

FIELD METHOD APPROACH IN THE FACTORIZATION OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the general solution of second-order nonlinear differential equations of Liénard type is obtained within the nonlinear factorization method of Rosu and Cornejo-Pérez by the so-called field method approach. This method is based on writing the factorization conditions in the dynamical systems form and requires to assume that the intermediate function Φ that occurs in the factorization be dependent not only on the dependent variable of the nonlinear equation, but also on the independent variable. The method is applied to several cases of Liénard type equations which are written in a commutative factorization form and their general solutions are obtained by solving Bernoulli differential equations.

Keywords: Liénard equation; commutative factorization; factorization condition; Sharma-Tasso-Olver equation; field method.

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I. INTRODUCTION

Nonlinear differential equations of the Liénard type [1, 2]

$$\ddot{x} + G(x)\dot{x} + F(x) = 0 , \quad (1)$$

where the dot denotes the derivative $D = d/dt$ and $G(x)$ and $F(x)$ are arbitrary real functions, serve as an important class of ordinary differential equations (ODE's) with widespread applications to a huge number of physical, chemical, biological, and engineering systems, see [3] and references therein. Even linear differential equations corresponding to $F(x) \sim x$ and $G(x) = \text{const.}$ can be included in the Liénard class.

The goal of this paper is to present a new approach to the nonlinear factorization method introduced by two of the authors a couple of decades ago [4, 5] which leads to exact solutions of many nonlinear equations in a simple way. The proposed method is similar to the so-called field method introduced by Vujanovic [6, 7], see also [8, 9], which states that for any nonconservative, holonomic, rheonomic, dynamical system whose evolution is given in the dynamical form

$$(\dot{x}_1, \dots, \dot{x}_n)^T = (X_1(t, x_1, \dots, x_n), \dots, X_n(t, x_1, \dots, x_n))^T , \quad (2)$$

we can choose one coordinate of the system, say x_1 , which can be selected as a *field* depending on time t and the rest of the coordinates

$$x_1 = \Phi(t, x_2, \dots, x_n) . \quad (3)$$

Taking the total time derivative with respect to time of Eq. (3) and using the last $n - 1$ equations of (2), one can write the dynamical system as a quasi-linear first order partial differential equation given by

$$\frac{\partial \Phi}{\partial t} + \sum_{j=2}^n \frac{\partial \Phi}{\partial x_j} X_j(t, \Phi, x_j) - X_1(t, \Phi, x_2, \dots, x_n) = 0 . \quad (4)$$

The complete solution of Eq. (4) is of the form

$$x_1 = f(t, x_j, C_1) , \quad (5)$$

where C_1 is an arbitrary constant. In the nonlinear factorization method, the solution (5) can be used to construct the general solution of (1).

In this paper, we implement this field approach into the factorization method and show that one can obtain the general solution of the problem for the commutative factorization setting of nonlinear ODE's in an easy and straightforward way.

Section II of the paper describes the implementation of the field method approach in the framework of commutative factorizations. In section III, we present some specific examples of the field method approach as applied to factored nonlinear ODE's, including a unifying treatment of the examples. Finally, the conclusions are summarized in the last section.

II. FACTORIZATION AND FIELD METHOD APPROACH

Equation (1) can be factorized in the form

$$[D - \phi_2(x)][D - \phi_1(x)]x = 0 , \quad (6)$$

under the conditions [4, 5]

$$\phi_2 + \frac{d(\phi_1 x)}{dx} = -G(x) , \quad (7)$$

$$\phi_1 \phi_2 x = F(x) . \quad (8)$$

From (8), one can see that if $F(x)$ is a homogeneous polynomial then one can choose the factorization functions ϕ_1, ϕ_2 as complementary pairs in the set of divisors of the nonhomogeneous polynomial $F(x)/x$, a hint that reveals the efficiency of this factorization method in the case of polynomial nonlinearities. The commutative factorization setting can be achieved by demanding the following conditions

$$\phi_1 + \phi_2 + x \frac{d\phi_1}{dx} = -G(x) , \quad (9)$$

$$\phi_1 + \phi_2 + x \frac{d\phi_2}{dx} = -G(x) , \quad (10)$$

$$\phi_1 \phi_2 = \frac{f(x)}{x} , \quad (11)$$

which implies that the two factorization functions differ only by a constant, i.e. $\phi_2 = \phi_1 + C$.

