

COUNTABLY COMPACT EXTENSIONS AND CARDINAL CHARACTERISTICS OF THE CONTINUUM

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ABSTRACT. In this paper we show that the existence of certain first-countable compact-like extensions is equivalent to the equality between corresponding cardinal characteristics of the continuum. For instance, $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ if and only if every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a regular first-countable countably compact space.

1. INTRODUCTION

The main motivation for investigation of regular first-countable countably compact spaces is the following problem of Nyikos which is listed among 20 central problems in Set-theoretic Topology by Hrušák and Moore [16].

Problem 1.1 (Nyikos). *Does ZFC imply the existence of a regular separable first-countable countably compact non-compact space?*

Following [3], a regular separable first-countable countably compact space is called a *Nyikos space*. Consistent examples of noncompact Nyikos spaces were constructed by Franklin and Rajagopalan [14] (under $\omega_1 = \mathfrak{t}$) and by Ostaszewski [25] (under $\mathfrak{b} = \mathfrak{c}$). Weiss [35] showed that under MA every perfectly normal Nyikos space is compact. Proper Forcing Axiom or, simply, PFA is a stronger version of Martin's Axiom. Basic information about PFA can be found in [20]. For fruitful applications of PFA in Topology see [10, 11, 12, 30, 34]. The following result was proved by Nyikos and Zdomskyy [24].

Theorem 1.2. (PFA) *Every normal Nyikos space is compact.*

A space X is called $\mathbb{R}$ -*rigid* if each continuous real-valued function on X is constant. For more about rigid spaces see [2, 17, 31, 32, 33]. Another problem related to Nyikos spaces appeared in [29, Problem C65].

Problem 1.3 (Tzannes). *Does there exist a regular (separable, first countable) countably compact $\mathbb{R}$ -rigid space?*

Taking into account the properties in brackets, Tzannes problem can be considered as an ultimate version of Nyikos problem, as $\mathbb{R}$ -rigid spaces, being not Tychonoff, are even not locally compact.

Bardyla and Osipov [2] constructed a ZFC example of a regular countably compact $\mathbb{R}$ -rigid space. Bardyla and Zdomskyy [3] obtained the following answer to Problem 1.3 under the consistent assumption ($\heartsuit$): “ $\omega_1 = \mathfrak{t} < \mathfrak{b} = \mathfrak{c}$ and there exists a simple $P_{\mathfrak{c}}$ -point in $\beta(\omega)$ ”.

Theorem 1.4 (Bardyla, Zdomskyy). ($\heartsuit$) *There exists an $\mathbb{R}$ -rigid Nyikos space.*

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† Sadly, Peter Nyikos passed away in February 2024, when this work was in the final stage of the preparation. Therefore the rest of the authors decided to finish the research and to submit the paper with his name as coauthor. This is our tribute to our dear friend Peter.

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The proof of Theorem 1.4 had two steps. During the first one we constructed a regular separable first-countable $\mathbb{R}$ -rigid space X of cardinality $< \mathfrak{c}$. The second step was to embed densely the space X into a first-countable countably compact space Y . Note that Y will be automatically an $\mathbb{R}$ -rigid Nyikos space. This was done using the following result proved in [3] under the assumption $(\heartsuit)$: “ $\mathfrak{b} = \mathfrak{c}$ and there exists a simple P_c -point in $\beta(\omega)$ ”.

Theorem 1.5 (Bardyla, Zdomskyy). $(\heartsuit)$ *Each regular first-countable space of cardinality $< \mathfrak{c}$ embeds densely into a regular first-countable countably compact space.*

Similar problems concerning embedding of topological spaces into first-countable compact-like spaces are known in General Topology. For instance, the most relevant to this paper is the following one posed in [27].

Problem 1.6 (Stephenson). *Does every locally feebly compact first-countable regular space embed densely into a feebly compact first-countable regular space?*

Problem 1.6 was solved affirmatively by Simon and Tironi [26]. In case of Tychonoff spaces the following result was obtained in [28].

Theorem 1.7 (Terada, Terasawa). *Each Tychonoff first-countable locally pseudocompact space embeds densely into a Tychonoff first-countable pseudocompact space.*

The mentioned results motivate the following general question which we address in this paper.

Question 1.8. *When does space X embed into a regular first-countable compact-like space?*

2. PRELIMINARIES

By $\mathbb{N}$ we denote the set of positive integers, i.e. $\mathbb{N} = \omega \setminus \{0\}$. The cardinality of the continuum is denoted by $\mathfrak{c}$. The set of all infinite subsets of a countable set A is denoted by $[A]^\omega$. The set of all finite subsets of a set A is denoted by $[A]^{<\omega}$. A family $\mathcal{A} \subset [\omega]^\omega$ is called *almost disjoint* if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$. A family $\mathcal{S} \subseteq [\omega]^\omega$ is called *splitting* if for any $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ which splits A , i.e. $|A \cap S| = |A \cap (\omega \setminus S)| = \omega$. For two subsets A and B of ω we write $A \subseteq^* B$ or $B \supseteq^* A$ if $|A \setminus B| < \omega$. A family $\mathcal{T} \subset [\omega]^\omega$ is called a *maximal tower* if it is well-ordered by the relation $\supseteq^*$ and there is no set $A \in [\omega]^\omega$ such that $A \subseteq^* T$ for all $T \in \mathcal{T}$. For any functions $f, g \in \omega^\omega$ we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A family $\Phi \subseteq \omega^\omega$ is called *unbounded* if there exists no $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in \Phi$. We shall use the following cardinal characteristics of the continuum:

- $\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subset [\omega]^\omega \text{ is a splitting family}\};$
- $\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \subset [\omega]^\omega \text{ is a maximal tower}\};$
- $\mathfrak{b} = \min\{|\Phi| : \Phi \subset \omega^\omega \text{ is an unbounded family}\}.$

It is well-known that $\omega_1 \leq \mathfrak{t} \leq \min\{\mathfrak{s}, \mathfrak{b}\}$.

An ultrafilter $\mathcal{F}$ on ω is called a simple P_c point if $\mathcal{F}$ has a base which forms a maximal tower of cardinality $\mathfrak{c}$. The existence of such an ultrafilter is consistent with ZFC and, in particular, follows from $\mathfrak{t} = \mathfrak{c}$.

Let X, Y be topological spaces. A map $f : X \rightarrow Y$ is called an *embedding of X into Y* if the map $f' : X \rightarrow f(X)$, obtained by restricting the range of f , is a homeomorphism. A weight of a topological space X is the smallest cardinal $w(X)$ such that X has a base of cardinality $w(X)$.

A space X is called

- *countably compact* if each closed discrete subset of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function is bounded;
- *feebly compact* if every locally finite family of open subsets of X is finite.

It is known that countable compactness implies feeble compactness. In the case of Tychonoff spaces pseudocompactness is equivalent to feeble compactness.

Pixley–Roy hyperspace $\text{PR}(X)$ of a space X is the set $[X]^{<\omega}$ of all finite subsets of X endowed with the topology generated by the base consisting of the sets

$$[F, U] = \{A \in [X]^{<\omega} : F \subseteq A \subseteq U\},$$

where $F \in [X]^{<\omega}$ and U is open in X . It can be easily checked that $\text{PR}(X)$ is Hausdorff and zero-dimensional, if X is Hausdorff. Moreover, if X is first-countable, then so is $\text{PR}(X)$.

The notions used but not defined in this paper are standard and can be found in [7, 8, 13, 19].

3. MAIN RESULTS

The majority of the results in this section are equivalences, each being based on an “embedding theorem” combined with a “non-embedding theorem”, which are proved in sections 4 and 5, respectively.

Theorem 3.1. *The following assertions are equivalent:*

- (1) $\omega_1 = \mathfrak{c}$.
- (2) *Every first-countable Tychonoff space of weight $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.*
- (3) *Each separable first-countable locally compact normal space of cardinality $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.*

Theorem 3.2. *The following assertions are equivalent.*

- (1) $\mathfrak{b} = \mathfrak{c}$.
- (2) *Every Hausdorff, locally compact, first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a Hausdorff first-countable locally compact countably compact space.*
- (3) *Every Hausdorff, locally compact, first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a Hausdorff first-countable countably compact space.*

Observe that $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ follows from $(\heartsuit)$. So, the following result generalizes Theorem 1.5.

Theorem 3.3. *The following assertions are equivalent.*

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) *Every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.*
- (3) *Every regular first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.*

Theorem 3.3 has the following zero-dimensional counterpart.

Theorem 3.4. *The following assertions are equivalent.*

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) *Every Hausdorff zero-dimensional first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a Hausdorff zero-dimensional first-countable countably compact space.*
- (3) *Every Hausdorff zero-dimensional first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a Hausdorff zero-dimensional first-countable countably compact space.*

Combining Theorem 3.3 with the results from [3] we get the following.

Theorem 3.5. *The following assertions are equivalent:*

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) *Every regular separable first-countable non-normal space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$ -rigid Nyikos space.*

- (3) *Every regular separable first-countable non-normal space of cardinality $< \mathfrak{c}$ embeds into an $\mathbb{R}$ -rigid Nyikos space.*

Theorem 3.5 allows us to prove the following generalization of Theorem 1.4

Theorem 3.6. $(\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c})$ *There exists an $\mathbb{R}$ -rigid Nyikos space.*

Theorems 1.2 and 3.5 imply the following corollary which shows a profound contrast in the behavior of normal and non-normal Nyikos spaces under PFA.

Corollary 3.7. (PFA) *The following assertions hold:*

- (1) *Each normal Nyikos space is compact.*
- (2) *Each regular separable first-countable space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$ -rigid Nyikos space.*

The following consistency result complements Theorem 1.5.

Theorem 3.8. $(\heartsuit)$ *Every Tychonoff first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a Tychonoff first-countable countably compact space.*

Turning to embeddings into pseudocompact spaces we obtain the following characterization.

Theorem 3.9. *The following assertions are equivalent:*

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) *Every first-countable zero-dimensional Hausdorff space of weight $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.*
- (3) *Every first-countable zero-dimensional Hausdorff space of cardinality $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.*

Recall that a subspace A of the Cantor space is called a λ -set if each countable subset of A is G_δ . As a by product we obtain the following characterization of λ -subsets of the Cantor space.

