# Mal'tsev products of varieties, II 

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#### Abstract

The Mal'tsev product of two varieties of the same similarity type is not in general a variety, because it can fail to be closed under homomorphic images. In the previous paper we provided a new sufficient condition for such a product to be a variety. In this paper we extend that result by weakening the assumptions regarding the two varieties. We also explore the various special cases of our new result and provide a number of examples of its application.


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## 1. Introduction

This paper is a continuation of the paper [8] by the same authors. The reader should consult [8] for further background and all notions that are not explicitly defined.

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same similarity type $\tau: \Omega \rightarrow \mathbb{N}$. The Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ of $\mathcal{V}$ and $\mathcal{W}$ consists of all algebras $A$ of type $\tau$ with a congruence $\theta$, such that $A / \theta$ belongs to $\mathcal{W}$ and every congruence class of $\theta$ that is a subalgebra of $A$ belongs to $\mathcal{V}$. Each algebra $A$ of type $\tau$ has the smallest congruence such that the corresponding quotient algebra belongs to $\mathcal{W}$. This congruence is called its $\mathcal{W}$-replica congruence and will be denoted by $\varrho$. (See e.g. [11, Ch. 3].) The congruence $\theta$ in the definition of the Mal'tsev product may be taken to be the $\mathcal{W}$-replica congruence $\varrho$ of $A$. (See [6].) Thus the definition of the Mal'tsev product of varieties becomes

$$
\begin{equation*}
\mathcal{V} \circ \mathcal{W}=\{A \mid(\forall a \in A)(a / \varrho \leq A \Rightarrow a / \varrho \in \mathcal{V})\} \tag{1.1}
\end{equation*}
$$

By results of Mal'tsev [6, Ths. 1, 2], it is known that the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is closed under the formation of subalgebras and of direct products. However in general, it is not closed under homomorphic images. We are interested in sufficient conditions for the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety.

If the factor $\mathcal{W}$ of the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is idempotent, then each $\mathcal{W}$-replica congruence class $a / \varrho$ of any algebra $A$ in $\mathcal{V} \circ \mathcal{W}$ is a subalgebra of
$A$, and

$$
\mathcal{V} \circ \mathcal{W}=\{A \mid(\forall a \in A)(a / \varrho \in \mathcal{V})\}
$$

In this case, we say that $A$ is a $\mathcal{W}$-sum of $\mathcal{V}$-algebras. (See [8, Sec. 1]). In this paper we extend the main result of [8], in which the second factor $\mathcal{W}$ of the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is required to be an idempotent variety. Now it is allowed to be a member of a wider class of varieties that we named term idempotent varieties. This change forces us to pay more attention to the role of idempotent elements (or just idempotents) in the theory of Mal'tsev products. Recall that for an algebra $A$ with a congruence $\theta$, a congruence class $a / \theta$ is a subalgebra of $A$ iff $a / \theta$ is an idempotent element of the quotient algebra $A / \theta$. Thus for $A \in \mathcal{V} \circ \mathcal{W}$, the congruence classes $a / \varrho$ of the $\mathcal{W}$ replica congruence $\varrho$ that are subalgebras of $A$ are precisely those that are idempotents of $A / \varrho \in \mathcal{W}$. Some special terms are used to keep track of those congruence classes. Let $\mathcal{V}$ be a variety of type $\tau$ and $t$ be a term of this type. If $\mathcal{V}$ satisfies the identities

$$
\begin{equation*}
\omega(t, \ldots, t)=t \tag{1.2}
\end{equation*}
$$

for all basic operations $\omega \in \Omega$, then we say that $t$ is a term idempotent of $\mathcal{V}$. (See [8, Sec. 1].) (Iskander [4] and [5] uses the name unit term for a unary term idempotent.) To justify the name, first recall that the free $\mathcal{V}$-algebra over $X$ may be represented as the quotient $X \Omega / \varrho$, where $X \Omega$ is the absolutely free algebra $X \Omega$ over $X$ and $\varrho$ is its $\mathcal{V}$-replica congruence. (See e.g. [11, Ch. 3].) Then note that a term $t$, with variables in a set $X$, is a term idempotent of $\mathcal{V}$ precisely if $t / \varrho$ is an idempotent of the free $\mathcal{V}$-algebra over $X$. A motivating example is provided by the term $t(x):=x x^{-1}$ in a variety of groups or of inverse semigroups. Note that a variety $\mathcal{V}$ is idempotent if a variable $x$ is a term idempotent of $\mathcal{V}$. Moreover, if a variety $\mathcal{V}$ is idempotent, then every term of type $\tau$ is a term idempotent of $\mathcal{V}$. Note also that if $t$ is a term idempotent of $\mathcal{V}$ and $A \in \mathcal{V}$, then for each $a \in A$, the element $t(a)$ is an idempotent of $A$.

We will restrict our attention to types with no symbols of nullary operations. It is a reasonable assumption when dealing with Mal'tsev products for the following reasons. First, note that if the type of $\mathcal{V}$ and $\mathcal{W}$ contains symbols of nullary operations, then each algebra $A$ in $\mathcal{V} \circ \mathcal{W}$ has only one congruence class of $\varrho$ that is a subalgebra, namely the one containing the constants. If additionally $\mathcal{W}$ is idempotent, then $\mathcal{V} \circ \mathcal{W}$ is just the variety $\mathcal{V}$. Then, if the type contains a symbol $c$ of a nullary operation, then one can replace it by a symbol of a constant unary basic operation $c(x)$, and in this way obtain equivalent varieties $\mathcal{V}^{\prime}$ and $\mathcal{W}^{\prime}$ of a type without constants. The varieties $\mathcal{V}^{\prime}$ and $\mathcal{W}^{\prime}$ satisfy the identity $c(x)=c(y)$, and the unary operation is constant on all algebras of these varieties.

This paper is organized as follows. Section 2 contains a summary of earlier sufficient conditions for a Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety. In Section 3] we introduce term idempotent varieties and investigate their properties. Section 4 contains the main result of this paper, Theorem 4.1, providing a sufficient condition for the Mal'tsev product of a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ to be a variety. This theorem extends several
earlier results and has a number of interesting consequences and applications that are discussed in Section 5. Finally, in Section 6, we investigate a subclass of term idempotent varieties $\mathcal{W}$ consisting of the so-called polarized varieties. They have a rather special property that $\mathcal{V} \circ \mathcal{W}$ is a variety for any variety $\mathcal{V}$. This result is not a consequence of the main theorem of Section 4

With the exception of some examples, we usually assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of the same finitary type $\tau: \Omega \rightarrow \mathbb{N}$ without symbols of nullary operations, and that all varieties, algebras and terms are of this type. If a variety $\mathcal{V}$ satisfies an identity $p=q$, then the terms $p$ and $q$ are called equivalent in $\mathcal{V}$ or $\mathcal{V}$-equivalent. An identity is trivial if it is of the form $p=p$. A term $t\left(x_{1}, \ldots, x_{n}\right)$ is called constant in $\mathcal{V}$, if $\mathcal{V}$ satisfies the identity $t\left(x_{1}, \ldots, x_{n}\right)=t\left(y_{1}, \ldots, y_{n}\right)$.

We usually abbreviate lists of variables $x_{1}, \ldots, x_{n}$ as $\mathbf{x}$. For a term $t=t\left(x_{1}, \ldots, x_{n}\right)$ we also write $t(\mathbf{x})$. Note that $t(\mathbf{x})$ need not necessarily involve the full set $x_{1}, \ldots, x_{n}$ of variables from $\mathbf{x}$. Similarly, we abbreviate lists of elements $a_{1}, \ldots, a_{n}$ of some algebra as $\mathbf{a}$, and we write $t(\mathbf{a})$ for $t\left(a_{1}, \ldots, a_{n}\right)$. With the exception of this special notation, we follow the usage of notation and conventions similar to those of [3] ,8] and [11].

For further information regarding Mal'tsev products, we refer the reader to [6] and [7]. For universal algebra, see [2] and [11].

