# A relative homology criteria of smoothness

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#### Abstract

We investigate the relationship between smoothness and the relative global dimension. We prove that a smooth ring map  $B \rightarrow A$  between commutative rings implies the finiteness of the relative global dimension gldim(A, B). Conversely, we identify a sufficient condition on B such that the finiteness of gldim(A, B) implies the smoothness of the map  $B \rightarrow A$ .

**Keywords:** relative homology, smooth morphisms, commutative rings, global dimension.

### 1 Introduction

Smoothness is a fundamental concept in algebraic geometry, providing a key link between geometric and algebraic properties of varieties. A fundamental result due to Auslander-Buchsbaum and Serre (see [AB56, Ser56]) claims that if *V* is an affine algebraic variety over a perfect field *k* with coordinate ring *A*, then the global dimension of *A* is finite if and only if *V* is smooth. Instead of a map  $k \rightarrow A$ , one can consider a more general case in which *k* is replaced by a commutative ring *B*. A well-established criterion of smoothness in this case (see [Lod92]) has a number of homological characterizations (see, for instance, [Rod90, AI00] and references therein).

In this manuscript, we provide a characterization of smoothness via relative global homology developed by Hochshild [Hoc56], specifically focusing on the relative global dimension gldim(A, B). Surprisingly, there is little literature on relative homological algebra for associative algebras. In contrast, a vast body of work exists for relative homological algebra in the context of representations of finite groups — results that have had a profound impact on modular representation theory and block theory of finite groups (see, for instance, [Lin18]). However, there has been a recent surge in interest in the application of relative homological algebra to attack certain homological conjectures for finite-dimensional algebras,

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such as the finitistic dimension conjecture and Han's conjecture. Interested readers can explore these recent developments and related work in [XX13, CLMS22, IM21] and references therein.

The manuscript is structured as follows: In Section 2, we review basic notions in relative homological algebra, alongside an overview of graded and filtered algebras and modules. Section 3.1 is devoted to proving the following result: given a smooth commutative *B*-algebra *A*, we establish that the relative global dimension, gldim(A, B), is finite. Finally, in Section 3.2, we study the converse. We present a sufficient condition on *B* such that the finiteness of gldim(A, B) implies the smoothness of the map  $B \rightarrow A$  of commutative rings. Together with the result of Section 3.1, this leads to the following conclusion (Corollary 8): if *k* is a perfect field, *B* a finitely generated *k*-algebra, and *A* a flat Noetherian *B*-algebra, locally of finite type, then  $B \rightarrow A$  is smooth if and only if gldim(A, B) is finite.

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## 2 Preliminaries

In this section, we introduce some notation and review basic definitions and results needed for proving our main theorems. We work within the category of associative commutative rings with identity. Given a ring A, we denote by A-Mod the category of A-modules. Throughout the text, for a ring map  $B \rightarrow A$ , we denote by  $A^e$  the ring  $A \otimes_B A$  and treat A as  $A^e$ -module via the canonical surjective map  $\mu : A \otimes_B A \rightarrow A$ ,  $a \otimes a' \mapsto aa'$ .

### 2.1 Relative (co)homology

In order to describe relative homological algebra (cf. [Hoc56]), we first recall the notion of relatively projective modules. Given a homomorphism between associative rings  $B \rightarrow A$ , an *A*-module *M* is called *relatively B-projective*, or (A, B)-*projective*, if it satisfies either of the following equivalent conditions:

- (*i*) the multiplication map  $\mu_M : A \otimes_B M \to M$  is a split epimorphism of *A*-modules;
- (*ii*) *M* is isomorphic to a direct summand of the induced module  $A \otimes_B V$ , for *V* some *B*-module;
- (*iii*) if ever an *A*-module homomorphism onto *M* splits as a *B*-module homomorphism, then it splits as an *A*-module homomorphism.

An exact sequence of *A*-module homomorphisms:

$$\cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to 0$$

is called (A, B)-exact if, for each  $i \ge n$ , the kernel of  $f_i$  is a direct B-module summand of  $M_i$ (cf. [Hoc56, Section 1]). One may check that a sequence of morphisms  $\{f_i : M_i \to M_{i-1} \mid i \ge n\}$  is (A, B)-exact if, and only if,

- 1)  $f_i \circ f_{i+1} = 0$  for all i > n,
- 2) there exists a contracting *B*-homotopy: that is, a sequence of *B*-module homomorphisms  $h_i : M_i \to M_{i+1}$ ,  $(i \ge n-1)$  such that  $f_{i+1}h_i + h_{i-1}f_i$  is the identity map on  $M_i$ .

