# Transposing cartesian and other structure in double categories 

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#### Abstract

The cartesian structure possessed by morphisms like relations, spans, and profunctors is elegantly expressed by universal properties in double categories. Though cartesian double categories were inspired in part by the older program of cartesian bicategories, the precise relationship between the double-categorical and bicategorical approaches has so far remained mysterious, except in special cases. We provide a formal connection by showing that every double category with iso-strong finite products, and in particular every cartesian equipment, has an underlying cartesian bicategory. To do so, we develop broadly applicable techniques for transposing natural transformations and adjunctions between double categories, extending a line of previous work rooted in the concepts of companions and conjoints.


## 1 Introduction

The quest to axiomatize the elusive cartesian structure possessed by relations, spans, and profunctors has been a long one. Bicategories of such morphisms have an evident set-theoretic cartesian product, but it is not the bicategorical product. What, then, is its category-theoretic nature?

An early attempt to solve this puzzle was made by Carboni and Walters [CW87], who axiomatized 'a bicategory of relations' as a locally posetal, symmetric monoidal bicategory in which each object carries the structure of a commutative comonoid, subject to several axioms. Due in part to the uncertain status of symmetric monoidal bicategories at the time, Carboni and Walters were able to define a cartesian bicategory only in the locally posetal case. Two decades later, Carboni, Kelly, Walters, and Wood proposed a general definition of a cartesian bicategory [Car+08]. Their later approach has the virtue of making it more obvious that being cartesian is a property of, not a structure on, a bicategory, yet the definition is a complicated one that takes much of the paper to state completely.

Further progress depended on realizing that the cartesian structure of relations, spans, and profunctors is most simply expressed by a universal property not in bicategories, but in double categories. Building on the double limits introduced by Grandis and Paré [GP99], Aleiferi defined a cartesian double category to be a double category with binary and nullary double products [Ale18]. Recently, reviving an idea by Paré [Par09], the author developed a stronger notion of a double category with finite products [Pat24], capturing products of various shapes, including local products. Behind both approaches is the insight that, using double adjunctions [GP04], products in double categories can be defined as right adjoints to diagonals, just as in ordinary category theory.

Yet a lack of clarity still prevails as few formal connections have been made between cartesian bicategories and double categories with finite products. Lambert has shown that the underlying bicategory of any locally posetal cartesian equipment is cartesian [Lam22, Proposition 3.1]. The general situation is entangled with issues of bicategorical coherence, and no formal results are known. Our original impetus for this work was to close this gap. We do so by showing that the underlying bicategory of any double category with iso-strong finite products is cartesian (Theorem 5.7). It follows that the underlying bicategory of a cartesian equipment is cartesian (Corollary 5.8).

The technique that we use to prove this result is, perhaps, more interesting than the result itself, as it sheds light on the relationship between double categories and bicategories generally. In fact, only in the paper's final Section 5 do we study cartesian and cocartesian structure. In the rest of the paper, we explore a more nebulous question: how can structure in double categories be "transposed" from the strict (2-categorical) direction to the weak (bicategorical) direction?

Systematic methods to transpose structure in double categories were first devised by Garner and Gurski [GG09] and by Shulman [Shu10], with antecedents in [Shu08, Appendix B] and subsequent improvements made by Hansen and Shulman [HS19]. All of these works seek expedient ways to construct tricategorical structures that avoid long calculations verifying coherence axioms. Garner and Gurski construct tricategories from "locally-double bicategories," whereas Hansen and Shulman construct symmetric monoidal bicategories from symmetric monoidal double categories. At the heart of such constructions are companions assumed to exist in a base double category.

Companions and conjoints are the basic tool for transposing morphisms in a double category [BS76; GP04; Shu08]. A companion of an arrow in a double category is a universal choice of proarrow with the same domain and codomain; a conjoint is similar, except it has the opposite orientation. Companions and conjoints are carefully reviewed in Section 2, as they play an indispensable role in this work as in previous ones.

The utility of companions as a general tool is vastly increased by lifting the property of having a companion from individual morphisms in a double category to natural transformations between double functors. The companion of a natural transformation, if it exists, should be a transformation of another kind, whose components at objects are proarrows rather than arrows [GP99; Gra19]. We will call them protransformations; they have also be called "loose transformations" or, under our orientation convention, "horizontal transformations," and they generalize the pseudonatural transformations familiar from bicategory theory.

But there is a twist: the companion of a natural transformation whose component arrows have companions is usually only an oplax protransformation (Theorem 3.8). Dually, the conjoint of a natural transformation is usually only a lax protransformation. These generalize the oplax and lax natural transformations from bicategory theory. The appearance of laxness is not an accident but a fundamental aspect of transposing structure in double categories, and we will see that it explains why cartesian bicategories take their peculiar form. Under very special conditions, the companion or conjoint of a natural transformation is a pseudo protransformation (Corollary 3.9), as also recently noticed by Gambino, Garner, and Vasilakopoulou [GGV22, §3].

Having found an environment in which to exhibit companions of natural transformations, we can apply the universal property of companions to deduce powerful results for transposing structure with a minimum of calculation (Section 4). Most importantly, we can transpose a double adjunction, turning the (strict) triangle identities that hold between natural transformations into triangle identities between oplax protransformations that hold only up to coherent isomorphism (Theorem 4.4). Due to the oplaxity, the latter is not quite a biadjunction as usually understood but is rather a kind of lax adjunction [Gra74].

From here, the passage to cartesian bicategories is quite direct (Section 5). A cartesian bicategory does not obtain its cartesian product as a right biadjoint to the diagonal; if it did, the cartesian product would simply be the bicategorical product, the failure of which was the impetus to invent cartesian bicategories in the first place. But Trimble has shown that the axioms for a cartesian bicategory can be reformulated using lax adjunctions, such that the cartesian product is a right lax adjoint to the diagonal, built out of pseudofunctors, oplax natural transformations, and invertible modifications [Tri09]. Thus, from a double category with finite products, we can produce a cartesian bicategory simply by transposing the double adjunction between diagonals and products into a lax adjunction (Theorem 5.7). It appears that all known, genuine examples of cartesian bicategories
arise in this way.
The asymmetry between products and coproducts in a typical double category such as that of relations or spans is magnified upon passage to the underlying bicategory. Using the terminology of [Pat24], products in a double category tend to be at most iso-strong, meaning essentially that parallel products of proarrows commute with external composition, as in a cartesian double category. By contrast, coproducts in a well-behaved double category are strong, meaning that arbitrary span-indexed coproducts of proarrows commute with external composition. That is enough to imply that the coprojection and codiagonal transformations each have companions and conjoints that are pseudo protransformations, hence that the double adjunction defining the coproducts transposes to give simultaneous bicategorical products and coproducts (Theorem 5.10), also known as direct sums [LWW10]. For instance, the bicategories of relations and of spans inherit their direct sums, based on set-theoretic coproducts, in this way.

Our outlook is that double categories offer a unifying language for two-dimensional category theory. Recent works have shown that two-dimensional products and coproducts, when defined using double adjunctions, attain universal properties that are simple to state and use [Ale18; Pat24]. In this work, we have shown that the bicategorical formulations, found in cartesian bicategories and bicategories with direct sums, are obtained by transposition. But even this bicategorical structure is better viewed as double-categorical because the oplax or lax protransformations furnishing it have universal properties as companions or conjoints-universal properties that become invisible upon passing to the underlying oplax or lax natural transformations. In the future, one might hope to construct other complicated bicategorical structures, such as compact closed bicategories [Car95; Sta16], using similar techniques.

Conventions. Double categories and double functors are assumed to be pseudo unless otherwise stated. Categories C, D, .. are written in sans-serif font, 2 -categories and bicategories $\mathbf{B}, \mathbf{C}, \ldots$ in bold font, and double categories $\mathbb{D}, \mathbb{E}, \ldots$ in blackboard bold font. Composites of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in a category are written variously in applicative order as $g \circ f$ or in diagrammatic order as $f \cdot g$. Composites of proarrows $x \xrightarrow{m} y \xrightarrow{n} z$ in a double category are always written in diagrammatic order as $m \odot n$.

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## 2 Companions and conjoints

Companions and conjoints provide the basic means to transpose structure in double categories. As such, they have played a central role in previous works on extracting structured bicategories from structured double categories [Shu10; HS19]. In this section, we review the definitions of companions and conjoints and the facts about them we will need. Most of the results are known, if not always stated explicitly in the literature. An exception is the final result recognizing companions as a biadjoint, which refines an adjunction observed earlier by Grandis and Paré [GP04].

Companions and conjoints were introduced by Grandis and Paré under the names "orthogonal companions" and "orthogonal adjoints" [GP04], with antecedents in Brown and Spencer's early work on "connections" in a double category [BS76]. The close relation between companions, conjoints, and proarrow equipments was discovered by Shulman [Shu08].

Definition 2.1 (Companions and conjoints). A companion of an arrow $f: x \rightarrow y$ in a double category consists of a proarrow $f_{!}: x \rightarrow y$ and a pair of cells

the unit and the counit, satisfying the equations

where in the first equation we suppress the unit isomorphisms $\operatorname{id}_{x} \odot f_{!} \cong f_{!}$and $f_{!} \odot \operatorname{id}_{y} \cong f_{!}$following our usual convention.

Dually, a conjoint of an arrow $f: x \rightarrow y$ consists of a proarrow $f^{*}: y \rightarrow x$ and a pair of cells

satisfying the equations $\varepsilon \odot \eta=1_{f^{*}}$ and $\eta \cdot \varepsilon=\mathrm{id}_{f}$.
Remark 2.2 (Duality). The two definitions are indeed dual; they are even dual according to both forms of double-categorical duality. A conjoint of an arrow $f: x \rightarrow y$ in a double category $\mathbb{D}$ is just a companion of $f: x \rightarrow y$ in $\mathbb{D}^{\text {rev }}$, where the reverse double category $\mathbb{D}^{\text {rev }}$ is obtained from $\mathbb{D}$ by swapping the source and target functors $s, t: \mathbb{D}_{1} \rightrightarrows \mathbb{D}_{0}$. Alternatively, a conjoint of an arrow $f: x \rightarrow y$ in $\mathbb{D}$ is exactly a companion of $f: y \rightarrow x$ in $\mathbb{D}^{\text {op }}$, where the opposite double category $\mathbb{D}^{\text {op }}$ has opposite underlying categories $\left(\mathbb{D}^{\text {op }}\right)_{i}=\left(\mathbb{D}_{i}\right)^{\text {op }}$ for $i=0,1$. Either way, any statement about companions has a dual statement about conjoints and vice versa.

Defining companions and conjoints equationally, analogous to defining an adjunction by unit and counit cells obeying the triangle identities, is useful for performing calculations. Alternatively, companions and conjoints can each be defined by either of two universal properties, recognizing the unit or counit as an opcartesian or cartesian cell, respectively. This fundamental three-way equivalence is stated in [GP04, $\S \S 1.2-1.3]$ and extended to equipments in [Shu08, Theorem 4.1]. As usual, the universal properties imply that companions and conjoints are unique up to canonical isomorphism whenever they exist, which can also be proved equationally [Shu10, Lemma 3.8].