Theorem 3.10. *A subspace X of the Cantor space is a λ -set if and only if the Pixley-Roy hyperspace $\text{PR}(X)$ embeds densely into a first-countable pseudocompact space.*

4. EMBEDDING THEOREMS

We start with embeddings into countably compact spaces.

As we mentioned in the previous section, the assumption $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$ is formally weaker than $\heartsuit$. Indeed, fix a simple $P_\mathfrak{c}$ -point p and a splitting family $\mathcal{C}$. To derive a contradiction, assume that $|\mathcal{C}| < \mathfrak{c}$. Consider the family $\mathcal{D} = \mathcal{C} \cap p$ and find an arbitrary pseudointersection $E \in p$ of the family $\mathcal{D} \cup \{\omega \setminus C : C \in \mathcal{C} \setminus p\}$. It is easy to check that there exists no $C \in \mathcal{C}$ which splits E . This contradiction implies that $\mathfrak{s} = \mathfrak{c}$. On the other hand in the Mathias model $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$, but there are no simple $P_\mathfrak{c}$ -points, see [6, Theorem 8]. Hence the next theorem generalizes Theorem 1.5.

Proposition 4.1. $(\mathfrak{s} = \mathfrak{b} = \mathfrak{c})$ *Let X be a regular first-countable space of weight $\kappa < \mathfrak{c}$. Then X can be densely embedded into a regular first-countable countably compact space.*

Proof. Without loss of generality we can assume that the underlying set of X is disjoint with $\mathfrak{c}$. The first countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathcal{D} \cup [\mathfrak{c}]^\omega \rightarrow \mathfrak{c}$ such that $h(a) \geq \sup(a)$ for any $a \in [\mathfrak{c}]^\omega$. It is easy to see that such a bijection exists. Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on $X_\alpha \subseteq X \cup \alpha$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α . At the end, we will show that the space $Y_\mathfrak{c}$ is regular first-countable countably compact and contains X as a dense subspace.

Let $X_0 = X$. Since X is first-countable and regular there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} B_0^x$ of the topology on X , where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is an open neighborhood base at x . With no loss of generality we can assume that $|\mathcal{B}_0| < \mathfrak{c}$; $U_{0,0}^x = X$ for each $x \in X$; and $\overline{U_{n+1,0}^x} \subset U_{n,0}^x$ for every $n \in \omega$ and $x \in X$.

Assume that for each $\alpha < \xi$ regular first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the topology τ_α , where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is an open neighborhood base at x . Additionally assume that $X_\alpha \subseteq X_\beta$ for any $\alpha < \beta$ and the family $\mathcal{B}_\alpha$ satisfies the following conditions:

- (1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
- (2) $U_{0,\alpha}^x = X_\alpha$ for each $x \in X_\alpha$;
- (3) $\text{cl}_{Y_\alpha}(U_{n+1,\alpha}^x) \subset U_{n,\alpha}^x$ for every $n \in \omega$ and $x \in X_\alpha$;
- (4) for every $\alpha < \beta < \xi$, $n \in \omega$ and $x \in X_\alpha$ we have that $U_{n,\alpha}^x = U_{n,\beta}^x \cap X_\alpha$.

There are three cases to consider:

- 1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ ;
- 2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ ;
- 3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$ and $\mathcal{B}_\xi = \mathcal{B}_\gamma$.

2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_n\}_{n \in \omega}$. For each $U_{n,\gamma}^x \in \mathcal{B}_\gamma$ consider the set

$$A_{U_{n,\gamma}^x} = \{k \in \omega : z_k \in U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x\}.$$

Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{s}$ we get that the family $\{A_{U_{n,\gamma}^x} : U_{n,\gamma}^x \in \mathcal{B}_\gamma\}$ is not splitting, i.e. there exists a subset $A \subset \omega$ such that for each $U_{n,\gamma}^x \in \mathcal{B}_\gamma$ the set $d_\gamma = \{z_n : n \in A\}$ is either almost contained in $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$ or almost disjoint with $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$. Since the set $\{z_n : n \in \omega\}$ is closed in Y_γ for each $x \in X_\gamma$ there exists $n \in \omega$ such that $|U_{n,\gamma}^x \cap \{z_k : k \in \omega\}| \leq 1$. Since $U_{0,\gamma}^x = X_\gamma$ for all $x \in X_\gamma$, we obtain that for each $x \in X_\gamma$ there exists a unique $m(x) \in \omega$ such that $d_\gamma \subset^* U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x$.

For each $U_{m(x),\gamma}^x \in \mathcal{B}_\gamma$ consider the function $f_{U_{m(x),\gamma}^x} \in \omega^A$ defined as follows: if $z_n \in U_{m(x),\gamma}^x \setminus \text{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x)$, then

$$f_{U_{m(x),\gamma}^x}(n) = \min\{k : U_{k,\gamma}^{z_n} \subset U_{m(x),\gamma}^x \setminus \text{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x)\},$$

and $f_{U_{m(x),\gamma}^x}(n) = 0$, otherwise. Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{b}$ there exists a function $f \in \omega^A$ such that $f \geq^* f_{U_{m(x),\gamma}^x}$ for every $U_{m(x),\gamma}^x \in \mathcal{B}_\gamma$. With no loss of generality we can additionally assume that $U_{f(n),\gamma}^{z_n} \cap U_{f(m),\gamma}^{z_m} = \emptyset$ for each distinct $n, m \in A$. It is easy to see that the sequence $\{U_{f(n),\gamma}^{z_n} : n \in A\}$ is locally finite in Y_γ .

Next we define an open neighborhood base at the point $\gamma \in X_\xi$: Put

$$U_{0,\xi}^\gamma = X_\xi \quad \text{and} \quad U_{k,\xi}^\gamma = \bigcup_{n \in A \setminus k} U_{f(n)+k,\gamma}^{z_n} \cup \{\gamma\}$$

for all $k \in \mathbb{N}$.

Now we are going to define an open neighborhood base $\mathcal{B}_\xi^x$ at each point $x \in X_\gamma$. Let $U_{0,\xi}^x = X_\xi$. For each $n \in \mathbb{N}$ let $U_{n,\xi}^x = U_{n,\gamma}^x$ if $n \geq m(x) + 1$ and $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$ if $n \leq m(x)$. It is easy to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ , and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ .

At this point it is a tedious routine to check that the family $\mathcal{B}_\xi$ satisfies conditions (1)–(4).

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$, and for every $n \in \omega$ let us put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x.$$

It is quite straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ , and the family $\mathcal{B}_\xi$ satisfies conditions (1), (2) and (4). In order to check the validity of condition (3) we need the following auxiliary claim.

Claim 4.2. *For any $\gamma \in \xi$, $n, m \in \omega$ and distinct points $y_0, y_1 \in Y_\gamma$, if $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$, then $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} = \emptyset$.*

Proof. Seeking a contradiction, assume that $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} \neq \emptyset$, but $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$ for some $\gamma \in \xi$. It is easy to see that $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} \subset \xi \setminus \gamma$. Let $\delta = \min U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1}$. It follows that $U_{n,\delta}^{y_0} \cap U_{m,\delta}^{y_1} = \emptyset$ and $U_{n,\delta+1}^{y_0} \cap U_{m,\delta+1}^{y_1} = \{\delta\}$. Then the set d_δ (see case 2 above) is closed and discrete in Y_δ . Since $\delta \in U_{n,\delta+1}^{y_0} \cap U_{m,\delta+1}^{y_1}$, the definition of $U_{n,\delta+1}^{y_0}$ and $U_{m,\delta+1}^{y_1}$ implies that $d_\delta \subset^* U_{n,\delta}^{y_0} \cap U_{m,\delta}^{y_1} = \emptyset$, which contradicts our assumption. $\square$

Fix any $x \in X_\xi$, $n \in \omega$ and $z \in \overline{U_{n+1,\xi}^x}$. There exists an ordinal $\gamma \in \xi$ such that $x, z \in X_\gamma$. We claim that $z \in \text{cl}_{Y_\gamma}(U_{n+1,\gamma}^x)$. Indeed, otherwise, there exists $m \in \omega$ such that $U_{m,\gamma}^z \cap U_{n+1,\gamma}^x = \emptyset$. The above claim implies that $U_{m,\xi}^z \cap U_{n+1,\xi}^x = \emptyset$, which contradicts the choice of z . Thus $z \in \text{cl}_{Y_\gamma}(U_{n+1,\gamma}^x) \subset U_{n,\gamma}^x \subset U_{n,\xi}^x$, which establishes condition (3).

By the construction, the space $Y_\mathfrak{c}$ is regular, first-countable and contains X as a dense subspace. Let A be any countable subset of $Y_\mathfrak{c}$. If the set $B = A \cap \mathfrak{c}$ is infinite, then consider $h(B) \in \mathfrak{c}$. By the construction, either B has an accumulation point in $Y_{h(B)}$ or $h(B)$ is an accumulation point of B in $Y_{h(B)+1}$. In both cases B has an accumulation point in $Y_\mathfrak{c}$. If $A \subset^* X$, then either it has an accumulation point in X , or A is closed and discrete in X . In the latter case either A has an accumulation point in $Y_{h(A)}$ or $h(A)$ is an accumulation point of A in $Y_{h(A)+1}$. Thus $Y_\mathfrak{c}$ is countably compact. $\square$

Proposition 4.3. ($\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$) *Let X be a Hausdorff zero-dimensional first-countable space of weight $\kappa < \mathfrak{c}$. Then X embeds densely into a Hausdorff zero-dimensional first-countable countably compact space.*

Proof. The proof of this theorem is very similar to one of Theorem 4.1, so we give only a sketch of it. Let $\mathcal{D}$ and h be such as in the proof of Theorem 4.1.

Let $X_0 = X$. Since X is first-countable and zero-dimensional there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} \mathcal{B}_0^x$ of the topology on X consisting of clopen sets, where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is a nested open neighborhood base at x . With no loss of generality we can assume that $|\mathcal{B}_0| < \mathfrak{c}$ and $U_{0,0}^x = X$ for each $x \in X$.