One may now develop the concepts of relative projective dimension and relative global dimension. Given an *A*-module *M*, we define the *relative projective* dimension of *M* to be the minimal number *n*, denoted by  $pd_{(A,B)}M$ , such that there is an (A, B)-exact sequence (called (A, B)-projective resolution of *M*)

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

where the  $P_i$  are (A, B)-projectives. If such an exact sequence does not exist, the relative projective dimension of M is infinite. This definition is equivalent to the corresponding definition from [XX13, Section 2]. Having relative projective resolutions, the relative derived functors  $\operatorname{Tor}_n^{(A,B)}$  and  $\operatorname{Ext}_{(A,B)}^n$  can be defined, and we refer to [Hoc56] for the details.

*Remark* 2.1. Given an  $M \in A$ -Mod, one produces the *standard* (A, B)-*projective resolution* (check [Hoc56, Section 2]) by splicing the short (A, B)-exact sequences

$$0 \to \ker \mu_n \to A \otimes_B K_n \xrightarrow{\mu_n} K_n \to 0$$

with  $K_1 = M$  and  $K_{i+1} = \ker \mu_i$ , when  $i \ge 1$ .

The *relative global dimension* gldim(A, B) of the extension  $B \rightarrow A$ , denoted by gldim(A, B), is defined as:

$$\operatorname{gldim}(A,B) = \sup\{\operatorname{pd}_{(A,B)}M \mid M \in A - \operatorname{Mod}\}$$

if this number exists, and infinity otherwise. The *relative global cohomological dimension* is defined as

$$\operatorname{cdim}(A, B) = \sup\{n \mid \operatorname{Ext}_{(B,S)}^n(A, Y) \neq 0, \text{ for some } Y \in R-\operatorname{Mod}\}$$

in which  $R = A \otimes_B A$ ,  $S = B \otimes_B A$  with the natural map  $S \rightarrow R$  and A with the natural structure of *R*-module. These dimensions are related as follows.

*Remark* 2.2. From [Hoc56, Corollary 1] we have that  $\operatorname{cdim}(A, B)$  can also be calculated as  $\operatorname{gldim}(A \otimes_B A, B \otimes_B A)$ . Moreover, it is clear from the definition that

$$\operatorname{cdim}(A,B) = \operatorname{pd}_{(A \otimes_B A, B \otimes_B A)} A.$$

Therefore applying  $- \otimes_A M$  to a  $(A \otimes_B A, B \otimes_B A)$ -resolution of A for each A-module M one gets that (see also [Hoc56, Corollary 1]):

$$\operatorname{gldim}(A, B) \leq \operatorname{cdim}(A, B).$$
 (2.3)

#### 2.2 Graded and filtered algebras

Recall the basic definition about filtered and graded algebras (cf. [MR87, Chapters 7, 12] and [NvO82]). An *filtered A*-algebra *R* is defined as an *A*-algebra satisfying the condition

$$R = \bigcup_{i \in \mathbb{N}} R_i,$$

where  $R_i$  are *R*-ideals, subject to the following properties:

- (1)  $R_i R_j \subseteq R_{i+j}$ ;
- (2)  $R_{i+1} \subseteq R_i$ ;
- (3)  $A = R_0/R_1$ .

Furthermore, if the  $R_i$  are flat A-modules for every *i*, then R is referred to as a *flat filtered* A-algebra. Given a filtered A-algebra R, a *filtration* of an R-module M is defined as a collection of R-modules  $M_i$ , satisfying:

(1)  $M = \bigcup_{i \in \mathbb{N}} M_i$ 

(2) 
$$R_i M_j \subseteq M_{i+j}$$

A homomorphism  $\phi : M \to N$  between filtered *R*-modules  $M = \bigcup_{i \in \mathbb{N}} M_i$  and  $N = \bigcup_{i \in \mathbb{N}} N_i$  is said to be a *filtered morphism* if  $\phi(M_i) \subseteq N_i$ .

