QUANTIFIER ALTERNATION DEPTH IN UNIVERSAL BOOLEAN DOCTRINES

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Abstract. We introduce the notion of a quantifier-stratified universal Boolean doctrine. This notion requires additional structure on a universal Boolean doctrine, accounting for the quantifier alternation depth of formulas. After proving that every Boolean doctrine over a small base category admits a quantifier completion, we show how to freely add the first layer of quantifier alternation depth to these doctrines. To achieve this, we characterize, within the doctrinal setting, the classes of quantifier-free formulas whose universal closure is valid in some common model.

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1. Introduction

The source of inspiration for this work has been M. Gehrke's talk [7] at the conference "Category theory 20→21". We now briefly recall the setting.

The set $\mathcal{F}$ of all formulas in a given first-order language can be decomposed in two different ways. The first option is to distinguish the formulas based on their quantifier depth, i.e. the depth of nesting of quantifiers. This gives a stratification $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ of $\mathcal{F}$, where $\mathcal{F}_n$ consists of the formulas whose quantifier depth is at most n. For example, all atomic formulas are in $\mathcal{F}_0$, the formula $(\forall x\, R(x,y)) \vee (\forall x\, S(x))$ (with R and S predicate symbols) belongs to $\mathcal{F}_1$, and $\forall x\, \neg \forall x\, R(x,y)$ belongs to $\mathcal{F}_2$. This is the "Noetherian induction" way of looking at $\mathcal{F}$.

This stratification is instrumental in proofs that use induction on the quantifier depth.

The second way of decomposing formulas is by looking only at the set of free variables. This second decomposition is naturally present in the categorical approach to logic initiated by F. W. Lawvere and relies on the notion of a hyperdoctrine. However, in the doctrinal approach we lose all the information about the quantifier depths of formulas.

In this paper, we wish to make the first steps in taking up on the invitation at the end of Gehrke's talk [7, minute 55]:

> "What I wanted to say, mainly, is that I wish you would try to make some nice mathematics [...] or some nice category theory out of this decoupage of the formulas [= the Noetherian induction] rather than just this [= the Lawverian way]. I mean, [the Lawverian way] is important, but [the Noetherian induction] is very useful, technically. Thank you."

It is worth mentioning that, in our work, we make a slight deviation from the usual notion of depth of nesting of quantifiers, and we consider the notion of *quantifier alternation depth* instead. For example, we place the formula $\forall x \forall y R(x,y)$ in $\mathcal{F}_1$ and not, in general, in $\mathcal{F}_2$. To be more precise, given $n \in \mathbb{N}$, we consider $\mathcal{F}_{n+1}$ to be the set of Boolean combinations of formulas of the form $\forall x_1 \dots \forall x_m\, \alpha(x_1, \dots, x_m, y_1, \dots, y_l)$ where $m, l \in \mathbb{N}$ and $\alpha(x_1, \dots, x_m, y_1, \dots, y_l) \in \mathcal{F}_n$; in particular, we stress that m ranges among *all* natural numbers. So, in general, $\forall x \forall y\, R(x,y)$ would have quantifier alternation depth 1, and $\forall x \forall y \neg \forall z\, S(x,y,z)$ would have quantifier alternation depth 2, while the classical depths of nesting of quantifiers would be 2 and 3, respectively. The name "quantifier alternation depth" is motivated by the fact that we are counting how many *alternations* of existential and universal quantifiers appear in a given formula. For example, $\forall x \forall y\, \neg \forall z\, S(x,y,z)$ is equivalent to $\forall x \forall y\, \exists z\, \neg S(x,y,z)$ (or one might consider $\neg \exists x \exists y \forall z\, S(x,y,z)$, as well), in which there are two alternating layers of universal and existential quantifier.

We use this approach because, in the doctrinal setting, one has an abstract notion of a finite set of variables in which one cannot count the number of variables.

We work in the setting of *universal Boolean doctrines*, which are a variation of Lawvere's hyperdoctrines [13, 14, 15]. The attribute "Boolean" refers to the fact that $\mathcal{F}$ can be endowed with a structure of a Boolean algebra, while "universal" refers to the fact that in $\mathcal{F}$ there are universal quantifications of formulas. Of course, in this Boolean case, the universal and existential quantifiers are interdefinable, so both quantifiers are considered even if we only mention one of them.

As hinted above, in the categorical interpretation of first-order logic given by universal Boolean doctrines, we don't have any information about the quantifier alternation depth of a formula. We address this issue by proposing a modification of the notion of a universal Boolean doctrine that takes the quantifier alternation depth into account. To this end, we give three definitions, which carry the same information: we define…

(1) …a *quantifier-free fragment* of a universal Boolean doctrine. Roughly speaking, and using the notation above, the set $\mathcal{F}$ of all first-order formulas is given, and the quantifier-free fragment specifies the set $\mathcal{F}_0$ of all quantifier-free formulas;
(2) …a *quantifier stratification* of a universal Boolean doctrine. Roughly speaking, the set $\mathcal{F}$ of all first-order formulas is given, and the quantifier stratification axiomatizes the sequence $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots$;
(3) …a *quantifier-stratified universal Boolean doctrine*. Roughly speaking, the set $\mathcal{F}$ of all first-order formulas is not given anymore, and the quantifier-stratified universal Boolean doctrine provides directly the stratification $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots$ The set $\mathcal{F}$ can then be obtained as the directed union of all the layers.

These definitions are in Section 3, where we also show their equivalence.

After this, we turn to the question: if the relevant structure on $\mathcal{F}$ is the structure of a universal Boolean doctrine, what is the intrinsic relevant structure on $\mathcal{F}_0$? It is easily seen that any quantifier-free fragment is a *Boolean doctrine*, i.e. a version of a universal Boolean doctrine which does not require the existence of quantifiers. Conversely, we prove that every Boolean doctrine satisfying a certain smallness assumption is the quantifier-free fragment of a universal Boolean doctrine. To do so, we show that we can freely add quantifiers to any Boolean doctrine $\mathbf{P}$ over a small base category; in other words, $\mathbf{P}$ admits a *quantifier completion* $\mathbf{P}^{\forall}$. This is done in Section 4.

The rest of the paper is motivated by the following question: given $\mathcal{F}_0$, how can one construct the set $\mathcal{F}_1$ obtained by freely adding one layer of quantification? Rephrasing formally: given a Boolean doctrine $\mathbf{P}_0$, letting $\mathbf{P}_0^{\forall}$ be its quantifier completion, and letting $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots$ be the quantifier stratification of $\mathbf{P}_0^{\forall}$ associated to the quantifier-free fragment $\mathbf{P}_0$ of $\mathbf{P}_0^{\forall}$, how can one construct $\mathbf{P}_1$ in terms of $\mathbf{P}_0$?

To answer this question, we characterize, within the doctrinal setting, when a finite conjunction of universal closures of quantifier-free formulas entails a finite disjunction of universal closures of quantifier-free formulas modulo a quantifier-free theory $\mathcal{T}$ (Theorem 6.6). Thanks to some basic properties of Boolean algebras, this is enough to completely characterize when a Boolean combination of universal closures of quantifier-free formulas entails another Boolean combination of universal closures of quantifier-free formulas modulo $\mathcal{T}$ (Corollary 6.10). In turn, this characterization gives the recipe for the construction of $\mathbf{P}_1$ in terms of $\mathbf{P}_0$ (Section 6.2).

To achieve these results, which are in Section 6, we need a detour about models (Section 5). This detour has its own interest and contains the most technical part of the paper. Its main result is the following: given a Boolean doctrine $\mathbf{P}$, we characterize the classes of formulas in $\mathbf{P}$ whose universal closure is valid in some Boolean model of $\mathbf{P}$ (Theorem 5.28). The characterization is reminiscent of the notion of an ultrafilter, and so we call *universal ultrafilters* the classes satisfying it (Definition 5.12). In the conclusion of Section 5, we use this characterization to obtain what we need for Section 6: given a quantifier-free theory $\mathcal{T}$, we characterize when two finite lists $(\alpha_1, \dots, \alpha_{\bar{\imath}})$ and $(\beta_1, \dots, \beta_{\bar{\jmath}})$ of quantifier-free formulas are such that every model satisfying the universal closures of all α_i's satisfies the universal closure of at least one β_j; see Corollary 5.30 (and its generalization Theorem 5.38 in which one specifies a finite number of variables exempt from universal closure).

To sum up, in this paper we propose a modification of the notion of a universal Boolean doctrine that takes the quantifier alternation depth into account, we characterize the layer 0, and we show how to freely obtain the layer 1 from a given layer 0. We believe these to be the first steps for a doctrinal understanding of the quantifier alternation depth in Boolean doctrines. This investigation opens the way to several further questions, discussed in Section 7.

2. Preliminaries on doctrines

Hyperdoctrines were introduced by F. W. Lawvere in a series of papers [13, 14, 15] to interpret both syntax and semantics of first-order theories in the same categorical setting. Lawvere's investigation in categorical logic "permits an invariant algebraic treatment of the essential problem of proof theory, though most of the later work by proof theorists still relies on presentation-dependent formulations" [16, Author's commentary]. In this paper we consider Boolean doctrines and universal Boolean doctrines, which are variations of Lawvere's hyperdoctrines; points of departure are, among others, the fact that we impose all the axioms of Boolean algebras and that we do not require the equality predicate. Boolean doctrines contain enough structure to interpret all logical connectives, while universal Boolean doctrines require further structure allowing to interpret also quantifiers. Lawvere's fundamental intuition was that quantifiers in logic are interpreted as certain adjoints.

Lawvere's doctrinal setting is amenable to a number of generalizations different from ours; for the interested reader we mention primary doctrines (where one can interpret finite conjunctions) [18, 19, 6], existential doctrines (finite conjunctions and existential quantifier) [18], universal doctrines (finite conjunctions and universal quantifier) [19], elementary doctrines (finite conjunctions and equality) [17, 6] and first-order doctrines (all logical connectives with the axioms of Heyting algebras and quantifiers) [6].

Notation 2.1. We let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers, including 0.

Notation 2.2. We let BA denote the category of Boolean algebras and Boolean homomorphisms, and Pos the category of partially ordered sets and order-preserving functions.

Definition 2.3 (Boolean doctrine)**.** For a category C with finite products, a *Boolean doctrine over* C is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. The category C is called the *base category of* $\mathbf{P}$. For each $X \in \mathsf{C}$, $\mathbf{P}(X)$ is called a *fiber*. For each morphism $f\colon X' \to X$, the function $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ is called the *reindexing along* f.

Definition 2.4 (Boolean doctrine morphism)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ be two Boolean doctrines. A *Boolean doctrine morphism* from $\mathbf{P}$ to $\mathbf{R}$ is a pair $(M, \mathfrak{m})$ where $M\colon \mathsf{C} \to \mathsf{D}$ is a functor that preserves finite products and $\mathfrak{m}\colon \mathbf{P} \to \mathbf{R} \circ M^{\mathrm{op}}$ is a natural transformation.

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[rr, "M^{\mathrm{op}}"] \arrow[dr, "\mathbf{P}"'] & \arrow[d, phantom, "\overset{\mathfrak{m}}{\to}"] & \mathsf{D}^{\mathrm{op}} \arrow[dl, "\mathbf{R}"] \\
& \mathsf{BA} &
\end{tikzcd}$$

Given Boolean doctrine morphisms $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ and $(N, \mathfrak{n})\colon \mathbf{R} \to \mathbf{S}$,

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "M^{\mathrm{op}}"] \arrow[dr, "\mathbf{P}"'] & \mathsf{D}^{\mathrm{op}} \arrow[r, "N^{\mathrm{op}}"] \arrow[d, "\mathbf{R}"] & \mathsf{E}^{\mathrm{op}} \arrow[dl, "\mathbf{S}"] \\
& \mathsf{BA} &
\end{tikzcd}$$

their composite $(N, \mathfrak{n}) \circ (M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{S}$ is the pair $(N \circ M, \mathfrak{n} \circ \mathfrak{m})\colon \mathbf{P} \to \mathbf{S}$, where $N \circ M$ is the composite of the functors between the base categories, and the component at $X \in \mathsf{C}$ of the natural transformation $\mathfrak{n} \circ \mathfrak{m}$ is defined as $(\mathfrak{n} \circ \mathfrak{m})_X = \mathfrak{n}_{M(X)} \circ \mathfrak{m}_X$, i.e. the composite of the following functions:

$$\mathbf{P}(X) \xrightarrow{\mathfrak{m}_X} \mathbf{R}(M(X)) \xrightarrow{\mathfrak{n}_{M(X)}} \mathbf{S}(NM(X)).$$

Definition 2.5. We let $\mathsf{Doct}_{\mathsf{BA}}$ denote the category of Boolean doctrines and Boolean doctrine morphisms between them.

Although 2-categorical aspects of doctrines would be very natural, we omit them for simplicity.

Definition 2.6 (Universal Boolean doctrine)**.** Given a category C with finite products, a *universal Boolean doctrine over* C is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the following properties.

(1) (Universal) For all $X, Y \in \mathsf{C}$, letting $\mathrm{pr}_1\colon X \times Y \to X$ denote the projection onto the first coordinate, the function
$$\mathbf{P}(\mathrm{pr}_1)\colon \mathbf{P}(X) \to \mathbf{P}(X \times Y),$$
has a right adjoint $\forall_X^Y$ (as an order-preserving map between posets). This means that for every $\beta \in \mathbf{P}(X \times Y)$ there is a (necessarily unique) element $\forall_X^Y \beta \in \mathbf{P}(X)$ such that, for every $\alpha \in \mathbf{P}(X)$,
$$\alpha \leq \forall_X^Y \beta \text{ in } \mathbf{P}(X) \text{ iff } \mathbf{P}(\mathrm{pr}_1)(\alpha) \leq \beta \text{ in } \mathbf{P}(X \times Y),$$

(2) (Beck-Chevalley condition) For any morphism $f\colon X' \to X$ in C, the following diagram in Pos commutes.
$$\begin{tikzcd}
X & \mathbf{P}(X \times Y) \arrow[r, "\forall_X^Y"] \arrow[d, "\mathbf{P}(f \times \mathrm{id}_Y)"'] & \mathbf{P}(X) \arrow[d, "\mathbf{P}(f)"] \\
X' \arrow[u, "f"] & \mathbf{P}(X' \times Y) \arrow[r, "\forall_{X'}^Y"'] & \mathbf{P}(X')
\end{tikzcd}$$

Definition 2.7 (Universal Boolean doctrine morphism)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ be two universal Boolean doctrines. A *universal Boolean doctrine morphism* from $\mathbf{P}$ to $\mathbf{R}$ is a Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ such that for every $X, Y \in \mathsf{C}$ the following diagram commutes.

$$\begin{tikzcd} \mathbf{P}(X \times Y) \arrow[r, "\mathfrak{m}_{X\times Y}"] \arrow[d, "\forall_X^Y"'] & \mathbf{R}(M(X) \times M(Y)) \arrow[d, "\forall_{M(X)}^{M(Y)}"] \\ \mathbf{P}(X) \arrow[r, "\mathfrak{m}_X"'] & \mathbf{R}(M(X)) \end{tikzcd} \tag{2.1}$$

Remark 2.8. In condition (1), instead of asking for a right adjoint, we can ask for the existence of a left adjoint $\exists_X^Y\colon \mathbf{P}(X \times Y) \to \mathbf{P}(X)$ of the function $\mathbf{P}(\mathrm{pr}_1)$ (again with the Beck-Chevalley condition). This property is called *existentiality*. In this Boolean case, existentiality is equivalent to universality, because the existential and the universal quantifiers are interdefinable: $\forall = \neg\exists\neg$ and $\exists = \neg\forall\neg$.

Next, we describe the leading example: the universal Boolean doctrine that describes a first-order theory.

Example 2.9 (Syntactic doctrine)**.** Fix a first-order language $\mathcal{L} = (\mathbb{F}, \mathbb{P})$ (without equality) and a theory $\mathcal{T}$ in the language $\mathcal{L}$. We define a universal Boolean doctrine

$$\mathsf{LT}^{\mathcal{T}}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA},$$

called the *syntactic doctrine of ($\mathcal{L}$ and) $\mathcal{T}$*, as follows. An object of the base category is a finite list of distinct variables and a morphism between two lists $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_m)$ is an m-tuple

$$(t_1(\vec{x}), \dots, t_m(\vec{x}))\colon (x_1, \dots, x_n) \to (y_1, \dots, y_m)$$

of terms in the context $\vec{x}$. The empty list () is the terminal object in Ctx. The product of two lists $\vec{x}$ and $\vec{y}$ in Ctx is any list whose length is the sum of the lengths of $\vec{x}$ and $\vec{y}$; if the variables in the two lists are all distinct, we can write their product as the juxtaposition $\langle \vec{x}; \vec{y}\rangle = (x_1 \dots, x_n, y_1, \dots, y_m)$. The functor $\mathsf{LT}^{\mathcal{T}}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ sends each list of variables to the poset reflection of the preordered set of well-formed formulas written with at most those variables ordered by provable consequence in $\mathcal{T}$ (so that two formulas $\alpha(\vec{x})$ and $\beta(\vec{x})$ are identified in $\mathsf{LT}^{\mathcal{T}}(\vec{x})$ if and only if $\alpha \dashv\vdash_{\mathcal{T}} \beta$); the order on $\mathsf{LT}(\vec{x})$ is $\vdash_{\mathcal{T}}$ for any pair of representatives. Moreover, $\mathsf{LT}^{\mathcal{T}}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ sends a morphism $\vec{t}(\vec{x})\colon \vec{x} \to \vec{y}$ to the substitution $[\vec{t}(\vec{x})/\vec{y}]$, which maps the equivalence class of a formula $\alpha(\vec{y})$ in $\mathsf{LT}(\vec{y})$ to the equivalence class of the formula $\alpha(\vec{t}(\vec{x})/\vec{y})$ in $\mathsf{LT}^{\mathcal{T}}(\vec{x})$.

The functor $\mathsf{LT}^{\mathcal{T}}$ is a universal Boolean doctrine. Indeed, each $\mathsf{LT}^{\mathcal{T}}(\vec{x})$ is a Boolean algebra, and every substitution preserves the Boolean structure. Moreover, given finite lists of variables $\vec{x}$ and $\vec{y}$, the right adjoint to $\mathsf{LT}^{\mathcal{T}}(\mathrm{pr}_1)\colon \mathsf{LT}^{\mathcal{T}}(\vec{x}) \to \mathsf{LT}^{\mathcal{T}}(\langle \vec{x}; \vec{y}\rangle)$ (which maps the equivalence class of a formula with variables from $\vec{x}$ to itself) is

$$\forall y_1 \dots \forall y_m\colon \mathsf{LT}^{\mathcal{T}}(\langle \vec{x}; \vec{y}\rangle) \to \mathsf{LT}^{\mathcal{T}}(\vec{x}).$$

The Beck-Chevalley condition follows from the properties of admissible substitutions of variables.

In the rest of the paper, if there is no confusion we usually omit the superscript and write LT instead of $\mathsf{LT}^{\mathcal{T}}$.

With this example in mind, given a universal Boolean doctrine, we suggest the reader thinking of the objects of the base category as lists of variables, the morphisms as terms, the fibers as sets of formulas, the reindexings as substitutions, the Boolean operations as logical connectives, and the adjunctions between fibers as quantifiers.

Remark 2.10. The setting of universal Boolean doctrines encompasses also *many-sorted* first-order theories. Indeed, any many-sorted first-order theory gives rise to a syntactic doctrine essentially in the same way as described in Example 2.9 above.

The following example is useful in defining models of a universal Boolean doctrine.

Example 2.11 (Subsets doctrine)**.** For a set X, we let $\mathscr{P}(X)$ denote the power set Boolean algebra of X. This gives rise to a universal Boolean doctrine $\mathscr{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{BA}$, called the *subsets doctrine*. The functor $\mathscr{P}$ maps an object to its power set, and maps a function $f\colon X' \to X$ to the preimage function

$$\mathscr{P}(f) \coloneqq f^{-1}[-]\colon \mathscr{P}(X) \to \mathscr{P}(X').$$

Moreover, given two sets X and Y, the right adjoint $\forall_X^Y$ to the order-preserving function $\mathrm{pr}_1^{-1}\colon \mathscr{P}(X) \to \mathscr{P}(X \times Y)$ is the function

$$\begin{aligned}\forall_X^Y\colon \mathscr{P}(X \times Y) &\longrightarrow \mathscr{P}(X)\\ S &\longmapsto \{x \in X \mid \text{for all } y \in Y,\ (x,y) \in S\}.\end{aligned}$$

3. Quantifier alternation depth for doctrines

We modify the notion of a universal Boolean doctrine so that the depth of alternation of quantifiers of the formulas is taken into account.

Consider the example of first-order formulas in Example 2.9. For every $n \in \mathbb{N}$ and every context $\vec{x}$ we define the Boolean algebra $\mathsf{LT}_n(\vec{x})$ (of "formulas with quantifier alternation depth less than or equal to n") inductively on n, as follows.

(1) We define $\mathsf{LT}_0(\vec{x}) \subseteq \mathsf{LT}(\vec{x})$ as the set of equivalence classes of quantifier-free first-order formulas with free variables in $\vec{x}$.
(2) For $n \geq 0$, $\mathsf{LT}_{n+1}(\vec{x})$ is the Boolean subalgebra of $\mathsf{LT}(\vec{x})$ generated by the elements $\forall \vec{y}\, \alpha(\vec{x}, \vec{y})$ for $\vec{y}$ ranging among contexts and $\alpha(\vec{x}, \vec{y})$ ranging in $\mathsf{LT}_n(\vec{x}, \vec{y})$.

Moreover, we have a chain of inclusions

$$\mathsf{LT}_0(\vec{x}) \subseteq \mathsf{LT}_1(\vec{x}) \subseteq \mathsf{LT}_2(\vec{x}) \subseteq \dots$$

because $\forall \vec{y}\, \alpha(\vec{x}, \vec{y})$ is equivalent to $\alpha(\vec{x})$ whenever $\vec{y}$ is the empty list.

Furthermore, every first-order formula $\alpha(\vec{x})$ belongs to $\mathsf{LT}_n(\vec{x})$ for some $n \in \mathbb{N}$. (This can be proved by induction on the complexity of $\alpha(\vec{x})$.) In other words,

$$\mathsf{LT}(\vec{x}) = \bigcup_{n \in \mathbb{N}} \mathsf{LT}_n(\vec{x}).$$

The *quantifier alternation depth* of a formula $\alpha(\vec{x}) \in \mathsf{LT}(\vec{x})$ is the least n such that $\alpha(\vec{x}) \in \mathsf{LT}_n(\vec{x})$.

Example 3.1. (1) Consider a language consisting of a unary relation symbol R and the theory with no axioms. Then $R(x)$ belongs to $\mathsf{LT}_0(x)$, and hence its quantifier alternation depth is 0.
(2) The formula $(\forall x\, R(x)) \wedge P(y)$ belongs to $\mathsf{LT}_1(x, y)$, and hence its quantifier alternation depth is less than or equal to 1.
(3) The formula $\exists x\, (P(x) \wedge \exists y Q(x, y))$ belongs to $\mathsf{LT}_1(x, y)$ because it is equivalent to $\exists x\, \exists y\, (P(x) \wedge Q(x, y))$. Therefore, its quantifier depth is less than or equal to 1.

For each $n \in \mathbb{N}$, the assignment LT_n can be extended to a functor

$$\mathsf{LT}_n\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}, \tag{3.1}$$

defining the reindexing $\mathsf{LT}_n(\vec{t}(\vec{x}))\colon \mathsf{LT}_n(\vec{y}) \to \mathsf{LT}_n(\vec{x})$ along the tuple of terms $\vec{t}(\vec{x})\colon \vec{x} \to \vec{y}$ as the restriction of $\mathsf{LT}(\vec{t}(\vec{x}))$.

The stratification of LT into the sequence LT_0, LT_1, … is the motivating example for the definitions that follow. Since the notion of a universal Boolean doctrine is blind to the quantifier alternation depth of a formula, we add further structure to take it into account. We propose three definitions, which we will prove to carry the same information. The most intrinsic one is the third one.

(1) In Definition 3.2 we define a *quantifier-free fragment* of a universal Boolean doctrine. A quantifier-free fragment consists of the class of formulas considered to be quantifier-free. This is enough to derive the quantifier alternation depth of all formulas, leading to the next definition.
(2) In Definition 3.7 we define a *quantifier stratification* of a universal Boolean doctrine: we rephrase the first definition in a way that takes all layers of the quantifier alternation depth as part of the structure.

(3) In Definition 3.12 we define a *quantifier-stratified universal Boolean doctrine*: we provide only the stratification, and the doctrine can be obtained by taking the directed union of all the layers.

3.1. Definitions of quantifier-stratifications.

Definition 3.2 (Quantifier-free fragment)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine. A *quantifier-free fragment of* $\mathbf{P}$ is a functor $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the following properties.

(1) For every $X \in \mathsf{C}$, $\mathbf{P}_0(X)$ is a Boolean subalgebra of $\mathbf{P}(X)$.
(2) For every morphism $f\colon X' \to X$ in C, the function $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ restricts to the function $\mathbf{P}_0(f)\colon \mathbf{P}_0(X) \to \mathbf{P}_0(X')$.
(3) For each object X in C, $\mathbf{P}(X) = \bigcup_{n\in\mathbb{N}} \mathbf{P}_n(X)$, where $\mathbf{P}_n(X)$ is the Boolean subalgebra of $\mathbf{P}(X)$ defined inductively on n as follows. The poset $\mathbf{P}_0(X)$ is already defined; for $n \geq 0$, $\mathbf{P}_{n+1}(X)$ is the Boolean subalgebra of $\mathbf{P}(X)$ generated by the union of the images of $\mathbf{P}_n(X \times Y)$ under $\forall^Y_X\colon \mathbf{P}(X \times Y) \to \mathbf{P}(X)$, for Y ranging among the objects of C.

Remark 3.3. For every $n \geq 0$, $\mathbf{P}_n(X) \subseteq \mathbf{P}_{n+1}(X)$. Indeed, X is a particular product of X with the terminal object $\mathbf{t}$, and the first projection $\mathrm{pr}_1\colon X = X \times \mathbf{t} \to X$ is the identity. The function $\mathbf{P}(\mathrm{pr}_1)\colon \mathbf{P}(X) \to \mathbf{P}(X)$ is the identity, and thus its right adjoint $\forall^{\mathbf{t}}_X\colon \mathbf{P}(X) \to \mathbf{P}(X)$ is also the identity. Thus, the image of $\mathbf{P}_n(X)$ under $\forall^{\mathbf{t}}_X$ is $\mathbf{P}_n(X)$, which is then contained in $\mathbf{P}_{n+1}(X)$.

Example 3.4. Consider the example of first-order formulas in Example 2.9. The functor $\mathsf{LT}_0\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ as in (3.1) is a quantifier-free fragment of LT.

Example 3.5. Given a universal Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, the functor $\mathbf{P}$ is a quantifier-free fragment of $\mathbf{P}$.

Example 3.6. We sketch an example which is both of the type of Example 3.4 and of Example 3.5. Consider a first-order theory $\mathcal{T}$. For each first-order formula $\alpha(\vec{x})$ add a relation symbol R_α with variables in $\vec{x}$, and add the axiom $\forall\vec{x}\,(R_\alpha(\vec{x}) \leftrightarrow \alpha(\vec{x}))$. This gives a new language and a theory $\mathcal{T}'$ in this language, in which every first-order formula is equivalent to a quantifier-free formula. Let $\mathsf{LT}^{\mathcal{T}'}$ be the syntactic doctrine defined from this new language and this new theory. Its quantifier-free fragment $\mathsf{LT}^{\mathcal{T}'}_0$ is $\mathsf{LT}^{\mathcal{T}'}$ itself.

Definition 3.7 (Quantifier stratification)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine. A *quantifier stratification of* $\mathbf{P}$ is a sequence of functors $\mathbf{P}_n\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the following properties.

(1) For every $X \in \mathsf{C}$ and every $n \in \mathbb{N}$, $\mathbf{P}_n(X)$ is an Boolean subalgebra of $\mathbf{P}(X)$.
(2) For every morphism $f\colon X' \to X$ in C and every $n \in \mathbb{N}$, the function $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ restricts to $\mathbf{P}_n(f)\colon \mathbf{P}_n(X) \to \mathbf{P}_n(X')$.
(3) (Chain of inclusions) For every $X \in \mathsf{C}$ we have a chain of inclusions of Boolean subalgebras
$$\mathbf{P}_0(X) \subseteq \mathbf{P}_1(X) \subseteq \mathbf{P}_2(X) \subseteq \dots.$$
(4) (Directed union) For every $X \in \mathsf{C}$,
$$\mathbf{P}(X) = \bigcup_{n\in\mathbb{N}} \mathbf{P}_n(X).$$
(5) (Restriction of universal) For every projection $\mathrm{pr}_1\colon X \times Y \to X$ in C, and every $n \in \mathbb{N}$, the function $\forall^Y_X\colon \mathbf{P}(X \times Y) \to \mathbf{P}(X)$ restricts to a function
$$\forall^Y_{X,n}\colon \mathbf{P}_n(X \times Y) \to \mathbf{P}_{n+1}(X).$$
(6) (Generation) For all $X \in \mathsf{C}$ and $n \in \mathbb{N}$, the Boolean algebra $\mathbf{P}_{n+1}(X)$ is generated by the union of the images of the functions $\forall^Y_{X,n}\colon \mathbf{P}_n(X \times Y) \to \mathbf{P}_{n+1}(X)$ for Y ranging in C.

Example 3.8. Consider the example of first-order formulas in Example 2.9. The sequence $\mathsf{LT}_n\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ defined in (3.1) is a quantifier stratification of LT.

Example 3.9. Given a universal Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, the constant sequence $(\mathbf{P}_n = \mathbf{P})_{n\in\mathbb{N}}$ is a quantifier stratification of $\mathbf{P}$.

Example 3.10. Given a quantifier stratification $(\mathbf{P}_n)_{n\in\mathbb{N}}$ of $\mathbf{P}$, and given $k \in \mathbb{N}$, $(\mathbf{P}_{k+n})_{n\in\mathbb{N}}$ is a quantifier stratification of $\mathbf{P}$.

Remark 3.11. A universal Boolean doctrine might have distinct stratifications. For example, take any first-order theory $\mathcal{T}$ in which not every formula is equivalent to a quantifier-free formula. Then the quantifiers-stratification in Example 3.8 of the doctrine corresponding to $\mathcal{T}$ differs from the constant quantifiers-stratification in Example 3.5. However, by Definition 3.7(6), a quantifier stratification $(\mathbf{P}_n)_n$ of a given Boolean doctrine $\mathbf{P}$ is completely determined by $\mathbf{P}_0$ (see Proposition 3.15 below).

A direct axiomatization (without the universal Boolean doctrine as part of the structure) is as follows.

Definition 3.12 (Quantifier-stratified universal Boolean doctrine)**.** A *quantifier-stratified universal Boolean doctrine* is a sequence of functors $(\mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{n\in\mathbb{N}}$ where C is a category with finite products, such that, for every $X \in \mathsf{C}$ and $n \in \mathbb{N}$, $\mathbf{P}_n(X)$ is a Boolean subalgebra of $\mathbf{P}_{n+1}(X)$, for every morphism $f \colon X' \to X$ in C and every $n \in \mathbb{N}$, the function $\mathbf{P}_{n+1}(f) \colon \mathbf{P}_{n+1}(X) \to \mathbf{P}_{n+1}(X')$ extends the function $\mathbf{P}_n(f) \colon \mathbf{P}_n(X) \to \mathbf{P}_n(X')$, and the following conditions hold.

(1) (Universal) For every projection $\mathrm{pr}_1 \colon X \times Y \to X$ in C, $n \in \mathbb{N}$, and $\beta \in \mathbf{P}_n(X \times Y)$ there is an element $\forall^Y_{X,n}\beta \in \mathbf{P}_{n+1}(X)$ such that, for every $\alpha \in \mathbf{P}_{n+1}(X)$, we have (denoting with $i_{X\times Y,n}$ the inclusion of $\mathbf{P}_n(X \times Y)$ into $\mathbf{P}_{n+1}(X \times Y)$)
$$\alpha \le \forall^Y_{X,n}\beta \text{ in } \mathbf{P}_{n+1}(X) \iff \mathbf{P}_{n+1}(\mathrm{pr}_1)(\alpha) \le i_{X\times Y,n}(\beta) \text{ in } \mathbf{P}_{n+1}(X \times Y).$$
(Note that one such element $\forall^Y_{X,n}\beta$ is unique, as we have described its principal downset.)

(2) (Beck-Chevalley) For every morphism $f \colon X' \to X$ in C and $n \in \mathbb{N}$ the following diagram in Pos commutes.
$$\begin{tikzcd} \mathbf{P}_n(X' \times Y) \arrow[r, "\forall^Y_{X',n}"] \arrow[d, "\mathbf{P}_n(f\times \mathrm{id}_Y)"'] & \mathbf{P}_{n+1}(X') \arrow[d, "\mathbf{P}_{n+1}(f)"] \\ \mathbf{P}_n(X \times Y) \arrow[r, "\forall^Y_{X,n}"'] & \mathbf{P}_{n+1}(X) \end{tikzcd}$$

(3) (Restriction of universal) For all $X, Y \in \mathsf{C}$ and every $n \in \mathbb{N}$, the map $\forall^Y_{X,n+1}$ restricts to $\forall^Y_{X,n}$, i.e. the following diagram in Pos commutes.
$$\begin{tikzcd} \mathbf{P}_n(X \times Y) \arrow[r, "\forall^Y_{X,n}"] \arrow[d, hook, "i_{X\times Y,n}"'] & \mathbf{P}_{n+1}(X) \arrow[d, hook, "i_{X,n+1}"] \\ \mathbf{P}_{n+1}(X \times Y) \arrow[r, "\forall^Y_{X,n+1}"'] & \mathbf{P}_{n+2}(X) \end{tikzcd}$$

(4) (Generation) For all $X \in \mathsf{C}$ and $n \in \mathbb{N}$, the Boolean algebra $\mathbf{P}_{n+1}(X)$ is generated by the union of the images of the functions $\forall^Y_{X,n} \colon \mathbf{P}_n(X \times Y) \to \mathbf{P}_{n+1}(X)$ for Y ranging in C.

3.2. **Equivalence between the definitions of quantifier-stratifications.** We next show the equivalence between Definitions 3.2, 3.7 and 3.12.

Lemma 3.13. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine, and let $\mathbf{P}_0$ be a quantifier-free fragment of $\mathbf{P}$. For all $n \in \mathbb{N}$ and $X \in \mathsf{C}$, the assignment $\mathbf{P}_n(X)$ can be extended to a functor $\mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ such that the sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is a quantifier stratification of $\mathbf{P}$.*

Proof. First of all, we prove inductively that $\mathbf{P}_n$ can be extended to a functor: for any morphism $f \colon X' \to X$ in C, let $\mathbf{P}_n(f) \colon \mathbf{P}_n(X) \to \mathbf{P}_n(X')$ be the restriction of $\mathbf{P}(f) \colon \mathbf{P}(X) \to \mathbf{P}(X')$. We show inductively that this restriction is well-defined. The function $\mathbf{P}_0(f)$ is the restriction of $\mathbf{P}(f)$ by definition. Then suppose that $\mathbf{P}_n(f) \colon \mathbf{P}_n(X) \to \mathbf{P}_n(X')$ is the restriction of $\mathbf{P}(f)$. Take a generator $\forall^Y_X\alpha \in \mathbf{P}_{n+1}(X)$ for some Y in C and some $\alpha \in \mathbf{P}_n(X \times Y)$. Then, using the Beck-Chevalley condition and the inductive hypothesis,
$$\mathbf{P}(f)(\forall^Y_X\alpha) = \forall^Y_{X'}\mathbf{P}(f \times \mathrm{id}_Y)(\alpha) = \forall^Y_{X'}\mathbf{P}_n(f \times \mathrm{id}_Y)(\alpha),$$

and hence $\mathbf{P}(f)(\forall_X^Y \alpha) \in \mathbf{P}_{n+1}(X')$. Since $\mathbf{P}(f)$ is a Boolean homomorphism, it restricts to the function $\mathbf{P}_{n+1}(f)\colon \mathbf{P}_{n+1}(X) \to \mathbf{P}_{n+1}(X')$. The sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is a quantifier stratification of $\mathbf{P}$; indeed, all the needed properties follow by definition, except for the inclusions $\mathbf{P}_n(X) \subseteq \mathbf{P}_{n+1}(X)$, which follow from Remark 3.3. □

Lemma 3.14. *Let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a quanitfier stratification of a universal Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. Then, $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}$.*

Proof. For every $X \in \mathsf{C}$, $\mathbf{P}_0(X)$ is a Boolean subalgebra of $\mathbf{P}(X)$ and, for any $f\colon X' \to X$, the reindexing $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ restricts to $\mathbf{P}_0(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ by (4). Then use (5) and (6) inductively to observe that for each $n \in \mathbb{N} \setminus \{0\}$, the Boolean subalgebra of $\mathbf{P}(X)$ generated by the union of the images of $\mathbf{P}_{n-1}(X)$ under $\forall_X^Y$ for Y ranging in C is $\mathbf{P}_n(X)$, and then by (4) we have that $\mathbf{P}(X) = \bigcup_{n\in\mathbb{N}} \mathbf{P}_n(X)$, as claimed. □

Proposition 3.15. *There is a 1:1 correspondence between quantifier-free fragments and quantifier stratifications of a given universal Boolean doctrine. The mutually inverse assignments are described in Lemmas 3.13 and 3.14.*

Proof. Let $\mathbf{P}$ be a universal Boolean doctrine.

Let $\mathbf{P}_0$ be a quantifier-free fragment. Apply Lemma 3.13 to define a quantifier stratification $(\mathbf{P}_n)_{n\in\mathbb{N}}$, and then apply Lemma 3.14 to obtain the quantifier-free fragment $\mathbf{P}_0$, which is the same quantifier-free fragment we started with.

Conversely, let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a quantifier stratification of $\mathbf{P}$. Consider the quantifier-free fragment $\mathbf{P}_0$, and then apply Lemma 3.13 to define a quantifier stratification $(\mathbf{P}'_n)_{n\in\mathbb{N}}$, where $\mathbf{P}'_n(X)$ is defined inductively in Definition 3.2. We prove inductively that the functors $\mathbf{P}_n$ and $\mathbf{P}'_n$ coincide. The base case is immediate. Let $n \in \mathbb{N}$ and suppose that $\mathbf{P}_n$ and $\mathbf{P}'_n$ coincide. The Boolean subalgebras $\mathbf{P}'_{n+1}(X)$ and $\mathbf{P}_{n+1}(X)$ of $P(X)$ have the same set of generators, and the reindexings are defined in both cases as the restrictions of the reindexings of $\mathbf{P}$. Therefore, the functors $\mathbf{P}_{n+1}$ and $\mathbf{P}'_{n+1}$ coincide. □

Lemma 3.16. *Let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a quantifier stratification of a universal Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. Then, $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is a quantifier-stratified universal Boolean doctrine.*

Proof. Let $\alpha \in \mathbf{P}_{n+1}(X)$ and $\beta \in \mathbf{P}_n(X \times Y)$. We have

$$
\begin{aligned}
&\alpha \leq \forall_{X,n}^Y \beta && \text{(in } \mathbf{P}_{n+1}(X))\\
&\iff \alpha \leq \forall_X^Y \beta && \text{(in } \mathbf{P}(X))\\
&\iff \mathbf{P}(\mathrm{pr}_1)(\alpha) \leq \beta && \text{(in } \mathbf{P}(X \times Y))\\
&\iff \mathbf{P}_{n+1}(\mathrm{pr}_1)(\alpha) \leq i_{X\times Y,n}(\beta) && \text{(in } \mathbf{P}_{n+1}(X \times Y)).
\end{aligned}
$$

This proves condition (1) in Definition 3.12. The diagram in Definition 3.12(2) is commutative because it is the restriction of the diagram defining the Beck-Chevalley condition for $\mathbf{P}$. The diagram in Definition 3.12(3) is commutative because both $\forall_{X,n}^Y$ and $\forall_{X,n+1}^Y$ are restrictions of the same function $\forall_X^Y$. Finally, condition (4) in Definition 3.12 follows from Definition 3.7(6). □

Lemma 3.17. *Let $(\mathbf{P}_n\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{n\in\mathbb{N}}$ be a quantifier-stratified universal Boolean doctrine. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be the functor defined as follows. For $X \in \mathsf{C}$, we set $\mathbf{P}(X) = \bigcup_{n\in\mathbb{N}} \mathbf{P}_n(X)$, with the obvious structure of a Boolean algebra. For a morphism $f\colon X' \to X$ in C, the function $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ is the obvious one. Then $\mathbf{P}$ is a universal Boolean doctrine, and $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is a quantifier stratification of $\mathbf{P}$.*

Proof. We first prove that $\mathbf{P}$ is a universal Boolean doctrine. For every projection $\mathrm{pr}_1\colon X \times Y \to X$, define the function $\forall_X^Y\colon \mathbf{P}(X \times Y) \to \mathbf{P}(X)$ as $\forall_X^Y \coloneqq \bigcup_{n\in\mathbb{N}} \forall_{X,n}^Y$. This is well-defined by Definition 3.12(3). To check that $\forall_X^Y$ is the right adjoint to $\mathbf{P}(\mathrm{pr}_1)$, let $\alpha \in \mathbf{P}(X)$ and $\beta \in \mathbf{P}(X \times Y)$. There is $n \in \mathbb{N}$ large enough so that $\alpha \in \mathbf{P}_{n+1}(X)$ and $\beta \in \mathbf{P}_n(X \times Y)$. Then,

$$\begin{aligned}
&\alpha \leq \forall_X^Y \beta && (\text{in } \mathbf{P}(X))\\
&\iff \alpha \leq \forall_{X,n}^Y \beta && (\text{in } \mathbf{P}_{n+1}(X))\\
&\iff i_{X\times Y,n}(\alpha) \leq \mathbf{P}_{n+1}(\mathrm{pr}_1)(\beta) && (\text{in } \mathbf{P}_{n+1}(X \times Y))\\
&\iff \mathbf{P}(\mathrm{pr}_1)(\alpha) \leq \beta && (\text{in } \mathbf{P}(X \times Y)).
\end{aligned}$$

Hence, $\forall_X^Y$ is the right adjoint of $\mathbf{P}(\mathrm{pr}_1)$. The Beck-Chevalley condition follows from (2). So $\mathbf{P}$ is indeed a universal Boolean doctrine.

The sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is easily seen to be a quantifier stratification of $\mathbf{P}$, as all conditions in Definition 3.7 follow directly from Definition 3.12. □

Proposition 3.18. *There is a 1:1 correspondence between universal Boolean doctrines equipped with a quantifier stratification and quantifier-stratified universal Boolean doctrines. The mutually inverse assignments are described in Lemmas 3.16 and 3.17.*

Proof. Let $(\mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{n\in\mathbb{N}}$ be a quantifier-stratified universal Boolean doctrine. By Lemma 3.17, $\bigcup_{n\in\mathbb{N}} \mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ is a universal Boolean doctrine and $(\mathbf{P}_n)_{n\in\mathbb{N}}$ a stratification of it. From this, using Lemma 3.16, we get the quantifier-stratified universal Boolean doctrine $(\mathbf{P}_n)_{n\in\mathbb{N}}$, which is the one we started from.

Conversely, let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a quantifier stratification of a universal Boolean doctrine $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. Use Lemma 3.16 to get a quantifier-stratified universal Boolean doctrine $(\mathbf{P}_n)_{n\in\mathbb{N}}$. By Lemma 3.17, $\bigcup_{n\in\mathbb{N}} \mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ is a universal Boolean doctrine and $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is a stratification of it. Moreover, $\bigcup_{n\in\mathbb{N}} \mathbf{P}_n = \mathbf{P}$ by assumption, and hence we have obtained again the universal Boolean doctrine $\mathbf{P}$ with its quantifier stratification $(\mathbf{P}_n)_{n\in\mathbb{N}}$. □

For each $n \in \mathbb{N}$, what are the properties satisfied by the tuples of the form $(\mathbf{P}_0, \dots, \mathbf{P}_n)$ for some quantifier-stratified universal Boolean doctrine $(\mathbf{P}_n)_{n\in\mathbb{N}}$? The answer for $n = 0$ (for the case of a small base category) is in the next section: $\mathbf{P}_0$ is a Boolean doctrine. The remaining cases seem to require much more effort and are left to future work; see Sections 7.1 and 7.2.

4. Quantifier completion of a Boolean doctrine

Via standard techniques of universal many-sorted algebra, we show that we can freely add quantifiers to any Boolean doctrine $\mathbf{P}$ over a small base category; in other words, $\mathbf{P}$ admits a *quantifier completion* $\mathbf{P}^\forall$ (Corollary 4.23). Towards the aim of showing that $\mathbf{P}$ embeds in $\mathbf{P}^\forall$, we establish a *completeness theorem for Boolean doctrines* (Theorem 4.9); this states that distinct elements in a common fiber of a Boolean doctrine (i.e., distinct formulas in a common context) are separated by Boolean models. Then, using the universal property of $\mathbf{P}^\forall$ and the fact that models are morphisms towards the subsets doctrine $\mathscr{P}$, which is universal, the completeness theorem allows us to prove that $\mathbf{P}$ embeds in $\mathbf{P}^\forall$.

All this shows that every Boolean doctrine is a quantifier-free fragment of its quantifier completion (Proposition 4.15). Consequently, Boolean doctrines are precisely the quantifier-free fragments of some universal Boolean doctrine. This answers the question at the end of Section 3 for the case $n = 0$.

4.1. Completeness theorem for Boolean doctrines.

Definition 4.1 (Boolean model)**.** Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *Boolean model of* $\mathbf{P}$ is a Boolean doctrine morphism $(M, \mathfrak{m}) \colon \mathbf{P} \to \mathscr{P}$, where $\mathscr{P}$ is the subsets doctrine.

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[rr, "M^{\mathrm{op}}"] \arrow[dr, "\mathbf{P}"'] & {} \arrow[d, phantom, "\xrightarrow{\mathfrak{m}}"] & \mathsf{Set}^{\mathrm{op}} \arrow[dl, "\mathscr{P}"] \\
& \mathsf{BA} &
\end{tikzcd}$$

Definition 4.2 (Universal Boolean model)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine. A *universal Boolean model of* $\mathbf{P}$ is a universal Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$.

In the syntactic context, a universal Boolean model $(M, \mathfrak{m})$ of $\mathsf{LT}^{\mathcal{T}}$ corresponds precisely to a model of the theory $\mathcal{T}$ in the classical sense. The assignment of the functor M on objects encodes the underlying set of the model, the assignment of M on morphisms encodes the interpretation of the function symbols, and the natural transformation $\mathfrak{m}$ encodes the interpretation of the predicate symbols. In detail, the underlying set $\mathbb{M}$ of the model is the value of the functor M at the object (x) (the context with only one variable). The interpretation $\mathbb{I}(f)\colon \mathbb{M}^n \to \mathbb{M}$ of a function symbol f of arity n is the value of the functor M at the morphism $f\colon (x_1, \dots, x_n) \to (y)$. The interpretation $\mathbb{I}(Q) \subseteq \mathbb{M}^n$ of an atomic formula Q of arity n is $\mathfrak{m}_{(x_1,\dots,x_n)}(Q(x_1, \dots, x_n)) \in \mathscr{P}(M(x_1, \dots, x_n))$.

Note that in Definitions 4.1 and 4.2 we admit the functor M to assign the empty set to some objects of C. In the syntactic context, this means that we allow the empty model.

Theorem 4.3. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine and F a subset of $\mathbf{P}(\mathbf{t})$. The following conditions are equivalent.*

(1) There is a Boolean model $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ such that $F = \{\alpha \in \mathbf{P}(\mathbf{t}) \mid \mathfrak{m}_{\mathbf{t}}(\alpha) = M(\mathbf{t})\}$.
(2) F is an ultrafilter of $\mathbf{P}(\mathbf{t})$.

Proof. (1) $\Rightarrow$ (2). The functor M preserves finite products, and hence $M(\mathbf{t})$ is a singleton. Thus, $\mathscr{P}(M(\mathbf{t}))$ is a two-element Boolean algebra. The set F is an ultrafilter because it is the preimage under the Boolean homomorphism $\mathfrak{m}_{\mathbf{t}}\colon \mathbf{P}(\mathbf{t}) \to \mathscr{P}(M(\mathbf{t}))$ of the top element of the two-element Boolean algebra $\mathscr{P}(M(\mathbf{t}))$.

(2) $\Rightarrow$ (1). Set $M \coloneqq \mathsf{Hom}(\mathbf{t}, -)\colon \mathsf{C} \to \mathsf{Set}$, and let $\mathfrak{m}\colon \mathbf{P} \to \mathscr{P} \circ M^{\mathrm{op}}$ be the natural transformation whose component at $X \in \mathsf{C}$ is the function

$$\begin{aligned} \mathfrak{m}_X\colon \mathbf{P}(X) &\longrightarrow \mathscr{P}(\mathsf{Hom}(\mathbf{t}, X)) \\ \alpha &\longmapsto \{c\colon \mathbf{t} \to X \mid \mathbf{P}(c)(\alpha) \in F\}. \end{aligned}$$

The fact that $\mathfrak{m}_X$ is a Boolean homomorphism is easily proved using that F is an ultrafilter and that $\mathbf{P}(c)$ is a Boolean homomorphism for each $c\colon \mathbf{t} \to X$. We prove naturality of $\mathfrak{m}$: let $X, X' \in \mathsf{C}$, let $\alpha \in \mathbf{P}(X)$, and let $f\colon X' \to X$ and $c\colon \mathbf{t} \to X'$ be morphisms in C. We have

$$\begin{aligned} c \in \mathfrak{m}_{X'}(\mathbf{P}(f)(\alpha)) &\iff \mathbf{P}(c)\mathbf{P}(f)(\alpha) \in F \\ &\iff \mathbf{P}(f \circ c)(\alpha) \in F \\ &\iff f \circ c \in \mathfrak{m}_X(\alpha) \\ &\iff c \in (f \circ -)^{-1}[\mathfrak{m}_X(\alpha)]. \end{aligned}$$

Finally, we prove $F = \{\alpha \in \mathbf{P}(\mathbf{t}) \mid \mathfrak{m}_{\mathbf{t}}(\alpha) = \mathsf{Hom}(\mathbf{t}, \mathbf{t})\}$. Let $\alpha \in \mathbf{P}(\mathbf{t})$. We have

$$\mathfrak{m}_{\mathbf{t}}(\alpha) = \mathsf{Hom}(\mathbf{t}, \mathbf{t}) \iff \mathrm{id}_{\mathbf{t}} \in \mathfrak{m}_{\mathbf{t}}(\alpha) \iff \mathbf{P}(\mathrm{id}_{\mathbf{t}})(\alpha) \in F \iff \alpha \in F. \qquad \square$$

Remark 4.4. We translate Theorem 4.3 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a countable set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let F be a set of closed formulas (i.e. formulas with no free variables) which are quantifier-free. The following conditions are equivalent.

(1) There is a model M of $\mathcal{T}$ such that F is the set of quantifier-free closed formulas α such that $M \vDash \alpha$.
(2) The following conditions hold.
   (a) For all quantifier-free closed formulas α, β, if $\alpha \in F$ and $\alpha \vdash_{\mathcal{T}} \beta$, then $\beta \in F$.
   (b) For all $\alpha_1, \alpha_2 \in F$ we have $\alpha_1 \wedge \alpha_2 \in F$.
   (c) $\top \in F$.
   (d) For all quantifier-free closed formulas α_1 and α_2, if $\alpha_1 \vee \alpha_2 \in F$, then $\alpha_1 \in F$ or $\alpha_2 \in F$.
   (e) $\bot \notin F$.

Remark 4.5 (Adding constants to a doctrine)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $S \in \mathsf{C}$. We review from [9] the construction that freely adds a constant of type S.

Let C_S be the Kleisli category for the reader comonad $S \times -\colon \mathsf{C} \to \mathsf{C}$. The category C_S has the same objects as the category C. For a pair of objects X, Y in C_S (or, equivalently, in C) a morphism $f\colon X \rightsquigarrow Y$ in C_S is a morphism $f\colon S \times X \to Y$ in C. The composition of $f\colon X \rightsquigarrow Y$ and $g\colon Y \rightsquigarrow Z$ is $g \circ \langle \mathrm{pr}_1, f\rangle\colon X \rightsquigarrow Z$:

$$S \times X \xrightarrow{\langle \mathrm{pr}_1, f\rangle} S \times Y \xrightarrow{g} Z.$$

The identity on the object X in C_S is the morphism $X \rightsquigarrow X$ corresponding to the projection over X in C:

$$S \times X \xrightarrow{\mathrm{pr}_2} X.$$

We remark that in this new category, there is a morphism $\mathbf{t} \rightsquigarrow S$, corresponding to $\mathrm{id}_S\colon S \to S$, by choosing S as a product of S and $\mathbf{t}$.

The new doctrine $\mathbf{P}_S\colon {\mathsf{C}_S}^{\mathrm{op}} \to \mathsf{BA}$ is defined as follows:

$$\text{The reindexing of} \quad \begin{matrix} Y \\ \Big\uparrow\!\wr\, f \\ X \end{matrix} \quad \text{is} \quad \begin{matrix} \mathbf{P}(S \times Y) \\ \Big\downarrow \mathbf{P}(\langle \mathrm{pr}_1, f\rangle) \\ \mathbf{P}(S \times X). \end{matrix}$$

The Boolean doctrine $\mathbf{P}_S$ comes with a canonical Boolean doctrine morphism $(L_S, \mathfrak{l}_S)\colon \mathbf{P} \to \mathbf{P}_S$. The functor $L_S\colon \mathsf{C} \to \mathsf{C}_S$ maps a morphism $f\colon X \to Y$ to the morphism $f \circ \mathrm{pr}_2\colon X \rightsquigarrow Y$. For an object X, the corresponding component of the natural transformation is the following:

$$\begin{aligned} (\mathfrak{l}_S)_X\colon \mathbf{P}(X) &\longrightarrow \mathbf{P}_S(X) = \mathbf{P}(S \times X) \\ \alpha &\longmapsto \mathbf{P}(\mathrm{pr}_2)(\alpha). \end{aligned}$$

The fibers of $\mathbf{P}_S$ inherit the Boolean structure of the fibers of $\mathbf{P}$.

If the starting Boolean doctrine $\mathbf{P}$ is universal, then $\mathbf{P}_S$ is also universal, and the morphism $(L_S, \mathfrak{l}_S)$ is universal. The universal structure is defined as follows: for a pair of objects $X, Y \in \mathsf{C}_S$, the universal quantifier $(\forall_S)^Y_X\colon \mathbf{P}_S(X \times Y) \to \mathbf{P}_S(X)$ is

$$\forall^Y_{S\times X}\colon \mathbf{P}(S \times X \times Y) \to \mathbf{P}(S \times X).$$

All these facts are proved in [9, Section 5].

The Boolean doctrine momorphism $(L_S, \mathfrak{l}_S)\colon \mathbf{P} \to \mathbf{P}_S$ and the morphism $\mathrm{id}_S\colon \mathbf{t} \rightsquigarrow S$ in C_S have the following universal property [9, Theorems 6.2 and 6.3]:

> For every Boolean doctrine $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$, Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ and morphism $c\colon \mathbf{t}_{\mathsf{D}} \to M(S)$ in D, there is a unique Boolean doctrine morphism $(N, \mathfrak{n})\colon \mathbf{P}_S \to \mathbf{R}$ such that $(N, \mathfrak{n}) \circ (L_S, \mathfrak{l}_S) = (M, \mathfrak{m})$ and $N(\mathrm{id}_S\colon \mathbf{t} \rightsquigarrow S) = c\colon \mathbf{t}_{\mathsf{D}} \to M(S)$.

Definition 4.6 (Boolean model at an object)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $S \in \mathsf{C}$. A *Boolean model of* $\mathbf{P}$ *at* S is a triple $(M, \mathfrak{m}, s)$ where $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ is a Boolean doctrine morphism from $\mathbf{P}$ to the subsets doctrine $\mathscr{P}$, and $s \in M(S)$.

Roughly speaking, a Boolean model of $\mathbf{P}$ at S is a Boolean model of $\mathbf{P}$ together with a value assignment of S in the model.

Lemma 4.7. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, let $S \in \mathsf{C}$, let $\mathbf{P}_S$ be the Boolean doctrine obtained from $\mathbf{P}$ by adding a constant of type S and let $(L_S, \mathfrak{l}_S)\colon \mathbf{P} \to \mathbf{P}_S$ be the canonical Boolean doctrine morphism. Let $(M, \mathfrak{m}, s)$ be a Boolean model of $\mathbf{P}$ of S and let $(N, \mathfrak{n})\colon \mathbf{P}_S$ the unique Boolean model of $\mathbf{P}_S$ such that $(N, \mathfrak{n}) \circ (L_S, \mathfrak{l}_S) = (M, \mathfrak{m})$ and $N(\mathrm{id}_S\colon \mathbf{t} \rightsquigarrow S) = \{*\} \to M(S)$ is the function that sends $*$ to $s \in M(S)$. Let $Y \in \mathsf{C}$ and let $\alpha \in \mathbf{P}(S \times Y) = \mathbf{P}_S(Y)$. Then, for all $y \in M(Y)$ we have*

$$y \in \mathfrak{n}_Y(\alpha) \iff (s, y) \in \mathfrak{m}_{S\times Y}(\alpha).$$

Proof. Consider the naturality diagram of $\mathfrak{n}$ with respect to the morphism $\mathrm{id}_{S\times Y}\colon Y \rightsquigarrow S \times Y$ in C_S:

$$\begin{array}{ccc} \mathbf{P}_S(S \times Y) & \xrightarrow{\mathfrak{n}_{S\times Y}} & \mathscr{P}(N(S) \times N(Y)) \\ {\scriptstyle \mathbf{P}_S(\mathrm{id}_{S\times Y})}\Big\downarrow & & \Big\downarrow {\scriptstyle N(\mathrm{id}_{S\times Y})^{-1}[-]} \\ \mathbf{P}_S(Y) & \xrightarrow[\mathfrak{n}_Y]{} & \mathscr{P}(N(S)) \end{array} . \tag{4.1}$$

Since L_S is the identity on objects, M and N have the same value assignments on objects. The diagram on the left-hand side below commutes in C_S, because the diagram on the right-hand side commutes in C.

$$
\begin{tikzcd}
 & Y \arrow[dl, rightsquigarrow, "!_{S\times Y}"'] \arrow[d, rightsquigarrow, "\mathrm{id}_{S\times Y}"] \arrow[ddr, rightsquigarrow, bend left, "\mathrm{pr}_2"] & \\
\mathbf{t} \arrow[d, rightsquigarrow, "\mathrm{id}_S"'] & S\times Y \arrow[dl, rightsquigarrow, "\mathrm{pr}_2"] \arrow[dr, rightsquigarrow, "\mathrm{pr}_3"'] & \\
S & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S\times Y \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2\rangle"] \arrow[ddr, bend left, "\mathrm{pr}_2"] & \\
S \arrow[d, "\mathrm{id}_S"'] & S\times S\times Y \arrow[dl, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_3"'] & \\
S & & Y
\end{tikzcd}
$$

Since N preserves finite products, the function $N(\mathrm{id}_{S\times Y})\colon N(Y) \to N(S) \times N(Y)$ maps $y \in N(Y)$ to $(s, y) \in N(S) \times N(Y)$. Moreover, observe that

$$\mathbf{P}_S(\mathrm{id}_{S\times Y})((\mathfrak{l}_S)_{S\times Y}(\alpha)) = P(\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2\rangle)(P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\alpha)) = \alpha. \tag{4.2}$$

By (4.1), (4.2) and since $\mathfrak{m} = \mathfrak{n} \circ \mathfrak{l}_S$, we have the following commuting diagram.

$$
\begin{tikzcd}
P(S\times Y) \arrow[dr, "(\mathfrak{l}_S)_{S\times Y}"] \arrow[drr, bend left, "\mathfrak{m}_{S\times Y}"] \arrow[ddr, bend right, "\mathrm{id}_{\mathbf{P}(S\times Y)}"'] & & \\
 & \mathbf{P}_S(S\times Y) \arrow[r, "\mathfrak{n}_{S\times Y}"] \arrow[d, "\mathbf{P}_S(\mathrm{id}_{S\times Y})"] & \mathscr{P}(M(S)\times M(Y)) \arrow[d, "N(\mathrm{id}_{S\times Y})^{-1}[-]"] \\
 & \mathbf{P}_S(Y) \arrow[r, "\mathfrak{n}_Y"'] & \mathscr{P}(M(S))
\end{tikzcd}
$$

Thus,

$$
\begin{aligned}
y \in \mathfrak{n}_Y(\alpha) &\iff y \in \mathfrak{n}_Y(\mathbf{P}_S(\mathrm{id}_{S\times Y})((\mathfrak{l}_S)_{S\times Y}(\alpha))) \\
&\iff N(\mathrm{id}_{S\times Y})^{-1}[\mathfrak{m}_{S\times Y}(\alpha)] \\
&\iff (s, y) \in \mathfrak{m}_{S\times Y}(\alpha).
\end{aligned}
$$
□

Theorem 4.8. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, $S \in \mathsf{C}$ and F a subset of $\mathbf{P}(S)$. The following conditions are equivalent.*

(1) There is a Boolean model $(M, \mathfrak{m}, s)$ of $\mathbf{P}$ at S such that $F = \{\alpha \in \mathbf{P}(S) \mid s \in \mathfrak{m}_S(\alpha)\}$.
(2) F is an ultrafilter of $\mathbf{P}(S)$.

Proof. This follows from Theorem 4.3 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S (see Remark 4.5 for the construction). Indeed, since $\mathbf{P}(S) = \mathbf{P}_S(\mathbf{t})$, F is an ultrafilter of $\mathbf{P}(S)$ if and only if F is an ultrafilter of $\mathbf{P}_S(\mathbf{t})$. By Theorem 4.3 this is equivalent to the existence of a Boolean model $(N, \mathfrak{n})\colon \mathbf{P}_S \to \mathscr{P}$ such that $F = \{\alpha \in \mathbf{P}_S(\mathbf{t}) \mid \mathfrak{n}_{\mathbf{t}}(\alpha) = N(\mathbf{t})\}$. By the universal property of $\mathbf{P}_S$, the Boolean model $(N, \mathfrak{n})$ is uniquely determined by its precomposition $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ with the canonical morphism $(L_S, \mathfrak{l}_S)\colon \mathbf{P} \to \mathbf{P}_S$ and by the evaluation of $\mathrm{id}_S\colon \mathbf{t} \rightsquigarrow S$ through N, namely $s \in N(S) = M(S)$. To conclude, we take $\alpha \in \mathbf{P}_S(\mathbf{t}) = \mathbf{P}(S)$. By Lemma 4.7, we have

$$s \in \mathfrak{m}_S(\alpha) \iff \mathfrak{n}_{\mathbf{t}}(\alpha) = N(\mathbf{t}).$$
□

Theorem 4.9 (Completeness for Boolean doctrines)**.** *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, let $S \in \mathsf{C}$ and let $\varphi, \psi \in \mathbf{P}(S)$ be such that $\varphi \nleq \psi$. Then there is a Boolean model $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ such that $\mathfrak{m}_S(\varphi) \nsubseteq \mathfrak{m}_S(\psi)$.*

Proof. By the ultrafilter lemma for Boolean algebras, there is an ultrafilter $F \subseteq \mathbf{P}(S)$ such that $\varphi \in F$ and $\psi \notin F$. By Theorem 4.8 there are a Boolean model $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ and $s \in M(S)$ such that $F = \{\alpha \in \mathbf{P}(S) \mid s \in \mathfrak{m}_S(\alpha)\}$. Since $\varphi \in F$, $s \in \mathfrak{m}_S(\varphi)$. Since $\psi \notin F$, $s \notin \mathfrak{m}_S(\psi)$. Thus, $\mathfrak{m}_S(\varphi) \nsubseteq \mathfrak{m}_S(\psi)$. □

4.2. **Existence of the quantifier completion.** In this subsection, we show that the forgetful functor from the category of universal Boolean doctrines with a small base category to the category of Boolean doctrines with a small base category has a left adjoint.

Definition 4.10. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. The *quantifier completion of* $\mathbf{P}$ is a Boolean doctrine morphism $(I, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$, where $\mathbf{P}^{\forall}\colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$ is a universal Boolean doctrine, with the following universal property: for every universal Boolean doctrine $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ and for every Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ there is a unique universal Boolean doctrine morphism $(N, \mathfrak{n})\colon \mathbf{P}^{\forall} \to \mathbf{R}$ such that $(M, \mathfrak{m}) = (N, \mathfrak{n}) \circ (I, \mathfrak{i})$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "{(I,\mathfrak{i})}"] \arrow[dr, "{(M,\mathfrak{m})}"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "{(N,\mathfrak{n})}"] \\
& \mathbf{R}
\end{tikzcd} \tag{4.3}$$

Remark 4.11 (Change of base)**.** Let $\mathbf{P}, \mathbf{R}$ be Boolean doctrines and $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ a Boolean doctrine morphism. We can factor $(M, \mathfrak{m})$ as the composition of two Boolean doctrine morphisms as follows:

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "\mathrm{id}_{\mathsf{C}}^{\mathrm{op}}"] \arrow[dr, "\mathbf{P}"'] \arrow[dr, phantom, bend left=10, "\mathfrak{m}"] & \mathsf{C}^{\mathrm{op}} \arrow[r, "M^{\mathrm{op}}"] \arrow[d, "\mathbf{R}\circ M^{\mathrm{op}}" description] \arrow[dr, phantom, bend right=10, "\mathrm{id}"] & \mathsf{D}^{\mathrm{op}} \arrow[dl, "\mathbf{R}"] \\
& \mathsf{BA}. &
\end{tikzcd}$$

Remark 4.12. In Remark 4.11, if we additionally ask for the Boolean doctrine $\mathbf{R}$ to be universal, it is easy to see that the Boolean doctrine $\mathbf{R} \circ M^{\mathrm{op}}$ is universal, and that in the factorization of $(M, \mathfrak{m}) = (M, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{m})$ the Boolean doctrine morphism $(M, \mathrm{id})\colon \mathbf{R} \circ M^{\mathrm{op}} \to \mathbf{R}$ is universal. Moreover, if also $\mathbf{P}$ and $(M, \mathfrak{m})$ are universal, then $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})$ is universal.

Proposition 4.13. *Let* $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine and suppose it has a quantifier completion* $(I, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$*. Then the functor* $I\colon \mathsf{C} \to \mathsf{C}'$ *is an isomorphism of categories.*

Proof. Consider the factorisation of $(I, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$ as $(I, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$, with $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall} \circ I^{\mathrm{op}}$ and $(I, \mathrm{id})\colon \mathbf{P}^{\forall} \circ I^{\mathrm{op}} \to \mathbf{P}^{\forall}$ as in Remark 4.11. By the universal property of the quantifier completion applied to $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$, there is a unique universal Boolean doctrine morphism $(G, \mathfrak{g})\colon \mathbf{P}^{\forall} \to \mathbf{P}^{\forall} \circ I^{\mathrm{op}}$ such that $(G, \mathfrak{g}) \circ (I, \mathfrak{i}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "{(I,\mathfrak{i})}"] \arrow[dr, "{(\mathrm{id}_{\mathsf{C}},\mathfrak{i})}"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "{\exists!(G,\mathfrak{g})}"] \\
& \mathbf{P}^{\forall} \circ I^{\mathrm{op}}
\end{tikzcd}$$

Now we use the universal property again, this time applied to $(I, \mathfrak{i})$: the identity $(\mathrm{id}_{\mathsf{C}'}, \mathrm{id})\colon \mathbf{P}^{\forall} \to \mathbf{P}^{\forall}$ is the unique universal Boolean doctrine morphism $(N, \mathfrak{n})\colon \mathbf{P}^{\forall} \to \mathbf{P}^{\forall}$ such that $(N, \mathfrak{n}) \circ (I, \mathfrak{i}) = (I, \mathfrak{i})$. Since (I, id) is a universal Boolean doctrine morphism and since

$$(I, \mathrm{id}) \circ (G, \mathfrak{g}) \circ (I, \mathfrak{i}) = (I, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i}) = (I, \mathfrak{i}),$$

we have $(I, \mathrm{id}) \circ (G, \mathfrak{g}) = (\mathrm{id}_{\mathsf{C}'}, \mathrm{id})$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "{(I,\mathfrak{i})}"] \arrow[dr, "{(\mathrm{id}_{\mathsf{C}},\mathfrak{i})}" description] \arrow[ddr, bend right, "{(I,\mathfrak{i})}"'] & \mathbf{P}^{\forall} \arrow[d, "{(G,\mathfrak{g})}" description] \arrow[dd, bend left=60, "{(\mathrm{id}_{\mathsf{C}'},\mathrm{id})}"] \\
& \mathbf{P}^{\forall} \circ I^{\mathrm{op}} \arrow[d, "{(I,\mathrm{id})}" description] \\
& \mathbf{P}^{\forall}
\end{tikzcd}$$

So, looking at the functors between the base categories, we obtain $G \circ I = \mathrm{id}_{\mathsf{C}}$ and $I \circ G = \mathrm{id}_{\mathsf{C}'}$, as desired. $\square$

In Proposition 4.13 above we proved that the functor between the base categories in a quantifier completion is an isomorphism of categories, so from now on we will suppose the quantifier completion (when it exists) to be of the form $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$.

Lemma 4.14. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine and suppose that it admits a quantifier completion $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$. Then every component of the natural transformation $\mathfrak{i}$ is injective.*

Proof. Let $X \in \mathsf{C}$ and $\varphi, \psi \in \mathbf{P}(X)$ be such that $\varphi \not\leq \psi$. By Theorem 4.9 there is a Boolean model $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ such that $\mathfrak{m}_X(\varphi) \not\subseteq \mathfrak{m}_X(\psi)$. By the universal property of $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$ with respect to $(M, \mathfrak{m})$ there is a universal Boolean model $(N, \mathfrak{n})\colon \mathbf{P}^{\forall} \to \mathscr{P}$ such that $(N, \mathfrak{m}) = (M, \mathfrak{n}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$. It follows that

$$\mathfrak{n}_X(\mathfrak{i}_X(\varphi)) = \mathfrak{m}_X(\varphi) \not\subseteq \mathfrak{m}_X(\psi) = \mathfrak{n}_X(\mathfrak{i}_X(\psi)),$$

thus $\mathfrak{i}_X(\varphi) \not\subseteq \mathfrak{i}_X(\psi)$. ☐

With a bit of set-theoretic stretching, we can thus assume that the components of the natural transformation $\mathfrak{i}$ are inclusions of Boolean subalgebras.

Proposition 4.15. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine and suppose it has a quantifier completion $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$. Then $\mathbf{P}$ is a quantifier-free fragment of $\mathbf{P}^{\forall}$.*

Proof. In light of Lemma 4.14, we are left to prove (3) in Definition 3.2. Define the doctrine $\mathbf{R}$ as follows. The base category of $\mathbf{R}$ is C. For each $X \in \mathsf{C}$, set $\mathbf{R}(X)$ as the Boolean subalgebra $\bigcup_{n\in\mathbb{N}} \mathbf{R}_n(X)$ of $\mathbf{P}^{\forall}(X)$, where $\mathbf{R}_n(X)$ is the Boolean subalgebra of $\mathbf{P}^{\forall}(X)$ defined by induction on n as follows. We set $\mathbf{R}_0(X) \coloneqq \mathbf{P}(X)$. For $n \geq 0$, $\mathbf{R}_{n+1}(X)$ is the Boolean subalgebra of $\mathbf{P}^{\forall}(X)$ generated by the union of the images of $\mathbf{R}_n(X \times Y)$ under $\forall^Y\colon \mathbf{P}^{\forall}(X \times Y) \to \mathbf{P}^{\forall}(X)$, for Y ranging among the objects of C. This is a universal Boolean doctrine. We also have a morphism $(\mathrm{id}_{\mathsf{C}}, \iota)\colon \mathbf{R} \to \mathbf{P}^{\forall}$ that, componentwise, is the inclusion: condition (3) in Definition 3.2 is equivalent to asking that ι is componentwise surjective.

By the universal property of $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$, there is a unique universal Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P}^{\forall} \to \mathbf{R}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})"] \arrow[dr, "(\mathrm{id}_{\mathsf{C}}, i_{|})"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "(M, \mathfrak{m})"] \\
& \mathbf{R}
\end{tikzcd}$$

Since both the composite $\mathbf{P}^{\forall} \xrightarrow{(M,m)} \mathbf{R} \xrightarrow{(\mathrm{id}_C, \iota)} \mathbf{P}^{\forall}$ and the identity of $\mathbf{P}^{\forall}$ make the outer triangle below commute,

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})"] \arrow[dr, "(\mathrm{id}_{\mathsf{C}}, i_{|})"] \arrow[ddr, bend right, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})"'] & \mathbf{P}^{\forall} \arrow[d, "(M, \mathfrak{m})"] \\
& \mathbf{R} \arrow[d, "(\mathrm{id}_{\mathsf{C}}, \iota)"] \\
& \mathbf{P}^{\forall}
\end{tikzcd}$$

by the universal property of $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$, the composition $(\mathrm{id}_{\mathsf{C}}, \iota) \circ (M, \mathfrak{m})$ is the identity of $\mathbf{P}^{\forall}$. It follows that ι is surjective, as desired. ☐

In the following, we show that it is enough to check the universal property of a quantifier completion $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$ over universal Boolean doctrines based on C having the identity as functor between the base categories.

Lemma 4.16. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Let $\mathbf{P}^{\forall}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine and $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$ be a Boolean doctrine morphism with the following universal property: for every universal Boolean doctrine $\mathbf{R}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and for every Boolean doctrine morphism of the form*

$(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ *there is a unique universal Boolean morphism* $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n})\colon \mathbf{P}^{\forall} \to \mathbf{R}$ *such that* $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}},\mathfrak{i})"] \arrow[dr, "(\mathrm{id}_{\mathsf{C}},\mathfrak{m})"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "(\mathrm{id}_{\mathsf{C}},\mathfrak{n})"] \\
& \mathbf{R}
\end{tikzcd}$$

Then, $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$ *is a quantifier completion of* $\mathbf{P}$.

Proof. Let $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ be universal Boolean doctrine and let $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ be a Boolean doctrine morphism. By Remarks 4.11 and 4.12 we can factor $(M, \mathfrak{m}) = (M, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{m})$, where $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R} \circ M^{\mathrm{op}}$ is a Boolean doctrine morphism and $(M, \mathrm{id})\colon \mathbf{R} \circ M^{\mathrm{op}} \to \mathbf{R}$ is a universal Boolean doctrine morphism. By the universal property of $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$ with respect to the Boolean doctrine morphism $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})$, there is a unique universal Boolean doctrine morphism $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n})\colon \mathbf{P}^{\forall} \to \mathbf{R} \circ M^{\mathrm{op}}$ such that $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}},\mathfrak{i})"] \arrow[dr, "(\mathrm{id}_{\mathsf{C}},\mathfrak{m})"] \arrow[ddr, bend right, "(M,\mathfrak{m})"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "(\mathrm{id}_{\mathsf{C}},\mathfrak{n})"] \\
& \mathbf{R} \circ M^{\mathrm{op}} \arrow[d, "(M,\mathrm{id})"] \\
& \mathbf{R}
\end{tikzcd}$$

We consider $(M, \mathfrak{n}) = (M, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{n})$. This is a universal Boolean doctrine morphism making (4.3) commute. We conclude by proving uniqueness. Let $(N', \mathfrak{n}')\colon \mathbf{P}^{\forall} \to \mathbf{R}$ be a universal Boolean doctrine morphism making (4.3) commute. Observe that $N' = M \circ \mathrm{id}_{\mathsf{C}} = M$. Moreover, we can factor $(N', \mathfrak{n}') = (M, \mathfrak{n}')$ as $(M, \mathrm{id}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}')$, so it is enough to prove that $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}')$. By Remark 4.12, $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n})$ is a universal Boolean doctrine morphism, and thus, by the universal property, the equality $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}')$ holds if and only if the equality $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}') \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$ holds, but the latter follows from the equality $(M, \mathfrak{m}) = (M, n') \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$. □

In the remainder of this subsection, we prove that every Boolean doctrine over a small base category has a quantifier completion. The idea for the proof is that a universal boolean doctrine is defined by quasi-equations (also known as quasi-identities, or implications), and classes defined by quasi-equations have free algebras.

Definition 4.17. Let C be a category with finite products.

(1) We let $\mathsf{Doct}^{\mathsf{C}}_{\mathsf{BA}}$ denote the category whose objects are Boolean doctrines over C and whose morphisms are natural transformations.
(2) We let $\mathsf{Doct}^{\mathsf{C}}_{\forall\mathsf{BA}}$ denote the category whose objects are universal Boolean doctrines over C and whose morphisms from $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ to $\mathbf{R}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ are natural transformations $\mathfrak{m}\colon \mathbf{P} \to \mathbf{R}$ such that the following diagram commutes for all $X, Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathbf{P}(X \times Y) \arrow[r, "\mathfrak{m}_{X\times Y}"] \arrow[d, "\forall^Y_X"'] & \mathbf{R}(X \times Y) \arrow[d, "\forall^Y_X"] \\
\mathbf{P}(X) \arrow[r, "\mathfrak{m}_X"'] & \mathbf{R}(X)
\end{tikzcd} \tag{4.4}$$

Remark 4.18 (Universal Boolean doctrines as many-sorted algebras)**.** Let C be a small category with finite products. We present $\mathsf{Doct}^{\mathsf{C}}_{\forall\mathsf{BA}}$ as a quasi-variety of many-sorted algebras (and homomorphism).

First, we describe a many-sorted algebraic language $\mathcal{L}_{\mathsf{C}}$. The set of objects of C is taken as the set of sorts. We equip each sort with the signature of a Boolean algebra. Moreover, for each morphism $f\colon X \to Y$ in C, we consider a unary function symbol f from sort Y to sort X. Finally, for each binary product diagram $X \xleftarrow{\mathrm{pr}_1} Z \xrightarrow{\mathrm{pr}_2} Y$, we consider a unary function symbol $\forall^Y_{X,\mathrm{pr}_1,\mathrm{pr}_2}$ from sort Z to sort X.

Next, we let $\mathcal{V}$ be the class of many-sorted algebras $\mathbf{P}$ in the language $\mathcal{L}_{\mathsf{C}}$ satisfying the following quasi-equational axioms (we write $\mathbf{P}_X$ for the value of $\mathbf{P}$ at the sort X, and we write $\mathbf{P}_f$ for the interpretation of the function symbol $f\colon X \to Y$ in $\mathbf{P}$).

(1) Each sort satisfies the axioms of a boolean algebra.
(2) For each morphism $f\colon X \to Y$, the function symbol $\mathbf{P}_f$ satisfies the axioms of a boolean homomorphism, i.e. for each Boolean function symbol $g(x_1, \dots, x_n)$, we have the axiom
$$\text{For all } \alpha_1, \dots, \alpha_n \in \mathbf{P}_Y,\ \mathbf{P}_f(g_{\mathbf{P}_Y}(\alpha_1, \dots, \alpha_n)) = g_{\mathbf{P}_X}(\mathbf{P}_f(\alpha_1), \dots, \mathbf{P}_f(\alpha_n)).$$
(3) For each object X,
$$\text{For all } \alpha \in \mathbf{P}_X,\ \mathbf{P}_{\mathrm{id}_X}(\alpha) = \alpha.$$
(4) Given two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have the axiom
$$\text{For all } \alpha \in \mathbf{P}_Z,\ \mathbf{P}_f(\mathbf{P}_g(\alpha)) = \mathbf{P}_{g\circ f}(\alpha).$$
(5) For each binary product diagram $X \xleftarrow{\mathrm{pr}_1} Z \xrightarrow{\mathrm{pr}_2} Y$, we have the axioms
$$\text{For all } \alpha \in \mathbf{P}_X \text{ and all } \beta \in \mathbf{P}_Z,\ \alpha \leq \forall^Y_{X,\mathrm{pr}_1,\mathrm{pr}_2}(\beta) \iff \mathbf{P}_{\mathrm{pr}_1}(\alpha) \leq \beta.$$
(6) For all objects X, X', Y, Z, Z', for every morphism $f\colon X' \to X$, and for all binary product diagrams $X \xleftarrow{\mathrm{pr}_1} Z \xrightarrow{\mathrm{pr}_2} Y$ and $X' \xleftarrow{\mathrm{pr}'_1} Z \xrightarrow{\mathrm{pr}'_2} Y$, we have the axiom
$$\text{For all } \alpha \in \mathbf{P}_Z,\ \forall^Y_{X',\mathrm{pr}'_1,\mathrm{pr}'_2}(\mathbf{P}_{f\times \mathrm{id}_Y}(\alpha)) = \mathbf{P}_f(\forall^Y_{X,\mathrm{pr}_1,\mathrm{pr}_2}(\alpha)).$$

A many-sorted algebra in this signature satisfying the axioms above is the same thing as a universal Boolean doctrine over C. Indeed,

- (1) guarantees that we have an assignment on objects from C to BA.
- (2) guarantees that we have an assignment on morphisms from C to BA.
- (3) guarantees that the identity is preserved, and (4) guarantees that the composition is preserved, so that we have a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$.
- (5) guarantees that the universal quantifier is a right adjoint.
- (6) guarantees that the Beck-Chevalley condition is satisfied.

A homomorphism $\mathfrak{m}$ of many-sorted algebras in $\mathcal{V}$ is the same thing as a morphism in $\mathsf{Doct}^{\mathsf{C}}_{\forall\mathsf{BA}}$. Indeed,

- The preservation of the Boolean function symbols guarantees that $\mathfrak{m}_X$ is a Boolean homomorphism for each $X \in \mathsf{C}$.
- The preservation of the unary function symbols associated to the morphisms of C guarantees the naturality of $\mathfrak{m}$.
- The preservation of the unary function symbols $\forall^Y_{X,\mathrm{pr}_1,\mathrm{pr}_2}$ guarantee the commutativity of (4.4).

Remark 4.19. Using the notation of Remark 4.18, we let $\mathcal{L}'$ be the sublanguage of $\mathcal{L}_{\mathsf{C}}$ consisting of all function symbols of $\mathcal{L}_{\mathsf{C}}$ excluding quantifiers. Then, a quantifier-free fragment of a universal Boolean doctrine $\mathbf{P}$ (Definition 3.2) is just a $\mathcal{L}'$-subalgebra of $\mathbf{P}$ that $\mathcal{L}_{\mathsf{C}}$-generates $\mathbf{P}$.