Rosu and Cornejo-Pérez considered initially only $[D - \phi_1(x)]x = 0$, but also the extension to $[D - \phi_1(x)]x = \Phi(x)$ has been discussed in the literature [10, 11]. Borrowing from the field method, we assume now $[D - \phi_1(x)]x = \Phi(x, t)$, which yields the following coupled ODE's for the factorized Eq. (6),

$$\dot{x} - \phi_1(x)x = \Phi(x, t) , \quad (12)$$

$$\dot{\Phi} - \phi_2(x)\Phi = 0 , \quad (13)$$

which can be also written in the matrix dynamical systems form

$$\begin{pmatrix} \dot{x} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} \phi_1 & 1 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} x \\ \Phi \end{pmatrix} . \quad (14)$$

According to the chain rule, the system of equations (12)-(13) can be rewritten as the quasi-linear first order partial differential equation

$$\frac{\partial \Phi}{\partial x} \dot{x} + \frac{\partial \Phi}{\partial t} \equiv \frac{\partial \Phi}{\partial x} (\Phi + x\phi_1) + \frac{\partial \Phi}{\partial t} = \phi_2 \Phi , \quad (15)$$

which can be solved by proposing the *ansatz* $\Phi(x, t) = x\zeta(t)$ [6, 12]. Then, for the function ζ , one obtains the first-order differential equation

$$\frac{d\zeta(t)}{dt} + \zeta^2(t) = (\phi_2 - \phi_1)\zeta(t) . \quad (16)$$

If the commutative factorization condition is considered, then the subtraction of factorization functions $\phi_2 - \phi_1 = -2c$ holds for $c \equiv \text{const.}$, and (16) reduces to

$$\frac{d\zeta(t)}{dt} + 2c\zeta(t) = -\zeta^2(t) , \quad (17)$$

which is a Bernoulli differential equation of nonlinear order two. Its solution is given as follows

$$\zeta(t) = \frac{1}{t - t_0} \quad \text{for } c = 0 , \quad (18)$$

$$\zeta(t) = -c [1 - \tanh(ct + \delta)] \quad \text{for } c \in \mathbb{R} \setminus \{0\} , \quad (19)$$

$$\zeta(t) = -\tilde{c} [i + \tan(\tilde{c}t + \delta)] \quad \text{for } c = i\tilde{c} \in \mathbb{I} \setminus \{i0\} , \quad (20)$$

where δ is an integration constant. This last result allows to rewrite the first-order ODE (12) in the form

$$\dot{x} - \zeta(t)x = \phi_1(x)x , \quad (21)$$

which is a Bernoulli differential equation with the nonlinearity of one order higher than the order of the ϕ 's. Its solution provides the general solution of Eq. (1) factored in the (commutative) form given in Eq. (6).

It is worth mentioning that for the complex case $\phi_2 - \phi_1 = -2i\tilde{c}$, the following symmetric factorization of Eq. (1) is necessary

$$[D - \phi_1(x) + c][D - \phi_1(x) - c]x = 0 , \quad (22)$$

from where the following compatible first order ODE is obtained

$$\dot{x} - (\phi_1(x) + c)x = \zeta(t)x , \quad (23)$$

or by using Eq. (20), we obtain for this case the following Bernoulli differential equation

$$\dot{x} + \tilde{c} \tan(\tilde{c}t + \delta)x = \phi_1(x)x . \quad (24)$$

From the expressions of $\zeta(t)$, one can see that only the one containing the tangent function is trigonometric. Therefore, taking into account that the general solutions of the Bernoulli equations can be written as logarithmic derivatives, then one may obtain periodic (isochronous) solutions of Eq. (21) only in the case of the trigonometric $\zeta(t)$.

III. SOME APPLICATIONS

To show how the field method works in practice, we discuss three simple Liénard equations of the cubic oscillator type, one quintic oscillator case, the first corresponding to the inverse power $\zeta(t)$, and the others corresponding to the trigonometric $\zeta(t)$. The general higher-order nonlinear Liénard equations for which the field method can be applied is also presented.