**Example 1.** Consider a homomorphism  $B \to A$ . Let  $R = A^e$  and  $J = \ker(\mu : A^e \to A)$ . Then R is a filtered A-algebra with  $R_i = J^i$  and  $R_0 = A^e$ . Assuming that  $B \to A$  is a flat ring map and  $J^i/J^{i+1}$  are flat A-modules, it follows that R is a flat filtered A-algebra.

In conjunction with the concept of filtered algebras and modules, we require the definitions of graded algebras and modules. In our context, we focus specifically on nonnegatively graded algebras.

A graded A-algebra R is defined as an A-algebra  $R = \bigoplus_{i \in \mathbb{N}} R_i$ , where each  $R_i$  is an additive subgroup of R, satisfying the properties  $R_i R_j \subseteq R_{i+j}$  and  $R_0 = A$ . Similarly,

a graded *R*-module *M* is a module decomposed as  $M = \bigoplus_{i \in \mathbb{N}} M_i$ , where  $M_i$  are additive subgroups of *M* with  $R_i M_j \subseteq M_{i+j}$ .

The elements in  $R_i$  are referred to as *homogeneous of degree i*. Additionally, an ideal I of a graded A-algebra R is called a *homogeneous ideal* if it can be generated by homogeneous elements. Furthermore, R is called *graded local* if it possesses only one maximal homogeneous ideal. We denote the homogeneous ideal  $\bigoplus_{i>0} R_i$  by  $R_+$ .

*Remark* 2.4. There exists a natural construction to pass from a filtered ring to a graded ring, achieved through the grading functor gr. Given a filtered *A*-algebra *R*, one constructs the *associated graded ring*  $gr(R) = \bigoplus_{i \in \mathbb{N}} R_i/R_{i+1}$ . This is naturally a graded *A*-algebra. Moreover, if *A* is a local ring with maximal ideal m, then gr(R) becomes a graded local *A*-algebra with maximal ideal m  $\oplus R_+$ .

For any filtered *R*-module *M*, we can also construct  $\operatorname{gr}(M) = \bigoplus_{i \in \mathbb{N}} M_i/M_{i+1}$ , equipped with a  $\operatorname{gr}(R)$ -module structure. If  $\phi : M \to N$  is a homomorphism of filtered *R*-modules, then there exists a  $\operatorname{gr}(R)$ -homomorphism  $\operatorname{gr}(\phi) : \operatorname{gr}(M) \to \operatorname{gr}(N)$ .

**Lemma 2.** Given a flat filtered A-algebra R with  $R = \bigcup_{i \in \mathbb{N}} R_i$  such that  $R_i/R_{i+1}$  are flat A-modules, for each i, one has

$$\operatorname{gr}(R \otimes_A M) = \operatorname{gr}(R) \otimes_A \operatorname{gr}(M)$$

for any filtered *R*-module *M*.

*Proof.* For a filtered *R*-module *M*, there exists a natural filtration of  $R \otimes_A M$ :

$$(R \otimes_A M)_n = \sum_{i+j=n} R_i \otimes_A M_j.$$

The fact that each summand on the right-hand side is an *R*-submodule of  $R \otimes_A M$  follows from *R* being a flat filtration, with  $R_i/R_{i+1}$  also being flat. Furthermore, we have the natural exact sequence:

$$0 \to \sum_{i+j=n+1} R_i \otimes_A M_j \to \sum_{i+j=n} R_i \otimes_A M_j \to \bigoplus_{i+j=n} R_i/R_{i+1} \otimes_A M_j/M_{j+1} \to 0,$$

which concludes the proof.

*Remark* 2.5. In what follows, we utilize two types of localization functors with respect to a given prime ideal in a graded *A*-algebra. Let  $\mathfrak{p}$  be a prime ideal in the graded *A*-algebra *R*. Define two sets:  $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$  and  $S_{(\mathfrak{p})} = h(R) \setminus \mathfrak{p}$ , where h(R) denotes the set of homogeneous elements in *R*, and consider the *A*-algebras  $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$  and  $R_{(\mathfrak{p})} = S_{(\mathfrak{p})}^{-1}R$ . Then  $R_{\mathfrak{p}}$  is a local ring with maximal ideal  $\mathfrak{p}R_{\mathfrak{p}}$ , while  $R_{(\mathfrak{p})}$  is a local graded ring with homogeneous maximal ideal  $(\mathfrak{p})_g R_{(\mathfrak{p})}$ , where  $(\mathfrak{p})_g$  denotes the homogeneous ideal contained in  $\mathfrak{p}$  such that no other homogeneous ideal contained in  $\mathfrak{p}$  contains it. For further details, refer to [NvO82, Chp B, III-1 to 3].

