THE STRUCTURE OF ÉTALE BOOLEAN RIGHT RESTRICTION MONOIDS

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Abstract. In this paper, we describe étale Boolean right restriction monoids in terms of Boolean inverse monoids.

1. Introduction

The goal of this paper is to describe the structure of étale Boolean right restriction monoids in terms of Boolean inverse monoids motivated by [6, 8, 9]. Etale Boolean right restriction monoids are interesting because these are precisely the Boolean right restriction monoids whose associated categories, under non-commutative Stone duality, are in fact groupoids [9, Theorem 5.2].

Our starting point is the definition of the set of partial units[1] of a Boolean right restriction monoid S. They form a Boolean inverse monoid $\mathsf{Inv}(S)$ [27]; this was proved in [6, 9], although we also give a full proof of this result in Lemma 3.3. A Boolean right restriction monoid S is said to be étale if every element is a join of a finite number of partial units. Etale Boolean right restriction monoid were first defined in [9] though a special class of such monoids was actually used in [21]. A more general notion of 'étale' was defined in [6].

From the definition, we can see that there is a close connection between étale Boolean right restriction monoids and Boolean inverse monoids. This is made precise in Section 5. In Theorem 5.2, we show that the Boolean inverse monoid of partial units of an étale Boolean right restriction monoid determines the structure of that monoid. Our main theorem, Theorem 5.9, shows how to manufacture an étale Boolean right restriction monoid T from a Boolean inverse monoid S using tools developed in Section 4. We call T constructed in this way, the 'companion' of S. Section 6 provides some concrete examples of the theory we have developed including a discussion of the classical Thompson-Higman groups $G_{n,1}$.

The rest of this introduction is given over to outlining some of the background needed to read this paper.

On every right restriction monoid is defined a partial order called the natural partial order, which plays an important role in determining the structure of that monoid. For this reason, we shall need some definitions and notation from the theory of posets. Let $(X, \leq)$ be a poset. If $Y \subseteq X$ define

$$Y^{\uparrow} = \{x \in X \colon \exists y \in Y; y \leq x\} \text{ and } Y^{\downarrow} = \{x \in X \colon \exists y \in Y; x \leq y\}.$$

In the case where $Y = \{y\}$, we write $y^{\uparrow}$ and $y^{\downarrow}$ instead of $\{y\}^{\uparrow}$ and $\{y\}^{\downarrow}$, respectively. If $Y = Y^{\downarrow}$ we say that Y is an *order-ideal*. The subset Y is said to be *downwards directed* if $x, y \in Y$ implies that there is $z \in Y$ such that $z \leq x, y$. If $Y = Y^{\uparrow}$ we say that Y is *closed upwards*.

The usual order on the set of idempotents of any semigroup is defined by $e \leq f$ if $e = ef = fe$.

We shall use some basic topology in this paper [28]. Let X be a set. A set $\beta = \{U_i \colon i \in I\}$ of subsets of X is called a *base* if it satisfies two conditions: the

[1] We prefer this term to that of 'partial isomorphism' used by Cockett and Garner [6].

first is that $X = \bigcup_{U \in \beta} U$ and the second is that if $x \in U_1 \cap U_2$, where $U_1, U_2 \in \beta$ then there exists $U \in \beta$ such that $x \in U \subseteq U_1 \cap U_2$. Bases are used to *generate* topologies on X. A space X is said to be 0-*dimensional* if it has a base consisting of clopen sets. A compact, Hausdorff, 0-dimensional space is said to be *Boolean*. It is important to distinguish *partial homeomorphisms* and *local homeomorphisms*. By a partial homeomorphism we mean a homeomorphism between two open subsets of a topological space. A local homeomorphism is a union of partial homeomorphisms.

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2. Right restriction semigroups

Semigroups generalizing inverse semigroups were studied by a number of authors at various times, work nicely summarized in [12]. In addition, category theorists also became interested in categorical analogues of such semigroups, motivated by a desire to axiomatize categories of partial functions, notably in the work of Grandis [11] and Cockett (and his collaborators) [5]. We shall focus on monoids in this paper. The following well-known example is key and serves to motivate the class of semigroups we shall study in this paper.

Example 2.1. Functions will always be computed from right to left. Denote by $\mathsf{PT}(X)$ the set of all partial functions defined on the (non-empty) set X. See [10]. An element of $\mathsf{PT}(X)$ has the form $f \colon A \to X$ where $A \subseteq X$. We call the subset A the *domain of definition* of f; this set will be denoted by $\mathrm{dom}(f)$. Denote by f^* (called f-*star*) the identity function defined on $\mathrm{dom}(f)$. Observe that $ff^* = f$. Whereas identity functions defined on subsets of X are idempotents, it is not true that all idempotents have this form. The set of all those idempotents which are identities defined on subsets is denoted by $\mathsf{Proj}(\mathsf{PT}(X))$ and is called the set of *projections*. Partial functions f and g can be compared using subset inclusion. In fact, $f \subseteq g$ precisely when $f = gf^*$. With respect to this order, the set of projections forms a Boolean algebra. If $f, g \in \mathsf{PT}(X)$ then $f \cup g \in \mathsf{PT}(X)$ since $f \cup g$ precisely when $fg^* = gf^*$; in this case, we say that f and g are *left-compatible*. Observe also that $f^*g = g(fg)^*$ which expresses the fact that we are dealing with partial *functions*. We may regard $\mathsf{PT}(X)$ as an algebra of type $(1,2)$ equipped with the star operation and the semigroup binary operation.

The above example is a special case of the following definition. We define a semigroup S to be a *right restriction semigroup* if it is equipped with a unary operation $a \mapsto a^*$ satisfying the following axioms:

(RR1). $(s^*)^* = s^*$.
(RR2). $(s^*t^*)^* = s^*t^*$.
(RR3). $s^*t^* = t^*s^*$.
(RR4). $ss^* = s$.
(RR5). $(st)^* = (s^*t)^*$.
(RR6). $t^*s = s(ts)^*$.

The unary operation $s \mapsto s^*$ is called *star*. Denote by $\mathsf{Proj}(S)$ those elements a such that $a^* = a$, called *projections*. Let S and T be right restriction semigroups. A *homomorphism* $\theta \colon S \to T$ of right restriction semigroups is a semigroup homomorphism such that $\theta(a^*) = \theta(a)^*$. Such homomorphisms map projections to projections. The lemma below is well-known but is included for context. The proofs follow quickly from the axioms.

Lemma 2.2. *Let S be a right restriction semigroup.*

(1) *Each projection is an idempotent.*
(2) *$ae = a$ implies that $a^* \leq e$ whenever e is a projection.*
(3) *If S is a monoid then $1^* = 1$.*
(4) *$(ab)^* \leq b^*$ for all elements $a, b \in S$.*
(5) *If S has a zero which is a projection, then $a = 0$ if and only if $a^* = 0$.*
(6) *The product of projections is a projection.*

Remark 2.3. There is a Cayley-type representation theorem which says that given any right restriction semigroup S there is an embedding of right restriction semigroups into $\mathsf{PT}(S)$. Define $\phi \colon S \to \mathsf{PT}(S)$ where $\phi(a)$ is the partial function with domain of definition a^*S such that $\phi(a)(x) = ax$. This was first proved in [26].[2]

In a right restriction semigroup, define a binary relation $a \leq b$ on S by $a = ba^*$. The following are useful. Again, these results are well-known and are included for context. The proofs are easy.

Lemma 2.4. *Let S be a right restriction semigroup.*

(1) *If $a = be$, where e is a projection, then $a \leq b$.*
(2) *If $a = eb$, where e is a projection, then $a \leq b$.*
(3) *If $a \leq b$ then $a^* \leq b^*$.*
(4) *The relation $\leq$ is a partial order.*
(5) *The semigroup S is partially ordered with respect to $\leq$.*
(6) *The set of projections forms an order-ideal.*

We call $\leq$ the *natural partial order*. This will be the only partial order we consider on a right restriction semigroup. Observe that the natural partial order, when restricted to the projections, is the usual order on idempotents. The following results are well-known and easy to prove.

Lemma 2.5. *Let S be a right restriction semigroup.*

(1) *If $a, b \leq c$ and $a^* = b^*$ then $a = b$.*
(2) *If $a, b \leq c$ then $ab^* = ba^*$.*

Part (2) of the above lemma motivates the following definition. Define $a \sim_l b$, and say that a and b are *left-compatible*, if $ab^* = ba^*$.

Remark 2.6. Homomorphisms of right restriction semigroups preserve the natural partial order and left-compatibility.

The following is included for the sake of completeness.

Lemma 2.7. *In an inverse semigroup, we have that $a \sim_l b$ if and only if ab^{-1} is an idempotent.*

Proof. Suppose that $a \sim_l b$. Then $ab^{-1}b = ba^{-1}a$. Thus $ab^{-1} = (ba^{-1}a)a^{-1} = a(b^{-1}b)a^{-1}$, which is an idempotent. Conversely, suppose that ab^{-1} is an idempotent. Then $ab^{-1}b, ba^{-1}a \leq a, b$. But $(ab^{-1}b)^* = (ba^{-1}a)^*$. Thus $ab^{-1}b = ba^{-1}a$ by Lemma 2.5. $\square$

In an inverse semigroup, we define *right-compatibility* by $a \sim_r b$ if and only if $bb^{-1}a = aa^{-1}b$; this is equivalent to $a^{-1}b$ being an idempotent by the dual of Lemma 2.7. In an inverse semigroup, we say that a and b are *compatible* if they are both left-compatible and right-compatible. The proof of the following is immediate

[2] My thanks to Victoria Gould for supplying this reference.