Assume that for each $\alpha < \xi$ Hausdorff zero-dimensional first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the topology τ_α on a set $X_\alpha \subseteq X \cup \alpha$, where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is a nested open neighborhood base at x consisting of clopen sets. Additionally assume that $X_\alpha \subseteq X_\beta$ for any $\alpha < \beta$ and the family $\mathcal{B}_\alpha$ satisfies the following conditions:

- (1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
- (2) $U_{0,\alpha}^x = X_\alpha$ for each $x \in X_\alpha$;
- (3) for every $\alpha < \beta < \xi$, $n \in \omega$ and $x \in X_\alpha$ we have that $U_{n,\alpha}^x = U_{n,\beta}^x \cap X_\alpha$.

There are three cases to consider:

- 1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ ;
- 2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ ;

3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$ and $\mathcal{B}_\xi = \mathcal{B}_\gamma$.

2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_n\}_{n \in \omega}$. Similarly as in the proof of Theorem 4.1, one can find a subset $A \subset \omega$ such that for each $U_{n,\gamma}^x \in \mathcal{B}_\gamma$ the set $d_\gamma = \{z_n : n \in A\}$ is either almost contained in $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$ or almost disjoint with $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$. Thus, for each $x \in X_\gamma$ there exists a unique $m(x) \in \omega$ such that $d_\gamma \subset^* U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x$.

For each $U_{m(x),\gamma}^x \in \mathcal{B}_\gamma$ consider the function $f_{U_{m(x),\gamma}^x} \in \omega^A$ defined as follows: if $z_n \in U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x$, then

$$f_{U_{m(x),\gamma}^x}(n) = \min\{k : U_{k,\gamma}^{z_n} \subset U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x\},$$

and $f_{U_{m(x),\gamma}^x}(n) = 0$, otherwise. Note that the function $f_{U_{m(x),\gamma}^x}$ is well-defined, as the set $U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x$ is open for every $x \in X_\gamma$. Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{b}$ we get that there exists a function $f \in \omega^A$ such that $f \geq^* f_{U_{m(x),\gamma}^x}$ for every $U_{m(x),\gamma}^x \in \mathcal{B}_\gamma$. We can additionally assume that $U_{f(n),\gamma}^{z_n} \cap U_{f(m),\gamma}^{z_m} = \emptyset$ for each distinct $n, m \in A$. It is easy to see that the family $\{U_{f(n),\gamma}^{z_n} : n \in A\}$ is locally finite in Y_γ .

Next we define an open neighborhood base at the point $\gamma \in X_\xi$: Put

$$U_{0,\xi}^\gamma = X_\xi \quad \text{and} \quad U_{k,\xi}^\gamma = \bigcup_{n \in A \setminus k} U_{f(n)+k,\gamma}^{z_n} \cup \{\gamma\}$$

for all $k \in \mathbb{N}$. Since the family $\{U_{f(n),\gamma}^{z_n} : n \in A\}$ is locally finite, the set $U_{k,\xi}^\gamma$ is clopen for each $k \in \omega$.

Now we are going to define an open neighborhood base $\mathcal{B}_\xi^x$ at each point $x \in X_\gamma$. Let $U_{0,\xi}^x = X_\xi$ for every $x \in X_\gamma$. For each $x \in X_\gamma$ and $n \in \mathbb{N}$ let $U_{n,\xi}^x = U_{n,\gamma}^x$ if $n \geq m(x) + 1$ and $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$ if $n \leq m(x)$. It is easy to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ , and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms a nested open neighborhood base at x in Y_ξ . Moreover, the space Y_ξ is Hausdorff and zero-dimensional.

At this point it is a tedious routine to check that the family $\mathcal{B}_\xi$ satisfies conditions (1)–(3).

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$. For each $x \in X_\xi$ and $n \in \omega$ put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x.$$

It is straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ , satisfies conditions (1)–(3) and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ . Hausdorffness of the space Y_ξ follows from the next claim which can be proved similarly as Claim 4.2.

Claim 4.4. *For any $\gamma \in \xi$, $n, m \in \omega$ and distinct points $y_0, y_1 \in Y_\gamma$, if $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$, then $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} = \emptyset$.*

It remains to check that the space Y_ξ is zero-dimensional which is done in the following claim.

Claim 4.5. *For any $x \in X_\xi$ and $n \in \omega$ the set $U_{n,\xi}^x$ is closed.*

Proof. Fix any $y \in X_\xi \setminus U_{n,\xi}^x$ and find $\gamma < \xi$ such that $x, y \in X_\gamma$. It follows that $y \notin U_{n,\gamma}^x$. By the inductive assumption the set $U_{n,\gamma}^x$ is closed, implying the existence of $m \in \omega$ such that $U_{m,\gamma}^y \cap U_{n,\gamma}^x = \emptyset$. Then the previous claim implies that $U_{m,\xi}^y \cap U_{n,\xi}^x = \emptyset$, witnessing that the set $U_{n,\xi}^x$ is closed. $\square$

Similarly as in the proof of Theorem 4.1 it can be checked that the space Y_ξ is Hausdorff, zero-dimensional, first-countable, countably compact and contains X as a dense subspace. $\square$

Let D be a subset of a set S . An *expansion* of D is a family $\{A_d : d \in D\}$ of subsets of S such that $d \in A_d$ and $A_d \cap D = \{d\}$ for all $d \in D$. A topological space X satisfies (or has) *Property D* if every countable closed discrete subspace $D \subset X$ has an expansion to a discrete collection of open sets. A space X is called *pseudonormal* if, whenever F_0 and F_1 are disjoint closed sets, one of which is countable, there are disjoint open sets U_0 and U_1 containing F_0 and F_1 respectively. We will also need the following facts proved in [21, Proposition 3.6] and [21, Theorem 3.7], respectively.

Proposition 4.6 (Nyikos). *Every pseudonormal space X has property D.*

Proposition 4.7 (Nyikos). *Every first-countable regular space of Lindelöf number $< \mathfrak{b}$ is pseudonormal.*

Propositions 4.6 and 4.7 imply the following useful corollary.

Corollary 4.8. *Every first-countable regular space of weight $< \mathfrak{b}$ has property D.*

The proof of the following result uses techniques developed in [1, 21, 22, 25].

Proposition 4.9. ($\mathfrak{b} = \mathfrak{c}$) *Each first-countable Hausdorff locally compact space of weight $< \mathfrak{c}$ embeds densely into a first-countable Hausdorff locally compact countably compact space.*

Proof. Fix any Hausdorff locally compact first-countable space X of weight $< \mathfrak{c} = \mathfrak{b}$. Without loss of generality we can assume that the underlying set of X is disjoint with $[0, 1] \times \mathfrak{c}$. First countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D}_0 = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathfrak{c} \times \mathfrak{c} \rightarrow \mathfrak{c}$ such that $h((a, b)) \geq a$.

Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on X_α which will be a union of X and some pairwise disjoint family of copies of the half interval $[0, 1]$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α . At the end, we will show that the space $Y_\mathfrak{c}$ has the desired properties.

Let $X_0 = X$ and fix any bijection $g_0 : \mathcal{D}_0 \rightarrow \{0\} \times \mathfrak{c}$. Note that by the definition of h , $g_0^{-1}(h^{-1}(0)) \in \mathcal{D}_0$. By Corollary 4.8, the space X has property D. It follows that the closed discrete set $\{z_n : n \in \omega\} = g_0^{-1}(h^{-1}(0))$ can be expanded to a discrete family $\{U_n : n \in \omega\}$ of open sets such that $z_n \in U_n$. Since X is locally compact, we lose no generality assuming that $\text{cl}_X(U_n)$ is compact for every $n \in \omega$. For each $n \in \omega$ fix a continuous function f_n such that $f(z_n) = 0$ and $f(X \setminus U_n) = 1$. Let $X_1 = X_0 \cup ([0, 1] \times \{1\})$, and τ_1 be a topology on X_1 which satisfies the following conditions:

- (1) X_0 is an open subspace of X_1 ;
- (2) the sets

$$W_n(0, 1) = [0, 1/n] \times \{1\} \cup \bigcup_{m \geq n} f_m^{-1}([0, 1/n]),$$

$n \in \mathbb{N}$ form an open neighborhood base at the point $(0, 1) \in X_1$;

- (3) For $0 < x < 1$ the sets

$$W_n(x, 1) = (x - 1/n, x + 1/n) \times \{1\} \cup \bigcup_{m \geq n} f_m^{-1}((x - 1/n, x + 1/n)),$$

where $n \in \mathbb{N}$ satisfies $(x - 1/n, x + 1/n) \subseteq (0, 1)$, form an open neighborhood base at the point $(x, 1) \in X_1$.

It is easy to check that Y_1 is a first-countable Hausdorff space of weight $< \mathfrak{b}$. Observe that $(0, 1)$ is an accumulation point of the set $\{z_n : n \in \omega\} = g_0^{-1}(h^{-1}(0))$ in Y_1 . Clearly, Y_1 is locally compact at each point $x \in X_0$. Let us show that Y_1 is locally compact at $(0, 1)$. We claim that the set $\text{cl}_{Y_1}(W_2(0, 1))$

is compact. Since the family $\{U_n : n \in \omega\}$ is discrete and the sets U_n , $n \in \omega$ have compact closure in X , it is easy to see that the set

$$\text{cl}_{Y_1}(W_2(0, 1)) \subseteq [0, 1/2] \times \{1\} \cup \bigcup_{n \in \omega \setminus \{0, 1\}} \text{cl}_X(U_n)$$

is σ -compact. At this point it is enough to show that $\text{cl}_{Y_1}(W_2(0, 1))$ is countably compact. Fix any infinite subset $A = \{a_n : n \in \omega\} \subset \text{cl}_{Y_1}(W_2(0, 1))$. If for some $n \in \omega$ the set $A \cap \text{cl}_X(U_n)$ is infinite or the set $A \cap ([0, 1/2] \times \{1\})$ is infinite, then A has an accumulation point in Y_1 . So, let us assume that all the aforementioned intersections are finite. Shrinking the set A if necessary, we can assume that $A \subset X$ and $|A \cap U_n| \leq 1$ for all $n \in \omega$. For each $n \in \omega$ let $m(n)$ be such that $a_n \in U_{m(n)}$. Put $y_n = f_{m(n)}(a_n)$. Since $\{y_n : n \in \omega\} \subset [0, 1/2]$ and the interval $[0, 1/2]$ is compact there exists $z \in [0, 1/2]$ and a subsequence $\{y_{n_k} : k \in \omega\}$ which converges to z . It is easy to see that $(z, 1)$ is an accumulation point of the set A in Y_1 . Hence the σ -compact set $\text{cl}_{Y_1}(W_2(0, 1))$ is countably compact and thus compact. Similarly one can check that Y_1 is locally compact at $(x, 1)$ for each $0 < x < 1$. Let

$$\mathcal{D}_1 = \{A \in [X_1]^\omega : A \text{ is closed and discrete in } Y_1\}.$$

Fix any bijection $g_1 : \mathcal{D}_1 \rightarrow \{1\} \times \mathfrak{c}$.