Theorem 4.20 ([4, Corollary 1, p. 129])**.** *A class of many-sorted algebras closed under subalgebras and products has all free algebras.*

The theorem above means the following. Let $\mathcal{L}$ be a many-sorted language, with S as the set of sorts, and let $\mathcal{A}$ be a class of algebras closed under subalgebras and products. The forgetful functor $\mathcal{A} \to \mathsf{Set}^S$ has a left adjoint. As in the one-sorted case [3, Chapter VI, Section 7], the free $\mathcal{A}$-algebra over a many-sorted set $X = (X_i)_{i\in S}$ is the image of the word algebra W over X (which might fail to belong to $\mathcal{A}$) under the homomorphism $W \to \prod_{i\in I} A_i$, for A_i ranging among the quotients of W belonging to $\mathcal{A}$. Although [3, Chapter VI, Section 7] assumes all the sorts to be nonempty, the argument goes through without this assumption, which is not required, for example, in the textbook [1].

Theorem 4.20 has the following generalization.

Theorem 4.21. *Let $\mathcal{L}^+$ be a many-sorted language and $\mathcal{L}^-$ be a sublanguage of $\mathcal{L}^+$. Let $\mathcal{A}^+$ be a class of $\mathcal{L}^+$-algebras closed under products and subalgebras and let $\mathcal{A}^-$ be a class of $\mathcal{L}^-$-algebras containing all the $\mathcal{L}^-$-reducts of the algebras in $\mathcal{A}^+$. The forgetful functor $\mathcal{A}^+ \to \mathcal{A}^-$ has a left adjoint.*

Proof. Let S denote the set of sorts of the language $\mathcal{L}^+$. Let $(A_i)_{i\in S} \in \mathcal{A}^-$. Consider the many-sorted language $\mathcal{L}'$, with the same set of sorts as $\mathcal{L}^+$ and with function symbols the same as those of $\mathcal{L}^+$ and additionally, for each sort $i \in S$ and each $a \in A_i$, a constant symbol c_a at sort i. Let $\mathcal{A}'$ be the class of algebras $A = (A_i)_{i\in S}$ in the language $\mathcal{L}'$ such that the $\mathcal{L}^+$-reduct of A belongs to $\mathcal{A}^+$ and such that, for every function symbol f in $\mathcal{L}^-$, and every $a_1, \dots, a_n$ in appropriate sorts we have $f(c_{a_1}, \dots, c_{a_n}) = c_{f(a_1, dots, a_n)}$. It is easy to see that $\mathcal{A}'$ is closed under subalgebras and products. Therefore, by Theorem 4.20, the forgetful functor $\mathcal{A}' \to \mathsf{Set}$ has a left adjoint. Let $F = (F_i)_{i\in S}$ be the free $\mathcal{A}'$-algebra over the empty S-sorted set $(\emptyset)_{i\in S}$. The $\mathcal{L}^+$-reduct of F belongs to $\mathcal{A}^+$, and the $\mathcal{L}^-$-reduct of F belongs to $\mathcal{A}^-$ because $\mathcal{A}^-$ contains all the $\mathcal{L}^-$-reducts of the algebras in $\mathcal{A}^+$. We have an obvious many-sorted map from A to F assigning to a the constant c_a. This is a $\mathcal{L}^-$-homomorphism from A to the $\mathcal{L}^-$-reduct of F. It has the desired universal property. □

Theorem 4.22. *Let C be a small category with finite products and let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. There is a universal Boolean doctrine $\mathbf{P}^\forall\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and a Boolean doctrine morphism $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^\forall$ with the following universal property: for every universal Boolean doctrine $\mathbf{R}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and for every Boolean doctrine morphism $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ there is a unique universal Boolean morphism $(\mathrm{id}_{\mathsf{C}}, \mathfrak{n})\colon \mathbf{P}^\forall \to \mathbf{R}$ such that $(\mathrm{id}_{\mathsf{C}}, \mathfrak{m}) = (\mathrm{id}_{\mathsf{C}}, \mathfrak{n}) \circ (\mathrm{id}_{\mathsf{C}}, \mathfrak{i})$.*

$$\begin{tikzcd} \mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})"] \arrow[dr, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{m})"'] & \mathbf{P}^\forall \arrow[d, dashed, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{n})"] \\ & \mathbf{R} \end{tikzcd}$$

Proof. In the one-sorted context, it is a standard fact that any class defined by quasi-equations is closed under products and subalgebras (see e.g. [5, Theorem 2.25, (a) ⇒ (c)]). The same is true in the many-sorted context. Apply Theorem 4.21 to the forgetful functor $\mathsf{Doct}^{\mathsf{C}}_{\forall\mathsf{BA}} \to \mathsf{Doct}^{\mathsf{C}}_{\mathsf{BA}}$. □

Corollary 4.23. *Let C be a small category with finite products and let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Then $\mathbf{P}$ has a quantifier completion.*

Proof. It follows from Theorem 4.22 and Lemma 4.16. □

Remark 4.24. Let C be a small category with finite products. Using the notation of Remark 4.18, we let $\mathcal{L}'$ be the sublanguage of $\mathcal{L}_{\mathsf{C}}$ consisting of all function symbols of $\mathcal{L}_{\mathsf{C}}$ excluding quantifiers. Theorem 4.22 shows that Boolean doctrines over C are precisely the $\mathcal{L}'$-subreducts of universal Boolean doctrines over C, i.e. the $\mathcal{L}'$-algebras obtained as $\mathcal{L}'$-subalgebras of some universal Boolean doctrine over C.

Example 4.25. Let $\mathcal{L}$ be a first-order language and $\mathcal{T}$ a theory whose axioms are all quantifier-free. Let $\mathsf{LT}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ be the syntactic doctrine as in Example 2.9 and LT_0 be the quantifier-free fragment of LT as in Example 3.4. We prove that $(\mathrm{id}_{\mathsf{Ctx}}, i)\colon \mathsf{LT}_0 \to \mathsf{LT}$ is a quantifier completion of LT_0. Let $\mathbf{R}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine and $(\mathrm{id}_{\mathsf{Ctx}}, \mathfrak{m})\colon \mathsf{LT}_0 \to \mathbf{R}$ be a Boolean doctrine morphism. Define $\mathfrak{n}\colon \mathsf{LT} \to \mathbf{R}$ as follows.

$$\begin{aligned} \mathfrak{n}_{\vec{x}}\colon \mathsf{LT}(\vec{x}) &\longrightarrow \mathbf{R}(\vec{x}) \\ \forall\vec{y}\,\beta(\vec{x},\vec{y}) &\longmapsto \forall^{\vec{y}}_{\vec{x}}\mathfrak{m}_{(\vec{x},\vec{y})}(\beta(\vec{x},\vec{y})) \qquad \text{for } \beta(\vec{x},\vec{y}) \in \mathsf{LT}_0(\vec{x},\vec{y}), \end{aligned}$$

extended by induction on the complexity of the formulas. In particular, if $\alpha(\vec{x}) \in \mathsf{LT}_0(\vec{x})$, we have

$$\mathfrak{n}_{\vec{x}}(i_{\vec{x}}(\alpha(\vec{x}))) = \mathfrak{n}_{\vec{x}}(\forall()\,\alpha(\vec{x})) = \forall^{()}_{\vec{x}}\mathfrak{m}_{\vec{x}}(\alpha(\vec{x})) = \mathfrak{m}_{\vec{x}}(\alpha(\vec{x})),$$

and thus $\mathfrak{n} \circ i = \mathfrak{m}$. Since every axiom in $\mathcal{T}$ is quantifier-free, $\mathfrak{n}$ is well-defined on the equivalence classes of the formulas: for every $\varphi \in \mathcal{T}$, the function $\mathfrak{n}_{()}\colon \mathsf{LT}() \to \mathbf{R}()$ maps φ to $\mathfrak{m}_{()}(\varphi) = \top_{\mathbf{R}()}$. The naturality of $\mathfrak{n}$ follows from the properties of admissible substitutions of variables. We check universality, i.e. that the following diagram is commutative.

$$\begin{tikzcd} \mathsf{LT}(\vec{x},\vec{z}) \arrow[r, "\mathfrak{n}_{(\vec{x},\vec{z})}"] \arrow[d, "\forall\vec{z}"'] & \mathbf{R}(\vec{x},\vec{z}) \arrow[d, "\forall^{\vec{z}}_{\vec{x}}"] \\ \mathsf{LT}(\vec{x}) \arrow[r, "\mathfrak{n}_{\vec{x}}"'] & \mathbf{R}(\vec{x}) \end{tikzcd}$$

To do so, at first we check that it commutes on every element of the form $\forall\vec{y}\,\beta(\vec{x},\vec{z},\vec{y})$ with $\beta(\vec{x},\vec{z},\vec{y}) \in \mathsf{LT}_0(\vec{x},\vec{z},\vec{y})$. Then, by induction on the complexity of formulas in $\mathsf{LT}(\vec{x},\vec{z})$, we can show the commutativity of the diagram. We proved that there is a Boolean morphism $(\mathrm{id}_{\mathsf{Ctx}},\mathfrak{n})\colon \mathsf{LT} \to \mathbf{R}$ such that $(\mathrm{id}_{\mathsf{Ctx}},\mathfrak{m}) = (\mathrm{id}_{\mathsf{Ctx}},\mathfrak{n}) \circ (\mathrm{id}_{\mathsf{Ctx}}, i)$. To prove uniqueness, let $(\mathrm{id}_{\mathsf{Ctx}},\mathfrak{n}')\colon \mathsf{LT} \to \mathbf{R}$ be a universal Boolean morphism such that $(\mathrm{id}_{\mathsf{Ctx}},\mathfrak{m}) = (\mathrm{id}_{\mathsf{Ctx}},\mathfrak{n}') \circ (\mathrm{id}_{\mathsf{Ctx}}, i)$. Let $\beta(\vec{x},\vec{y})$ be in $\mathsf{LT}_0(\vec{x},\vec{y})$:

$$\begin{aligned} \mathfrak{n}'_{\vec{x}}(\forall\vec{y}\,\beta(\vec{x},\vec{y})) &= \forall^{\vec{y}}_{\vec{x}}\mathfrak{n}'_{(\vec{x},\vec{z})}\beta(\vec{x},\vec{y}) && \text{by universality}\\ &= \forall^{\vec{y}}_{\vec{x}}\mathfrak{m}_{(\vec{x},\vec{z})}\beta(\vec{x},\vec{y}) && \text{since } \mathfrak{n}' \circ i = \mathfrak{m}\\ &= \mathfrak{n}_{\vec{x}}(\forall\vec{y}\,\beta(\vec{x},\vec{y})). \end{aligned}$$

Since $\mathfrak{n}_{\vec{x}}$ and $\mathfrak{n}'_{\vec{x}}$ have the same value on generators and are both Boolean homomorphisms, they coincide.

By Lemma 4.16, $(\mathrm{id}_{\mathsf{Ctx}}, i)\colon \mathsf{LT}_0 \to \mathsf{LT}$ is a quantifier completion of LT_0.

Remark 4.26 (The existential completion of a primary doctrine)**.** The quantifier completion of a Boolean doctrine $\mathbf{P}$ is not the same thing as the existential completion (in the sense of [22]) of $\mathbf{P}$ seen as a primary doctrine. We give more details for the reader interested in the difference between the two completions.

A *primary doctrine* is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{InfSL}$ where C is a category with finite products and InfSL is the category of inf-semilattices. The existential completion of a primary doctrine $\mathbf{P}$ consists of an existential (and thus primary) doctrine $\mathbf{P}^e$ and a primary doctrine morphism $(\mathrm{id}_{\mathsf{C}},\iota)\colon \mathbf{P} \to \mathbf{P}^e$ satisfying a suitable universal property. Roughly speaking, $\mathbf{P}^e$ freely adds the existential quantifier to $\mathbf{P}$.

In our setting, we want to add the universal quantifier to Boolean doctrines. Of course, adding the universal or the existential quantifier to a Boolean doctrine is the same process. However, to obtain the quantifier completion of a Boolean doctrine we cannot apply D. Trotta's construction (recalled below) of the existential completion of a primary doctrine [22, Section 4].

Before recalling Trotta's construction, we start with the logical intuition. Let $\mathcal{A}$ be the set of formulas in the $\{\wedge,\top\}$-fragment of a given first-order theory. The set of (equivalence classes of) formulas of the form $\exists\vec{x}\,\alpha(\vec{x},\vec{y})$ with $\alpha \in \mathcal{A}$ is closed under finite meets and existential quantification, and contains $\mathcal{A}$. With this observation in mind, we recall the construction of the existential completion of a primary doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{InfSL}$ with C small. For every object $A \in \mathsf{C}$ consider the set $\mathcal{E}_A \coloneqq \{(C,\alpha) \mid C \in \mathsf{C},\ \alpha \in \mathbf{P}(A\times C)\}$ equipped with the preorder

$$(C,\alpha) \leq (C',\alpha') \iff \text{there is } g\colon A\times C \to C' \text{ such that } \alpha \leq \mathbf{P}(\langle \mathrm{pr}_1, g\rangle)(\alpha') \text{ in } \mathbf{P}(A\times C).$$

Then, for every object $A \in \mathsf{C}$, $\mathbf{P}^e(A)$ is the poset reflection of the preorder. With an abuse of notation, we do not distinguish between elements and their equivalence classes. This assignment can be extended to morphisms of C to obtain an existential doctrine $\mathbf{P}^e$. The intuition in the syntactic setting is that an element $(C,\alpha) \in \mathcal{E}_A$ shall be understood as the formula $\exists C\,\alpha(A,C)$.

If we start from a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, by forgetting the Boolean structure we obtain a primary doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{InfSL}$. However, when we compute the existential completion $(\mathrm{id}_{\mathsf{C}},\iota)\colon \mathbf{P} \to \mathbf{P}^e$ of $\mathbf{P}$, the primary doctrine $\mathbf{P}^e$ is not necessarily a Boolean doctrine, nor, even when it is, the primary doctrine morphism $(\mathrm{id}_{\mathsf{C}},\iota)$ is necessarily a Boolean doctrine morphism. From a logical perspective, this failure can be observed when we consider formulas such as $\neg\exists x\,\alpha(x)$, or $\exists x\,\neg\exists y\,\alpha(x,y)$, which are in general not equivalent to existential closures of quantifier-free formulas. At a formal level, we propose two counterexamples: in Example 4.27 we provide a Boolean doctrine $\mathbf{P}$ such that $\mathbf{P}^e$ is not a Boolean doctrine. In Example 4.28 we provide a Boolean doctrine $\mathbf{P}$ such that $\mathbf{P}^e$ is a Boolean doctrine but the doctrine morphism $(\mathrm{id}_{\mathsf{C}},\iota)\colon \mathbf{P} \to \mathbf{P}^e$ does not preserve the bottom element.

Example 4.27. Let C be the two-object posetal category $0 < 1$. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be the Boolean doctrine with $\mathbf{P}(0) = \{*\}$, $\mathbf{P}(1) = \{\bot, \top\}$, and $\mathbf{P}(0 < 1)$ the unique function $\mathbf{P}(1) \to \mathbf{P}(0)$. The doctrine $\mathbf{P}^e\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{InfSL}$ has the following assignment on the object 1:

$$\mathbf{P}^e(1) = \{(0,*) \lneq (1,\bot) \lneq (1,\top)\}.$$

The three-element chain $\mathbf{P}^e(1)$ is not a Boolean algebra, and thus $\mathbf{P}^e$ is not a Boolean doctrine.

Example 4.28. Let C be the two-object posetal category $0 < 1$. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be the Boolean doctrine with $\mathbf{P}(0) = \mathbf{P}(1) = \{*\}$, and $\mathbf{P}(0 < 1) = \mathrm{id}_{\{*\}}$. The primary doctrine $\mathbf{P}^e\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{InfSL}$ has the following assignment on objects and morphisms.

$$\begin{array}{ccc} \mathbf{P}^e(1) & = & \{(0,*) \lneq (1,*)\} \\ {\scriptstyle !}\big\downarrow & & \\ \mathbf{P}^e(0) & = & \{(0,*) = (1,*)\} \end{array}$$

The functor $\mathbf{P}^e$ is a Boolean doctrine. However, the component at the object 1 of the natural transformation $\iota\colon \mathbf{P} \to \mathbf{P}^e$ does not preserve the bottom element:

$$\begin{aligned} \iota_1\colon \mathbf{P}(1) &\longrightarrow \mathbf{P}^e(1) \\ * &\longmapsto (1,*) \neq (0,*). \end{aligned}$$

We conclude the section by introducing the problems addressed in the next ones. Let $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. By Corollary 4.23, $\mathbf{P}_0$ has a quantifier completion $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P}_0 \to \mathbf{P}_0^\forall$. By Proposition 4.15, $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}_0^\forall$. For each $S \in \mathsf{C}$, define $\mathbf{P}_1(S)$ as the Boolean subalgebra of $\mathbf{P}_0^\forall(S)$ generated by the union of the images of $\mathbf{P}_0(S \times Y)$ under $\forall_S^Y\colon \mathbf{P}_0^\forall(S \times Y) \to \mathbf{P}_0^\forall(S)$, for Y ranging in C, as in Definition 3.2(3). This gives a subfunctor $\mathbf{P}_1$ of $\mathbf{P}^\forall$. Intuitively, $\mathbf{P}_1$ freely adds one layer of quantification to $\mathbf{P}_0$.

One of the main remaining goals, reached only in Section 6, is to describe $\mathbf{P}_1$ explicitly. For example, when should a formula $(\forall x\, \alpha(x)) \wedge (\forall y\, \beta(y))$ be below another formula $(\forall z\, \gamma(z)) \vee (\forall w\, \delta(w))$? Our approach to the question is via models. In this light, one possible answer is: when every model of $\mathbf{P}_0$ satisfying $\forall x\, \alpha(x)$ and $\forall y\, \beta(y)$ also satisfies $\forall z\, \gamma(z)$ or $\forall w\, \delta(w)$; equivalently, when there is no model of $\mathbf{P}_0$ that satisfies $\forall x\, \alpha(x)$, $\forall y\, \beta(y)$, $\neg\forall z\, \gamma(z)$ and $\neg\forall w\, \delta(w)$. In the next two sections we take a detour to investigate the relationship between models of Boolean doctrines and one layer of quantification. To this end, we will introduce the notions of *universal filters* and *universal ideals*. Universal filters axiomatize classes of universally valid formulas (in some family of models), while universal ideals axiomatize classes of universally invalid formulas (in some family of models). The answer above can then be formulated as: when the universal filter generated by α and β does not intersect the universal ideal generated by γ and δ. In turn, this amounts to an explicit condition about the Boolean doctrine $\mathbf{P}_0$ and the elements α, β, γ and δ.

5. Characterization of classes of universally valid formulas

This section contains the mathematical heart of the paper. To illustrate its main result, we introduce the following notation.

Notation 5.1. Let $(M, \mathfrak{m})$ be a Boolean model of a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. For each $X \in \mathsf{C}$, define

$$F_X^{(M,\mathfrak{m})} := \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\}.$$

Remark 5.2. We translate Notation 5.1 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a countable set of variables, $\mathcal{L}$ a language, $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$, and M a model for $\mathcal{T}$. To further simplify, instead of taking contexts as arbitrary tuples of distinct variables, we consider only contexts of the type $(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define

$$F_n^M := \{\alpha(x_1, \dots, x_n) \text{ quantifier-free} \mid M \vDash \forall x_1 \dots \forall x_n\, \alpha(x_1, \dots, x_n)\}.$$

The end result of this section (Theorem 5.28) is a characterization of the families of the form $(F_X^{(M,\mathfrak{m})})_{X\in\mathsf{C}}$ for some model $(M,\mathfrak{m})$, at least in the case where the base category C is small; these families are captured axiomatically by the notion of a *universal ultrafilter*, introduced in Definition 5.12 below.

To obtain Theorem 5.28, we will need the auxiliary notions of universal filters and of universal ideals. To motivate these, we extend Notation 5.1 as follows.

Notation 5.3. Let $\mathcal{M}$ be a class of Boolean models of a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$. For each $X \in \mathsf{C}$ define

$$F_X^{\mathcal{M}} \coloneqq \{\alpha \in \mathbf{P}(X) \mid \text{for all } (M,\mathfrak{m}) \in \mathcal{M},\ \text{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\},$$
$$I_X^{\mathcal{M}} \coloneqq \{\alpha \in \mathbf{P}(X) \mid \text{for all } (M,\mathfrak{m}) \in \mathcal{M},\ \text{not all } x \in M(X) \text{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

Roughly speaking,

- $F^{\mathcal{M}}$ consists of all the formulas whose "universal closure" is valid in all elements of $\mathcal{M}$,
- $I^{\mathcal{M}}$ consists of all the formulas whose "universal closure" is invalid in all elements of $\mathcal{M}$.

Remark 5.4. We translate Notation 5.3 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a countable set of variables, $\mathcal{L}$ a language, $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$ and $\mathcal{M}$ a class of models for $\mathcal{T}$. To further simplify, instead of taking contexts as arbitrary tuples of distinct variables, we consider only contexts of the type $(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define

$$F_n^{\mathcal{M}} \coloneqq \{\alpha(x_1,\dots,x_n) \text{ quantifier-free} \mid \text{for all } M \in \mathcal{M},\ M \vDash \forall x_1 \dots \forall x_n\, \alpha(x_1,\dots,x_n)\},$$
$$I_n^{\mathcal{M}} \coloneqq \{\alpha(x_1,\dots,x_n) \text{ quantifier-free} \mid \text{for all } M \in \mathcal{M},\ M \nvDash \forall x_1 \dots \forall x_n\, \alpha(x_1,\dots,x_n)\}.$$

In the next pages, we introduce the notions of *universal filters* and *universal ideals*. The notion of a universal filter characterizes the families of the form $F^{\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models, i.e. the classes of formulas that are universally true in all members of a certain class of models. The notion of a universal ideal characterizes the classes of the form $I^{\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models, i.e. the classes of formulas that are universally invalid in all members of a certain class of models. While we use the notions of universal filters and ideals to prove the main result of this section (Theorem 5.28), the mentioned characterization of universal filters, resp. ideals) as classes of the form $F^{\mathcal{M}}$, resp. $I^{\mathcal{M}}$ is not needed, and so we postpone it to the appendix: see Theorems A.5 and A.9.

5.1. **Universal filters, ideals and ultrafilters.** We introduce *universal filters*, meant to characterize the families of the form $F^{\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models (see Notation 5.3).

Definition 5.5 (Universal filter)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *universal filter for* $\mathbf{P}$ is a family $(F_X)_{X\in\mathsf{C}}$, with $F_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$, with the following properties.

1. For all $f\colon X \to Y$ and $\alpha \in F_Y$, $\mathbf{P}(f)(\alpha) \in F_X$.
2. For all $X \in \mathsf{C}$, F_X is a filter of $\mathbf{P}(X)$.

Remark 5.6. We translate Definition 5.5 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal filter for* $\mathcal{T}$ is a family $(F_n)_{n\in\mathbb{N}}$, with F_n a set of quantifier-free $\mathcal{L}$-formulas with $x_1,\dots,x_n$ as (possibly dummy) free variables, with the following properties.

1. For all $n, m \in \mathbb{N}$, every $\alpha(x_1,\dots,x_m) \in F_m$ and every m-tuple $(f_i(x_1,\dots,x_n))_{i=1,\dots,m}$ of n-ary terms,
$$\alpha(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)) \in F_n.$$
2. For all $n \in \mathbb{N}$,
   (a) for all quantifier-free formulas $\alpha(x_1,\dots,x_n)$ and $\beta(x_1,\dots,x_n)$, if $\alpha(x_1,\dots,x_n) \in F_n$ and $\alpha(x_1,\dots,x_n) \vdash_{\mathcal{T}} \beta(x_1,\dots,x_n)$, then $\beta(x_1,\dots,x_n) \in F_n$;
   (b) for all $\alpha_1(x_1,\dots,x_n), \alpha_2(x_1,\dots,x_n) \in F_n$ we have $\alpha_1(x_1,\dots,x_n) \wedge \alpha_2(x_1,\dots,x_n) \in F_n$;
   (c) $\top(x_1,\dots,x_n) \in F_n$ (where $\top(x_1,\dots,x_n)$ is the constant "true" with n dummy variables).

If the family $(F_n^{\mathcal{M}})_{n\in\mathbb{N}}$ is defined by a class $\mathcal{M}$ of models as in Remark 5.4, it is easy to check that $(F_n^{\mathcal{M}})_{n\in\mathbb{N}}$ satisfies the conditions above.

Example 5.7. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. The family $(\{\top_{\mathbf{P}(X)}\})_{X\in\mathsf{C}}$ is a universal filter for $\mathbf{P}$.

An arbitrary (componentwise) intersection of universal filters for a given Boolean doctrine $\mathbf{P}$ is a universal filter. Therefore, we can define the *universal filter generated by a family* $(A_X)_{X\in\mathsf{C}}$ as the smallest universal filter containing the family.

Lemma 5.8 (Description of the universal filter generated by a family)**.** *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Let $(A_X)_{X\in\mathsf{C}}$ be a family with $A_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The universal filter for $\mathbf{P}$ generated by $(A_X)_{X\in\mathsf{C}}$ is the family $(F_X)_{X\in\mathsf{C}}$ where for each $X \in \mathsf{C}$ F_X is the set of $\varphi \in \mathbf{P}(X)$ such that there are $Y_1,\dots,Y_n \in \mathsf{C}$, $(\alpha_i \in A_{Y_i})_{i=1,\dots,n}$ and $(f_i\colon X \to Y_i)_{i=1,\dots,n}$ such that, in $\mathbf{P}(X)$,*

$$\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_i) \leq \varphi.$$

Proof. It is easily seen that family $F = (F_X)_{X\in\mathsf{C}}$ contains $A = (A_X)_{X\in\mathsf{C}}$ and is contained in any universal filter containing A. We are left to show that F is a universal filter. The family F is closed under reindexings because, if $g\colon Y \to X$ is a morphism in C and $\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_i) \leq \varphi$, then

$$\bigwedge_{i=1}^{n} \mathbf{P}(f_i \circ g)(\alpha_i) = \mathbf{P}(f)\left(\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_i)\right) \leq \mathbf{P}(f)(\varphi).$$

It is easy to see that, for each $X \in \mathsf{C}$, F_X is upward closed, is closed under binary meets, and contains $\top_{\mathbf{P}(X)}$ (take $n = 0$). ☐

We introduce *universal ideals*, meant to characterize the families of the form $I^{\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models (see Notation 5.3).

Definition 5.9 (Universal ideal)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *universal ideal for* $\mathbf{P}$ is a family $(I_X)_{X\in\mathsf{C}}$, with $I_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$, with the following properties.

(1) For all $m \in \mathbb{N}$, $(f_j\colon X \to Y)_{j=1,\dots,m}$ and $\alpha \in \mathbf{P}(Y)$, if $\bigwedge_{j=1}^{m} \mathbf{P}(f_i)(\alpha) \in I_X$ then $\alpha \in I_Y$.
(2) For all $X \in \mathsf{C}$, I_X is downward closed.
(3) For all $\alpha_1 \in I_{X_1}$ and $\alpha_2 \in I_{X_2}$, $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \in I_{X_1\times X_2}$.
(4) $\perp_{\mathbf{P}(\mathbf{t})} \in I_{\mathbf{t}}$.

Remark 5.10. We translate Definition 5.9 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal ideal for* $\mathcal{T}$ is a family $(I_n)_{n\in\mathbb{N}}$, with I_n a set of quantifier-free $\mathcal{L}$-formulas with $x_1,\dots,x_n$ as (possibly dummy) free variables, with the following properties.

(1) For all $p, q, m \in \mathbb{N}$, every $(m \cdot q)$-tuple $(f_{j,k}(x_1,\dots,x_p))_{j\in\{1,\dots,m\},\, k\in\{1,\dots,q\}}$ of p-ary terms and every quantifier-free formula $\alpha(x_1,\dots,x_q)$, if

$$\bigwedge_{j=1}^{m} \alpha(f_{j,1}(x_1,\dots,x_p),\dots,f_{j,p}(x_1,\dots,x_p)) \in I_p,$$

then

$$\alpha(x_1,\dots,x_q) \in I_q.$$

(2) For all $n \in \mathbb{N}$, all quantifier-free formulas $\alpha(x_1,\dots,x_n)$ and $\beta(x_1,\dots,x_n)$, if $\beta(x_1,\dots,x_n) \in I_n$ and $\alpha(x_1,\dots,x_n) \vdash_{\mathcal{T}} \beta(x_1,\dots,x_n)$, then $\alpha(x_1,\dots,x_n) \in I_n$.
(3) For all $n_1, n_2 \in \mathbb{N}$, $\alpha_1(x_1,\dots,x_{n_1}) \in I_{n_1}$ and $\alpha_2(x_1,\dots,x_{n_2}) \in I_{n_2}$, we have

$$\alpha_1(x_1,\dots,x_{n_1}) \vee \alpha_2(x_{n_1+1},\dots,x_{n_1+n_2}) \in I_{n_1+n_2};$$

(4) For all $n \in \mathbb{N}$, $\perp(x_1,\dots,x_n) \in I_n$ (where $\perp(x_1,\dots,x_n)$ is the constant "false" with n dummy variables).

These conditions are satisfied by any family $(I_n^{\mathcal{M}})_{n\in\mathbb{N}}$ defined by a class $\mathcal{M}$ of models as in Remark 5.4.

An arbitrary intersection of universal ideals for a given Boolean doctrine $\mathbf{P}$ is a universal ideal. Therefore, we can define the *universal ideal generated by a family* $(A_X)_{X\in\mathsf{C}}$ as the smallest universal ideal containing the family.

Lemma 5.11 (Description of the universal ideal generated by a family)**.** *Let* $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine. Let* $(A_X)_{X\in\mathsf{C}}$ *be a family with* $A_X \subseteq \mathbf{P}(X)$ *for each* $X \in \mathsf{C}$*. The universal ideal for* $\mathbf{P}$ *generated by* $(A_X)_{X\in\mathsf{C}}$ *is the family* $(I_X)_{X\in\mathsf{C}}$ *where for each* $X \in \mathsf{C}$ I_X *is the set of* $\varphi \in \mathbf{P}(X)$ *such that there are* $Y_1,\dots,Y_n \in \mathsf{C}$*,* $\alpha_1 \in A_{Y_1},\dots,\alpha_n \in A_{Y_n}$ *and* $(f_j\colon \prod_{i=1}^n Y_i \to X)_{j=1,\dots,m}$ *such that, in* $\mathbf{P}(\prod_{i=1}^n Y_i)$*,*

$$\bigwedge_{j=1}^{m} \mathbf{P}(f_j)(\varphi) \leq \bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\alpha_i).$$

Proof. The family $(I_X)_{X\in\mathsf{C}}$ contains $(A_X)_{X\in\mathsf{C}}$: indeed, for $X \in \mathsf{C}$ and $\varphi \in A_X$, take $n = 1$, $m = 1$ and f the identity on X. Moreover, it is easily seen that any universal ideal containing $(A_X)_{X\in\mathsf{C}}$ contains $(I_X)_{X\in\mathsf{C}}$.

We are left to show that $(I_X)_{X\in\mathsf{C}}$ is a universal ideal.

Let $(f_j\colon X \to Y)_{j=1,\dots,m}$ and $\alpha \in \mathbf{P}(Y)$ with $\bigwedge_{j=1}^m \mathbf{P}(f_i)(\alpha) \in I_X$. There are $Z_1,\dots,Z_n \in \mathsf{C}$, $\beta_1 \in A_{Z_1},\dots,\beta_n \in A_{Z_n}$ and $(g_k\colon \prod_{i=1}^n Z_i \to X)_{k=1,\dots p}$ such that

$$\bigwedge_{k=1}^{p} \mathbf{P}(g_k)\left(\bigwedge_{j=1}^{m} \mathbf{P}(f_i)(\alpha)\right) \leq \bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\beta_i).$$

Therefore,

$$\bigwedge_{j\in\{1,\dots,m\},\, k\in\{1,\dots,p\}} \mathbf{P}(f_i \circ g_k)(\alpha) \leq \bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\beta_i).$$

By (3) and (4) in Definition 5.9, $\bigvee_{i=1}^n \mathbf{P}(\mathrm{pr}_i)(\beta_i) \in I_{\prod_{i=1}^n Z_i}$. By Definition 5.9(2), universal ideals are downward closed; thus, $\bigwedge_{j\in\{1,\dots,m\},\, k\in\{1,\dots,p\}} \mathbf{P}(f_i \circ g_k)(\alpha) \in I_{\prod_{i=1}^n Z_i}$. By Definition 5.9(1), $\alpha \in I_Y$.

For each $X \in \mathsf{C}$, I_X is clearly downward closed.

Let $\alpha_1 \in I_{X_1}$ and $\alpha_2 \in I_{X_2}$. By definition of $(I_X)_{X\in\mathsf{C}}$, there are objects $\{Y_i\}_{i=1}^n$, $\{Z_k\}_{k=1}^p$, elements $\{\beta_i \in A_{Y_i}\}_{i=1}^n$, $\{\gamma_k \in A_{Z_k}\}_{k=1}^p$ and morphisms $(f_j\colon \prod_{i=1}^n Y_i \to X_1)_{j=1,\dots,m}$, $(g_h\colon \prod_{k=1}^p Z_k \to X_2)_{h=1,\dots,q}$ such that

$$\bigwedge_{j=1}^{m} \mathbf{P}(f_j)(\alpha_1) \leq \bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\beta_i) \quad \text{and} \quad \bigwedge_{h=1}^{q} \mathbf{P}(g_h)(\alpha_2) \leq \bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\gamma_k).$$

Then $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \in I_{X_1\times X_2}$: indeed, consider the morphisms $(f_j \times g_h\colon (\prod_{i=1}^n Y_i) \times (\prod_{k=1}^p Z_k) \to X_1 \times X_2)_{j\in\{1,\dots,m\},\, h\in\{1\dots,q\}}$ and compute:

$$\begin{aligned}
\bigwedge_{j=1}^{m}\bigwedge_{h=1}^{q} \mathbf{P}(f_j \times g_h)\big(\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2)\big) &= \bigwedge_{j=1}^{m}\bigwedge_{h=1}^{q} \big(\mathbf{P}(\mathrm{pr}_1)\mathbf{P}(f_j)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)\mathbf{P}(g_h)(\alpha_2)\big) \\
&= \mathbf{P}(\mathrm{pr}_1)\left(\bigwedge_{j=1}^{m} \mathbf{P}(f_j)(\alpha_1)\right) \vee \mathbf{P}(\mathrm{pr}_2)\left(\bigwedge_{h=1}^{q} \mathbf{P}(g_h)(\alpha_2)\right) \\
&\leq \mathbf{P}(\mathrm{pr}_1)\left(\bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\beta_i)\right) \vee \mathbf{P}(\mathrm{pr}_2)\left(\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\gamma_k)\right) \\
&= \left(\bigvee_{i=1}^{n} \mathbf{P}(\mathrm{pr}_i)(\beta_i)\right) \vee \left(\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_{n+k})(\gamma_k)\right).
\end{aligned}$$

Finally, $\bot_{\mathbf{P}(\mathbf{t})}$ belongs to $I_{\mathbf{t}}$: take $n = 0$, $m = 1$ and $f_1 = \mathrm{id}_{\mathbf{t}}$. ∎

The following notion of a *universal ultrafilter* is meant to characterize the classes of the form $(F_X^{(M,\mathfrak{m})})_{X\in\mathsf{C}}$ for some Boolean model $(M,\mathfrak{m})$ of $\mathbf{P}$, where we recall from Notation 5.1, that, for each $X \in \mathsf{C}$,

$$F_X^{(M,\mathfrak{m})} \coloneqq \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

This characterization is the main result of this section and will be proved in Theorem 5.28.

Definition 5.12 (Universal ultrafilter)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *universal ultrafilter for* $\mathbf{P}$ is a family $(F_X)_{X\in\mathsf{C}}$, with $F_X \subseteq \mathbf{P}(X)$ for all $X \in \mathsf{C}$, with the following properties.

(1) For all $f\colon X \to Y$ and $\alpha \in F_Y$, $\mathbf{P}(f)(\alpha) \in F_X$.
(2) For all $X \in \mathsf{C}$, F_X is a filter of $\mathbf{P}(X)$.
(3) For all $\alpha_1 \in \mathbf{P}(X_1)$ and $\alpha_2 \in \mathbf{P}(X_2)$, if $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \in F_{X_1\times X_2}$ then $\alpha_1 \in F_{X_1}$ or $\alpha_2 \in F_{X_2}$.
(4) $\bot_{\mathbf{P}(\mathbf{t})} \notin F_{\mathbf{t}}$.

Remark 5.13. We translate Definition 5.12 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal ultrafilter for* $\mathcal{T}$ is a family $(F_n)_{n\in\mathbb{N}}$, with F_n a set of quantifier-free $\mathcal{L}$-formulas with $x_1, \dots, x_n$ as (possibly dummy) free variables, with the following properties.

(1) For all $n, m \in \mathbb{N}$, every $\alpha(x_1, \dots, x_m) \in F_m$ and every m-tuple $(f_i(x_1, \dots, x_n))_{i=1,\dots,n}$ of n-ary terms,
$$\alpha(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \in F_n.$$
(2) For all $n \in \mathbb{N}$,
  (a) for all quantifier-free formulas $\alpha(x_1, \dots, x_n), \beta(x_1, \dots, x_n)$, if $\alpha(x_1, \dots, x_n) \vdash_{\mathcal{T}} \beta(x_1, \dots, x_n)$ and $\alpha(x_1, \dots, x_n) \in F_n$, then $\beta(x_1, \dots, x_n) \in F_n$;
  (b) for all $\alpha_1(x_1, \dots, x_n), \alpha_2(x_1, \dots, x_n) \in F_n$ we have $\alpha_1(x_1, \dots, x_n) \wedge \alpha_2(x_1, \dots, x_n) \in F_n$;
  (c) $\top(x_1, \dots, x_n) \in F_n$ (where $\top(x_1, \dots, x_n)$ is the constant "true" with n dummy variables).
(3) For all $n_1, n_2 \in \mathbb{N}$, for all quantifier-free formulas $\alpha_1(x_1, \dots, x_{n_1})$ and $\alpha_2(x_1, \dots, x_{n_2})$, if
$$\alpha_1(x_1, \dots, x_{n_1}) \vee \alpha_2(x_{n_1+1}, \dots, x_{n_1+n_2}) \in F_{n_1+n_2},$$
then $\alpha_1(x_1, \dots, x_{n_1}) \in F_{n_1}$ or $\alpha_2(x_1, \dots, x_{n_2}) \in F_{n_2}$.
(4) For all $n \in \mathbb{N}$, $\bot(x_1, \dots, x_n) \notin F_n$.

For every model M of $\mathcal{T}$, it is easy to check that the family $(F_n)_{n\in\mathbb{N}}$ defined by
$$F_n \coloneqq \{\alpha(x_1, \dots, x_n) \text{ quantifier-free} \mid M \vDash \forall x_1 \dots \forall x_n\, \alpha(x_1, \dots, x_n)\},$$
is a universal filter in the sense above.

Remark 5.14. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $(F_X)_{X\in\mathsf{C}}$ be a universal ultrafilter for $\mathbf{P}$. By (2), (3) and (4) in Definition 5.12, $F_{\mathbf{t}}$ is a filter of $\mathbf{P}(\mathbf{t})$ whose complement is an ideal, and so it is an ultrafilter of $\mathbf{P}(\mathbf{t})$ (in the classic sense).

Definition 5.15 (Universal ultraideal)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *universal ultraideal for* $\mathbf{P}$ is a family $(I_X)_{X\in\mathsf{C}}$, with $I_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$, such that

(1) For all $f\colon X \to Y$ and $\alpha \in \mathbf{P}(Y)$, if $\mathbf{P}(f)(\alpha) \in I_X$ then $\alpha \in I_Y$.
(2) For all $X \in \mathsf{C}$, $\mathbf{P}(X) \setminus I_X$ is a filter of $\mathbf{P}(X)$.
(3) For all $\alpha_1 \in I_{X_1}$ and $\alpha_2 \in I_{X_2}$, we have $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \in I_{X_1\times X_2}$.
(4) $\bot_{\mathbf{P}(\mathbf{t})} \in F_{\mathbf{t}}$.

Lemma 5.16. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, let $A \coloneqq (A_X)_{X\in\mathsf{C}}$ be a family with $A_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$, and set $B \coloneqq (\mathbf{P}(X) \setminus A_X)_{X\in\mathsf{C}}$. The following conditions are equivalent.*

(1) The family A is a universal ultrafilter.
(2) The family A is a universal filter and the family B is a universal ideal.
(3) The family B is a universal ultraideal.

Proof. Immediate. ☐

Lemma 5.17. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, let $A = (A_X)_{X\in\mathsf{C}}$ and $B = (B_X)_{X\in\mathsf{C}}$ be families with $A_X \subseteq \mathbf{P}(X)$ and $B_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) The universal filter for $\mathbf{P}$ generated by A intersects the universal ideal for $\mathbf{P}$ generated by B.