Lemma 2.8. *In an inverse semigroup, we have that $a \sim_l b$ if and only if $a^{-1} \sim_r b^{-1}$, and $a \sim_r b$ if and only if $a^{-1} \sim_l b^{-1}$,*

Lemma 2.9. *Let S be a right restriction semigroup in which $a \sim_l b$.*
(1) *If $a \sim_l b$ and $c \sim_l d$ then $ac \sim_l bd$.*
(2) *If $a \sim_l b$ and $x \leq a$ and $y \leq b$ then $x \sim_l y$.*

Proof. (1) We are given that $a \sim_l b$ and $c \sim_l d$. This means that $ab^* = ba^*$ and $cd^* = dc^*$. Thus $ab^*cd^* = ba^*dc^*$. Now apply the axioms for a right restriction semigroup to get the result.

(2) We have that $ab^* = ba^*$ and $x = ax^*$ and $y = by^*$. We have that $xy^* = ax^*b^*y^* = ba^*x^*y^*$ and $yx^* = by^*a^*x^*$. Since projections commute, we have shown that $x \sim_l y$. $\square$

The following result tells us that being compatible is a property of the poset and the star operation alone and not the semigroup structure.

Lemma 2.10. *Let S be a right restriction semigroup. Then $a \sim_l b$ if and only if $a \wedge b$ exists and $(a \wedge b)^* = a^*b^*$.*

Proof. Suppose first $a \sim_l b$. Put $x = ab^* = ba^*$. Clearly, $x \leq a, b$ and $x^* = a^*b^*$. Suppose that $z \leq a, b$. Then $z^* \leq a^*, b^*$. By the definition of the natural partial order, we have that $z = az^* = bz^* \leq ab^* = x$. We now prove the converse. We have that $a \wedge b \leq a$ and so $a \wedge b = a(a \wedge b)^* = ab^*$. By symmetry, we have that $a \wedge b = ba^*$. It follows that $ab^* = ba^*$ and so a and b are left compatible. $\square$

Remark 2.11. If $a \sim_l b$ then $a \wedge b = ab^*$ and so this meet is *algebraically defined*. It is therefore preserved under any homomorphism of right restriction semigroups.

Let S be a right restriction semigroup. An element $a \in S$ is said to be a *partial unit* if there is an element $b \in S$ such that $ba = a^*$ and $ab = b^*$. Clearly, every projection is a partial unit. The set of all partial units of S is denoted by $\mathsf{Inv}(S)$.

Lemma 2.12. *Let S be a right restriction semigroup and let $a \in \mathsf{Inv}(S)$. Suppose that $ax = x^*$ and $xa = a^*$, and $ay = y^*$ and $ya = a^*$. Then $x = y$.*

Proof. We have that $xa = ya$. Thus $xay = yay$. It follows that $y = xy^*$ and so $y \leq x$. By symmetry, $x \leq y$ and so $x = y$. $\square$

Let S be a right restriction semigroup. If $a \in \mathsf{Inv}(S)$ then we shall often denote by a^{-1} the unique element guaranteed by Lemma 2.12 such that $aa^{-1} = (a^{-1})^*$ and $a^{-1}a = a^*$. We shall now say more about the set $\mathsf{Inv}(S)$. Most of the following was first proved as [6, Lemma 2.14]. We give proofs anyway.

Lemma 2.13. *Let S be a right restriction semigroup.*
(1) *If $a, b \in \mathsf{Inv}(S)$ then $ab \in \mathsf{Inv}(S)$.*
(2) *If $a, b \in \mathsf{Inv}(S)$ then $(ab)^{-1} = b^{-1}a^{-1}$.*
(3) *If $a, b \in \mathsf{Inv}(S)$ then $a \leq b$ if and only if $a = aa^{-1}b$.*
(4) *$\mathsf{Inv}(S)$ is an inverse semigroup with set of idempotents $\mathsf{Proj}(S)$.*
(5) *$\mathsf{Inv}(S)$ is an order-ideal.*

Proof. (1) We have that $aa^{-1} = (a^{-1})^*$ and $a^{-1}a = a^*$ and $bb^{-1} = (b^{-1})^*$ and $b^{-1}b = b^*$. Observe that $ab(b^{-1}a^{-1}) = (b^{-1}a^{-1})^*$ and $(b^{-1}a^{-1})ab = (ab)^*$.

(2) Immediate from (1) above.

(3) Only one direction needs proving. Suppose that $a \leq b$. Then $a = ba^*$. Now, a has inverse a^{-1} but a^*b^{-1} is also an inverse. So, by the uniqueness of inverses guaranteed by Lemma 2.12, we have that $a^{-1} = a^*b^{-1}$. Thus $a^{-1} \leq b^{-1}$ and so $a^{-1} = b^{-1}(a^{-1})^*$ from the definition of the natural partial order. Taking inverses of both sides again, we get that $a = (a^{-1})^*b$ and so $a = aa^{-1}b$, as required.

(4) By (1) above, it follows that $\mathsf{Inv}(S)$ is closed under products. Suppose, now, that $a \in \mathsf{Inv}(S)$ is an idempotent. Let $b \in S$ be such that $ab = b^*$, $ba = a^*$. Since a is an idempotent, we have that $a = (aba)(aba) = a(ba)(ab)a$. But projections commute. Thus $a = (a^2b)(ba^2) = (ab)(ba)$. It follows that a is the product of two projections and so is itself a projection. Since the projections commute, we have proved that $\mathsf{Inv}(S)$ is an inverse semigroup.

(5) Suppose that $a \in \mathsf{Inv}(S)$ and that $b \leq a$. We prove that $b \in \mathsf{Inv}(S)$. By definition, $b = ab^*$. Observe that $b^*a^{-1}b = a^*b^* = (ab^*)^*$ and $bb^*a^{-1} = ab^*a^{-1} = (a^{-1})(b^*a^{-1})^* = (b^*a^{-1})^*$. This proves that $b \in \mathsf{Inv}(S)$. □

3. Order completeness properties of right restriction semigroups

We shall study right restriction semigroups which satisfy some order completeness properties with respect to the natural partial order. A set of elements in a right restriction monoid is said to be *left-compatible* if each pair of elements is left-compatible. We say that a right restriction semigroup is *complete* if every left-compatible set of elements has a join and multiplication distributes over such joins *from the right*. We say that a right restriction semigroup is *distributive* if each pair of left-compatible elements has a join, multiplication distributes over binary joins *from the right*, and the projections form a distributive lattice. A distributive right restriction semigroup is *Boolean* if the set of projections actually forms a generalized Boolean algebra.

We can make similar definitions for inverse semigroups, but require compatibility rather than left-compatibility.

In the following result, part (1) is proved in [4, Proposition 2.14(i)], part (2) is a slightly expanded version of [16, Lemma 2.15], part (3) is proved in [4, Proposition 2.14(iii)], and parts (4) and (5) are the analogues of parts (3) and (4) of [20, Lemma 2.5] with almost identical proofs.

Lemma 3.1. *Let S be a right restriction semigroup.*

1. *If $\bigvee_{j\in I} a_j$ exists then $a_i = \left(\bigvee_{j\in I} a_j\right) a_i^*$ for each $i \in I$.*
2. *If both $\bigvee_{i\in I} a_i$ and $\bigvee_{i\in I} a_i^*$ exist then $\bigvee_{i\in I} a_i^*$ is a projection and $\left(\bigvee_{i\in I} a_i\right)^* = \bigvee_{i\in I} a_i^*$.*
3. *Let S be a complete right restriction semigroup and if I is finite then we may assume that S is only a distributive right restriction semigroup. Suppose that $\bigvee_{i\in I} a_i$ is defined. Then $c\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} ca_i$.*
4. *Let S be a distributive right restriction semigroup. Suppose that $\bigvee_{i=1}^m a_i$ and $c \wedge (\bigvee_{i=1}^m a_i)$ both exist. Then all meets $c \wedge a_i$ exist, the join $\bigvee_{i=1}^m c \wedge a_i$ exists, and $c \wedge (\bigvee_{i=1}^m a_i) = \bigvee_{i=1}^m c \wedge a_i$.*
5. *Let S be a distributive right restriction semigroup. Suppose that $b = \bigvee_{i=1}^m b_i$ exists, and all meets $a \wedge b_i$ exist. Then the meet $a \wedge b$ exists, and is equal to $\bigvee_{i=1}^m a \wedge b_i$.*

Remark 3.2. The above lemma tells us that although complete or distributive right restriction monoids were defined in terms of multiplication distributing over any joins on the right, in fact, multiplication in such monoids distributes over any joins also from the left.

The following result is expected. It was first proved in [6].

Lemma 3.3. *In a Boolean right restriction monoid, the set of partial units forms a Boolean inverse monoid.*

Proof. It is enough to prove that if a and b are partial units which are compatible then $a \vee b$ is a partial unit. Since a and b are compatible, so too are a^{-1} and b^{-1}.

It follows that the element $a^{-1} \vee b^{-1}$ is defined. We calculate $(a \vee b)(a^{-1} \vee b^{-1})$. This equals $aa^{-1} \vee ab^{-1} \vee ba^{-1} \vee bb^{-1}$. By assumption, both ab^{-1} and ba^{-1} are idempotents. Observe that $ab^{-1} \leq aa^{-1}$ and $ba^{-1} \leq bb^{-1}$ Thus $(a \vee b)(a^{-1} \vee b^{-1}) = aa^{-1} \vee bb^{-1}$. This is a join of projections and so is a projection by Lemma 3.1. Dually, we have that $(a^{-1} \vee b^{-1})(a \vee b)$ is a projection itself. $\square$

We say that a Boolean right restriction monoid is *étale* if every element is a join of a finite number of partial units.