1. A pure cubic anharmonic oscillator with nonlinear damping

Let us consider the following cubic anharmonic oscillator equation

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0 , \quad (25)$$

which admits the commutative factorization

$$\left(\frac{d}{dt} + kx \right) \left(\frac{d}{dt} + kx \right) x = 0 , \quad (26)$$

where $\phi_1 = \phi_2 = -kx$, and $c = 0$. Then, by substituting the corresponding function $\zeta(t)$ from Eq. (18) into Eq. (21), we obtain the Bernoulli equation

$$\dot{x} - \frac{1}{t - t_0}x = -kx^2 , \quad (27)$$

whose general solution is given as follows

$$x(t) = \frac{2(t - t_0)}{k(t - t_0)^2 - C} , \quad (28)$$

where C is an integration constant, while t_0 appears as a parameter in the Bernoulli differential equation, which, from Eq. (28), can be also interpreted as a constant fixed through the initial conditions applied to Eq. (25). In Fig. 1, a particular case of the solution given in Eq. (28) is displayed.

2. Same cubic anharmonic oscillator with added harmonic term

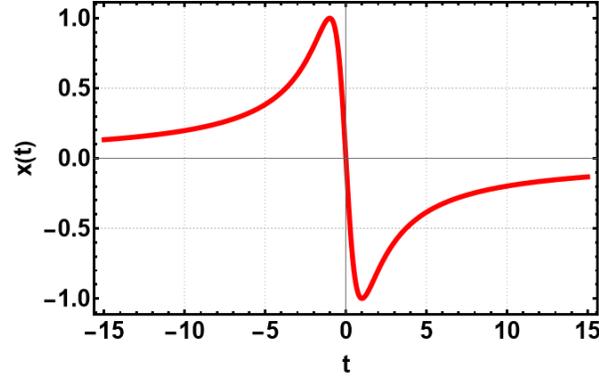


Figure 1: The solution $x(t)$ of the pure cubic oscillator with non linear damping for the following values of the constants: $C = 1$, $t_0 = 0$, $k = -1$.

Let us consider now the more general nonlinear ODE

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + \omega^2x = 0, \quad (29)$$

which admits the commutative factorization

$$\left(\frac{d}{dt} + kx + i\omega\right) \left(\frac{d}{dt} + kx - i\omega\right) x = 0, \quad (30)$$

where $\phi_1 = -kx + i\omega$, $\phi_2 = -kx - i\omega$, and $c = i\omega$ is an imaginary constant. Thus for this case, (24) takes the form

$$\dot{x} + \omega \tan(\omega t + \delta)x = -kx^2, \quad (31)$$

with solution given as follows

$$x(t) = \frac{\omega \cos(\omega t + \delta)}{A\omega + k \sin(\omega t + \delta)}, \quad (32)$$

where A is an integration constant, while δ has a role similar to t_0 of the previous case. If $A \notin [-k/\omega, +k/\omega]$, the general solution (32) is an isochronous solution of frequency ω . In Fig. 2, we present the plot corresponding to the solution given in Eq. (32) for $A = 1$, $\delta = 0$, $\omega = 3$, and $k = 2$.

3. Travelling wave solution of the Sharma-Tasso-Olver equation

The Sharma–Tasso–Olver (STO) equation is an odd order nonlinear partial differential equation of the Burgers hierarchy that plays a very important role in sciences and engineering fields, for

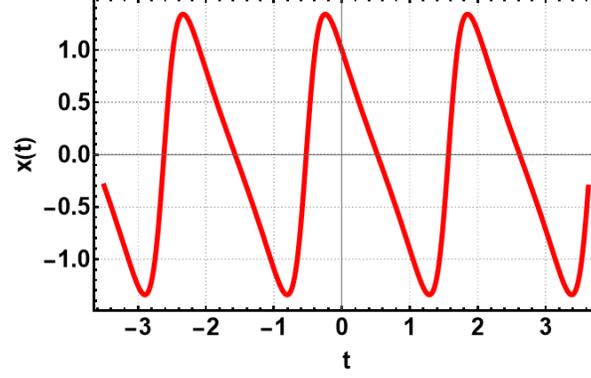


Figure 2: A particular solution $x(t)$ of the anharmonic oscillator given in equation (29) for the following values of the constants: $A = 1$, $\omega = 3$, $\delta = 0$ and $k = 2$.