(2) There are $Y_1, \dots, Y_n, Z_1, \dots, Z_m \in \mathsf{C}$, $\alpha_1 \in A_{Y_1}$, ..., $\alpha_n \in A_{Y_n}$, $\beta_1 \in B_{Z_1}$, ..., $\beta_m \in B_{Z_m}$, $(f_i \colon \prod_{j=1}^m Z_j \to Y_i)_{i=1,\dots,n}$ such that (in $\mathbf{P}(\prod_{j=1}^m Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_i) \leq \bigvee_{j=1}^{m} \mathbf{P}(\mathrm{pr}_j)(\beta_j).$$

Proof. (1) $\Rightarrow$ (2). Suppose that the filter F generated by A intersects the ideal I generated by B, i.e. there are $X \in \mathsf{C}$ and $\varphi \in F_X \cap I_X$. By Lemma 5.8, from $\varphi \in F_X$ we deduce the existence of $W_1, \dots, W_n \in \mathsf{C}$, $\gamma_1 \in A_{W_1}, \dots, \gamma_q \in A_{W_q}$ and $(h_i \colon X \to W_i)_{i=1,\dots,q}$ such that, in $\mathbf{P}(X)$,

$$\bigwedge_{i=1}^{q} \mathbf{P}(h_i)(\gamma_i) \leq \varphi.$$

By Lemma 5.11, from $\varphi \in I_X$ we deduce the existence of $Z_1, \dots, Z_m \in \mathsf{C}$, $\beta_1 \in B_{Z_1}, \dots, \beta_m \in B_{Z_m}$ and $(g_k \colon \prod_{j=1}^m Z_j \to X)_{k=1,\dots,p}$ such that, in $\mathbf{P}(\prod_{j=1}^m Z_j)$,

$$\bigwedge_{k=1}^{p} \mathbf{P}(g_k)(\varphi) \leq \bigvee_{j=1}^{m} \mathbf{P}(\mathrm{pr}_j)(\beta_j).$$

Therefore, in $\mathbf{P}(\prod_{j=1}^m Z_j)$,

$$\bigwedge_{k=1}^{p} \bigwedge_{i=1}^{q} \mathbf{P}(h_i \circ g_k)(\gamma_i) = \bigwedge_{k=1}^{p} \mathbf{P}(g_k) \left(\bigwedge_{i=1}^{q} \mathbf{P}(h_i)(\gamma_i) \right) \leq \bigwedge_{k=1}^{p} \mathbf{P}(g_k)(\varphi) \leq \bigvee_{j=1}^{m} \mathbf{P}(\mathrm{pr}_j)(\beta_j).$$

We set n as pq, the sequence $Y_1, \dots, Y_n$ as

$$\underbrace{W_1, \dots, W_1}_{p \text{ times}},\ \underbrace{W_2, \dots, W_2}_{p \text{ times}},\ \dots,\ \underbrace{W_q, \dots, W_q}_{p \text{ times}},$$

the sequence $\alpha_1, \dots, \alpha_n$ as

$$\underbrace{\gamma_1, \dots, \gamma_1}_{p \text{ times}},\ \underbrace{\gamma_2, \dots, \gamma_2}_{p \text{ times}},\ \dots,\ \underbrace{\gamma_q, \dots, \gamma_q}_{p \text{ times}},$$

and the sequence $f_1, \dots, f_n$ as

$$h_1 \circ g_1, \dots, h_1 \circ g_p,\ h_2 \circ g_1, \dots, h_2 \circ g_p,\ \dots,\ h_q \circ g_1, \dots, h_q \circ g_p.$$

(2) $\Rightarrow$ (1). This is straightforward from the closure properties of universal filters and universal ideals. ☐

Lemma 5.18. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, $F = (F_X)_{X \in \mathsf{C}}$ a universal filter and $I = (I_X)_{X \in \mathsf{C}}$ a universal ideal. Let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y)$.*

(1) The universal filter generated by F and α intersects I (at some fiber) if and only if there are $X \in \mathsf{C}$, $n \in \mathbb{N}$, $(f_i \colon X \to Y)_{i=1,\dots,n}$ and $\beta \in F_X$ such that $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in I_X$.

(2) The universal ideal generated by I and α intersects F (at some fiber) if and only if there is $X \in \mathsf{C}$ with $I_X \cap F_X \neq \varnothing$ or there are $Z \in \mathsf{C}$, $\gamma \in I_Z$ such that $\mathbf{P}(\mathrm{pr}_1)(\alpha) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) \in F_{Y \times Z}$.

Proof. This is straightforward from Lemma 5.17 and the closure properties of universal filters and universal ideals. ☐

Lemma 5.19. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Let $\bar{i}, \bar{j} \in \mathbb{N}$, let $Y_1, \dots, Y_{\bar{i}}, Z_1, \dots, Z_{\bar{j}} \in \mathsf{C}$, let $(\alpha_i \in \mathbf{P}(Y_i))_{i=1,\dots,\bar{i}}$, and let $(\beta_j \in \mathbf{P}(Z_j))_{j=1,\dots,\bar{j}}$. The following conditions are equivalent.*

(1) The universal filter generated by $\alpha_1, \dots, \alpha_{\bar{i}}$ intersects the universal ideal generated by $\beta_1, \dots, \beta_{\bar{j}}$.

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, and $(g_i \colon \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ such that (in $\mathbf{P}(\prod_{j=1}^{\bar{j}} Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(g_i)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\mathrm{pr}_j)(\beta_j).$$

Proof. By Lemma 5.17, the universal filter generated by $\alpha_1, \dots, \alpha_{\bar{i}}$ intersects the universal ideal generated by $\beta_1, \dots, \beta_{\bar{j}}$ if and only if there are $n, m \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, $k_1, \dots, k_m \in \{1, \dots, \bar{j}\}$, and $(f_i \colon \prod_{h=1}^m Z_{k_h} \to Y_{l_i})_{i=1,\dots,n}$, such that (in $\prod_{h=1}^m Z_{k_h}$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_{l_i}) \leq \bigvee_{h=1}^{m} \mathbf{P}(\mathrm{pr}_h)(\beta_{k_h}).$$

Therefore, the implication (2) $\Rightarrow$ (1) in the statement of the lemma holds (take $m = \bar{j}$, $k_h = h$ and $g_i = f_i$). For the implication (1) $\Rightarrow$ (2), suppose

$$\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_{l_i}) \leq \bigvee_{h=1}^{m} \mathbf{P}(\mathrm{pr}_h)(\beta_{k_h}). \tag{5.1}$$

For each $i = \{1, \dots, n\}$, set $g_i \colon \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i}$ as the composite

$$\prod_{j=1}^{\bar{j}} Z_j \xrightarrow{\langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle} \prod_{h=1}^{m} Z_{k_h} \xrightarrow{f_i} Y_{l_i}$$

Therefore, we get, in $\mathbf{P}(\prod_{j=1}^{\bar{j}} Z_j)$,

$$\begin{aligned}
\bigwedge_{i=1}^{n} \mathbf{P}(g_i)(\alpha_{l_i}) &= \bigwedge_{i=1}^{n} \mathbf{P}(f_i \circ \langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)(\alpha_{l_i}) \\
&= \bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)(\mathbf{P}(f_i)(\alpha_{l_i})) \\
&= \mathbf{P}(\langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)\left(\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha_{l_i})\right) \\
&\leq \mathbf{P}(\langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)\left(\bigvee_{h=1}^{m} \mathbf{P}(\mathrm{pr}_h)(\beta_{k_h})\right) && \text{by (5.1)} \\
&= \bigvee_{h=1}^{m} \mathbf{P}(\langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)(\mathbf{P}(\mathrm{pr}_h)(\beta_{k_h})) \\
&= \bigvee_{h=1}^{m} \mathbf{P}(\mathrm{pr}_h \circ \langle \mathrm{pr}_{k_1}, \dots, \mathrm{pr}_{k_m} \rangle)(\beta_{k_h}) \\
&= \bigvee_{h=1}^{m} \mathbf{P}(\mathrm{pr}_{k_h})(\beta_{k_h}) \\
&\leq \bigvee_{j=1}^{q} \mathbf{P}(\mathrm{pr}_j)(\beta_j).
\end{aligned}$$

□

Remark 5.20. For the reader interested in the translation of the condition (2) in Lemma 5.19 to the classic syntactic setting, we refer to Remark 5.31 below.

5.2. **Universal ultrafilter lemma.** One version of the classical ultrafilter lemma is: in a Boolean algebra, every filter disjoint from an ideal I can be extended to a prime filter disjoint from I (see [21, Thm. 6] in the larger context of lattices, and see also the earlier result by G. Birkhoff [2, Thm. 21.1]). We give an analogous version in our context.

Theorem 5.21 (Universal ultrafilter lemma)**.** *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, $(F_X)_{X \in \mathsf{C}}$ be a universal filter and $(I_X)_{X \in \mathsf{C}}$ be a universal ideal. Suppose that, for all $X \in \mathsf{C}$, $F_X \cap I_X = \varnothing$. There is a universal ultrafilter that, componentwise, extends F and is disjoint from I.*

Proof. Let $\mathcal{A}$ be the class of pairs $((G_X)_{X\in\mathsf{C}},(J_X)_{X\in C})$ such that G is a universal filter of $\mathbf{P}$ that extends F componentwise, J is a universal ideal of $\mathbf{P}$ that extends I componentwise, and G and J are componentwise disjoint. We order $\mathcal{A}$ by componentwise inclusion. Any nonempty chain of $\mathcal{A}$ admits an upper bound: the componentwise union. Therefore, by Zorn's lemma (for classes), (F,I) admits an upper bound (G,J) that is maximal in $\mathcal{A}$.

We prove that G and J are componentwise complementary. Since G and J are componentwise disjoint, it is enough to show $\mathbf{P}(Y)=G_Y\cup J_Y$ for all $Y\in\mathsf{C}$. By way of contradiction, suppose this is not the case. So there are $Y\in\mathsf{C}$ and $\alpha\in\mathbf{P}(Y)$ such that $\alpha\notin G_Y\cup J_Y$. Let $(G'_X)_X$ be the universal filter generated by G and α, and $(J'_X)_X$ the universal ideal generated by J and α. By maximality of (G,J) in $\mathcal{A}$, G' intersects J and G intersects J'. By Lemma 5.18(1), there are $X\in\mathsf{C}$, $(f_i\colon X\to Y)_{i=1,\dots,n}$, $\beta\in G_X$ such that

$$\beta\wedge\bigwedge_{i=1}^{n}\mathbf{P}(f_i)(\alpha)\in J_X.$$

By Lemma 5.18(2), and since G and J are componentwise disjoint, there are $Z\in\mathsf{C}$ and $\gamma\in J_Z$ such that

$$\mathbf{P}(\mathrm{pr}_1)(\alpha)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma)\in G_{Y\times Z}.$$

The universal filter G is closed under reindexings: thus for every $i=1,\dots,n$,

$$\mathbf{P}(\mathrm{pr}_1)\mathbf{P}(f_i)(\alpha)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma)=\mathbf{P}(f_i\times\mathrm{id}_Z)(\mathbf{P}(\mathrm{pr}_1)(\alpha)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma))\in G_{X\times Z},$$

where we have reindexed $\mathbf{P}(\mathrm{pr}_1)(\alpha)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma)$ along $f_i\times\mathrm{id}_Z\colon X\times Z\to Y\times Z$. Moreover, since $\beta\in G_X$,

$$\mathbf{P}(\mathrm{pr}_1)(\beta)\in G_{X\times Z}.$$

By distributivity of the lattice $\mathbf{P}(X\times Z)$, in $\mathbf{P}(X\times Z)$ we have

$$\begin{aligned}
&\mathbf{P}(\mathrm{pr}_1)(\beta)\wedge\bigwedge_{i=1}^{n}\big(\mathbf{P}(\mathrm{pr}_1)\mathbf{P}(f_i)(\alpha)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma)\big) &(5.2)\\
&=\mathbf{P}(\mathrm{pr}_1)(\beta)\wedge\big(\mathbf{P}(\mathrm{pr}_1)\big(\bigwedge_{i=1}^{n}\mathbf{P}(f_i)(\alpha)\big)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma)\big)\\
&=\big(\mathbf{P}(\mathrm{pr}_1)(\beta)\wedge\mathbf{P}(\mathrm{pr}_1)\big(\bigwedge_{i=1}^{n}\mathbf{P}(f_i)(\alpha)\big)\big)\vee\big(\mathbf{P}(\mathrm{pr}_1)(\beta)\wedge\mathbf{P}(\mathrm{pr}_2)(\gamma)\big)\\
&=\mathbf{P}(\mathrm{pr}_1)\big(\beta\wedge\bigwedge_{i=1}^{n}\mathbf{P}(f_i)(\alpha)\big)\vee\big(\mathbf{P}(\mathrm{pr}_1)(\beta)\wedge\mathbf{P}(\mathrm{pr}_2)(\gamma)\big).\\
&\leq\mathbf{P}(\mathrm{pr}_1)\big(\beta\wedge\bigwedge_{i=1}^{n}\mathbf{P}(f_i)(\alpha)\big)\vee\mathbf{P}(\mathrm{pr}_2)(\gamma). &(5.3)
\end{aligned}$$

Since $G_{X\times Z}$ is a filter, the conjunction in (5.2) belongs to $G_{X\times Z}$. Moreover, the element in (5.3) belongs to $J_{X\times Z}$. Since $J_{X\times Z}$ is downward closed, the element in (5.2) belongs to $J_{X\times Z}$, as well. Therefore, $G_{X\times Z}\cap J_{X\times Z}\neq\varnothing$, a contradiction. ☐

Remark 5.22. In our proof we did not use the classical ultrafilter lemma for Boolean algebras. In turn, the latter follows from Theorem 5.21: take C as the trivial category with one object and one morphism.

5.3. **Richness of a Boolean doctrine with respect to a universal ultrafilter.** We want to build models out of universal ultrafilters. This (as everything in life) is easy if we have richness. Recall that a maximally consistent deductively closed first-order theory $\mathcal{T}$ is *rich* if for every formula $\exists x\,\beta(x)\in\mathcal{T}$ there is a 0-ary term c (a "witness") such that $\beta(c)\in\mathcal{T}$. Given such a theory $\mathcal{T}$, we can easily find a model of $\mathcal{T}$—namely, the set of all 0-ary terms, with the obvious interpretation of the function and predicate symbols. We take inspiration from this definition to define *richness* for a doctrine with respect to a universal ultrafilter. Recall that the formulas in the universal ultrafilter are meant to be those whose universal closure is valid (in a certain model). Richness says that, for every formula $\alpha(x)$ whose universal closure is not valid (i.e.

which does not belong to the universal ultrafilter), there is a constant c witnessing this failure (i.e. such that $\alpha(c)$ does not belong to the universal ultrafilter).

Definition 5.23 (Richness)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine and let $(F_X)_{X\in\mathsf{C}}$ be a universal ultrafilter for $\mathbf{P}$. We say that $\mathbf{P}$ is *rich with respect to* $(F_X)_{X\in\mathsf{C}}$ if, for all $X \in \mathsf{C}$ and $\alpha \in \mathbf{P}(X) \setminus F_X$, there is $c\colon \mathbf{t} \to X$ such that $\mathbf{P}(c)(\alpha) \notin F_{\mathbf{t}}$.

Remark 5.24. We translate Definition 5.23 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language, $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$, and $(F_n)_{n\in\mathbb{N}}$ a universal ultrafilter for $\mathcal{T}$ in the sense of Remark 5.13. The theory $\mathcal{T}$ is *rich with respect to* $(F_n)_{n\in\mathbb{N}}$ if, for every $n \in \mathbb{N}$ and every quantifier-free formula $\alpha(x_1, \dots, x_n)$ not belonging to F_n, there are 0-ary terms (i.e. term-definable constants) $c_1, \dots, c_n$ such that $\alpha(c_1, \dots, c_n)$ does not belong to F_0.

The following shows how to obtain a model out of a universal ultrafilter in the rich case.

Proposition 5.25. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine and let $(F_X)_{X\in\mathsf{C}}$ be a universal ultrafilter for $\mathbf{P}$ such that $\mathbf{P}$ is rich with respect to $(F_X)_{X\in\mathsf{C}}$. There is a Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$ such that, for all $X \in \mathsf{C}$,*

$$F_X = \{\alpha \in \mathbf{P}(X) \mid \textit{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\}.$$

Proof. By Remark 5.14, $F_{\mathbf{t}}$ is an ultrafilter of the Boolean algebra $\mathbf{P}(\mathbf{t})$. Thus, as in the proof of the implication (2) $\Rightarrow$ (1) of Theorem 4.3, we obtain a Boolean model $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ by setting M as $\mathsf{Hom}(\mathbf{t}, -)\colon \mathsf{C} \to \mathsf{Set}$, and $\mathfrak{m}\colon \mathbf{P} \to \mathscr{P} \circ M^{\mathrm{op}}$ as the natural transformation whose component at $X \in \mathsf{C}$ is

$$\begin{aligned} \mathfrak{m}_X\colon \mathbf{P}(X) &\longrightarrow \mathscr{P}(\mathsf{Hom}(\mathbf{t}, X)) \\ \alpha &\longmapsto \{c\colon \mathbf{t} \to X \mid \mathbf{P}(c)(\alpha) \in F_{\mathbf{t}}\}. \end{aligned}$$

To conclude, let $X \in \mathsf{C}$, and let us prove $F_X = \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\}$. We first prove the left-to-right inclusion. Let $\alpha \in F_X$, and let $x \in M(X) = \mathsf{Hom}(\mathbf{t}, X)$. We shall prove $x \in \mathfrak{m}_X(\alpha)$. We have $\mathfrak{m}_X(\alpha) = \{c\colon \mathbf{t} \to X \mid \mathbf{P}(c)(\alpha) \in F_{\mathbf{t}}\}$. So, it is enough to prove $\mathbf{P}(x)(\alpha) \in F_{\mathbf{t}}$. This follows from $\alpha \in F_X$ since F is closed under reindexings by Definition 5.12(1). Conversely, suppose $\alpha \notin F_X$. By definition of richness, there is $c\colon \mathbf{t} \to X$ such that $\mathbf{P}(c)(\alpha) \notin F_{\mathbf{t}}$, so that $c \notin \mathfrak{m}_X(\alpha)$. □

We want to obtain a model out of a universal ultrafilter also in the non-rich case. To do so, we will produce a rich theory out of the non-rich one. For this, the idea is to extend the language C, adding new constants meant to witness the failure of universal closures of the formulas not belonging to the universal ultrafilter F. Once we have added the constants, we interpret the formulas in F in the extended language C', producing a new class of formulas G. However, G might fail to be a universal ultrafilter in the extended language because of the new formulas involving the new constants. To remedy this, we will use the universal ultrafilter lemma (relying on the axiom of choice) to extend G to a universal ultrafilter F' in the extended language (Lemma 5.26). However, we might lack some witnesses for the new formulas (involving the new constants) not belonging to F'. This calls for an iterative process, where at each step we add new constants and use the universal ultrafilter lemma: the rich theory will be obtained as a colimit after ω steps (Theorem 5.27).

In the following lemma, we address a single iteration of this process. This, together with the universal ultrafilter lemma, is the main technical lemma of the paper.

Lemma 5.26 (Extension to richness: first step)**.** *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $(F_X)_{X\in\mathsf{C}}$ be a universal ultrafilter for $\mathbf{P}$. There are a small category C' with the same objects of C, a Boolean doctrine $\mathbf{P}'\colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$, a Boolean doctrine morphism $(R, \mathfrak{r})\colon \mathbf{P} \to \mathbf{P}'$ such that $R\colon \mathsf{C} \to \mathsf{C}'$ is the identity on objects, and a universal ultrafilter $(F'_X)_{X\in\mathsf{C}'}$ for $\mathbf{P}'$ with the following properties.*

1. *For every $X \in \mathsf{C}$, $F_X = \mathfrak{r}_X^{-1}[F'_X]$.*
2. *For every $X \in \mathsf{C}$ and for every $\alpha \in \mathbf{P}(X) \setminus F_X$, there is a morphism $c\colon \mathbf{t}_{\mathsf{C}'} \to X$ in C' such that $\mathbf{P}'(c)(\mathfrak{r}_X(\alpha)) \notin F'_{\mathbf{t}_{\mathsf{C}'}}$.*

Before starting the proof, we give an informal outline of it. For every context $X \in \mathsf{C}$ and every formula $\sigma \in I_X \coloneqq \mathbf{P}(X) \setminus F_X$, we will add to the language a constant c_σ in the context X. We will denote by G the universal filter generated by the formulas in F seen in this new language. Moreover, we will denote by J the universal ideal generated by the formulas belonging to I_X (for X ranging among contexts), seen in this new language, together with the formulas $\sigma(c_\sigma/X)$ obtained by substituting c_σ to X in σ, for each context $X \in \mathsf{C}$ and formula $\sigma \in I_X$. We will prove that G and J are disjoint, and then we will extend G to a universal ultrafilter F' disjoint from J.

Proof of Lemma 5.26. Let $\mathcal{A}$ be the set of finite subsets of $\{(X, \alpha) \mid X \in \mathsf{C},\ \alpha \in \mathbf{P}(X) \setminus F_X\}$, partially ordered by inclusion. The poset $\mathcal{A}$ is directed.

Let $\bar{X} = \big\{(X_1, \alpha_1), \ldots, (X_n, \alpha_n)\big\}$ and $\bar{Y} = \big\{(Y_1, \beta_1), \ldots, (Y_m, \beta_m)\big\}$. Whenever $\bar{X} \subseteq \bar{Y}$ in $\mathcal{A}$, there exists a unique function $\tau \colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ induced by the inclusion such that $(X_i, \alpha_i) = (Y_{\tau(i)}, \beta_{\tau(i)})$ for all $i = 1, \ldots, n$.

Define the following diagram on $\mathcal{A}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "D"] & & \mathsf{Doct}_{\mathsf{BA}} \\
\varnothing \arrow[rr, mapsto] \arrow[d, phantom, "\cap" rotate=-90] & & \mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA} \arrow[d, "{(L_{\bar{X}}, \mathfrak{l}_{\bar{X}})}"] \\
\bar{X} = \big\{(X_1, \alpha_1), \ldots, (X_n, \alpha_n)\big\} \arrow[rr, mapsto] \arrow[d, phantom, "\cap" rotate=-90] & & \mathbf{P}_{\Pi_{a=1}^n X_a} \colon \mathsf{C}^{\mathrm{op}}_{\Pi_{a=1}^n X_a} \to \mathsf{BA} \arrow[d, "{(L_{\bar{X}\bar{Y}}, \mathfrak{l}_{\bar{X}\bar{Y}})}"] \\
\bar{Y} = \big\{(Y_1, \beta_1), \ldots, (Y_m, \beta_m)\big\} \arrow[rr, mapsto] & & \mathbf{P}_{\Pi_{b=1}^m Y_b} \colon \mathsf{C}^{\mathrm{op}}_{\Pi_{b=1}^m Y_b} \to \mathsf{BA}
\end{tikzcd}$$

where $D(\bar{X})$ is the Boolean doctrine $\mathbf{P}_{\Pi_{a=1}^n X_a} \colon \mathsf{C}^{\mathrm{op}}_{\Pi_{a=1}^n X_a} \to \mathsf{BA}$ obtained from $\mathbf{P}$ by adding a constant of type $\prod_{a=1}^n X_a$, and $D(\varnothing \subseteq \bar{X})$ is the canonical Boolean doctrine morphism $(L_{\bar{X}}, \mathfrak{l}_{\bar{X}}) \colon \mathbf{P} \to \mathbf{P}_{\Pi_a X_a}$, and $D(\bar{X} \subseteq \bar{Y})$ is the unique Boolean doctrine morphism $(L_{\bar{X}\bar{Y}}, \mathfrak{l}_{\bar{X}\bar{Y}}) \colon \mathbf{P}_{\Pi_a X_a} \to \mathbf{P}_{\Pi_b Y_b}$ such that

$$(L_{\bar{X}\bar{Y}}, \mathfrak{l}_{\bar{X}\bar{Y}}) \circ (L_{\bar{X}}, \mathfrak{l}_{\bar{X}}) = (L_{\bar{Y}}, \mathfrak{l}_{\bar{Y}}) \quad \text{ and } \quad L_{\bar{X}\bar{Y}}(\mathrm{id}_{\Pi_a X_a} \colon \mathbf{t} \rightsquigarrow \Pi_a X_a) = \langle \mathrm{pr}_{\tau(1)}, \ldots, \mathrm{pr}_{\tau(n)} \rangle \colon \mathbf{t} \rightsquigarrow \Pi_a X_a.$$

defined by the universal property of $\mathbf{P}_{\Pi_a X_a}$. Here $\langle \mathrm{pr}_{\tau(1)}, \ldots, \mathrm{pr}_{\tau(n)} \rangle$ is the projection on the corresponding components from the object $\prod_{b=1}^m Y_b$ to the object $\prod_{a=1}^n X_a$, since X_i appears as the $\tau(i)$-th component of $\bar{Y}$. We refer the reader to Remark 4.5 to have more details about these constructions.

The diagram $D \colon \mathcal{A} \to \mathsf{Doct}_{\mathsf{BA}}$ is a directed.

Take the colimit of D in $\mathsf{Doct}_{\mathsf{BA}}$, $\mathbf{P}' \colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$, computed as in [10, Sections 2.2 and 3.1]. Objects in the base category C' are the same as C, since for every $\bar{X}, \bar{Y} \in \mathcal{A}$ the functor $L_{\bar{X}\bar{Y}}$ acts like the identity on objects. A morphism $[f, \bar{X}]$ in C' from A to B—written as $[f, \bar{X}] \colon A \dashrightarrow B$—is the equivalence class of a morphism $f \colon \prod_{a=1}^n X_a \times A \to B$ for some fixed $\bar{X} = \{(X_1, \alpha_1), \ldots, (X_n, \alpha_n)\} \in \mathcal{A}$. We have $[(f, \bar{X})] = [(f', \bar{Y})]$ in C', for some $f' \colon \prod_{b=1}^m Y_b \times A \to B$ with $\bar{Y} = \{(Y_1, \beta_1), \ldots, (Y_m, \beta_m)\} \in \mathcal{A}$, if and only if there is $\bar{Z} \in \mathcal{A}$ such that $\bar{X} \subseteq \bar{Z} \supseteq \bar{Y}$ making the following diagram commute.

$$\begin{tikzcd}
 & \prod_{a=1}^n X_a \times A \arrow[dr, "f"] & \\
\prod_{c=1}^s Z_c \times A \arrow[ur, "\langle \mathrm{pr}_{\tau(1)}, \ldots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A"] \arrow[dr, "\langle \mathrm{pr}_{\tau'(1)}, \ldots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_A"'] & & B \\
 & \prod_{b=1}^m Y_b \times A \arrow[ur, "f'"'] &
\end{tikzcd}$$

Here τ and τ' are induced by $\bar{X} \subseteq \bar{Z}$ and $\bar{Y} \subseteq \bar{Z}$ in $\mathcal{A}$ respectively.

For any object A, the fiber $\mathbf{P}'(A)$ is the colimit of D in BA restricted to the fibers: $\mathbf{P}'(A)$ is the set of equivalence classes of the form $[(\varphi, \bar{X})]$ for some $\varphi \in \mathbf{P}(\prod_{a=1}^n X_a \times A)$, where $[(\varphi, \bar{X})] = [(\varphi', \bar{Y})]$, for $\varphi' \in P(\prod_{b=1}^m Y_b \times A)$ if and only if there is $\bar{Z} \in \mathcal{A}$ such that $\bar{X} \subseteq \bar{Z} \subseteq \bar{Y}$ with induced function τ and τ' such that $\mathbf{P}(\langle \mathrm{pr}_{\tau(1)}, \ldots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)(\varphi) = \mathbf{P}(\langle \mathrm{pr}_{\tau'(1)}, \ldots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_A)(\varphi')$ in $\mathbf{P}(\prod_{c=1}^s Z_c \times A)$. Reindexing is defined in a common list of $\mathcal{A}$: if $[(f, \bar{X})]\colon A \dashrightarrow B$ and $[(\psi, \bar{Y})] \in \mathbf{P}'(B)$, take $\bar{X} \subseteq \bar{Z} \supseteq \bar{Y}$; then

$$\mathbf{P}'([(f, \bar{X})])[(\psi, \bar{Y})] = [\mathbf{P}(\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)}, f \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\rangle)(\psi), \bar{Z}].$$

$$\begin{tikzcd}
\prod Z_c \times A \arrow[r, "\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A"] \arrow[d, "\langle \mathrm{pr}_1, \dots, \mathrm{pr}_s, f \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\rangle"'] & \prod X_a \times A \arrow[r, "f"] & B \\
\prod Z_c \times B \arrow[r, "\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_B"'] & \prod Y_b \times B &
\end{tikzcd}$$

We call $(R, \mathfrak{r})\colon \mathbf{P} \to \mathbf{P}'$ the colimit map from $D(\varnothing) = \mathbf{P}$ to the colimit $\mathbf{P}'$. Recall once again that the objects of C and C' are the same.

For each $X \in \mathsf{C}$, set $I_X \coloneqq \mathbf{P}(X) \setminus F_X$. Let G be the universal filter for $\mathbf{P}'$ generated by $(\mathfrak{r}_X[F_X])_{X \in \mathsf{C}}$. For $X \in \mathsf{C}$ and $\sigma \in I_X$, we let c_σ denote the morphism $[\mathrm{id}_X, \{(X, \sigma)\}]\colon \mathbf{t} \dashrightarrow X$ in C'. Let J be the universal ideal generated by $(\mathfrak{r}_X[I_X])_{X \in \mathsf{C}}$ and by the following subset of $\mathbf{P}'(\mathbf{t})$:

$$\{\mathbf{P}'(c_\alpha)(\mathfrak{r}_X(\alpha)) \mid X \in \mathsf{C},\ \alpha \in I_X\}.$$

To show that G and J are componentwise disjoint, let us suppose that G and J intersect, and let us prove a contradiction.

By Lemma 5.17, there are $Y_1, \dots, Y_n, Z_1, \dots, Z_m, S_1, \dots, S_p \in \mathsf{C}$, $(\alpha_i \in F_{Y_i})_{i=1,\dots,n}$, $(\gamma_j \in I_{Z_j})_{j=1,\dots,m}$, $(\sigma_k \in I_{S_k})_{k=1,\dots,p}$, $([f_i, \bar{U}^i]\colon \prod_{j=1}^m Z_j \dashrightarrow Y_i)_{i=1,\dots,n}$ such that in $\mathbf{P}'(\prod_{j=1}^m Z_j)$

$$\bigwedge_{i=1}^n \mathbf{P}'([f_i, \bar{U}^i])(\mathfrak{r}_{Y_i}(\alpha_i)) \leq \bigvee_{j=1}^m \mathbf{P}'(\mathrm{pr}_j)(\mathfrak{r}_{Z_j}(\gamma_j)) \vee \bigvee_{k=1}^p \mathbf{P}'(!_{\prod_j Z_j})\mathbf{P}'(c_{\sigma_k})(\mathfrak{r}_{S_k}(\sigma_k)). \tag{5.4}$$

We prove that it is enough to take $n = 1$. Indeed, let $\bar{W} = \{(W_1, \omega_1), \dots, (W_w, \omega_w)\}$ be the union of all entries in $\bar{U}^i$ for $i = 1, \dots, n$, where each $\bar{U}^i$ is $\{(U_1^i, \mu_1^i), \dots, (U_{u_i}^i, \mu_{u_i}^i)\}$. In particular, for every $i = 1, \dots, n$ there is a unique function $\tau^i\colon \{1, \dots, u_i\} \to \{1, \dots, w\}$ such that $(U_h^i, \mu_h^i) = (W_{\tau^i(h)}, \omega_{\tau^i(h)})$ for $h = 1, \dots, u_i$. We can now compute the conjunction in a common fiber.

$$\left(\prod_{a=1}^w W_a\right) \times \left(\prod_{j=1}^m Z_j\right) \xrightarrow{\langle \mathrm{pr}_{\tau^i(1)}, \dots, \mathrm{pr}_{\tau^i(u_i)} \rangle \times \mathrm{id}_{\prod_j Z_j}} \left(\prod_{h=1}^{u_i} U_h^i\right) \times \left(\prod_{j=1}^m Z_j\right) \xrightarrow{f_i} Y_i$$

Set $\alpha \coloneqq \bigwedge_{i=1}^n \mathbf{P}(\langle \mathrm{pr}_{\tau^i(1)}, \dots, \mathrm{pr}_{\tau^i(u_i)} \rangle \times \mathrm{id}_{\prod_j Z_j})\mathbf{P}(f_i)(\alpha_i)$ in $\mathbf{P}((\prod_{a=1}^z W_a) \times (\prod_{j=1}^m Z_j))$. We have

$$\begin{aligned}
\bigwedge_{i=1}^n \mathbf{P}'([f_i, \bar{U}^i])(\mathfrak{r}_{Y_i}(\alpha_i)) &= \bigwedge_{i=1}^n \mathbf{P}'([f_i, \bar{U}^i])([\alpha_i, \varnothing])) \\
&= \bigwedge_{i=1}^n [\mathbf{P}(f_i)(\alpha_i), \bar{U}^i] \\
&= \bigwedge_{i=1}^n [\mathbf{P}(\langle \mathrm{pr}_{\tau^i(1)}, \dots, \mathrm{pr}_{\tau^i(u_i)} \rangle \times \mathrm{id}_{\prod_j Z_j})\mathbf{P}(f_i)(\alpha_i), \bar{W}] \\
&= \left[\bigwedge_{i=1}^n \mathbf{P}(\langle \mathrm{pr}_{\tau^i(1)}, \dots, \mathrm{pr}_{\tau^i(u_i)} \rangle \times \mathrm{id}_{\prod_j Z_j})\mathbf{P}(f_i)(\alpha_i), \bar{W}\right] \\
&= [\alpha, \bar{W}] \\
&= \mathbf{P}'([\mathrm{id}_{\prod_a W_a \times \prod_j Z_j}, \bar{W}])([\alpha, \varnothing]) \\
&= \mathbf{P}'([\mathrm{id}_{\prod_a W_a \times \prod_j Z_j}, \bar{W}])(\mathfrak{r}_{\prod_a W_a \times \prod_j Z_j}(\alpha)).
\end{aligned}$$

We can replace all the Y_i's, α_i's and $[f_i, \bar{U}^i]$'s with the single object $Y \coloneqq \prod_a W_a \times \prod_j Z_j$, the single element α (which belongs to F_Y since F is closed under reindexing and conjunctions), and the single morphism

$[\mathrm{id}_{\prod_a W_a \times \prod_j Z_j}, \bar{W}] \colon \prod_j Z_j \dashrightarrow Y$. This shows that we can take $n = 1$. So we replace (5.4) with

$$\mathbf{P}'([f,\bar{U}])(\mathfrak{r}_Y(\alpha)) \leq \bigvee_{j=1}^{m} \mathbf{P}'(\mathrm{pr}_j)(\mathfrak{r}_{Z_j}(\gamma_j)) \vee \bigvee_{k=1}^{p} \mathbf{P}'(!_{\prod_j Z_j})\mathbf{P}'(c_{\sigma_k})(\mathfrak{r}_{S_k}(\sigma_k)). \tag{5.5}$$

We next show that it is enough to take $m = 1$, as well. Indeed, using the fact that R preserves products and using naturality of $\mathfrak{r}$:

$$\bigvee_{j=1}^{m} \mathbf{P}'(\mathrm{pr}_j)(\mathfrak{r}_{Z_j}(\gamma_j)) = \bigvee_{j=1}^{m} \mathbf{P}'(R(\mathrm{pr}_j))(\mathfrak{r}_{Z_j}(\gamma_j)) = \bigvee_{j=1}^{m} \mathfrak{r}_{\prod_j Z_j}\mathbf{P}(\mathrm{pr}_j)(\gamma_j) = \mathfrak{r}_{\prod_j Z_j}\left(\bigvee_{j=1}^{m} \mathbf{P}(\mathrm{pr}_j)(\gamma_j)\right).$$

We can replace all the Z_j's and γ_j's with the single object $Z \coloneqq \prod_j Z_j$ and the single element $\gamma \coloneqq \bigvee_{j=1}^{m} \mathbf{P}(\mathrm{pr}_j)(\gamma_j)$, which belongs to I_Z by (3) and (4) in Definition 5.9. This shows that it is enough to take $m = 1$. We rewrite (5.5) as

$$\mathbf{P}'([f,\bar{U}])(\mathfrak{r}_Y(\alpha)) \leq \mathfrak{r}_Z(\gamma) \vee \bigvee_{k=1}^{p} \mathbf{P}'(!_Z)\mathbf{P}'(c_{\sigma_k})(\mathfrak{r}_{S_k}(\sigma_k)). \tag{5.6}$$

Without loss of generality, we can suppose that the pairs $(S_1,\sigma_k),\dots,(S_p,\sigma_p)$ are pairwise distinct. We define $\bar{S} \coloneqq \{(S_1,\sigma_1),\dots,(S_p,\sigma_p)\}$. We compute the disjunction $\bigvee_{k=1}^{p} \mathbf{P}'(!_Z)\mathbf{P}'(c_{\sigma_k})(\mathfrak{r}_{S_k}(\sigma_k))$ on the right-hand side of (5.6):

$$\begin{aligned}
\bigvee_{k=1}^{p} \mathbf{P}'(!_Z)\mathbf{P}'(c_{\sigma_k})(\mathfrak{r}_{S_k}(\sigma_k)) &= \bigvee_{k=1}^{p} \mathbf{P}'(!_Z)\mathbf{P}'([\mathrm{id}_{S_k},\{(S_k,\sigma_k)\}])([\sigma_k,\varnothing]) \\
&= \bigvee_{k=1}^{p} \mathbf{P}'(!_Z)([\sigma_k,\{(S_k,\sigma_k)\}]) \\
&= \mathbf{P}'(!_Z)\left(\bigvee_{k=1}^{p} [\sigma_k,\{(S_k,\sigma_k)\}]\right) \\
&= \mathbf{P}'(!_Z)\left(\left[\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\sigma_k),\bar{S}\right]\right).
\end{aligned}$$

We rewrite (5.6) as

$$\mathbf{P}'([f,\bar{U}])(\mathfrak{r}_Y(\alpha)) \leq \mathfrak{r}_Z(\gamma) \vee \mathbf{P}'(!_Z)\left(\left[\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\sigma_k),\bar{S}\right]\right). \tag{5.7}$$

We compute the left-hand and right-hand side of (5.7) in any upper bound $\bar{E} = \{(E_1,\varepsilon_1),\dots,(E_e,\varepsilon_e)\}$ in $\mathcal{A}$ of $\bar{U}$ and $\bar{S}$. Given such an $\bar{E}$, there is a function $\lambda \colon \{1,\dots,u\} \to \{1,\dots,e\}$ induced by the inclusion, which is the unique one such that $(U_h,\mu_h) = (E_{\lambda(h)},\varepsilon_{\lambda(h)})$ for $h = 1,\dots,u$. Similarly, there is a unique function $\theta \colon \{1,\dots,p\} \to \{1,\dots,e\}$ such that $(S_k,\sigma_k) = (E_{\theta(k)},\varepsilon_{\theta(k)})$ for $k = 1,\dots,p$.

$$\prod_{k=1}^{p} S_k \xleftarrow{\langle \mathrm{pr}_{\theta(1)},\dots,\mathrm{pr}_{\theta(p)}\rangle} \left(\prod_{l=1}^{e} E_l\right) \times Z \xrightarrow{\langle \mathrm{pr}_{\lambda(1)},\dots,\mathrm{pr}_{\lambda(u)}\rangle \times \mathrm{id}_Z} \left(\prod_{h=1}^{u} U_h\right) \times Z \xrightarrow{f} Y$$

We compute the left-hand side of (5.7):

$$\mathbf{P}'([f,\bar{U}])(\mathfrak{r}_Y(\alpha)) = \mathbf{P}'([f,\bar{U}])([\alpha,\varnothing]) = \left[\mathbf{P}(\langle \mathrm{pr}_{\lambda(1)},\dots,\mathrm{pr}_{\lambda(u)}\rangle \times \mathrm{id}_Z)\mathbf{P}(f)(\alpha),\bar{E}\right].$$

We compute the right-hand side of (5.7):

$$\begin{aligned}\mathfrak{r}_Z(\gamma) \vee \mathbf{P}'(!_Z)\left(\left[\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\sigma_k), \bar{S}\right]\right) &= [\gamma, \varnothing] \vee \mathbf{P}'(!_Z)\left(\left[\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\sigma_k), \bar{S}\right]\right)\\ &= [\mathbf{P}(\mathrm{pr}_{e+1})(\gamma), \bar{E}] \vee \left[\mathbf{P}(\langle \mathrm{pr}_{\theta(1)}, \dots, \mathrm{pr}_{\theta(p)}\rangle)\left(\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_k)(\sigma_k)\right), \bar{E}\right]\\ &= \left[\mathbf{P}(\mathrm{pr}_{e+1})(\gamma) \vee \bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_{\theta(k)})(\sigma_k), \bar{E}\right].\end{aligned}$$

Therefore, we rewrite (5.7) as

$$\left[\mathbf{P}(\langle \mathrm{pr}_{\lambda(1)}, \dots, \mathrm{pr}_{\lambda(u)}\rangle \times \mathrm{id}_Z)\mathbf{P}(f)(\alpha), \bar{E}\right] \leq \left[\mathbf{P}(\mathrm{pr}_{e+1})(\gamma) \vee \bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_{\theta(k)})(\sigma_k), \bar{E}\right].$$

It follows that, for a sufficiently large upper bound $\bar{E}$ of $\bar{U}$ and $\bar{S}$, in the fiber $\mathbf{P}((\prod_{l=1}^{e} E_l) \times Z)$ we have

$$\mathbf{P}(\langle \mathrm{pr}_{\lambda(1)}, \dots, \mathrm{pr}_{\lambda(u)}\rangle \times \mathrm{id}_Z)\mathbf{P}(f)(\alpha) \leq \mathbf{P}(\mathrm{pr}_{e+1})(\gamma) \vee \bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_{\theta(k)})(\sigma_k). \tag{5.8}$$

Since $\alpha \in F_Y$ and since F is closed under reindexings, it follows that the left-hand side of (5.8) belongs to $F_{(\prod_l E_l) \times Z}$. Then, since F is upward closed, also the right-hand side of (5.8) belongs to $F_{(\prod_l E_l) \times Z}$. This disjunction equals

$$\mathbf{P}(\mathrm{pr}_{e+1})(\gamma) \vee \left(\bigvee_{k=1}^{p} \mathbf{P}(\mathrm{pr}_{\theta(k)})(\sigma_k)\right) \vee \left(\bigvee_{i \in \{1,\dots,e\} \setminus \mathrm{Im}(\theta)} \mathbf{P}(\mathrm{pr}_i)(\bot_{\mathbf{P}(E_i)})\right).$$

This is a disjunction of reindexings of elements of I along projections. Indeed, $\gamma \in I_Z$, $\sigma_k \in I_{S_k}$, and for all $i \in \{1, \dots, e\} \setminus \mathrm{Im}(\theta)$ we have $\bot_{\mathbf{P}(E_i)} \in I_{E_i}$ because $\bot_{\mathbf{P}(E_i)} \leq \varepsilon_i \in I_{E_i}$. Therefore, by Definition 5.9(3), the element in the right-hand side of (5.8) belongs to $I_{\prod_l E_l \times Z}$. Since it also belongs to $F_{\prod_l E_l \times Z}$, we have reached a contradiction with the assumption that F and I are componentwise disjoint. This proves our claim that G and J are componentwise disjoint.

