Non-existence of tensor t-structures on singular noetherian schemes

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Abstract

We show that there are no non-trivial tensor t-structures on the category of perfect complexes of a singular irreducible finite-dimensional noetherian scheme. To achieve this, we establish some technical results on Thomason filtrations and corresponding tensor t-structures.

In this note, we prove that there exist no non-trivial tensor t-structures on the derived category of perfect complexes of a singular irreducible noetherian scheme of finite dimension. This theorem (Theorem 2.6) is a generalisation of the affine case, which was proven by Smith, see [Smi22, Theorem 6.5]. Both the affine result of Smith, and the non-affine result we obtain in this document, are proved via the classification of compactly generated tensor t-structures in terms of Thomason filtrations. The classification of compactly generated t-structures on $\mathbf{D}(R)$ for a commutative noetherian ring R in terms of Thomason filtrations was done in [ATJLS10, Theorem 3.11]. More recently, a generalisation of this classification to compactly generated tensor t-structures on $\mathbf{D}_{\mathrm{qc}}(X)$ for a noetherian scheme X has been obtained in [DS23, Theorem 4.11], which is what our proof utilises to extend to the non-affine case.

Note that Theorem 2.6 cannot be true in the non-affine setting without some additional hypothesis on the class of t-structures (in our case, tensor compatibility is the additional hypothesis) as there are known examples of non-trivial t-structures on $\mathbf{D}^{\mathrm{perf}}(X)$, for X a singular variety, which arise from semi-orthogonal decompositions. This is unlike the affine case, since tensor compatibility is trivially satisfied by any t-structure on $\mathbf{D}^{\mathrm{perf}}(R)$.

1 Background and Notation

In this section we briefly recall some notation, definitions, and results used in this paper.

Let X be a noetherian scheme. Throughout this note, $\mathbf{D}_{\mathrm{qc}}(X)$ denotes the unbounded derived category of cochain complexes of $\mathcal{O}_X$-modules with quasicoherent cohomology, and $\mathbf{D}^{\mathrm{perf}}(X)$ denotes the derived category of perfect complexes on X. Let $Z \subseteq X$ be a closed subset, then we denote the corresponding full subcategories of complexes whose cohomology is supported on Z by $\mathbf{D}_{\mathrm{qc},Z}(X)$ and $\mathbf{D}^{\mathrm{perf}}_Z(X)$ respectively. Moreover, for a commutative ring R, $\mathbf{D}(R)$ denotes the unbounded derived category of cochain complexes of R-modules, and $\mathbf{D}^{\mathrm{perf}}(R)$ denotes the derived category of perfect complexes of R-modules.

Definition 1.1. Let X be a noetherian scheme, and let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$. We say that $(\mathcal{U}, \mathcal{V})$ is a *tensor t-structure* if $\mathbf{D}_{\mathrm{qc}}(X)^{\leq 0} \otimes \mathcal{U} \subseteq \mathcal{U}$, where $\mathbf{D}_{\mathrm{qc}}(X)^{\leq 0}$ denotes the aisle of the standard t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$. Moreover, a tensor t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$ is a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ which restricts to a t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$.

Note that when $X = \operatorname{Spec} R$, every compactly generated t-structure on $\mathbf{D}(R)$ is automatically a tensor t-structure, since the standard t-structure is compactly generated by the tensor unit R.

We now state the classification of compactly generated tensor t-structures on $\mathbf{D}_{\mathrm{qc}}(X)$ for a noetherian scheme X. Towards this end, we begin by recalling the definition of Thomason subsets and Thomason filtrations. Note that Thomason sets and filtrations can be defined in a more general setting, but we only state the definition for noetherian schemes.

Definition 1.2. Let X be a noetherian scheme, then a *Thomason subset* of X is precisely a specialisation closed subset. A *Thomason filtration* is a sequence $\Phi = \{Z^i\}_{i\in\mathbb{Z}}$ such that each Z^i is a Thomason subset of

X, and $Z^i \supseteq Z^{i+1}$ for all $i \in \mathbb{Z}$.

In [DS23], the authors obtain the following classification theorem for compactly generated tensor t-structures on noetherian schemes, generalising the classification of compactly generated t-structures on a noetherian ring from [ATJLS10].