### 3 Relative global dimension and smoothness

#### 3.1 Relative global dimension of smooth algebras

For commutative noetherian algebras, the notion of smooth algebras is well-established (cf. [Lod92, Appendix E] and references therein). An ideal J of a ring R is said to be a *locally complete intersection* if, for every maximal ideal m of R, the ideal  $J_m$  is generated by an  $A_m$ -regular sequence. Moreover, for a noetherian ring B and a B-algebra A of essentially finite type, A is called *smooth* if the ring map  $B \to A$  is flat and the ideal ker( $\mu : A^e \to A$ ) is a locally complete intersection. In this case, we say that A is a smooth commutative noetherian B-algebra. Throughout this section, denote by J the kernel of multiplication map  $\mu : A^e \to A$ , and by  $\Omega_{A|B}$  the A-bimodule  $J/J^2$ .

*Remark* 3.1. Observe that there are two types of maximal ideals in  $A^e$ : A first type of the form  $\mu^{-1}(\mathfrak{m})$  for some maximal ideal  $\mathfrak{m}$  of A. Furthermore, the localization of A, as an  $A^e$ -module, at these ideals is equal to  $A_{\mathfrak{m}}$ . The second type consists of those  $\mathfrak{n}$  where the localization of A as an  $A^e$ -module is trivial, that is,  $\mu(\mathfrak{n}) = A$ . Moreover, localizing  $J^k/J^{k+1}$  as an  $A^e$ -module at a maximal ideal of the form  $\mu^{-1}(\mathfrak{m})$  is equivalent to localizing it as an A-module at  $\mathfrak{m}$ .

*Remark* 3.2. One consequence of the definition of smoothness above, which is sometimes used as part of the definition of smoothness for noetherian *B*-algebras, is that the *A*-module  $\Omega_{A|B}$  is finite and projective. Another consequence is that if we consider the filtration as the one in Example 1 then  $gr(A_{\mu^{-1}(\mathfrak{m})}^{e})$ , where  $\mathfrak{m}$  is a maximal ideal of *A*, is isomorphic to  $A_{\mathfrak{m}}[x_{1},...,x_{n}]$ , where the  $x_{i}$  are determined by the regular sequence generating the ideal  $J_{\mu^{-1}(\mathfrak{m})}$ .

**Proposition 3.** A smooth Noetherian B-algebra A satisfies the following conditions

- (i)  $\operatorname{pd}_{\operatorname{gr}(A^e)} A$  is finite;
- (ii)  $gr(A^e)$  is a projective A-module.

*Proof.* [(*i*)] A homogeneous maximal ideal m in

$$\operatorname{gr}(A^e) = A^e/J \oplus J/J^2 \oplus \cdots \oplus J^i/J^{i+1} \oplus \dots$$

always has the form

$$\mathfrak{m} = \mathfrak{n} \oplus J/J^2 \oplus \cdots \oplus J^i/J^{i+1} \oplus \ldots,$$

where n is a maximal ideal in R. This shows that  $gr(A^e)_{(\mathfrak{m})} = (R/J)_{\mathfrak{n}}[x_1, ..., x_n]$ . For ideals of the form  $\mathfrak{n} = \mu^{-1}(\tilde{\mathfrak{m}})$  for some maximal ideal  $\tilde{\mathfrak{m}}$  in A we have  $(A^e/J)_{\mathfrak{n}} = A_{\tilde{\mathfrak{m}}}$  and, for those which are not of this form, then  $gr(A^e)_{(\mathfrak{m})} = 0$ . Therefore, from this point on, we only consider those maximal ideals of  $A^e$  that are the inverse image of a maximal ideal in A by the multiplication.