Since G and J are disjoint, by Theorem 5.21 there is a universal ultrafilter F' for $\mathbf{P}'$ that extends G and is disjoint from J. We check that F' satisfies the desired properties. Since F' extends G and G is generated by $(\mathfrak{r}_X[F_X])_{X \in \mathsf{C}}$, for every $X \in \mathsf{C}$ we have $F_X \subseteq \mathfrak{r}_X^{-1}[F'_X]$. For the converse inclusion, let $\alpha \in \mathbf{P}(X)$ and suppose $\alpha \notin F_X$, i.e. $\alpha \in I_X$. Then $\mathfrak{r}_X(\alpha) \in J_X$, which implies $\mathfrak{r}_X(\alpha) \notin F'_X$ because J_X and F'_X are disjoint. This proves that for all $X \in \mathsf{C}$ we have $F_X = \mathfrak{r}_X^{-1}[F'_X]$, which is condition (1) in the statement. We are left to check the condition (2) of the statement, i.e. that for all $X \in \mathsf{C}$ and $\alpha \in \mathbf{P}(X) \setminus F_X$ there is a morphism $c \colon \mathbf{t} \dashrightarrow X$ in C' such that $\mathbf{P}'(c)(\mathfrak{r}_X(\alpha)) \notin F'_{\mathbf{t}}$. Since $\alpha \in I_X$, we can take $c \coloneqq c_\alpha = [\mathrm{id}_X, \{(X, \alpha)\}]$. Recall that $\mathbf{P}'(c_\alpha)(\mathfrak{r}_X(\alpha)) \in \mathbf{P}'(\mathbf{t})$ is a generator of J, which is disjoint from F'; hence $\mathbf{P}'(c_\alpha)(\mathfrak{r}_X(\alpha)) \notin F'_{\mathbf{t}}$, as desired. □

In the following theorem, we show how to produce a rich theory from an arbitrary one. We accomplish this using Lemma 5.26 ω times. The desired rich theory is obtained as the colimit. Note that this rich theory is not canonically determined by the original one, because in each step we use the axiom of choice (in the form of the universal ultrafilter lemma).

Theorem 5.27 (Extension to richness)**.** *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $(F_X)_{X \in \mathsf{C}}$ be a universal ultrafilter for $\mathbf{P}$. There are a category C' with the same objects of C, a Boolean doctrine $\mathbf{P}' \colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$, a Boolean doctrine morphism $(R, \mathfrak{r}) \colon \mathbf{P} \to \mathbf{P}'$ such that $R \colon \mathsf{C} \to \mathsf{C}'$ is the identity on objects, and a universal ultrafilter $(F'_X)_{X \in \mathsf{C}'}$ for $\mathbf{P}'$ such that $\mathbf{P}'$ is rich with respect to $(F'_X)_{X \in \mathsf{C}'}$, and moreover, for all $X \in \mathsf{C}$, $F_X = \mathfrak{r}_X^{-1}[F'_X]$.*

Proof. We define a sequence $(\mathbf{P}^n \colon (\mathsf{C}^n)^{\mathrm{op}} \to \mathsf{BA})$ of Boolean doctrines with C^n small and having the same objects of C, together with a sequence $((R^n, \mathfrak{r}^n) \colon \mathbf{P}^n \to \mathbf{P}^{n+1})_{n \in \mathbb{N}}$ of Boolean doctrine morphisms where each $R^n\colon \mathsf{C} \to \mathsf{C}'$ is the identity on objects, and a sequence $(F^n)_{n\in\mathbb{N}}$ with F_n a universal ultrafilter for $\mathbf{P}^n$ with the following properties.

(1) $\mathsf{C}^0 = \mathsf{C}$, $\mathbf{P}^0 = \mathbf{P}$ and $F^0 = F$.
(2) For all $n \in \mathbb{N}$ and $X \in \mathsf{C}^n$, $F^n_X = (\mathfrak{r}^n_X)^{-1}[F^{n+1}_X]$.
(3) For all $n \in \mathbb{N}$, $X \in \mathsf{C}^n$ and $\alpha \in \mathbf{P}^n(X) \setminus F^n_X$, there is a morphism $c\colon \mathbf{t}_{\mathsf{C}^{n+1}} \to X$ in C^{n+1} such that $\mathbf{P}^{n+1}(c)(\mathfrak{r}^n_X(\alpha)) \notin F^{n+1}_{\mathbf{t}_{\mathsf{C}^{n+1}}}$.

We define these sequences inductively (with the aid of the axiom of dependent choice). For the base case, set $\mathsf{C}^0 \coloneqq \mathsf{C}$, $\mathbf{P}^0 \coloneqq \mathbf{P}$ and $F^0 \coloneqq F$. For the inductive case, for any $n \in \mathbb{N}$, given C^n, $\mathbf{P}^n$ and F^n, apply Lemma 5.26 to find C^{n+1}, $\mathbf{P}^{n+1}$, $(R^n, \mathfrak{r}^n)$ and F^{n+1}. This gives us the sequences with the desired properties. The sequence allows us to define a directed diagram of Boolean doctrines indexed by the poset $(\mathbb{N}, \leq)$ of natural numbers:

$$\mathbf{P}^0 \xrightarrow{(R^0,\mathfrak{r}^0)} \mathbf{P}^1 \xrightarrow{(R^1,\mathfrak{r}^1)} \mathbf{P}^2 \to \cdots \to \mathbf{P}^n \xrightarrow{(R^n,\mathfrak{r}^n)} \mathbf{P}^{n+1} \to \dots$$

For every $n \leq m \in \mathbb{N}$, we call $(R^{n;m}, \mathfrak{r}^{n;m})$ the composite

$$(R^{m-1}, \mathfrak{r}^{m-1}) \circ \cdots \circ (R^{n+1}, \mathfrak{r}^{n+1}) \circ (R^n, \mathfrak{r}^n)\colon \mathbf{P}^n \longrightarrow \mathbf{P}^m.$$

Let $\mathbf{P}'\colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$ be the colimit of this diagram, and call $(Q^n, \mathfrak{q}^n)\colon \mathbf{P}^n \to \mathbf{P}'$ the colimit morphism for every $n \in \mathbb{N}$. In particular we define the desired Boolean docrtrine morphism $(R, \mathfrak{r})\colon \mathbf{P} \to \mathbf{P}'$ from the statement as $(R, \mathfrak{r}) \coloneqq (Q^0, \mathfrak{q}^0)$.

We collect here some properties of this colimit. We can choose the colimit in a way such that C' has the same objects of C, and for each n the functor $Q^n\colon \mathsf{C} \to \mathsf{C}'$ is the identity on objects. Since the colimit category C' is computed in **Cat**, by [10, Section 2.2], for every morphism $g\colon X \to Y$ in C' there are $n \in \mathbb{N}$ and $f\colon X \to Y$ in C^n such that $g = Q^n(f)$.

$$\begin{tikzcd}
\mathbf{P}^0(X) \arrow[r, "\mathfrak{r}^0_X"] \arrow[rr, bend left, "\mathfrak{r}^{0;2}_X"] \arrow[drr, "\mathfrak{q}^0_X"'] & \mathbf{P}^1(X) \arrow[r, "\mathfrak{r}^1_X"] \arrow[dr, "\mathfrak{q}^1_X"] & \mathbf{P}^2(X) \arrow[r, "\mathfrak{r}^2_X"] \arrow[d, "\mathfrak{q}^2_X"] & \dots \arrow[r, "\mathfrak{r}^{n-1}_X"] & \mathbf{P}^n(X) \arrow[r, "\mathfrak{r}^n_X"] \arrow[dll, "\mathfrak{q}^n_X"] & \dots \\
 & & \mathbf{P}'(X) & & &
\end{tikzcd}$$

Moreover, for every $X \in \mathsf{C}$ and every $\beta \in \mathbf{P}'(X)$, there are $n \in \mathbb{N}$ and $\alpha \in \mathbf{P}^n(X)$ such that $\beta = \mathfrak{q}^n_Y(\alpha)$. Morever, for every $\alpha \in \mathbf{P}^n(X)$ and $\beta \in \mathbf{P}^m(X)$, we have

$$\mathfrak{q}^n_X(\alpha) \leq \mathfrak{q}^m_X(\beta) \text{ if and only if there is } k \geq n, m \text{ such that } \mathfrak{r}^{n;k}_X(\alpha) \leq \mathfrak{r}^{m;k}_X(\beta).$$

For every $X \in \mathsf{C}$, let $F'_X \coloneqq \bigcup_{n\in\mathbb{N}} \mathfrak{q}^n_X[F^n_X]$. Similarly, for every $X \in \mathsf{C}$, let $I'_X \coloneqq \bigcup_{n\in\mathbb{N}} \mathfrak{q}^n_X[I^n_X]$, where for every $X \in \mathsf{C}$ and every $n \in \mathbb{N}$, $I^n_X \coloneqq \mathbf{P}^n(X) \setminus F^n_X$.

We are going to prove that F' is a universal filter and I' is a universal ideal. Roughly speaking, these facts hold because universal filters and universal ideals are defined by closure conditions involving finitely many elements, and thus they are preserved in a directed colimit.

We first prove that F' is a universal filter. Let $f\colon X \to Y$ be a morphism in C' and $\alpha \in F'_Y$. We prove that $\mathbf{P}'(f)(\alpha) \in F'_X$. There are $n, m \in \mathbb{N}$, $g\colon X \to Y$ in C^n and $\beta \in F^m_Y$ such that $Q^n(g) = f$ and $\mathfrak{q}^m_Y(\beta) = \alpha$. We take $k \geq n, m$ and compute

$$\mathbf{P}'(f)(\alpha) = \mathbf{P}'(Q^n(g))(\mathfrak{q}^m_Y(\beta)) = \mathbf{P}'(Q^k(R^{n;k}(g)))(\mathfrak{q}^k_Y(\mathfrak{r}^{m;k}_Y(\beta))) = \mathfrak{q}^k_X(\mathbf{P}^k(R^{n;k}(g))(\mathfrak{r}^{m;k}_Y(\beta))).$$

By (2), $\mathfrak{r}^{m;k}_Y(\beta) \in F^k_Y$. Then, since F^k is closed under reindexing, $\mathbf{P}^k(R^{n;k}(g))(\mathfrak{r}^{m;k}_Y(\beta)) \in F^k_X$; hence, by definition of F'_X, $\mathbf{P}'(f)(\alpha) = \mathfrak{q}^k_X(\mathbf{P}^k(R^{n;k}(g))(\mathfrak{r}^{m;k}_Y(\beta))) \in F'_X$.

We show that, for every $X \in \mathsf{C}$, F'_X is a filter. Since $\top_{\mathbf{P}(X)} \in F^0_X$, we have $\top_{\mathbf{P}'(X)} = \mathfrak{q}^0_X(\top_{\mathbf{P}(X)}) \in F'_X$. Let $n, m \in \mathbb{N}$, $\alpha \in F^n_X$ and $\beta \in F^m_X$. For $k \geq n, m$ we have

$$\mathfrak{q}^n_X(\alpha) \wedge \mathfrak{q}^m_X(\beta) = \mathfrak{q}^k_X(\mathfrak{r}^{n;k}_X(\alpha)) \wedge \mathfrak{q}^k_X(\mathfrak{r}^{m;k}_X(\beta)) = \mathfrak{q}^k_X(\mathfrak{r}^{n;k}_X(\alpha) \wedge \mathfrak{r}^{m;k}_X(\beta)). \tag{5.9}$$

By (2) we deduce that both $\mathfrak{r}_X^{n;k}(\alpha)$ and $\mathfrak{r}_X^{m;k}(\beta)$ belong to F_X^k. Since F_X^k is a filter, the conjunction in (5.9) belongs to F_X^k. Therefore, F'_X is closed under binary meets. We next show that F'_X is upward closed. Let $n, m \in \mathbb{N}$, $\alpha \in F_X^n$ and $\beta \in \mathbf{P}^m(X)$ be such that $\mathfrak{q}_X^n(\alpha) \leq \mathfrak{q}_X^m(\beta)$. There is $k \geq n, m$ such that $\mathfrak{r}_X^{n;k}(\alpha) \leq \mathfrak{r}_X^{m;k}(\beta)$. By (2) we have $\mathfrak{r}_X^{n;k}(\alpha) \in F_X^k$, and hence $\mathfrak{r}_X^{m;k}(\beta) \in F_X^k$. Thus, $\mathfrak{q}_X^m(\beta) = \mathfrak{q}_X^k(\mathfrak{r}_X^{m;k}(\beta)) \in F'_X$, as desired. This shows that F' is a universal filter.

We now show that I' is a universal ideal. Let $n, m \in \mathbb{N}$, $X, Y \in \mathsf{C}$, $(f_j \colon X \to Y)_{j=1,\dots,m}$ morphisms in C', $\alpha \in \mathbf{P}^n(Y)$ such that $\bigwedge_{j=1}^m \mathbf{P}'(f_j)(\mathfrak{q}^n(\alpha)) \in I'_X$. We prove that $\mathfrak{q}_Y^n(\alpha) \in I'_Y$. For every $j = 1, \dots, m$, there are $n_j \in \mathbb{N}$ and a morphism $g_j \colon X \to Y$ in C^{n_j} such that $f_j = Q^{n_j}(g_j)$. So for every $k \geq n, n_1, \dots, n_m$ we compute:

$$\begin{aligned}
\bigwedge_{j=1}^m \mathbf{P}'(f_j)(\mathfrak{q}^n(\alpha)) &= \bigwedge_{j=1}^m \mathbf{P}'(Q^{n_j}(g_j))(\mathfrak{q}_Y^n(\alpha)) \\
&= \bigwedge_{j=1}^m \mathbf{P}'(Q^k(R^{n_j;k}(g_j)))(\mathfrak{q}_Y^k(\mathfrak{r}_Y^{n;k}(\alpha))) \\
&= \bigwedge_{j=1}^m \mathfrak{q}_X^k(\mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))) \\
&= \mathfrak{q}_X^k\left(\bigwedge_{j=1}^m \mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))\right).
\end{aligned}$$

Since $\mathfrak{q}_X^k\left(\bigwedge_{j=1}^m \mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))\right) \in I_X^k$, there are $t \in \mathbb{N}$ and $\beta \in I_X^t$ such that

$$\mathfrak{q}_X^k\left(\bigwedge_{j=1}^m \mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))\right) = \mathfrak{q}_X^t(\beta)$$

in $\mathbf{P}'(X)$. Hence there is $s \geq k, t$ such that

$$\mathfrak{r}_X^{k;s}\left(\bigwedge_{j=1}^m \mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))\right) = \mathfrak{r}_X^{t;s}(\beta).$$

Moreover,

$$\begin{aligned}
\mathfrak{r}_X^{k;s}\left(\bigwedge_{j=1}^m \mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))\right) &= \bigwedge_{j=1}^m \mathfrak{r}_X^{k;s}(\mathbf{P}^k(R^{n_j;k}(g_j))(\mathfrak{r}_Y^{n;k}(\alpha))) \\
&= \bigwedge_{j=1}^m \mathbf{P}^s(R^{k;s}(R^{n_j;k}(g_j)))(\mathfrak{r}_Y^{k;s}(\mathfrak{r}_Y^{n;k}(\alpha))) \\
&= \bigwedge_{j=1}^m \mathbf{P}^s(R^{n_j;s}(g_j))(\mathfrak{r}_Y^{n;s}(\alpha)).
\end{aligned}$$

Observe that $\mathfrak{r}_X^{t;s}(\beta) \in I_X^s$. Indeed, if $\mathfrak{r}_X^{t;s}(\beta) \notin I_X^s$, we would have $\mathfrak{r}_X^{t;s}(\beta) \in F_X^s$, and hence by (2) $\beta \in F_X^t$, a contradiction.

It follows that $\bigwedge_{j=1}^m \mathbf{P}^s(R^{n_j;s}(g_j))(\mathfrak{r}_Y^{n;s}(\alpha)) \in I_X^s$. Since I^s is a universal ideal for $\mathbf{P}^s$, we get that $\mathfrak{r}_Y^{n;s}(\alpha) \in I_Y^s$. Hence $\mathfrak{q}_Y^n(\alpha) = \mathfrak{q}_Y^s(\mathfrak{r}_Y^{n;s}(\alpha)) \in I'_Y$, as desired.

The proof that I' is componentwise downward closed is similar to the proof that F' is componentwise upward closed seen above. To check the condition (3) in Definition 5.9, we take $\alpha_1 \in I_{X_1}^n$ and $\alpha_2 \in I_{X_2}^m$, and we prove that $\mathbf{P}'(\mathrm{pr}_1)(\mathfrak{q}_{X_1}^n(\alpha_1)) \vee \mathbf{P}'(\mathrm{pr}_2)(\mathfrak{q}_{X_2}^m(\alpha_2)) \in I'_{X_1 \times X_2}$. Let $k \geq n, m$. Using again condition (2) we have $\mathfrak{r}_{X_1}^{n;k}(\alpha_1) \in I_{X_1}^k$ and $\mathfrak{r}_{X_2}^{m;k}(\alpha_2) \in I_{X_2}^k$. Since I^k is a universal ideal for $\mathbf{P}^k$, it follows that

$\mathbf{P}^k(\mathrm{pr}_1)(\mathfrak{r}^{n;k}_{X_1}(\alpha_1)) \vee \mathbf{P}^k(\mathrm{pr}_2)(\mathfrak{r}^{m;k}_{X_2}(\alpha_2)) \in I^k_{X_1\times X_2}$, and hence

$$\mathfrak{q}^k_{X_1\times X_2}(\mathbf{P}^k(\mathrm{pr}_1)(\mathfrak{r}^{n;k}_{X_1}(\alpha_1)) \vee \mathbf{P}^k(\mathrm{pr}_2)(\mathfrak{r}^{m;k}_{X_2}(\alpha_2))) \in I'_{X_1\times X_2}.$$

Moreover,

$$\begin{aligned}
&\mathfrak{q}^k_{X_1\times X_2}(\mathbf{P}^k(\mathrm{pr}_1)(\mathfrak{r}^{n;k}_{X_1}(\alpha_1)) \vee \mathbf{P}^k(\mathrm{pr}_2)(\mathfrak{r}^{m;k}_{X_2}(\alpha_2)))\\
&= \mathfrak{q}^k_{X_1\times X_2}(\mathbf{P}^k(\mathrm{pr}_1)(\mathfrak{r}^{n;k}_{X_1}(\alpha_1))) \vee \mathfrak{q}^k_{X_1\times X_2}(\mathbf{P}^k(\mathrm{pr}_2)(\mathfrak{r}^{m;k}_{X_2}(\alpha_2)))\\
&= \mathbf{P}'(\mathrm{pr}_1)(\mathfrak{q}^k_{X_1}(\mathfrak{r}^{n;k}_{X_1}(\alpha_1))) \vee \mathbf{P}'(\mathrm{pr}_2)(\mathfrak{q}^k_{X_2}(\mathfrak{r}^{m;k}_{X_2}(\alpha_2)))\\
&= \mathbf{P}'(\mathrm{pr}_1)(\mathfrak{q}^n_{X_1}(\alpha_1)) \vee \mathbf{P}'(\mathrm{pr}_2)(\mathfrak{q}^m_{X_2}(\alpha_2)).
\end{aligned}$$

To conclude, the proof that condition (4) in Definition 5.9 is met is similar to the proof that F' contains the top elements of each fiber. This shows that I' is indeed a universal ideal for $\mathbf{P}'$.

We next show that, for every $X \in \mathsf{C}$, F'_X and I'_X are complementary. Fix $X \in \mathsf{C}$. To prove $F'_X \cup I'_X = \mathbf{P}'(X)$, let $\alpha \in \mathbf{P}'(X)$. There are $n \in \mathbb{N}$ and $\beta \in \mathbf{P}^n(X)$ such that $\alpha = \mathfrak{q}^n_X(\beta)$. Since F^n_X and I^n_X are complementary in $\mathbf{P}^n(X)$, $\beta \in F^n_X$ or $\beta \in I^n_X$, and hence $\alpha \in F'_X$ or $\alpha \in I'_X$. This proves $F'_X \cup I'_X = \mathbf{P}'(X)$. We now show that $F'_X \cap I'_X = \varnothing$. We take $\alpha \in F'_X \cap I'_X$ and we seek a contradiction. Since $\alpha \in F'_X$, there are $n \in \mathbb{N}$ and $\beta \in F^n_X$ such that $\alpha = \mathfrak{q}^n_X(\beta)$. Since $\alpha \in I'_X$, there are $m \in \mathbb{N}$ and $\gamma \in I^m_X$ such that $\alpha = \mathfrak{q}^m_X(\gamma)$. Since $\mathfrak{q}^n_X(\beta) = \mathfrak{q}^m_X(\gamma)$, there is $k \geq n, m$ such that $\mathfrak{r}^{n;k}_X(\beta) = \mathfrak{r}^{m;k}_X(\gamma)$, contradicting $F^k_X \cap I^k_X = \varnothing$.

This proves that F' is a universal ultrafilter (see Lemma 5.16).

We prove that $\mathbf{P}'$ is rich with respect to F'. Let $X \in \mathsf{C}$ and $\alpha \in \mathbf{P}'(X) \setminus F'_X$. We seek a morphism $c\colon \mathbf{t} \to X$ in C' such that $\mathbf{P}'(c)(\alpha) \notin F'_{\mathbf{t}}$. Since $\alpha \notin F'_X$, we have $\alpha \in I'_X$. So there are $n \in \mathbb{N}$ and $\beta \in I^n_X$ such that $\alpha = \mathfrak{q}^n_X(\beta)$. Since $\beta \in \mathbf{P}^n(X) \setminus F^n_X$, we use property (3) to get a morphism $d\colon \mathbf{t} \to X$ in C^n such that $\mathbf{P}^n(d)(\beta) \notin F^n_{\mathbf{t}}$, so $\mathbf{P}^n(d)(\beta) \in I^n_{\mathbf{t}}$. Setting $c := Q^n(d)$, we have

$$\mathbf{P}'(c)(\alpha) = \mathbf{P}'(Q^n(d))(\mathfrak{q}^n_X(\beta)) = \mathfrak{q}^n_{\mathbf{t}}(\mathbf{P}^n(d)(\beta)) \in I'_{\mathbf{t}}.$$

Therefore, $\mathbf{P}'(c)(\alpha) \notin F'_{\mathbf{t}}$, as desired.

Finally, we prove $F_X = \mathfrak{r}_X^{-1}[F'_X]$ for each $X \in \mathsf{C}$. Let $\alpha \in \mathbf{P}(X)$. If $\alpha \in F_X$, then $\mathfrak{r}_X(\alpha) = \mathfrak{q}^0_X(\alpha) \in F'_X$ by definition. Conversely, if $\alpha \notin F_X$, then $\alpha \in I_X$, so $\mathfrak{r}_X(\alpha) = \mathfrak{q}^0_X(\alpha) \in I'_X$, and thus $\mathfrak{r}_X(\alpha) \notin F'_X$. ☐

5.4. **Characterization of classes of universally valid formulas.** We have all the ingredients to prove the main theorem of this section, which shows that universal ultrafilters are precisely the classes of universally valid formulas.

Theorem 5.28. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $F = (F_X)_{X\in\mathsf{C}}$ be a family with $F_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$ such that, for every $X \in \mathsf{C}$,

$$F_X = \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\}.$$

(2) F is a universal ultrafilter for $\mathbf{P}$.

Proof. (1) ⇒ (2). We check that the conditions in Definition 5.12 are satisfied.

First, we prove that F is closed under reindexings. Take a morphism $f\colon X \to Y$ and $\alpha \in F_Y$, i.e. $\mathfrak{m}_Y(\alpha) = M(Y)$. Then $\mathbf{P}(f)(\alpha) \in F_X$ if and only if $\mathfrak{m}_X(\mathbf{P}(f)(\alpha)) = M(X)$. By naturality of $\mathfrak{m}$,

$$\mathfrak{m}_X(\mathbf{P}(f)(\alpha)) = M(f)^{-1}[\mathfrak{m}_Y(\alpha)] = M(f)^{-1}[M(Y)] = M(X).$$

Thus, F is closed under reindexing.

Second, we prove that F is fiberwise a filter. For every $X \in \mathsf{C}$ we have $F_X = \mathfrak{m}_X^{-1}[\{M(X)\}] = \mathfrak{m}_X^{-1}[\{\top_{\mathscr{P}(M(X))}\}]$, and this is a filter since it is the preimage under the Boolean homomorphism $\mathfrak{m}_X$ of the filter $\{M(X)\}$ of $\mathscr{P}(M(X))$.

Next, let $\alpha_1 \in \mathbf{P}(X_1) \setminus F_{X_1}$ and $\alpha_2 \in \mathbf{P}(X_2) \setminus F_{X_2}$. Then, there are $x_1 \in M(X_1) \setminus \mathfrak{m}_{X_1}(\alpha_1)$ and $x_2 \in M(X_2) \setminus \mathfrak{m}_{X_2}(\alpha_2)$. We show that $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \notin F_{X_1\times X_2}$. We have

$$\begin{aligned}
\mathfrak{m}_{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2)) &= \mathfrak{m}_{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)) \cup \mathfrak{m}_{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_2)(\alpha_2))\\
&= \mathrm{pr}_1^{-1}[\mathfrak{m}_{X_1}(\alpha_1)] \cup \mathrm{pr}_2^{-1}[\mathfrak{m}_{X_2}(\alpha_2)].
\end{aligned}$$

Observe that $(x_1, x_2) \in M(X_1) \times M(X_2) = M(X_1 \times X_2)$ does not belong to $\mathfrak{m}_{X_1 \times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2))$.

Finally, we have $\bot_{\mathbf{P}(\mathbf{t})} \notin F_{\mathbf{t}}$ since $\mathfrak{m}_{\mathbf{t}}(\bot_{\mathbf{P}(\mathbf{t})}) = \varnothing$.

(2) $\Rightarrow$ (1). By Theorem 5.27, there are a category C', a Boolean doctrine $\mathbf{P}' \colon \mathsf{C}'^{\mathrm{op}} \to \mathsf{BA}$, a Boolean doctrine morphism $(R, \mathfrak{r}) \colon \mathbf{P} \to \mathbf{P}'$ and a universal ultrafilter $(F'_Y)_{Y \in \mathsf{C}'}$ for $\mathbf{P}'$ such that $\mathbf{P}'$ is rich with respect to $(F'_Y)_{Y \in \mathsf{C}'}$, and moreover, for all $X \in \mathsf{C}$, $F_X = \mathfrak{r}_X^{-1}[F'_{R(X)}]$. By Proposition 5.25, there is a Boolean model $(M', \mathfrak{m}')$ of $\mathbf{P}'$ such that, for all $X \in \mathsf{C}$,

$$F'_Y = \{\alpha \in \mathbf{P}(Y) \mid \text{for all } x \in M'(Y),\, x \in \mathfrak{m}'_Y(\alpha)\}.$$

Let $(M, \mathfrak{m})$ be the composite $(M', \mathfrak{m}') \circ (R, \mathfrak{r})$ of the morphisms $(M', \mathfrak{m}')$ and $(R, \mathfrak{r})$. Clearly, $(M, \mathfrak{m})$ is a Boolean model of $\mathbf{P}$. Moreover, for every $X \in \mathsf{C}$ and every $\alpha \in \mathbf{P}(X)$, we have

$$\begin{aligned} \alpha \in F_X &\iff \mathfrak{r}_X(\alpha) \in F'_{R(X)} \\ &\iff \text{for all } x \in M'(R(X)),\, x \in \mathfrak{m}'_{R(X)}(\mathfrak{r}_X(\alpha)) \\ &\iff \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha). \end{aligned}$$

□

Note that the model obtained from the universal ultrafilter is not canonical: indeed, its existence was established using the extension to richness, which uses the axiom of choice.

Remark 5.29. We translate Theorem 5.28 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $(F_n)_{n \in \mathbb{N}}$ be a family with F_n a set of quantifier-free $\mathcal{L}$-formulas with $x_1, \dots, x_n$ as free (possibly dummy) variables. The following conditions are equivalent.

(1) There is a model M of $\mathcal{T}$ such that, for every $n \in \mathbb{N}$,

$$F_n = \{\alpha(x_1, \dots, x_n) \text{ quantifier-free} \mid M \vDash \forall x_1 \dots \forall x_n\, \alpha(x_1, \dots, x_n)\}.$$

(2) $(F_n)_{n \in \mathbb{N}}$ is a universal ultrafilter for $\mathcal{T}$ (in the sense of Remark 5.13).

Corollary 5.30. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and suppose C to be small. Let $\bar{i}, \bar{j} \in \mathbb{N}$, $Y_1, \dots, Y_{\bar{i}}, Z_1, \dots, Z_{\bar{j}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(Y_i))_{i=1,\dots,\bar{i}}$, and $(\beta_j \in \mathbf{P}(Z_j))_{j=1,\dots,\bar{j}}$. The following conditions are equivalent.*

(1) For every Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$, if for every $i \in \{1, \dots, \bar{i}\}$ we have that for every $y \in M(Y_i)$ $y \in \mathfrak{m}_{Y_i}(\alpha_i)$, then there is $j \in \{1, \dots, \bar{j}\}$ such that for every $z \in M(Z_j)$ $z \in \mathfrak{m}_{Z_j}(\beta_j)$.

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, and $(g_i \colon \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ such that (in $\mathbf{P}(\prod_{j=1}^{\bar{j}} Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(g_i)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\mathrm{pr}_j)(\beta_j).$$

Proof. By Theorem 5.28, condition (1) is equivalent to

(1') For every universal ultrafilter $(F_X)_{X \in \mathsf{C}}$, if for every $i \in \{1, \dots, \bar{i}\}$ we have $\alpha_i \in F_{Y_i}$, then there is $j \in \{1, \dots, \bar{j}\}$ such that $\beta_j \in F_{Z_j}$.

By Lemma 5.19, condition (2) is equivalent to

(2') The universal filter generated by $\alpha_1, \dots, \alpha_{\bar{i}}$ intersects the universal ideal generated by $\beta_1, \dots, \beta_{\bar{j}}$.

We prove that (1') is equivalent to (2'). To do so we prove that the negation of (1') is equivalent to the negation of (2').

(¬1') There is a universal ultrafilter $(F_X)_{X \in \mathsf{C}}$ such that for all $i \in \{1, \dots, \bar{i}\}$ we have $\alpha_i \in F_{Y_i}$ and for all $j \in \{1, \dots, \bar{j}\}$ we have $\beta_j \notin F_{Z_j}$.

(¬2') The universal filter generated by $\alpha_1, \dots, \alpha_{\bar{i}}$ and the universal ideal generated by $\beta_1, \dots, \beta_{\bar{j}}$ are fiberwise disjoint.

(¬1') $\Rightarrow$ (¬2'). This is immediate since a universal ultrafilter is a universal filter whose fiberwise complement is a universal ideal.

(¬2') $\Rightarrow$ (¬1'). This follows from Theorem 5.21. □

Remark 5.31. We translate Corollary 5.30 to the classic syntactic setting. Let $\{x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $\bar{i}, \bar{j} \in \mathbb{N}$, let $p_1, \dots, p_{\bar{i}}, q_1, \dots, q_{\bar{j}} \in \mathbb{N}$, let $(\alpha_i(x_1, \dots, x_{p_i}))_{i=1,\dots,\bar{i}}$ and $(\beta_j(x_1, \dots, x_{q_j}))_{j=1,\dots,\bar{j}}$ be tuples of quantifier-free $\mathcal{L}$-formulas. The following conditions are equivalent.

(1) For every model M of $\mathcal{T}$ we have

$$M \vDash \left(\bigwedge_{i=1}^{\bar{i}} \forall x_1 \dots \forall x_{p_i}\, \alpha(x_1, \dots, x_{p_i}) \right) \to \left(\bigvee_{j=1}^{\bar{j}} \forall x_1 \dots \forall x_{q_j}\, \beta_j(x_1, \dots, x_{q_j}) \right).$$

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ and terms $(g_i^h(x_1, \dots, x_{\Sigma_j q_j}))_{i \in \{1,\dots,n\},\, h \in \{1,\dots,p_{l_i}\}}$ such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(g_i^1(x_1, \dots, x_{\Sigma_j q_j}), \dots, g_i^{p_{l_i}}(x_1, \dots, x_{\Sigma_j q_j})) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \beta_j(x_{1+\Sigma_{t=1}^{j-1} q_t}, \dots, x_{\Sigma_{t=1}^{j} q_t}). \tag{5.10}$$

The unpleasant game of subscripts in (2) is just a way to ensure that the disjuncts β_j on the right-hand side of (5.10) have no variables in common. Note that, by the soundness and completeness theorems for first-order logic, (1) is equivalent to

$$\bigwedge_{i=1}^{\bar{i}} \forall x_1 \dots \forall x_{p_i}\, \alpha(x_1, \dots, x_{p_i}) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \forall x_1 \dots \forall x_{q_j}\, \beta_j(x_1, \dots, x_{q_j}).$$

Remark 5.31 characterizes when a finite conjunction of universal closures of quantifier-free formulas implies a finite disjunction of universal closures of quantifier-free formulas modulo a quantifier-free theory.

Example 5.32. For example, if $\alpha(x)$ is a quantifier-free formula and β is a quantifier-free closed formula, when does $\forall x\, \alpha(x)$ imply β modulo a given quantifier-free theory $\mathcal{T}$? To be more precise, we are in the setting of Remark 5.31 with $\bar{i} = 1$, $p_1 = 1$, $\bar{j} = 1$, $q_1 = 0$, $\alpha(x_1)$ a quantifier-free formula, and β a closed quantifier-free formula. Remark 5.31 tells us that $\forall x\, \alpha(x) \vdash_{\mathcal{T}} \beta$ occurs precisely when there are $n \in \mathbb{N}$ and 0-ary terms $g_1, \dots, g_n$ (i.e. term-definable constants) such that

$$\bigwedge_{i=1}^{n} \alpha(g_i) \vdash_{\mathcal{T}} \beta.$$

In other words, if we know that $\alpha(x)$ holds for all x, the only way to prove β is to instantiate $\alpha(x)$ on a finite number of constants $g_1, \dots, g_n$ and then prove β from $\alpha(g_1) \dots, \alpha(g_n)$.

Remark 5.33. Note that, in Example 5.32 above, it is important that n can also be given the value 0. For example, if $\beta = \top$ and the language has no term-definable constants, it is true that $\forall x\, \alpha(x)$ implies $\top$, but we cannot instantiate $\alpha(x)$ in any term-definable constants, and so we need permission to take $n = 0$.

Remark 5.34. Note that, in Example 5.32 above, it is also important that we are allowed to take $n \geq 2$. For example, let $\mathcal{L}$ be the language with two constant symbols $\{a, b\}$ and a unary predicate symbol R, and let $\mathcal{T} = \{R(a) \vee R(b)\}$. When does the theory $\mathcal{T}$ prove the formula $\exists x\, R(x)$? Or equivalently, when does $\forall x\, \neg R(x)$ imply $\bot$ modulo $\mathcal{T}$? By Example 5.32 with $\alpha = \neg R(x)$ and $\beta = \bot$, the ways to prove $\bot$ would be:

(1) $\top \vdash_{\mathcal{T}} \bot$;
(2) $\neg R(a) \vdash_{\mathcal{T}} \bot$;
(3) $\neg R(b) \vdash_{\mathcal{T}} \bot$;
(4) $\neg R(a) \wedge \neg R(b) \vdash_{\mathcal{T}} \bot$.

However,

(1) does not hold (as witnessed by the model $M = \{\bar{a}\}$, $\mathbb{I}(a) = \mathbb{I}(b) = \bar{a}$, $\mathbb{I}(R) = \{\bar{a}\}$);
(2) is equivalent to $\neg R(a) \wedge (R(a) \vee R(b)) \vdash \bot$, which in turn is equivalent to $R(b) \vdash R(a)$, which does not hold (take $M = \{\bar{a}, \bar{b}\}$, $\mathbb{I}(a) = \bar{a}$, $\mathbb{I}(b) = \bar{b}$, $\mathbb{I}(R) = \{\bar{b}\}$);

(3) is equivalent to $\neg R(b) \wedge (R(a) \vee R(b)) \vdash \bot$, which in turn is equivalent to $R(a) \vdash R(b)$, which does not hold (take $M = \{\bar{a}, \bar{b}\}$, $\mathbb{I}(a) = \bar{a}$, $\mathbb{I}(b) = \bar{b}$, $\mathbb{I}(R) = \{\bar{a}\}$);

(4) is equivalent to $\neg R(a) \wedge \neg R(b) \wedge (R(a) \vee R(b)) \vdash \bot$, which in turn is equivalent to $R(a) \vee R(b) \vdash R(a) \vee R(b)$, which holds.

So, in this example, it was necessary to take $n \geq 2$.

Example 5.35. For example, if $\alpha(x)$ and $\beta(y)$ are quantifier-free formulas, when does $\forall x\, \alpha(x)$ imply $\forall y\, \beta(y)$ modulo a given quantifier-free theory $\mathcal{T}$? To be more precise, we are in the setting of Remark 5.31 with $\bar{\imath} = 1$, $p_1 = 1$, $\bar{\jmath} = 1$, $q_1 = 1$, $\alpha(x_1), \beta(x_1)$ quantifier-free formulas. Remark 5.31 tells us that $\forall x\, \alpha(x) \vdash_{\mathcal{T}} \forall y\, \beta(y)$ occurs precisely when there are $n \in \mathbb{N}$ and unary terms $g_1(y), \dots, g_n(y)$ such that

$$\bigwedge_{i=1}^{n} \alpha(g_i(y)) \vdash_{\mathcal{T}} \beta(y).$$

In other words, if we know that $\alpha(x)$ holds for all x, the only way to prove $\beta(y)$ for an arbitrary y is to instantiate $\alpha(x)$ on a finite number of terms $g_1(y), \dots, g_n(y)$ depending solely on y and then prove $\beta(y)$ from $\alpha(g_1(y)) \dots, \alpha(g_n(y))$.

Example 5.36. For example, if $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ are quantifier-free formulas, when does $\forall x\, \alpha(x)$ imply $\forall y\, \beta(y) \vee \forall z\, \gamma(z)$ modulo a given quantifier-free theory $\mathcal{T}$? To be more precise, we are in the setting of Remark 5.31 with $\bar{\imath} = 1$, $p_1 = 1$, $\bar{\jmath} = 2$, $q_1 = q_2 = 1$, $\alpha(x_1), \beta(x_1), \gamma(x_1)$ quantifier-free formulas. Remark 5.31 tells us that $\forall x\, \alpha(x) \vdash_{\mathcal{T}} \forall y\, \beta(y) \vee \forall z\, \gamma(z)$ occurs precisely when there are $n \in \mathbb{N}$ and binary terms $g_1(y,z), \dots, g_n(y,z)$ such that

$$\bigwedge_{i=1}^{n} \alpha(g_i(y,z)) \vdash_{\mathcal{T}} \beta(y) \vee \gamma(z),$$

where here it is important that y and z are distinct variables. Note that $\forall y\, \beta(y) \vee \forall z\, \gamma(z)$ is equivalent to $\forall y \forall z\, \beta(y) \vee \gamma(z)$ (using that y and z are distinct). Then, if we know that $\alpha(x)$ holds for all x, the only way to prove $\beta(y) \vee \gamma(z)$ for arbitrary y and z is to instantiate $\alpha(x)$ on a finite number of terms $g_1(y,z), \dots, g_n(y,z)$ depending solely on y and z and then prove $\beta(y) \vee \gamma(z)$ from $\alpha(g_1(y,z)) \dots, \alpha(g_n(y,z))$.