Assume that locally compact Hausdorff spaces $Y_\alpha = (X_\alpha, \tau_\alpha)$ with weight $< \mathfrak{b}$ are constructed for all $\alpha \in \xi$ such that Y_α is an open dense subspace of Y_η for all $\alpha \in \eta \in \xi$. Additionally assume that if $X_{\alpha+1} \setminus X_\alpha \neq \emptyset$, then $X_{\alpha+1} \setminus X_\alpha = [0, 1] \times \{\alpha + 1\}$ and for each $\alpha \in \xi$ it is defined a bijection $g_\alpha : \mathcal{D}_\alpha \rightarrow \{\alpha\} \times \mathfrak{c}$, where

$$\mathcal{D}_\alpha = \{A \in [X_\alpha]^\omega : A \text{ is closed and discrete in } Y_\alpha\}.$$

If ξ is a limit ordinal, then put $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$ and $\tau_\xi = \bigcup_{\alpha \in \xi} \tau_\alpha$. Clearly, Y_ξ is first-countable, Hausdorff, locally compact with weight less than $\mathfrak{b}$. Fix any bijection $g_\xi : \mathcal{D}_\xi \rightarrow \{\xi\} \times \mathfrak{c}$.

Assume that $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$. Consider the pair $h^{-1}(\gamma) = (a, b) \in \mathfrak{c} \times \mathfrak{c}$. If the set $g_a^{-1}(h^{-1}(\gamma)) \in \mathcal{D}_a$ already has an accumulation point in Y_γ , then put $Y_\xi = Y_\gamma$. If the set $g_a^{-1}(h^{-1}(\gamma))$ is closed and discrete in Y_γ , then we construct a Hausdorff locally compact first-countable space Y_ξ of weight $< \mathfrak{b}$ similarly as we constructed Y_1 . In particular, $X_\xi = X_\gamma \cup [0, 1] \times \{\xi\}$, and $(0, \xi)$ is an accumulation point of the set $g_a^{-1}(h^{-1}(\gamma))$.

Obviously, Y_ξ is a Hausdorff locally compact first-countable space. To derive a contradiction, assume that Y_ξ is not countably compact. Then Y_ξ contains an infinite countable discrete closed subset A . Since the cardinal $\mathfrak{b} = \mathfrak{c}$ is regular, there exists an ordinal $\xi \in \mathfrak{c}$ such that $A \subset X_\xi$. Obviously, A is an infinite closed discrete subset of Y_β for each $\beta \geq \xi$. Then $g_\xi(A) = (\xi, \delta)$ for some $\delta \in \mathfrak{c}$. Let $\mu \in \mathfrak{c}$ be such that $\xi \leq \mu$ and $h((\xi, \delta)) = \mu$. By the construction, the point $(0, \mu + 1) \in X_{\mu+1}$ is an accumulation point of A in Y_μ , which implies the desired contradiction. Hence the space Y_ξ is countably compact. $\square$

Next we turn to embeddings into pseudocompact spaces. We adopt the notations from [5].

Let $S = \{S_n : n \in \omega\}$ and $T = \{T_n : n \in \omega\}$ be sequences of subsets of a set X . We say that S *refines* T if there exists an injection $f \in \omega^\omega$ such that $S_n \subseteq T_{f(n)}$ for all $n \in \omega$. We write that the sequence S is *eventually contained* in a subset $A \subset X$ if there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \subset A$. The sequences S and T are called *eventually disjoint* if there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \cap \bigcup_{n \geq k} T_n = \emptyset$. A family $\mathcal{R}$ of sequences of subsets of X is called *eventually disjoint* if any distinct elements $S, T \in \mathcal{R}$ are eventually disjoint.

Let X be a zero-dimensional space and $\mathcal{B}$ be a base of X consisting of clopen sets. By $\mathbf{D}(\mathcal{B})$ we denote the set of all locally finite sequences of elements of $\mathcal{B}$. A sequence $S = \{S_n : n \in \omega\} \in \mathbf{D}(\mathcal{B})$ is called $\mathcal{B}$ -*exact* if for each $B \in \mathcal{B}$ which intersects infinitely many S_n there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \subset B$. Let $\mathbf{E}(\mathcal{B}) = \{S \in \mathbf{D}(\mathcal{B}) : S \text{ is } \mathcal{B}\text{-exact}\}$.

For any eventually disjoint family $\mathcal{R} \subset \mathbf{E}(\mathcal{B})$ we define a topology τ on the set $X \cup \mathcal{R}$ as follows: for each $B \in \mathcal{B}$ let $B^* = B \cup \{R \in \mathcal{R} : R \text{ is eventually contained in } B\}$. For every $R = \{R_n : n \in \omega\} \in \mathcal{R}$ and $n \in \omega$ let $U_n(R) = \{R\} \cup \bigcup_{m \geq n} R_m^*$. The topology τ is generated by the base $\mathcal{B}^* = \{B^* : B \in \mathcal{B}\} \cup \{U_n(R) : R \in \mathcal{R}, n \in \omega\}$. By $X(\mathcal{R})$ we denote the space $(X \cup \mathcal{R}, \tau)$. It is easy to check that the space $X(\mathcal{R})$ is Hausdorff and zero-dimensional.

The following result is proved by Bell in [5, Claim 3.2].

Proposition 4.10. *Let X be a first-countable zero-dimensional space, $\mathcal{B}$ be a base of X consisting of clopen sets and for every $T \in \mathbf{D}(\mathcal{B})$ there exists $S \in \mathbf{E}(\mathcal{B})$ such that S refines T . Then for every maximal eventually disjoint subfamily $\mathcal{R}$ of $\mathbf{E}(\mathcal{B})$ the space $X(\mathcal{R})$ is zero-dimensional, first-countable and pseudocompact.*

Proposition 4.11. *Each first-countable zero-dimensional Hausdorff space X of weight $< \min\{\mathfrak{s}, \mathfrak{b}\}$ embeds densely into a first-countable zero-dimensional pseudocompact space.*

Proof. Fix a base $\mathcal{B}$ of X of size $< \min\{\mathfrak{s}, \mathfrak{b}\}$ consisting of clopen sets. We lose no generality assuming that $\mathcal{B} = \{U_n^x : x \in X, n \in \omega\}$, where for every $x \in X$ the family $\{U_n^x : n \in \omega\}$ forms a nested open neighborhood base at x . By Proposition 4.10 it is enough to check that for every $T \in \mathbf{D}(\mathcal{B})$ there exists $S \in \mathbf{E}(\mathcal{B})$ such that S refines T . Consider an arbitrary sequence $T = \{T_n : n \in \omega\} \in \mathbf{D}(\mathcal{B})$ and for each $n \in \omega$ fix a point $x_n \in T_n$. To each $B \in \mathcal{B}$ we assign a function $f_B \in \omega^\omega$ defined as follows: if $x_n \in B$, then put $f_B(n) = \min\{k \in \omega : U_k^{x_n} \subset B \cap T_n\}$; otherwise put $f_B(n) = \min\{k \in \omega : U_k^{x_n} \subset T_n \text{ and } U_k^{x_n} \cap B = \emptyset\}$. Since $|\mathcal{B}| < \mathfrak{b}$, there exists a function $f \in \omega^\omega$ such that $f \geq^* f_B$ for every $B \in \mathcal{B}$. Consider the refinement $\{U_{f(n)}^{x_n} : n \in \omega\}$ of T . Clearly, for each $B \in \mathcal{B}$ there exists $k \in \omega$ such that for every $n \geq k$ either $U_{f(n)}^{x_n} \cap B = \emptyset$ or $U_{f(n)}^{x_n} \subset B$. So, to each $B \in \mathcal{B}$ we can assign a set $C_B = \{n \in \omega : U_{f(n)}^{x_n} \subset B\} \subset \omega$. Since $|\mathcal{B}| < \mathfrak{s}$ we get that the family $\{C_B : B \in \mathcal{B}\}$ is not splitting. Hence there exists an infinite subset A of ω possessing the following property: for every $B \in \mathcal{B}$ there exists $k \in \omega$ such that either $U_{f(n)}^{x_n} \subset B$ for each $n \in A \setminus k$ or $U_{f(n)}^{x_n} \cap B = \emptyset$ for each $n \in A \setminus k$. Hence the sequence $S = \{U_{f(n)}^{x_n} : n \in A\}$ is a $\mathcal{B}$ -exact refinement of T . $\square$

5. NON-EMBEDDING THEOREMS

To each almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ corresponds a *Mrowka space* $\psi(\mathcal{A}) = (\mathcal{A} \cup \omega, \tau)$, where the topology τ is defined as follows: points of ω are isolated and for each $A \in \mathcal{A}$ the family $\{\{A\} \cup (A \setminus n) : n \in \omega\}$ is an open neighborhood base at A in τ . We refer the reader to [15] for basic properties of Mrowka spaces.

Proposition 5.1. *There exists an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{b}$ such that the corresponding Mrowka space $\psi(\mathcal{A})$ doesn't embed into first-countable Hausdorff countably compact spaces.*

Proof. First we construct an unbounded subset $\{b_\alpha : \alpha \in \mathfrak{b}\} \subset \omega^\omega$ which satisfies the following conditions:

- (1) $b_\alpha(\omega) \cap b_\beta(\omega) =^* \emptyset$ for any $\alpha \neq \beta$;
- (2) the function b_α is increasing for every $\alpha \in \mathfrak{b}$;
- (3) $b_\alpha \leq^* b_\beta$ whenever $\alpha \in \beta$.