Now, observe the compatibility of the following localizations, for a given homogeneous maximal ideal  $\mathfrak{m}$  of  $gr(A^e)$ :

$$(\operatorname{gr}(A^e))_{\mathfrak{m}} = ((\operatorname{gr}(A^e))_{(\mathfrak{m})})_{\mathfrak{m}^e},$$

where  $\mathfrak{m}^e$  is the extension of  $\mathfrak{m}$  in  $\operatorname{gr}(A^e)_{(\mathfrak{m})}$ . From our description of  $\operatorname{gr}(A^e)_{(\mathfrak{m})}$  and [BH93, Proposition 1.5.15], we have  $\operatorname{pd}_{\operatorname{gr}(A^e)_{\mathfrak{m}}} A_{\mu(\mathfrak{m})} \leq n$ . Note that the number of variables in the polynomial ring  $\operatorname{gr}(A^e)_{(\mathfrak{m})}$  is equal to the rank of  $J_{\mu^{-1}(\mathfrak{m})}/J^2_{\mu^{-1}(\mathfrak{m})}$ . Since  $\Omega_{A|B}$  is a finite and projective *A*-module, its rank is locally constant [Sta18, Tag 00NV], and therefore

$$\operatorname{pd}_{\operatorname{gr}(A^e)} A = \sup\{\operatorname{pd}_{\operatorname{gr}(A^e)_{\mathfrak{m}}} A_{\mu(\mathfrak{m})}\} \leqslant l,$$

with the supremum taken for  $\mathfrak{m}$  being in the set of graded maximal ideals of  $\operatorname{gr}(A^e)$  and l the maximal rank of the free  $A_{\mu(\mathfrak{m})}$ -modules  $(\Omega_{A|B})_{\mu(\mathfrak{m})}$ .

[(*ii*)] This follows from Remark 3.2. As gr(R) is an *A*-direct sum of the *A*-modules  $J^i/J^{i+1}$ , which, when localized at each maximal ideal of *A*, become the  $A_{\mathfrak{m}}$ -modules of the form  $J^k_{\mu^{-1}(\mathfrak{m})}/J^{i+1}_{\mu^{-1}(\mathfrak{m})}$ . These modules are projective  $A_{\mathfrak{m}}$ -modules for every *i*, and moreover, they are free by the isomorphism given in [Eis95, Ex.17.16]. Therefore, each  $J^i/J^{i+1}$  is *A*-projective and we get the claim.

*Remark* 3.3. One immediate consequence of the proposition is that, for commutative rings, being a smooth noetherian *B*-algebra implies *homological smoothness*, that is, *A* has a finite projective resolution of finitely generated  $A^e$ -modules.

**Theorem 4.** The relative global dimension gldim(A, B) is finite for a smooth gradually finite *B*-algebra. Moreover, it is bounded by the projective dimension of *A* as a  $gr(A^e)$ -module.

*Proof.* We begin the proof by constructing a standard  $(A^e, A)$ -projective resolution of A, where each term of the resolution is of the form  $A^e \otimes_A V$  for some A-module V. Note that the multiplication map  $\mu : A^e \to A$  induces the  $(A^e, A)$ -exact sequence

$$0 \to J \to A^e \xrightarrow{\mu} A \to 0, \tag{3.4}$$

where  $J = \text{ker}(\mu)$ . Then proceeding as in Remark 2.1, we obtain the long  $(A^e, A)$ -exact sequence by splicing the corresponding sequence of short  $(A^e, A)$ -exact sequences:



Observe that we can view this sequence as a resolution of filtered *A*-modules, where we induce the filtration inductively from the filtration of (3.4) and by taking the product filtration, as in the proof of Lemma 2.

Now, we consider the graded resolution associated with this filtered resolution. Since  $gr(A^e)$  is a projective *A*-module and by Lemma 2, we conclude that this is also a gr(R)-projective resolution of *A*. Using the fact that *A* has finite projective dimension as a  $gr(A^e)$ -module, there exists a positive integer *n* such that truncating the resolution at degree *n* yields

a projective  $gr(A^e)$ -module  $gr K_n$ . Applying [MR87, Cor 12.2.9], one can verify that  $gr K_n$  is isomorphic to  $gr A^e \otimes_A Q$ , where Q is a projective A-module graded by A. Using the same argument as in the proof of [MR87, Theorem 12.3.4], we conclude that the corresponding term  $K_n$  is  $(A^e, A)$ -projective, as it is isomorphic to  $A^e \otimes_A Q$ . Therefore,  $pd_{(A^e, A)} A$  is finite, and by (2.3), gldim(A, B) is also finite.

#### 3.2 Smoothness of algebras with finite (co)homological dimension

**Theorem 5.** Suppose A is a noetherian flat B-algebra locally of finite type. If cdim(A, B) is finite, then A is a smooth B-algebra.

