reduces to the following question.

Problem 5. *Given a homogeneous polynomial P with integer coefficients, determine whether equation (1) has an integer solution $(x_1, \dots, x_n) \neq (0, \dots, 0)$.*

The smallest homogeneous equation for which Problem 5 is open is the equation

$$x^4 + x^3y - y^4 + y^3z + z^4 = 0 \tag{26}$$

of size $H = 80$. The next-smallest ones are the equations

$$x^3 + 2x^2y + y^3 + y^2z + xz^2 + z^3 + t^2x - t^2y - t^2z + tyz + t^3 = 0 \tag{27}$$

$$2x^3 + x^2y + xy^2 + y^3 + y^2z - 2xz^2 - z^3 + t^2y + tz^2 - t^3 = 0 \tag{28}$$

and

$$2x^3 + 2x^2y + 2y^3 + y^2z - 2xz^2 - z^3 + tz^2 - t^3 = 0 \tag{29}$$

of size $H = 96$.

One may also study Problem 5 for the equations of the form (20), or, more generally, of the form

$$a_1x_1^d + a_2x_2^d + \dots + a_nx_n^d = 0. \tag{30}$$

The smallest equations of the form (30) for which Problem 5 is open are listed in Table 8. The smallest open equations of the form (20) are listed in Table 9.

6 Existence of integer solutions

Maybe the most famous question about Diophantine equations is Hilbert 10th problem:

Problem 6. *Given a Diophantine equation, determine whether it has an integer solution.*

H	Equation	H	Equation
288	$11x^4+4y^4+2z^4-t^4=0$	288	$7x^4+6y^4+3z^4-2t^4=0$
288	$8x^4+7y^4+2z^4-t^4=0$	288	$7x^4+5y^4+4z^4-2t^4=0$
288	$7x^4+6y^4+4z^4-t^4=0$	288	$8x^4+7y^4+z^4-2t^4=0$

Table 8: Open equations of the form (30) of size $H \leq 288$

H	Equation
704	$14x^5+5y^5+3z^5=0$
832	$17x^5+6y^5+3z^5=0$
928	$14x^5+13y^5+2z^5=0$
960	$21x^5+5y^5+4z^5=0$
992	$17x^5+9y^5+5z^5=0$

Table 9: The smallest equations of the form (20) with open Problem 5

H	Equation	H	Equation	H	Equation
32	$y^3+xy=x^4+4$	34	$y^3-y^2=x^4+2x+2$	34	$y^3+xy-y=x^4-4$
32	$y^3+xy=x^4+x+2$	34	$y^3+y^2=x^4+2x-2$	34	$y^3+xy=x^4+x+4$
32	$y^3+y=x^4+x+4$	34	$y^3-y^2+xy=x^4+2$	34	$y^3+xy=x^4-x-4$
32	$y^3-y=x^4+2x-2$	34	$y^3+y=x^4+x+6$	34	$y^3+xy=x^4+x^2+2$
33	$y^3+y^2+xy=x^4+1$	34	$y^3+y=x^4+x-6$	34	$y^3-y^2+y=x^4+x+2$
33	$y^3+xy=x^4+x+3$	34	$y^3+2y=x^4+x+4$	34	$y^3+y^2+y=x^4+x+2$
33	$y^3+xy=x^4-x-3$	34	$y^3+y=x^4+2x+4$	34	$y^3+y^2-y=x^4+x+2$
33	$y^3+xy=x^4+x-3$	34	$y^3+y=x^4+2x-4$	34	$y^3+xy-y=x^4+x+2$
33	$y^3+xy+y=x^4-x+1$	34	$y^3+xy+y=x^4+4$	34	$y^3-y=x^4-x^2+x+2$
34	$y^3+y^2=x^4+x+4$	34	$y^3+xy-y=x^4+4$	34	$y^3-y=x^4-x^2+x-2$

Table 10: Two-variable equations of size $H \leq 34$ with open Problem 6

H	Equation	H	Equation
33	$1-x+x^3+x^2y^2+z+z^2=0$	34	$6+x^3-y^2+y^2z^2=0$
34	$2+2x+x^3-y^2-xy^2+xz^2=0$	34	$6+x^3+xy^2z+z^2=0$
34	$2+x^3+xy+y^2+y^2z^2=0$		

Table 11: Equations in $n \geq 3$ variables of size $H \leq 34$ with open Problem 6.

H	Equation	H	Equation
34	$2+2x+x^3-y^2-xy^2+xz^2=0$	35	$3x+x^2+x^3+y^2z+yz^2-1=0$
35	$-2x+x^3+y^2-xy^2-xz^2+3=0$	35	$-2x+x^2+x^3+y+y^3+yz^2-1=0$

Table 12: Cubic equations of size $H \leq 36$ with open Problem 6.

The smallest equations with open Problem 6 are the ones listed in Tables 10 and 11. As you can see, the smallest open ones are four equations of size $H = 32$ from Table 10. These are also the smallest open two-variable equations. The smallest open cubic one is the equation

$$2 + 2x + x^3 - y^2 - xy^2 + xz^2 = 0$$

of size $H = 34$ from Table 11. All open cubic equations of size $H \leq 36$ are listed in Table 12.

The smallest open equations with independent monomials are the equations

$$x^4 + y^3 + z^3 = -4 \tag{31}$$

and

$$x^4 + y^3 + z^3 = 4 \tag{32}$$

of size $H = 36$. The smallest open symmetric equations is

$$x^3 + x + y^3 + y + z^3 + z = xyz + 1 \tag{33}$$

of size $H = 39$. The next-smallest open ones are the equations

$$x^3 + 2x + y^3 + 2y + z^3 + 2z = xyz + 1$$

and

$$x^3 - x + y^3 - y + z^3 - z = xyz + 7$$

of size $H = 45$. These are also the smallest open cyclic equations. The smallest three-monomial equation with open Problem 6 is the equation

$$x^3y^2 = z^3 + 6 \tag{34}$$

of size $H = 46$.