Theorem 1.3 [DS23, Theorem 4.11]. *Let X be a noetherian scheme. Then, there is a bijective correspondence between the collection of compactly generated tensor t-structures on $\mathbf{D}_{\mathrm{qc}}(X)$ and the collection of Thomason filtrations on X.*

We give one of the maps of the bijective correspondence explicitly in the following remark, as we will make use of it later.

Remark 1.4. Let $\Phi = \{Z^i\}$ be a Thomason filtration, then the aisle of the corresponding t-structure is given by

$$\mathcal{U}_\Phi := \{E \in \mathbf{D}_{\mathrm{qc}}(X) \mid \mathrm{Supp}(\mathcal{H}^i(E)) \subseteq Z^i \text{ for all } i \in \mathbb{Z}\}.$$

Given a Thomason filtration Φ we will denote the corresponding tensor t-structure by $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$ and the truncation functors by $\tau_\Phi^{\leq 0}$ and $\tau_\Phi^{\geq 1}$.

We recall the following definition from [DS23].

Definition 1.5. Given a collection of objects $\mathcal{A} \subset \mathcal{T}$, we denote the smallest cocomplete preaisle of $\mathcal{T}$ containing $\mathcal{A}$ by $\langle \mathcal{A} \rangle^{\leq 0}$. Let X be a scheme and $j : U \hookrightarrow X$ an open immersion. Let $\mathcal{U}$ be a preaisle on X, then we define the restriction of the preaisle to U by $\mathcal{U}|_U := \langle j^*\mathcal{U} \rangle^{\leq 0}$.

Lemma 1.6 [DS23, Lemma 4.6]. *Let X be a noetherian scheme and $(\mathcal{U}, \mathcal{V})$ a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$, and U an open affine subscheme. Then, $(\mathcal{U}|_U, \mathcal{V}|_U)$ is a t-structure on $\mathbf{D}_{\mathrm{qc}}(U)$. Further, for any $F \in \mathbf{D}_{\mathrm{qc}}(X)$, the truncation triangle with respect to this t-structure is given by,*

$$j^*(\tau_{\mathcal{U}}^{\leq 0} F) \to j^*(F) \to j^*(\tau_{\mathcal{U}}^{\geq 1} F) \to j^*(\tau_{\mathcal{U}}^{\leq 0} F)[1]$$

where $\tau_{\mathcal{U}}^{\leq 0}$ and $\tau_{\mathcal{U}}^{\geq 1}$ are the truncation functors for $(\mathcal{U}, \mathcal{V})$.

We now state some results from [Smi22] which we will need in the following section.

Lemma 1.7 [Smi22, Lemma 6.2]. *Let R be a noetherian ring, $\mathfrak{p}$ a prime ideal, and $\Phi = \{Z^i\}$ a Thomason filtration such that $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$ restricts to a t-structure on $\mathbf{D}^{\mathrm{perf}}(R)$. We define the Thomason filtration $\Psi = \{Z^i \cap \mathrm{Spec}\, R_\mathfrak{p}\}$ on $\mathrm{Spec}\, R_\mathfrak{p}$. Then, the t-structure $(\mathcal{U}_\Psi, \mathcal{V}_\Psi)$ restricts to $\mathbf{D}^{\mathrm{perf}}(R_\mathfrak{p})$.*

Lemma 1.8 [Smi22, Lemma 6.3]. *Let R be a noetherian ring and $\Phi = \{Z^i\}$ a Thomason filtration such that $Z^i = \varnothing$ for large i, and there exists an integer j such that Z^j has a non-trivial intersection with the singular locus of $\mathrm{Spec}\, R$. Then, $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$ does not restrict to a t-structure on $\mathbf{D}^{\mathrm{perf}}(R)$.*

We recall the following well-known definition.

Definition 1.9. Let X be a scheme, and $Z \subset X$ be any subset. We define the *height* of Z, denoted by $\mathrm{height}(Z)$, to be the infimum over the dimensions of the local rings at all $x \in Z$.

The following is a combination of [Smi22, Proposition 5.3 & Corollary A.5]. Also see the proof of [Smi22, Theorem 6.4].