instance plasma, nonlinear optics, quantum fields with many solution methods available in the literature [13, 14]. The STO equation is given by

$$u_t + 3\alpha u_x^2 + 3\alpha u^2 u_x + 3\alpha u u_{xx} + \alpha u_{xxx} = 0. \quad (33)$$

If we consider Eq. (33) in the moving reference frame under the transformation $z = x + vt$ such that $u(x, t) = u(z)$ and integrating once we have the ODE

$$u'' + 3uu' + u^3 + \frac{v}{\alpha}u + \frac{I}{\alpha} = 0, \quad (34)$$

where I is an integration constant. Next, we make the following shift transformation $u(z) = U(z) + \epsilon$, where ϵ is a constant that satisfies the following cubic equation

$$\epsilon^3 + \frac{v}{\alpha}\epsilon + \frac{I}{\alpha} = 0. \quad (35)$$

The roots of this equation are classified according to the discriminant

$$\Delta_3 = \frac{I^2}{\alpha^2} + \frac{4}{27} \left(\frac{v}{\alpha}\right)^3.$$

If $\Delta_3 > 0$, there is only one real root, and if $\Delta_3 \leq 0$, all the three roots are real.

Through the shift transformation, one obtains the following second order nonlinear ODE

$$U'' + 3(\epsilon + U)U' + U^3 + 3\epsilon U^2 + \left(3\epsilon^2 + \frac{v}{\alpha}\right)U = 0. \quad (36)$$

The latter equation can be factored in the following form

$$\left(\frac{d}{dz} + U + \frac{3\alpha\epsilon + i\sqrt{3\alpha^2\epsilon^2 + 4\alpha v}}{2\alpha}\right) \left(\frac{d}{dz} + U + \frac{3\alpha\epsilon - i\sqrt{3\alpha^2\epsilon^2 + 4\alpha v}}{2\alpha}\right) U = 0, \quad (37)$$

only for the case of the real roots of the cubic equation (35). For the special case when we choose the integration constant $I = -2\alpha \left((1-b)\sqrt{v/\alpha} \right)^3$, where $b = 0, 1, 2$, $\alpha > 0$ and $v > 0$, we have the real roots for the cubic equation given by $\epsilon = (1-b)\sqrt{v/\alpha}$. Therefore, from the factorization given in Eq. (37), we have that $\phi_2 - \phi_1 = -i\sqrt{3\epsilon^2\alpha^2 + 4v\alpha}/\alpha$, which implies $\tilde{c} = \sqrt{v\alpha(3(1-b)^2 + 4)}/2\alpha$. Consequently, the general solution of Eq. (37) is obtained by solving the following Bernoulli differential equation

$$\frac{dU}{dz} + \left(\tilde{c} \tan(\tilde{c}z + \delta) + \frac{3(1-b)}{2} \sqrt{\frac{v}{\alpha}} \right) U = -U^2, \quad (38)$$

with solution given as follows

$$U(z) = \frac{2(1 + 3(1-b)^2)v \cos(\tilde{c}z + \delta)}{2Av(1 + 3(1-b)^2)e^{3(1-b)\sqrt{vz}/2\sqrt{\alpha}} - 3(1-b)\sqrt{v\alpha} \cos(\tilde{c}z + \delta) + 2a\tilde{c} \sin(\tilde{c}z + \delta)}, \quad (39)$$

where δ and A are integration constants. Because of the exponential function in the denominator, the solutions with periodic singularities in at least one direction can be avoided only in the case $b = 1$ for which an isochronous solution similar to the isochronous solution (32) is obtained if $A \notin [-\sqrt{\alpha/v}, +\sqrt{\alpha/v}]$. In Fig. 3, the plots corresponding to the particular STO solution obtained from Eq. (39) for $\delta = 0$, $A = 1$, $v = 1$, $\alpha = 0.1$, and the three values of the shift parameter ϵ are displayed, showing the various types of solutions that can be obtained in this case.