## 2. A brief summary of earlier results

We proceed with a brief summary of the earlier sufficient conditions for $\mathcal{V} \circ \mathcal{W}$ to be a variety. However let us first recall a result about the identities true in $\mathcal{V} \circ \mathcal{W}$.

Definition 2.1. [8, Def. 2.1] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, and let $\Sigma$ be an equational base for $\mathcal{V}$. We define the following set $\Sigma^{\mathcal{W}}$ of identities:

$$
\begin{aligned}
\Sigma^{\mathcal{W}}:= & \left\{u\left(r_{1}, \ldots, r_{n}\right)=v\left(r_{1}, \ldots, r_{n}\right) \mid\right. \\
& (u=v) \in \Sigma, \\
& \forall i=1, \ldots, n-1, \quad \mathcal{W} \models r_{i}=r_{i+1} \\
& \left.\forall \omega \in \Omega, \quad \mathcal{W} \models \omega\left(r_{1}, \ldots, r_{1}\right)=r_{1}\right\} .
\end{aligned}
$$

The last two conditions of this definition imply that $\mathcal{W} \models \omega\left(r_{i}, \ldots, r_{i}\right)=r_{i}$ for all $\omega \in \Omega$ and each $i=1, \ldots, n-1$, and thus all $r_{i}$ are term idempotents of $\mathcal{W}$. One can say that every identity in $\Sigma^{\mathcal{W}}$ is obtained from some identity of $\Sigma$ by substituting for its variables pairwise $\mathcal{W}$-equivalent term idempotents of $\mathcal{W}$.

Theorem 2.2. [8, Lem. 2.2] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, and let $\Sigma$ be an equational base for $\mathcal{V}$. Then the variety $\mathrm{H}(\mathcal{V} \circ \mathcal{W})$ generated by the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is defined by the identities $\Sigma^{\mathcal{W}}$.
Corollary 2.3. If the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety, then $\Sigma^{\mathcal{W}}$ is an equational base for $\mathcal{V} \circ \mathcal{W}$.

Theorem 2.4. [8, Th. 3.3] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, and let $\mathcal{W}$ be idempotent. If there exist terms $f(x, y)$ and $g(x, y)$ such that
(a) $\mathcal{V} \models f(x, y)=x \quad$ and $\mathcal{V} \models g(x, y)=y$,
(b) $\mathcal{W} \models f(x, y)=g(x, y)$,
then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
An identity is called regular if both its sides contain precisely the same variables; otherwise it is irregular. Furthermore an identity is called strongly irregular, if it is of the form $t(x, y)=x$, where $t(x, y)$ is a binary term containing both variables $x$ and $y$. A variety is called strongly irregular if it satisfies a strongly irregular identity. For a plural type $\tau$, i.e. one with no nullary operations and at least one non-unary operation, the variety $\mathcal{S}_{\tau}$ of $\tau$-semilattices is the unique variety of type $\tau$ that is equivalent to the variety $\mathcal{S}$ of semilattices. This variety satisfies precisely all the regular identities of type $\tau$. (See [10] and [3] for details.) As a corollary of Theorem 2.4 one obtains the following theorem.

Theorem 2.5. 3, Th. 6.3] If $\mathcal{V}$ is a strongly irregular variety of a plural type $\tau$, then $\mathcal{V} \circ \mathcal{S}_{\tau}$ is a variety.

Algebras in $\mathcal{V} \circ \mathcal{S}_{\tau}$ are called semilattice sums of $\mathcal{V}$-algebras.
The main result of this paper is a common generalization of Theorem 2.4 and of the following theorem of Bergman.

Theorem 2.6. [1, Cor. 2.3] If $\mathcal{V}$ and $\mathcal{W}$ are idempotent subvarieties of a congruence permutable variety, then $\mathcal{V} \circ \mathcal{W}$ is a variety.

## 3. Term idempotent varieties

We start this section with a special property of term idempotents. The set $X \Omega$ of terms of a given type $\tau$ (without constants) over a countably infinite set $X$ of variables is preordered by the following relation: $p\left(x_{1} \ldots x_{n}\right) \preceq q$ iff there exist terms $t_{1}, \ldots, t_{n}$ of type $\tau$ such that $q=p\left(t_{1} \ldots t_{n}\right)$. Note that if $p \preceq q$ and $q \preceq p$, then $p$ and $q$ are the same to within a renaming of the variables. Using the rules of equational logic one easily obtains the following lemma.

Lemma 3.1. If $p$ is a term idempotent of a variety $\mathcal{V}$, and $p \preceq q$, then $q$ is also a term idempotent of $\mathcal{V}$.

It is known that for a preordered set $(P, \preceq)$, the relation $\alpha$ defined on $P$ by

$$
(p, q) \in \alpha \text { iff } p \preceq q \text { and } q \preceq p
$$

is an equivalence relation. Furthermore, the relation $\leq$ defined on $P / \alpha$ by

$$
p / \alpha \leq q / \alpha \text { iff } p \preceq q
$$

is an order relation. An upper set of a preordered set $(P, \preceq)$ can be defined similarly as in the case of an ordered set. A subset $Q$ of $P$ is an upper set, if
whenever $p \in Q, q \in P$ and $p \preceq q$, then $q \in Q$. Thus Lemma 3.1 shows that term idempotents of a given variety form an upper set of ( $X \Omega, \preceq$ ). It is easy to see that the variables of $X$ form one class of $\alpha$. This class is the minimum of the ordered set $(X \Omega / \alpha, \leq)$ and obviously each variable of $X$ is related by $\preceq$ with any other element of $X \Omega$. Thus if a variable is a term idempotent, then all terms are term idempotents. In other words, a variety is idempotent iff the upper set of its term idempotents contains all terms.

Definition 3.2. A nontrivial identity $p=q$ satisfied in a variety $\mathcal{V}$ will be called term idempotent, if both $p$ and $q$ are term idempotents of $\mathcal{V}$. A variety $\mathcal{V}$ will be called term idempotent, if every nontrivial identitity it satisfies is term idempotent.

Note that every idempotent variety is term idempotent. Below we provide some examples of term idempotent varieties that are not idempotent.

Example 3.3. Let $\mathcal{C S}$ be the variety of constant semigroups, i.e. the variety of groupoids (magmas, binars) defined by the identity $x y=z t$. A nontrivial identity $p=q$ is satisfied in $\mathcal{C S}$ precisely if neither $p$ nor $q$ is a variable. (Cf [8, Ex. 2.5].) In particular $\mathcal{C S}$ satisfies $p \cdot p=p$ for every term $p$ different from a variable. Consequently, all such terms are term idempotents of $\mathcal{C S}$, and so $\mathcal{C S}$ is a term idempotent variety.

Example 3.4. The variety $\mathcal{C}_{\tau}$ of constant algebras of type $\tau$ is defined by the identities

$$
\omega\left(x_{1}, \ldots, x_{n}\right)=\varphi\left(y_{1}, \ldots, y_{m}\right)
$$

for all $\omega, \varphi \in \Omega$. It satisfies precisely the nontrivial identities whose neither side is a variable. Every term different from a variable is constant in $\mathcal{C}_{\tau}$. If type $\tau$ consists of a single binary operation, then $\mathcal{C}_{\tau}$ is just the variety $\mathcal{C S}$. For any other type, $\mathcal{C}_{\tau}$ is equivalent to $\mathcal{C S}$. An argument analogous to that of Example 3.3 shows that $\mathcal{C}_{\tau}$ is a term idempotent variety.

Example 3.5. Let $\mathcal{R S}$ be the variety of semigroups defined by the identities

$$
\begin{equation*}
(x y) z=x z=x(y z) \tag{3.1}
\end{equation*}
$$

The subvariety of $\mathcal{R S}$ defined by the idempotent law $x x=x$ is the variety of rectangular bands. Algebras in $\mathcal{R S}$ will be called rectangular semigroups. Recall that in any variety of semigroups each term $t$ is equivalent to a product of variables $t=x_{1} \cdots x_{n}$. If $t$ is different from a variable, then (3.1) implies that $\mathcal{R S} \models t=x_{1} x_{n}$. Thus $\mathcal{R S}$ satisfies

$$
t \cdot t=x_{1} x_{n} x_{1} x_{n}=x_{1} x_{n}=t
$$

Consequently, all terms different from a variable are term idempotents of $\mathcal{R S}$. Since all nontrivial identities derivable from (3.1) have both sides different from a variable, it follows that $\mathcal{R S}$ is term idempotent.