The goal of this paper is to describe étale Boolean right restriction monoids in terms of their Boolean inverse monoids of partial units.

4. Complete right restriction monoids

The material in this section generalizes the notion of nucleus to be found in [13, Chapter II, Section 2] by way of what we did in [22, Section 4].

Let S be a right restriction monoid. If $A \subseteq S$, define $A^* = \{a^* \colon a \in A\}$. The proof of the following is straightforward; for the proof of part (3) use Lemma 2.9.

Lemma 4.1. *Let S be a right restriction monoid.*

(1) *If A and B are order-ideals then AB is an order-ideal.*
(2) *If A is an order-ideal then A^* is an order-ideal.*
(3) *If A and B are both left-compatible sets then AB is a left-compatible set.*

We shall generalize [24] and prove that every right restriction monoid can be embedded in a complete right restriction monoid, although the same construction can be found in [6]. We say that a subset of S is *acceptable* if it is a left-compatible order-ideal. Put $\mathsf{R}(S)$ equal to the set of all acceptable subsets of S. Observe that subsets of S of the form $a^{\downarrow}$ are acceptable by Lemma 2.5. We may therefore define a function $\iota \colon S \to \mathsf{R}(S)$ by $\iota(a) = a^{\downarrow}$.

Proposition 4.2. *Let S be a right restriction monoid. Then $\mathsf{R}(S)$ is a complete right restriction monoid in which the natural partial order is subset inclusion and the projections are the order-ideals of $\mathsf{Proj}(S)$. In addition, the function $\iota \colon S \to \mathsf{R}(S)$ is an embedding of right restriction monoids.*

Proof. We first show that $\mathsf{R}(S)$ is a right restriction monoid. Using parts (1) and (3) of Lemma 4.1, the set $\mathsf{R}(S)$ is a semigroup under subset multiplication. The set of all order-ideals of $\mathsf{Proj}(S)$ is a set of idempotents for $\mathsf{R}(S)$. Observe that the set of all projections, $1^{\downarrow}$, is an identity for $\mathsf{R}(S)$ which is therefore a monoid. If we define a unary map on $\mathsf{R}(S)$ by $A \mapsto A^*$ then this is well-defined by part (2) of Lemma 4.1. It remains to check that $\mathsf{R}(S)$ is a right restriction monoid with respect to these operations. The proofs of axioms (RR1), (RR2) and (RR3) are immediate. To prove that axiom (RR4) holds, it is immediate that $A \subseteq AA^*$. The proof that the reverse inclusion holds follows from the fact that A is an order-ideal. The proof that axiom (RR5) holds is immediate. It remains to show that axiom (RR5) holds. It is immediate that $A^*B \subseteq B(AB)^*$ by axiom (RR6). We now prove the reverse inclusion. Let $b(ab_1)^*$ be such that $b, b_1 \in B$ and $a \in A$. We shall prove that this is an element of A^*B. Because $b, b_1 \in B$, an acceptable set, we have that $bb_1^* = b_1b^*$. We have that

$$b(ab_1)^* = b(ab_1)^*b_1^* = bb_1^*(ab_1)^* = b_1b^*(ab_1)^* = b_1(ab_1)^*b^*.$$

Thus

$$b(ab_1)^* = b_1(ab_1)^*b^* = a^*b_1b^*.$$

But $b_1b^* \in B$ because B is an order-ideal. We have therefore proved that $B(AB)^* \subseteq A^*B$ This completes the proof that $\mathsf{R}(S)$ is a right restriction monoid.

Claim: if A and B are acceptable sets then $A \leq B$ in $\mathsf{R}(S)$ if and only if $A \subseteq B$. We now prove the claim. Suppose first that $A \leq B$. By definition $A = BA^*$. Let $a \in A$. Then $a = bc^*$ where $b \in B$ and $c \in A$. But $a \leq b$ and $b \in B$ an order-ideal. It follows that $a \in B$ because B is an order-ideal. We have proved that $A \subseteq B$. We now prove the converse. Suppose that $A \subseteq B$. We prove that $A = BA^*$. Observe that $A \subseteq BA^*$. Let $x \in BA^*$. Then $x = bc^*$ where $c \in A$. We have that $A \subseteq B$ and so $b \sim_l c$. It follows that $bc^* = cb^*$. Thus $x = cb^*$ and so $x \leq c$. But $c \in A$ and A is an order-ideal and so $c \in A$, as required.

Claim: if A and B are acceptable sets then $A \sim_l B$ in $\mathsf{R}(S)$ if and only if $A \cup B \in \mathsf{R}(S)$. We now prove the claim. Suppose first that $A \cup B \in \mathsf{R}(S)$. We prove that $A \sim_l B$. In fact, we shall prove that $AB^* \subseteq BA^*$ and then appeal to symmetry. Let $ab^* \in AB^*$. By assumption, $A \cup B$ is an acceptable set and so, in particular, $ab^* = ba^*$. It follows that $ab^* \in BA^*$. Suppose now that $A \sim_l B$. We shall prove that $A \cup B \in \mathsf{R}(S)$. It is enough to prove that if $a \in A$ and $b \in B$ then $a \sim_l b$. We are given that $AB^* = BA^*$. We have that $ab^* \in AB^*$ and so $ab^* = b_1 a_1^*$ where $b_1 \in B$ and $a_1 \in A$. By assumption, $a \sim a_1$ and $b \sim b_1$. We claim that $ab^* \leq ba^*$ and symmetry delivers the result. To prove the claim, we use the fact that $ab^* = b_1 a_1^*$. Thus

$$ab^* = b_1 a_1^* b^* a^* = b_1 b^* a_1^* a^* = b b_1^* (a_1^* a^*).$$

With these two results, we can now prove that $\mathsf{R}(S)$ is a *complete* right restriction monoid. Let $\{A_i : i \in I\}$ be a left-compatible subset of $\mathsf{R}(S)$. We claim that $A = \bigcup_{i \in I} A_i \in \mathsf{R}(S)$. We therefore have to prove that A is acceptable. It is clearly an order-ideal and so we have to show that any two elements of A are left-compatible. Without loss of generality, suppose that $a \in A_i$ and $b \in A_j$. Then, by the above, $A_i \cup A_j$ is acceptable and so a and b are left-compatible. We have proved that A is an acceptable set. Also, by what we proved above we have that $A_i \leq A$ for any $i \in I$. Suppose that $A_i \leq B$ for any $i \in I$ where B is acceptable. Then $A_i \subseteq B$ for any $i \in I$, by what we proved above. Thus $A \subseteq B$ and so $A \leq B$. We have therefore proved that $\mathsf{R}(S)$ has joins of left-compatible subsets. Now, let $\{A_i : i \in I\}$ be a left-compatible subset of $\mathsf{R}(S)$ and let B be any acceptable set. We have to prove that $\left(\bigcup_{i \in I} A_i\right) B = \bigcup_{i \in I} A_i B$. However, this is true on set-theoretic grounds alone. This completes the proof that $\mathsf{R}(S)$ is a complete right restriction monoid.

It remains to prove that the function ι is a homomorphism of right restriction monoids. Observe that $a^{\downarrow} b^{\downarrow} = (ab)^{\downarrow}$; this is true since if $a' \leq a$ and $b' \leq b$ then $a'b' \leq ab$ and if $x \leq ab$ then $x = a(bx^*) = a'b'$ where $a' = a$ and $b' = bx^* \leq b$. We therefore have a homomorphism of semigroups which is also a monoid homomorphism since the identity of $\mathsf{R}(S)$ is $1^{\downarrow}$. Thus we finish if we show that ι is a homomorphism of right restriction monoids. This requires us to show that $\iota(a^*) = \iota(a)^*$. Let $x \in \iota(a^*)$. Then $x \leq a^*$ and so is a projection. Consider the element ax. Because x is a projection, we have that $ax \leq a$ and so $ax \in \iota(a)$. But $(ax)^* = a^* x = x$. It follows that $x \in \iota(a)^*$. On the other hand, let $b^* \in \iota(a)^*$ where $b \leq a$. It follows that $b^* \leq a^*$ and so $b^* \in \iota(a^*)$. □

The above procedure can be applied, *inter alia*, when S is a Boolean inverse monoid. This is the only case that will interest us in Section 5.

We can say more about the function $\iota \colon S \to \mathsf{R}(S)$ and so the construction of $\mathsf{R}(S)$. A *homomorphism* between complete right restriction semigroups is a right restriction homomorphism that preserves joins.