### 2.1 Reshaping cells by sliding

Companions and conjoints allow cells in a double category to be reshaped by "sliding" arrows around corners, turning them into proarrows. Since this is the basic technique on which the paper rests, we will present it in detail. The following lemma restates the rules for sliding companions and conjoints, as found in sources such as [GP04, §1.6], [GG09, Proposition 20], [Shu11, Proposition 5.13], and [Par23, §1].

Lemma 2.3 (Sliding). Suppose $x \xrightarrow{f} x^{\prime} \xrightarrow{f^{\prime}} x^{\prime \prime}$ and $y \xrightarrow{g} y^{\prime} \xrightarrow{g^{\prime}} y^{\prime \prime}$ are arrows in a double category. If the arrows $f^{\prime}$ and $g$ have companions and a choice of them is made, then there is a bijective correspondence between cells as on the left


Dually, if the arrows $f$ and $g^{\prime}$ have chosen conjoints, then there is a bijective correspondence between cells as on the right.

Proof. The first bijection, involving companions, is given by the assignment
and conversely

That these assignments are mutually inverse follows directly from the defining equations for a companion. The second bijection, involving conjoints, is dual.

A particular case of the lemma is useful enough to state independently. It furnishes the globular cells available in a bicategory.

Lemma 2.4 (Sliding, globular). Suppose $f: x \rightarrow w$ and $g: y \rightarrow z$ are arrows with companions or conjoints in a double category. Then, for any choice of companions or conjoints of $f$ and $g$, there is one or the other bijective correspondence between cells:


Proof. The first bijection follows by taking $f$ and $g^{\prime}$ to be identities in Lemma 2.3 and the second follows by taking $f^{\prime}$ and $g$ to be identities.

Remark 2.5 (Mates). A companion-conjoint pair of proarrows in a double category is, in particular, an adjunction in the underlying bicategory by [GP04, Proposition 1.4] or [Shu10, Lemma 3.21]. Under the bijection in Lemma 2.4, the cells $\alpha!$ and $\alpha^{*}$ are then mates with respect to the adjunctions $f_{!} \dashv f^{*}$ and $g!\dashv g^{*}$. The sliding rules for companions and conjoints thus extend the calculus of mates for adjunctions in a bicategory, a perspective taken in [Shu11].
Remark 2.6 (Commuters). Following [Par23, Definition 8.1], we say that a cell $\alpha$ as in Lemma 2.4 is a commuter if $\alpha_{!}$is an isomorphism. The interpretation is that $\alpha$ represents a square of arrows and proarrows that "commutes" up to isomorphism. Dually, a cell $\alpha$ is a cocommuter if $\alpha^{*}$ is an isomorphism. Note that being a commuter or a cocommuter are independent properties, so that a cell can represent an up-to-iso "commutative square" in two different senses. For a cell to be a commuter or cocommuter, it need not itself be an isomorphism, although, as the next lemma shows, that is a sufficient condition.
 arrows $f$ and $g$ have companions and $\alpha$ is an isomorphism (hence $f$ and $g$ are also isomorphisms), then $\alpha$ is a commuter.

Dually, if $f$ and $g$ have conjoints and $\alpha$ is an isomorphism, then $\alpha$ is a cocommuter.
Proof. As observed in [Shu10, Lemma 3.20], if $f$ is an isomorphism with a companion $f_{!}$, then $f_{!}$is also a conjoint of the inverse $f^{-1}$, via the unit and counit cells


It is then straightforward to show that an inverse to the cell

is provided by the cell

The sliding operations are functorial and natural in every sense that is well typed. Such properties have often been tacitly used in the literature. For the sake of completeness, we record the properties of sliding that we will need in a series of simple but useful lemmas.

Lemma 2.8 (External functoriality of sliding, globular). Suppose $f, g$, and $h$ are arrows with chosen companions in a double category. Then, under the bijection in Lemma 2.4, we have

In particular, the external identity cell $\mathrm{id}_{f}$ is a commuter.
Proof. Compose the cell $\alpha \odot \beta$ on the left with $\eta$ and on the right with $\varepsilon$, and insert the identity $\mathrm{id}_{g}=\eta \cdot \varepsilon$ in the middle, to obtain the equation

Comparing with the assignment $\alpha \mapsto \alpha_{!}$in Equation (2.1) proves the first statement. The second statement amounts to the defining equation $\eta \odot \varepsilon=f_{!}$.

A transpose of this lemma is also true:
Lemma 2.9 (Internal functoriality of sliding, special). Suppose $h, h^{\prime}$, and $h^{\prime \prime}$ are arrows with chosen companions in a double category. Then, under the bijection in Lemma 2.3, we have


Proof. Dually to Lemma 2.8, the first statement is proved by pre-composing the cell $\alpha_{!} \cdot \alpha_{!}^{\prime}$ with $\eta$, post-composing with $\varepsilon$, and inserting the identity $1_{h_{!}^{\prime}}=\eta \odot \varepsilon$ in the middle. The second statement amounts to the defining equation $\eta \cdot \varepsilon=\mathrm{id}_{h}$.

The next two lemmas use the functoriality of companions.
Lemma 2.10 (External functoriality of sliding, special). Suppose $x \xrightarrow{k} y \xrightarrow{\ell} z$ and $x^{\prime} \xrightarrow{k^{\prime}} y^{\prime} \xrightarrow{\ell^{\prime}} z^{\prime}$ are arrows with chosen companions in a double category, and choose the companions of their composites
to be $k_{!} \odot \ell_{!}$and $k_{!}^{\prime} \odot \ell_{!}^{\prime}$, respectively. Also, choose the companions of the identities $1_{x}$ and $1_{x^{\prime}}$ to be $\mathrm{id}_{x}$ and $\mathrm{id}_{x^{\prime}}$. Then, under the bijection in Lemma 2.3, we have

Proof. By [Shu10, Lemma 3.13], the composite $k_{!} \odot \ell$ is indeed a companion for $k \cdot \ell$, with unit and counit cells

and


The first statement is proved by pre-composing the cell $\alpha_{!} \odot \beta_{!}$with the unit for $k_{!} \odot \ell$, post-composing with the counit for $k_{!}^{\prime} \odot \ell_{!}^{\prime}$, and using the inverse assignment $\alpha!\mapsto \alpha$ in Equation (2.2). The second statement is proved similarly in view of [Shu10, Lemma 3.12].

Lemma 2.11 (Internal functoriality of sliding, globular). Suppose $x \xrightarrow{f} x^{\prime} \xrightarrow{f^{\prime}} x^{\prime \prime}$ and $y \xrightarrow{g} y^{\prime} \xrightarrow{g^{\prime}} y^{\prime \prime}$ are arrows with chosen companions in a double category, and choose the companions of their composites to be the composites of their companions, and likewise for identities. Then, under the bijection in Lemma 2.4, we have


The proof this lemma is similar to that of the previous one and is omitted.
Lemma 2.12 (Naturality of sliding). Suppose $f: x \rightarrow w$ and $g: y \rightarrow z$ are arrows with companions in a double category. Then an equation between cells as on the left

implies the equation between cells on the right.

Proof. Follows directly by composing on the left and right with $\eta$ and $\varepsilon$ as in Equation (2.1).

### 2.2 Transposition and companions as biadjoints

The operation of sliding and its many functoriality properties can be distilled into a single result, characterizing companions in double categories as a right biadjoint to transposing strict double categories. Let us first recall the double-categorical transpose.

Construction 2.13 (Transpose). The transpose $\mathbb{D}^{\top}$ of a strict double category $\mathbb{D}$ is the strict double category that exchanges the arrows and proarrows of $\mathbb{D}$, hence transposes the cells of $\mathbb{D}$ [Gra19, §3.2.2]. Transposition is the object part of a 2 -functor

$$
(-)^{\top}: \mathbf{S t r D b l}_{\text {pro }} \rightarrow \mathbf{S t r D b l},
$$

where $\mathbf{S t r D b l}$ is the 2-category of strict double categories, strict double functors, and (tight) natural transformations, and $\mathbf{S t r D b l}$ pro is the same 2-category, except that its 2 -morphisms are now loose or horizontal transformations, which will be called strict protransformations in Definition 3.1.

Construction 2.14 (Companions as a 2-functor). Let StrDbl $_{!\text {,pro }}$ be the 2-category of strict double categories equipped with a functorial choice of companion for each arrow, strict double functors preserving the chosen companions, and strict protransformations. Then the transposition 2 -functor can be restricted/extended to a 2 -functor

$$
(-)^{\top}: \mathbf{S t r D b l}_{!\text {,pro }} \rightarrow \mathbf{D b l} .
$$

Here, as usual, Dbl is the 2-category of (pseudo) double categories, (pseudo) double functors, and natural transformations.

Going in the other direction, there is a 2 -functor

$$
\mathbb{C o m p}: \mathbf{D b l} \rightarrow \text { StrDbl }_{!, \text {pro }}
$$

that sends a double category $\mathbb{D}$ to the strict double category $\mathbb{C o m p}(\mathbb{D})$ whose objects are those of $\mathbb{D}$, arrows are arrows in $\mathbb{D}$ that have companions, proarrows are arbitrary arrows in $\mathbb{D}$, and cells

as on the left are special cells in $\mathbb{D}$ as on the right. Cells in $\mathbb{C o m p}(\mathbb{D})$ compose via the evident pastings. By construction, every arrow in $\mathbb{C o m p}(\mathbb{D})$ has a canonical choice of companion, namely itself, with the external identity in $\mathbb{D}$ giving both binding cells.

To complete the construction, a double functor $F: \mathbb{D} \rightarrow \mathbb{E}$ induces a strict double functor $\mathbb{C o m p} F: \mathbb{C o m p} \mathbb{D} \rightarrow \mathbb{C o m p} \mathbb{E}$ that acts as the underlying 2-functor of $F$. Finally, given double functors $F, G: \mathbb{D} \rightarrow \mathbb{E}$, a natural transformation $\alpha: F \Rightarrow G$ induces a strict protransformation $\mathbb{C o m p} \alpha: \mathbb{C o m p} F \Rightarrow \mathbb{C o m p} G$ comprising

- for each object $x$ in $\mathbb{D}$, the component $(\mathbb{C o m p} \alpha)_{x}:=\alpha_{x}$, which is an arrow in $\mathbb{E}$, hence a proarrow in $\mathbb{C o m p} \mathbb{E}$;
- for each arrow $f: x \rightarrow y$ in $\mathbb{C o m p} \mathbb{D}$, which is an arrow in $\mathbb{D}$ with a companion, the component
the external identity in $\mathbb{E}$ on the naturality square of $\alpha$ at $f$.
The 2-functors just constructed are biadjoint to each other:
Theorem 2.15 (Companions as a biadjoint). The 2-category of strict double categories with a functorial choice of companions and the 2-category of double categories are related by a biadjunction


The component of the counit at a double category $\mathbb{D}$ is a double functor

$$
(\mathbb{C o m p} \mathbb{D})^{\top} \rightarrow \mathbb{D},
$$

that is the identity on objects and arrows and sends proarrows (arrows in $\mathbb{D}$ that have companions) to choices of companions in $\mathbb{D}$.