Example 3.6. For $n \geq 0$, let $\mathcal{U}_{n}$ be the variety of monounary algebras $(A, f)$ defined by the identity

$$
\begin{equation*}
f\left(f^{n}(x)\right)=f^{n}(x) \tag{3.2}
\end{equation*}
$$

Clearly, $f^{n}(x)$ is a term idempotent of $\mathcal{U}_{n}$. Recall that each term of monounary type has the form $f^{m}(x)$ for some $m \geq 0$ (with $f^{0}(x)$ being just $x$ ). Then by (3.2), if $m \geq n$, then $\mathcal{U}_{n} \models f^{m}(x)=f^{n}(x)$. Every nontrivial identity derivable from (3.2) is of the form $f^{k}(x)=f^{l}(x)$ for different $k, l \geq n$. So both sides of such an identity are term idempotents of $\mathcal{U}_{n}$, and hence $\mathcal{U}_{n}$ is term idempotent. Observe that $\mathcal{U}_{0}$ is idempotent, in $\mathcal{U}_{1}$ all terms different from a variable are term idempotent, and if $n \geq 2$, then in $\mathcal{U}_{n}$ not all terms different from a variable are term idempotents.

Some regular varieties provide further examples of term idempotent varieties. Recall that the regularization $\widetilde{\mathcal{V}}$ of a variety $\mathcal{V}$ of a plural type $\tau$ is the variety defined by all the regular identities satisfied in $\mathcal{V}$. Equivalently, $\widetilde{\mathcal{V}}$ can be defined as the join $\mathcal{V} \vee \mathcal{S}_{\tau}$ of $\mathcal{V}$ and the variety $\mathcal{S}_{\tau}$ of $\tau$-semilattices. It is known that if $\mathcal{V}$ is irregular, then each algebra in $\widetilde{\mathcal{V}}$ is a semilattice sum of $\mathcal{V}$-algebras. If $\mathcal{V}$ is strongly irregular, then $\widetilde{\mathcal{V}}$ coincides with the class of Płonka sums of $\mathcal{V}$-algebras. (See e.g. 9], [11, Ch. 4], [10].)

Proposition 3.7. Let $\mathcal{V}$ be a variety of a plural type $\tau$. If $\mathcal{V}$ is term idempotent, then $\widetilde{\mathcal{V}}$ is also term idempotent.

Proof. Let $u=v$ be a nontrivial identity satisfied in $\widetilde{\mathcal{V}}$. Then $u=v$ is also satisfied in $\mathcal{V}$, and hence $u$ and $v$ are term idempotents of $\mathcal{V}$. Thus $\mathcal{V}$ satisfies the identities $\omega(u, \ldots, u)=u$ and $\omega(v, \ldots, v)=v$ for all $\omega \in \Omega$. Since these identities are regular, they are also satisfied in $\widetilde{\mathcal{V}}$. Therefore $u$ and $v$ are term idempotents of $\widetilde{\mathcal{V}}$, and so $\widetilde{\mathcal{V}}$ is a term idempotent variety.

Note that none of the examples of term idempotent varieties which are not idempotent that we provided so far are strongly irregular. The next proposition shows that this is not a coincidence.

Proposition 3.8. If a variety $\mathcal{V}$ is term idempotent and strongly irregular, then $\mathcal{V}$ is idempotent.

Proof. The variety $\mathcal{V}$ satisfies a strongly irregular identity $t(x, y)=x$. This identity is nontrivial, so it is term idempotent, and thus in particular its right-hand side $x$ is a term idempotent of $\mathcal{V}$. Therefore $\mathcal{V}$ is idempotent.

We conclude this section with a characterization of term idempotent varieties in terms of replica congruences. First recall a very useful description of a replica congruence which will also be used in the proof of the main result in Section 4.

Definition 3.9. Let $\mathcal{W}$ be a variety, and let $A$ be an algebra of the same type as $\mathcal{W}$. We define a binary relation $\varrho^{0}$ on the universe of $A$ as follows: $(a, b) \in \varrho^{0}$ if and only if there are an identity $p(\mathbf{x})=q(\mathbf{x})$ satisfied in $\mathcal{W}$, and elements $\mathbf{d}$ of $A$, such that $a=p(\mathbf{d})$ and $b=q(\mathbf{d})$.

Note that the relation $\varrho^{0}$ is reflexive and symmetric.

Proposition 3.10. [8, Prop. 3.2] Let $\mathcal{W}$ be a variety, and let $A$ be an algebra of the same type as $\mathcal{W}$. The $\mathcal{W}$-replica congruence relation $\varrho$ of $A$ coincides with the transitive closure of $\varrho^{0}$.

Proposition 3.11. Let $\mathcal{W}$ be a variety of type $\tau$. Then $\mathcal{W}$ is term idempotent if and only if, for every algebra $A$ of type $\tau$, every congruence class a/@ of the $\mathcal{W}$-replica congruence of $A$ which is not an idempotent of $A / \varrho$, is a singleton.

Proof. $(\Rightarrow)$ Assume that $\mathcal{W}$ is a term idempotent variety. Let $a / \varrho$ be a congruence class with more than one element. We will show that $a / \varrho$ is an idempotent of $A / \varrho$. Let $b \in a / \varrho$ be an element different from $a$. By Proposition 3.10, $\varrho$ is the transitive closure of $\varrho^{0}$. Since $(a, b) \in \varrho$, there is an element $c \in a / \varrho$ different from $a$, such that $(a, c) \in \varrho^{0}$. This means that there is a nontrivial identity $p(\mathbf{x})=q(\mathbf{x})$ true in $\mathcal{W}$, and elements $\mathbf{d}$ of $A$, such that $a=p(\mathbf{d})$ and $c=q(\mathbf{d})$. Since $\mathcal{W}$ is a term idempotent variety, $p$ and $q$ are term idempotents of $\mathcal{W}$. Thus $a$, being a value of a term idempotent, is an idempotent of $A$. It follows that for each $\omega \in \Omega$,

$$
a=\omega(a, \ldots, a),
$$

and hence

$$
a / \varrho=\omega(a / \varrho, \ldots, a / \varrho) .
$$

Therefore $a / \varrho$ is an idempotent of $A / \varrho$.
$(\Leftarrow)$ Now assume that for every algebra $A$ of type $\tau$, any congruence class $a / \varrho$ which is not an idempotent of $A / \varrho$, has exactly one element. In particular, this is true for the absolutely free algebra $X \Omega$ of type $\tau$ over a countably infinite set $X$. Recall that the quotient $X \Omega / \varrho$ of $X \Omega$ by its $\mathcal{W}$ replica congruence $\varrho$, is the free $\mathcal{W}$-algebra over $X$. (See e.g. [11, Ch. 3].) By the definition of term idempotents, for a term $t$ of type $\tau$, the congruence class $t / \varrho$ is an idempotent of $X \Omega / \varrho$ precisely if $t$ is a term idempotent of $\mathcal{W}$. Now let $p=q$ be a nontrivial identity satisfied in $\mathcal{W}$. Then $p$ and $q$ are different elements of the same congruence class $C:=p / \varrho=q / \varrho$. Since $C$ is not a singleton, it is an idempotent of $X \Omega / \varrho$. Hence $p$ and $q$ are term idempotents of $\mathcal{W}$, and therefore $\mathcal{W}$ is term idempotent.

As a corollary, one obtains a result on the structure of algebras in $\mathcal{V} \circ \mathcal{W}$ for a term idempotent variety $\mathcal{W}$.

Corollary 3.12. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same type, and let $\mathcal{W}$ be term idempotent. If $A \in \mathcal{V} \circ \mathcal{W}$, then each congruence class of the $\mathcal{W}$-replica congruence $\varrho$ of $A$ is either a subalgebra of $A$ or a singleton.