Theorem 1.10. *Let R be a noetherian ring. Let Φ be a Thomason filtration on $\mathrm{Spec}\, R$ such that $\mathrm{height}(Z^i) = h \geq 1$ for all $a \leq i \leq a + h$. Then, $H^{a+h}(\tau_\Phi^{\leq 0} R[-a])$ is not finitely generated as an R-module.*

2 Proof of the main result

Lemma 2.1. *Let X be a notherian scheme and U an open affine subscheme. Let $\Phi = \{Z^i\}$ be a Thomason filtration on X. Let $\Phi' = \{Z^i \cap U\}$ be the restricted Thomason filtration on U. Then, $\mathcal{U}_{\Phi'} = \mathcal{U}_\Phi|_U$.*

Proof. We first prove the easier inclusion, $\mathcal{U}_{\Phi'} \supseteq \mathcal{U}_{\Phi}|_U$. It is easy to see that that $j^*(\mathcal{U}_\Phi) \subseteq \mathcal{U}_{\Phi'}$ as the support condition is satisfied trivially, see Remark 1.4. As $\mathcal{U}_{\Phi'}$ is an aisle, it further contains $\mathcal{U}_\Phi|_U$, which is the cocomplete pre-aisle generated by $j^*(\mathcal{U}_\Phi)$.

Let $U = \operatorname{Spec}(R)$. Then, for the other inclusion, it is enough to show that for each prime $\mathfrak{p} \in Z^i \cap U \subseteq \operatorname{Spec}(R)$, there is a complex $K \in \mathbf{D}^{\mathrm{perf}}(X)$ with $K[-i] \in \mathcal{U}_\Phi$ such that $j^*(K) = K(\mathfrak{p})$. Note that the Koszul complex $K(\mathfrak{p})$ is supported on the closed subset $V(\mathfrak{p}) \subseteq U$. Hence, it lies in $\mathbf{D}_{\mathrm{qc},V(\mathfrak{p})}(U)$. Now, by [Nee, Corollary 3.5 and Remark 3.6], there is an equivalence $j_* : \mathbf{D}_{\mathrm{qc},V(\mathfrak{p})}(U) \leftrightarrows \mathbf{D}_{\mathrm{qc},V(\mathfrak{p})}(X) : j^*$, which restricts on the compacts to the equivalence, $j_! = j_* : \mathbf{D}^{\mathrm{perf}}_{V(\mathfrak{p})}(U) \leftrightarrows \mathbf{D}^{\mathrm{perf}}_{V(\mathfrak{p})}(X) : j^*$. We define K to be $j_*(K(\mathfrak{p}))$, which is a perfect complex by the above discussion. Then, $K[-i]$ lies in $\mathcal{U}_\Phi$. Finally, note that $K(\mathfrak{p}) \cong j^*(j_*(K(\mathfrak{p}))) = j^*(K)$, which is what we needed. $\square$

Lemma 2.2. *Let X be a noetherian scheme. Let $\Phi = \{Z^i\}$ be a Thomason filtration on X corresponding to a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ which restricts to a t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$. Let U be an affine open subset of X. Then the t-structure corresponding to the filtration $\Phi' = \{Z^i \cap \operatorname{Spec} R\}$ on $\mathbf{D}(R)$ restricts to $\mathbf{D}^{\mathrm{perf}}(R)$.*

Proof. By Lemma 1.6, $\mathcal{U}_\Phi|_U$ is an aisle and, for any $F \in \mathbf{D}_{\mathrm{qc}}(X)$, the truncation triangle with respect to this aisle is given by, $j^*(\tau_\Phi^{\leq 0} F) \to j^*(F) \to j^*(\tau_\Phi^{\geq 1} F) \to j^*(\tau_\Phi^{\leq 0} F)[1]$. By Lemma 2.1, we know that $\mathcal{U}_\Phi|_U = \mathcal{U}_{\Phi'}$, where Φ' is the Thomason filtration given by $\{Z^i \cap U\}$.