Example 5.37. For example, if $\alpha_1(x)$ and $\alpha_2(y)$ are quantifier-free formulas and β is a quantifier-free closed formula, when does $\forall x\, \alpha_1(x) \wedge \forall y\, \alpha_2(y)$ imply β modulo a given quantifier-free theory $\mathcal{T}$? To be more precise, we are in the setting of Remark 5.31 with $\bar{\imath} = 2$, $p_1 = p_2 = 1$, $\bar{\jmath} = 1$, $q_1 = 0$, $\alpha_1(x_1), \alpha_2(x_1)$ quantifier-free formulas, and β a closed quantifier-free formula. Remark 5.31 tells us that $\forall x\, \alpha_1(x) \wedge \forall y\, \alpha_2(y) \vdash_{\mathcal{T}} \beta$ occurs precisely when there are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{0, 1\}$ and 0-ary terms $(g_i)_{i \in \{1,\dots,n\}}$ (i.e. term-definable constants) such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(g_i) \vdash_{\mathcal{T}} \beta.$$

Equivalently, this happens when there are $n_1, n_2 \in \mathbb{N}$, and 0-ary terms $(f_i)_{i \in \{1,\dots,n_1\}}$, $(f'_j)_{j \in \{1,\dots,n_2\}}$ (i.e. term-definable constants) such that

$$\bigwedge_{i=1}^{n_1} \alpha_1(f_i) \wedge \bigwedge_{j=1}^{n_2} \alpha_2(f'_j) \vdash_{\mathcal{T}} \beta.$$

In other words, if we know that $\alpha_1(x)$ holds for all x and that $\alpha_2(y)$ holds for all y, the only way to prove β is to instantiate $\alpha_1(x)$ on a finite number of constants $f_1, \dots, f_{n_1}$ and $\alpha_2(y)$ on a finite number of constants $f'_1, \dots, f'_{n_2}$ and then prove β from $\alpha_1(f_1) \dots, \alpha_1(f_{n_1}), \alpha_2(f'_1) \dots, \alpha_2(f'_{n_2})$.

Theorem 5.38. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and suppose C to be small. Let $\bar{\imath}, \bar{\jmath} \in \mathbb{N}$, $S, Y_1, \dots, Y_{\bar{\imath}}, Z_1, \dots, Z_{\bar{\jmath}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(S \times Y_i))_{i=1,\dots,\bar{\imath}}$, and $(\beta_j \in \mathbf{P}(S \times Z_j))_{j=1,\dots,\bar{\jmath}}$. The following conditions are equivalent.*

(1) *For every Boolean model $(M, \mathfrak{m}, s)$ of $\mathbf{P}$ at S, if for every $i \in \{1, \dots, \bar{i}\}$ we have that for every $y \in M(Y_i)$ $(s, y) \in \mathfrak{m}_{S \times Y_i}(\alpha_i)$, then there is $j \in \{1, \dots, \bar{j}\}$ such that for every $z \in M(Z_j)$ $(s, z) \in \mathfrak{m}_{S \times Z_j}(\beta_j)$.*

(2) *There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, and $(g_i \colon S \times \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ such that (in $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}} Z_j)$)*

$$\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i \rangle)(\alpha_{l_i}) \le \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1} \rangle)(\beta_j).$$

Proof. This follows from Corollary 5.30 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S and from Lemma 4.7. □

Remark 5.39. We translate Theorem 5.38 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \dots, s_k, x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $\bar{i}, \bar{j} \in \mathbb{N}$, let $p_1, \dots, p_{\bar{i}}, q_1, \dots, q_{\bar{j}} \in \mathbb{N}$, for each $i = 1, \dots, \bar{i}$ let $\alpha_i(s_1, \dots, s_k, x_1, \dots, x_{p_i})$ be a quantifier-free formula, and for each $j = 1, \dots, \bar{j}$ let $\beta_j(s_1, \dots, s_k, x_1, \dots, x_{q_j})$ be a quantifier-free formula. The following conditions are equivalent.

(1) For every model M of $\mathcal{T}$ and for every $c_1, \dots, c_k \in M$ the formula

$$\left(\bigwedge_{i=1}^{\bar{i}} \forall x_1 \dots \forall x_{p_i}\, \alpha_i(s_1, \dots, s_k, x_1, \dots, x_{p_i}) \right) \to \left(\bigvee_{j=1}^{\bar{j}} \forall x_1 \dots \forall x_{q_j}\, \beta_j(s_1, \dots, s_k, x_1, \dots, x_{q_j}) \right).$$

is valid in M under the variable assignment $[(s_i \mapsto c_i)_{i=1,\dots,k}]$.

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ and terms $(g_i^h(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}))_{i \in \{1,\dots,n\},\, h \in \{1,\dots,p_{l_i}\}}$ such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(s_1, \dots, s_k, g_i^1(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}), \dots, g_i^{p_{l_i}}(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}))$$
$$\vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \beta_j(s_1, \dots, s_k, x_{1+\Sigma_{t=1}^{j-1} q_t}, \dots, x_{\Sigma_{t=1}^{j} q_t}).$$

6. Free one-step construction

In this section, we describe how to freely add one layer of quantification to a Boolean doctrine $\mathbf{P}_0 \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ over a small base category C. In this way, we accomplish the goal announced at the end of Section 4. To be more precise, let $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i}) \colon \mathbf{P}_0 \to \mathbf{P}_0^{\forall}$ be the quantifier completion of $\mathbf{P}_0$, and let $\mathbf{P}_1$ be the subfunctor of $\mathbf{P}_0^{\forall}$ defined as in Definition 3.2(3): for every $S \in \mathsf{C}$, the fiber $\mathbf{P}_1(S)$ is the Boolean subalgebra of $\mathbf{P}_0^{\forall}(S)$ generated by the union of the images of $\mathbf{P}_0(S \times Y)$ under $\forall_S^Y \colon \mathbf{P}_0^{\forall}(S \times Y) \to \mathbf{P}_0^{\forall}(S)$, for Y ranging in C. Intuitively, $\mathbf{P}_1$ freely adds one layer of quantification to $\mathbf{P}_0$. In the first part of this section we explicitly describe $\mathbf{P}_1$ (Corollary 6.10). In the second part of this section we use this result to construct $\mathbf{P}_1$ via generators and relations (see Remark 6.19).

6.1. Fragment of depth 1 of the quantifier completion.

The main results of this subsection are Theorem 6.6 and Corollary 6.10, that characterize the order in the fibers $\mathbf{P}_1(S)$ in terms of relations on $\mathbf{P}_0$. To this aim, we use results from our "detour" in Section 5.

To get back to the question "When should a formula $(\forall x\, \alpha(x)) \wedge (\forall y\, \beta(y))$ be below another formula $(\forall z\, \gamma(z)) \vee (\forall w\, \delta(w))$?" proposed at the end of Section 4, we will obtain the following answer:

$$(\forall x\, \alpha(x)) \wedge (\forall y\, \beta(y)) \le (\forall z\, \gamma(z)) \vee (\forall w\, \delta(w))$$
$$\Updownarrow$$
every model of $\mathbf{P}_0$ satisfying $\forall x\, \alpha(x)$ and $\forall y\, \beta(y)$ also satisfies $\forall z\, \gamma(z)$ or $\forall w\, \delta(w)$
$$\Updownarrow$$

there are terms $t_1(z,w),\dots,t_n(z,w),s_1(z,w),\dots,s_m(z,w)$ such that

$$\bigwedge_{i=1}^{n}\alpha(t_i(z,w))\wedge\bigwedge_{j=1}^{m}\beta(s_j(z,w))\leq\gamma(z)\vee\delta(w).$$

(It is important here that z and w are distinct variables.)

Lemma 6.1. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{BA}$ be a universal Boolean doctrine. Let $X,Y\in\mathsf{C}$, $\alpha,\gamma\in\mathbf{P}(X)$ and $\beta\in\mathbf{P}(X\times Y)$. Then*

$$\alpha\leq\gamma\vee\forall_X^Y\beta \text{ in } \mathbf{P}(X)\iff\mathbf{P}(\mathrm{pr}_1)(\alpha)\leq\mathbf{P}(\mathrm{pr}_1)(\gamma)\vee\beta \text{ in } \mathbf{P}(X\times Y).$$

Proof.

$$\begin{aligned}
&\alpha\leq\gamma\vee\forall_X^Y\beta && \text{in } \mathbf{P}(X)\\
&\iff\alpha\wedge\neg\gamma\leq\forall_X^Y\beta && \text{in } \mathbf{P}(X)\\
&\iff\mathbf{P}(\mathrm{pr}_1)(\alpha\wedge\neg\gamma)\leq\beta && \text{in } \mathbf{P}(X\times Y)\\
&\iff\mathbf{P}(\mathrm{pr}_1)(\alpha)\wedge\neg\mathbf{P}(\mathrm{pr}_1)(\gamma)\leq\beta && \text{in } \mathbf{P}(X\times Y)\\
&\iff\mathbf{P}(\mathrm{pr}_1)(\alpha)\leq\mathbf{P}(\mathrm{pr}_1)(\gamma)\vee\beta && \text{in } \mathbf{P}(X\times Y).
\end{aligned}$$ □

Lemma 6.2. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{BA}$ be a universal Boolean doctrine, let $X,Y\in\mathsf{C}$, let $\alpha\in\mathbf{P}(Y)$ and let $f\colon X\to Y$ be a morphism in C. In $\mathbf{P}(X)$ we have*

$$\forall_X^Y\mathbf{P}(\mathrm{pr}_2)(\alpha)\leq\mathbf{P}(f)(\alpha).$$

Proof. By adjointness, for every $\gamma\in\mathbf{P}(X\times Y)$, we have $\mathbf{P}(\mathrm{pr}_1)\forall_X^Y(\gamma)\leq\gamma$ (in $\mathbf{P}(X\times Y)$). For $\gamma=\mathbf{P}(\mathrm{pr}_2)(\alpha)$ we get, in $\mathbf{P}(X\times Y)$,

$$\mathbf{P}(\mathrm{pr}_1)(\forall_X^Y\mathbf{P}(\mathrm{pr}_2)(\alpha))\leq\mathbf{P}(\mathrm{pr}_2)(\alpha).$$

Applying on both sides the reindexing $\mathbf{P}(\langle\mathrm{id}_X,f\rangle)\colon\mathbf{P}(X\times Y)\to\mathbf{P}(X)$ along $\langle\mathrm{id}_X,f\rangle\colon X\to X\times Y$, we obtain, in $\mathbf{P}(X)$,

$$\forall_X^Y\mathbf{P}(\mathrm{pr}_2)(\alpha)=\mathbf{P}(\langle\mathrm{id}_X,f\rangle)(\mathbf{P}(\mathrm{pr}_1)(\forall_X^Y\mathbf{P}(\mathrm{pr}_2)(\alpha)))\leq\mathbf{P}(\langle\mathrm{id}_X,f\rangle)(\mathbf{P}(\mathrm{pr}_2)(\alpha))=\mathbf{P}(f)(\alpha).$$ □

Lemma 6.3 ($\forall$ distributes over $\bigvee$ with disjoint variables)**.** *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{BA}$ be a universal Boolean doctrine, let $X_1,\dots,X_n\in\mathsf{C}$, and let $\alpha_i\in\mathbf{P}(X_i)$ for $i=1,\dots,n$. Then in $\mathbf{P}(\mathbf{t})$*

$$\bigvee_{i=1}^{n}\forall_{\mathbf{t}}^{X_i}\alpha_i=\forall_{\mathbf{t}}^{\Pi_iX_i}\left(\bigvee_{i=1}^{n}\mathbf{P}(\mathrm{pr}_i)(\alpha_i)\right).$$

Proof. For $n=0$, we shall check that $\bot_{\mathbf{P}(\mathbf{t})}=\forall_{\mathbf{t}}^{\mathbf{t}}\bot_{\mathbf{P}(\mathbf{t})}$, but this follows from Remark 3.3 ($\forall_{\mathbf{t}}^{\mathbf{t}}$ is the right adjoint of the identity of $\mathbf{P}(\mathbf{t})$, and hence it is the identity).

For $n=1$ the statement is trivially true.

So let $n=2$. We begin with the inequality ($\leq$). For $i=1,2$, from $\mathbf{P}(!_{X_i})(\forall_{\mathbf{t}}^{X_i}\alpha_i)\leq\alpha_i$ we get in $\mathbf{P}(X_1\times X_2)$

$$\mathbf{P}(!_{X_1\times X_2})(\forall_{\mathbf{t}}^{X_i}\alpha_i)=\mathbf{P}(\mathrm{pr}_i)\mathbf{P}(!_{X_i})(\forall_{\mathbf{t}}^{X_i}\alpha_i)\leq\mathbf{P}(\mathrm{pr}_i)(\alpha_i).$$

So in $\mathbf{P}(X_1\times X_2)$ we get

$$\mathbf{P}(!_{X_1\times X_2})(\forall_{\mathbf{t}}^{X_i}\alpha_i)\leq\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2),$$

which is equivalent to

$$\forall_{\mathbf{t}}^{X_i}\alpha_i\leq\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2))$$

in $\mathbf{P}(\mathbf{t})$. Hence

$$\forall_{\mathbf{t}}^{X_1}\alpha_1\vee\forall_{\mathbf{t}}^{X_2}\alpha_2\leq\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2)),$$

as desired.

We now prove the inequality ($\geq$), i.e. that in $\mathbf{P}(\mathbf{t})$

$$\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2))\leq\forall_{\mathbf{t}}^{X_1}\alpha_1\vee\forall_{\mathbf{t}}^{X_2}\alpha_2.$$

By Lemma 6.1, this is equivalent to

$$\mathbf{P}(!_{X_1})(\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2)))\leq\alpha_1\vee\mathbf{P}(!_{X_1})(\forall_{\mathbf{t}}^{X_2}\alpha_2) \tag{6.1}$$

in $\mathbf{P}(X_1)$. We now use the Beck-Chevalley condition:

$$\begin{tikzcd} \mathbf{t} & \mathbf{P}(X_2) \arrow[r, "\forall_{\mathbf{t}}^{X_2}"] \arrow[d, "\mathbf{P}(\mathrm{pr}_2)"'] & \mathbf{P}(\mathbf{t}) \arrow[d, "\mathbf{P}(!_{X_1})"] \\ X_1 \arrow[u, "!_{X_1}"] & \mathbf{P}(X_1\times X_2) \arrow[r, "\forall_{X_1}^{X_2}"'] & \mathbf{P}(X_1). \end{tikzcd}$$

So we can rewrite (6.1) as

$$\mathbf{P}(!_{X_1})(\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2)))\leq\alpha_1\vee\forall_{X_1}^{X_2}\mathbf{P}(\mathrm{pr}_2)(\alpha_2).$$

By Lemma 6.1 again, this is equivalent to the following inequality in $\mathbf{P}(X_1\times X_2)$:

$$\mathbf{P}(\mathrm{pr}_1)(\mathbf{P}(!_{X_1})(\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2))))\leq\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2),$$

which we rewrite as

$$\mathbf{P}(!_{X_1\times X_2})\forall_{\mathbf{t}}^{X_1\times X_2}(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2))\leq\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2).$$

We use again the Beck-Chevalley condition:

$$\begin{tikzcd} \mathbf{t} & \mathbf{P}(X_1\times X_2) \arrow[r, "\forall_{\mathbf{t}}^{X_1\times X_2}"] \arrow[d, "\mathbf{P}(\langle\mathrm{pr}_3,\mathrm{pr}_4\rangle)"'] & \mathbf{P}(\mathbf{t}) \arrow[d, "\mathbf{P}(!_{X_1\times X_2})"] \\ X_1\times X_2 \arrow[u, "!_{X_1\times X_2}"] & \mathbf{P}(X_1\times X_2\times X_1\times X_2) \arrow[r, "\forall_{X_1\times X_2}^{X_1\times X_2}"'] & \mathbf{P}(X_1\times X_2), \end{tikzcd}$$

and so we are left to prove

$$\forall_{X_1\times X_2}^{X_1\times X_2}\mathbf{P}(\langle\mathrm{pr}_3,\mathrm{pr}_4\rangle)(\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2)))\leq\mathbf{P}(\mathrm{pr}_1)(\alpha_1)\vee\mathbf{P}(\mathrm{pr}_2)(\alpha_2).$$

This follows from Lemma 6.2 with $f=\mathrm{id}_{X_1\times X_2}\colon X_1\times X_2\to X_1\times X_2$.

The statement (for an arbitrary n) follows by induction. □

Lemma 6.4 ($\forall$ distributes over $\bigvee$ with disjoint variables, over fixed free variables)**.** *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{BA}$ be a universal Boolean doctrine, let $S,X_1,\dots,X_n\in\mathsf{C}$, and let $\alpha_i\in\mathbf{P}(S\times X_i)$ for $i=1,\dots,n$. Then in $\mathbf{P}(S)$*

$$\bigvee_{i=1}^{n}\forall_S^{X_i}\alpha_i=\forall_S^{\prod_i X_i}\left(\bigvee_{i=1}^{n}\mathbf{P}(\langle\mathrm{pr}_1\,\mathrm{pr}_{i+1}\rangle)(\alpha_i)\right).$$

Proof. This follows from Lemma 6.3 applied to the universal Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S. □

Lemma 6.5. *Let $\mathbf{P}\colon\mathsf{C}^{\mathrm{op}}\to\mathsf{BA}$ be a Boolean doctrine, let $\mathbf{R}\colon\mathsf{D}^{\mathrm{op}}\to\mathsf{BA}$ be a universal Boolean doctrine and let $(M,\mathfrak{m})\colon\mathbf{P}\to\mathbf{R}$ be a Boolean doctrine morphism. Let $\bar{i},\bar{j}\in\mathbb{N}$, let $S,Y_1,\dots,Y_{\bar{i}},Z_1,\dots,Z_{\bar{j}}\in\mathsf{C}$, $(\alpha_i\in\mathbf{P}(S\times Y_i))_{i=1,\dots,\bar{i}}$ and $(\beta_j\in\mathbf{P}(S\times Z_j))_{j=1,\dots,\bar{j}}$. Suppose there are $n\in\mathbb{N}$, $l_1,\dots,l_n\in\{1,\dots,\bar{i}\}$ and $(g_i\colon S\times\prod_{j=1}^{\bar{j}}Z_j\to Y_{l_i})_{i=1,\dots,n}$ such that (in $\mathbf{P}(S\times\prod_{j=1}^{\bar{j}}Z_j)$)*

$$\bigwedge_{i=1}^{n}\mathbf{P}(\langle\mathrm{pr}_1,g_i\rangle)(\alpha_{l_i})\leq\bigvee_{j=1}^{\bar{j}}\mathbf{P}(\langle\mathrm{pr}_1,\mathrm{pr}_{j+1}\rangle)(\beta_j).$$

Then in $\mathbf{R}(M(S))$ we have

$$\bigwedge_{i=1}^{\bar{i}}\forall_{M(S)}^{M(Y_i)}\mathfrak{m}_{S\times Y_i}(\alpha_i)\leq\bigvee_{j=1}^{\bar{j}}\forall_{M(S)}^{M(Z_j)}\mathfrak{m}_{S\times Z_j}(\beta_j).$$

Proof. First, observe that in $\mathbf{R}(M(S))$ we have

$$\bigwedge_{i=1}^{\bar{i}} \forall_{M(S)}^{M(Y_i)} \mathfrak{m}_{S\times Y_i}(\alpha_i) \leq \bigwedge_{i=1}^{n} \forall_{M(S)}^{M(Y_{l_i})} \mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i}). \tag{6.2}$$

For every $i = 1, \dots, n$, by the adjunction $\mathbf{R}(\mathrm{pr}_1) \dashv \forall_{M(S)}^{M(Y_{l_i})}$ we have in $\mathbf{R}(M(S) \times M(Y_{l_i}))$

$$\mathbf{R}(\mathrm{pr}_1)(\forall_{M(S)}^{M(Y_{l_i})}(\mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i}))) \leq \mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i}).$$

Then, apply $\mathbf{R}(\langle \mathrm{pr}_1, M(g_i)\rangle)$ to both sides to get in $\mathbf{R}(M(S) \times \prod_{j=1}^{\bar{j}} M(Z_j))$

$$\begin{aligned}\mathbf{R}(\mathrm{pr}_1)(\forall_{M(S)}^{M(Y_{l_i})}(\mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i}))) &= \mathbf{R}(\langle \mathrm{pr}_1, M(g_i)\rangle)(\mathbf{R}(\mathrm{pr}_1)(\forall_{M(S)}^{M(Y_{l_i})}(\mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i})))) \\ &\leq \mathbf{R}(\langle \mathrm{pr}_1, M(g_i)\rangle)(\mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i})) \\ &= \mathfrak{m}_{S\times \Pi_j Z_j}(\mathbf{P}(\langle \mathrm{pr}_1, g_i\rangle)(\alpha_{l_i})). \end{aligned} \tag{6.3}$$

It follows that in $\mathbf{R}(M(S) \times \prod_{j=1}^{\bar{j}} M(Z_j))$

$$\begin{aligned}\mathbf{R}(\mathrm{pr}_1)\left(\bigwedge_{i=1}^{\bar{i}} \forall_{M(S)}^{M(Y_i)} \mathfrak{m}_{S\times Y_i}(\alpha_i)\right) &\leq \mathbf{R}(\mathrm{pr}_1)\left(\bigwedge_{i=1}^{n} \forall_{M(S)}^{M(Y_{l_i})} \mathfrak{m}_{S\times Y_{l_i}}(\alpha_{l_i})\right) && \text{by (6.2)} \\ &\leq \bigwedge_{i=1}^{n} \mathfrak{m}_{S\times \Pi_j Z_j}(\mathbf{P}(\langle \mathrm{pr}_1, g_i\rangle)(\alpha_{l_i})) && \text{by (6.3)} \\ &\leq \mathfrak{m}_{S\times \Pi_j Z_j}\left(\bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\beta_j)\right) && \text{by assumption} \\ &= \bigvee_{j=1}^{\bar{j}} \mathbf{R}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\mathfrak{m}_{S\times Z_j}(\beta_j)) && \text{by naturality of } \mathfrak{m}. \end{aligned}$$

By the adjunction $\mathbf{R}(\mathrm{pr}_1) \dashv \forall_{M(S)}^{\Pi_j M(Z_j)}$, we get in $\mathbf{R}(M(S))$

$$\bigwedge_{i=1}^{\bar{i}} \forall_{M(S)}^{M(Y_i)} \mathfrak{m}_{S\times Y_i}(\alpha_i) \leq \forall_{M(S)}^{\Pi_j M(Z_j)}\left(\bigvee_{j=1}^{\bar{j}} \mathbf{R}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\mathfrak{m}_{S\times Z_j}(\beta_j))\right). \tag{6.4}$$

Then apply Lemma 6.4 to the left-hand side of (6.4) to conclude the proof. $\square$

Theorem 6.6. *Let* $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine with* C *small, and let* $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P} \to \mathbf{P}^{\forall}$ *be a quantifier completion of* $\mathbf{P}$. *For all* $S, Y_1, \dots, Y_{\bar{i}}, Z_1, \dots, Z_{\bar{j}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(S\times Y_i))_{i=1,\dots,\bar{i}}$ *and* $(\beta_j \in \mathbf{P}(S\times Z_j))_{j=1,\dots,\bar{j}}$, *the following conditions are equivalent.*

(1) *In* $\mathbf{P}^{\forall}(S)$ *we have*

$$\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathfrak{i}_{S\times Y_i}(\alpha_i) \leq \bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathfrak{i}_{S\times Z_j}(\beta_j).$$

(2) *There are* $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ *and* $(g_i\colon S \times \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ *such that (in* $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}} Z_j)$*)*

$$\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i\rangle)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\beta_j).$$

Proof. (1) $\Rightarrow$ (2). We prove the contrapositive. Suppose (2) does not hold.

By Theorem 5.38, there is a Boolean model $(M, \mathfrak{m}, s)$ of $\mathbf{P}$ at S such that for all $i = 1, \ldots \bar{i}$ and for all $y \in M(Y_i)$ we have $(s, y) \in \mathfrak{m}_{S \times Y_i}(\alpha_i)$ and there is no $j \in \{1, \ldots, \bar{j}\}$ such that for all $z \in M(Z_j)$ $(s, z) \in \mathfrak{m}_{S \times Z_j}(\beta_j)$. Thus

$$\begin{aligned}&\bigcap_{i=1}^{\bar{i}} \{s' \in M(S) \mid \text{for all } y \in M(Y_i),\, (s', y) \in \mathfrak{m}_{S \times Y_i}(\alpha_i)\} \\ &\nsubseteq \bigcup_{j=1}^{\bar{j}} \{s' \in M(S) \mid \text{for all } z \in M(Z_j),\, (s', z) \in \mathfrak{m}_{S \times Z_j}(\beta_j)\},\end{aligned} \tag{6.5}$$

because s belongs to the intersection on the left of (6.5) but not to the union on the right.

By the universal property of the quantifier completion, there is a unique universal Boolean doctrine morphism $(M, \mathfrak{n})$ such that the following triangle commutes:

$$\begin{tikzcd} \mathbf{P} \arrow[r, "(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})"] \arrow[dr, "(M,\mathfrak{m})"'] & \mathbf{P}^{\forall} \arrow[d, dashed, "(M,\mathfrak{n})"] \\ & \mathscr{P}. \end{tikzcd}$$

The condition in (6.5) is equivalent to

$$\bigcap_{i=1}^{\bar{i}} \forall_{M(S)}^{M(Y_i)} \mathfrak{m}_{S \times Y_i}(\alpha_i) \nsubseteq \bigcup_{j=1}^{\bar{j}} \forall_{M(S)}^{M(Z_j)} \mathfrak{m}_{S \times Z_j}(\beta_j),$$

which we rewrite as

$$\bigcap_{i=1}^{\bar{i}} \forall_{M(S)}^{M(Y_i)} \mathfrak{n}_{S \times Y_i}(\mathfrak{i}_{S \times Y_i}(\alpha_i)) \nsubseteq \bigcup_{j=1}^{\bar{j}} \forall_{M(S)}^{M(Z_j)} \mathfrak{n}_{S \times Z_j}(\mathfrak{i}_{S \times Z_j}(\beta_j)),$$

which, by the commutativity of (2.1) in the definition of a universal Boolean doctrine morphism (Definition 2.7) is equivalent to

$$\bigcap_{i=1}^{\bar{i}} \mathfrak{n}_S(\forall_S^{Y_i} \mathfrak{i}_{S \times Y_i}(\alpha_i)) \nsubseteq \bigcup_{j=1}^{\bar{j}} \mathfrak{n}_S(\forall_S^{Z_j} \mathfrak{i}_{S \times Z_j}(\beta_j)),$$

which, since $\mathfrak{n}_S$ is a Boolean homomorphism, is equivalent to

$$\mathfrak{n}_S\left(\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathfrak{i}_{S \times Y_i}(\alpha_i)\right) \nsubseteq \mathfrak{n}_S\left(\bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathfrak{i}_{S \times Z_j}(\beta_j)\right).$$

Thus, by monotonicity of $\mathfrak{n}_S$, in $\mathbf{P}^{\forall}(S)$ we have

$$\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathfrak{i}_{S \times Y_i}(\alpha_i) \nleq \bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathfrak{i}_{S \times Z_j}(\beta_j).$$

(2) $\Rightarrow$ (1). This implication follows from Lemma 6.5. $\square$

Remark 6.7. We translate Theorem 6.6 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \ldots, s_k, x_1, x_2, x_3, \ldots\}$ be a set of variables, $\mathcal{L}$ a language, and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $\bar{i}, \bar{j} \in \mathbb{N}$, let $p_1, \ldots, p_{\bar{i}}, q_1, \ldots, q_{\bar{j}} \in \mathbb{N}$, for each $i = 1, \ldots, \bar{i}$ let $\alpha_i(s_1, \ldots, s_k, x_1, \ldots, x_{p_i})$ be a quantifier-free formula, and for each $j = 1, \ldots, \bar{j}$ let $\beta_j(s_1, \ldots, s_k, x_1, \ldots, x_{q_j})$ be a quantifier-free formula. The following conditions are equivalent.

(1)

$$\bigwedge_{i=1}^{\bar{i}} \forall x_1 \ldots \forall x_{p_i}\, \alpha_i(s_1, \ldots, s_k, x_1, \ldots, x_{p_i}) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \forall x_1 \ldots \forall x_{q_j}\, \beta_j(s_1, \ldots, s_k, x_1, \ldots, x_{q_j}).$$

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ and terms $(g_i^h(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}))_{i \in \{1,\dots,n\},\, h \in \{1,\dots,p_{l_i}\}}$ such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(s_1, \dots, s_k, g_i^1(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}), \dots, g_i^{p_{l_i}}(s_1, \dots, s_k, x_1, \dots, x_{\Sigma_j q_j}))$$
$$\vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \beta_j(s_1, \dots, s_k, x_{1+\Sigma_{t=1}^{j-1} q_t}, \dots, x_{\Sigma_{t=1}^{j} q_t}).$$

Theorem 6.8. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine with C small, and let $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i}) \colon \mathbf{P} \to \mathbf{P}^{\forall}$ be a quantifier completion of $\mathbf{P}$. For all $S, Y_1, \dots, Y_{\bar{i}}, W_1, \dots, W_{\bar{h}}, Z_1, \dots, Z_{\bar{j}}, V_1, \dots, V_{\bar{k}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(S \times Y_i))_{i=1,\dots,\bar{i}}$, $(\gamma_h \in \mathbf{P}(S \times W_h))_{h=1,\dots,\bar{h}}$, $(\beta_j \in \mathbf{P}(S \times Z_j))_{j=1,\dots,\bar{j}}$ and $(\delta_k \in \mathbf{P}(S \times V_k))_{k=1,\dots,\bar{k}}$, the following conditions are equivalent.*

(1) In $\mathbf{P}^{\forall}(S)$ we have

$$\left(\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathfrak{i}_{S \times Y_i}(\alpha_i)\right) \wedge \left(\bigwedge_{h=1}^{\bar{h}} \exists_S^{W_h} \mathfrak{i}_{S \times W_h}(\gamma_h)\right) \leq \left(\bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathfrak{i}_{S \times Z_j}(\beta_j)\right) \vee \left(\bigvee_{k=1}^{\bar{k}} \exists_S^{V_k} \mathfrak{i}_{S \times V_k}(\delta_k)\right).$$

(2) There are $n, n' \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, $l'_1, \dots, l'_n \in \{1, \dots, \bar{k}\}$, $(g_i \colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to Y_{l_i})_{i=1,\dots,n}$ and $(g'_k \colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to V_{l'_k})_{k=1,\dots,n'}$ such that (in $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}} Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i \rangle)(\alpha_{l_i}) \wedge \bigwedge_{h=1}^{\bar{h}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{h+\bar{j}+1} \rangle)(\gamma_h) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1} \rangle)(\beta_j) \vee \bigvee_{k=1}^{n'} \mathbf{P}(\langle \mathrm{pr}_1, g'_k \rangle)(\delta_{l'_k}).$$

Proof. Item (1) holds if and only if

$$\left(\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathfrak{i}_{S \times Y_i}(\alpha_i)\right) \wedge \left(\bigwedge_{k=1}^{\bar{k}} \forall_S^{V_k} \mathfrak{i}_{S \times V_k}(\neg\delta_k)\right) \leq \left(\bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathfrak{i}_{S \times Z_j}(\beta_j)\right) \vee \left(\bigvee_{h=1}^{\bar{h}} \forall_S^{W_h} \mathfrak{i}_{S \times W_h}(\neg\gamma_h)\right).$$

Applying Theorem 6.6, this is equivalent to the existence of $n, n' \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, $l'_1, \dots, l'_n \in \{1, \dots, \bar{k}\}$, $(g_i \colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to Y_{l_i})_{i=1,\dots,n}$ and $(g'_k \colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to V_{l'_k})_{k=1,\dots,n'}$ such that (in $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}} Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i \rangle)(\alpha_{l_i}) \wedge \bigwedge_{k=1}^{n'} \mathbf{P}(\langle \mathrm{pr}_1, g'_k \rangle)(\neg\delta_{l'_k}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1} \rangle)(\beta_j) \vee \bigvee_{h=1}^{\bar{h}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{h+\bar{j}+1} \rangle)(\neg\gamma_h),$$

which is equivalent to (2). ☐

Roughly speaking, this means that, given a quantifier-free theory $\mathcal{T}$,

$$\left(\bigwedge_{i=1}^{\bar{i}} \forall y_i\, \alpha_i(y_i)\right) \wedge \left(\bigwedge_{h=1}^{\bar{h}} \exists w_h\, \gamma_h(w_h)\right) \vdash_{\mathcal{T}} \left(\bigvee_{j=1}^{\bar{j}} \forall z_j\, \beta_j(z_j)\right) \vee \left(\bigvee_{k=1}^{\bar{k}} \exists v_k\, \delta_k(v_k)\right)$$

holds (where all the α_i's, γ_h's, β_j's and δ_k's are quantifier-free) if and only if, fixing arbitrary w_h and z_j (and supposing all these variables to be distinct), there are finitely many instantiations $(g_i)_{i=1,\dots,n}$ of the α_i's and finitely many instantiations $(g'_k)_{k=1,\dots,n'}$ of the δ_k's such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(g_i) \wedge \bigwedge_{h=1}^{\bar{h}} \gamma_h(w_h) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \beta_j(z_j) \vee \bigvee_{k=1}^{n'} \delta_{l'_k}(g'_k).$$

Example 6.9 (Herbrand's theorem)**.** Let $\mathcal{T}$ be a quantifier-free theory and let $\delta(x_1, \dots, x_v)$ be a quantifier-free formula, where $v \in \mathbb{N}$. By Theorem 6.8, the condition

$$\top \vdash_{\mathcal{T}} \exists x_1 \dots \exists x_v\, \delta(x_1, \dots, x_v).$$

holds if and only if there are $n \in \mathbb{N}$ and lists of 0-ary terms $(g_k^1, \dots, g_k^v)_{k=1,\dots,n}$ such that

$$\top \vdash_{\mathcal{T}} \bigvee_{k=1}^{n} \delta(g_k^1, \dots, g_k^v).$$

This is the classic Herbrand's theorem [11].

The results obtained so far allow us to characterize when a Boolean combination of universal quantifications of quantifier-free formulas implies another Boolean combination of universal quantifications of quantifier-free formulas. Modulo rewriting a Boolean combination in disjunctive/conjunctive normal form, this is illustrated in the following result.

Corollary 6.10. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine with C small, and let $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i}) \colon \mathbf{P} \to \mathbf{P}^{\forall}$ be its quantifier completion. For all $S, Y_1^p, \dots, Y_{\bar{i}_p}^p, W_1^p, \dots, W_{\bar{h}_p}^p \in \mathsf{C}$ (for $p = 1, \dots \bar{p}$), for all $Z_1^q, \dots, Z_{\bar{j}_q}^q, V_1^q, \dots V_{\bar{k}_q}^q \in \mathsf{C}$ (for $q = 1, \dots \bar{q}$), for all $(\alpha_i^p \in \mathbf{P}(S \times Y_i^p))_{p \in \{1,\dots,\bar{p}\}, i \in \{1,\dots,\bar{i}_p\}}$, $(\gamma_h^p \in \mathbf{P}(S \times W_h^p))_{p \in \{1,\dots,\bar{p}\}, h \in \{1,\dots,\bar{h}_p\}}$, $(\beta_j^q \in \mathbf{P}(S \times Z_j^q))_{q \in \{1,\dots,\bar{q}\}, j \in \{1,\dots,\bar{j}_q\}}$, $(\delta_k^q \in \mathbf{P}(S \times V_k^q))_{q \in \{1,\dots,\bar{q}\}, k \in \{1,\dots,\bar{k}_q\}}$ the following conditions are equivalent.*

(1) In $\mathbf{P}^{\forall}(S)$ we have

$$\bigvee_{p=1}^{\bar{p}} \left(\left(\bigwedge_{i=1}^{\bar{i}_p} \forall_S^{Y_i^p} \mathfrak{i}_{S \times Y_i^p}(\alpha_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg \forall_S^{W_h^p} \mathfrak{i}_{S \times W_h^p}(\gamma_h^p) \right) \right)$$
$$\leq \bigwedge_{q=1}^{\bar{q}} \left(\left(\bigvee_{j=1}^{\bar{j}_q} \forall_S^{Z_j^q} \mathfrak{i}_{S \times Z_j^q}(\beta_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg \forall_S^{V_k^q} \mathfrak{i}_{S \times V_k^q}(\delta_k^q) \right) \right).$$

(2) For all $p = 1, \dots, \bar{p}$ and $q = 1, \dots, \bar{q}$ there are $n, n' \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}_p\}$, $l'_1, \dots, l'_{n'} \in \{1, \dots, \bar{k}_q\}$, $(g_i \colon S \times \prod_{j=1}^{\bar{j}_q} Z_j^q \times \prod_{h=1}^{\bar{h}_p} W_h^p \to Y_{l_i}^p)_{i=1,\dots,n}$ and $(g'_k \colon S \times \prod_{j=1}^{\bar{j}_q} Z_j^q \times \prod_{h=1}^{\bar{h}_p} W_h^p \to V_{l'_i}^q)_{k=1,\dots,n'}$ such that (in $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}_q} Z_j^q \times \prod_{h=1}^{\bar{h}_p} W_h^p)$)

$$\left(\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i \rangle)(\alpha_{l_i}^p) \right) \wedge \left(\bigwedge_{k=1}^{n'} \mathbf{P}(\langle \mathrm{pr}_1, g'_k \rangle)(\delta_{l'_k}^q) \right)$$
$$\leq \left(\bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1} \rangle)(\beta_j^q) \right) \vee \left(\bigvee_{h=1}^{\bar{h}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{h+\bar{j}+1} \rangle)(\gamma_h^p) \right).$$

Proof. Item (1) holds if and only if for all $p = 1, \dots, \bar{p}$ and $q = 1, \dots, \bar{q}$ we have

$$\left(\bigwedge_{i=1}^{\bar{i}_p} \forall_S^{Y_i^p} \mathfrak{i}_{S \times Y_i^p}(\alpha_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg \forall_S^{W_h^p} \mathfrak{i}_{S \times W_h^p}(\gamma_h^p) \right) \leq \left(\bigvee_{j=1}^{\bar{j}_q} \forall_S^{Z_j^q} \mathfrak{i}_{S \times Z_j^q}(\beta_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg \forall_S^{V_k^q} \mathfrak{i}_{S \times V_k^q}(\delta_k^q) \right). \quad (6.6)$$

In turn, (6.6) holds if and only if

$$\left(\bigwedge_{i=1}^{\bar{i}_p} \forall_S^{Y_i^p} \mathfrak{i}_{S \times Y_i^p}(\alpha_i^p) \right) \wedge \left(\bigwedge_{k=1}^{\bar{k}_q} \forall_S^{V_k^q} \mathfrak{i}_{S \times V_k^q}(\delta_k^q) \right) \leq \left(\bigvee_{j=1}^{\bar{j}_q} \forall_S^{Z_j^q} \mathfrak{i}_{S \times Z_j^q}(\beta_j^q) \right) \vee \left(\bigvee_{h=1}^{\bar{h}_p} \forall_S^{W_h^p} \mathfrak{i}_{S \times W_h^p}(\gamma_h^p) \right).$$

Apply Theorem 6.6. □

Example 6.11. Let $\mathcal{T}$ be a quantifier-free theory. Let $\alpha(x)$ be a quantifier-free formula with free variable x and β be a closed quantifier-free formula. We write $\beta(x)$ when we consider x to be a dummy variable of β.

$$(\exists x\, \alpha(x)) \vee \beta \vdash_{\mathcal{T}} \exists x(\alpha(x) \vee \beta(x)) \iff \exists x\, \alpha(x) \vdash_{\mathcal{T}} \exists x(\alpha(x) \vee \beta(x)) \text{ and } \beta \vdash_{\mathcal{T}} \exists x(\alpha(x) \vee \beta(x))$$
$$\iff \beta \vdash_{\mathcal{T}} \exists x(\alpha(x) \vee \beta(x)).$$

By Theorem 6.8, seeing β as $\exists()\beta$, this is equivalent to the existence of $n \in \mathbb{N}$ and 0-ary terms $(f_i)_{i\in\{1,\dots,n\}}$ such that

$$\beta \vdash_{\mathcal{T}} \bigvee_{i=1}^{n} (\alpha(f_i) \vee \beta).$$

This trivially holds if $\beta \vdash_{\mathcal{T}} \bot$ (take $n = 0$). If instead $\beta \nvdash_{\mathcal{T}} \bot$ (so that we cannot take $n = 0$), this holds if and only if there is at least a 0-ary term in the language. Semantically, the fact that $(\exists x\, \alpha(x)) \vee \beta \vdash_{\mathcal{T}} \exists x(\alpha(x) \vee \beta(x))$ might fail is explained by the fact that we allow the empty model.

Finally, we mention that the converse direction

$$\exists x(\alpha(x) \vee \beta(x)) \vdash_{\mathcal{T}} (\exists x\, \alpha(x)) \vee \beta$$

always holds.

6.2. **The construction.** We now exhibit how to freely add one layer of quantification to a Boolean doctrine over a small base category via generators and relations. Let $\mathbf{P}_0 \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $S \in \mathsf{C}$. Let B_S be the free Boolean algebra over $A_S \coloneqq \bigsqcup_{Y\in\mathsf{C}} \mathbf{P}_0(S \times Y)$, and, for each $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}_0(S \times Y)$, let $\forall_S^Y \alpha$ denote the image of α under the free map $A_S \to B_S$. Let $\sim_S$ be the Boolean congruence on B_S generated by the following relations: for each $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ and $(g_i \colon S \times \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ such that (in $\mathbf{P}_0(S \times \prod_{j=1}^{\bar{j}} Z_j)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}_0(\langle \mathrm{pr}_1, g_i\rangle)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}_0(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\beta_j). \tag{6.7}$$

we impose the relation

$$\left[\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \alpha_i\right] \leq \left[\bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \beta_j\right]$$

in $B_S/{\sim_S}$.

Notation 6.12 (Free one-step construction on objects)**.** We let

$$\mathrm{Free}_1^{\mathbf{P}_0}(S)$$

denote the quotient $B_S/{\sim_S}$. (For an intrinsic description of $\sim$, we refer to Corollary 6.10.)