In the definition of a term idempotent variety $\mathcal{V}$ we require that all nontrivial identities true in $\mathcal{V}$ are term idempotent. One might wonder if this property is equivalent to the requirement that the set of identities used to define $\mathcal{V}$ be term idempotent. This is not the case however, since a term idempotent identity may entail nontrivial identities that are not term idempotent. As an example consider the identity $x x^{-1}=y y^{-1}$ true in the variety of groups. Both of its sides are term idempotents. However this identity implies
the nontrivial identity $\left(x x^{-1}\right) z=\left(y y^{-1}\right) z$ whose both sides are equivalent to $z$ which is not a term idempotent.

## 4. A new sufficient condition for $\mathcal{V} \circ \mathcal{W}$ to be a variety

We are now ready to state and prove our generalization of Theorem 2.4.
Theorem 4.1. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same type, and let $\mathcal{W}$ be term idempotent. If there exist terms $f(x, y, z)$ and $g(x, y, z)$ such that
(a) $\mathcal{V} \models f(x, y, y)=x$ and $\mathcal{V} \models g(x, x, y)=y$,
(b) $\mathcal{W} \models f(x, x, y)=g(x, x, y)$,
(c) $f(x, x, y)$ is a term idempotent of $\mathcal{W}$,
then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
Proof. We need to show that $\mathrm{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{V} \circ \mathcal{W}$. The proof will be divided into several parts. In what follows we assume that $A \in \mathrm{H}(\mathcal{V} \circ \mathcal{W})$, i.e. $A$ is a quotient of an algebra belonging to $\mathcal{V} \circ \mathcal{W}$.
A. The $\mathcal{W}$-replica congruence $\varrho$ of $A$ coincides with the relation $\varrho^{0}$.

By Proposition 3.10 we have to show that the relation $\varrho^{0}$ is transitive. Let $a, b, c \in A$ and $a \varrho^{0} b \varrho^{0} c$. If either two of the elements $a, b, c$ are equal, then by reflexivity and symmetry of $\varrho^{0}$ one obtains $a \varrho^{0} c$. So let us assume that these elements are pairwise different. Then there exist nontrivial identities $p_{1}\left(\mathbf{x}_{1}\right)=q_{1}\left(\mathbf{x}_{1}\right)$ and $p_{2}\left(\mathbf{x}_{2}\right)=q_{2}\left(\mathbf{x}_{2}\right)$ satisfied in $\mathcal{W}$, and sets of elements $\mathbf{d}_{1}$ and $\mathbf{d}_{2}$ in $A$, such that

$$
\begin{aligned}
a=p_{1}\left(\mathbf{d}_{1}\right), & b=q_{1}\left(\mathbf{d}_{1}\right), \\
& b=p_{2}\left(\mathbf{d}_{2}\right), \quad c=q_{2}\left(\mathbf{d}_{2}\right)
\end{aligned}
$$

Since $\mathcal{W}$ is term idempotent, terms $p_{1}, q_{1}, p_{2}, q_{2}$ are term idempotents of $\mathcal{W}$. Let

$$
p\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right):=f\left(p_{1}, q_{1}, p_{2}\right) \quad \text { and } \quad q\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right):=g\left(q_{1}, q_{1}, q_{2}\right) .
$$

By (b), $\mathcal{W}$ satisfies the identity $p=q$. By Theorem 2.2 $A$ satisfies the identities $f\left(p_{1}, q_{1}, q_{1}\right)=p_{1}$ and $g\left(p_{2}, p_{2}, q_{2}\right)=q_{2}$. It follows that

$$
\begin{aligned}
p\left(\mathbf{d}_{1}, \mathbf{d}_{2}\right) & =f\left(p_{1}\left(\mathbf{d}_{1}\right), q_{1}\left(\mathbf{d}_{1}\right), p_{2}\left(\mathbf{d}_{2}\right)\right) \\
& =f\left(p_{1}\left(\mathbf{d}_{1}\right), q_{1}\left(\mathbf{d}_{1}\right), q_{1}\left(\mathbf{d}_{1}\right)\right)=p_{1}\left(\mathbf{d}_{1}\right)=a
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
q\left(\mathbf{d}_{1}, \mathbf{d}_{2}\right) & =g\left(q_{1}\left(\mathbf{d}_{1}\right), q_{1}\left(\mathbf{d}_{1}\right), q_{2}\left(\mathbf{d}_{2}\right)\right) \\
& =g\left(p_{2}\left(\mathbf{d}_{2}\right), p_{2}\left(\mathbf{d}_{2}\right), q_{2}\left(\mathbf{d}_{2}\right)\right)=q_{2}\left(\mathbf{d}_{2}\right)=c
\end{aligned}
$$

Thus $a \varrho^{0} c$, and hence $\varrho^{0}$ is transitive.
B. If $C$ is a congruence class of $\varrho$ which is a subalgebra of $A$, then $C$ satisfies the identities of (a).

Let $u(x, y)=v(x, y)$ be an identity satisfied in $\mathcal{V}$. If $C$ has only one element $a$, then $u(a, a)=a=v(a, a)$. So $C$ satisfies the identity $u=v$. Now assume that $C$ has more than one element. Let $a, b \in C$ with $a \neq b$. Then
$(a, b) \in \varrho^{0}$, so there exist a nontrivial identity $p(\mathbf{x})=q(\mathbf{x})$ satisfied in $\mathcal{W}$ and elements $\mathbf{d}$ of $A$, such that $a=p(\mathbf{d})$ and $b=q(\mathbf{d})$. The terms $p$ and $q$ are term idempotents of $\mathcal{W}$. Hence by Theorem 2.2, $A$ satisfies the identities $u(p, q)=v(p, q)$ and $u(p, p)=v(p, p)$. Therefore

$$
\begin{aligned}
& u(a, b)=u(p(\mathbf{d}), q(\mathbf{d}))=v(p(\mathbf{d}), q(\mathbf{d}))=v(a, b) \\
& u(a, a)=u(p(\mathbf{d}), p(\mathbf{d}))=v(p(\mathbf{d}), p(\mathbf{d}))=v(a, a)
\end{aligned}
$$

It follows that $C$ satisfies any identity in at most two variables valid in $\mathcal{V}$. In particular

$$
\begin{equation*}
C \models f(x, y, y)=x \text { and } C \models g(x, x, y)=y . \tag{4.1}
\end{equation*}
$$

C. Assume that $\mathcal{W} \models p_{i}\left(\mathbf{z}_{i}\right)=q_{i}\left(\mathbf{z}_{i}\right)$ for $i=1, \ldots, n-1$. Then for each $i=1, \ldots, n$, define terms $t_{i, j}$ recursively for $j=0, \ldots, n-1$ by

$$
t_{i, j}:= \begin{cases}p_{1} & \text { for } j=0  \tag{4.2}\\ f\left(q_{j}, p_{j}, t_{i, j-1}\right) & \text { for } 0<j<i, \\ g\left(q_{j}, q_{j}, t_{i, j-1}\right) & \text { for } j \geq i\end{cases}
$$

Set

$$
\begin{equation*}
t_{i}:=t_{i, n-1} \tag{4.3}
\end{equation*}
$$

Then $\mathcal{W} \models t_{i}=t_{i+1}$ for $i=1, \ldots, n-1$.
Let $1 \leq i \leq n-1$. Then by (4.2) we have the following equalities

$$
\begin{aligned}
& t_{i, 0}=p_{1}=t_{i+1,0} \\
& t_{i, 1}=f\left(q_{1}, p_{1}, t_{i, 0}\right)=f\left(q_{1}, p_{1}, t_{i+1,0}\right)=t_{i+1,1} \\
& t_{i, 2}=f\left(q_{2}, p_{2}, t_{i, 1}\right)=f\left(q_{2}, p_{2}, t_{i+1,1}\right)=t_{i+1,2} \\
& \vdots \\
& t_{i, i-1}=f\left(q_{i-1}, p_{i-1}, t_{i, i-2}\right)=f\left(q_{i-1}, p_{i-1}, t_{i+1, i-2}\right)=t_{i+1, i-1} .
\end{aligned}
$$