For this fix any almost disjoint family $\mathcal{D} = \{D_\xi : \xi \in \mathfrak{b}\} \subset [\omega]^\omega$ and an unbounded family $\{g_\xi : \xi \in \mathfrak{b}\} \subset \omega^\omega$ such that $g_\alpha \leq^* g_\beta$ for all $\alpha \leq \beta$. Assume that for each $\alpha \in \xi$ we already constructed an increasing function $b_\alpha \in \omega^\omega$ such that $b_\alpha(\omega) \subseteq D_\alpha$, $b_\alpha \geq^* g_\alpha$ and $b_\alpha \leq^* b_\beta$ whenever $\alpha \leq^* \beta$. Since $\xi < \mathfrak{b}$, the family $B_\xi = \{b_\alpha : \alpha \in \xi\} \cup \{g_\xi\}$ is not bounded, i.e. there exists $\phi \in \omega^\omega$ such that $\phi \geq^* h$ for all $h \in B_\xi$. Next we inductively construct a subset $\{d_i : i \in \omega\}$ of D_ξ . Let d_0 be an arbitrary element of D_ξ such that $d_0 \geq \phi(0)$. Assume that for some $n \geq 0$ we constructed a subset $\{d_i : i \in n\}$ of D_ξ . Let $d_n \in D_\xi$ be arbitrary such that $d_n > \max\{d_{n-1}, \phi(n)\}$. Put $b_\xi = \{(n, d_n) : n \in \omega\}$.

Clearly, $b_\xi(\omega) \subseteq D_\xi$, $b_\xi \geq^* g_\xi$ and $b_\xi \geq^* b_\alpha$ for every $\alpha \in \xi$. Thus, after completing $\mathfrak{b}$ -many steps of the induction we obtain an unbounded subset $\{b_\alpha : \alpha \in \mathfrak{b}\}$ which satisfies conditions (1)–(3).

For $i \in \omega$ set $v_i = \{(i, n) : n \in \omega\}$. Let $\mathcal{A} \stackrel{\text{def}}{=} \{b_\alpha : \alpha \in \mathfrak{b}\} \cup \{v_i : i \in \omega\}$. Clearly, $\mathcal{A}$ is an almost disjoint family of subsets of $\omega \times \omega$.

To derive a contradiction, assume that there exists a first-countable Hausdorff countably compact space X which contains the Mrowka space $\psi(\mathcal{A})$ as a subspace. Find an infinite subset C of ω such that the sequence $\{v_n : n \in C\}$ converges to a point $x \in X$. Fix an open neighborhood base $\{U_n : n \in \omega\}$ at x . Without loss of generality we can assume that $\{v_k : k \in C \setminus n\} \subset U_n$ for all $n \in \omega$. Observe that for each $n \in \omega$ there exists a function $f_n \in \omega^C$ such that $\{(k, m) : k \in C \setminus n, m \geq f_n(k)\} \subset U_n$. Since the functions b_α , $\alpha \in \mathfrak{b}$ are increasing (see condition 2), the family $\{b_\alpha \upharpoonright_C : \alpha \in \mathfrak{b}\}$ is unbounded. Then for each $n \in \omega$ there exists $\alpha_n \in \mathfrak{b}$ such that $f_n \not\geq^* b_{\alpha_n} \upharpoonright_C$. It follows that for each $n \in \omega$ and $\xi \geq \alpha_n$,

$$|\{(k, m) : k \in C \setminus n, m \geq f_n(k)\} \cap b_\xi| = \omega.$$

Thus for $\mu = \sup\{\alpha_n : n \in \omega\}$ the set $b_\mu \cap \{(k, m) : k \in C \setminus n, m \geq f_n(k)\}$ is infinite for all $n \in \omega$. Hence $b_\mu \in \overline{U_n}$ for all $n \in \omega$, which contradicts the Hausdorffness of X . $\square$

The following result is essentially based on the work of van Douwen and Przymusiński [9].

Proposition 5.2. *There exists a separable first-countable normal Lindelöf space of cardinality $\mathfrak{s}$ which cannot be densely embedded into a first-countable regular feebly compact space.*

Proof. By $\mathbb{Q}$ we denote the set of rational numbers from the interval $[0, 1]$. Enumerate the set of all subsets of ω as $\{A_x : x \in [0, 1]\}$, and for each $x \in [0, 1]$ and $m \in \mathbb{N}$ choose $q_x(m) \in \mathbb{Q}$ such that $0 < |x - q_x(m)| < 1/m$ and put $Q_x = \{q_x(m) : m \in \mathbb{N}\}$. Let X be the set $[0, 1] \times \{0\} \cup \mathbb{Q} \times \mathbb{N}$ endowed with a topology τ satisfying the following conditions:

- $\mathbb{Q} \times \mathbb{N}$ is the set of isolated points;
- if $x \in \mathbb{Q}$, then basic neighborhoods of a point $(x, 0) \in [0, 1] \times \{0\}$ have the form

$$B_m(x) = \{(a, b) \in X : |x - a| < 1/m\} \setminus (Q_x \times A_x \cup \{x\} \times \mathbb{N}) \quad \text{for } m \in \mathbb{N};$$

- if $x \in [0, 1] \setminus \mathbb{Q}$, then basic neighborhoods of a point $(x, 0) \in [0, 1] \times \{0\}$ have the form

$$B_m(x) = \{(a, b) \in X : |x - a| < 1/m\} \setminus (Q_x \times A_x) \quad \text{for } m \in \mathbb{N}.$$

Clearly, the space¹ X is regular, separable and first-countable.

Fix an arbitrary splitting family $\mathcal{S} \subset [\omega]^\omega$ of cardinality $\mathfrak{s}$ and consider the subspace

$$Y = \{x \in [0, 1] : A_x \in \mathcal{S}\} \times \{0\} \cup \mathbb{Q} \times \mathbb{N}$$

of X . It is clear that Y is regular and first-countable. Taking into account that the subspace $([0, 1] \times \{0\}) \cap Y$ of Y is second-countable, we deduce that Y is separable, Lindelöf and subsequently normal.

Let us show that Y doesn't embed densely into a first-countable regular feebly compact space. Assuming the contrary, fix any first-countable regular feebly compact space Z which contains Y as a dense subspace. The following claim is crucial.

Claim 5.3. *For each $B \in [\mathbb{N}]^\omega$ there exists $q \in \mathbb{Q}$ such that $q \times B$ is not a convergent sequence in Z .*

Proof. Fix an arbitrary infinite subset $B \subseteq \mathbb{N}$ and find $S \in \mathcal{S}$ that splits B . Let $x \in [0, 1]$ be such that $A_x = S$. Fix an arbitrary open neighborhood W of $(x, 0)$ in Z such that $W \cap Y \subseteq B_2(x)$. By the regularity of Z there exists an open neighborhood V of $(x, 0)$ in Z such that $\text{cl}_Z(V) \subseteq W$. Find $m \in \mathbb{N}$ such that $B_m(x) \subset V$. Assume that the set $\{q_x(m)\} \times B$ converges to some $b \in Z$. Then $b \in \text{cl}_Z(V) \subseteq W$. Recall that $W \cap Y \subseteq B_2(x)$ and $B_2(x) \cap (\{q_x(m)\} \times \mathbb{N}) = \{q_x(m)\} \times (\mathbb{N} \setminus S)$. Since

¹This space was originally constructed by van Douwen and Przymusiński [9] in order to investigate remainders of Čech-Stone compactifications.

the set $B \cap S$ is infinite, the open set W contains no open neighborhood of b which contradicts the choice of W . $\square$

Fix any enumeration $\mathbb{Q} = \{q_n : n \in \omega\}$. Since the set $Q_0 \stackrel{\text{def}}{=} \{q_0\} \times \mathbb{N}$ is open and discrete in a feebly compact space Z , there exists an accumulation point z_0 of Q_0 . Since Z is first-countable, there exists $A_0 \subseteq \mathbb{N}$ such that the sequence $\{(q_0, n) : n \in A_0\}$ converges to z_0 . Assume that for each $i \leq n$ we constructed a subset A_i of $\mathbb{N}$ satisfying:

- (1) $A_i \subset A_j$ for every $i \leq j$;
- (2) the sequence $\{(q_i, n) : n \in A_i\}$ converges to some $z_i \in Z$.

Consider the discrete open set $Q_{n+1} \stackrel{\text{def}}{=} \{q_{n+1}\} \times A_n$. Since Z is first-countable and feebly compact, there exists an infinite subset $A_{n+1} \subset A_n$ such that the sequence $\{(q_{n+1}, n) : n \in A_{n+1}\}$ converges to some point $z_{n+1} \in Z$.

After completing the induction we obtain a decreasing chain $\{A_i : i \in \omega\}$ of infinite subsets of $\mathbb{N}$ and a sequence of points $\{z_i : i \in \omega\}$ such that z_i is the limit of the sequence $\{(q_i, n) : n \in A_i\}$. Let A be any pseudointersection of the family $\{A_i : i \in \omega\}$, i.e. $A \subseteq^* A_i$ for all $i \in \omega$. It is easy to see, that the sequence $\{(q_i, n) : n \in A\}$ converges to z_i for each $i \in \omega$. But this contradicts Claim 5.3. $\square$

The following trivial fact will be useful in the next proposition:

Lemma 5.4. *Suppose that $A \cup B$ is a G_δ subset of a Hausdorff space X . If B is countable, then A is G_δ .*

Proof. Let $B \setminus A = \{b_n : n \in \omega\}$ and $A \cup B = \bigcap_{n \in \omega} U_n$, where U_n is open for each $n \in \omega$. Then the set $V_n \stackrel{\text{def}}{=} U_n \setminus \{b_n\}$ is open for each $n \in \omega$, and $\bigcap_{n \in \omega} V_n = A$, i.e. A is G_δ . $\square$

For each $i \in \omega$ let $\omega^{\leq i} = \bigcup_{n < i} \omega^n$ and $\omega^{< i} = \bigcup_{n < i} \omega^n$. Clearly, the sets $\omega^{\leq i}$ and $\omega^{< i}$ ordered by the inclusion are subtrees of the complete ω -ary tree. Recall that the definition of a Pixley-Roy hyperspace is given in Section 2.

The proof of the next result uses the ideas of Bell [5].