Proposition 4.3. *The map $\iota \colon S \to \mathsf{R}(S)$ is universal for right restriction monoid homomorphisms to complete right restriction monoids.*

Proof. Let T be a complete right restriction monoid and let $\alpha \colon S \to T$ be a monoid homorphism of right restriction monoids. Define $\beta \colon \mathsf{R}(S) \to T$ by $\beta(A) = \bigvee_{a \in A} \alpha(a)$. This makes sense since the elements of A are pairwise left-compatible and left-compatibility is preserved by homomorphisms of right restriction semigroups. We now calculate $\beta(\iota(a))$. By definition this is $\beta(a^{\downarrow})$ which is $\bigvee_{x \leq a} \alpha(x)$. Observe that $x \leq a$ implies that $\alpha(x) \leq \alpha(a)$. It follows that $\beta(\iota(a)) = \alpha(a)$. We show that β is a right restriction monoid homomorphism. It is immediate from the definitions that this is a monoid homomorphism. We need to prove that it is a homomorphism of right restriction semigroups. Let A be an acceptable set. Then, by definition,

$$\beta(A^*) = \bigvee_{a^* \in A^*} \alpha(a^*).$$

But α is a homomorphism of right restriction semigroups. Thus $\alpha(a^*) = \alpha(a)^*$. Now apply part (2) of Lemma 3.1 to get

$$\beta(A^*) = \left(\bigvee_{a^* \in A^*} \alpha(a) \right)^* .$$

We now use the fact that $a \in A$ if and only if $a^* \in A$. This gives us

$$\beta(A^*) = \left(\bigvee_{a \in A} \alpha(a) \right)^* = \beta(A)^*.$$

We have therefore shown that β is a homomorphism of right restriction semigroups. We show that β preserves arbitrary left-compatible joins. Let $\{A_i \colon i \in I\}$ be a left-compatible set in $\mathsf{R}(S)$. Put $A = \bigcup_{i \in I} A_i$, the join of the A_i in $\mathsf{R}(S)$. By definition

$$\beta(A) = \bigvee_{a \in A} \alpha(a) = \bigvee_{a \in A_i, i \in I} \alpha(a) = \bigvee_{i \in I} \left(\bigvee_{a \in A_i} \alpha(a) \right).$$

But this is equal to

$$\bigvee_{i \in I} \beta(A_i).$$

We finish off by proving the categorical property we need. Observe that for any acceptable set A we have that $A = \bigcup_{a \in A} a^{\downarrow}$, because A is an order-ideal. It now follows that if β' is a homomorphism of complete right restriction monoids such that $\alpha = \beta' \iota$ then $\beta' = \beta$ $\square$

Proposition 4.2 tells us how to manufacture complete right restriction monoids from monoids that are merely right restriction monoids. For the rest of this section, we shall work with an *arbitrary* complete right restriction monoid S. A function $\nu \colon S \to S$ is called a *nucleus* if it satisfies the following six conditions:

(N1). $a \leq \nu(a)$.
(N2). $a \leq b$ implies that $\nu(a) \leq \nu(b)$.
(N3). $\nu^2(a) = \nu(a)$.
(N4). $\nu(a)\nu(b) \leq \nu(ab)$.
(N5). If e is a projection then $\nu(e)$ is a projection.
(N6). $\nu(a^*) = \nu(\nu(a)^*)$.

This clearly generalizes to a non-commutative setting the classical notion of nucleus, to be found in, say, [13]. We shall use nuclei to construct new complete restriction monoids from old ones.

Lemma 4.4. *Let ν be a nuclus defined on the complete right restriction monoid S. Then*

$$\nu(ab) = \nu(a\nu(b)) = \nu(\nu(a)b) = \nu(\nu(a)\nu(b)).$$

Proof. By (N1), we have that $a \leq \nu(a)$ and $b \leq \nu(b)$. In particular, $ab \leq \nu(a)\nu(b)$. Thus by (N2), we have that $\nu(ab) \leq \nu(\nu(a)\nu(b))$. But by (N4), we have that $\nu(a)\nu(b) \leq \nu\nu(ab)$. Thus by (N2), we have that $\nu(\nu(a)\nu(b)) \leq \nu^2(ab)$. But by (N3), we have that $\nu(\nu(a)\nu(b)) \leq \nu(ab)$. We have therefore proved that $\nu(ab) = \nu(\nu(a)\nu(b))$. The other cases are proved similarly. □

Let S complete right restriction monoid equipped with a nucleus ν. Define

$$S_\nu = \{a \in S \colon \nu(a) = a\},$$

the set of ν-*closed* elements. Define $\cdot$ on S_ν by

$$a \cdot b = \nu(ab).$$

The following result tells us exactly how to build a new complete right restriction monoid from an old one equipped with a nucleus.

Proposition 4.5. *Let ν be a nucleus defined on the complete right restriction monoid S. Then $(S_\nu, \cdot)$ is also a complete right restriction monoid.*

Proof. By Lemma 4.4, $(S_\nu, \cdot)$ is a semigroup. It is, in fact, a monoid with identity $\nu(1)$ since $a \cdot \nu(1) = \nu(a\nu(1))$ where $a \in S_\nu$. By Lemma 4.4, we have that $\nu(a\nu(1)) = \nu(a1) = \nu(a) = a$. We have proved that $\nu(1)$ is a right identity. It is a left identity by symmetry.

We now prove that S_ν is a right restriction monoid. Put $\mathsf{Proj}(S_\nu) = \{\nu(e) \colon e \in \mathsf{Proj}(S)\}$. This is a set of projections of S by axiom (N5). Thus $\nu(1)$ is a projection. If $a \in S_\nu$, define

$$a^\star = \nu(a^*);$$

observe that on the left we have an honest-to-goodness star, whereas on the right we have an asterisk. It is a projection by (N5). We now show that the axioms for a right restriction semigroup hold. Let $a, b \in S_\nu$.

(RR1) holds: $(a^\star)^\star = \nu(\nu(a^*)^*) = \nu(\nu(a^*)) = \nu(a^*) = a^\star$ by (N5) and (N3).

(RR2) holds: $(a^\star \cdot b^\star)^\star = (\nu(a^\star b^\star))^\star$. This is equal to $\nu(\nu(a^*)\nu(b^*))^\star = \nu(a^*b^*)^\star$ using Lemma 4.4. This is equal to $\nu((\nu(a^*b^*)^*) = \nu(a^*b^*)$ using (N6). Whereas $a^\star \cdot b^\star = \nu(a^\star b^\star) = \nu(\nu(a^*)\nu(b^*)) = \nu(a^*b^*)$ using Lemma 4.4.

(RR3) holds: $a^\star \cdot b^\star = \nu(a^\star b^\star) = \nu(b^\star a^\star) = b^\star \cdot a^\star$ where we have used (N5).

(RR4) holds: $a \cdot a^\star = \nu(aa^\star) = \nu(a\nu(a^*) = \nu(aa^*) = \nu(a) = a$ by Lemma 4.4.

(RR5) holds: $(a \cdot b)^\star = \nu(\nu(ab)^*) = \nu((ab)^*)$ by (N6). On the other hand, $(a^\star \cdot b)^\star = \nu(\nu(\nu(a^*)b)^*) = \nu(\nu(a^*b)^*) = \nu((a^*b)^*) = \nu((ab)^*)$ by (N3) and (N6).

(RR6) holds: $b^\star \cdot a = \nu(b^\star a) = \nu(\nu(b^*)a) = \nu(b^*a)$ by Lemma 4.4. Whereas $a \cdot (b \cdot a)^\star = \nu(a(b \cdot a)^\star) = \nu(a\nu(ba)^\star) = \nu(a\nu(\nu(ba)^*)) = \nu(a(ba)^*)$ by (N6) and Lemma 4.4. Thus, we have proved that $(S_\nu, \cdot)$ is a right restriction monoid.

Let $a, b \in S_\nu$. Denote the natural partial order on S_ν by $\preceq$. Claim: $a \preceq b$ if and only if $a \leq b$. Proof of claim. Suppose first that $a \preceq b$. Then $a = b \cdot a^\star$. Thus $a = \nu(ba^\star) = \nu(b\nu(a^*)) = \nu(ba^*)$ using Lemma 4.4. But $ba^* \leq b$ and so $\nu(ba^*) \leq \nu(b) = b$ by (N2). Thus $a \leq b$. Suppose now that $a \leq b$. This means that $a = ba^*$. Thus $a = \nu(ba^*) = \nu(b\nu(a^*)) = b \cdot a^\star$ by Lemma 4.4. Whence we have proved that $a \preceq b$.

Claim: $a \sim_l b$ in S_ν if and only if $a \sim_l b$ in S. Proof of claim. Suppose, first, that $a \sim_l b$ in S_ν. Then $a \cdot b^\star = b \cdot a^\star$. This means that $\nu(a\nu(b^*)) = \nu(b\nu(a^*))$. Consequently, we have that $\nu(ab^*) = \nu(ba^*)$ by Lemma 4.4. But $ab^* \leq \nu(ab^*)$ by (N1). Similarly, $ba^* \leq \nu(ba^*)$. It follows that $ab^* \sim_l ba^*$ by Lemma 2.5. Thus $ab^*(ba^*)^* = ba^*(ab^*)^*$. Whence $ab^* = ba^*$ and so $a \sim_l b$ in S. Now, suppose that $a \sim_l b$ in S. This means that $ab^* = ba^*$. But $a \cdot \nu(b^*) = a \cdot b^\star$ and $a \cdot \nu(b^*) = \nu(a\nu(b^*)) = \nu(ab^*)$ by Lemma 4.4. It follows that $a \sim_l b$ in S.

Claim: if $X = \{a_i : i \in I\}$ is a left-compatible set in S_ν then the join in S_ν of X exists, it is denoted by $\bigsqcup_{i\in I} a_i$, and is equal to $\nu\left(\bigvee_{i\in I} a_i\right)$. Proof of claim. By the above, this is a left-compatible set in S. It therefore has a join a in S. We claim that $\nu(a)$ is the join of X in S_ν. It is an element of S_ν by (N2). We have that $a_i \leq a$ for all i. Thus $a_i \leq \nu(a)$ for all i by (N2). Let $a_i \leq b$ for all i where $b \in S_\nu$. This means that $a_i \leq b$ in S. Thus $a \leq b$. It follows that $\nu(a) \leq b$ in S_ν. Thus the join of X exists in S_ν. It follows that all joins of left-compatible subsets of S_ν exists.