Since the identities $p_{i}=q_{i}$ and $f(x, x, y)=g(x, x, y)$ are valid in $\mathcal{W}$, it follows that

$$
\mathcal{W} \models t_{i, i}=g\left(q_{i}, q_{i}, t_{i, i-1}\right)=f\left(q_{i}, p_{i}, t_{i+1, i-1}\right)=t_{i+1, i} .
$$

Hence, again by (4.2)

$$
\begin{aligned}
\mathcal{W} & \models t_{i, i+1}=g\left(q_{i+1}, q_{i+1}, t_{i, i}\right)=g\left(q_{i+1}, q_{i+1}, t_{i+1, i}\right)=t_{i+1, i+1} \\
\mathcal{W} & \models t_{i, i+2}=g\left(q_{i+1}, q_{i+1}, t_{i, i+1}\right)=g\left(q_{i+1}, q_{i+1}, t_{i+1, i+1}\right)=t_{i+1, i+2}, \\
& \vdots \\
\mathcal{W} & \models t_{i, n-1}=g\left(q_{n-1}, q_{n-1}, t_{i, n-2}\right)=g\left(q_{n-1}, q_{n-1}, t_{i+1, n-2}\right)=t_{i+1, n-1} .
\end{aligned}
$$

Consequently $\mathcal{W} \models t_{i}=t_{i+1}$.
D. The term $t_{1}$ is a term idempotent of $\mathcal{W}$.

By (b) and (c), $g(x, x, y)$ is a term idempotent of $\mathcal{W}$. Then we have

$$
g(x, x, y) \preceq g\left(q_{n-1}, q_{n-1}, t_{1, n-2}\right)=t_{1, n-1}=t_{1} .
$$

By Lemma 3.1, it follows that $t_{1}$ is a term idempotent of $\mathcal{W}$.
E. Let $a_{1}, \ldots, a_{n} \in C$. There exist pairwise $\mathcal{W}$-equivalent term idempotents $t_{1}, \ldots, t_{n}$ of $\mathcal{W}$ and elements $\mathbf{c}$ of $A$, such that $a_{i}=t_{i}(\mathbf{c})$ for each $i=1, \ldots, n$.

Let $1 \leq i \leq n-1$. Since $a_{i} \varrho^{0} a_{i+1}$, there is an identity $p_{i}\left(\mathbf{z}_{i}\right)=q_{i}\left(\mathbf{z}_{i}\right)$ true in $\mathcal{W}$, and elements $\mathbf{c}_{i}$ of $A$, such that

$$
a_{i}=p_{i}\left(\mathbf{c}_{i}\right) \text { and } a_{i+1}=q_{i}\left(\mathbf{c}_{i}\right)
$$

Denote the list $\mathbf{c}_{1}, \ldots, \mathbf{c}_{n}$ by $\mathbf{c}$. Define terms $t_{1}, \ldots, t_{n}$ by (4.2) and (4.3). Then $\mathbf{C}$ and $\mathbf{D}$ imply that the terms $t_{1}, \ldots, t_{n}$ are $\mathcal{W}$-equivalent term idempotents of $\mathcal{W}$. By (4.1), one obtains the following equalities:

$$
\begin{aligned}
& t_{i, 0}(\mathbf{c})=p_{1}\left(\mathbf{c}_{1}\right)=a_{1}, \\
& t_{i, 1}(\mathbf{c})=f\left(q_{1}\left(\mathbf{c}_{1}\right), p_{1}\left(\mathbf{c}_{1}\right), t_{i, 0}(\mathbf{c})\right)=f\left(a_{2}, a_{1}, a_{1}\right)=a_{2} \\
& t_{i, 2}(\mathbf{c})=f\left(q_{2}\left(\mathbf{c}_{2}\right), p_{2}\left(\mathbf{c}_{2}\right), t_{i, 1}(\mathbf{c})\right)=f\left(a_{3}, a_{2}, a_{2}\right)=a_{3}, \\
& \quad \vdots \\
& t_{i, i-1}(\mathbf{c})=f\left(q_{i-1}\left(\mathbf{c}_{i-1}\right), p_{i-1}\left(\mathbf{c}_{i-1}\right), t_{i, i-2}(\mathbf{c})\right)=f\left(a_{i}, a_{i-1}, a_{i-1}\right)=a_{i}, \\
& t_{i, i}(\mathbf{c})=g\left(q_{i}\left(\mathbf{c}_{i}\right), q_{i}\left(\mathbf{c}_{i}\right), t_{i, i-1}(\mathbf{c})\right)=g\left(a_{i+1}, a_{i+1}, a_{i}\right)=a_{i}, \\
& t_{i, i+1}(\mathbf{c})=g\left(q_{i+1}\left(\mathbf{c}_{i+1}\right), q_{i+1}\left(\mathbf{c}_{i+1}\right), t_{i, i}(\mathbf{c})\right)=g\left(a_{i+2}, a_{i+2}, a_{i}\right)=a_{i} . \\
& \quad \vdots \\
& t_{i, n-1}(\mathbf{c})=g\left(q_{n-1}\left(\mathbf{c}_{n-1}\right), q_{n-1}\left(\mathbf{c}_{n-1}\right), t_{i, n-2}(\mathbf{c})\right)=g\left(a_{n}, a_{n}, a_{i}\right)=a_{i} .
\end{aligned}
$$

Therefore $a_{i}=t_{i, n-1}(\mathbf{c})=t_{i}(\mathbf{c})$ for each $i=1, \ldots, n$.
F. The subalgebra $C$ of $A$ satisfies any identity

$$
u\left(x_{1}, \ldots, x_{n}\right)=v\left(x_{1}, \ldots, x_{n}\right)
$$

valid in $\mathcal{V}$.
Let $a_{1}, \ldots, a_{n} \in C$. Let $t_{1}, \ldots, t_{n}$ be terms and $\mathbf{c}$ be elements of $A$ satisfying the condition of $\mathbf{E}$. By Theorem [2.2, the identity $u\left(t_{1}, \ldots, t_{n}\right)=$ $v\left(t_{1}, \ldots, t_{n}\right)$ is valid in $A$. Hence

$$
u\left(a_{1}, \ldots, a_{n}\right)=u\left(t_{1}(\mathbf{c}), \ldots, t_{n}(\mathbf{c})\right)=v\left(t_{1}(\mathbf{c}), \ldots, t_{n}(\mathbf{c})\right)=v\left(a_{1}, \ldots, a_{n}\right)
$$

G. The Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

By $\mathbf{F}$, we conclude that $C \in \mathcal{V}$, and consequently that $A \in \mathcal{V} \circ \mathcal{W}$. Thus the inclusion $\mathrm{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{V} \circ \mathcal{W}$ holds, and so $\mathcal{V} \circ \mathcal{W}$ is a variety.

## 5. Consequences and examples

Theorem 4.1 has a number of interesting consequences. First note that since every term in an idempotent variety is a term idempotent, one easily obtains the following corollary.
Corollary 5.1. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same type, and let $\mathcal{W}$ be idempotent. If there exist terms $f(x, y, z)$ and $g(x, y, z)$ such that
(a) $\mathcal{V} \models f(x, y, y)=x$ and $\mathcal{V} \models g(x, x, y)=y$,
(b) $\mathcal{W} \models f(x, x, y)=g(x, x, y)$,
then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
Another special case is when the terms $f(x, y, z)$ and $g(x, y, z)$ do not depend on the middle variable.

Corollary 5.2. Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial varieties, and let $\mathcal{W}$ be term idempotent. If there exist terms $f(x, y)$ and $g(x, y)$ such that
(a) $\mathcal{V} \models f(x, y)=x$ and $\mathcal{V} \models g(x, y)=y$,
(b) $\mathcal{W} \models f(x, y)=g(x, y)$,
then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
Proof. We only need to show that the condition (c) of Theorem4.1]is satisfied, i.e. $f(x, y)$ is a term idempotent of $\mathcal{W}$. First note that the terms $f(x, y)$ and $g(x, y)$ cannot coincide. Otherwise, the condition (a) would imply that $\mathcal{V} \models x=y$, contradicting the nontriviality of $\mathcal{V}$. So the identity of (b) is nontrivial, and hence $f(x, y)$ is a term idempotent of $\mathcal{W}$.