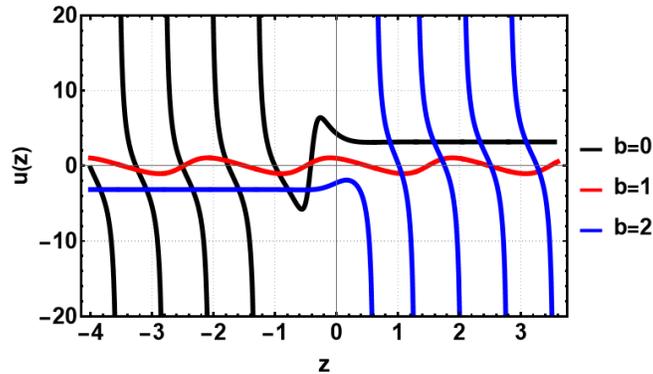


Figure 3: The solutions $u(z)$ of the STO equation (33) for $\delta = 0$, $A = 1$, $v = 1$, $\alpha = 0.1$, and values of $b = 0, 1, 2$, as black, red, and blue curves, respectively.

4. Wilson polynomial Liénard equation

The Wilson polynomial Liénard equation [15] is the following particular case of quintic-cubic Duffing-van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \frac{\mu^2}{16}x^3(x^2 - 4) + x = 0. \quad (40)$$

It was the first example of Liénard equation with an algebraic hyperelliptic limit cycle for $|\mu| < 2$ and also admits the commutative factorization

$$\left[D_t \pm i \frac{\sqrt{4 - \mu^2}}{2} + \frac{\mu}{2} \left(\frac{x^2}{2} - 1 \right) \right] \left[D_t \mp i \frac{\sqrt{4 - \mu^2}}{2} + \frac{\mu}{2} \left(\frac{x^2}{2} - 1 \right) \right] x = 0 . \quad (41)$$

For this case the constant \tilde{c} is

$$\tilde{c} \equiv \tilde{c}_\mu = \pm \frac{\sqrt{4 - \mu^2}}{2} , \quad 0 < |\mu| < 2 \quad (42)$$

and the Bernoulli equation (21) has the trigonometric form

$$\dot{x} + \left(\tilde{c}_\mu \tan(\tilde{c}_\mu t + \delta) - \frac{\mu}{2} \right) x = -\frac{\mu}{4} x^3 \quad (43)$$

whose general solution is

$$x(t) = \pm \frac{2 \cos(\tilde{c}_\mu t + \delta)}{\left\{ 1 + 4k_2 e^{-\mu t} + \frac{\mu}{2} \left\{ \frac{\mu}{2} \cos[2(\tilde{c}_\mu t + \delta)] + \tilde{c}_\mu \sin[2(\tilde{c}_\mu t + \delta)] \right\} \right\}^{1/2}} , \quad (44)$$

where k_2 and δ are integration constants. In Fig. 4, for $\mu = 1.8$ and $\delta = 0$, one can observe the

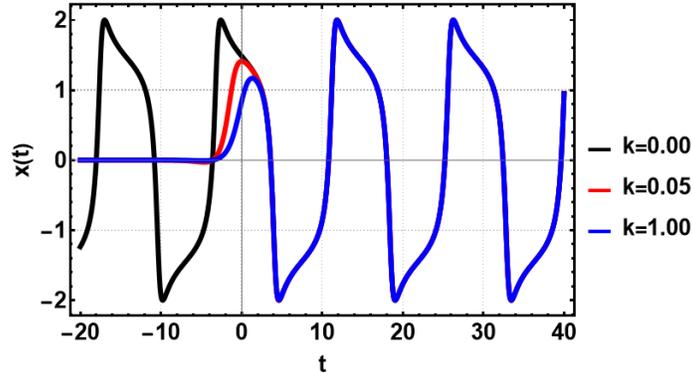


Figure 4: The solution (44) of the Wilson polynomial Liénard equation: of isochronous type for $k_2 = 0$, and non isochronous type for $k_2 = 0.5$ and $k_2 = 1$. In all cases, $\mu = 1.8$ and $\delta = 0$.

isochronous solution (44) when $k_2 = 0$, and the way in which the isochronicity is lost for $k_2 = 0.5$ and $k_2 = 1$.