For each $S \in \mathsf{C}$, $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ is a Boolean algebra generated by the image of A_S under the function $A_S \to \mathrm{Free}_1^{\mathbf{P}_0}(S)$ that maps $\alpha \in \mathbf{P}_0(S \times Y) \subseteq A_S$ to $[\forall_S^Y \alpha] \in \mathrm{Free}_1^{\mathbf{P}_0}(S)$. We know another Boolean algebra generated by A_S. Since C is small, there is a quantifier completion $(\mathrm{id}_C, \mathfrak{i}) \colon \mathbf{P}_0 \to \mathbf{P}_0^{\forall}$ of $\mathbf{P}_0$. By Proposition 4.15, $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}_0^{\forall}$, and thus we can define $\mathbf{P}_1(S)$ the Boolean subalgebra of $\mathbf{P}_0^{\forall}(S)$ generated by the union of the images of $\mathbf{P}_0(S \times Y)$ under $\forall_S^Y \colon \mathbf{P}_0^{\forall}(S \times Y) \to \mathbf{P}_0^{\forall}(S)$, for Y ranging in C, as in Definition 3.2(3). So $\mathbf{P}_1(S)$ is generated by the image of the function $A_S \to \mathbf{P}_1(S)$ that maps $\alpha \in \mathbf{P}_0(S \times Y) \subseteq A_S$ to $\forall_S^Y \mathfrak{i}_{S\times Y}(\alpha) \in \mathbf{P}_1(S)$. In particular, both $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ and $\mathbf{P}_1(S)$ are quotients of the free algebra B_S over A_S. In the following, we prove that these two quotients are "the same", meaning that $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ and $\mathbf{P}_1(S)$ are isomorphic and the two quotient maps have the same kernel congruences.

$$\begin{tikzcd}
 & & \mathbf{P}_0^{\forall}(S \times Y) \arrow[rrr, "\forall_S^Y"] \arrow[rrrd, "\forall_S^Y"] & & & \mathbf{P}_0^{\forall}(S) & & \\
\alpha & \in & \mathbf{P}_0(S \times Y) \arrow[u, "\mathfrak{i}_{S\times Y}"] \arrow[r, dashed, "\subseteq"] \arrow[rr, bend right, "\forall_S^Y"'] & A_S \arrow[r] & B_S \arrow[r, two heads] \arrow[rd, two heads] & \mathbf{P}_1(S) \arrow[u, dashed, "\subseteq"] \arrow[d, dashed, leftrightarrow] & \ni & \forall_S^Y \mathfrak{i}_{S\times Y}(\alpha) \\
 & & & & & \mathrm{Free}_1^{\mathbf{P}_0}(S) & \ni & [\forall_S^Y \alpha]
\end{tikzcd}$$

Lemma 6.13. *Let B and P be boolean algebras, let A be a set, let $\iota\colon A \to B$ be a function such that $\iota[A]$ generates B, and let $\pi\colon B \to P$ be a boolean homomorphism. The kernel congruence of π coincides with the congruence $\sim$ generated by the following set of relations: for every $x_1, \dots, x_n, y_1, \dots, y_m \in A$ such that $\pi\iota(x_1) \wedge \dots \wedge \pi\iota(x_n) \leq \pi\iota(y_1) \vee \dots \vee \pi\iota(y_m)$ in P, take the relation $\iota(x_1) \wedge \dots \wedge \iota(x_n) \leq_\sim \iota(y_1) \vee \dots \vee \iota(y_n)$.*

Proof. Let $\approx$ be the kernel congruence of π, and let $\sim$ be the congruence generated by the relations in the statement. The inclusion $\sim \subseteq \approx$ holds because every generator of $\sim$ belongs to $\approx$. For the converse inclusion, let $\omega_1, \omega_2 \in B$ be such that $[\omega_1] \leq_\approx [\omega_2]$, and let us prove $[\omega_1] \leq_\sim [\omega_2]$. Since B is generated by $\iota[A]$, we can write

$$\omega_1 = \bigvee_{p=1}^{\bar{p}} \left(\left(\bigwedge_{i=1}^{\bar{i}_p} \iota(x_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg\iota(z_h^p) \right) \right)$$

and

$$\omega_2 = \bigwedge_{q=1}^{\bar{q}} \left(\left(\bigvee_{j=1}^{\bar{j}_q} \iota(y_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg\iota(w_k^q) \right) \right)$$

for appropriate elements $x_i^p, z_h^p, y_j^q, w_k^q \in A$. The condition $[\omega_1] \leq_\approx [\omega_2]$ means that, in P,

$$\bigvee_{p=1}^{\bar{p}} \left(\left(\bigwedge_{i=1}^{\bar{i}_p} \pi\iota(x_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg\pi\iota(z_h^p) \right) \right) \leq \bigwedge_{q=1}^{\bar{q}} \left(\left(\bigvee_{j=1}^{\bar{j}_q} \pi\iota(y_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg\pi\iota(w_k^q) \right) \right),$$

i.e. that, for every $p = 1, \dots, \bar{p}$ and every $q = 1, \dots \bar{q}$, in P,

$$\left(\bigwedge_{i=1}^{\bar{i}_p} \pi\iota(x_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg\pi\iota(z_h^p) \right) \leq \left(\bigvee_{j=1}^{\bar{j}_q} \pi\iota(y_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg\pi\iota(w_k^q) \right),$$

or, equivalently,

$$\left(\bigwedge_{i=1}^{\bar{i}_p} \pi\iota(x_i^p) \right) \wedge \left(\bigwedge_{k=1}^{\bar{k}_q} \pi\iota(w_k^q) \right) \leq \left(\bigvee_{j=1}^{\bar{j}_q} \pi\iota(y_j^q) \right) \vee \left(\bigvee_{h=1}^{\bar{h}_p} \pi\iota(z_h^p) \right).$$

By definition of $\sim$, for every $p = 1, \dots, \bar{p}$ and every $q = 1, \dots \bar{q}$

$$\left[\left(\bigwedge_{i=1}^{\bar{i}_p} \iota(x_i^p) \right) \wedge \left(\bigwedge_{k=1}^{\bar{k}_q} \iota(w_k^q) \right) \right] \leq_\sim \left[\left(\bigvee_{j=1}^{\bar{j}_q} \iota(y_j^q) \right) \vee \left(\bigvee_{h=1}^{\bar{h}_p} \iota(z_h^p) \right) \right],$$

and hence

$$\left[\bigvee_{p=1}^{\bar{p}} \left(\left(\bigwedge_{i=1}^{\bar{i}_p} \iota(x_i^p) \right) \wedge \left(\bigwedge_{h=1}^{\bar{h}_p} \neg\iota(z_h^p) \right) \right) \right] \leq_\sim \left[\bigwedge_{q=1}^{\bar{q}} \left(\left(\bigvee_{j=1}^{\bar{j}_q} \iota(y_j^q) \right) \vee \left(\bigvee_{k=1}^{\bar{k}_q} \neg\iota(w_k^q) \right) \right) \right],$$

i.e., $[\omega_1] \leq_\sim [\omega_2]$. □

Lemma 6.14. *Let P be a Boolean algebra, A a set, and $g\colon A \to P$ a function. Let $\iota\colon A \to B$ be the free Boolean algebra map over X, and let $\pi\colon B \to P$ be the Boolean morphism induced by the universal property of free algebras. The kernel congruence of π coincides with the congruence $\sim$ generated by the following relations: for every $x_1, \dots, x_n, y_1, \dots, y_m \in A$ such that $g(x_1) \wedge \dots \wedge g(x_n) \leq g(y_1) \vee \dots \vee g(y_m)$ in P, take the relation $\iota(x_1) \wedge \dots \wedge \iota(x_n) \leq_\sim \iota(y_1) \vee \dots \vee \iota(y_n)$. As a consequence, if $g[A]$ generates P, π induces an isomorphism between $B/{\sim}$ and P.*

Proof. By Lemma 6.13. □

Proposition 6.15. *Let $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, let $(\mathrm{id}_{\mathsf{C}}, \mathfrak{i})\colon \mathbf{P}_0 \to \mathbf{P}_0^\forall$ be a quantifier completion of $\mathbf{P}_0$. Let $\mathbf{P}_1$ be defined from $\mathbf{P}_0$ and $\mathbf{P}^\forall$ as in Definition 3.2(3). Then for every $S \in \mathsf{C}$ the Boolean algebras $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ and $\mathbf{P}_1(S)$ are isomorphic.*

Proof. First of all, recall from Proposition 4.15 that $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}_0^{\forall}$, so we can indeed define $\mathbf{P}_1$ as in the statement.

Apply Lemma 6.14 with $A = A_S$, $P = \mathbf{P}_1(S)$, $g\colon A_S \to \mathbf{P}_1(S)$ is the function that maps $\alpha \in \mathbf{P}_0(S \times Y) \subseteq A_S$ to $\forall_S^Y \mathbf{i}_{S\times Y}(\alpha) \in \mathbf{P}_1(S)$, so that $B = B_S$, the free Boolean algebra over A_S. Then $\mathbf{P}_1(S)$ is isomorphic to the quotient of B_S by the congruence $\sim$ defined by the following set of relations: for all $Y_1, \dots, Y_{\bar{i}}, Z_1, \dots, Z_{\bar{j}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(S \times Y_i))_{i=1,\dots,\bar{i}}$ and $(\beta_j \in \mathbf{P}(S \times Z_j))_{j=1,\dots,\bar{j}}$ such that in $\mathbf{P}_1(S)$ we have

$$\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathbf{i}_{S\times Y_i}(\alpha_i) \leq \bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathbf{i}_{S\times Z_j}(\beta_j). \tag{6.8}$$

take the relation

$$\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \alpha_i \leq_{\sim} \bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \beta_j.$$

By Theorem 6.6, (6.8) is equivalent to (6.7), and so $\sim$ coincides with $\sim_S$. Therefore, $\mathbf{P}_1(S)$ is isomorphic to $B_S/{\sim} = B_S/{\sim_S} = \mathrm{Free}_1^{\mathbf{P}_0}(S)$. □

Remark 6.16. Similarly to [22] (see Remark 4.26), we could have constructed $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ as the poset reflection of a certain preordered set. For example, as a preordered set we may take $\mathscr{P}_\omega(\mathscr{P}_\omega(A_S \sqcup A_S))$, where $\mathscr{P}_\omega(Y)$ denotes the set of finite subsets of Y, and with a preorder suggested by Corollary 6.10 according to the intuition that an element $\mathcal{A} \in \mathscr{P}_\omega(\mathscr{P}_\omega(A_S \sqcup A_S))$ represents

$$\bigvee_{A\in\mathcal{A}} \left(\left(\bigwedge_{a\in A_S \,:\, \mathrm{inl}\, a\in A} \forall_S^{Y_a} a \right) \wedge \left(\bigwedge_{a\in A_S \,:\, \mathrm{inr}\, a\in A} \neg\forall_S^{Y_a} a \right) \right),$$

where $\mathrm{inl}, \mathrm{inr}\colon A_S \to A_S \sqcup A_S$ denote the two inclusions. We are afraid we are not able to describe this in a digestible way.

We can extend the assignment $\mathrm{Free}_1^{\mathbf{P}_0}$ to morphisms of C using the isomorphisms described in Proposition 6.15: let S, S' be objects in C and let $f\colon S \to S'$ be a morphism. We let

$$\mathrm{Free}_1^{\mathbf{P}_0}(f)\colon \mathrm{Free}_1^{\mathbf{P}_0}(S') \to \mathrm{Free}_1^{\mathbf{P}_0}(S)$$

denote the Boolean homomorphism that closes the square below:

$$\begin{tikzcd}
S' & \mathrm{Free}_1^{\mathbf{P}_0}(S') \arrow[r, "\sim"] \arrow[d, dashed, "\mathrm{Free}_1^{\mathbf{P}_0}(f)"'] & \mathbf{P}_1(S') \arrow[d, "\mathbf{P}_1(f)"] \\
S \arrow[u, "f"] & \mathrm{Free}_1^{\mathbf{P}_0}(S) \arrow[r, "\sim"'] & \mathbf{P}_1(S).
\end{tikzcd}$$

The Boolean homomorphism $\mathbf{P}_1(f)$ is defined because $\mathbf{P}_1$ is a functor (see Lemma 3.13). It follows that $\mathrm{Free}_1^{\mathbf{P}_0}$ is a functor, naturally isomorphic to $\mathbf{P}_1$.

Lemma 6.17. *For every $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}_0(S' \times S)$, the Boolean homomorphism $\mathrm{Free}_1^{\mathbf{P}_0}(f)$ maps the generator $[\forall_{S'}^Y \alpha]$ to $[\forall_S^Y(\mathbf{P}_0(f \times \mathrm{id}_Y)(\alpha))]$.*

Proof. For the reader's convenience, we insert the following commutative diagram, which includes some relevant morphisms.

$$
\begin{tikzcd}
 & B_{S'} \arrow[d, two heads] \arrow[dr, two heads] & & \\
S' & \mathrm{Free}_1^{\mathbf{P}_0}(S') \arrow[r, "\sim"] \arrow[d, dashed, "\mathrm{Free}_1^{\mathbf{P}_0}(f)"'] & \mathbf{P}_1(S') \arrow[r, "\subseteq"] \arrow[d, "\mathbf{P}_1(f)"] & \mathbf{P}_0^{\forall}(S') \arrow[d, "\mathbf{P}_0^{\forall}(f)"] \\
S \arrow[u, "f"] & \mathrm{Free}_1^{\mathbf{P}_0}(S) \arrow[r, "\sim"'] & \mathbf{P}_1(S) \arrow[r, "\subseteq"'] & \mathbf{P}_0^{\forall}(S) \\
 & B_S \arrow[u, two heads] \arrow[ur, two heads] & &
\end{tikzcd}
\tag{6.9}
$$

Let $[\forall_{S'}^{Y}\alpha]$ be a generator in $\mathrm{Free}_1^{\mathbf{P}_0}(S')$ for $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}_0(S' \times Y)$, which corresponds to the element $\forall_{S'}^{Y}\mathfrak{i}_{S \times Y'}(\alpha)$ in $\mathbf{P}_1(S')$ under the isomorphism $\mathbf{P}_1(S') \cong \mathrm{Free}_1^{\mathbf{P}_0}(S')$. In $\mathbf{P}_1(S)$ we have

$$
\begin{aligned}
\mathbf{P}_1(f)(\forall_{S'}^{Y}\mathfrak{i}_{S' \times Y}(\alpha)) &= \mathbf{P}_0^{\forall}(f)(\forall_{S'}^{Y}\mathfrak{i}_{S' \times Y}(\alpha)) && \text{by (6.9) on the right} \\
&= \forall_S^Y \mathbf{P}_0^{\forall}(f \times \mathrm{id}_Y)(\mathfrak{i}_{S' \times Y}(\alpha)) && \text{by the Beck-Chevalley condition} \\
&= \forall_S^Y \mathfrak{i}_{S \times Y}(\mathbf{P}_0(f \times \mathrm{id}_Y)(\alpha)) && \text{by naturality of } \mathfrak{i},
\end{aligned}
$$

which, under the isomorphism $\mathbf{P}_1(S) \cong \mathrm{Free}_1^{\mathbf{P}_0}(S)$, corresponds to the element $[\forall_S^Y \mathbf{P}_0(f \times \mathrm{id}_Y)(\alpha)]$ in $\mathrm{Free}_1^{\mathbf{P}_0}(S)$, as desired. □

Then, the following is a direct definition of $\mathrm{Free}_1^{\mathbf{P}_0}(f)$ without going through the isomorphism with $\mathbf{P}_1$.

Notation 6.18 (Free one-step construction on morphisms)**.** Let S, S' be objects in C and let $f\colon S \to S'$ be a morphism. We let

$$\mathrm{Free}_1^{\mathbf{P}_0}(f)\colon \mathrm{Free}_1^{\mathbf{P}_0}(S') \to \mathrm{Free}_1^{\mathbf{P}_0}(S)$$

denote the unique Boolean homomorphism that, for all $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}_0(S' \times Y)$, maps the generator $[\forall_{S'}^{Y}\alpha]$ to $[\forall_S^Y(\mathbf{P}_0(f \times \mathrm{id}_Y)(\alpha))]$.

We skip a direct proof of well-definedness of the map in Notation 6.18; at any rate, using the isomorphism between $\mathrm{Free}_1^{\mathbf{P}_0}(S) \cong \mathbf{P}_1(S)$, this follows from Lemma 6.17.

Remark 6.19. As a recap, let $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. To add freely a layer of quantifiers via generators and relations, define the functor $\mathrm{Free}_1^{\mathbf{P}_0}$ on objects as in Notation 6.12, on morphisms as in Notation 6.18, and define the connecting natural transformation $\mathbf{P}_0 \to \mathrm{Free}_1^{\mathbf{P}_0}$ by defining the component at $S \in \mathsf{C}$ as

$$
\begin{aligned}
\mathbf{P}_0(S) &\longrightarrow \mathrm{Free}_1^{\mathbf{P}_0}(S) \\
\alpha &\longmapsto [\forall_S^{\mathbf{t}}\alpha].
\end{aligned}
$$

In light of the isomorphism between $\mathrm{Free}_1^{\mathbf{P}_0}$ and the fragment $\mathbf{P}_1$ of $\mathbf{P}_0^{\forall}$ (Proposition 6.15), Corollary 6.10 provides an explicit description of the order on $\mathrm{Free}_1^{\mathbf{P}_0}$.

7. Future work

One long-term goal is to provide a stepwise construction of the quantifier completion $\mathbf{P}^{\forall}$ of a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with C small. In this paper we have addressed the first step.

Let $(\mathbf{P}_n)_{n \in \mathbb{N}}$ be a quantifier-stratified universal Boolean doctrine. For each $n \in \mathbb{N}$, what are the properties satisfied by the tuple $(\mathbf{P}_0, \ldots, \mathbf{P}_n)$? In this paper, we only addressed the case $\mathbf{P}_0$. This corresponds precisely to the following, for the case of a small base category: $\mathbf{P}_0$ is a Boolean doctrine. This follows from the existence of the quantifier completion and the completeness theorem for Boolean doctrines (Theorem 4.9).

7.1. **Completeness for the first layer.**

Definition 7.1 (One-step quantifier Boolean doctrine)**.** A *one-step quantifier Boolean doctrine* is an ordered pair of functors $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1}$ where C is a category with finite products, such that, for every $X \in \mathsf{C}$, $\mathbf{P}_0(X)$ is a Boolean subalgebra of $\mathbf{P}_1(X)$, for every morphism $f\colon X \to X'$ in C the function $\mathbf{P}_1(f)\colon \mathbf{P}_1(X') \to \mathbf{P}_1(X)$ extends the function $\mathbf{P}_0(f)\colon \mathbf{P}_0(X') \to \mathbf{P}_0(X)$, and the following conditions hold.

(1) (One-step universal) For every projection $\mathrm{pr}_1\colon X \times Y \to X$ in C and $\beta \in \mathbf{P}_0(X \times Y)$ there is an element $\forall_X^Y \beta \in \mathbf{P}_1(X)$ such that, for all $\alpha \in \mathbf{P}_1(X)$, we have (denoting with $i_{X\times Y}$ the inclusion $\mathbf{P}_0(X \times Y) \hookrightarrow \mathbf{P}_1(X \times Y)$)
$$\alpha \leq \forall_X^Y \beta \text{ in } \mathbf{P}_1(X) \iff \mathbf{P}_1(\mathrm{pr}_1)(\alpha) \leq i_{X\times Y}(\beta) \text{ in } \mathbf{P}_1(X \times Y).$$
(Note that one such element $\forall_X^Y \beta$ is unique.)

(2) (One-step Beck-Chevalley) For every morphism $f\colon X \to X'$ in C the following diagram in Pos commutes.
$$\begin{tikzcd}
\mathbf{P}_0(X' \times Y) \arrow[r, "\forall_{X'}^Y"] \arrow[d, "\mathbf{P}_0(f\times \mathrm{id}_Y)"'] & \mathbf{P}_1(X') \arrow[d, "\mathbf{P}_1(f)"] \\
\mathbf{P}_0(X \times Y) \arrow[r, "\forall_X^Y"'] & \mathbf{P}_1(X)
\end{tikzcd}$$

(3) (One-step generation) For all $X \in \mathsf{C}$, the Boolean algebra $\mathbf{P}_1(X)$ is generated by the union of the images of the functions $\forall_X^Y\colon \mathbf{P}_0(X \times Y) \to \mathbf{P}_1(X)$ for Y ranging in C.

The idea that led us to this definition was to take the axioms from Definition 3.12 that only involve $n = 0, 1$. It seems a reasonable conjecture that Definition 7.1 captures all the properties satisfied by $(\mathbf{P}_0, \mathbf{P}_1)$ in a quantifier-stratified universal Boolean doctrine $(\mathbf{P}_n)_{n\in\mathbb{N}}$. One way to check this would be by using models. There is a natural way to define a *one-step quantifier Boolean model* of $(\mathbf{P}_0, \mathbf{P}_1)$ as the Boolean models of $\mathbf{P}_1$ preserving all the relevant structure. The fact that Definition 7.1 is the correct one then would be guaranteed by the following:

Conjecture 7.2 (Completeness for one-step quantifier Boolean doctrines)**.** *Let $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1}$ be a one-step quantifier Boolean doctrine, with C small, let $S \in \mathsf{C}$ and let $\varphi, \psi \in \mathbf{P}_1(S)$ be such that $\varphi \nleq \psi$. There is a one-step quantifier Boolean model $(M, \mathfrak{m})$ of $(\mathbf{P}_0, \mathbf{P}_1)$ such that $\mathfrak{m}_S(\varphi) \nsubseteq \mathfrak{m}_S(\psi)$.*

We refrained from introducing one-step quantifier Boolean doctrines in the body of the paper, as we are waiting until we prove Conjecture 7.2 (or until we find possible additional conditions to be added to the definition of one-step quantifier Boolean doctrine that make the conjecture true). At any rate, whatever the correct conditions on a pair $(\mathbf{P}_0, \mathbf{P}_1)$ are, the construction in Section 6.2 provides the free construction over a Boolean doctrine over a small base category.

7.2. **Step from 0 and 1 to 2, and beyond.** Similarly to what we did in Section 6, the next goal is to provide a free construction of $\mathbf{P}_2$ given a one-step quantifier Boolean doctrine $(\mathbf{P}_0, \mathbf{P}_1)$. Then we conjecture that the following definition (obtained from Definition 3.12 that only involve $n = 0, 1, 2$) captures all the properties satisfied by $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2)$ in a quantifier-stratified universal Boolean doctrine $(\mathbf{P}_n)_{n\in\mathbb{N}}$.

Definition 7.3 (Two-step quantifier Boolean doctrine)**.** A *two-step quantifier Boolean doctrine* is an ordered triple of functors $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,2}$ where C is a category with finite products, such that, for every $X \in \mathsf{C}$ and $n \in \{0, 1\}$, $\mathbf{P}_n(X)$ is a Boolean subalgebra of $\mathbf{P}_{n+1}(X)$, for every morphism $f\colon X' \to X$ in C and every $n \in \{0, 1\}$, the function $\mathbf{P}_{n+1}(f)\colon \mathbf{P}_{n+1}(X) \to \mathbf{P}_{n+1}(X')$ extends the function $\mathbf{P}_n(f)\colon \mathbf{P}_n(X) \to \mathbf{P}_n(X')$, and the following conditions hold.

(1) (Two-step universal) For every projection $\mathrm{pr}_1\colon X \times Y \to X$ in C, $n \in \{0, 1\}$, and $\beta \in \mathbf{P}_n(X \times Y)$ there is an element $\forall_{X,n}^Y \beta \in \mathbf{P}_{n+1}(X)$ such that, for every $\alpha \in \mathbf{P}_{n+1}(X)$, we have (denoting with $i_{X\times Y,n}$ the inclusion of $\mathbf{P}_n(X \times Y)$ into $\mathbf{P}_{n+1}(X \times Y)$)
$$\alpha \leq \forall_{X,n}^Y \beta \text{ in } \mathbf{P}_{n+1}(X) \iff \mathbf{P}_{n+1}(\mathrm{pr}_1)(\alpha) \leq i_{X\times Y,n}(\beta) \text{ in } \mathbf{P}_{n+1}(X \times Y).$$
(Note that one such element $\forall_{X,n}^Y \beta$ is unique)

(2) (One-step Beck-Chevalley) For every morphism $f\colon X' \to X$ in C and $n \in \{0,1\}$ the following diagram in Pos commutes.

$$\begin{tikzcd}
\mathbf{P}_n(X' \times Y) \arrow[r, "\forall^Y_{X',n}"] \arrow[d, "\mathbf{P}_n(f \times \mathrm{id}_Y)"'] & \mathbf{P}_{n+1}(X') \arrow[d, "\mathbf{P}_{n+1}(f)"] \\
\mathbf{P}_n(X \times Y) \arrow[r, "\forall^Y_{X,n}"'] & \mathbf{P}_{n+1}(X)
\end{tikzcd}$$

(3) (Restriction of universal) For all $X, Y \in \mathsf{C}$, the map $\forall^Y_{X,1}$ restricts to $\forall^Y_{X,0}$, i.e. the following diagram in Pos commutes.

$$\begin{tikzcd}
\mathbf{P}_0(X \times Y) \arrow[r, "\forall^Y_{X,n}"] \arrow[d, hook, "i_{X \times Y,0}"'] & \mathbf{P}_1(X) \arrow[d, hook, "i_{X,1}"] \\
\mathbf{P}_1(X \times Y) \arrow[r, "\forall^Y_{X,1}"'] & \mathbf{P}_2(X)
\end{tikzcd}$$

(4) (Generation) For all $X \in \mathsf{C}$ and $n \in \{0,1\}$, the Boolean algebra $\mathbf{P}_{n+1}(X)$ is generated by the union of the images of the functions $\forall^Y_{X,n}\colon \mathbf{P}_n(X \times Y) \to \mathbf{P}_{n+1}(X)$ for Y ranging in C.

Again, there is a natural way to define a *two-step quantifier Boolean model* of $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2)$ as the Boolean models of $\mathbf{P}_2$ preserving all the relevant structure. The fact that Definition 7.3 is the correct one then would be guaranteed by the following analogue of Conjecture 7.2:

Conjecture 7.4 (Completeness for two-step quantifier Boolean doctrines)**.** *Let $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,2}$ be a two-step quantifier Boolean doctrine, with C small, let $S \in \mathsf{C}$ and let $\varphi, \psi \in \mathbf{P}_2(S)$ be such that $\varphi \nleq \psi$. There is a two-step quantifier Boolean model $(M, \mathfrak{m})$ of $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2)$ such that $\mathfrak{m}_S(\varphi) \nsubseteq \mathfrak{m}_S(\psi)$.*

The work above might be enough to obtain the free construction of $\mathbf{P}_n$ from $(\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{n-1})$ for any $n \geq 3$, by applying the free construction of the second layer to the one-step quantifier Boolean doctrine $(\mathbf{P}_{n-1}, \mathbf{P}_n)$. This is because we conjecture the following to be the correct axiomatization of the tuples $(\mathbf{P}_0, \ldots, \mathbf{P}_n)$ arising from a quantifier-stratified universal Boolean doctrine.

Definition 7.5 (n-step quantifier Boolean doctrine)**.** An *n-step quantifier Boolean doctrine* is an ordered list of functors $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,\ldots,n}$, where C is a category with finite products, such that, for all $i = 0, \ldots, n-2$, $(\mathbf{P}_i, \mathbf{P}_{i+1}, \mathbf{P}_{i+2})$ is a two-step quantifier Boolean doctrine.

Once more, there is a natural way to define an *n-step quantifier Boolean model* of $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,\ldots,n}$ as the Boolean models of $\mathbf{P}_n$ preserving all the relevant structure. The fact that Definition 7.5 is the correct one then would be guaranteed by the following analogues of Conjectures 7.2 and 7.4:

Conjecture 7.6 (Completeness for n-step quantifier Boolean doctrines)**.** *Let $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,\ldots,n}$ be an n-step quantifier Boolean doctrine, with C small, let $S \in \mathsf{C}$ and let $\varphi, \psi \in \mathbf{P}_n(S)$ be such that $\varphi \nleq \psi$. There is an n-step quantifier Boolean model $(M, \mathfrak{m})$ of $(\mathbf{P}_i\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA})_{i=0,1,\ldots,n}$ such that $\mathfrak{m}_S(\varphi) \nsubseteq \mathfrak{m}_S(\psi)$.*

7.3. **Bounded distributive lattices.** A direction of further research is to generalize from Boolean algebras to bounded distributive lattices. In this case, existential and universal quantifiers are not interdefinable, making the resulting theory somewhat more complicated.

7.4. **Polyadic spaces.** We recall that a *Stone space* (also known as a *Boolean space* or a *profinite space*) is a compact Hausdorff space in which distinct points are separated by closed open sets. We let Stone denote the category of Stone spaces and continuous functions between them. Stone duality [20] establishes a dual equivalence of categories between Stone and BA. There are two main advantages to utilizing duality. Firstly, duality theory often connects syntax and semantics. For instance, in classical propositional logic, the Lindenbaum-Tarski algebra is the free Boolean algebra on the set V of propositional variables, and its dual space is the Cantor space 2^V of all valuations over V. The second advantage is that it is often easier, technically, to solve a problem on the dual side.

Given a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, we obtain a functor $\mathbf{E}\colon \mathsf{C} \to \mathsf{Stone}$ by composing $\mathbf{P}$ with Stone duality. When the Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ is also universal, its pointwise dual $\mathbf{E}\colon \mathsf{C} \to \mathsf{Stone}$ is a *polyadic space*. This notion was introduced (in a more restrictive form) by Joyal in the preliminary report [12]. Joyal's terminology *polyadic space* is inspired by Halmos' *polyadic algebras*, and is not to be confused with the entirely different use of this term as a generalization of a "dyadic space".

The functor $\mathbf{E}$ associated to $\mathbf{P}$ has a very natural interpretation. For a context $S \in \mathsf{C}$, we recall that a model of $\mathbf{P}$ at S consists, roughly speaking, of a model M of $\mathbf{P}$ together with a value assignment s of S in M (Definition 4.6). Then, for each context $S \in \mathsf{C}$, the elements of the Stone space $\mathbf{E}(S)$ are—roughly speaking—the equivalence classes of models (M, s) of $\mathbf{P}$ at S with respect to the equivalence relation that identifies two models (M, s) and (M', s') if they satisfy the same first-order formulas in the context S (with the interpretation of the free variables as prescribed by the value assignments s and s').

Given a quantifier stratification $(\mathbf{P}_n)_{n\in\mathbb{N}}$ of a universal Boolean doctrine $\mathbf{P}$, there is a corresponding sequence $(\mathbf{E}_n\colon \mathsf{C} \to \mathsf{Stone})_{n\in\mathbb{N}}$ of functors (linked by componentwise surjective natural transformations $\mathbf{E} \twoheadrightarrow \mathbf{E}_{n+1} \twoheadrightarrow \mathbf{E}_n$), which also has a very natural interpretation. For all $n \in \mathbb{N}$ and $S \in \mathsf{C}$, the elements of the space $\mathbf{E}_n(S)$ are the equivalence classes of models $\mathbf{P}$ at S with respect to the equivalence relation that identifies two models if they satisfy the same formulas of quantifier depth less than or equal to n.

The study of polyadic spaces is, in a certain sense, the study of spaces of models.

We plan to dualize the notions in Section 3, and we plan to exhibit how to freely add one layer of quantifiers, dually. Our study of universal ultrafilters should make this easy. Recall that filters of a Boolean algebra correspond to closed subsets of the dual space. In light of this correspondence, we expect universal ultrafilters to dually correspond to quantifier points as defined in Definition 7.7 below. We use the notation $\mathcal{V}(X)$ for the Stone space of closed subsets of a Stone space X equipped with the Vietoris topology [23].

Definition 7.7. Let $\mathbf{E}\colon \mathsf{C} \to \mathsf{Stone}$ be a functor, where C is a category with finite products, and let $S \in \mathsf{C}$. A *quantifier point for* $\mathbf{E}$ *at* S is a family $(\rho_X)_{X\in\mathsf{C}} \in \prod_{X\in\mathsf{C}} \mathcal{V}(\mathbf{E}(S \times X))$ with the following properties.

(1) For all $X, Y \in \mathsf{C}$ and every morphism $g\colon S \times X \to Y$, $\mathbf{E}(\langle \mathrm{pr}_1, g\rangle)[\rho_X] \subseteq \rho_Y$.
(2) For all $X_1, X_2 \in \mathsf{C}$ and all $x_1 \in \rho_{X_1}$ and $x_2 \in \rho_{X_2}$, there is $y \in \rho_{X_1\times X_2}$ such that $\mathbf{E}(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(y) = x_1$ and $\mathbf{E}(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)(y) = x_2$.
(3) $\rho_{\mathbf{t}} \neq \varnothing$.

Let $\mathbf{E}_0\colon \mathsf{C} \to \mathsf{Stone}$ be a functor, where C is a small category with finite products. Let $\mathbf{E}_1\colon \mathsf{C} \to \mathsf{Stone}$ be the functor dual to the functor $\mathrm{Free}_1^{\mathbf{P}_0}$ obtained by adding one layer of quantification to the Boolean doctrine $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ dual to $\mathbf{E}_0$ (see Notations 6.12 and 6.18).

Conjecture 7.8. *For each $S \in \mathsf{C}$, the Stone space $\mathbf{E}_1(S)$ is isomorphic to the subspace of $\prod_{X\in\mathsf{C}} \mathcal{V}(\mathbf{E}_0(S \times X))$ consisting of all quantifier points for $\mathbf{E}_0$ at S.*

The reason why we believe this conjecture is that quantifier points at S should correspond to universal ultrafilters at S, which correspond to equivalence classes of models at S with respect to a certain equivalence relation (Theorem B.6), which in turn should correspond to points of $\mathbf{E}_1(S)$.

This would provide an answer, in the setting of Boolean doctrines, to the "notable obstacle to a full duality theoretic understanding of step-by-step quantification in predicate logics" mentioned in [8, Section 4.4, First paragraph]:

> "We saw in Sect. 4.3.1 that adding a layer of existential quantifier $\exists$ to a Boolean algebra B of first-order formulas (with free variables in $v_1, \ldots, v_n$) dually corresponds to taking the image of a continuous map $\beta(\mathrm{Mod}_n) \to \mathcal{V}(X) \times X$, where X is the dual Stone space of B. [...] This continuous map is defined in a canonical way, and ensures the *soundness* of the construction. But we do not know, so far, how to characterise the continuous maps $\beta(\mathrm{Mod}_n) \to \mathcal{V}(X) \times X$ arising in this manner, which would establish the *completeness* of the construction. This is a notable obstacle to a full duality theoretic understanding of step-by-step quantification in predicate logic."

Appendix A. Semantic characterizations of universal filters and ideals

Theorem 5.28 shows that the notion of universal ultrafilter is meaningful. To prove the theorem, we used universal filters and universal ideals. In this section we will show that these two notions are not just auxiliary technical notions, but are also meaningful since they have a semantic characterization. In Theorem A.5 we prove that universal filters are precisely the families of all formulas that are universally valid in all models of some class of models. Similarly, in Theorem A.9, we prove that universal ideals are precisely the families of all formulas that are universally invalid in all models of some family of models. For the sake of completeness, we also characterize the pairs consisting of a filter and an ideal that arise from a common family of models. This is obtained in Theorem A.16. Such pairs are called *filter-ideal pairs*.

In Appendix B we will generalize the results in Section 5 and in this appendix to the case where there are some fixed free variables exempt from universal closure.

Finally, let us clarify the dependency between the appendices and the previous sections: in the appendices we use notions and results from the rest of the manuscript, while results from the appendices are not needed in the main body of the paper, and are only mentioned there for motivational purposes.

A.1. Semantic characterization of universal filters.

Lemma A.1. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, let $F = (F_X)_{X\in\mathsf{C}}$ be a universal filter, let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y) \setminus F_Y$. There is a universal ultrafilter that extends F and does not contain α.*

Proof. Let $(I_X)_X$ be the universal ideal generated by $\alpha \in \mathbf{P}(Y)$. By Lemma 5.11, for each $X \in \mathsf{C}$ we have

$$I_X = \{\varphi \in \mathbf{P}(X) \mid \text{there is } f\colon Y \to X \text{ such that } \mathbf{P}(f)(\varphi) \leq \alpha\}.$$

Observe that F and I are componentwise disjoint: indeed suppose $\varphi \in I_X \cap F_X$, so that there is $f\colon Y \to X$ such that $\mathbf{P}(f)(\varphi) \leq \alpha$ in $\mathbf{P}(Y)$. Since F is closed under reindexing and upward closed, we get $\alpha \in F_Y$, a contradiction. By Theorem 5.21, there is a universal ultrafilter G extending F and disjoint from I. In particular α does not belong to G, as desired. ☐

Remark A.2. Lemma A.1 is similar to the version of the classical ultrafilter lemma stating that every filter not containing an element a can be extended to an ultrafilter not containing a.

Definition A.3. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ a Boolean doctrine. A universal filter $(F_X)_{X\in\mathsf{C}}$ for $\mathbf{P}$ is *consistent* if $\bot_{\mathbf{P}(\mathbf{t})} \notin F_{\mathbf{t}}$.

Lemma A.4. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ a Boolean doctrine. Every consistent universal filter $(F_X)_{X\in\mathsf{C}}$ for $\mathbf{P}$ can be extended to a universal ultrafilter.*

Proof. Apply Lemma A.1 with $Y = \mathbf{t}$ and $\alpha = \bot_{\mathbf{P}(\mathbf{t})}$. ☐

Theorem A.5. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $F = (F_X)_{X\in\mathsf{C}}$ be a family with $F_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class $\mathcal{M}$ of Boolean models of $\mathbf{P}$ such that, for every $X \in \mathsf{C}$,

$$F_X = \{\alpha \in \mathbf{P}(X) \mid \textit{for all } (M,\mathfrak{m}) \in \mathcal{M},\ \textit{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

(2) F is a universal filter for $\mathbf{P}$.

(3) F is the intersection of the universal ultrafilters for $\mathbf{P}$ containing F.

Equivalent are also the statements obtained by requiring, additionally: nonemptiness of $\mathcal{M}$ in (1), consistency of F in (2), and the existence of a universal ultrafilter for $\mathbf{P}$ containing F in (3).

Proof. (1) ⇒ (3). Let $\mathcal{G}$ be the family of universal ultrafilters containing F. Fix $X \in \mathsf{C}$. The inclusion $F_X \subseteq \bigcap_{G\in\mathcal{G}} G_X$ is immediate by definition of $\mathcal{G}$. For the converse inclusion, let $\alpha \in \bigcap_{G\in\mathcal{G}} G_X$. To prove $\alpha \in F_X$, we check that for all $(M,\mathfrak{m}) \in \mathcal{M}$ and $x \in M(X)$ we have $x \in \mathfrak{m}_X(\alpha)$. Let $(M,\mathfrak{m}) \in \mathcal{M}$. For all $Y \in \mathsf{C}$, set

$$H_Y = \{\beta \in \mathbf{P}(Y) \mid \text{for all } x \in M(Y),\, x \in \mathfrak{m}_Y(\beta)\}.$$

By Theorem 5.28, $(H_Y)_{Y\in\mathsf{C}}$ is a universal ultrafilter, and it is easy to see that it belongs to $\mathcal{G}$. Then, $\alpha \in \bigcap_{G\in\mathcal{G}} G_X \subseteq H_X$, as desired.

(3) ⇒ (1) Let $\mathcal{M}$ be the class of Boolean models $(M, \mathfrak{m})$ of $\mathbf{P}$ such that

$$F_X \subseteq \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

Let

$$\beta \in \bigcap_{(M,\mathfrak{m}) \in \mathcal{M}} \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

We show that $\beta \in F_X$, i.e. that β belongs to all universal ultrafilters G containing F. Let G be any such universal ultrafilter. By Theorem 5.28 there is a Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$ such that, for all $Y \in \mathsf{C}$, $G_Y = \{\gamma \in \mathbf{P}(Y) \mid \text{for all } y \in M(Y),\, y \in \mathfrak{m}_Y(\gamma)\}$. It is then easy to see that the Boolean model $(M, \mathfrak{m})$ belongs to $\mathcal{M}$. By hypothesis on β, we have $\beta \in G_X$, as desired.

(2) ⇒ (3). Clearly, F is contained in the intersection of the universal ultrafilters containing F. For the converse inclusion, let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y) \setminus F_Y$. By Lemma A.1, there is a universal ultrafilter extending F and not containing α.

(3) ⇒ (2). The componentwise intersection of universal (ultra)filters is a universal filter.

This proves that the statements (1), (2) and (3) are equivalent.

Let us now prove that the statements (1'), (2') and (3') obtained from (1), (2) and (3) as in the final paragraph of the theorem are equivalent.

(1') ⇒ (3'). Since $\mathcal{M}$ is nonempty, there is $(M, \mathfrak{m}) \in \mathcal{M}$. The family $(H_Y)_{Y \in \mathsf{C}}$ defined by

$$H_Y \coloneqq \{\beta \in \mathbf{P}(Y) \mid \text{for all } x \in M(Y),\, x \in \mathfrak{m}_Y(\beta)\}$$

is a universal ultrafilter for $\mathbf{P}$ (by Theorem 5.28) containing F.

(3') ⇒ (1'). The class $\mathcal{M}$ is nonempty because, if $\mathcal{M}$ were empty, we would have $F_{\mathbf{t}} = \mathbf{P}(\mathbf{t})$, contradicting the existence of a universal ultrafilter for $\mathbf{P}$ containing F.

(2') ⇒ (3'). This follows from Lemma A.4.