Example 5.3. Let $\mathcal{V}$ be the variety of groupoids defined by identities

$$
(x x) y=y=y(x x)
$$

and let $\mathcal{R S}$ be the variety from Example 3.5. Define

$$
f(x, y):=x(y y) \quad \text { and } \quad g(x, y):=(x x) y .
$$

It is easy to see that $\mathcal{V}$ satisfies the identities $f(x, y)=x$ and $g(x, y)=y$, and $\mathcal{R S}$ satisfies the identity $f(x, y)=g(x, y)$. Thus, by Corollary 5.2 the Mal'tsev product $\mathcal{V} \circ \mathcal{R S}$ is a variety.

Example 5.4. Let $\mathcal{V}$ be a strongly irregular variety of a plural type $\tau$ that satisfies a strongly irregular identity $t(x, y)=x$. By Example 3.4 the variety $\mathcal{C}_{\tau}$ of constant algebras of type $\tau$, is a term idempotent variety. Set $f(x, y):=$ $t(x, y)$ and $g(x, y):=t(y, x)$. Clearly $\mathcal{V}$ satisfies the identities $f(x, y)=x$ and $g(x, y)=y$. Since neither $f(x, y)$ nor $g(x, y)$ is a variable, it follows that $\mathcal{C}_{\tau} \models f(x, y)=g(x, y)$. Corollary 5.2 implies that the Mal'tsev product $\mathcal{V} \circ \mathcal{C}_{\tau}$ is a variety.

Replacing the variety $\mathcal{C}_{\tau}$ by its regularization $\widetilde{\mathcal{C}_{\tau}}$ one obtains a further example. First note that, by Proposition 3.7, the regularization $\widetilde{\mathcal{C}_{\tau}}$ of $\mathcal{C}_{\tau}$ is a term idempotent variety. Keep the same terms $f(x, y)$ and $g(x, y)$ as in the previous case. Then note that the identity $f(x, y)=g(x, y)$ is regular. Hence it is also satisfied in $\widetilde{\mathcal{C}_{\tau}}$. By Corollary 5.2 again, the Mal'tsev product $\mathcal{V} \circ \widetilde{\mathcal{C}_{\tau}}$ is a variety for any strongly irregular variety $\mathcal{V}$.

An additional assumption that the variety $\mathcal{W}$ of Corollary 5.2 is idempotent yields Theorem 2.4 as a special case. If we further set $g(x, y):=y$, then the condition (a) of Corollary 5.2 reduces to only one identity and one obtains the following corollary.

Corollary 5.5. Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial varieties, and let $\mathcal{W}$ be term idempotent. If there exists a term $f(x, y)$ such that
(a) $\mathcal{V} \models f(x, y)=x$,
(b) $\mathcal{W} \models f(x, y)=y$,
then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
The conditions (a) and (b) of Corollary 5.5 mean that the varieties $\mathcal{V}$ and $\mathcal{W}$ are independent. (See e.g. [11, §3.5].) The join $\mathcal{V} \vee \mathcal{W}$ of independent varieties $\mathcal{V}$ and $\mathcal{W}$ consists of (algebras isomorphic to) products $A \times B$ of $A \in \mathcal{V}$ and $B \in \mathcal{W}$, and obviously $\mathcal{V} \vee \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}$.
Example 5.6. The variety $\mathcal{L Z}$ of left-zero bands (defined by the identity $x y=x$ ) and the variety $\mathcal{R} \mathcal{Z}$ of right-zero bands (defined by the identity $x y=y$ ) are clearly independent. So, by Corollary 5.5 the Mal'tsev product $\mathcal{L Z} \circ \mathcal{R Z}$ is a variety. Its subvariety $\mathcal{L Z} \vee \mathcal{R} \mathcal{Z}$ is the variety of rectangular bands.

If we set $f$ and $g$ in Corollary5.1to be the same term, then the condition (b) is trivially satisfied, and the identities of (a) become Mal'tsev identities. So the variety $\mathcal{V}$ is congruence permutable. We thus obtain the following extension of Theorem 2.6.
Corollary 5.7. Let $\mathcal{V}$ be a congruence permutable variety and $\mathcal{W}$ be an idempotent variety. Then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Typical examples of congruence permutable (or Mal'tsev) varieties are given by varieties of groups, quasigroups, loops, rings or modules. If $\mathcal{V}$ is any such variety and $\mathcal{W}$ is an idempotent variety of the same type as $\mathcal{V}$, then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.
Example 5.8. Now we will consider the Mal'tsev product $\mathcal{G} \circ \mathcal{L}$ of the variety of groups and the variety of lattices. To do so we first have to describe them as varieties of the same type.

Here groups are defined as algebras $\left(G, \cdot,^{-1}\right)$ with one binary and one unary operations. (See e.g. [3, Ex. 7.6].) And the variety $\mathcal{G}$ of groups will be considered as the variety of algebras $\left(G,+, \cdot,{ }^{-1}\right)$, satisfying the usual identities of groups and the identity $x+y=x \cdot y$.

On the other hand, lattices will be considered as algebras of the same type as $\mathcal{G}$ with $x^{-1}:=x$. The variety of groups is a Mal'tsev variety, and the variety of lattices is an idempotent variety. By Corollary 5.7 the Mal'tsev product $\mathcal{G} \circ \mathcal{L}$ is a variety.

Note that in those Mal'tsev products $\mathcal{V} \circ \mathcal{W}$ considered so far which are actually varieties, the factor $\mathcal{V}$ was always strongly irregular. It is natural to ask if this strongly irregular variety $\mathcal{V}$ could be replaced by an irregular (but not strongly irregular) variety $\mathcal{V}$. We conclude this section with an example of a Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ that is not a variety, where the factor $\mathcal{V}$ is irregular, but not strongly irregular, and where the factor $\mathcal{W}$ is an idempotent (and hence term idempotent) variety. In particular, this example shows that in

Theorem [2.5, which is a corollary of Theorem 4.1, the assumption of strong irregularity cannot be replaced by irregularity alone.

Example 5.9. It will be shown that the Mal'tsev product $\mathcal{C S} \circ \mathcal{S}$ fails to be a variety. We provide an example of a groupoid that belongs to $\mathcal{C S} \circ \mathcal{S}$, but has a quotient that does not. Let $A$ be the groupoid defined by the following table

| $\cdot$ | $a$ | $e$ | $b$ | $f$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $e$ | $e$ | $b$ | $f$ |
| $e$ | $e$ | $e$ | $f$ | $f$ |
| $b$ | $b$ | $f$ | $f$ | $f$ |
| $f$ | $f$ | $f$ | $f$ | $f$ |

The groupoid $A$ is a member of $\mathcal{C S} \circ \mathcal{S}$. The semilattice replica congruence $\varrho$ of $A$ has two congruence classes $\{a, e\}$ and $\{b, f\}$ that can be seen to be constant semigroups with constant values $e$ and $f$ respectively. The congruence $\theta$ of $A$ with congruence classes $\{a\},\{b\}$ and $E:=\{e, f\}$ has the quotient $A / \theta$ given by the table

| $\cdot$ | $\{a\}$ | $\{b\}$ | $E$ |
| :---: | :---: | :---: | :---: |
| $\{a\}$ | $E$ | $\{b\}$ | $E$ |
| $\{b\}$ | $\{b\}$ | $E$ | $E$ |
| $E$ | $E$ | $E$ | $E$ |

The semilattice replica congruence of $A / \theta$ is the all relation with one congruence class containing all elements. However $A / \theta$ is not a constant semigroup, hence $A / \theta \notin \mathcal{C S} \circ \mathcal{S}$. Thus $\mathcal{C S} \circ \mathcal{S}$ is not closed under homomorphic images, and so it is not a variety.