Claim: $\nu\left(\bigvee_{i\in I} \nu(a_i)\right) = \nu(\bigvee_{i\in I} a_i)$, where the a_i are arbitrary elements of S which form a left-compatible set. Proof of claim. We have that $a_i \leq \bigvee_{i\in I} a_i$. Thus $\nu(a_i) \leq \nu\left(\bigvee_{i\in I} a_i\right)$ by (N2). Thus $\bigvee_{i\in I} \nu(a_i) \leq \nu\left(\bigvee_{i\in I} a_i\right)$. Whence $\nu\left(\bigvee_{i\in I} \nu(a_i)\right) \leq \nu\left(\bigvee_{i\in I} a_i\right)$ using (N2). To prove the reverse inequality, we start with $a_i \leq \nu(a_i)$ by (N1). It follows that $\bigvee_{i\in I} a_i \leq \bigvee_{i\in I} \nu(a_i)$ and so $\nu\left(\bigvee_{i\in I} a_i\right) \leq \nu\left(\bigvee_{i\in I} \nu(a_i)\right)$.

Claim: $\left(\bigsqcup_{i\in I} a_i\right)\cdot b = \bigsqcup_{i\in I} a_i\cdot b$ in S_ν. Proof of claim. We have that $\left(\bigsqcup_{i\in I} a_i\right)\cdot b = \nu\left(\bigvee_{i\in I} a_i b\right)$ and $\bigsqcup_{i\in I} a_i \cdot b = \nu\left(\bigvee_{i\in I} \nu(a_i b)\right)$. The result now follows by what we proved above. □

We are actually interested in constructing Boolean right restriction monoids. The following concept is just what we need to cut down from arbitrary joins to finitary ones. Let S be a complete right restriction monoid. An element $a \in S$ is said to be *finite* if whenever $a \leq \bigvee_{i\in I} a_i$ then $a \leq \bigvee_{i=1}^{m} a_i$, relabelling if necessary. Denote the set of finite elements of a complete right restriction monoid S by $\mathsf{fin}(S)$.

Lemma 4.6. *Let S be a complete right restriction monoid.*

1. *$a \in S$ is finite if and only if a^* is finite.*
2. *If a and b are finite and $a \sim_l b$ then $a \vee b$ is finite.*
3. *If the finite elements are closed under multiplication then they form a distributive right restriction semigroup.*

Proof. The proof of (1) follows from Lemma 3.1. To prove (2), suppose that a and b are both finite and $a \sim_l b$. We prove that $a \vee b$ is finite. Suppose that $a \vee b \leq \bigvee_{i\in I} a_i$. Since $a \leq a \vee b$ there is a finite subset I_1 of I such that $a \leq \bigvee_{i\in I_1} a_i$. Likewise, there is a finite subset I_2 of I such that $b \leq \bigvee_{i\in I_2} a_i$. It follows that $a \vee b \leq \bigvee_{i\in I_1\cup I_2} a_i$, and so $a \vee b$ is finite. The proof of (3) now follows from (1) and (2) above. □

There is no guarantee that the product of finite elements is finite though, as we shall see, this will hold in the case of interest to us.

5. The structure of étale Boolean right restriction monoids

The goal of this section is to show how to construct étale Boolean right restriction monoids from Boolean inverse monoids [27].

Remark 5.1. Observe that if S and T are isomorphic as inverse semigroups, then the partially ordered sets $(S, \leq)$ and $(T, \leq)$ are order isomomorphic. We shall use this observation below in the course of the proof of our first theorem.

Our first theorem below shows that the structure of an étale Boolean right restriction monoid is completely determined by the structure of its Boolean inverse monoid of partial units.

Theorem 5.2. *Let S and T be étale Boolean right restriction monoids. If $\mathsf{Inv}(S) \cong \mathsf{Inv}(T)$ then $S \cong T$ as right restriction semigroups.*

Proof. Let $\theta\colon \mathsf{Inv}(S) \to \mathsf{Inv}(T)$ be the isomorphism. Our goal is to extend θ to an isomorphism ϕ of right restriction monoids $\phi\colon S \to T$. Let $a \in S$. Then, under the assumption that S is étale, we may write

$$a = \bigvee_{i=1}^{m} a_i,$$

where the a_i are partial units such that $a_i \sim_l a_j$. From $a_i \sim_l a_j$, it follows that $\theta(a_i) \sim_l \theta(a_j)$ and so $\bigvee_{i=1}^{m} \theta(a_i)$ is defined in T. This means that we can define

$$\phi(a) = \bigvee_{i=1}^{m} \theta(a_i).$$

Of course, this appears to depend on our choice of partial units a_i. We prove that this is not the case. Accordingly, suppose that $a = \bigvee_{j=1}^{n} b_j$, where the b_j are partial units. Then

$$\bigvee_{i=1}^{m} a_i = \bigvee_{j=1}^{n} b_j.$$

Thus $a_i \leq \bigvee_{j=1}^{n} b_j$. It follows by Lemma 3.1, that $a_i = \bigvee_{j=1}^{n} a_i \wedge b_j$. Now, $a_i \wedge b_j$ is a partial unit by Lemma 3.3. In addition, $a_i \wedge b_j \leq a_i$. It follows that these elements are pairwise *compatible*. Thus the join is an honest-to-goodness join in $\mathsf{Inv}(S)$. It follows that $\theta(a_i) = \bigvee_{j=1}^{n} \theta(a_i \wedge b_j)$. But the elements a_i and b_j are partial units and we know that they are $\sim_l$-related (because they are all below a). It follows that the meet $a_i \wedge b_j$ is algebraic. Thus $\theta(a_i \wedge b_j) = \theta(a_i) \wedge \theta(b_j)$ and so $\theta(a_i) = \bigvee_{j=1}^{n} \theta(a_i) \wedge \theta(b_j)$. By Lemma 3.1, we deduce that $\theta(a_i) \leq \bigvee_{j=1}^{n} \theta(b_j)$. Thus $\bigvee_{i=1}^{m} \theta(a_i) \leq \bigvee_{j=1}^{n} \theta(b_j)$. By symmetry we get equality. This proves that ϕ is a well-defined function extending θ. The fact that ϕ is a semigroup homomorphism follows from the observation that if $a = \bigvee_{i=1}^{m} a_i$ and $b = \bigvee_{j=1}^{n} b_j$, where the a_i and b_j are partial units, then $ab = \bigvee_{i,j} a_i b_j$, where each product $a_i b_j$ is a partial unit by Lemma 3.3. We now prove that ϕ is a bijection. Let $a, b \in S$ such that $\phi(a) = \phi(b)$. We can write

$$a = \bigvee_{i=1}^{m} a_i \text{ and } b = \bigvee_{j=1}^{n} b_j,$$

where the a_i and b_j are partial units. By assumption

$$\bigvee_{i=1}^{m} \theta(a_i) = \bigvee_{j=1}^{n} \theta(b_j).$$

Thus $\theta(a_i) \leq \bigvee_{j=1}^{n} \theta(b_j)$. It follows by Lemma 3.1, that $\theta(a_i) = \bigvee_{j=1}^{n} \theta(a_i) \wedge \theta(b_j)$. The element $\theta(a_i) \wedge \theta(b_j)$ is a partial unit by Lemma 2.13. Since θ is an isomorphism (and so an order isomorphism) it follows that $a_i \wedge b_j$ exists and $\theta(a_i) \wedge \theta(b_j) = \theta(a_i \wedge b_j)$. Again, since θ is an isomorphism $\bigvee_{j=1}^{n} \theta(a_i \wedge b_j) = \theta\left(\bigvee_{j=1}^{n} a_i \wedge b_j\right)$. It follows that $a_i = \bigvee_{j=1}^{n} a_i) \wedge b_j$. Whence $a_i \leq b$. This means that $a \leq b$. By symmetry we have that $a = b$ and we have proved that ϕ is injective. We now prove that ϕ is surjective. Let $a' \in T$. Then $a' = \bigvee_{i=1}^{m} a_i'$ where the a_i' are partial units which are pairwise left-compatible. Let a_i be the partial unit of S such that $\theta(a_i) = a_i'$. The elements a_i are also left-compatible. Put $a = \bigvee_{i=1}^{m} a_i$. Then, by construction, $\phi(a) = a'$. We have therefore proved that ϕ is an isomorphism of semigroups. It remains to check that we have an isomorphism of right restriction semigroups. Suppose that $a = \bigvee_{i=1}^{m} a_i$, where the a_i are partial units. We claim that $a^* = \bigvee_{i=1}^{m} a_i^{-1} a_i$. But $a_i^* = a_i^{-1} a_i$. The result now follows. $\square$

The above result is purely theoretical in that it shows that Boolean inverse monoids determine the structure of étale Boolean right restriction monoids. We now show how to actually construct all étale Boolean right restriction monoids directly from Boolean inverse monoids.

Let S be a Boolean inverse monoid. Given any subset $B \subseteq S$, define $\overline{B}$ to be the set of all *compatible* joins of elements of B. We say that B is *closed* if $B = \overline{B}$. The following two results are for motivation.

Lemma 5.3. *Let S be an étale Boolean right restriction monoid. If $a \in S$ then the set $a^{\downarrow} \cap \mathsf{Inv}(S)$ has the following properties:*

(1) *It is acceptable.*
(2) *It is closed.*
(3) *There is a finite set of left-compatible partial units $\{b_1, \ldots, b_n\}$ in $a^{\downarrow} \cap \mathsf{Inv}(S)$ such that if $c \in a^{\downarrow} \cap \mathsf{Inv}(S)$ then c is a compatible join of partial units in $\{b_1, \ldots, b_n\}^{\downarrow}$.*

Proof. (1) $a^{\downarrow}$ is an order-ideal by construction, and $\mathsf{Inv}(S)$ is an order-ideal by part (2) of Lemma 3.3. Thus $a^{\downarrow} \cap \mathsf{Inv}(S)$ is an order-deal. The fact that the set is left-compatible follows by Lemma 2.5.