*Proof.* Consider the same resolution as in the proof of Theorem 4 from the natural exact sequence given by the multiplication morphism. Applying the additiveness of the Tor functor and the adjunction isomorphism

$$\operatorname{Tor}_{i}^{A^{e}}(A^{e}\otimes_{A}M, -) \cong \operatorname{Tor}_{i}^{A}(M, -), \quad i \ge 0$$

one checks that each  $(A^e, A)$ -projective module is A-flat and hence J and  $K_i$  are all A-flat (as kernels of flat modules). Since  $\operatorname{cdim}(A, B)$  is a finite module  $K_n$  is  $(A^e, A)$ -projective for some n, implying that  $\operatorname{fd}_{A^e} A$  is also finite (in which  $\operatorname{fd}_R M$  denotes the flat dimension of a R-module M). As was shown in [Rod90] if  $\operatorname{fd}_{A^e} A < \infty$  then  $B \to A$  has geometrically regular fibers. But, for B-algebras of finite type, this is equivalent to being smooth by [Sta18, Tag 038X].

Jointly with Theorem 4 we get the following

**Corollary 6.** Let A be a flat noetherian B-algebra locally of finite type. A is smooth if and only if  $\operatorname{cdim}(A, B)$  is finite.

It is natural to inquire whether the finiteness of gldim(A, B) implies that the *B*-algebra *A* is smooth. Notably, there are well-known counterexamples in the case where B = k and k is a non-perfect field. Indeed, let  $k = \mathbb{F}_p(t)$  be a transcendental extension of the finite field  $\mathbb{F}_p$ , the *k*-algebra  $A = k[x]/(x^p - t)$  has global dimension zero (being a field) but it is not a smooth *k*-algebra (because  $A \otimes_k A$  is a local ring with nilpotent elements and hence has infinite global dimension). This underscores the need for additional conditions on *B* to ensure the validity of the claim. Below we provide some sufficient conditions on *B* under which the question holds true.

**Theorem 7.** Let k be a perfect field, and B a finitely generated k-algebra. Suppose A is a flat noetherian B-algebra, locally of finite type, with finite gldim(A, B). Then A is smooth.

*Proof.* Using the fiberwise criterion of smoothness and the fact that smoothness is local on the target (meaning that it is sufficient to check it for closed points, see [Sta18, Tag 02G1] and [GW23, Proposition 6.15]), we prove that:

- (1) gldim $(A \otimes_B k(\mathfrak{m}))$  is finite,
- (2)  $k(\mathfrak{m})$  is a perfect field,

for every maximal ideal m in *B*. Item (2) is a direct consequence of the general Nullstellensatz theorem as stated in [WK16, Theorem 5.6.7], and the fact that algebraic extensions of a perfect field are also perfect. To prove (1), we consider an  $A \otimes k(\mathfrak{m})$ -module *M* treated as an *A*-module, and take the standard (*A*, *B*)-projective resolution of *M*:



As gldim(A, B) is finite, then  $K_n$  is (A, B)-projective for some n. Note that  $A \otimes_B M$ ,  $K_i$ , and  $A \otimes_B K_i$  all have structures as  $A \otimes_B k(\mathfrak{m})$ -modules. Furthermore, since all of these  $A \otimes_B k(\mathfrak{m})$ -modules can be viewed as  $k(\mathfrak{m})$ -vector spaces, we conclude that  $A \otimes_B M$  and  $A \otimes_B K_i$  are  $A \otimes_B k(\mathfrak{m})$ -free modules. The same argument shows that  $K_n$  is a  $A \otimes k(\mathfrak{m})$ projective module. This proves that  $gldim(A \otimes_B k(\mathfrak{m}))$  is finite.

To complete the proof, we utilize the fact that for perfect fields  $k(\mathfrak{m})$ , being geometrically regular over  $k(\mathfrak{m})$  is equivalent to being regular over  $k(\mathfrak{m})$  by [Bou22, Chp X, Section 6.4]. Additionally, being geometrically regular over a field is equivalent to the smoothness of the fiber over k(m), as per [Sta18, Tag 038X].

The previous theorem, together with Theorem 4, yield the following result.

**Corollary 8.** Let k be a perfect field, B a finitely generated k-algebra, and A a flat Noetherian Balgebra, locally of finite type. Then A is smooth if and only if gldim(A, B) is finite.

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