## 6. Purely polarized varieties

A variety $\mathcal{V}$ that has a constant unary term idempotent is called polarized. (See [6].) Such a term is called a polar term and is denoted by $p(x)$. The constant value $p$ of a polar term $p(x)$ in a given algebra $A \in \mathcal{V}$ is called the pole of this algebra. The pole of $A$ is the unique idempotent of $A$, and a class of a congruence on $A$ is a subalgebra of $A$ iff it is the congruence class of the pole. (See [8, Sec. 1].) Since the only possible value in $A$ of a term idempotent of $\mathcal{V}$ is the pole of $A$, it follows that all term idempotents of $\mathcal{V}$ are constant and pairwise $\mathcal{V}$-equivalent. In particular every unary term idempotent is a polar term.

Examples of polarized varieties are provided by varieties of groups with a polar term $p(x):=x x^{-1}$, varieties of loops with a polar term $p(x):=x / x$, and varieties of rings with a polar term $p(x):=x-x$. One more example is given by the variety $\mathcal{C}_{\tau}$ of constant algebras where every unary term different from a variable is a polar term.

A variety will be called purely polarized, if it is both polarized and term idempotent. The two conditions are independent. For example, the variety of groups is polarized but not term idempotent, and on the other hand, the variety $\mathcal{R S}$ of rectangular semigroups of Example 3.5 is term idempotent but
not polarized. However, if a variety satisfies both of these conditions, then their consequences are rather strong.
Proposition 6.1. Let $\mathcal{V}$ be a purely polarized variety with a polar term $p(x)$. Let $u$ and $v$ be different terms of the type of $\mathcal{V}$. Then

$$
\mathcal{V} \models u=v \quad \Longleftrightarrow \quad \mathcal{V} \models u=p(x) \quad \text { and } \quad \mathcal{V} \models v=p(x) .
$$

Proof. Suppose that $\mathcal{V} \models u=v$ for different terms $u$ and $v$. Since $\mathcal{V}$ is a term idempotent variety and the identity $u=v$ is nontrivial, both $u$ and $v$ are term idempotents of $\mathcal{V}$. Since $\mathcal{V}$ is polarized, they are $\mathcal{V}$-equivalent to $p(x)$. So $\mathcal{V} \models u=p(x)$ and $\mathcal{V} \models v=p(x)$.

Typical examples of purely polarized varieties are the variety $\mathcal{C}$ of constant semigroups and, more generally, the varieties $\mathcal{C}_{\tau}$ of constant algebras of type $\tau$. Next we show a method of extending these examples.

Example 6.2. Constant semigroups may be defined by the consequences of the identities $x_{1} \cdots x_{n}=y_{1} \cdots y_{n}$ for all $n \geq 2$. Now, for $k \geq 2$, consider the variety $\mathcal{C}_{k}$ of semigroups defined by the consequences of the identities $x_{1} \cdots x_{n}=y_{1} \cdots y_{n}$, where $n \geq k$. Note that $\mathcal{C}_{2}$ coincides with $\mathcal{C}$. The variety $\mathcal{C}_{k}$ satisfies all the identities $p=q$ of type $\tau$ such that neither $p$ nor $q$ is a product of less than $k$ variables. It is easy to see that both sides of every such identity are term idempotents of $\mathcal{C}_{k}$, and that $\mathcal{C}_{k}$ is polarized by the polar term $p(x)=x \cdots x$, where $x$ is repeated $k$ times. Hence $\mathcal{C}_{k}$ is purely polarized.

Recall the preordered set ( $X \Omega, \preceq$ ) of Section 3, Consider the case when $\Omega$ consists only of a single symbol of a binary operation. Note that the preorder $\preceq$ carries over to the free semigroup $X \mathcal{S G}$ over $X$. All term idempotents of $\mathcal{C}_{k}$ form an upper set of $(X \mathcal{S G}, \preceq)$ that is generated by the element $x_{1} \cdots x_{k}$.

Proposition 6.1 shows that both sides of every nontrivial identity satisfied in a purely polarized variety $\mathcal{V}$ are constant term idempotents of $\mathcal{V}$. Such identities will be called polar identities.
Corollary 6.3. A polarized variety $\mathcal{V}$ is purely polarized if and only if all nontrivial identities satisfied in $\mathcal{V}$ are polar.

A term $p(x)$ of type $\tau$ will be called a zero term of a variety $\mathcal{V}$, if it is constant and for all $\omega \in \Omega$ and every $1 \leq i \leq n$,

$$
\begin{equation*}
\mathcal{V} \models \omega\left(x_{1}, \ldots, x_{i-1}, p(x), x_{i+1}, \ldots, x_{n}\right)=p(x) \tag{6.1}
\end{equation*}
$$

Note that the identities of (6.1) imply

$$
\mathcal{V} \models \omega(p(x), \ldots, p(x))=p(x)
$$

Hence a zero term is also a polar term, and any variety with a zero term is polarized. The pole of any algebra $A$ in a variety with a zero term is the zero of $A$ (i.e. it forms a one-element $\operatorname{sink}$ of $A$ ). If $p(x)$ is a zero term of $\mathcal{V}$, then the identities of (6.1) may be easily generalized to all terms of type $\tau$. Indeed, for any term $t\left(x_{1}, \ldots, x_{n}\right)$ and every $1 \leq i \leq n$,

$$
\begin{equation*}
\mathcal{V} \models t\left(x_{1}, \ldots, x_{i-1}, p(x), x_{i+1}, \ldots, x_{n}\right)=p(x) . \tag{6.2}
\end{equation*}
$$

If the type $\tau$ is plural, then there exists a term $t(x, y)$ involving both variables $x$ and $y$, such that the identities of (6.1) imply

$$
\mathcal{V} \models p(x)=t(p(x), p(y))=p(y) .
$$

Thus if we restrict ourselves to plural types, a separate assumption that a zero term is constant is unnecessary. The following proposition provides a basic property of polar identities.

Proposition 6.4. Let $\mathcal{V}$ be a polarized variety. Let $u=v$ be a nontrivial polar identity true in $\mathcal{V}$. The following conditions are equivalent:
(1) Every nontrivial consequence of a set of polar identities true in $\mathcal{V}$ is also polar.
(2) Every nontrivial consequence of the identity $u=v$ is polar.
(3) $\mathcal{V}$ has a zero term.

Proof. (1) $\Rightarrow(2)$ This implication is obvious.
$(2) \Rightarrow(3)$ Assume (2). For any $\omega \in \Omega$ and every $1 \leq i \leq n$, the identity

$$
\begin{equation*}
\omega\left(x_{1}, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n}\right)=\omega\left(x_{1}, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_{n}\right) \tag{6.3}
\end{equation*}
$$

is a nontrivial consequence of $u=v$. So, by (2), it is polar. The left-hand sides of the polar identities $u=v$ and (6.3) are constant term idempotents. Hence they are $\mathcal{V}$-equivalent, because $\mathcal{V}$ is a polarized variety. Consequently, $\mathcal{V}$ satisfies the identity

$$
\omega\left(x_{1}, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n}\right)=u
$$

Therefore, the identities of (6.1) hold for the unary term $p(x):=u(x, \ldots, x)$, and so $p(x)$ is a zero term of $\mathcal{V}$.
$(3) \Rightarrow(1)$ Suppose that $\mathcal{V}$ has a zero term $p(x)$. (It is also a polar term of $\mathcal{V}$.) Consider a polar identity $u=v$ satisfied in $\mathcal{V}$. We will show that the consequences of this identity are also polar.

Let $t\left(x_{1}, \ldots, x_{n}\right)$ be a term and let $1 \leq i \leq n$. Then the identity

$$
\begin{equation*}
t\left(x_{1}, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n}\right)=t\left(x_{1}, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_{n}\right) \tag{6.4}
\end{equation*}
$$

is a consequence of $u=v$. Since $u$ and $v$ are constant term idempotents, they are both $\mathcal{V}$-equivalent to $p(x)$. Thus, for the left-hand side of (6.4), we have

$$
t\left(x_{1}, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n}\right)=t\left(x_{1}, \ldots, x_{i-1}, p(x), x_{i+1}, \ldots, x_{n}\right)=p(x)
$$

An analogous identity holds for its right-hand side. Hence both sides of (6.4) are constant term idempotents, and so (6.4) is a polar identity.