(2) Suppose that $\{a_1, \ldots, a_n\}$ is a compatible set of partial units each less than or equal to a. Then their join is less than or equal to a.

(3) By the definition of étale, we may write $a = \bigvee_{i=1}^{n} b_i$, where the b_i are a finite left-compatible set of partial units. Let c be any partial unit $c \leq a$. Then $c = \bigvee_{i=1}^{n} c \wedge b_i$ by Lemma 3.1. By part (2) of Lemma 3.3, each $c \wedge b_i$ is a partial unit and $c \wedge b_i \leq b_i$. □

We now prove that the data assumed in the lemma above is actually enough to determine elements of a étale Boolean right restriction monoid.

Lemma 5.4. *Let S be an étale Boolean right restriction monoid. Suppose that $a, b \in S$ are such that $a^{\downarrow} \cap \mathsf{Inv}(S) = b^{\downarrow} \cap \mathsf{Inv}(S)$. Then $a = b$.*

Proof. We prove first that $a \leq b$; the result then follows by symmetry. By defintion, $a = \bigvee_{i=1}^{m} a_i$ where the $a_i \in a^{\downarrow} \cap \mathsf{Inv}(S)$ and so are elements of $b^{\downarrow} \cap \mathsf{Inv}(S)$. It follows that $a_i \leq b$ from which it follows that $a \leq b$. □

Lemma 5.3 and Lemma 5.4 motivate what we now do. Given a Boolean inverse monoid S, we shall construct an étale Boolean right restriction monoid $\mathsf{Etale}(S)$, called the *(right restriction) companion* of S, such that $S \cong \mathsf{Inv}(\mathsf{Etale}(S))$. It is here that we use the results from Section 4. By Proposition 4.2, the semigroup $\mathsf{R}(S)$ is a complete right restriction monoid. We shall now show how to define a nucleus on $\mathsf{R}(S)$.

Lemma 5.5. *Let S be a Boolean inverse monoid.*

(1) *If A is acceptable in S then $\overline{A}$ is acceptable in S and $A \subseteq \overline{A}$.*
(2) *The function $A \mapsto \overline{A}$ is a nucleus on $\mathsf{R}(S)$.*

Proof. (1) We prove that $\overline{A}$ is acceptable. We prove first that $\overline{A}$ is an order-ideal. A typical element of $\overline{A}$ has the form $\bigvee_{i=1}^{m} a_i$ where $\{a_1, \ldots, a_m\}$ is a compatible subset of A. Suppose that $x \leq \bigvee_{i=1}^{m} a_i$. Then by Lemma 3.1, we have that $x = \bigvee_{i=1}^{m} a_i \wedge x$. Because A is an order-ideal, we have that $a_i \wedge x \in A$. Thus $x \in \overline{A}$. We prove first that $\overline{A}$ is left-compatible. Suppose that $x, y \in \overline{A}$. Then $x = \bigvee_{i=1}^{m} x_i$ and $y = \bigvee_{j=1}^{n} y_j$ where $x_i, y_j \in A$. By assumption, $x_i \sim_l y_j$ for all i and j then $x \sim_l y$. Thus by Lemma 2.7, we have that $x_i y_j^{-1}$ is an idempotent. We have that $xy^{-1} = \bigvee_{i,j} x_i y_j^{-1}$. Thus xy^{-1} is an idempotent and so by Lemma 2.7 we have proved that x and y are left-compatible. Thus $\overline{A}$ is acceptable. It is immediate that $A \subseteq \overline{A}$.

(2) The function is well-defined by part (1) above. The proof that the properties (N1), (N2) and (N3) of a nucleus hold are immediate. It is easy to verify that (N4) and (N5) hold. We prove that (N6) holds. Only one direction needs proving. Let $x \in \overline{(\overline{A})^*}$. Suppose that $x = \bigvee_{i=1}^{m} x_i$ where $x_i \in (\overline{A})^*$. Now, each $x_i = (\bigvee_{j_i} y_{j_i})^*$ where $y_{j_i} \in A$. Thus $x = \bigvee_{i=1}^{m} (\bigvee_{j_i} y_{j_i})^*$ where $y_{j_i} \in A$. It follows that $x = (\bigvee_i \bigvee_{j_i} y_{j_i})^*$ where $y_{j_i} \in A$. This is just $x = \bigvee_i \bigvee_{j_i} y_{j_i}^*$ where $y_{j_i} \in A$. Whence $x \in \overline{(A^*)}$. $\square$

Put $\nu(A) = \overline{A}$. Then we have the following by Proposition 4.5 and Lemma 5.5.

Lemma 5.6. *Let S be a Boolean inverse monoid. Then $\mathsf{R}(S)_\nu$ is a complete right restriction monoid.*

The monoid $\mathsf{R}(S)_\nu$, constructed in Lemma 5.6, is too big for our purposes and so we shall cut it down. Define

$$\mathsf{Etale}(S) = \mathsf{fin}(\mathsf{R}(S)_\nu),$$

the finite elements of $\mathsf{R}(S)_\nu$. Currently, this is just a set. Our first job, therefore, is to describe the finite elements in $\mathsf{R}(S)_\nu$.

Lemma 5.7. *Let S be a Boolean inverse monoid. The finite elements of $\mathsf{R}(S)_\nu$ are the subsets of the form $\overline{\{a_1, \ldots, a_m\}^{\downarrow}}$, where $\{a_1, \ldots, a_m\}$ is a left-compatible subset of S.*

Proof. Put $A = \{a_1, \ldots, a_m\}$. Then $A^{\downarrow}$ is an acceptable set, using Lemma 2.9, and so a well-defined element of $\mathsf{R}(S)$. Observe that if $X = \overline{\{a_1, \ldots, a_m\}^{\downarrow}}$, then $X = a_1^{\downarrow} \sqcup \ldots \sqcup a_1^{\downarrow}$ by the proof of Proposition 4.5. It is easy to check that each set of the form $a^{\downarrow}$, where $a \in S$, is finite. By part (2) of Lemma 4.6, it follows that X is finite. We now show that all finite elements have this form. Suppose that X is a finite element of $\mathsf{R}(S)_\nu$. Thus X is a closed acceptable subset. We may write $X = \bigsqcup_{a \in A} a^{\downarrow}$. But we have assumed that X is finite. Thus $X = \bigsqcup_{i=1}^{m} a_i^{\downarrow}$ for some m. $\square$

The proof of the following is now easy by Lemma 5.7.

Lemma 5.8. *In $\mathsf{R}(S)_\nu$, the product of finite elements is a finite element.*

We can now state and prove the main theorem of this paper. This shows us how to construct all étale Boolean right restriction monoids from Boolean inverse monoids.

Theorem 5.9. *Let S be a Boolean inverse monoid. Then $\mathsf{Etale}(S)$ is an étale Boolean right restriction monoid whose semigroup of partial units is isomorphic to S.*

Proof. The product of two finite elements is finite by Lemma 5.8. It follows by part (3) of Lemma 4.6, that $\mathsf{Etale}(S)$ is a distributive right restriction monoid. If A is finite then $A^\star$ is finite by Lemma 4.6. However, $A^\star$ is of the form $\overline{\{a_1^* \ldots a_m^*\}^{\downarrow}}$ which is equal to $(a_1^* \vee \ldots a_m^*)^{\downarrow}$. We deduce that the projections of $\mathsf{Etale}(S)$ are the principal order ideals of the idempotents of S and so form a meet-semilattice isomorphic with the set of idempotents of S, which is a Boolean algebra. It follows that $\mathsf{Etale}(S)$ is, in fact, a Boolean right restriction monoid. We now locate some of the partial units. Elements of the form $a^{\downarrow}$, for some $a \in S$, are partial units because $a^{\downarrow}(a^{-1})^{\downarrow} = (aa^{-1})^{\downarrow}$ and $(a^{-1})^{\downarrow} a^{\downarrow} = (a^{-1}a)^{\downarrow}$. However, every element of $\mathsf{Etale}(S)$ can be written $\sqcup_{i=1}^{m} a_i^{\downarrow}$, for some left-compatible subset $\{a_1, \ldots, a_m\}$.