(3') ⇒ (2'). This is immediate. □

A.2. **Semantic characterization of universal ideals.**

Lemma A.6. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, $I = (I_X)_{X \in \mathsf{C}}$ a universal ideal, $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y) \setminus I_Y$. There is a universal ultraideal that extends I and does not contain α.*

Proof. Let $(F_X)_X$ be the universal filter generated by $\alpha \in \mathbf{P}(Y)$. By the description in Lemma 5.8, for every $X \in \mathsf{C}$

$$F_X = \left\{\beta \in \mathbf{P}(X) \mid \text{there are } (f_i \colon X \to Y)_{i=1,\dots,n} \text{ such that } \bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha) \leq \beta\right\}.$$

Observe that F and I are componentwise disjoint: indeed suppose $\beta \in I_X \cap F_X$, so that there are $(f_i \colon X \to Y)_{i=1,\dots,n}$ such that $\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha) \leq \beta$ in $\mathbf{P}(X)$. Since I is doward closed, we get $\bigwedge_{i=1}^{n} \mathbf{P}(f_i)(\alpha) \in I_X$. Therefore, by Definition 5.9(1), we obtain $\alpha \in I_Y$, a contradiction. □

Definition A.7. Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A universal ideal $(I_X)_{X \in \mathsf{C}}$ for $\mathbf{P}$ is *consistent* if $\top_{\mathbf{P}(\mathbf{t})} \notin I_{\mathbf{t}}$.

Lemma A.8. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Every consistent universal ideal $(I_X)_{X \in \mathsf{C}}$ for $\mathbf{P}$ can be extended to a universal ultraideal.*

Proof. Applying Lemma A.6 (with $Y = \mathbf{t}$ and $\alpha = \top_{\mathbf{P}(\mathbf{t})} \notin I_{\mathbf{t}}$), we obtain that there is a universal ultraideal that extends I (and does not contain $\top_{\mathbf{P}(\mathbf{t})}$). □

Theorem A.9. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $I = (I_X)_{X \in \mathsf{C}}$ be a family with $I_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class $\mathcal{M}$ of Boolean models of $\mathbf{P}$ such that, for every $X \in \mathsf{C}$,

$$I_X = \{\alpha \in \mathbf{P}(X) \mid \text{for all } (M, \mathfrak{m}) \in \mathcal{M},\ \text{not all } x \in M(X) \text{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

(2) I is a universal ideal for $\mathbf{P}$.

(3) I is the intersection of the universal ultraideals for $\mathbf{P}$ containing I.

Equivalent are also the three statements obtained by requiring, additionally: in (1) *nonemptiness of* $\mathcal{M}$*, in* (2) *consistency of* I*, and in* (3) *the existence of a universal ultraideal for* $\mathbf{P}$ *containing* I*.*

Proof. (1) $\Rightarrow$ (3). Let $\mathcal{J}$ be the family of universal ultraideals containing I. Fix $X \in \mathsf{C}$. The inclusion $I_X \subseteq \bigcap_{J\in\mathcal{J}} J_X$ is immediate by definition of $\mathcal{J}$. For the converse inclusion, let $\alpha \in \bigcap_{J\in\mathcal{J}} J_X$. To prove that $\alpha \in I_X$, we check that, for all $(M, \mathfrak{m}) \in \mathcal{M}$, not all $x \in M(X)$ satisfy $x \in \mathfrak{m}_X(\alpha)$. Let $(M, \mathfrak{m}) \in \mathcal{M}$. For all $Y \in \mathsf{C}$, set

$$L_Y = \{\beta \in \mathbf{P}(Y) \mid \text{not all } x \in M(Y) \text{ satisfy } x \in \mathfrak{m}_Y(\beta)\}.$$

By Theorem 5.28 and Lemma 5.16, $(L_Y)_{Y\in\mathsf{C}}$ is a universal ultraideal, and it is easy to see that it belongs to $\mathcal{J}$. Then, $\alpha \in \bigcap_{J\in\mathcal{J}} J_X \subseteq L_X$, as desired.

(3) $\Rightarrow$ (1). Let $\mathcal{M}$ be the class of Boolean models $(M, \mathfrak{m})$ of $\mathbf{P}$ such that

$$I_X \subseteq \{\alpha \in \mathbf{P}(X) \mid \text{not all } x \in M(X) \text{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

Let

$$\beta \in \bigcap_{(M,\mathfrak{m})\in\mathcal{M}} \{\alpha \in \mathbf{P}(X) \mid \text{not all } x \in M(X) \text{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

We show $\beta \in I_X$, i.e. that β belongs to all universal ultraideal J containing I. Let J be any such universal ultraideal and G the componentwise complement of J. In particular G is a universal ultrafilter. By Theorem 5.28 there is a Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$ such that, for all $Y \in \mathsf{C}$, $G_Y = \{\gamma \in \mathbf{P}(Y) \mid \text{for all } y \in M(Y),\ y \in \mathfrak{m}_Y(\gamma)\}$. It is then easy to see that the Boolean model $(M, \mathfrak{m})$ belongs to $\mathcal{M}$. By hypothesis on β, we have $\beta \notin G_X$ and hence $\beta \in J_X$, as desired.

(2) $\Rightarrow$ (3). It is easy to see that I is contained in the intersection of the universal ultraideals that contain I. For the converse inclusion, let $Y \in \mathsf{C}$ and let $\alpha \in \mathbf{P}(Y) \setminus I_Y$. Apply Lemma A.6 to get a universal ultraideal J that extends I and does not contain α, as desired.

(3) $\Rightarrow$ (2). The componentwise intersection of universal (ultra)ideals is a universal ideal.

This proves that the statements (1), (2) and (3) are equivalent.

Let us now prove that the statements (1'), (2') and (3') obtained from (1), (2) and (3) as in the final paragraph of the theorem are equivalent.

(1') $\Rightarrow$ (3'). Since $\mathcal{M}$ is nonempty, there is $(M, \mathfrak{m}) \in \mathcal{M}$. The family $(L_Y)_{Y\in\mathsf{C}}$ defined by

$$L_Y = \{\beta \in \mathbf{P}(Y) \mid \text{not all } x \in M(Y) \text{ satisfy } x \in \mathfrak{m}_Y(\beta)\}.$$

is a universal ultraideal (by Theorem 5.28 and Lemma 5.16) containing I.

(3') $\Rightarrow$ (1'). The class $\mathcal{M}$ is nonempty because, if $\mathcal{M}$ were empty, we would have $I_{\mathbf{t}} = \mathbf{P}(\mathbf{t})$, contradicting the existence of a universal ultraideal containing I.

(2') $\Rightarrow$ (3'). This follows from Lemma A.8.

(3') $\Rightarrow$ (2'). This is immediate. $\square$

A.3. **Semantic characterization of universal filter-ideal pairs.**

Definition A.10 (Universal filter-ideal pair)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A *universal filter-ideal pair for* $\mathbf{P}$ is a pair (F, I) where $F = (F_X)_{X\in\mathsf{C}}$ is a universal filter for $\mathbf{P}$, $I = (I_X)_{X\in\mathsf{C}}$ is a universal ideal for $\mathbf{P}$, and the following conditions hold for all $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y)$.

(1) For all $X \in \mathsf{C}$, $n \in \mathbb{N}$, $(f_i\colon X \to Y)_{i=1,\dots,n}$ and $\beta \in F_X$, if $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in I_X$, then $\alpha \in I_Y$.
(2) For all $Z \in \mathsf{C}$ and $\gamma \in I_Z$, if $\mathbf{P}(\mathrm{pr}_1)(\alpha) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) \in F_{Y\times Z}$, then $\alpha \in F_Y$.

Definition A.11. We say that a universal filter-ideal pair (F, I) is *consistent* when for every $X \in \mathsf{C}$ we have $F_X \cap I_X = \varnothing$. Otherwise, we say it is *inconsistent*.

Lemma A.12. *Let* (F, I) *be an inconsistent universal filter-ideal pair. For every* $Y \in \mathsf{C}$*,* $F_Y = I_Y = \mathbf{P}(Y)$*.*

Proof. By inconsistency, there are $Z \in \mathsf{C}$ and $\gamma \in F_Z \cap I_Z$. Since $\gamma \in I_Z$ and

$$\mathbf{P}(\mathrm{pr}_1)(\perp_{\mathbf{P}(\mathbf{t})}) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) = \gamma \in F_Y = F_{\mathbf{t}\times Z},$$

we have $\perp_{\mathbf{P}(\mathbf{t})} \in F_{\mathbf{t}}$ by Definition A.10(2). Let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y)$. The fact that $\alpha \in F_Y$ follows from $\perp_{\mathbf{P}(\mathbf{t})} \in F_{\mathbf{t}}$ and the properties of universal filters. Since $\perp_{\mathbf{P}(\mathbf{t})} \in F_{\mathbf{t}} \cap I_{\mathbf{t}}$, by Definition A.10(1) (applied with $n = 0$ and $\beta = \perp_{\mathbf{P}(\mathbf{t})}$), $\alpha \in I_Y$. $\square$

The following lemma explains the purpose of (1) and (2) in Definition A.10.

Lemma A.13. *Let (F, I) be a universal filter-ideal pair, let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y)$.*

(1) If $\alpha \notin I_Y$, then I is componentwise disjoint from the universal filter generated by F and α.
(2) If $\alpha \notin F_Y$, then F is componentwise disjoint from the universal ideal generated by I and α.

Proof. (1). We prove the contrapositive. Suppose that I intersects the universal filter generated by F and α at some fiber. By Lemma 5.18(1), there are $X \in \mathsf{C}$, $n \in \mathbb{N}$, $(f_i \colon X \to Y)_{i=1,\dots,n}$ and $\beta \in F_X$ such that $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in I_X$. By Definition A.10(1), $\alpha \in I_Y$.

(2). We prove the contrapositive. Suppose that F intersects the universal ideal generated by I and α. By Lemma 5.18(2), there is $X \in \mathsf{C}$ such that $I_X \cap F_X \neq \varnothing$ or there are $Z \in \mathsf{C}$ and $\gamma \in I_Z$ such that $\mathbf{P}(\mathrm{pr}_1)(\alpha) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) \in F_{Y \times Z}$. In the first case, (F, I) is inconsistent, and thus $\alpha \in F_Y$ by Lemma A.12. In the second case, by Definition A.10(2), $\alpha \in F_Y$. □

Lemma A.14. *Let (F, I) be a universal filter-ideal pair, let $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(Y)$.*

(1) If $\alpha \notin I_Y$, then there is a universal ultrafilter that extends F, contains α and is disjoint from I.
(2) If $\alpha \notin F_Y$, then there is a universal ultrafilter that extends F, does not contain α, and is disjoint from I.

Proof. (1). By Lemma A.13, I is componentwise disjoint from the universal filter F' generated by F and α. By Theorem 5.21, there is a universal filter F'' that, componentwise, extends F' and is disjoint from I.

(2) is proved analogously. □

Lemma A.15. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. Let $(F_X)_{X \in \mathsf{C}}$ be a universal ultrafilter, and for each $X \in \mathsf{C}$ set $I_X \coloneqq \mathbf{P}(X) \setminus F_X$. The pair $((F_X)_{X \in \mathsf{C}}, (I_X)_{X \in \mathsf{C}})$ is a universal filter-ideal pair.*

Proof. The family $(I_X)_{X \in \mathsf{C}}$ is a universal ideal (as already mentioned in Lemma 5.16).

We prove the conditions (1) and (2) in Definition A.10.

(1). Let $X, Y \in \mathsf{C}$, $\alpha \in \mathbf{P}(Y)$, let $n \in \mathbb{N}$, let $(f_i \colon X \to Y)_{i=1,\dots,n}$, let $\beta \in F_X$, and suppose $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in I_X$. We shall prove $\alpha \in I_Y$, i.e., $\alpha \notin F_Y$. We suppose $\alpha \in F_Y$ and we seek a contradiction. From $\alpha \in F_Y$ we deduce that for all $i = 1, \dots, n$, $\mathbf{P}(f_i)(\alpha) \in F_X$, and hence $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in F_X$. Thus $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(f_i)(\alpha) \in F_X \cap I_X$, a contradiction.

(2). Let $Y, Z \in \mathsf{C}$, $\alpha \in \mathbf{P}(Y)$, $\gamma \in I_Z$ and suppose $\mathbf{P}(\mathrm{pr}_1)(\alpha) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) \in F_{Y \times Z}$. We check $\alpha \in F_Y$. We suppose $\alpha \notin F_Y$ and we seek a contradiction. From $\gamma \in I_Z$ we deduce $\gamma \notin F_Z$. Use condition (3) in Definition 5.12 we obtain $\mathbf{P}(\mathrm{pr}_1)(\alpha) \vee \mathbf{P}(\mathrm{pr}_2)(\gamma) \notin F_{Y \times Z}$, a contradiction. □

Theorem A.16. *Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $F = (F_X)_{X \in \mathsf{C}}$ and $I = (I_X)_{X \in \mathsf{C}}$ be families with $F_X \subseteq \mathbf{P}(X)$ and $I_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class $\mathcal{M}$ of Boolean models of $\mathbf{P}$ such that, for every $X \in \mathsf{C}$,

$$F_X = \{\alpha \in \mathbf{P}(X) \mid \textit{for all } (M, \mathfrak{m}) \in \mathcal{M},\ \textit{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\},$$
$$I_X = \{\alpha \in \mathbf{P}(X) \mid \textit{for all } (M, \mathfrak{m}) \in \mathcal{M},\ \textit{not all } x \in M(X) \textit{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

(2) (F, I) is a universal filter-ideal pair for $\mathbf{P}$.
(3) F is the intersection of the universal ultrafilters for $\mathbf{P}$ containing F and disjoint from I, and I is the intersection of the universal ultraideals for $\mathbf{P}$ containing I and disjoint from F.

Equivalent are also the three statements obtained by requiring, additionally: in (1) nonemptiness of $\mathcal{M}$, in (2) consistency of (F, I), and in (3) the existence of a universal ultrafilter for $\mathbf{P}$ containing F and disjoint from I.

Proof. (1) ⇒ (3). We prove that F is the componentwise intersection of the family $\mathcal{G}$ of all universal ultrafilters containing F and disjoint from I. Fix $X \in \mathsf{C}$. The inclusion $F_X \subseteq \bigcap_{G \in \mathcal{G}} G_X$ is immediate by definition of $\mathcal{G}$. For the converse inclusion, let $\alpha \in \bigcap_{G \in \mathcal{G}} G_X$. To prove that $\alpha \in F_X$, we check that for all $(M, \mathfrak{m}) \in \mathcal{M}$ and $x \in M(X)$ we have $x \in \mathfrak{m}_X(\alpha)$. Let $(M, \mathfrak{m}) \in \mathcal{M}$. For all $Y \in \mathsf{C}$, set

$$H_Y = \{\beta \in \mathbf{P}(Y) \mid \text{for all } x \in M(Y),\ x \in \mathfrak{m}_Y(\beta)\}$$

By Theorem 5.28, $(H_Y)_{Y\in\mathsf{C}}$ is a universal ultrafilter, and it is easy to see that it belongs to $\mathcal{G}$. We have $\alpha \in \bigcap_{G\in\mathcal{G}} G_X \subseteq H_X$, as desired.

A similar argument shows that I is the intersection of the universal ultraideals for $\mathbf{P}$ containing I.

(3) $\Rightarrow$ (1). Let $\mathcal{M}$ be the class of Boolean models $(M,\mathfrak{m})$ of $\mathbf{P}$ such that

$$F_X \subseteq \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}, \text{ and}$$
$$I_X \subseteq \{\alpha \in \mathbf{P}(X) \mid \text{not all } x \in M(X) \text{ satisfy } x \in \mathfrak{m}_X(\alpha)\}.$$

Let

$$\beta \in \bigcap_{(M,\mathfrak{m})\in\mathcal{M}} \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

We show that $\beta \in F_X$, i.e. that β belongs to all universal ultrafilters G containing F and disjoint from I. Let G be any such universal ultrafilter. By Theorem 5.28 there is a Boolean model $(M,\mathfrak{m})$ of $\mathbf{P}$ such that, for all $Y \in \mathsf{C}$, $G_Y = \{\gamma \in \mathbf{P}(Y) \mid \text{for all } y \in M(Y),\, y \in \mathfrak{m}_Y(\gamma)\}$. It is then easy to see that the Boolean model $(M,\mathfrak{m})$ belongs to $\mathcal{M}$. By hypothesis on β, we have $\beta \in G_X$, as desired.

A similar argument shows the desired condition on I.

(2) $\Rightarrow$ (3). It is easy to see that F is contained in the intersection of the universal ultrafilters for $\mathbf{P}$ that contain F and are disjoint from I. The converse inclusion is precisely Lemma A.14(2). Analogously, I is the intersection of the universal ultraideals for $\mathbf{P}$ containing I and disjoint from F.

(3) $\Rightarrow$ (2). By Lemma A.15, if $(G_X)_{X\in\mathsf{C}}$ is a universal ultrafilter and for each $X \in \mathsf{C}$ we set $J_X \coloneqq \mathbf{P}(X)\setminus G_X$, then the pair $((G_X)_{X\in\mathsf{C}}, (J_X)_{X\in\mathsf{C}})$ is a universal filter-ideal pair. Thus, (F,I) is an intersection of universal filter-ideal pairs (G,J) (with the property that G is a universal ultrafilter and J is componentwise complementary), and so it is a universal filter-ideal pair.

This proves that the statements (1), (2) and (3) are equivalent.

Let us now prove that the statements (1'), (2') and (3') obtained from (1), (2) and (3) as in the final paragraph of the theorem are equivalent.

(1') $\Rightarrow$ (3'). Since $\mathcal{M}$ is nonempty, there is $(M,\mathfrak{m}) \in \mathcal{M}$. The family $(H_Y)_{Y\in\mathsf{C}}$ defined by

$$H_Y \coloneqq \{\beta \in \mathbf{P}(Y) \mid \text{for all } x \in M(Y),\, x \in \mathfrak{m}_Y(\beta)\}$$

is a universal ultrafilter (by Theorem 5.28) containing F and disjoint from I.

(3') $\Rightarrow$ (1'). The class $\mathcal{M}$ is nonempty because, if $\mathcal{M}$ were empty, we would have $F_X = \mathbf{P}(X)$ for all $X \in \mathsf{C}$, contradicting the existence of a universal ultrafilter for $\mathbf{P}$ containing F.

(2') $\Rightarrow$ (3'). This follows from the universal ultrafilter lemma (Theorem 5.21).

(3') $\Rightarrow$ (2'). This is immediate. ☐

Appendix B. Semantic characterizations over fixed free variables

In Section 5, we characterized the classes of formulas whose universal closure (with respect to *all* free variables) is valid in some fixed model. In this appendix, we do something similar, but we fix some free variables that are exempt from universal closure. To illustrate this, we introduce the following notation.

B.1. Semantic characterization of universal ultrafilters over fixed free variables.

Notation B.1. Let $(M,\mathfrak{m},s)$ be a Boolean model of a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ at an object $S \in \mathsf{C}$ (see Definition 4.6). For each $X \in \mathsf{C}$, define

$$F_X^{S,(M,\mathfrak{m},s)} \coloneqq \{\alpha \in \mathbf{P}(S\times X) \mid \text{for all } x \in M(X),\, (s,x) \in \mathfrak{m}_{S\times X}(\alpha)\},$$

where we made implicit use of the isomorphism between $M(S\times X)$ and $M(S)\times M(X)$ in writing $(s,x) \in \mathfrak{m}_{S\times X}(\alpha)$.

Remark B.2. We translate Notation B.1 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1,\dots,s_k,x_1,x_2,x_3,\dots\}$ be a set of variables, $\mathcal{L}$ a language, $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$, M a model of $\mathcal{T}$, and $c_1,\dots,c_k \in M$. For each $n \in \mathbb{N}$ we define

$$F_n^{k,M,c_1,\dots,c_k} \coloneqq \{\alpha(s_1,\dots,s_k,x_1,\dots,x_n) \text{ q.-free} \mid M, [(s_i \mapsto c_i)_i] \vDash \forall x_1 \dots \forall x_n\, \alpha(s_1,\dots,s_k,x_1,\dots,x_n)\},$$

where by $M, [(s_i \mapsto c_i)_i] \vDash \forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)$ we mean that, under the variable assignment $[(s_i \mapsto c_i)_{i=1,\dots,k}]$, the formula $\forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)$ is valid in M.

In Theorem B.6 below we characterize the families of the form $(F_X^{S,(M,\mathfrak{m},s)})_{X \in \mathsf{C}}$ for some model $(M, \mathfrak{m}, s)$ at S, at least in the case where the base category C is small; these families are captured axiomatically by the notion of a *universal ultrafilter at* S, introduced in Definition B.3 below.

Definition B.3 (Universal ultrafilter at an object)**.** Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $S \in \mathsf{C}$. A *universal ultrafilter for* $\mathbf{P}$ *at* S is a family $(F_X)_{X \in \mathsf{C}}$, with $F_X \subseteq \mathbf{P}(S \times X)$ for each $X \in \mathsf{C}$, with the following properties.

1. For all $f \colon S \times X \to Y$ and $\alpha \in F_Y$, we have $\mathbf{P}(\langle \mathrm{pr}_1, f \rangle)(\alpha) \in F_X$.
2. For all $X \in \mathsf{C}$, F_X is a filter of $\mathbf{P}(S \times X)$.
3. For all $\alpha_1 \in \mathbf{P}(S \times X_1) \setminus F_{X_1}$ and $\alpha_2 \in \mathbf{P}(S \times Z_2) \setminus F_{Z_2}$, in $\mathbf{P}(S \times X_1 \times X_2)$ we have $\mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)(\alpha_1) \vee \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)(\alpha_2) \notin F_{X_1 \times X_2}$.
4. $\bot_{\mathbf{P}(S)} \notin F_{\mathbf{t}}$.

Remark B.4. We translate Definition B.3 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \dots, s_k, x_1, x_2, x_3, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal ultrafilter for* $\mathcal{T}$ *at* k is a family $(F_n)_{n \in \mathbb{N}}$, with F_n a set of quantifier-free $\mathcal{L}$-formulas with $s_1, \dots, s_k, x_1, \dots, x_n$ as (possibly dummy) free variables, with the following properties.

1. For every $n, m \in \mathbb{N}$, for every formula $\alpha(s_1, \dots, s_k, x_1, \dots, x_m) \in F_m$ and for every m-tuple $(f_i(s_1, \dots, s_k, x_1, \dots, x_n))_{i=1,\dots,n}$ of $(k+n)$-ary terms,
$$\alpha(f_1(s_1, \dots, s_k, x_1, \dots, x_n), \dots, f_m(s_1, \dots, s_k, x_1, \dots, x_n)) \in F_n.$$
2. For all $n \in \mathbb{N}$,
   (a) for all quantifier-free formulas $\alpha(s_1, \dots, s_k, x_1, \dots, x_n)$ and $\beta(s_1, \dots, s_k, x_1, \dots, x_n)$, if we have $\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \in F_n$ and $\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \vdash_{\mathcal{T}} \beta(x_1, \dots, x_n)$, then we have $\beta(x_1, \dots, x_n) \in F_n$;
   (b) for all $\alpha_1(s_1, \dots, s_k, x_1, \dots, x_n), \alpha_2(s_1, \dots, s_k, x_1, \dots, x_n) \in F_n$, we have
$$\alpha_1(s_1, \dots, s_k, x_1, \dots, x_n) \wedge \alpha_2(s_1, \dots, s_k, x_1, \dots, x_n) \in F_n;$$
   (c) $\top(s_1, \dots, s_k, x_1, \dots, x_n) \in F_n$.
3. For every $n_1, n_2 \in \mathbb{N}$ and for every pair of quantifier-free formulas $\alpha_1(s_1, \dots, s_k, x_1, \dots, x_{n_1})$ and $\alpha_2(s_1, \dots, s_k, x_1, \dots, x_{n_2})$, if
$$\alpha_1(s_1, \dots, s_k, x_1, \dots, x_{n_1}) \vee \alpha_2(s_1, \dots, s_k, x_{n_1+1}, \dots, x_{s_1,\dots,s_k,n_1+n_2}) \in F_{n_1+n_2},$$
then $\alpha_1(s_1, \dots, s_k, x_1, \dots, x_{n_1}) \in F_{n_1}$ or $\alpha_2(s_1, \dots, s_k, x_1, \dots, x_{n_2}) \in F_{n_2}$.
4. For all $n \in \mathbb{N}$, $\bot(s_1, \dots, s_k, x_1, \dots, x_n) \notin F_n$.

For every model M of $\mathcal{T}$ and for every $c_1, \dots, c_k \in M$, it is easy to check that the family $(F_n)_{n \in \mathbb{N}}$ defined by
$$F_n \coloneqq \{\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \text{ quantifier-free} \mid M, [(s_i \mapsto c_i)_i] \vDash \forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)\},$$
is a universal filter in the sense above.

Remark B.5. Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A universal ultrafilter for $\mathbf{P}$ at S is a universal ultrafilter for the Boolean doctrine $\mathbf{P}_S \colon \mathsf{C}_S^{\mathrm{op}} \to \mathsf{BA}$ obtained by adding a constant of type S for $\mathbf{P}$ (see Remark 4.5).

Theorem B.6. *Let* $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine, with* C *small, and let* $S \in \mathsf{C}$. *Let* $F = (F_X)_{X \in \mathsf{C}}$ *be a family with* $F_X \subseteq \mathbf{P}(S \times X)$ *for each* $X \in \mathsf{C}$. *The following conditions are equivalent.*

1. *There is a Boolean model* $(M, \mathfrak{m}, s)$ *of* $\mathbf{P}$ *at* S *such that, for every* $X \in \mathsf{C}$,
$$F_X = \{\alpha \in \mathbf{P}(S \times X) \mid \textit{for all } x \in M(X),\ (s, x) \in \mathfrak{m}_{S \times X}(\alpha)\}.$$
2. F *is a universal ultrafilter for* $\mathbf{P}$ *at* S.

Proof. This follows from Theorem 5.28 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S and from Lemma 4.7. $\square$

Remark B.7. We translate Theorem B.6 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \dots, s_k, x_1, x_2, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $(F_n)_{n\in\mathbb{N}}$ be a family with F_n a set of quantifier-free $\mathcal{L}$-formulas with $s_1, \dots, s_k, x_1, \dots, x_n$ as free (possibly dummy) variables. The following conditions are equivalent.

(1) There are a model M of $\mathcal{T}$ and $c_1, \dots, c_k \in M$ such that, for every $n \in \mathbb{N}$,

$$F_n = \{\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \text{ quantifier-free} \mid M, [(s_i \mapsto c_i)_i] \vDash \forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)\}.$$

(2) $(F_n)_{n\in\mathbb{N}}$ is a universal ultrafilter for $\mathcal{T}$ at k (in the sense of Remark B.4).

B.2. **Semantic characterization of universal filters over fixed free variables.** Analogously to Section 5, we introduce the notions of *universal filters* and *universal ideals* at a given object. To motivate these notions, we extend Notation B.1 as follows.

Notation B.8. Let $\mathcal{M}$ be a class of models of a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ at an object $S \in \mathsf{C}$. For each $X \in \mathsf{C}$, define

$$F_X^{S,\mathcal{M}} \coloneqq \{\alpha \in \mathbf{P}(S \times X) \mid \text{for all } (M, \mathfrak{m}, s) \in \mathcal{M}, \text{ for all } x \in M(X),\ (s, x) \in \mathfrak{m}_{S\times X}(\alpha)\},$$

$$I_X^{S,\mathcal{M}} \coloneqq \{\alpha \in \mathbf{P}(S \times X) \mid \text{for all } (M, \mathfrak{m}, s) \in \mathcal{M}, \text{ not all } x \in M(X) \text{ satisfy } (s, x) \in \mathfrak{m}_{S\times X}(\alpha)\}.$$

Roughly speaking,

- $F^{S,\mathcal{M}}$ consists of all the formulas $\alpha(S, X)$ such that $M, [S \mapsto s] \vDash \forall X\, \alpha(S, X)$ is valid in all elements (M, s) of $\mathcal{M}$,
- $I^{S,\mathcal{M}}$ consists of all the formulas $\alpha(S, X)$ such that $M, [S \mapsto s] \vDash \neg(\forall X\, \alpha(S, X))$ is valid in all elements (M, s) of $\mathcal{M}$.

Remark B.9. We translate Notation B.8 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \dots, s_k, x_1, x_2, x_3, \dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. Let $\mathcal{M}$ be a class of tuples $(M, c_1, \dots, c_k)$ where M is a model of $\mathcal{T}$ and $c_1, \dots, c_k \in M$. For each $n \in \mathbb{N}$ we define

$$\begin{aligned} F_n^{k,\mathcal{M}} \coloneqq \{\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \text{ quantifier-free} \mid \text{for all } (M, c_1, \dots, c_k) \in \mathcal{M},\\ M, [(s_i \mapsto c_i)_i] \vDash \forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)\},\\ I_n^{k,\mathcal{M}} \coloneqq \{\alpha(s_1, \dots, s_k, x_1, \dots, x_n) \text{ quantifier-free} \mid \text{for all } (M, c_1, \dots, c_k) \in \mathcal{M},\\ M, [(s_i \mapsto c_i)_i] \nvDash \forall x_1 \dots \forall x_n\, \alpha(s_1, \dots, s_k, x_1, \dots, x_n)\}. \end{aligned}$$

We introduce *universal filters at an object* S, meant to characterize the families of the form $F^{S,\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models at S (see Notation B.8).

Definition B.10 (Universal filter at an object)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A *universal filter for* $\mathbf{P}$ *at* S is a family $(F_X)_{X\in\mathsf{C}}$, with $F_X \subseteq \mathbf{P}(S \times X)$ for each $X \in \mathsf{C}$, with the following properties.

(1) For all $f\colon S \times X \to Y$ and $\alpha \in F_Y$, $\mathbf{P}(\langle \mathrm{pr}_1, f\rangle)(\alpha) \in F_X$.
(2) For all $X \in \mathsf{C}$, F_X is a filter of $\mathbf{P}(S \times X)$.

Remark B.11. We translate Definition B.10 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1, \dots, s_k, x_1, x_2, x_3, \dots\}$ be a set of variables, let $\mathcal{L}$ a language and let $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal filter for* $\mathcal{T}$ *at* k is a family $(F_n)_{n\in\mathbb{N}}$, with F_n a set of quantifier-free $\mathcal{L}$-formulas with $s_1, \dots, s_k, x_1, \dots, x_n$ as (possibly dummy) free variables, with the following properties.

(1) For every $n, m \in \mathbb{N}$, for every formula $\alpha(s_1, \dots, s_k, x_1, \dots, x_m) \in F_m$ and for every m-tuple $(f_i(s_1, \dots, s_k, x_1, \dots, x_n))_{i=1,\dots,n}$ of $(k + n)$-ary terms,

$$\alpha(f_1(s_1, \dots, s_k, x_1, \dots, x_n), \dots, f_m(s_1, \dots, s_k, x_1, \dots, x_n)) \in F_n.$$

(2) For all $n \in \mathbb{N}$,

(a) for all quantifier-free formulas $\alpha(s_1,\dots,s_k,x_1,\dots,x_n)$ and $\beta(s_1,\dots,s_k,x_1,\dots,x_n)$, if we have $\alpha(s_1,\dots,s_k,x_1,\dots,x_n) \in F_n$ and $\alpha(s_1,\dots,s_k,x_1,\dots,x_n) \vdash_{\mathcal{T}} \beta(s_1,\dots,s_k,x_1,\dots,x_n)$, then we have $\beta(s_1,\dots,s_k,x_1,\dots,x_n) \in F_n$;

(b) for all $\alpha_1(s_1,\dots,s_k,x_1,\dots,x_n), \alpha_2(s_1,\dots,s_k,x_1,\dots,x_n) \in F_n$, we have

$$\alpha_1(s_1,\dots,s_k,x_1,\dots,x_n) \wedge \alpha_2(s_1,\dots,s_k,x_1,\dots,x_n) \in F_n;$$

(c) $\top(s_1,\dots,s_k,x_1,\dots,x_n) \in F_n$.

These conditions are satisfied by any family $(F_n^{k,\mathcal{M}})_{n\in\mathbb{N}}$ defined by a class $\mathcal{M}$ as in Remark B.9.

Remark B.12. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A universal filter for $\mathbf{P}$ at S is a universal filter for the Boolean doctrine $\mathbf{P}_S\colon \mathsf{C}_S^{\mathrm{op}} \to \mathsf{BA}$ obtained by adding a constant of type S for $\mathbf{P}$ (see Remark 4.5).

Definition B.13. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $S \in \mathsf{C}$. A universal filter $(F_X)_{X\in\mathsf{C}}$ for $\mathbf{P}$ at S is *consistent* if $\perp_{\mathbf{P}(S)} \notin F_{\mathbf{t}}$.

Theorem B.14. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $S \in \mathsf{C}$. Let $F = (F_X)_{X\in\mathsf{C}}$ be a family with $F_X \subseteq \mathbf{P}(S\times X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class (resp. nonempty class) $\mathcal{M}$ of Boolean models of $\mathbf{P}$ at S such that, for every $X \in \mathsf{C}$,

$$F_X = \{\alpha \in \mathbf{P}(S\times X) \mid \textit{for all } (M,\mathfrak{m},s) \in \mathcal{M},\ \textit{for all } x \in M(X),\ (s,x) \in \mathfrak{m}_{S\times X}(\alpha)\},$$

(2) F is a universal filter (resp. consistent universal filter) for $\mathbf{P}$ at S.

Proof. This follows from Theorem A.5 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S and from Lemma 4.7. □

B.3. **Semantic characterization of universal ideal over fixed free variables.** We introduce *universal ideals at an object* S, meant to characterize the families of the form $I^{S,\mathcal{M}}$ for $\mathcal{M}$ an arbitrary class of models at S (see Notation B.8).

Definition B.15 (Universal ideal at an object)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A *universal ideal for* $\mathbf{P}$ *at* S is a family $(I_X)_{X\in\mathsf{C}}$, with $I_X \subseteq \mathbf{P}(S\times X)$ for each $X \in \mathsf{C}$, with the following properties.

(1) For all $m \in \mathbb{N}$, $(f_j\colon S\times X \to Y)_{j=1,\dots,m}$ and $\alpha \in \mathbf{P}(S\times Y)$, if $\bigwedge_{j=1}^m \mathbf{P}(\langle \mathrm{pr}_1, f_i\rangle)(\alpha) \in I_X$ then $\alpha \in I_Y$.

(2) For all $X \in \mathsf{C}$, I_X is downward closed.

(3) For all $\alpha_1 \in I_{X_1}$ and $\alpha_2 \in I_{X_2}$, in $\mathbf{P}(S\times X_1\times X_2)$ we have $\mathbf{P}(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)(\alpha_1) \vee \mathbf{P}(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)(\alpha_2) \in I_{X_1\times X_2}$.

(4) $\perp_{\mathbf{P}(S)} \in I_{\mathbf{t}}$.

Remark B.16. We translate Definition B.15 to the classic syntactic setting. For this, we fix $k \in \mathbb{N}$. Let $\{s_1,\dots,s_k,x_1,x_2,x_3,\dots\}$ be a set of variables, $\mathcal{L}$ a language and $\mathcal{T}$ a quantifier-free theory in $\mathcal{L}$. A *universal ideal for* $\mathcal{T}$ *at* k is a family $(I_n)_{n\in\mathbb{N}}$, with I_n a set of quantifier-free $\mathcal{L}$-formulas with $s_1,\dots,s_k,x_1,\dots,x_n$ as (possibly dummy) free variables, with the following properties.

(1) For all $p,q,m \in \mathbb{N}$, every $(m\cdot q)$-tuple $(f_{j,k}(s_1,\dots,s_k,x_1,\dots,x_p))_{j\in\{1,\dots,m\},\,k\in\{1,\dots,q\}}$ of $(k+p)$-ary terms and every quantifier-free formula $\alpha(s_1,\dots,s_k,x_1,\dots,x_q)$, if

$$\bigwedge_{j=1}^{m} \alpha(f_{j,1}(s_1,\dots,s_k,x_1,\dots,x_p),\dots,f_{j,p}(s_1,\dots,s_k,x_1,\dots,x_p)) \in I_p,$$

then

$$\alpha(s_1,\dots,s_k,x_1,\dots,x_q) \in I_q.$$

(2) For all $n \in \mathbb{N}$, all quantifier-free formulas $\alpha(s_1,\dots,s_k,x_1,\dots,x_n)$ and $\beta(s_1,\dots,s_k,x_1,\dots,x_n)$, if $\beta(s_1,\dots,s_k,x_1,\dots,x_n) \in I_n$ and $\alpha(s_1,\dots,s_k,x_1,\dots,x_n) \vdash_{\mathcal{T}} \beta(s_1,\dots,s_k,x_1,\dots,x_n)$, then

$$\alpha(s_1,\dots,s_k,x_1,\dots,x_n) \in I_n.$$

(3) For all $n_1, n_2 \in \mathbb{N}$, $\alpha_1(s_1, \ldots, s_k, x_1, \ldots, x_{n_1}) \in I_{n_1}$ and $\alpha_2(s_1, \ldots, s_k, x_1, \ldots, x_{n_2}) \in I_{n_2}$, we have

$$\alpha_1(s_1, \ldots, s_k, x_1, \ldots, x_{n_1}) \vee \alpha_2(s_1, \ldots, s_k, x_{n_1+1}, \ldots, x_{n_1+n_2}) \in I_{n_1+n_2};$$

(4) For all $n \in \mathbb{N}$, $\bot(s_1, \ldots, s_k, x_1, \ldots, x_n) \in I_n$.

These conditions are satisfied by any family $(I_n^{k,\mathcal{M}})_{n\in\mathbb{N}}$ defined by a class $\mathcal{M}$ as in Remark B.9.

Remark B.17. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A universal ideal for $\mathbf{P}$ at S is a universal ideal for the Boolean doctrine $\mathbf{P}_S\colon \mathsf{C}_S^{\mathrm{op}} \to \mathsf{BA}$ obtained by adding a constant of type S for $\mathbf{P}$ (see Remark 4.5).

Definition B.18. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and let $S \in \mathsf{C}$. A universal ideal $(I_X)_{X\in\mathsf{C}}$ for $\mathbf{P}$ at S is *consistent* if $\top_{\mathbf{P}(S)} \notin I_{\mathbf{t}}$.

Theorem B.19. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $S \in \mathsf{C}$. Let $I = (I_X)_{X\in\mathsf{C}}$ be a family with $I_X \subseteq \mathbf{P}(S \times X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class (resp. nonempty class) $\mathcal{M}$ of Boolean models of $\mathbf{P}$ at S such that, for every $X \in \mathsf{C}$,

$$I_X = \{\alpha \in \mathbf{P}(S \times X) \mid \textit{for all } (M, \mathfrak{m}, s) \in \mathcal{M},\ \textit{not all } x \in M(X) \textit{ satisfy } (s, x) \in \mathfrak{m}_{S\times X}(\alpha)\}.$$

(2) I is a universal ideal (resp. consistent universal ideal) for $\mathbf{P}$ at S.

Proof. This follows from Theorem A.9 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S and from Lemma 4.7. ☐

B.4. **Semantic characterization of universal filter-ideal pairs over fixed free variables.**

Definition B.20 (Universal filter-ideal pair at an object)**.** Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A *universal filter-ideal pair for $\mathbf{P}$ at S* is a pair (F, I) where F is a universal filter for $\mathbf{P}$ at S, I is a universal ideal for $\mathbf{P}$ at S, and the following conditions hold for all $Y \in \mathsf{C}$ and $\alpha \in \mathbf{P}(S \times Y)$.

(1) For all $X \in \mathsf{C}$, $n \in \mathbb{N}$, $(f_i\colon S \times X \to Y)_{i=1,\ldots,n}$ and $\beta \in F_X$, if $\beta \wedge \bigwedge_{i=1}^n \mathbf{P}(\langle \mathrm{pr}_1, f_i\rangle)(\alpha) \in I_X$, then $\alpha \in I_Y$.

(2) For all $Z \in \mathsf{C}$ and $\gamma \in I_Z$, if $\mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\alpha) \vee \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)(\gamma) \in F_{Y\times Z}$, then $\alpha \in F_Y$.

Remark B.21. Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, and $S \in \mathsf{C}$. A universal filter-ideal for $\mathbf{P}$ at S is a universal filter-ideal pair for the Boolean doctrine $\mathbf{P}_S\colon \mathsf{C}_S^{\mathrm{op}} \to \mathsf{BA}$ obtained by adding a constant of type S for $\mathbf{P}$ (see Remark 4.5).

Theorem B.22. *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small, and let $S \in \mathsf{C}$. Let $F = (F_X)_{X\in\mathsf{C}}$ and $I = (I_X)_{X\in\mathsf{C}}$ be families with $F_X \subseteq \mathbf{P}(S \times X)$ and $I_X \subseteq \mathbf{P}(S \times X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.*

(1) There is a class (resp. nonempty class) $\mathcal{M}$ of Boolean models of $\mathbf{P}$ at S such that, for every $X \in \mathsf{C}$,

$$F_X = \{\alpha \in \mathbf{P}(S \times X) \mid \textit{for all } (M, \mathfrak{m}, s) \in \mathcal{M},\ \textit{for all } x \in M(X),\ (s, x) \in \mathfrak{m}_{S\times X}(\alpha)\},$$
$$I_X = \{\alpha \in \mathbf{P}(S \times X) \mid \textit{for all } (M, \mathfrak{m}, s) \in \mathcal{M},\ \textit{not all } x \in M(X) \textit{ satisfy } (s, x) \in \mathfrak{m}_{S\times X}(\alpha)\}.$$

(2) (F, I) is a universal filter-ideal pair (resp. consistent universal filter-ideal pair) for $\mathbf{P}$ at S.

Proof. This follows from Theorem A.16 applied to the Boolean doctrine $\mathbf{P}_S$ obtained from $\mathbf{P}$ by adding a constant of type S and from Lemma 4.7. ☐

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