If we substitute arbitrary terms for the variables of a constant term idempotent of $\mathcal{V}$, we again obtain a constant term idempotent. Thus an identity $u^{\prime}=v^{\prime}$, obtained from the polar identity $u=v$ by substituting some terms for its variables, is also polar.

If $u=v$ and $v=w$ are polar identities, then obviously $u=w$ is also a polar identity. Therefore all nontrivial consequences of any set of polar identities true in $\mathcal{V}$ are polar.

Corollary 6.5. In a purely polarized variety, all polar terms are zero terms.

Proof. This follows directly by Proposition 6.4 and Corollary 6.3
The following proposition provides an equational base for any purely polarized variety.

Proposition 6.6. A variety $\mathcal{V}$ of type $\tau$ is purely polarized if and only if it is defined by the identities
(a) $p(x)=p(y)$,
(b) $\omega\left(x_{1}, \ldots, x_{i-1}, p(x), x_{i+1}, \ldots, x_{n}\right)=p(x)$ for all $\omega \in \Omega$ and $1 \leq i \leq n$,
(c) $t_{i}=p(x)$ for all $i \in I$,
where $p(x)$ and $\left\{t_{i} \mid i \in I\right\}$ are some terms of type $\tau$.
Proof. $(\Rightarrow)$ Let $\mathcal{V}$ be a purely polarized variety with a polar term $p(x)$, defined by some identities $\left\{u_{i}=v_{i} \mid i \in I\right\}$. The polar term $p(x)$ satisfies the identity (a) and, by Corollary 6.5 also the identities of (b). By Proposition 6.1, an identity $u_{i}=v_{i}$, for $i \in I$, has the same consequences as the identities $u_{i}=p(x)$ and $v_{i}=p(x)$. Thus we can construct an equational base of the required form.
$(\Leftarrow)$ Assume that $\mathcal{V}$ is defined by the identities of (a), (b) and (c). Then (a) and (b) imply that $p(x)$ is a polar term, and hence $\mathcal{V}$ is polarized. Thus, the defining identities (a), (b) and (c) of $\mathcal{V}$ are all polar identities. The identities of (b) show that $p(x)$ is a zero term of $\mathcal{V}$. By Proposition 6.4, all consequences of the defining identities are also polar, and so by Corollary 6.3, $\mathcal{V}$ is purely polarized.

We can see that purely polarized varieties form a very special class of algebras. Additionally, we will find that they interact with Mal'tsev products in an interesting way. First, let us look at what Theorem 4.1 says about the special case when $\mathcal{W}$ is a purely polarized variety. If we set $f(x, y, z):=p(x)$ and $g(x, y, z):=p(z)$ for a unary term $p(x)$, then the conditions (a), (b) and (c) reduce to: (a) $\mathcal{V} \models p(x)=x$, (b) $\mathcal{W} \models p(x)=p(y)$ and (c) $p(x)$ is a term idempotent of $\mathcal{W}$. We thus obtain the following corollary.

Corollary 6.7. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, and let $\mathcal{W}$ be purely polarized with a polar term $p(x)$. If $\mathcal{V}$ satisfies $p(x)=x$, then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Since every unary term different from a variable is a polar term of $\mathcal{C}_{\tau}$, additionally, we have the following.

Corollary 6.8. Let $\mathcal{V}$ be a variety of type $\tau$. Let $u(x)$ be a unary term of type $\tau$, which is not just a variable. If $\mathcal{V}$ satisfies $u(x)=x$, then the Mal'tsev product $\mathcal{V} \circ \mathcal{C}_{\tau}$ is a variety.

The assumption on the variety $\mathcal{V}$ is rather weak. For example, it is satisfied by all idempotent varieties and all strongly irregular varieties. Surprisingly, it is possible to prove a much more general result. We conclude the paper by showing that the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ of any variety $\mathcal{V}$ and a purely polarized variety $\mathcal{W}$ is a variety.

This next lemma follows by Corollary 3.12 and the fact that each algebra in a purely polarized variety has exactly one idempotent.

Lemma 6.9. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, and let $\mathcal{W}$ be a purely polarized variety. If $A \in \mathcal{V} \circ \mathcal{W}$, then exactly one class of the $\mathcal{W}$-replica congruence of $A$ is a subalgebra of $A$, and all the other classes are singletons.

Theorem 6.10. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$. If $\mathcal{W}$ is a purely polarized variety, then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof. Let $A$ be a member of $\mathcal{V} \circ \mathcal{W}$. We will show that for any congruence $\theta$ of $A$, the quotient algebra $A / \theta$ is also a member of $\mathcal{V} \circ \mathcal{W}$.

Let $\varrho$ be the $\mathcal{W}$-replica congruence of $A$. By Lemma 6.9 exactly one class of $\varrho$, call it $E$, is a subalgebra of $A$. Moreover $E \in \mathcal{V}$, since $A \in \mathcal{V} \circ \mathcal{W}$. All the other classes of $\varrho$ are singletons. Let $\bar{\theta}:=\varrho \vee \theta$, and let $a, b \in A$. Since the class $E$ of $\varrho$ is a subalgebra of $A$ and other classes are singletons, it follows that

$$
\begin{aligned}
(a, b) \in \bar{\theta} \Longleftrightarrow & \text { either } a / \theta \cap E \neq \varnothing \neq b / \theta \cap E \\
& \text { or } a / \theta=b / \theta \text { and } a / \theta \cap E=\varnothing
\end{aligned}
$$

Hence, one class of $\bar{\theta}$, denoted by $E \theta$, contains $E$ and is a disjoint union of $\theta$-classes having a non-empty intersection with $E$, i.e.

$$
E \theta=\{a \in A \mid \exists e \in E, a \theta e\}=\bigcup_{e \in E} e / \theta .
$$

By the Second Isomorphism Theorem [11, Thm. 1.2.4], Et is a subalgebra of $A$ and

$$
\begin{equation*}
E \theta /\left(\theta \cap(E \theta)^{2}\right) \cong E /\left(\theta \cap E^{2}\right) \in \mathcal{V} \tag{6.5}
\end{equation*}
$$

All the other classes of $\bar{\theta}$ coincide with $\theta$-classes which are disjoint from $E$.
Now recall that $(a, b) \in \bar{\theta}$ iff $(a / \theta, b / \theta) \in \bar{\theta} / \theta$, and by the First Isomorphism Theorem [11, Thm. 1.2.3],

$$
\begin{equation*}
(A / \theta) /(\bar{\theta} / \theta) \cong A / \bar{\theta} \cong(A / \varrho) /(\bar{\theta} / \varrho) . \tag{6.6}
\end{equation*}
$$

Since $A / \varrho$ belongs to $\mathcal{W}$, it follows that its quotient $(A / \varrho) /(\bar{\theta} / \varrho)$, and hence also the quotients $A / \bar{\theta}$ and $(A / \theta) /(\bar{\theta} / \theta)$, are members of $\mathcal{W}$.

Under the first isomorphism of (6.6), the $\bar{\theta}$-class $E \theta$ corresponds to the $\bar{\theta} / \theta$-class $\bar{E}:=\{e / \theta \mid e \in E\}$, the unique $\bar{\theta} / \theta$-class which is a subalgebra of $A / \theta$. Each of the remaining $\bar{\theta} / \theta$-classes consists of one $\theta$-class disjoint from $E$. In particular, $\bar{\theta} / \theta$ is the $\mathcal{W}$-replica congruence of $A / \theta$. Note that, by (6.5), $\bar{E}=E \theta /\left(\theta \cap(E \theta)^{2}\right) \in \mathcal{V}$. It follows that $A / \theta \in \mathcal{V} \circ \mathcal{W}$, and hence $\mathrm{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{V} \circ \mathcal{W}$. Therefore $\mathcal{V} \circ \mathcal{W}$ is a variety.

Corollary 6.11. Let $\mathcal{V}$ be a variety of type $\tau$. Then the Mal'tsev product $\mathcal{V} \circ \mathcal{C}_{\tau}$ is a variety.

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