Now, we need to show that this aisle restricts to an aisle on $\mathbf{D}^{\mathrm{perf}}(R)$. That is, we need to show that the truncation triangles with respect to $\mathcal{U}_{\Phi'}$ respect perfect complexes. Let $U = \operatorname{Spec}(R)$. Then, we already know that $j^*(\tau_U^{\leq 0}\mathcal{O}_X) \to R \to j^*(\tau_U^{\geq 1}\mathcal{O}_X) \to j^*(\tau_U^{\leq 0}\mathcal{O}_X)[1]$ is the truncation triangle for R. As j^* preserves perfect complexes, we get that $j^*(\tau_U^{\leq 0}\mathcal{O}_X) = \tau_{\Phi'}^{\leq 0} R$ and $j^*(\tau_U^{\geq 1}\mathcal{O}_X) = \tau_{\Phi'}^{\geq 1} R$ are perfect complexes. Note that $\tau_{\Phi'}^{\leq 0}$ and $\tau_{\Phi'}^{\geq 1}$ respects summands, extensions, and shifts. As R is a classical generator for $\mathbf{D}^{\mathrm{perf}}(R)$, we get that the truncation triangles respect perfect complexes of $\mathbf{D}(R)$. $\square$

Lemma 2.3. *Let X be a noetherian scheme. Let $\Phi = \{Z^i\}$ be a Thomason filtration on X corresponding to a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ which restricts to $\mathbf{D}^{\mathrm{perf}}(X)$. Let $x \in X$ be some point. Then the filtration $\Phi' = \{Z^i \cap \operatorname{Spec} \mathcal{O}_x\}$ on $\operatorname{Spec} \mathcal{O}_x$ corresponds to a t-structure on $\mathbf{D}(\mathcal{O}_x)$ which restricts to $\mathbf{D}^{\mathrm{perf}}(\mathcal{O}_x)$.*

Proof. From Lemma 2.2 we can reduce the problem to the case where X is affine. Moreover, the affine case is already covered by 1.7. $\square$

Lemma 2.4. *Let X be a singular noetherian scheme. Let $\Phi = \{Z^i\}$ be a Thomason filtration on X such that there exists an r with $Z^r = \varnothing$ and an s with Z^s containing a singular point of X. Then the compactly generated tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ corresponding to this Thomason filtration does not restrict to $\mathbf{D}^{\mathrm{perf}}(X)$.*

Proof. Suppose for contradiction that the t-structure does restrict to the $\mathbf{D}^{\mathrm{perf}}(X)$. Now pick any affine open set $\operatorname{Spec} R$ containing the singular point in question. By Lemma 2.2 the filtration $\Phi' = \{Z^i \cap \operatorname{Spec} R\}$ restricts to $\mathbf{D}^{\mathrm{perf}}(R)$, so the question is local. Now we apply Lemma 1.8, which gives us a contradiction. $\square$

Lemma 2.5. *Let X be a noetherian scheme and let $\Phi = \{Z^i\}$ be a Thomason filtration on X corresponding to a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ which restricts to $\mathbf{D}^{\mathrm{perf}}(X)$. If $\dim X \geq h \geq 1$ then there are at most h many consecutive Z^i's of height h.*

Proof. Suppose for a contradiction that there are $h+1$ many consecutive Z^i's of height h, that is, there exists an a such that for i in the interval $[a, a+h]$, each Z^i has height $h \geq 1$. Take a point $x \in Z^{a+h}$ which is minimal, that is, the point x corresponds to a prime ideal $\mathfrak{p}$ in some affine open set $\operatorname{Spec} R$ such that $\operatorname{height}(\mathfrak{p}) = \operatorname{height}(Z^{a+h})$. Consider the Thomason filtration $\Phi' = \{Z^i \cap \operatorname{Spec} R\}$ on $\operatorname{Spec} R$. By our assumption combined with Lemma 2.2 we can see that the t-structure $(\mathcal{U}_{\Phi'}, \mathcal{V}_{\Phi'})$ restricts to $\mathbf{D}^{\mathrm{perf}}(R)$. The Thomason filtration $\Phi' = \{Z^i \cap \operatorname{Spec} R\}$ on $\operatorname{Spec} R$ also has the property that the height of $Z^i \cap \operatorname{Spec} R$ is equal to h for all $i \in [a, a+h]$, since the minimal point x of Z^{a+h} was assumed to be in $\operatorname{Spec} R$ and such filtrations are defined to be decreasing.