Proof. We first construct the unit of the biadjunction. Given a strict double category $\mathbb{C}$ with a functorial choice of companions, the component of the unit $\eta$ at $\mathbb{C}$ is the strict double functor

$$
\eta_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C o m p}\left(\mathbb{C}^{\top}\right)
$$

that is the identity on objects and proarrows; sends each arrow $f$ in $\mathbb{C}$ to the chosen companion $f_{!}$in


obtained by reshaping the cell $\alpha$ in $\mathbb{C}$ (Lemma 2.4). The strict functoriality of this assignment follows from the functorial choice of companions in $\mathbb{C}$ along with Lemmas 2.8 and 2.11.

We now construct the counit of the biadjunction. Given a double category $\mathbb{D}$, the component at $\mathbb{D}$ of the counit $\varepsilon$ is the double functor

$$
\varepsilon_{\mathbb{D}}:(\mathbb{C o m p} \mathbb{D})^{\top} \rightarrow \mathbb{D}
$$

that is the identity on objects and arrows; sends an arrow $h: x \rightarrow y$ in $\mathbb{D}$ with a companion to
 (Lemma 2.3):


The (pseudo) functoriality of this assignment follows from Lemmas 2.9 and 2.10.
It remains to show that the triangle identities hold, at least up to invertible modifications.


For any strict double category $\mathbb{C}$ with a functorial choice of companions, the first triangle can fail to commute because $\varepsilon_{\mathbb{C}^{\top}}$ need not choose the companion of an arrow $\eta_{\mathbb{C}}^{\top}(f)=f_{!}$in $\mathbb{C}^{\top}$ to be $f$ itself, but only some proarrow isomorphic to $f$ in $\mathbb{C}^{\top}$. Correcting this discrepancy fills the triangle with an invertible icon: a natural isomorphism whose component arrows are identities. For any double category $\mathbb{D}$, the second triangle actually commutes on the nose, because, although $\varepsilon_{\mathbb{D}}$ is pseudo, $\mathbb{C o m p} \varepsilon_{\mathbb{D}}$ uses only the 2 -functor underlying it.

This result refines Grandis and Parés earlier [GP04, Theorem 1.7] in several ways. First, the adjunction is between 2-categories rather than mere categories. Also, by using strict double categories rather than 2-categories on the left side of the adjunction, we can extract the arrows of a double category that have companions without discarding the other arrows along the way. Another difference is that, on the right side of the adjunction, we work with the property of having a companion, rather than the structure of a companion pair, and we allow pseudo double categories and double functors rather merely strict ones. This requires weakening the adjunction to a biadjunction.

## 3 Natural transformations and protransformations

A double category can be defined as a pseudocategory internal to Cat, the 2-category of categories. Likewise, a lax, colax, or pseudo double functor is respectively a lax, colax, or pseudo functor internal to Cat. Having adopted this perspective, one must sooner or later also consider natural transformations internal to Cat. These are not the usual (tight) natural transformations between double functors but a transposed notion, whose components on objects are proarrows rather than arrows. We will call them "protransformations;" another plausible name is "loose transformations."

### 3.1 Double categories of protransformations

In this section, we define lax protransformations between lax double functors and show how they assemble into the proarrows of a double category of lax functors. Although these notions are straightforward laxifications of pseudo notions long present in the literature on double categories
[GP99; Gra19], it seems worthwhile to give a detailed account since they are central to the present paper and we will need to reference the various axioms and operations.

Definition 3.1 (Protransformation). Let $F, G: \mathbb{D} \rightarrow \mathbb{E}$ be lax functors between double categories. A lax, oplax, or pseudo protransformation $\tau: F \Rightarrow G$ is respectively a lax, oplax, or pseudo natural transformation, internal to Cat, from $F$ to $G$.

We will immediately unwind this conceptual definition, but for the general definitions of pseudocategories, pseudofunctors, and pseudo natural transformations, see [Mar06].

Equivalently, a lax protransformation $\tau: F \nRightarrow G$ is seen to consist of

- for each object $x \in \mathbb{D}$, a proarrow $\tau_{x}: F x \rightarrow G x$ in $\mathbb{E}$, the component of $\tau$ at $x$;
- for each arrow $f: x \rightarrow y$ in $\mathbb{D}$, a cell $\tau_{f}$ in $\mathbb{E}$ of the form

$$
\begin{gathered}
F x \xrightarrow{\tau_{x}} G x \\
F f \downarrow \\
F y \xrightarrow[\tau_{y}]{\tau_{f}} G y
\end{gathered}
$$

called the component of $\tau$ at $f$;

- for each proarrow $m: x \rightarrow y$ in $\mathbb{D}$, a globular cell $\tau_{m}$ in $\mathbb{E}$ of the form

called the naturality comparison of $\tau$ at $m$.
The following axioms must be satisfied.
- Functoriality of components: $\tau_{f \cdot g}=\tau_{f} \cdot \tau_{g}$ for all arrows $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathbb{D}$, and also $\tau_{1_{x}}=1_{\tau_{x}}$ for all objects $x$ in $\mathbb{D}$.


- Coherence with respect to external composition: for every pair of composable proarrows $m: x \rightarrow y$ and $n: y \rightarrow z$ in $\mathbb{D}$,
- Coherence with respect to external identities: for every object $x$ in $\mathbb{D}$,


A oplax protransformation $\tau: F \Rightarrow G$ is defined similarly, except that the orientation of the naturality comparisons is reversed, giving them the form

for each proarrow $m: x \rightarrow y$ and changing the above axioms accordingly. Finally, a lax or oplax protransformation $\tau$ is pseudo if every comparison cell $\tau_{m}$ is invertible, and is strict if every comparison is an identity.

Unless otherwise stated, protransformations, like double categories and double functors, are assumed to be pseudo. Pseudo protransformations were first defined, under a different name, by Grandis and Paré; see [GP99, §7.4] or [Gra19, Definition 3.8.2].

Double functors between a fixed pair of double categories, and protransformations of those, generally do not form a category. Since their components are proarrows, protransformations have a composition that is associative and unital only up to isomorphism. But protransformations are the proarrows of a double category. The arrows of this double category are natural transformations and the cells are modifications.

Definition 3.2 (Modification). Let $F, G, H, K: \mathbb{D} \rightarrow \mathbb{E}$ be lax functors between double categories. A modification

bounded by lax protransformations $\sigma: F \Rightarrow G$ and $\tau: H \Rightarrow K$ and natural transformations $\alpha: F \Rightarrow H$ and $\beta: G \Rightarrow K$ consists of, for each object $x \in \mathbb{D}$, a cell in $\mathbb{E}$ of the form

the component of $\mu$ at $x$. Two axioms must be satisfied.

- Internal equivariance: for every arrow $f: x \rightarrow y$ in $\mathbb{D}$,

- External equivariance: for every proarrow $m: x \rightarrow y$ in $\mathbb{D}$,


Modifications between oplax protransformations are defined in nearly the same way, adjusting only the second axiom.

Modifications bounded by two natural transformations, optionally allowed to be pseudo, and by two pseudo protransformations were first defined by Grandis and Paré; see [GP99, §7.4] or [Gra19, Definition 3.8.3]. We prefer natural transformations to be strict, since they belong to the strict direction of the double categories, but we allow protransformations to be lax or oplax, since even strict natural transformations give rise to lax or oplax protransformations, as we will see.

Lax functors, natural transformations, protransformations, and modifications form a double category, as asserted in [GP99, §7.4] and [Gra19, Theorem 3.8.4]. The same is true if the protransformations are allowed to be either lax or oplax. For ease of reference, we state the construction in all three cases.

Construction 3.3 (Double categories of protransformations). Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. Then there are double categories

$$
\mathbb{L a x}_{\operatorname{lax}}(\mathbb{D}, \mathbb{E}), \quad \mathbb{L a x}_{\text {opl }}(\mathbb{D}, \mathbb{E}), \quad \text { and } \quad \mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})
$$

whose

- objects are lax double functors $\mathbb{D} \rightarrow \mathbb{E}$;
- arrows are natural transformations;
- proarrows are lax, oplax, or pseudo protransformations, respectively;
- cells are modifications.

In all three double categories, the composite $\sigma \odot \tau: F \Rightarrow H$ of protransformations $\sigma: F \Rightarrow G$ and $\tau: G \Rightarrow H$ is defined componentwise in $\mathbb{E}$, so that $(\sigma \odot \tau)_{x}:=\sigma_{x} \odot \tau_{x}$ for each object $x \in \mathbb{D}$ and

for each arrow $f: x \rightarrow y$ in $\mathbb{D}$, and (in the lax case) is defined on naturality comparisons by

for each proarrow $m: x \rightarrow y$ in $\mathbb{D}$. The identity protransformation $\mathrm{id}_{F}$ has components $\left(\mathrm{id}_{F}\right)_{x}:=\mathrm{id}_{F x}$ and $\left(\mathrm{id}_{F}\right)_{f}:=\mathrm{id}_{F f}$ and its (invertible) naturality comparisons are given by unitors in $\mathbb{E}$. Modifications compose componentwise in $\mathbb{E}$, in both directions. Finally, the associator and unitor modifications are also defined componentwise by associators and unitors in $\mathbb{E}$.

Lax and oplax protransformations are dual in that there is an isomorphism of double categories

$$
\mathbb{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})^{\mathrm{rev}} \cong \mathbb{L a x}_{\operatorname{lax}}\left(\mathbb{D}^{\mathrm{rev}}, \mathbb{E}^{\mathrm{rev}}\right)
$$

where $(-)^{\text {rev }}$ is the reversal duality from Remark 2.2 .
Protransformations, like natural transformations, can be pre-whiskered with lax functors. While that might seem unsurprising, it is worth examining with some care since protransformations generally cannot be post-whiskered with lax functors, only with pseudo ones.

Construction 3.4 (Pre-whiskering). Let $\tau: G \Rightarrow H: \mathbb{D} \rightarrow \mathbb{E}$ be a protransformation, possibly lax or oplax, between lax functors. The pre-whiskering of $\tau$ with another lax functor $F: \mathbb{C} \rightarrow \mathbb{D}$ is a protransformation

$$
\tau * F: G \circ F \Rightarrow H \circ F: \mathbb{C} \rightarrow \mathbb{E}
$$

of the same kind, defined to have components $(\tau * F)_{x}:=\tau_{F x}$ and $(\tau * F)_{f}:=\tau_{F f}$ at each object $x$ and arrow $f$ in $\mathbb{C}$ and naturality comparisons $(\tau * F)_{m}:=\tau_{F m}$ for each proarrow $m$ in $\mathbb{C}$.

Similarly, given lax functors $G, H, K, L: \mathbb{D} \rightarrow \mathbb{E}$, a modification as on the left

has a pre-whiskering with a lax functor $F: \mathbb{C} \rightarrow \mathbb{D}$, the modification on the right defined to have components $(\mu * F)_{x}:=\mu_{F x}$ at each object $x \in \mathbb{C}$.

Lemma 3.5 (Pre-whiskering is functorial). Given double categories $\mathbb{D}$ and $\mathbb{E}$, pre-whiskering with a lax double functor $F: \mathbb{C} \rightarrow \mathbb{D}$ defines a strict double functor

$$
(-) * F: \mathbb{L a x}(\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{L a x}_{p}(\mathbb{C}, \mathbb{E})
$$

whenever ' $p$ ' is replaced with any of 'lax', 'oplax', or 'pseudo'.
Proof. We first need to check that pre-whiskerings of protransformations and modifications are well-defined. The proof is essentially the same as in the bicategorical setting [JY21, Lemma 11.1.5] but we will record it anyway.