5. A unifying treatment of the previous examples

Liénard equations of the type

$$\ddot{x} + \left[A(n+1)x^{n-1} + B \right] \dot{x} + A^2 x^{2n-1} + ABx^n + Cx = 0 , \quad (45)$$

can be factored with

$$\phi_1(x) = - \left(Ax^{n-1} + \frac{B \pm \sqrt{\Delta}}{2} \right), \quad \phi_2(x) = - \left(Ax^{n-1} + \frac{B \mp \sqrt{\Delta}}{2} \right), \quad \Delta = B^2 - 4C, \quad (46)$$

which satisfy the commutative factorization condition of the form

$$\phi_2 - \phi_1 = \pm \sqrt{\Delta} \equiv \text{const.} \quad (47)$$

implying the three possible cases of $\zeta(t)$ as in (18)-(20), but with c replaced by $\mp \sqrt{\Delta}/2$.

The corresponding Bernoulli differential equation for $x(t)$ reads

$$\dot{x} + \left(\frac{B \pm \sqrt{\Delta}}{2} - \zeta(t) \right) x = -Ax^n, \quad n = 2, 3, \dots, \quad (48)$$

with the solution

$$x(t) = \sqrt[n-1]{\frac{e^{-(n-1) \int^t P(\tau) d\tau}}{K + (n-1)A \int^t e^{-(n-1) \int^\tau P(t') dt'} d\tau}}, \quad (49)$$

where $P(t) = \frac{B \pm \sqrt{\Delta}}{2} - \zeta(t)$ and K is an arbitrary integration constant. This solution can be also written in the logarithmic derivative form

$$x(t) = [(n-1)A]^{-\frac{1}{n-1}} \left[\text{dlog} \left(\frac{K}{(n-1)A} + \int^t e^{-(n-1) \int^\tau P(t') dt'} \right) \right]^{\frac{1}{n-1}}, \quad (50)$$

and represents the general Liénard solution of (45) since it depends on two arbitrary constants. The choice of a positive sign in front of the parenthesis for the factoring functions ϕ_1 and ϕ_2 in (46), yields a similar factorization scheme whose equivalence is obtained through $A \rightarrow -A$ and $B \rightarrow -B$, however the same form of general solution given in Eq. (49) is obtained.

The factoring functions in (46) cover all the afore discussed particular cases and some others as well. Indeed, for $A = 0, B = \gamma, C = \omega^2$, and $n = 1$, we have the linear damped oscillator case, for $A = k, B = C = 0$ and $n = 2$, the nonlinear oscillator of the second case is obtained, $A = k, B = 0, C = \omega^2$ and $n = 2$ corresponds to the third case, $A = 1, B = 3\epsilon, C = 3\epsilon^2 + \frac{\nu}{\alpha}$ and $n = 2$ provides the STO case, $A = 1, B = 0, C = 1$ and $n = 2m + 2$ corresponds to equation (4) in [16] which has as a particular case the equation (29), a case with all A, B, C nonzero and $n = 3$ corresponds to a cubic-quintic Duffing-van der Pol unforced equation [17], and the list can be continued.

IV. CONCLUSION

An alternative way of solving the nonlinear commutative factorization setting previously studied in [3] has been provided here. An intermediate function in the method is assumed to depend

explicitly on the independent variable of the factored equation, aside to the dependent variable. This leads to a quasi-linear differential equation which can be solved and allows one to obtain the general solution of the problem for many commutative factorization settings. When a separable form in the two variables of the intermediate function is assumed, the dependent variable of the nonlinear equation is found to satisfy a Bernoulli differential equation with parameters and nonlinear part that depend on the factorization functions which is easily solved and leads to the general solution of the nonlinear second-order differential equation. This alternative method has been applied to several illustrative cases of Liénard equations and can be used for many equations by employing polynomial factoring functions that differ by a constant.

CRedit authorship contribution statement

G. González: Conceptualization, Writing – original draft.

O. Cornejo-Pérez: Methodology, Validation.

H.C. Rosu: Formal analysis, Writing - review and editing.

Declaration of competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

No data have been used in this paper.

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