This immediately implies that $\mathsf{Etale}(S)$ is étale. We shall now prove that every partial unit $\mathsf{Etale}(S)$ is of the form $a^{\downarrow}$ from which it follows that the partial units of $\mathsf{Etale}(S)$ form a semigroup isomorphic to S. Let $A = \overline{\{a_1, \ldots, a_m\}^{\downarrow}}$ be a partial unit of $\mathsf{Etale}(S)$. Then there is an element $X = \overline{\{x_1, \ldots, x_n\}^{\downarrow}}$ of $\mathsf{Etale}(S)$ such that $A \cdot X = (x_1^* \vee \ldots \vee x_n^*)^{\downarrow}$ and $X \cdot A = (a_1^* \vee \ldots \vee a_m^*)^{\downarrow}$. Choose any a_i. Then $a_i^* \in X \cdot A$. It follows that $a_i^* = \bigvee_{j=1}^{s} y_j b_j$ where $y_j \in \{x_1, \ldots, x_n\}^{\downarrow}$ and $b_j \in \{a_1, \ldots, a_m\}^{\downarrow}$. Since each $b_j \in \{a_1, \ldots, a_m\}^{\downarrow}$, we may write $b_j = b_j b_j^{-1} a_j'$ where a_j' is one of the elements $a_1, \ldots, a_m$. It is therefore immediate that $y_j b_j = (y_j b_j b_j^{-1}) a_j'$. But $y_j b_j b_j^{-1} \in \{x_1, \ldots, x_n\}^{\downarrow}$ since this is an order-ideal. Thus we can write (relabelling if necessary) $a_i^{-1} a_i = \bigvee_{j=1}^{s} y_j a_j'$ where $y_j \in \{x_1, \ldots, x_n\}^{\downarrow}$ and the a_j' is one of the elements $a_1, \ldots, a_m$. It follows that $a_i^{-1} = \bigvee_{j=1}^{s} y_j (a_j' a_i^{-1})$ since $a_i^{-1} = a_i^{-1} a_i a_i^{-1}$. Because $\{a_1, \ldots, a_m\}$ is a left-compatible set the element $a_j' a_i^{-1}$ is always a projection/idempotent by Lemma 2.7. Thus $y_j (a_j' a_i^{-1}) \in \{x_1, \ldots, x_n\}^{\downarrow}$. We have therefore shown that $a_i^{-1} \in X$. The elements of X are left-compatible. It follows that a_i^{-1} and a_j^{-1} are left-compatible, and so, a_i and a_j are right-compatible by Lemma 2.8. Thus a_i and a_j are compatible. It follows that $A = \overline{\{a_1, \ldots, a_m\}^{\downarrow}} = (a_1 \vee \ldots \vee a_m)^{\downarrow}$. □

We can now determine for which class of Boolean inverse monoids S the companion is actually isomorphic to S.

Proposition 5.10. *Let S be a Boolean monoid. Then $\mathsf{Etale}(S) \cong S$ if and only if in S left-compatible elements are compatible.*

Proof. Suppose first that in S left-compatible elements are compatible. Then it is immediate from the way that $\mathsf{Etale}(S)$ is constructed that $\mathsf{Etale}(S) \cong S$. We now prove the converse. Suppose that $\mathsf{Etale}(T) \cong S$. Thus, we are given an étale Boolean right restriction monoid T such that $T \cong \mathsf{Inv}(S)$. Thus T is an étale Boolean right restriction monoid which is also inverse. We prove first that $T = \mathsf{Inv}(T)$. We therefore need to prove that every element of T is a partial unit. Let $a \in T$. Then there is a unique element $b \in T$ such that $a = aba$ and $b = bab$. By assumption

$$a = \bigvee_{i=1}^{m} a_i \text{ and } b = \bigvee_{j=1}^{m} b_j,$$

where the a_i and b_j are partial units. It follows that

$$ab = \bigvee_{1 \leq i \leq m, 1 \leq j \leq n} a_i b_j,$$

where $a_i b_j$ is a partial unit by part (1) of Lemma 2.13. But ab is an idempotent and so $a_i b_j$ is an idempotent. By part (3) of Lemma 2.13, it follows that $a_i b_j$ is a projection. By part (2) of Lemma 3.1, it follows that ab is a projection. By symmetry, ba is a projection. Thus a is a partial unit. We have therefore proved that $T = \mathsf{Inv}(T)$. We can now finish off the proof. Suppose that $a, b \in T$ are such that $a \sim_l b$. Then $a \vee b$ exists and, by assumption, is a partial unit. We have that $a, b \leq a \vee b$. It follows by part (2) of Lemma 2.13, that $a = aa^{-1}(a \vee b)$. By Lemma 3.1, we have that $a = a \vee aa^{-1}b$. Thus, in particular, $aa^{-1}b \leq a$. But $bb^{-1}a \leq a$. Now check that $(aa^{-1}b)^* = (bb^{-1}a)^*$, using the fact that projections commute. By part (1) of Lemma 2.5, we have that $aa^{-1}b = bb^{-1}a$. Thus $a \sim_r b$. We have therefore proved that $a \sim b$. □

The above result therefore highlights the class of inverse semigroups in which $\sim_l \subseteq \sim_r$. The E-reflexive inverse semigroups, discussed in [17, page 86], are examples.

6. Examples

In this section, we shall illustrate the theory developed in this paper by describing two examples.

6.1. **Partial functions on a set.** We return to Example 2.1. The monoid $\mathsf{PT}(X)$ is a Boolean right restriction monoid. We locate the partial units. We claim that these are precisely the elements $\mathsf{I}(X)$, the set of partial bijections on X. It is clear that $\mathsf{I}(X) \subseteq \mathsf{Inv}(X)$. Let $f \in \mathsf{PT}(X)$ be a partial unit. Then there is an element $g \in \mathsf{PT}(X)$ such that $fg = g^*$ and $gf = f^*$. Suppose that $f(x) = f(y)$ where $x, y \in \mathrm{dom}(f)$. Then $g(f(x)) = g(f(y))$. But gf is the identity function on $\mathrm{dom}(f)$. Thus $x = y$. We have proved that f is injective. Put $\mathrm{im}(f) = \mathrm{dom}(g)$. Let $y \in \mathrm{dom}(g)$. Then $(fg)(y) = y$. Put $x = g(y)$. Then $f(x) = (fg)(y) = y$. We therefore have a bijection from $\mathrm{dom}(f)$ onto $\mathrm{dom}(g)$. It follows that $f \in \mathsf{I}(X)$. We now specialize to the case where X is finite. Define the partial function g^x_y which has domain of definition $\{x\}$ and maps x to y. Clearly, g^x_y is a partial unit and each element of $\mathsf{PT}(X)$ is a finite join of left-compatible elements of the form g^x_y. Thus $\mathsf{PT}(X)$ is étale. It follows that in the case where X is finite, $\mathsf{PT}(X)$ arises from $\mathsf{I}(X)$ via the construction of this paper.

6.2. **A general example.** Let X be any Boolean space. Denote by $\mathsf{I}(X)$ the set of all partial homeomorphisms between the clopen sets of X. This is a Boolean inverse monoid by [20, Proposition 2.16, Proposition 5.2]. Let $\mathsf{S}(X)$ be the set of all local homeomorphisms $\theta \colon U \to X$ where U is clopen. This is a Boolean right restriction monoid [21, Example 5.6]. Observe that U is compact and so $\theta(U)$ is compact. But X is Hausdorff. It follows that $\theta(U)$ is also clopen. Thus $\mathsf{S}(X)$ is the set of all surjective local homeomorphisms between the clopen subsets of X. Clearly, $\mathsf{I}(X) \subseteq \mathsf{S}(X)$ and the set of partial units of $\mathsf{S}(X)$ is $\mathsf{I}(X)$. Let $\theta \colon U \to V$ be a surjective local homeomorphism between two clopen sets. For each $x \in U$, there is an open set $U_x \subseteq U$ such that θ restricted to this set is a homeomorphism. But the clopens form a base for the topology on X. So, without loss of generality, we may assume that U_x is clopen. Thus, the U_x cover U. But U is compact. It follows that we can write θ as a finite union of partial homeomorphisms. We have therefore shown that $\mathsf{S}(X)$ is étale. Since the set of partial units of $\mathsf{S}(X)$ is $\mathsf{I}(X)$, it follows by Theorem 5.2 that $\mathsf{S}(X)$ is the companion of $\mathsf{I}(X)$; that is, $\mathsf{S}(X) = \mathsf{Etale}(\mathsf{I}(X))$.

Although not needed, we shall show how to actually construct $\mathsf{S}(X)$ from $\mathsf{I}(X)$. Let $A = \overline{\{f_1, \ldots, f_m\}^{\downarrow}}$ be a finite closed acceptable set in $\mathsf{I}(X)$. We have that $f_1, \ldots, f_m \in \mathsf{I}(X)$ and to say that they are left-compatible simply means that $\theta = f_1 \cup \ldots \cup f_m$ is a well-defined partial function of X. Put $U = U_1 \cup \ldots \cup U_n$ where U_i is the domain of definition of f_i. Put $V = V_1 \cup \ldots \cup V_n$ where V_i is the range of f_i. The sets U and V are both clopen and $\theta \colon U \to V$ is a surjective local homeomorphism. We have therefore defined an element of $\mathsf{S}(X)$. We show that $A = \theta^{\downarrow} \cap \mathsf{I}(X)$. Let $f \in A$. Then f is a compatible join of elements of $\{f_1, \ldots, f_m\}^{\downarrow}$. But each element of $\{f_1, \ldots, f_m\}^{\downarrow}$ is below an element such as f_i and so $f \leq \theta$. In addition, a compatible join of partial homeomorphisms is itself a partial homeomorphism. We have proved that $A \subseteq \theta^{\downarrow} \cap \mathsf{I}(X)$. To prove the reverse inclusion, let f be any partial homeomorphism between the clopens of X where $f \subseteq \theta$. Then f is the compatible join of partial homemorphisms each of the form $f \cap f_i$. Thus $f \in A$. We have therefore demonstrated the construction.

6.3. **The classical Thompson-Higman groups $G_{n,1}$.** We need some preparation before we can describe our example.

String theory

We refer the reader to [1] for more on strings. Let A_n be an n-element alphabet. We shall always assume that $n \geq 2$. The free monoid on A_n is denoted by A_n^* with identity ε. If $x, y \in A_n^*$ then we say that x and y are *(prefix) incomparable* if $xA_n^* \cap yA_n^* = \varnothing$, otherwise they are said to be *(prefix) comparable*. A finite subset X of A_n^* is said to be a *prefix code* if the elements are pairwise prefix incomparable. A prefix code X is said to be a *maximal prefix code* if every element of A_n^* is comparable with an element of X. The smallest maximal prefix code is $\{\varepsilon\}$, which we call the *trivial maximal prefix code*.