Given, say, a lax protransformation $\tau: G \nRightarrow H$ between lax functors $G, H: \mathbb{D} \rightarrow \mathbb{E}$, the pre-whiskering $\tau * F$ has functorial components $\tau_{F f}$ by the functoriality of $F_{0}: \mathbb{C}_{0} \rightarrow \mathbb{D}_{0}$, and the naturality of the pre-whiskering $\tau * F$ at a cell $\alpha$ in $\mathbb{C}$ follows immediately from the naturality of $\tau$ at the cell $F \alpha$ in $\mathbb{D}$. The two coherence axioms are not quite as immediate. To prove coherence with respect to external composition, fix proarrows $x \xrightarrow{m} y \xrightarrow{n} z$ in $\mathbb{C}$ and calculate

where the first equation is the coherence of $\tau$ at the proarrows $F x \stackrel{F m}{\rightarrow} F y \xrightarrow{F n} F z$ in $\mathbb{D}$ and the second equation is the naturality of $\tau$ with respect to the cell $F_{m, n}$. Coherence with respect to external identities is proved similarly.

Finally, the equivariance axioms of a pre-whiskered modification $\mu * F$ at an arrow $f$ or a proarrow $m$ in $\mathbb{C}$ follow immediately from those of $\mu$ at $F f$ or $F m$, respectively, and the pre-whiskering operation $(-) * F$ is strictly functorial because it simply amounts to reindexing.

Turning to post-whiskering, we caution again that protransformations cannot be post-whiskered with lax double functors, only with pseudo ones. This problem is well known for (lax, oplax, or pseudo) natural transformations in the bicategorical setting [JY21, §11.1]; in fact, it is a good reason to consider double categories in the first place, since ordinary natural transformations between lax double functors do not suffer from this problem [Shu09].

Construction 3.6 (Post-whiskering). Let $\tau: F \Rightarrow G: \mathbb{C} \rightarrow \mathbb{D}$ be a lax protransformation between lax functors. The post-whiskering of $\tau$ with a double functor $H: \mathbb{D} \rightarrow \mathbb{E}$ is the lax protransformation

$$
H * \tau: H \circ F \Rightarrow H \circ G: \mathbb{C} \rightarrow \mathbb{E}
$$

with components $(H * \tau)_{x}:=H\left(\tau_{x}\right)$ and $(H * \tau)_{f}:=H\left(\tau_{f}\right)$ at each object $x$ and arrow $f$ in $\mathbb{C}$ and with natural comparisons

for each proarrow $m: x \rightarrow y$ in $\mathbb{C}$. Post-whiskering of an oplax or pseudo protransformation is defined similarly.

The post-whiskering $H * \mu$ of a modification $\mu$ is defined by applying $H$ componentwise, so that $(H * \mu)_{x}:=H\left(\mu_{x}\right)$ for each object $x \in \mathbb{C}$.

We omit the proof that post-whiskerings of protransformations are well-defined, as it is again essentially the same as in the bicategorical setting [JY21, Lemma 11.1.6].

Lemma 3.7 (Post-whiskering is functorial). Given double categories $\mathbb{C}$ and $\mathbb{D}$, post-whiskering with a (pseudo) double functor $H: \mathbb{D} \rightarrow \mathbb{E}$ defines a double functor

$$
H *(-): \mathbb{L a x} p(\mathbb{C}, \mathbb{D}) \rightarrow \mathbb{L a x} p(\mathbb{C}, \mathbb{E})
$$

whenever ' $p$ ' is replaced with any of 'lax', 'oplax', or 'pseudo'.

### 3.2 Companion and conjoint transformations

Among other possible uses, double categories of protransformations are environments in which to find companions and conjoints of natural transformations between double functors.

Theorem 3.8 (Companions and conjoints of natural transformations). A natural transformation $\alpha: F \Rightarrow G$ between lax double functors $F, G: \mathbb{D} \rightarrow \mathbb{E}$ has a companion in $\mathbb{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})$, which is an oplax protransformation $\alpha_{!}: F \Rightarrow G$, if and only if each component arrow $\alpha_{x}: F x \rightarrow G x$ has a companion in $\mathbb{E}$.

Dually, the natural transformation $\alpha: F \Rightarrow G$ has a conjoint in $\mathbb{L a x}_{\operatorname{lax}}(\mathbb{D}, \mathbb{E})$, which is a lax protransformation $\alpha^{*}: G \Rightarrow F$, if and only if each component $\alpha_{x}$ has a conjoint in $\mathbb{E}$.

Proof. Supposing each component of $\alpha: F \Rightarrow G$ has a companion, we define an oplax protransformation $\alpha_{!}: F \Rightarrow G$ as follows. The component of $\alpha_{!}$at an object $x \in \mathbb{D}$ is any choice of companion $\left(\alpha_{!}\right)_{x}:=\left(\alpha_{x}\right)_{!}: F x \rightarrow G x$ of the corresponding component $\alpha_{x}: F x \rightarrow G x$ of $\alpha$. The component of
$\alpha_{!}$at an arrow $f: x \rightarrow y$ in $\mathbb{D}$ is defined by sliding arrows (Lemma 2.3) in the identity cell

on the left, induced by the naturality square of $\alpha$ at $f$. The naturality comparison of $\alpha$ ! at a proarrow $m: x \rightarrow y$ in $\mathbb{D}$ is defined by sliding arrows (Lemma 2.4) in the component of $\alpha$ at $m$ :


We thus have the data of an oplax protransformation $\alpha_{!}: F \Rightarrow G$.
The axioms now follow straightforwardly from the functoriality and naturality of sliding. First, for any arrows $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathbb{D}$, the equation between external identities

implies that $\left(\alpha_{!}\right)_{f \cdot g}=\left(\alpha_{!}\right)_{f} \cdot\left(\alpha_{!}\right)_{g}$, by Lemma 2.9. Also, $\left(\alpha_{!}\right)_{1_{x}}=1_{\left(\alpha_{!}\right)_{x}}$ by the same lemma, proving the functoriality of the components of $\alpha_{!}$. The coherence axioms of $\alpha$ imply those of $\alpha_{!}$. Specifically, for any proarrows $x \xrightarrow{m} y \xrightarrow{n} z$ in $\mathbb{D}$, the coherence equation
induces the corresponding coherence equation

by Lemmas 2.8 and 2.12. The coherence axiom for identity proarrows is proved similarly. To
 naturality equation for $\alpha$ :


Compose both sides on the left with $\operatorname{id}_{F f} \cdot \eta_{w}$, where $\eta_{w}$ is the unit cell for $\left(\alpha_{w}\right)$ !, and on the right with $\varepsilon_{y} \cdot \operatorname{id}_{G g}$, where $\varepsilon_{y}$ is the counit for $\left(\alpha_{y}\right)!$. Also, insert the identity $\operatorname{id}_{\alpha_{x}}=\eta_{x} \cdot \varepsilon_{x}$ in the middle of the left-hand side and insert the identity $\operatorname{id}_{\alpha_{z}}=\eta_{z} \cdot \varepsilon_{z}$ in the middle of the right-hand side. From the correspondence in Equation (2.1), we obtain the naturality equation

So $\alpha_{!}$is a well-defined oplax protransformation.
The unit and counit for the companion pair $(\alpha, \alpha!)$ in $\mathbb{L a x} \mathrm{opl}(\mathbb{D}, \mathbb{E})$ are the modifications

whose components are the units and counits
for the companion pairs $\left(\alpha_{x},\left(\alpha_{x}\right)\right.$ !) in $\mathbb{E}$, for each object $x \in \mathbb{D}$. Since modifications compose componentwise, the two axioms for $\eta$ and $\varepsilon$ follow immediately from those for each $\eta_{x}$ and $\varepsilon_{x}$, so we just need to show that the modifications are well defined.

We prove the two equivariance axioms for the modification $\eta$; the proofs for $\varepsilon$ are dual. The first equivariance axiom states that for every arrow $f: x \rightarrow y$ in $\mathbb{D}$,


Post-composing both sides with counit $\varepsilon_{y}$ and using the inverse correspondence in Equation (2.2) gives the true equation


Therefore, by the universal property possessed by $\varepsilon_{y}$ as a restriction cell, the original equation holds too. The other equivariance axiom states that for every proarrow $m: x \rightarrow y$ in $\mathbb{D}$,

Post-composing both sides with the cell $\varepsilon_{x} \odot 1_{G m}$ and again using the inverse correspondence in Equation (2.2) gives the true equation $\alpha_{m}=\alpha_{m}$. Since $\varepsilon_{x} \odot 1_{G m}$ is a restriction cell, namely the restriction of $G m$ along $\alpha_{x}$ and $1_{G y}$, the universal property of the restriction implies that the original equation holds too.

This completes the proof that $\left(\alpha, \alpha_{!}\right)$is a companion pair in $\mathbb{L} \mathrm{ax}_{\text {opl }}(\mathbb{D}, \mathbb{E})$, establishing the harder direction of the first equivalence. Conversely, if $(\alpha, \alpha!)$ is a companion pair in $\mathbb{L a x}_{\text {opl }}(\mathbb{D}, \mathbb{E})$ with binding modifications $\eta$ and $\varepsilon$, then for each object $x \in \mathbb{D}$, the components $\eta_{x}$ and $\varepsilon_{x}$ are immediately seen to make $\left(\alpha_{x},\left(\alpha_{!}\right)_{x}\right)$ into a companion pair in $\mathbb{E}$.

The second equivalence, about conjoints, is dual in view of Remark 2.2 and the isomorphism $\mathbb{L a x}_{\text {lax }}(\mathbb{D}, \mathbb{E})^{\text {rev }} \cong \mathbb{L a x}_{\text {opl }}\left(\mathbb{D}^{\text {rev }}, \mathbb{E}^{\text {rev }}\right)$. For the sake of concreteness, we note that if the natural transformation $\alpha: F \Rightarrow G$ has components with conjoints in $\mathbb{E}$, then the conjoint lax protransformation $\alpha^{*}: G \Rightarrow F$ has components $\left(\alpha^{*}\right)_{x}:=\left(\alpha_{x}\right)^{*}: G x \rightarrow F x$ at each object $x \in \mathbb{D}$ and has naturality
comparisons

induced by Lemma 2.4 for each proarrow $m: x \rightarrow y$ in $\mathbb{D}$.
Corollary 3.9 (Companions and conjoints, pseudo case). A natural transformation $\alpha: F \Rightarrow G$ between lax double functors $F, G: \mathbb{D} \rightarrow \mathbb{E}$ has a companion in $\mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})$, which is a (pseudo) protransformation $\alpha_{!}: F \Rightarrow G$, if and only if each component arrow $\alpha_{x}$ has a companion and each component cell $\alpha_{m}$ is a commuter (Remark 2.6).

Dually, the natural transformation $\alpha: F \Rightarrow G$ has a conjoint in $\mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})$, which is a protransformation $\alpha^{*}: G \Rightarrow F$, if and only if each component arrow $\alpha_{x}$ has a conjoint and each component cell $\alpha_{m}$ is cocommuter.