One may also study the problem of solution existence in positive integers.

Problem 7. *Given a Diophantine equation, determine whether it has a solution in* ***positive*** *integers.*

The smallest equations for which Problem 7 is open are

$$(a)\ y(x^3 - z^2) = z \qquad (b)\ x^2y^2 + x = z^3 \tag{35}$$

and

$$y(x^3 - z^2) = x \tag{36}$$

of size $H = 26$. As remarked in [1], equations (35) (a) and (35) (b) are equivalent.

7 The shortest open equations

Obviously, there are many other natural ways to order polynomial Diophantine equations. One possibility is to define the **length** of a polynomial P consisting of monomials of degrees $d_1, \dots, d_k$ with coefficients $a_1, \dots, a_k$ as

$$l(P) = \sum_{i=1}^{k} \log_2(|a_i|) + \sum_{i=1}^{k} d_i = \log_2\left(\prod_{i=1}^{k}\left(|a_i| \cdot 2^{d_i}\right)\right). \tag{37}$$

This can we rewritten as $l(P) = \log_2 L(p)$, where

$$L(P) = \prod_{i=1}^{k}\left(|a_i| \cdot 2^{d_i}\right). \tag{38}$$

Obviously, ordering polynomials by length $l(P)$ is equivalent to ordering them by $L(P)$.

l	Equation	l	Equation	l	Equation
8	$y(x^3-z^2)=z$	8	$y(x^3-z^2)=z+1$	8	$x^2y^2+xz^2+y-1=0$
8	$y(x^3-z^2)=x$	8	$2x^2-xyz-y^2-1=0$	8	$x^2y^2+xz^2+y+1=0$
8	$y(x^3-z^2)=x-1$	8	$2x^2-xyz+y^2+1=0$	8	$x^3y^2+z^2+y+1=0$
8	$y(x^3-z^2)=x+1$	8	$y^2+x^2y+z^2x+1=0$	8	$y^2+x^3y+z^2+1=0$

Table 13: Equations of length $l \leq 8$ for which Problem 2 is open.

l	Equation	l	Equation	l	Equation
8	$y^2+x^3y+z^2+1=0$	9	$z^2+y^2z+x^3+x+1=0$	9	$x^3y^2=z^3+2$
9	$y^2+x^3y+z^2-2=0$	9	$y^2+xyz+x^3+2=0$	9	$x^3y^2=z^3-z+1$
9	$y^2+2x^3y+z^2+1=0$	9	$y^2+xyz+2x^3+1=0$	9	$x^3y^2=z^3+z+1$
9	$y^2+x^3y+z^2+z-1=0$	9	$x^2y^2+x^2z-z^2-1=0$	9	$x^3y^2=2z^3+1$
9	$y^2+x^3y+z^2+z+1=0$	9	$z^2+y^2z+x^3y+1=0$	9	$x^3y^2=z^4+1$
9	$y^2+x^3y+y+z^2+1=0$	9	$x^2y+y^2z+z^2x=1$	9	$x^4y^3=z^2+1$
9	$y^2+z^2=x^5-1$	9	$x^4+y^3+z^2+1=0$	9	$y^3-y=x^4-x$
9	$z^2+y^2z+x^3-2=0$	9	$x^3+x^2y^2+z^2+1=0$	9	$y^3+y=x^4+x$
9	$z^2+y^2z+2x^3+1=0$	9	$y(x^3-z^2)=x^2+1$	9	$x^4+xy+y^3-1=0$
9	$z^2+y^2z+x^3-x-1=0$	9	$y(x^3-z^2)=2x-1$	9	$x^4+xy+y^3+1=0$
9	$z^2+y^2z+x^3+x-1=0$	9	$y(x^3-z^2)=2x+1$		

Table 14: Equations of length $l \leq 9$ for which Problem 4 is open.

The last seven equations from Table 1 have length $l=5$, and these are the shortest equations for which Problem 1 is open. If the problem is to describe all integer solutions in any "reasonable" way, then the smallest open equations are (3), (4), and possibly also the equations

$$x^3y^2=z^2\pm 1. \tag{39}$$

Here, "possibly" depends on whether you are satisfied with the description of integer solutions to equations (39) presented in [1]. All these equations have length $l=7$.

The shortest equations with open Problems 2, 4 and 6 are listed in Tables 13, 14, and 15, respectively.

The shortest cubic equation for which Problem 6 is open is the equation

$$6x^2z+z^2y+3y^3+1=0 \tag{40}$$

of length $10+2\log_2 3 \approx 13.2$. The next-shortest open cubic equations are the equation

$$y^2+10xyz+x^3-x-2=0$$

of length $11+\log_2 5 \approx 13.3$, and the equations

$$y^2+7xyz+3x^3-2=0$$

l	Equation	l	Equation	l	Equation
10	$2y^3+xy+x^4+1=0$	10.6	$x^3y^2=z^3+6$	10.6	$3x+x^2z^2+2y^2z+1=0$
10	$y^2+x^3y+z^4+1=0$	10.6	$x^3y^2=z^4-3$	10.6	$3x^2y+y^2z^2+2z-1=0$
10	$x^3y^2=z^4+2$	10.6	$x^4y^3=z^2+3$		

Table 15: Equations of length $l < 11$ for which Problem 6 is open.

and

$$7x^3 + 2y^3 = 3z^2 + 1$$

of length $l = 9 + \log_2 21 \approx 13.4$.

References

[1] Grechuk, B., *Polynomial Diophantine Equations. A systematic approach.* Springer, 2024.