Proposition 5.5. *Let X be a subspace of the Cantor space which is not a λ -set. Then the Pixley-Roy hyperspace $\text{PR}(X)$ is a zero-dimensional Tychonoff first-countable space that doesn't embed densely into first-countable regular feebly-compact spaces.*

Proof. Let A be a countable infinite subset of X which is not G_δ . Seeking a contradiction, assume that $\text{PR}(X)$ embeds densely into a regular first-countable feebly compact space Z . Fix any countable dense subset A' of X . Lemma 5.4 implies that $Q = A \cup A'$ is not G_δ . Fix an increasing sequence $\{F_k^\emptyset : k \in \omega\}$ of finite subsets of Q such that $\bigcup_{k \in \omega} F_k^\emptyset = Q$. Let $B_\emptyset \stackrel{\text{def}}{=} X$. Consider the family $\{\text{cl}_Z[F_k^\emptyset, X] : k \in \omega\}$. Taking into account that $\text{PR}(X)$ is dense in Z and $[F_k^\emptyset, B_\emptyset]$ is clopen in $\text{PR}(X)$, it is easy to check that $[F_k^\emptyset, B_\emptyset] \subset \text{int}_Z(\text{cl}_Z[F_k^\emptyset, B_\emptyset])$ for all $k \in \omega$. So, the family $\{\text{int}_Z(\text{cl}_Z[F_k^\emptyset, B_\emptyset]) : k \in \omega\}$ consists of open subsets of Z and is centered. By feeble compactness of Z , there exists

$$z_\emptyset \in \bigcap_{k \in \omega} \text{cl}_Z(\text{int}_Z(\text{cl}_Z[F_k^\emptyset, B_\emptyset])) = \bigcap_{k \in \omega} \text{cl}_Z[F_k^\emptyset, B_\emptyset].$$

Fix a countable open neighborhood base $\{O_k^\emptyset : k \in \omega\}$ at $z_\emptyset$. Since $\text{PR}(X)$ is dense in Z , for each $k \in \omega$ there exists a basic clopen subset $[G_k^\emptyset, W_k^\emptyset]$ of $\text{PR}(X)$ such that $[G_k^\emptyset, W_k^\emptyset] \subseteq O_k^\emptyset \cap [F_k^\emptyset, B_\emptyset]$. Clearly, $F_k^\emptyset \subseteq G_k^\emptyset$ and the sequence $\{G_k^\emptyset : k \in \omega\} \subset \text{PR}(X)$ converges to $z_\emptyset$.

Suppose that for some $i \geq 0$ we accomplished the following things:

- (1) constructed a subtree $\mathbb{T}_i$ of $\omega^{\leq i}$;
- (2) constructed a family $\{B_s : s \in \mathbb{T}_i\}$ of clopen subsets of X satisfying the following conditions:

- (2.1) for every $s \in \mathbb{T}_i \cap \omega^{<i}$, $Q_s \stackrel{\text{def}}{=} Q \cap B_s \subseteq \bigcup_{n \in \beta_s} B_{s \smallfrown n} \subseteq B_s$, where $\beta_s \stackrel{\text{def}}{=} \{n \in \omega : s \smallfrown n \in \mathbb{T}_i\} \in \omega + 1$;
- (2.2) $B_{s \smallfrown n} \cap B_{s \smallfrown m} = \emptyset$ for every $s \in \mathbb{T}_i \cap \omega^{<i}$ and distinct $n, m \in \beta_s$;
- (2.3) for each $s \in \mathbb{T}_i$ the diameter of B_s doesn't exceed $1/|s|$.
- (3) $\forall s \in \mathbb{T}_i$ fixed an increasing sequence $\{F_k^s : k \in \omega\}$ of finite subsets of $Q_s \stackrel{\text{def}}{=} B_s \cap Q$ such that $\bigcup_{k \in \omega} F_k^s = Q_s$;
- (4) $\forall s \in \mathbb{T}_i$ fixed a point $z_s \in \bigcap_{k \in \omega} \text{cl}_Z[F_k^s, B_s]$ and a nested open neighborhood base $\{O_k^s : k \in \omega\}$ at z_s ;
- (5) $\forall s \in \mathbb{T}_i$, $\forall k \in \omega$ fixed a subset $[G_k^s, W_k^s] \subset O_k^s \cap [F_k^s, B_s]$ such that the following condition holds:
 - (5.1) $\forall s \in \omega^{<i} \cap \mathbb{T}_i$, $\forall n \in \beta_s$, $\exists k(s, n) \geq |s| + 1$ such that $B_{s \smallfrown n} \subseteq \bigcap_{j \leq |s|} W_{k(s, n)}^{s \smallfrown j}$.

Fix any $s \in \mathbb{T}_i \cap \omega^i$. Since B_s is open in X and Q is dense, the set $Q_s \stackrel{\text{def}}{=} Q \cap B_s$ is not empty. Note that since X could contain isolated points, it is possible that $|Q_s| < \omega$. So, let us fix an enumeration $Q_s = \{q_n^s : n \in \alpha_s\}$, where $\alpha_s \in \omega + 1$. Observe that $q_0^s \in Q \cap B_{s \smallfrown j}$ for all $j \leq |s|$. Since $\{F_k^{s \smallfrown j} : k \in \omega\}$ is an increasing sequence of finite subsets of $Q \cap B_{s \smallfrown j}$, there exists $n_j \in \omega$ such that $q_0^s \in F_{n_j}^{s \smallfrown j}$ for all $n \geq n_j$. Put $k(s, 0) = \max\{n_j : j \leq |s| + 1\}$ and note that the open set $\bigcap_{j \leq |s|} W_{k(s, 0)}^{s \smallfrown j}$ contains q_0^s . So fix an arbitrary clopen set $B_{s \smallfrown 0} \subseteq \bigcap_{j \leq |s|} W_{k(s, 0)}^{s \smallfrown j}$ which contains q_0^s and has diameter $\leq 1/(i + 1)$. By condition (5), we get that $W_{k(s, 0)}^{s \smallfrown 0} \subseteq B_s$ which implies that $B_{s \smallfrown 0} \subseteq B_s$. If $Q_s \subset B_{s \smallfrown 0}$, we put $\beta_s = 1$ and move to another $s \in \mathbb{T}_i \cap \omega^i$. If $Q_s \setminus B_{s \smallfrown 0} \neq \emptyset$, find minimal $m \in \omega$ such that $q_m^s \notin B_{s \smallfrown 0}$. Similarly as above there exists $k(s, 1) \geq |s| + 1$ such that the open set $\bigcap_{j \leq |s|} W_{k(s, 1)}^{s \smallfrown j}$ contains q_m^s . Since the set $B_{s \smallfrown 0}$ is clopen and doesn't contain q_m^s , we can find a clopen neighborhood $B_{s \smallfrown 1}$ of q_m^s of diameter $\leq 1/(i + 1)$ which is contained in $(B_s \setminus B_{s \smallfrown 0}) \cap \bigcap_{j \leq |s|} W_{k(s, 1)}^{s \smallfrown j}$. If $Q_s \subset B_{s \smallfrown 0} \cup B_{s \smallfrown 1}$, we put $\beta_s = 2$ and move to another $s \in \mathbb{T}_i \cap \omega^i$. Otherwise, we can find minimal $m \in \omega$ such that $q_m^s \in B_s \setminus (B_{s \smallfrown 0} \cup B_{s \smallfrown 1})$ and repeat the previous steps. Proceeding this way we shall construct a family $\{B_{s \smallfrown k} : k \in \beta_s \in \omega + 1\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3) and (5.1). Note that $\beta_s \leq \alpha_s$ can be finite. Let $\mathbb{T}_{i+1}^s = \mathbb{T}_i \cup \{s \smallfrown k : k \in \beta_s\}$.

After making the above construction for all $s \in \mathbb{T}_i \cap \omega^i$, we obtain a tree $\mathbb{T}_{i+1} = \bigcup_{s \in \mathbb{T}_i \cap \omega^i} \mathbb{T}_{i+1}^s$ and a family $\{B_{s \smallfrown k} : k \in \beta_s \text{ and } s \in \mathbb{T}_i \cap \omega^i\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3) and (5.1).

For each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ fix an increasing sequence $\{F_k^s : k \in \omega\}$ of finite subsets of Q_s such that $\bigcup_{k \in \omega} F_k^s = Q_s$. Analogous arguments to those we used to find $z_\emptyset$ imply that for each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ there exists a point $z_s \in \bigcap_{k \in \omega} \text{cl}_Z[F_k^s, B_s]$. By the first-countability of Z , we can fix a nested open neighborhood base $\{O_k^s : k \in \omega\}$ at z_s . For each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ and $k \in \omega$ fix a subset $[G_k^s, W_k^s] \subset O_k^s \cap [F_k^s, B_s]$. This finishes step $i + 1$ of the induction.

Upon completing the induction we obtain a tree $\mathbb{T} = \bigcup_{i \in \omega} \mathbb{T}_i$, and a family $\{B_s : s \in \mathbb{T}\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3) and (5.1). Also, for each $s \in \mathbb{T}$ we found an element $z_s \in Z$ which satisfies condition (4) and constructed the families $\{W_k^s : k \in \omega, s \in \mathbb{T}\}$ (consisting of open subsets of X) and $\{G_k^s : k \in \omega, s \in \mathbb{T}\}$ (consisting of finite subsets of X), which satisfy conditions (5) and (5.1).

Let

$$P = \bigcap_{i \in \omega} \bigcup_{s \in \omega^i \cap \mathbb{T}} B_s.$$

By the construction, P is a G_δ -set containing Q . Since $|P| > \omega$, we can fix a point

$$(1) \quad p_\infty \in P \setminus \bigcup_{s \in \mathbb{T}} \bigcup_{k \in \omega} G_k^s.$$

It is easy to see that there exists $t \in \omega^\omega$ such that $p_\infty \in B_{t|i}$ for any $i \in \omega$. Note that since the diameter of $B_{t|i}$ doesn't exceed $1/i$, we get that the family $\{B_{t|i} : i \in \omega\}$ forms an open neighborhood base at p_∞ in X . It follows that the family $\{[\{p_\infty\}, B_{t|i}] : i \in \omega\}$ forms an open neighborhood base of $\{p_\infty\}$ in $\text{PR}(X)$. Put $R = \text{int}_Z(\text{cl}_Z([\{p_\infty\}, X]))$. One can check that $R \cap \text{PR}(X) = [\{p_\infty\}, X]$. By the regularity of Z , there exists an open in Z neighborhood W of $\{p_\infty\}$ such that $\text{cl}_Z(W) \subseteq R$. Thus, there exists $i \in \omega$ such that $\text{cl}_Z([\{p_\infty\}, B_{t|i}]) \subseteq R$.