Proof. This follows immediately from Theorem 3.8 and Equations (3.1) and (3.2) defining the naturality comparisons of the (op)lax protransformations.

The preceding corollary was stated recently as [GGV22, Proposition 3.10], with only a sketch of a proof. The next corollary is a further specialization.

Corollary 3.10 (Companions and conjoints of natural isomorphisms). Suppose $\alpha: F \Rightarrow G$ is a natural isomorphism between lax double functors $F, G: \mathbb{D} \rightarrow \mathbb{E}$ such that each component $\alpha_{x}$ has a companion. Then $\alpha$ has a companion $\alpha_{!}: F \Rightarrow G$ in $\mathbb{L a x}(\mathbb{D}, \mathbb{E})$. If, in addition, each component $\alpha_{x}$ has a conjoint, then $\alpha$ has a conjoint $\alpha^{*}: G \Rightarrow F$, and the two protransformations form an adjoint equivalence $\alpha!\dashv \alpha^{*}$ in $\mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})$.

Proof. Since $\alpha$ is a natural isomorphism, each component cell $\alpha_{m}$ is an isomorphism and hence is a commuter by Lemma 2.7. Thus, $\alpha$ has a companion in $\mathbb{L a x}(\mathbb{D}, \mathbb{E})$ by Corollary 3.9. If, moreover, each component arrow $\alpha_{x}$ has a conjoint, then each component cell $\alpha_{m}$ is a cocommuter and $\alpha$ has a conjoint in $\mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})$. That $\alpha_{!}$and $\alpha^{*}$ then form an adjoint equivalence is a general fact about the companion and the conjoint of an isomorphism [Shu10, Lemma 3.21].

The following lemma about natural transformations is a useful source of commuter cells.
Lemma 3.11 (Components at companions are commuters). Suppose $\alpha: F \Rightarrow G$ is a natural transformation between normal lax functors $F, G: \mathbb{D} \rightarrow \mathbb{E}$. If $f: x \rightarrow y$ is an arrow in $\mathbb{D}$ with a companion $f_{!}: x \rightarrow y$, then under sliding (Lemma 2.3) the component $\alpha_{f_{!}}$corresponds to the external identity on the naturality square for $f$ :


Moreover, if the components $\alpha_{x}$ and $\alpha_{y}$ also have companions, then the cell $\alpha_{f!}$ is a commuter. In fact, the reshaped cell $\left(\alpha_{f_{!}}\right)$! (Lemma 2.4) is the canonical isomorphism between companions of $F y \cdot \alpha_{y}=\alpha_{x} \cdot G f$.

Proof. Only as a convenience, assume that $F$ and $G$ are unitary lax functors, i.e., strictly preserve external identities. So, if $\left(f, f_{!}, \eta, \varepsilon\right)$ is a companion pair, then its image under $F$ is again a companion pair ( $F f, F f_{!}, F \eta, F \epsilon$ ) without inserting identity comparisons, and likewise for the image under $G$. Now calculate


Performing the calculation in the other direction exhibits the right-hand side as equal to $\mathrm{id}_{\alpha_{x} \cdot G f}$. Either way, we find that $\alpha_{f_{!}}$corresponds under sliding to an external identity.

Moreover, if $\alpha_{x}$ and $\alpha_{y}$ have companions, then the reshaped cell $\left(\alpha_{f}\right)$ ! is the canonical isomorphism between companions, as seen by comparing with the formula in [Shu10, Lemma 3.8].

## 4 From double categories to bicategories

Combining results from the previous two sections, we obtain procedures for transposing structure in double categories. This includes as a special case transferring structure from a double category to its underlying bicategory. Much of the work has already been done in Theorem 3.8 and its corollaries by recognizing the companion of a natural transformation between lax double functors as an oplax or pseudo protransformation. We now simply apply generalities about companions from Section 2 to this situation.

### 4.1 Transposing natural transformations

Given double categories $\mathbb{D}$ and $\mathbb{E}$, there is a strict double category $\mathbb{L a x}_{\mathrm{c}}(\mathbb{D}, \mathbb{E})$ having as objects, lax double functors $\mathbb{D} \rightarrow \mathbb{E}$; as arrows, natural transformations whose component arrows have
 of the form $\mu: \beta \circ \gamma \Rightarrow \delta \circ \alpha$. Define the strict double category $\mathbb{L a x}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})$ in the same way, except that its arrows are natural transformations whose component arrows have companions and whose component cells are commuters. ${ }^{1}$

Theorem 4.1. For any double categories $\mathbb{D}$ and $\mathbb{E}$, there are double functors

$$
\operatorname{Lax}_{\mathrm{c}}(\mathbb{D}, \mathbb{E})^{\top} \rightarrow \mathbb{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E}) \quad \text { and } \quad \mathbb{L a x}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})^{\top} \rightarrow \mathbb{L a x} \mathrm{xps}(\mathbb{D}, \mathbb{E})
$$

that act as identities on objects and arrows, and send proarrows (which are natural transformations satisfying extra properties) to oplax or pseudo protransformations, respectively.

[^0]Proof. Using the biadjunction $(-)^{\top} \dashv$ Comp in Theorem 2.15, the two components of the counit at the double categories $\mathbb{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})$ and $\mathbb{L a x}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})$ are double functors

$$
\operatorname{Comp}\left(\operatorname{Lax}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})\right)^{\top} \rightarrow \mathbb{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E}) \quad \text { and } \quad \operatorname{Comp}\left(\operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})\right)^{\top} \rightarrow \mathbb{L} \mathrm{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})
$$

with the claimed properties. But Theorem 3.8 and Corollary 3.9 say precisely that

$$
\operatorname{Comp}\left(\mathbb{L a x}{ }_{\mathrm{opl}}(\mathbb{D}, \mathbb{E})\right)=\operatorname{Lax}_{\mathrm{c}}(\mathbb{D}, \mathbb{E}) \quad \text { and } \quad \operatorname{Comp}\left(\operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})\right)=\operatorname{Lax}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})
$$

Deducing a result about bicategories is, from a high level, as simple as passing from double categories to their underlying bicategories. But this procedure is not without subtleties since, notoriously, there is no 2-category (or 3-category) of bicategories, lax functors, and lax, oplax, or pseudo natural transformations [Shu09]. A fortiori, there is no 2-category of double categories, lax functors, and lax, oplax, or pseudo protransformations, and so there can be no forgetful 2-functor from the latter to the former. Let us take a closer look at the passage from double categories to bicategories.

Construction 4.2 (Underlying bicategory). First of all, there are mere categories $\mathrm{Dbl}_{\text {lax }}$ and Bicat $_{\text {lax }}$ that have double categories and bicategories as objects, respectively, and have lax functors as morphisms. The underlying bicategory functor

$$
\text { B : } \mathrm{Dbl}_{\mathrm{lax}} \rightarrow \text { Bicat }_{\text {lax }}
$$

sends

- each (pseudo) double category $\mathbb{D}$ to its underlying bicategory $\mathbf{B}(\mathbb{D})$, comprising the objects, proarrows, and globular cells of $\mathbb{D}$; and
- each lax double functor to the lax functor between underlying bicategories that has the same action on objects, proarrows, and globular cells and the same comparison cells.

This forgetful functor restricts on pseudofunctors to a functor $\mathbf{B}: \mathrm{Dbl} \rightarrow$ Bicat; see, for instance, [Shu10, Theorem 4.1].

Turning to protransformations, notice that the double-categorical notion of a lax protransformation generalizes the bicategorical notion of a lax natural transformation. To be more precise, when bicategories $\mathbf{B}$ and $\mathbf{C}$ are regarded as double categories with only identity arrows, a lax or oplax protransformation (Definition 3.1) between lax functors $F, G: \mathbf{B} \rightarrow \mathbf{C}$ is exactly a lax or oplax natural transformation between $F, G$, as defined in bicategory theory [JY21, §§4.2-4.3]. Moreover, a modification between protransformations (Definition 3.2) with identity source and target is precisely a modification in the bicategorical sense [JY21, §4.4].

Conversely, for fixed double categories $\mathbb{D}$ and $\mathbb{E}$, there are forgetful 2-functors

$$
\text { B : }\left\{\begin{array}{l}
\operatorname{Lax}_{\text {lax }}(\mathbb{D}, \mathbb{E}) \rightarrow \operatorname{Lax}_{\mathrm{lax}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E})) \\
\operatorname{Lax}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E}) \rightarrow \operatorname{Lax}_{\mathrm{opl}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E})) \\
\operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E}) \rightarrow \operatorname{Lax}_{\mathrm{ps}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E}))
\end{array}\right.
$$

from the 2-categories of lax double functors $\mathbb{D} \rightarrow \mathbb{E}$, (lax, oplax, or pseudo) protransformations, and modifications to the 2-categories of lax functors $\mathbf{B}(\mathbb{D}) \rightarrow \mathbf{B}(\mathbb{E})$, (lax, oplax, or pseudo) natural transformations, and modifications. The action on lax functors is as above, whereas the action on protransformations is simply to forget the component cells, while keeping the component proarrows and the naturality comparisons.

We need a final bit of notation to state the next result. Given double categories $\mathbb{D}$ and $\mathbb{E}$, let $\operatorname{Lax}_{c}(\mathbb{D}, \mathbb{E})$ be the 2-category of lax double functors $\mathbb{D} \rightarrow \mathbb{E}$, natural transformations whose component arrows have companions, and modifications. Let $\operatorname{Lax}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})$ be the 2-category with the same objects and cells, but with morphisms being the natural transformations whose component arrows have companions and whose component cells are commuters.

Corollary 4.3. For any double categories $\mathbb{D}$ and $\mathbb{E}$, there are pseudofunctors

$$
\operatorname{Lax}_{\mathrm{c}}(\mathbb{D}, \mathbb{E})^{\mathrm{co}} \rightarrow \operatorname{Lax}_{\mathrm{opl}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E})) \quad \text { and } \quad \mathbf{L a x}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})^{\mathrm{co}} \rightarrow \operatorname{Lax}_{\mathrm{ps}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E}))
$$

that send

- lax double functors $\mathbb{D} \rightarrow \mathbb{E}$ to the lax functors between the underlying bicategories;
- natural transformations (satisfying extra properties) to oplax or pseudo natural transformations, respectively; and
- modifications to modifications, reversing the orientation.

Proof. Applying the forgetful functor B: Dbl $\rightarrow$ Bicat to the double functors from Theorem 4.1 gives pseudofunctors

$$
\mathbf{L a x}_{\mathrm{c}}(\mathbb{D}, \mathbb{E})^{\mathrm{co}} \rightarrow \mathbf{L a x}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E}) \quad \text { and } \quad \operatorname{Lax}_{\mathrm{cc}}(\mathbb{D}, \mathbb{E})^{\mathrm{co}} \rightarrow \operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E})
$$

To complete the proof, post-compose these with the forgetful 2 -functors

$$
\operatorname{Lax}_{\mathrm{opl}}(\mathbb{D}, \mathbb{E}) \xrightarrow{\mathbf{B}} \operatorname{Lax}_{\mathrm{opl}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E})) \quad \text { and } \quad \operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{E}) \xrightarrow{\mathrm{B}} \operatorname{Lax}_{\mathrm{ps}}(\mathbf{B}(\mathbb{D}), \mathbf{B}(\mathbb{E})) .
$$

The corollary slightly strengthens Hansen and Shulman's [HS19, Theorem 4.6], which is proved directly without passing through protransformations; see also Shulman's earlier [Shu10, Theorem 4.6]. We think the abstract perspective offered here is valuable even when the extra flexibility of our Theorem 4.1 is not needed.