Let $X \subseteq A_n^*$. If $x \in A_n^*$ then $x^{-1}X$ is the set of all elements y such that $xy \in X$.

Denote by A_n^ω the set of all right-infinite strings over the n-element alphabet A_n. This set is equipped with the topology in which the open sets of X are the subsets XA_n^ω where $X \subseteq A_n^*$. With this topology A_n^ω is the *Cantor space* and is a Boolean space. The clopen subsets are precisely those where X is finite. The following is [14, Lemma 3.16].

Lemma 6.1. *Let A_n be a finite alphabet such that $n \geq 2$. Then for finite strings x and y we have that $xA_n^\omega = yA_n^\omega$ implies $x = y$.*

Observe that requiring $n \geq 2$ is necessary in the lemma above, since over a 1-element alphabet the above result is not true; for example, $a\{a\}^\omega = aa\{a\}^\omega$.

The following is [14, Lemma 3.15].

Lemma 6.2. *$xA_n^\omega \cap yA_n^\omega \neq \varnothing$ if and only if x and y are prefix comparable.*

We have the following, which is well-known but a proof can be deduced from [18, Lemma 4.3].

Lemma 6.3. *Let Z be a prefix code in A_n^*. Then Z is a maximal prefix code if and only if $ZA_n^\omega = A_n^\omega$.*

Lemma 6.4. *Suppose that X is a finite set of finite strings. Then $XA_n^\omega = YA_n^\omega$ where $Y \subseteq X$ and Y is a prefix code.*

Proof. Suppose that $x, x' \in X$ are prefix-comparable. Without loss of generality, we assume that $x = x'u$. Let w be a right-infinite string. Then $xw = x'uw$. Thus $X' = X \setminus \{x\}$ has the property that $XA_n^\omega = X'A_n^\omega$ and $X' \subseteq X$. This process can be repeated until we have whittled X down to a prefix code. □

The Boolean right restriction monoid H_n

This section bears an analogous relationship to [3], as [2] does to my original paper [18].[3] We now apply the constructions of this paper to the Boolean inverse monoids that arise in constructing the Thompson-Higman groups $G_{n,1}$. These Boolean inverse monoids were first described in [18] and then in [23] with a more general perspective provided by [21]. According to [25], the starting point for constructing the classical Thompson-Higman groups involves free actions [15][4] of free monoids A_n^*. We shall simplify things by considering only those Boolean inverse monoids that arise in the construction of the groups $G_{n,1}$. This involves free monoids alone. To give a little perspective, we shall specialize what we did in the previous subsection by taking as our Boolean space $X = A_n^\omega$.

Assumption: *in this section, we shall always want to think of finite sets with elements having a specific order and where repetitions are allowed. Thus if X is such a 'set' its elements are $x_1, \ldots, x_p$ where the order is as shown by the subscript*

[3] I am grateful to Richard Garner for reminding me of this.

[4] My thanks to Victoria Gould for supplying this reference.

and where we do not rule out the possibility the $x_i = x_j$.

We shall define some functions on the clopen subsets of A_n^ω. If $x \in A_n^*$ then the function $\lambda_x \colon A_n^\omega \to xA_n^\omega$ is defined by $w \mapsto xw$ where $w \in A_n^\omega$. This is a partial bijection and, therefore, has a partial inverse λ_x^{-1}. We shall denote the function $\lambda_x \lambda_y^{-1}$ by xy^{-1}. This means that $x\varepsilon^{-1}$ denotes the function λ_x. Let X and Y be finite sets of finite strings having the same number of elements (and bear in mind our assumption). We shall always assume that X is a prefix code. Define a function $f_Y^X : XA_n^\omega \to YA_n^\omega$ by $f_Y^X(x_i w) = y_i w$ where $w \in A_n^\omega$. We can equally denote this function by $\bigcup_{i=1}^p y_i x_i^{-1}$ if $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_p\}$. We shall usually denote $\bigcup$ by $\bigvee$. The set of all such functions is denoted by H_n. The set of all functions of the form f_Y^X where both X and Y are prefix codes is denoted by C_n.

We shall prove that H_n is a Boolean right restriction monoid, that C_n is a Boolean inverse monoid, that $\mathsf{Inv}(H_n) = C_n$, that H_n is étale and that $\mathsf{Etale}(C_n) = H_n$. To do all of this, we shall first of all describe the building blocks of the elements of H_n. Let x and y be finite strings. Define the function xy^{-1}, as above, from yA_n^ω to xA_n^ω by $(xy^{-1})(yw) = xw$ where w is any right-infinite string.

Lemma 6.5. *The function xy^{-1} is a well-defined partial homeomorphism.*

Proof. This is well-defined by Lemma 6.1 and is clearly a bijection. It remains to prove that this is a homeomorphism. Let $XA_n^\omega \subseteq xA_n^\omega$. Then all strings in X begin with x. It follows that the inverse image of XA_n^ω under the function xy^{-1} is the set $y(x^{-1}X)A_n^*$ which is also open. It follows that xy^{-1} is a partial homeomorphism and is the function $f_{\{x\}}^{\{y\}}$. □

We call functions of the form xy^{-1} *basic*. Observe that $(xy^{-1})^{-1} = yx^{-1}$.

Lemma 6.6.

(1) $(xy^{-1})(uv^{-1}) = 0$ *if and only if y and u are prefix incomparable.*
(2) $(xy^{-1})(uv^{-1}) = x(vz)^{-1}$ *if $y = uz$.*

Proof. (1) Suppose that $(xy^{-1})(uv^{-1}) = 0$. Then $yA_n^{-1} \cap uA_n^\omega = \varnothing$. Thus y and u are prefix incomparable by Lemma 6.2. The converse is proved similarly.

(2) We use the fact that $u^{-1}u = 1$. □

Lemma 6.7. *Suppose that the elements xy^{-1} and uv^{-1} satisfy neither $xy^{-1} \leq uv^{-1}$ nor $uv^{-1} \leq xy^{-1}$. Then $xy^{-1} \sim_l uv^{-1}$ if and only if y and v are prefix incomparable.*

Proof. Suppose that $xy^{-1} \sim_l uv^{-1}$. Then $xy^{-1}vv^{-1} = uv^{-1}yy^{-1}$. If y and v are prefix comparable then, by symmetry, we can suppose that $y = vz$. Then $xy^{-1}vv^{-1} = x(vz)^{-1}$ and $uv^{-1}yy^{-1} = uz(vz)^{-1}$. Thus $x = uz$. Thus $xy^{-1} \leq uv^{-1}$. But this contradicts our assumptions. The proof of the converse is immediate by Lemma 6.6. □

Suppose now that we have a join of left-compatible basic functions $\bigvee_{i=1}^m x_i y_i^{-1}$ We do not change the join if we eliminate the smaller of two elements. We shall assume this has been done. Put $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$. Thus we can assume, by Lemma 6.7, that X is a prefix code. We therefore have that

$$\bigvee_{i=1}^m y_i x_i^{-1} = f_Y^X.$$

Thus H_n is a well-defined monoid. So, too, is C_n whose elements have the form f_Y^X where both X and Y are prefix codes. Observe that both of these monoids contain all the projections of $\mathsf{S}(A_n^\omega)$. Thus we have a Boolean algebra of projections in both cases.

Lemma 6.8. *The semigroup H_n is closed under left-compatible joins.*

Proof. Let the two elements be $f = \bigvee_{i=1}^{p} x_i y_i^{-1}$ and $g = \bigvee_{j=1}^{q} x_j y_j^{-1}$ and we are given that $f \sim_l g$. We may form $f \vee g$ in $\mathsf{S}(A_n^\omega)$. By Lemma 2.9, we have that $x_i y_i^{-1} \sim_l x_j y_j^{-1}$. In the join $f \vee g$ we may eliminate any basic functions which are smaller than other basic functions in the join. If we do that, then by Lemma 6.7, we obtain a function of the form f_Y^X where X is a prefix code. $\square$

The proofs of the following are now immediate; for the second lemma, recall what we proved in the previous subsection.

Lemma 6.9. *C_n is a Boolean inverse monoid with group of units the Thompson-Higman group $G_{n,1}$.*

Lemma 6.10. *The monoid H_n is an étale Boolean right restriction monoid with monoid of partial units isomorphic to C_n.*

We can restrict the result we obtained in the previous subsection to dedcuce that $\mathsf{Etale}(C_n) = H_n$.

Here is a broader descrption of what we have accomplished. Let P_n be the polycyclic monoid on n generators [17]. Then there is an injective homomorphism of semigroups $P_n \to \mathsf{I}(A_n^\omega)$ whose isomorphic image consists of the basic functions and zero. This is an example of what we called a *strong representation* of the polycyclic monoid [19]. This was our original approach to constructing the Thompson-Higman groups. See [21] for a retrospective. A parallel, but more general, approach was pioneered by [7]. See, for example, [7, Example 4.1]. Observe that Hughes works from geometry whereas we work from language theory. The connection with the theory of inverse semigroups is slightly obscured by the approach Hughes adopts, but his [7, Definition 3.1] is really the definition of a particular kind of inverse semigroup of which the polycyclic inverse monoids are special cases. Hughes is working with ultrametric spaces in which closed balls are also open and if two balls intersect then one must be contained in the other. This parallels what happens in free monoids in that if two finite strings are comparable then one must be the prefix of the other.

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