Claim 5.6. $z_{t|i} \in \text{cl}_Z([\{p_\infty\}, B_{t|i}])$.

Proof. Pick any integer $l > i$. Using condition (5.1) for $s = t|l$ and $n = t(l)$ we can find an integer $k(t|l, t(l)) \geq l + 1$ such that $B_{t|(l+1)} \subseteq W_{k(t|l, t(l))}^{t|i}$. Then $p_\infty \in W_k^{t|i}$ for infinitely many $k \in \omega$. It follows that $[\{p_\infty\}, B_{t|i}] \cap [G_k^{t|i}, W_k^{t|i}] \neq \emptyset$ for infinitely many k . Since for each $k \in \omega$, $[G_k^{t|i}, W_k^{t|i}] \subset O_k^{t|i}$ and the family $\{O_k^{t|i} : k \in \omega\}$ forms an open neighborhood base at $z_{t|i}$, we deduce that $z_{t|i} \in \text{cl}_Z([\{p_\infty\}, B_{t|i}])$. $\square$

Since $\text{cl}_Z([\{p_\infty\}, B_{t|i}]) \subseteq R$, Claim 5.6 implies that $z_{t|i} \in R$. Condition (5) implies that for each $s \in \mathbb{T}$ the sequence $\{G_k^s : k \in \omega\} \subseteq \text{PR}(X)$ converges to z_s . Since the set R is an open neighborhood of $z_{t|i}$, there exists $k \in \omega$ such that $G_k^{t|i} \in R \cap \text{PR}(X) = [\{p_\infty\}, X]$, which is impossible, as $p_\infty \notin \bigcup_{s \in \mathbb{T}} \bigcup_{k \in \omega} G_k^s$ (see Equation 1). $\square$

By [7, Corollary 8.51] there exists a subspace A of the Cantor space which is not a λ -set and has cardinality $\mathfrak{b}$. Note that $|\text{PR}(A)| = |A| = \mathfrak{b}$. Hence Proposition 5.5 implies the following:

Corollary 5.7. *There exists a Tychonoff zero-dimensional first-countable space X of cardinality $\mathfrak{b}$ which doesn't embed densely into regular first-countable feebly compact spaces.*

6. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. We need to show that the following assertions are equivalent:

- (1) $\omega_1 = \mathfrak{c}$.
- (2) Every first-countable Tychonoff space of weight $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.
- (3) Each separable first-countable locally compact normal space of cardinality $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.

Proof. Implication (1) $\Rightarrow$ (2) follows from the fact that Each Tychonoff second-countable space is a subspace of the Tychonoff cube $[0, 1]^\omega$. Implication (2) $\Rightarrow$ (3) follows from the fact that the weight of a first-countable space doesn't exceed its cardinality. Proposition 2.11 from [4] implies the existence of a separable normal locally compact first-countable space X which contains a homeomorphic copy of the cardinal ω_1 endowed with the order topology. Since the only compactification of ω_1 is $\omega_1 + 1$, which is not first-countable, we deduce that X cannot be embedded into a Hausdorff first-countable compact space. $\square$

Proof of Theorem 3.2. We need to show that the following statements are equivalent.

- (1) $\mathfrak{b} = \mathfrak{c}$.
- (2) Every Hausdorff, locally compact, first-countable space of weight $< \mathfrak{c}$ can be (densely) embedded in a Hausdorff first-countable locally compact countably compact space.
- (3) Every Hausdorff, locally compact, first-countable space of cardinality $< \mathfrak{c}$ can be (densely) embedded in a Hausdorff first-countable countably compact space.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 4.9. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication (2) $\Rightarrow$ (3) follows. Taking into account

that Mrowka spaces are first-countable and locally compact, implication (3) $\Rightarrow$ (1) follows from Proposition 5.1. $\square$

Proof of Theorem 3.3. We need to show that the following assertions are equivalent.

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) Every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.
- (3) Every regular first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.

Proof. Implication (1) $\Rightarrow$ (2) follows from Proposition 4.1. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication (2) $\Rightarrow$ (3) holds true.

(3) $\Rightarrow$ (1). Proposition 5.1 implies that $\mathfrak{b} = \mathfrak{c}$. Proposition 5.2 implies that $\mathfrak{s} = \mathfrak{c}$. $\square$

Proof of Theorem 3.4. We need to show that the following assertions are equivalent.

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) Every Hausdorff zero-dimensional first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a Hausdorff zero-dimensional first-countable countably compact space.
- (3) Every Hausdorff zero-dimensional first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded into a Hausdorff zero-dimensional first-countable countably compact space.

Proof. Implication (1) $\Rightarrow$ (2) follows from Proposition 4.3. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication (2) $\Rightarrow$ (3) holds true.

(3) $\Rightarrow$ (1) Proposition 5.1 implies that $\mathfrak{b} = \mathfrak{c}$. Recall that a Tychonoff first-countable space Y constructed in the proof of Proposition 5.2 has cardinality $\mathfrak{s}$. If $\mathfrak{s} < \mathfrak{c}$, then the space Y is zero-dimensional (as every Tychonoff space of cardinality $< \mathfrak{c}$ is zero-dimensional), and Y doesn't embed into regular first-countable countably compact spaces, which contradicts assertion (3). Hence $\mathfrak{s} = \mathfrak{c}$, i.e. assertion (1) holds. $\square$

In order to prove Theorem 3.5 we need the following auxiliary lemma.

Lemma 6.1. *There exists a regular zero-dimensional separable first-countable non-normal space of cardinality ω_1 .*

Proof. We shall use ideas of Nyikos and Vaughan [23]. Consider a Hausdorff (ω_1, ω_1) gap, i.e. a family $\{A_\alpha : \alpha \in \omega_1\} \cup \{B_\alpha : \alpha \in \omega_1\} \subset [\omega]^\omega$ which satisfies the following conditions:

- $A_\alpha \subset^* B_\beta$ for any $\alpha, \beta \in \omega_1$;
- $A_\alpha \subset^* A_\beta$ for any $\alpha < \beta < \omega_1$;
- $B_\alpha \subset^* B_\beta$ for any $\beta < \alpha < \omega_1$;
- there exists no $C \in [\omega]^\omega$ such that $A_\alpha \subset^* C \subset^* B_\alpha$ for all $\alpha \in \omega_1$.

For a finite set F and ordinals $\alpha < \beta < \omega_1$ let us introduce two sorts of sets:

$$A(\alpha, \beta, F) = \{A_\xi \mid \alpha < \xi \leq \beta\} \cup ((A_\beta \setminus A_\alpha) \setminus F) \quad \text{and}$$

$$B(\alpha, \beta, F) = \{B_\xi \mid \alpha < \xi \leq \beta\} \cup ((B_\alpha \setminus B_\beta) \setminus F).$$

Let N be the set $\{A_\alpha : \alpha \in \omega_1\} \cup \{B_\alpha : \alpha \in \omega_1\} \cup \omega$ endowed with the topology τ defined by the subbase:

$$\mathcal{B} = \{\{n\} : n \in \omega\} \cup \{A_0\} \cup \{B_0\} \cup \{A(\alpha, \beta, F) : \alpha < \beta < \omega_1, F \in [\omega]^{<\omega}\} \cup \{B(\alpha, \beta, F) : \alpha < \beta < \omega_1, F \in [\omega]^{<\omega}\}.$$

It is straightforward to check that the space N is regular, separable, first-countable and has cardinality ω_1 . Also, the closed sets $\{A_\alpha : \alpha \in \omega_1\}$ and $\{B_\alpha : \alpha \in \omega_1\}$ cannot be separated by disjoint open sets. Hence the space N is not normal. $\square$

Proof of Theorem 3.5. We need to show the equivalence of the following assertions:

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) Every regular separable first-countable non-normal space of weight $< \mathfrak{c}$ embeds into a Nyikos $\mathbb{R}$ -rigid space.
- (3) Every regular separable first-countable non-normal space of cardinality $< \mathfrak{c}$ embeds into a Nyikos $\mathbb{R}$ -rigid space.

Proof. (1) $\Rightarrow$ (2). Let X be a regular separable first countable non-normal space of weight $< \mathfrak{c}$. Applying Jones machine (see [3] or [18] for details) we embed X into a first-countable regular separable space $J(X)$ of weight $< \mathfrak{c}$ containing two points a, b which cannot be separated by any real-valued continuous function. Applying the modified van Douwen's extension $E_{a,b}(J(X))$ of $J(X)$ (see Section 5 in [3]) we embed $J(X)$ into a regular separable first-countable $\mathbb{R}$ -rigid space of weight $< \mathfrak{c}$. By Theorem 4.1 we get a regular first-countable countably compact space Z which contains the separable $\mathbb{R}$ -rigid space $E_{a,b}(J(X))$ as a dense subspace. Hence Z is an $\mathbb{R}$ -rigid Nyikos space which contains a homeomorphic copy of X .

The implication (2) $\Rightarrow$ (3) holds true, because the weight of an infinite first-countable space doesn't exceed its cardinality.

(3) $\Rightarrow$ (1). Let N_1 be the topological sum of the space N constructed in Lemma 6.1 and the Mrowka space $\psi(\mathcal{A})$ constructed in Proposition 5.1. It is clear that N_1 is a regular separable first-countable non-normal space of cardinality $\mathfrak{b}$ which doesn't embed into Nyikos spaces. This example implies that $\mathfrak{b} = \mathfrak{c}$.

Let N_2 be the topological sum of the space N constructed in Lemma 6.1 and the space Y constructed in Proposition 5.2. It can be checked that N_2 is a separable regular first-countable non-normal space of cardinality $\mathfrak{s}$ which doesn't embed into Nyikos spaces. Hence $\mathfrak{s} = \mathfrak{c}$. $\square$

Proof of Theorem 3.6. Assuming that $\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ we need to construct an $\mathbb{R}$ -rigid Nyikos space.

Proof. Consider the space N constructed in Lemma 6.1 and apply item (3) of Theorem 3.5 to it. $\square$

Proof of Corollary 3.7. We need to show that under PFA the following assertions hold:

- (1) Every normal Nyikos space is compact.
- (2) Every regular separable first-countable space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$ -rigid Nyikos space.