### 4.2 Transposing adjunctions

If natural transformations between double functors can be transposed, it stands to reason that double adjunctions can be too. And they can, but not without a few subtleties. First, since the companion of a natural transformation is generally an oplax protransformation, we cannot have an ordinary biadjunction but must have something looser. There are many such notions, depending on whether the transformations are lax, oplax, or pseudo and whether the modifications are invertible, but they are all described by the same axioms, due to Gray [Gra74, §I.7]. We call all such situations lax adjunctions and will make clear from context what kinds of cells are involved.

Theorem 4.4. Suppose $(\eta, \varepsilon): F \dashv G: \mathbb{D} \rightleftarrows \mathbb{E}$ is a double adjunction such that $F$ and $G$ are pseudo and the component arrows of the unit and counit

$$
\eta: 1_{\mathbb{D}} \Rightarrow G \circ F \quad \text { and } \quad \varepsilon: F \circ G \Rightarrow 1_{\mathbb{E}}
$$

have companions in $\mathbb{D}$ and $\mathbb{E}$, respectively. Then this data extends to a lax adjunction

$$
\left(\eta_{!}, \varepsilon_{!}, s, t\right): F \dashv G: \mathbb{D} \rightleftarrows \mathbb{E}
$$

comprising double functors, oplax protransformations, and invertible modifications.
If, moreover, the component cells of the unit and counit are commuters, then there is a biadjunction comprising double functors, protransformations, and invertible modfications.

Proof. By assumption, the triangle identities

hold on the nose and, by Theorem 3.8 , the unit and counit have companions $\eta$ ! and $\varepsilon!$ in $\mathbb{L a x}{ }_{\text {opl }}(\mathbb{D}, \mathbb{D})$ and $\mathbb{L a x}_{\mathrm{opl}}(\mathbb{E}, \mathbb{E})$, respectively. Now, since double functors preserve companions, Lemma 3.5 implies that the pre-whiskerings $\eta!* G$ and $\varepsilon!* F$ of companions are companions of the pre-whiskerings $\eta * G$ and $\varepsilon * F$. Similarly, by Lemma 3.7, the post-whiskerings $F * \eta$ ! and $G * \varepsilon_{!}$of companions are companions of the post-whiskerings $F * \eta$ and $G * \varepsilon$. Thus, there are unique globular isomorphisms

the triangulators, witnessing the functoriality and essential uniqueness of companions [Shu10, Lemmas 3.8 and 3.12-13].

It remains to show that the triangulators satisfy the two coherence axioms of a lax adjunction [Gra74, Definition I.7.1], sometimes called the swallowtail identities [BL03]. The first of these is

which becomes possibly more intelligible when expressed in components as


Either by Lemma 3.11 or by construction, all of these globular cells in $\mathbb{E}$ are canonical isomorphisms between choices of companions, hence the equation holds by the uniqueness of those isomorphisms. The other swallowtail identity is proved analogously.

Under the further assumption that the unit $\eta$ and counit $\varepsilon$ have component cells that are commuters, their companions $\eta$ ! and $\varepsilon$ ! are pseudo protransformations by Corollary 3.9 and we obtain a genuine biadjunction.

Corollary 4.5. Suppose $(\eta, \varepsilon): F \dashv G: \mathbb{D} \rightleftarrows \mathbb{E}$ is a double adjunction such that $F$ and $G$ are pseudo and $\eta$ and $\varepsilon$ have component arrows with companions. Then, denoting the underlying bicategories
of $\mathbb{D}$ and $\mathbb{E}$ by $\mathbf{D}$ and $\mathbf{E}$, there is a lax adjunction $\left(\eta_{!}, \varepsilon_{!}, s, t\right): F \dashv G: \mathbf{D} \rightleftarrows \mathbf{E}$ comprising pseudofunctors, oplax natural transformations, and invertible modifications.

If, moreover, $\eta$ and $\varepsilon$ have component cells that are commuters, there is a biadjunction comprising pseudofunctors, pseudonatural transformations, and invertible modifications.
Proof. Follows from the theorem by passing to bicategories and their cells (Construction 4.2).
These results reveal another subtlety about transposing adjunctions. In its most general form, a double adjunction is between a colax double functor on the left and a lax double functor on the right [GP04; Gra19]. Yet we cannot transpose colax-lax or even pseudo-lax double adjunctions, i.e., adjunctions in the 2 -category $\mathbf{D b l}_{\text {lax }}$, but must restrict still further to pseudo-pseudo double adjunctions, i.e., adjunctions in the 2 -category $\mathbf{D b l}$. The problem is that protransformations cannot be post-whiskered by lax functors, so the data of a lax adjunction seems to not even make sense in the more general contexts. Having a simple but flexible theory of two-dimensional adjunctions is an important virtue of double categories that is largely lost when passing to bicategories.

## 5 From double categories with products to cartesian bicategories

As an application of the theory developed so far, we show how to transpose the cartesian structure possessed by a double category with products, where we use "products" in the generalized sense considered first by Paré [Par09] and later by the author [Pat24]. This line of reasoning will culminate in a proof that every double category with finite products, and in particular every cartesian equipment, has an underlying cartesian bicategory [Car+08]. In this section, we assume acquaintance with double-categorical products [Pat24]. We avoid assuming much about cartesian bicategories by taking advantage of an alternative axiomatization due to Trimble [Tri09].

### 5.1 Structure proarrows and comonoid homomorphisms

First, we recall our notation for products and the structure maps between them [Pat24, §8]. The product of an $I$-indexed family of objects $\underline{x}$ in a double category, assuming it exists, is denoted $\Pi \underline{x}:=\prod_{i \in I} x_{i}$. An $I$-indexed family $\underline{x}$ can be reindexed along any function $f_{0}: J \rightarrow I$, yielding a $J$ indexed family of objects $f_{0}^{*}(\underline{x}):=\underline{x} \circ f_{0}$, comprising the objects $x_{f_{0}(j)}$ for each $j \in J$. The universal property of the product $\Pi f_{0}^{*} \underline{x}$ then furnishes a structure map, the unique arrow $\Pi\left(f_{0}\right)_{\underline{x}}: \Pi \underline{x} \rightarrow \Pi f_{0}^{*} \underline{x}$ satisfying $\pi_{j} \circ \Pi\left(f_{0}\right)_{\underline{x}}=\pi_{f_{0}(j)}$ for each $j \in J$. Similarly, an $I$-indexed parallel family of proarrows $\underline{m}: \underline{x} \rightarrow \underline{y}$ has a reindexing along a function $f_{0}: J \rightarrow I$, which is a $J$-indexed parallel family of proarrows $f_{0}^{*}(\underline{m}): f_{0}^{*}(\underline{x}) \rightarrow f_{0}^{*}(\underline{y})$. The universal property of the product again gives a structure map, now a cell of the form:

$$
\begin{aligned}
& \begin{array}{c}
\Pi \underline{x} \underset{\Gamma}{\Pi \underline{m}} \Pi \underline{y} \\
\Pi\left(f_{0}\right)_{\underline{x}} \downarrow \underset{\square}{\Pi\left(f_{0}\right)_{\underline{m}}} \quad \downarrow\left(f_{0}\right)_{\underline{y}} .
\end{array} \\
& \Pi f_{0}^{*} \underline{x} \underset{\Pi f_{0}^{*} \underline{\underline{m}}}{ } \Pi f_{0}^{*} \underline{y}
\end{aligned}
$$

Such arrows and cells between products can be transposed, by a straightforward application of Theorem 3.8.

Proposition 5.1 (Structure proarrows between products). Let $\mathbb{D}$ be a double category with normal lax (finite) products, and let $f_{0}: J \rightarrow I$ be a function between (finite) sets. Then for each $I$-indexed family of objects $\underline{x}$ in $\mathbb{D}$, the structure arrow $\Pi\left(f_{0}\right)_{\underline{x}}: \Pi \underline{x} \rightarrow \Pi f_{0}^{*} \underline{x}$ has a companion and a conjoint,

$$
\left(\Pi\left(f_{0}\right)_{\underline{x}}\right)_{!}: \Pi \underline{x} \rightarrow \Pi f_{0}^{*} \underline{x} \quad \text { and } \quad\left(\Pi\left(f_{0}\right)_{\underline{x}}\right)^{*}: \Pi f_{0}^{*} \underline{x} \rightarrow \Pi \underline{x} .
$$

For each I-indexed parallel family of proarrows $\underline{m}: \underline{x} \rightarrow \underline{y}$ in $\mathbb{D}$, the structure cell $\Pi\left(f_{0}\right)_{\underline{m}}$ can be reshaped into cells

(Lemma 2.4), which together form the components of oplax and lax protransformations


Moreover, whenever $f_{0}$ is a bijection, the two protransformations are pseudo and form an adjoint equivalence $\Pi\left(f_{0}\right)$ ! $\dashv\left(f_{0}\right)^{*}$ in $\operatorname{Lax}_{\mathrm{ps}}\left(\mathbb{D}^{I}, \mathbb{D}\right)$.

Proof. Fixing a function $f: J \rightarrow I$, the structure arrows and cells, $\Pi\left(f_{0}\right)_{\underline{x}}$ and $\Pi\left(f_{0}\right)_{\underline{m}}$, form the components of a natural transformation

between lax double functors. Indeed, for any arrows $f: \underline{x} \rightarrow \underline{y}$ in $\mathbb{D}^{I}$, the naturality square

commutes, its common composite being the unique arrow $\Pi\left(f_{0}, f\right): \Pi \underline{x} \rightarrow \Pi f_{0}^{*} \underline{y}$ in $\mathbb{D}$ satisfying $\pi_{j} \circ \Pi\left(f_{0}, f\right)=f_{f_{0}(j)} \circ \pi_{f_{0}(j)}$ for each $j \in J$. Naturality with respect to cells is perfectly analogous.

Now, since $\mathbb{D}$ has normal lax products, each component arrow $\Pi\left(f_{0}\right)_{\underline{x}}$ has both a companion and a conjoint [Pat24, Corollary 8.4]. Therefore, by Theorem 3.8, the natural transformation $\Pi\left(f_{0}\right)$ itself has a companion and a conjoint, which are oplax and lax protransformations, respectively. Moreover, when $f_{0}$ is bijection, $\Pi\left(f_{0}\right)$ is a natural isomorphism and hence, by Corollary 3.10 , the induced protransformations are pseudo and form an adjoint equivalence.

The proposition has several important special cases, applicable in any double category $\mathbb{D}$ with normal lax finite products.