From Theorem 1.10, we see that this property of the filtration implies that $H^{a+h}(\tau_{\Phi}^{\leq 0} R[-a])$ is infinitely generated over R, and thus $\tau_{\Phi}^{\leq 0} R[-a]$ cannot be a perfect complex. Therefore the t-structure $(\mathcal{U}_{\Phi'}, \mathcal{V}_{\Phi'})$ cannot restrict to $\mathbf{D}^{\mathrm{perf}}(R)$, which is a contradiction. □

Theorem 2.6. *Let X be a singular irreducible finite-dimensional noetherian scheme. Let $(\mathcal{U}, \mathcal{V})$ be a tensor t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$, then either $\mathcal{U} = \varnothing$ or $\mathcal{U} = \mathbf{D}^{\mathrm{perf}}(X)$.*

Proof. To achieve this, we classify the possible Thomason filtrations on X which can correspond to tensor t-structures on $\mathbf{D}_{\mathrm{qc}}(X)$ which restrict to $\mathbf{D}^{\mathrm{perf}}(X)$.

If $\dim X = 0$, then X is just a single point. In which case there are only three possible Thomason filtrations on X, the constant filtration associated to $\varnothing$, the constant filtration associated to X, and the standard filtration up to shifting. Lemma 2.4 contradicts the possibility of the standard filtration being associated to a tensor t-structure which restricts to $\mathbf{D}^{\mathrm{perf}}(X)$. The other two filtrations correspond to the two trivial tensor t-structures.

Let $d = \dim X \geq 1$, and let $\Phi = \{Z^i\}$ be a Thomason filtration on X corresponding to a tensor t-structure on $\mathbf{D}_{\mathrm{qc}}(X)$ which restricts to $\mathbf{D}^{\mathrm{perf}}(X)$. Since X is finite-dimensional, the heights of the non-empty Z^i must be bounded between 0 and $\dim X$. Lemma 2.5 shows us that there can only h many Z^i of height h for $\dim X \geq h \geq 1$, therefore we can see that there can only be at most $d(d+1)/2$ non-empty Z^i of height $\dim X \geq h \geq 1$. Combining these two observations we see that the Thomason filtration must be bounded, in the sense that it must start with either $\varnothing$ or X, and end with either $\varnothing$ or X. In other words, the only permitted Thomason filtrations are of one of three forms, the constant filtration associated to the empty set:

$$\cdots \supseteq \varnothing \supseteq \cdots \supseteq \varnothing \supseteq \ldots,$$

the constant filtration associated to the entire space:

$$\cdots \supseteq X \supseteq \cdots \supseteq X \supseteq \ldots,$$

or some intermediate filtration with finitely many terms in the middle

$$\cdots \supseteq X \supseteq Z^{a+d(d+1)/2} \supseteq \cdots \supseteq Z^{a+1} \supseteq \varnothing \supseteq \ldots.$$

Now since X is assumed to be singular, Lemma 2.4 excludes the possibility of this intermediate filtration, as it would give us a contradiction. Therefore we can see that the only possibilities are the constant filtrations associated to either $\varnothing$ or X. These filtrations correspond to the two trivial tensor t-structures, so we are done. □

Remark 2.7. If X is a finite-dimensional noetherian scheme and $Z \subseteq X$ is an irreducible closed subset, it is an obvious question to wonder if there are any tensor t-structures on the category $\mathbf{D}_Z^{\mathrm{perf}}(X)$ whenever Z is not contained in the regular locus of X. Indeed, the details work out nicely with suitable modifications, and are to appear in forthcoming work by the authors.

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References

[ATJLS10] Leovigildo Alonso Tarrío, Ana Jeremías López, and Manuel Saorín. Compactly generated t-structures on the derived category of a noetherian ring. *Journal of Algebra*, 324(3):313–346, 2010.

[DS23] Umesh V Dubey and Gopinath Sahoo. Compactly generated tensor t-structures on the derived categories of Noetherian schemes. *Math. Z.*, 303(4):Paper No. 100, 22pp, 2023.

[Nee] Amnon Neeman. Bounded t-structures on the category of perfect complexes. Acta Math (to appear). arxiv:2202.08861v3.

[Smi22] Harry Smith. Bounded t-structures on the category of perfect complexes over a noetherian ring of finite krull dimension. *Adv. Math*, 399:Paper No. 108241, 21pp, 2022.

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