- Diagonals and codiagonals: taking the unique function $f_{0}: 2 \xrightarrow{!} 1$, the diagonal $\Delta_{x}: x \rightarrow x^{2}$ at an object $x$ has a companion and a conjoint,

$$
\delta_{x}:=\left(\Delta_{x}\right)!: x \rightarrow x^{2} \quad \text { and } \quad \delta_{x}^{*}:=\left(\Delta_{x}\right)^{*}: x \rightarrow x^{2},
$$

and the diagonal $\Delta_{m}: m \rightarrow m^{2}$ at a proarrow $m: x \rightarrow y$ has reshapings into cells
which together form the components of oplax and lax protransformations

$$
\delta: 1_{\mathbb{D}} \Rightarrow \times \circ \Delta_{\mathbb{D}} \quad \text { and } \quad \delta^{*}: \times \circ \Delta_{\mathbb{D}} \Rightarrow 1_{\mathbb{D}} .
$$

- Deletion and creation: taking the unique function $f_{0}: 0 \xrightarrow{!} 1$, the deletion map $!_{x}: x \rightarrow 1$ at an object $x$ has a companion and a conjoint,

$$
\varepsilon_{x}:=(!x)!: x \rightarrow 1 \quad \text { and } \quad \varepsilon_{x}^{*}:=(!x)^{*}: 1 \rightarrow x,
$$

and the deletion cell $!_{m}: m \rightarrow 1$ at a proarrow $m: x \rightarrow y$ has reshapings into cells

which together form the components of oplax and lax protransformations

$$
\varepsilon: 1_{\mathbb{D}} \Rightarrow I_{\mathbb{D}} \circ!\mathbb{D} \quad \text { and } \quad \varepsilon^{*}: I_{\mathbb{D}} \circ!!_{\mathbb{D}} \Rightarrow 1_{\mathbb{D}} .
$$

- Symmetries: taking the swap function $f_{0}: 2 \xrightarrow{\cong} 2$, the symmetry isomorphism $\sigma_{x, x^{\prime}}: x \times x^{\prime} \rightarrow$ $x^{\prime} \times x$ at a pair of objects $x$ and $x^{\prime}$ has a companion and a conjoint,

$$
\tau_{x, x^{\prime}}:=\left(\sigma_{x, x^{\prime}}\right)!: x \times x^{\prime} \rightarrow x^{\prime} \times x \quad \text { and } \quad \tau_{x, x^{\prime}}^{*}:=\left(\sigma_{x, x^{\prime}}\right)^{*}: x^{\prime} \times x \rightarrow x \times x^{\prime},
$$

and the symmetry isomorphism $\sigma_{m, m^{\prime}}: m \times m^{\prime} \rightarrow m^{\prime} \times m$ at a pair of proarrows $m: x \rightarrow y$ and $m^{\prime}: x^{\prime} \rightarrow y^{\prime}$ has reshapings into invertible cells


These are the components of two protransformations, forming an adjoint equivalence


By combining these special cases, we obtain the comonoid and monoid structures that play such a central role in Carboni and Walters' original axioms for a (locally posetal) cartesian bicategory [CW87]. Any object $x$ in a double category $\mathbb{D}$ with lax finite products canonically has the structure of a commutative comonoid $\left(x, \Delta_{x},!_{x}\right)$ in $\mathbb{D}_{0}$. When $\mathbb{D}$ has normal lax finite products, these can be transposed into a commutative comonoid $\left(x, \delta_{x}, \varepsilon_{x}\right)$ and a commutative monoid $\left(x, \delta_{x}^{*}, \varepsilon_{x}^{*}\right)$ in the underlying bicategory $\mathbf{B} \mathbb{D}$, where the commutative (co)monoid laws now hold only up to isomorphism, by the pseudofunctoriality of companions and conjoints. The comonoid and monoid structures are also adjoint in the sense that $\delta_{x} \dashv \delta_{x}^{*}$ and $\varepsilon_{x} \dashv \varepsilon_{x}^{*}$ in $\mathbf{B} \mathbb{D}$. Finally, every proarrow $m$ in $\mathbb{D}$ is automatically an "oplax comonoid homomorphism," witnessed by the comparison cells $\delta_{m}$ and $\varepsilon_{m}$, and a "lax monoid homomorphism," witnessed by the comparison cells $\delta_{m}^{*}$ and $\varepsilon_{m}^{*}$. In the locally posetal case, these are essentially the axioms of a cartesian bicategory [CW87, Definition 1.2].

Such observations are easily cast into the form of a theorem but do not on their own suffice to give a cartesian bicategory in the general case. For the moment, we focus on a precise treatment of comonoid and monoid homomorphisms in a double category with products, an interesting topic in its own right.

Definition 5.2 (Comonoid homomorphisms). A proarrow $m$ in a double category with normal lax finite products is a comonoid homomorphism if the cells $\Delta_{m}: m \rightarrow m^{2}$ and $!_{m}: m \rightarrow 1$ are commuters, i.e., the globular cells $\delta_{m}$ and $\varepsilon_{m}$ in Equations (5.1) and (5.2) are invertible.

Dually, a proarrow $m$ is a monoid homomorphism if the cells $\Delta_{m}$ and $!_{m}$ are cocommuters, i.e., the globular cells $\delta_{m}^{*}$ and $\varepsilon_{m}^{*}$ are invertible.

Proposition 5.3 (Companions are comonoid homomorphisms). Any companion of an arrow in a double category with normal lax finite products is a comonoid homomorphism, and any conjoint is a monoid homomorphism.

Proof. Let $\mathbb{D}$ be a double category with normal lax finite products. Taking the functions $f_{0}: 2 \xrightarrow{!} 1$ and $g_{0}: 0 \xrightarrow{\text { b }} 1$ in the proof of Proposition 5.1 gives natural transformations

$$
\Delta:=\Pi\left(f_{0}\right): 1_{\mathbb{D}} \Rightarrow \times \circ \Delta_{\mathbb{D}} \quad \text { and } \quad!:=\Pi\left(g_{0}\right): 1_{\mathbb{D}} \Rightarrow 1 \circ!\mathbb{D}
$$

between normal lax functors. Now apply Lemma 3.11 and its dual for conjoints.
The converse is false: a comonoid homomorphism in a double category with products need not be a companion, nor a map (in the sense recalled in Definition 5.4 below). A counterexample in the double category of boolean-valued profunctors, which is even locally posetal, has been given by Todd Trimble [Tri23].

Commutative comonoids and comonoid homomorphisms, which figure so prominently in the original axioms for a locally posetal cartesian bicategory [CW87], make no appearance in the axioms for a general cartesian bicategory proposed much later [Car+08]. The central concept is now that of a map.

Definition 5.4 (Map). A morphism in a bicategory is a map if it has a right adjoint. Similarly, a proarrow in a double category $\mathbb{D}$ is map if it is a map in the underlying bicategory of $\mathbb{D}$.

A companion of an arrow in a double category is a map whenever the arrow also has a conjoint, in which case the conjoint is right adjoint to the companion (Remark 2.5). Again, the converse is not true: a double category can possess maps that are not companions. In the double category of profunctors, the statement that a profunctor $C \rightarrow D$ has a right adjoint if and only if it is a companion of a functor $\mathrm{C} \rightarrow \mathrm{D}$ is equivalent to the codomain D being Cauchy complete [Bor94, Volume 1, Theorem 7.9.3].

Remark 5.5 (Maps as structure versus property). In a few important double categories with products, such as $\mathbb{S p a n}$ and $\mathbb{R} e l$, companions, maps, and comonoid homomorphisms all coincide and capture the expected notion of "function-like" proarrow. This elegant equivalence motivates the theory of cartesian bicategories. However, the equivalence is a rather special situation. The facts cited above imply that in the double category Prof, the three classes of proarrows - companions, maps, and comonoid homomorphisms-separate, with only the first exactly capturing the profunctors isomorphic to functors. Apparently, it is more robust to treat the "function-like" morphisms not as "relation-like" morphisms satisfying special properties but rather as extra structure. That is, of course, what happens in a double category.

### 5.2 Transposing cartesian structure

To reduce complications with coherence, Carboni, Kelly, Walters, and Wood define a cartesian bicategory in two stages, taking advantage of bicategorical universal properties in the first stage. In outline, a bicategory $\mathbf{B}$ is precartesian when $\mathbf{B}$ has finite local products and Map $\mathbf{B}$, the locally full sub-bicategory of maps, has finite products (in the sense of bilimits) [Car +08 , Definition 3.1]. Using these universal properties, lax functors $\otimes: \mathbf{B}^{2} \rightarrow \mathbf{B}$ and $I: \mathbf{1} \rightarrow \mathbf{B}$ are constructed [Car+08, Theorem 3.15]. Finally, a precartesian bicategory $\mathbf{B}$ is defined be cartesian when the lax functors are pseudo [Car +08 , Definition 4.1]. This definition, while indirect, has the advantage of making it immediately clear that being cartesian is a property of a bicategory.

Nevertheless, it is more convenient for our purposes to use another definition, due to Todd Trimble [Tri09], of a cartesian structure on a bicategory, closer in spirit to the original CarboniWalters definition [CW87]. On the cited nLab page, Trimble sketches a proof that such a cartesian structure is essentially unique, and in particular recovers the universal properties of a cartesian bicategory.

Definition 5.6 (Cartesian bicategory à la Trimble). A cartesian structure on a bicategory B consists of:
(i) pseudofunctors $\otimes: \mathbf{B}^{2} \rightarrow \mathbf{B}$ and $I: \mathbf{1} \rightarrow \mathbf{B}$;
(ii) oplax natural transformations

that are map-valued, meaning that their components are maps;
(iii) invertible modifications

the triangulators, constituting lax adjunctions

$$
(\delta, \pi, s, t): \Delta_{\mathbf{B}} \dashv \otimes: \mathbf{B} \rightleftarrows \mathbf{B}^{2} \quad \text { and } \quad\left(\varepsilon, 1_{1_{1}}, 1_{1_{!_{\mathbf{B}}}}, u\right):!_{\mathbf{B}} \dashv I: \mathbf{1} \rightleftarrows \mathbf{B}
$$

We can now state and prove the main result of this section. The "iso-strong" condition on double products essentially says that parallel products commute with external composition up to isomorphism, as in a cartesian double category; for details, see [Pat24, §7].

Theorem 5.7 (Underlying cartesian bicategory). The underlying bicategory of a double category with iso-strong finite products is cartesian.

Proof. Let $\mathbb{D}$ be a double category with iso-strong finite products. We will exhibit a cartesian structure on the underlying bicategory of $\mathbb{D}$, a structure that is essentially unique [Tri09].
(i) The iso-strong double products in $\mathbb{D}$ restrict to double functors $\times: \mathbb{D}^{2} \rightarrow \mathbb{D}$ and $1: \mathbb{1} \rightarrow \mathbb{D}$. Underlying these are pseudofunctors $\otimes: \mathbf{D}^{2} \rightarrow \mathbf{D}$ and $I: \mathbf{1} \rightarrow \mathbf{D}$, where we write $\mathbf{D}$ for the underlying bicategory of $\mathbb{D}$.
(ii) As already observed, taking the functions $f_{0}: 2 \xrightarrow{!} 1$ and $g_{0}: 0 \stackrel{!}{\rightarrow} 1$ in Proposition 5.1 gives natural transformations

$$
\Delta:=\Pi\left(f_{0}\right): 1_{\mathbb{D}} \Rightarrow \times \circ \Delta_{\mathbb{D}} \quad \text { and } \quad!:=\Pi\left(g_{0}\right): 1_{\mathbb{D}} \Rightarrow 1 \circ!\mathbb{D}
$$

that have companions, which are map-valued, oplax protransformations

$$
\delta:=\Pi\left(f_{0}\right)!: 1_{\mathbb{D}} \Rightarrow \times \circ \Delta_{\mathbb{D}} \quad \text { and } \quad \varepsilon:=\Pi\left(g_{0}\right)!: 1_{\mathbb{D}} \Rightarrow 1 \circ!\mathbb{D}
$$

Similarly, taking each of the two inclusions $\iota_{1}, \iota_{2}: 1 \hookrightarrow 2$, we form the natural transformation

$$
\Pi:=\left(\Pi^{1}, \Pi^{2}\right):=\left(\Pi\left(\iota_{1}\right), \Pi\left(\iota_{2}\right)\right): \Delta_{\mathbb{D}} \circ \times \Rightarrow 1_{\mathbb{D}^{2}}
$$

whose components are the pairs of projections

$$
\Pi_{(x, y)}:=\left(\Pi_{x, y}^{1}, \Pi_{x, y}^{2}\right):(x \times y, x \times y) \rightarrow(x, y), \quad x, y \in \mathbb{D}
$$

It too has a companion, a map-valued, oplax protransformation $\pi:=\Pi_{!}: \Delta_{\mathbb{D}} \circ \times \Rightarrow 1_{\mathbb{D}^{2}}$. Underlying $\delta, \varepsilon$, and $\pi$ are the map-valued, oplax natural transformations that we need.
(iii) Since the double category $\mathbb{D}$ has iso-strong finite products, it is, in particular, a cartesian double category [Pat24, §7]. By the definition of the latter [Ale18], there are double adjunctions

$$
(\Delta, \Pi): \Delta_{\mathbb{D}} \dashv \times: \mathbb{D} \rightleftarrows \mathbb{D}^{2} \quad \text { and } \quad\left(!, 1_{1_{\mathbb{1}}}\right):!_{\mathbb{D}} \dashv 1: \mathbb{D} \rightleftarrows \mathbb{1}
$$

Applying Corollary 4.5 to these double adjunctions completes the proof.
Corollary 5.8. The underlying bicategory of a cartesian equipment is cartesian.
Proof. This follows from the theorem because cartesian equipments have iso-strong finite products [Pat24, Corollary 8.7].

Using a more abstract method of proof, the corollary strengthens Lambert's result that locally posetal cartesian equipments have underlying cartesian bicategories [Lam22, Proposition 3.1]. Nearly all of the cartesian bicategories envisaged by Carboni, Kelly, Walters, and Wood [Car+08, Example 3.2], including the prototypical bicategories of spans and of profunctors, are known to be cartesian equipments and thus inherit their cartesian structure from the corollary. ${ }^{2}$

[^1]
### 5.3 Transposing cocartesian structure

We turn now from cartesian to cocartesian structure. Of course, they are formally dual, but due to asymmetries in the main examples, namely that they tend to have strong coproducts but only iso-strong products, we can obtain stronger results about cocartesian structure. Loosely speaking, a double category has strong coproducts if arbitrary span-indexed coproducts of proarrows exist and commute with external composition; for details, see [Pat24, §4].

The following proposition dualizes Proposition 5.1 while strengthening its hypotheses and conclusions to account for strong coproducts.

Proposition 5.9 (Structure maps between strong coproducts). Let $\mathbb{D}$ be a double category with strong (finite) coproducts. For any function $f: I \rightarrow J$ between (finite) sets, the natural transformation

induced by the universal property of coproducts has both a companion $\Sigma\left(f_{0}\right)$ ! and a conjoint $\Sigma\left(f_{0}\right)^{*}$ in the double category $\mathbb{L a x}_{\mathrm{ps}}\left(\mathbb{D}^{J}, \mathbb{D}\right)$.

Proof. By Corollary 3.9, it is equivalent to show that, for each $J$-indexed family of objects $\underline{x}$ in $\mathbb{D}$, the component arrow $\Sigma\left(f_{0}\right)_{\underline{x}}: \Sigma f_{0}^{*} \underline{x} \rightarrow \Sigma \underline{x}$ has both a companion and a conjoint in $\mathbb{D}$, and for each $J$-indexed parallel family of proarrows $\underline{m}: \underline{x} \rightarrow \underline{y}$ in $\mathbb{D}$, the component cell

$$
\begin{aligned}
& \Sigma f_{0}^{*} \underline{x} \xrightarrow{\Sigma f_{p}^{*} \underline{m}} \Sigma f_{0}^{*} \underline{y} \\
& \Sigma\left(f_{0}\right)_{\underline{x}} \downarrow \quad \Sigma\left(f_{0}\right)_{\underline{m}} \quad \downarrow^{\Sigma\left(f_{0}\right)_{\underline{y}}} \\
& \Sigma \underline{x} \xrightarrow[\Sigma \underline{m}]{\stackrel{I}{y}} \Sigma \underline{y}
\end{aligned}
$$

is both a commuter and cocommuter. The existence of the companions and conjoints in $\mathbb{D}$ follows directly from the dual of [Pat24, Corollary 8.4].

We prove that the component cells are commuters; that they are cocommuters is proved dually. Given a $J$-indexed family $\underline{m}: \underline{x} \rightarrow \underline{y}$, we must show that the reshaped cell

is invertible. Using the notation of $[\operatorname{Pat} 24, \S 3]$, we have the equation in $\mathbb{F a m}(\mathbb{D})$

where the cells $\eta=\left(1_{I}, 1\right)$ and $\varepsilon=\left(f_{0}, 1\right)$ are indeed binding cells for companions in $\mathbb{F a m}(\mathbb{D})$ by [Pat24, Proposition 3.8]. Therefore, the composite of the images of the cells on the left-hand side
under the coproduct double functor $\Sigma: \mathbb{F a m}(\mathbb{D}) \rightarrow \mathbb{D}$ is the cell of interest, $\left(\Sigma\left(f_{0}\right)_{m}\right)$ !. But the image of the right-hand side is invertible, in fact is the identity. So, by the naturality and invertibility of the composition comparisons of the double functor $\Sigma: \mathbb{F a m}(\mathbb{D}) \rightarrow \mathbb{D}$, it follows that the cell $\left(\Sigma\left(f_{0}\right)_{\underline{m}}\right)$ ! is also invertible.

We will say that a bicategory has direct sums when it has finite bicategorical products and coproducts, and they coincide. ${ }^{3}$

Theorem 5.10. The underlying bicategory of a double category with strong finite coproducts has direct sums, induced from coproducts in the underlying 2-category by taking companions and conjoints.

Proof. Suppose $\mathbb{D}$ is a double category with strong finite coproducts. Then, for any finite set $I$, there is a double adjunction

$$
(\iota, \nabla): \Sigma \dashv \Delta_{\mathbb{D}}: \mathbb{D}^{I} \rightleftarrows \mathbb{D} .
$$

By Proposition 5.9 applied to the functions $f_{0}: 1 \stackrel{\iota_{i}}{\longrightarrow} I$, for $i \in I$, and $f_{0}: I \xrightarrow{!} 1$, both of the transformations $\iota$ and $\nabla$ have companion and conjoint protransformations. In particular, we have an adjunction $\nabla!\dashv \nabla^{*}$ between the codiagonals and diagonals in the bicategory $\operatorname{Lax}_{\mathrm{ps}}(\mathbb{D}, \mathbb{D})$.

Furthermore, by Corollary 4.5 and its dual for conjoints, there are biadjunctions

$$
\left(\iota, \nabla \nabla_{!}\right): \oplus \dashv \Delta_{\mathbf{D}}: \mathbf{D}^{I} \rightleftarrows \mathbf{D} \quad \text { and } \quad\left(\nabla^{*}, \iota^{*}\right): \Delta_{\mathbf{D}} \dashv \oplus: \mathbf{D} \rightleftarrows \mathbf{D}^{I},
$$

where $\oplus: \mathbf{D}^{I} \rightarrow \mathbf{D}$ is the pseudofunctor between bicategories underlying the coproduct double functor $\Sigma: \mathbb{D}^{I} \rightarrow \mathbb{D}$. Thus, the pseudofunctor $\otimes: \mathbf{D}^{I} \rightarrow \mathbf{D}$ is a choice of direct sums in $\mathbf{D}$.

We caution that the theorem says nothing about cocartesian equipments, which in general have only iso-strong finite coproducts. Here are two positive examples, beginning with the ur-example of spans, which have direct sums whenever the base category has finite coproducts that interact well with pullbacks.
Example 5.11 (Spans). For any extensive category $S$ with pullbacks, the double category of spans in S has strong finite coproducts [Pat24, Theorem 4.3]. So, by the theorem above, its underlying bicategory of spans has direct sums. In this way, we recover a result about bicategories of spans proved directly by Lack, Walters, and Wood [LWW10, Theorem 6.2].
Example 5.12 (Matrices). For any (infinitary) distributive monoidal category $\mathcal{V}$, the double category of $\mathcal{V}$-matrices has strong coproducts [Pat24, Proposition 4.5]. Thus, its underlying bicategory of $\mathcal{V}$-matrices has direct sums, giving a categorified "matrix calculus."
Remark 5.13 (Other notions of coproduct). The theorem makes connections with several related ideas in the literature. Through his work on proarrow equipments [Woo82; Woo85], Wood has explored structure and axioms on a bicategory that would make it a suitable environment for formal category theory. His first three axioms define a proarrow equipment [Woo85, Axioms 1-3], which is interchangeable with an equipment in the sense of double categories [Shu08, Appendix C]. His fourth axiom states that the 2-category of arrows has finite coproducts, which become bicategorical coproducts upon taking companions and bicategorical products upon taking conjoints [Woo85, Axiom 4]. So, it follows from Theorem 5.10 that an equipment with strong finite coproducts satisfies Wood's first four axioms. In addition, we obtain an instance of Garner and Shulman's "tight finite coproducts," a kind of "tight collage" in an equipment [GS16, Example 16.14]. When the double

[^2]category is strict, we likewise have an instance of Lack and Shulman's "tight finite coproducts" in an $\mathcal{F}$-category [LS12, §3.5.1].

Wood's fifth and final axiom [Woo85, Axiom 5], as well as Garner and Shulman's other kind of tight collage [GS16, Example 16.13], concern the existence of Kleisli objects and are beyond the scope of this paper. Nevertheless, the results presented here lend further support to the thesis, under active development by numerous category theorists, that double categories are an elegant foundation for formal category theory.

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[^0]:    ${ }^{1}$ Natural transformation satisfying both of these conditions are said to have loosely strong companions by Hansen and Shulman [HS19, Definition 4.10].

[^1]:    ${ }^{2}$ The only exception in [Car+08, Example 3.2] is the 2-category of categories with finite limits, finite-productpreserving functors, and natural transformations, which is better viewed as a 2-category than a bicategory. Indeed, it has finite products in the strict sense, by general results about algebras of 2-monads [Lac10, §§6.1-6.2].

[^2]:    ${ }^{3}$ By construction, our direct sums will be bicategorical products and coproducts with identical underlying objects; other authors allow the products and coproducts to be the same only up to equivalence in the bicategory [LWW10, §6].