Proof. Assertion (1) follows from Theorem 1.2. In order to prove assertion (2) fix any regular separable first-countable space X of weight $< \mathfrak{c}$. Let N be the space constructed in Lemma 6.1. Recall that PFA implies $\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c}$. Hence the topological sum $X \cup N$ is a regular separable first-countable non-normal space of weight $< \mathfrak{c}$. By Theorem 3.5(2), $X \cup N$ embeds into an $\mathbb{R}$ -rigid Nyikos space. $\square$

In order to prove Theorem 3.8 we shall need the following auxiliary lemma.

Lemma 6.2. *Let X be a first-countable Tychonoff space of weight κ . Then for each $x \in X$ there exists an open neighborhood base $\mathcal{B}_x = \{U_n^x : n \in \omega\}$ at x such that $U_{n+1}^x \subseteq U_n^x$ for any $n \in \omega$, and $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a base of X of size κ satisfying the following condition:*

- (i) *for each $x \in X$ and $U_n^x \in \mathcal{B}_x$ there exists $f_{U_n^x} \in C(X, [0, 1])$ such that $f_{U_n^x} \upharpoonright_{X \setminus U_n^x} \equiv 0$ and $f_{U_n^x} \upharpoonright_{U_{n+1}^x} \equiv 1$.*

Proof. Let $\mathcal{D}$ be a base of X of size κ . Put

$$\mathcal{Y} = \{\langle D_0, D_1 \rangle \in \mathcal{D}^2 : \exists f_{\langle D_0, D_1 \rangle} \in C(X, [0, 1]) \text{ such that } f_{\langle D_0, D_1 \rangle} \upharpoonright_{D_1} \equiv 1 \text{ and } f_{\langle D_0, D_1 \rangle} \upharpoonright_{X \setminus D_0} \equiv 0\}.$$

Claim 6.3. *For every $x \in X$ and open neighborhood U of x there exists $\langle D_0, D_1 \rangle \in \mathcal{Y}$ such that $x \in D_1 \subset D_0 \subset U$.*

Proof. Fix any $x \in X$ and an open neighborhood U of x . Since $\mathcal{D}$ is a base, there exists an open set $D_0 \in \mathcal{D}$ such that $x \in D_0 \subseteq U$. Then there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_{X \setminus D_0} \equiv 0$ and $h(x) = 1$. Since the set $h^{-1}([1/2, 1])$ contains x in its interior, there exists $D_1 \in \mathcal{D}$ such that $D_1 \subset h^{-1}([1/2, 1])$. Clearly, $f(z) \stackrel{\text{def}}{=} 2 \min\{h(z), 1/2\} \in C(X, [0, 1])$. It is easy to see that $f|_{D_1} \equiv 1$ and $f|_{X \setminus D_0} \equiv 0$, i.e. $\langle D_0, D_1 \rangle \in \mathcal{Y}$. $\square$

Fix any $x \in X$ and an open neighborhood base $\mathcal{V} = \{V_n : n \in \omega\} \subset \mathcal{D}$ of x such that $V_{n+1} \subset V_n$ for all $n \in \omega$. Wlog we can assume that $V_0 = X$. Let $U_0^x = X$ and $f_{U_0^x}$ be the constant map from X to $[0, 1]$ with value 1. Claim 6.3 implies that there exist a pair $\langle D_0^1, D_1^1 \rangle \in \mathcal{Y}$ such that $x \in D_0^1 \subset D_1^1 \subseteq V_1$ and a function $f_{\langle D_0^1, D_1^1 \rangle} \in C(X, [0, 1])$ such that $f_{\langle D_0^1, D_1^1 \rangle}|_{D_1^1} \equiv 1$ and $f_{\langle D_0^1, D_1^1 \rangle}|_{X \setminus D_0^1} \equiv 0$. Put $U_1^x = D_0^1$ and $f_{U_1^x} = f_{\langle D_0^1, D_1^1 \rangle}$. Using Claim 6.3 again we can find a pair $\langle D_0^2, D_1^2 \rangle \in \mathcal{Y}$ such that $x \in D_0^2 \subset D_1^2 \subseteq V_2 \cap D_1^1$ and a function $f_{\langle D_0^2, D_1^2 \rangle} \in C(X, [0, 1])$ such that $f_{\langle D_0^2, D_1^2 \rangle}|_{D_1^2} \equiv 1$ and $f_{\langle D_0^2, D_1^2 \rangle}|_{X \setminus D_0^2} \equiv 0$. Put $U_2^x = D_0^2$ and $f_{U_2^x} = f_{\langle D_0^2, D_1^2 \rangle}$. Proceeding this way, for each $x \in X$ we obtain an open neighborhood base $\{U_n^x : n \in \omega\}$ at x and a family $\{f_{U_n^x} : U_n^x \in \mathcal{B}_x\} \subset C(X, [0, 1])$ such that $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ satisfies condition (i). Since $\mathcal{B} \subseteq \mathcal{D}$ we get that $|\mathcal{B}| \leq \kappa$. $\square$

Recall that by $\heartsuit$ we denote the following consistent assumption: “ $\mathfrak{b} = \mathfrak{c}$ and there exists a simple $P_{\mathfrak{c}}$ -point in $\beta(\omega)$ ”.

Proof of Theorem 3.8. We need to show that assuming $(\heartsuit)$ every Tychonoff first-countable space X of weight $\kappa < \mathfrak{c}$ embeds densely into a Tychonoff first-countable countably compact space.

Proof. Let X be a first-countable Tychonoff space such that $w(X) < \mathfrak{c}$. Without loss of generality we can assume that the underlying set of X is disjoint with $\mathfrak{c}$. The first countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathcal{D} \cup [\mathfrak{c}]^\omega \rightarrow \mathfrak{c}$ such that $h(a) \geq \sup(a)$ for any $a \in [\mathfrak{c}]^\omega$. Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on $X_\alpha \subseteq X \cup \alpha$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α . At the end, we will show that the space $Y_{\mathfrak{c}}$ has the desired properties.

Let $X_0 = X$. By Lemma 6.2, there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} \mathcal{B}_0^x$ of size $< \mathfrak{c}$ of the topology on X , where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is a nested open neighborhood base at x and satisfies the following: $U_{0,0}^x = X$; for every $n > 0$ and $U_{n,0}^x \in \mathcal{B}_0$ there exists a continuous function $f_{U_{n,0}^x} : X \rightarrow [0, 1]$ such that $f_{U_{n,0}^x}|_{X \setminus U_{n,0}^x} \equiv 0$, and $f_{U_{n,0}^x}|_{U_{n+1,0}^x} \equiv 1$. Also put $f_{U_{0,0}^x} \equiv 1$.

Assume that for each $\alpha < \xi$ the Tychonoff first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the topology τ_α on a set $X_\alpha \subseteq X \cup \alpha$, where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is a nested open neighborhood base at x . Additionally assume that $X_\alpha \subset X_\beta$ for any $\alpha \in \beta$, and we have in parallel constructed a family $\Phi_\alpha = \{f_{U_{n,\alpha}^x} : U_{n,\alpha}^x \in \mathcal{B}_\alpha\}$, where for each $U_{n,\alpha}^x \in \mathcal{B}_\alpha$ the function $f_{U_{n,\alpha}^x} : Y_\alpha \rightarrow [0, 1]$ is continuous and $f_{U_{n,\alpha}^x}|_{X_\alpha \setminus U_{n,\alpha}^x} \equiv 0$. The families $\mathcal{B}_\alpha$ and Φ_α are presumed to satisfy the following conditions for all $\alpha < \xi$:

- (1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
- (2) $U_{0,\alpha}^x = X_\alpha$ for each $x \in X_\alpha$;
- (3) $f_{U_{0,\alpha}^x} \equiv 1$ for any $x \in X_\alpha$;
- (4) for every $x \in X_\alpha$ and $n \in \omega$ we have $f_{U_{n,\alpha}^x}|_{U_{n+1,\alpha}^x} \equiv 1$ and $f_{U_{n,\alpha}^x}|_{X_\alpha \setminus U_{n,\alpha}^x} \equiv 0$;
- (5) for every $\beta < \alpha$, $n \in \omega$ and $x \in X_\beta$ we have $U_{n,\beta}^x = U_{n,\alpha}^x \cap X_\beta$ and $\text{cl}_{Y_\alpha}(U_{n,\beta}^x) = \text{cl}_{Y_\alpha}(U_{n,\alpha}^x)$;
- (6) for every $\beta < \alpha$, $n \in \omega$ and $x \in X_\beta$ we have $f_{U_{n,\alpha}^x}|_{X_\beta} = f_{U_{n,\beta}^x}$.

There are three cases to consider:

- 1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ ;
- 2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ ;

3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$, $\mathcal{B}_\xi = \mathcal{B}_\gamma$ and $\Phi_\xi = \Phi_\gamma$.

2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_i\}_{i \in \omega}$. Fix any simple $P_\mathfrak{c}$ -point p which exists by the assumption. Since the set $h^{-1}(\gamma)$ is closed and discrete, for any $x \in X_\gamma$ there exists a positive integer $k(x)$ such that $|U_{k(x),\gamma}^x \cap h^{-1}(\gamma)| \leq 1$. Since $U_{0,\gamma}^x = X_\gamma$ and the ultrafilter p is free, for any $x \in X_\gamma$ there exists $m(x) < k(x)$ such that

$$F_x = \{i \in \omega : z_i \in U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x\} \in p.$$

Note that the set F_x actually depends on the pair $(U_{m(x),\gamma}^x, U_{m(x)+1,\gamma}^x)$.

Fix an arbitrary $f \in \Phi_\gamma$. By the compactness of $[0, 1]$, the ultrafilter $f(p)$ on $[0, 1]$ generated by the family $\{f(P) : P \in p\}$ converges to a point $b \in [0, 1]$. That is for each $n \in \mathbb{N}$

$$P_{f,n} = \{i \in \omega : f(z_i) \in (b - 1/n, b + 1/n)\} \in p.$$

Since p is a P -point, there exists $G_f \in p$ such that $G_f \subset^* P_{f,1/n}$ for every $n \in \mathbb{N}$. It is easy to see that the sequence $\{f(z_i) : i \in G_f\}$ converges to b .

Since $|\mathcal{B}_\gamma \cup \Phi_\gamma| < \mathfrak{c}$ we obtain that $|\{F_x : x \in X_\gamma\} \cup \{G_f : f \in \Phi_\gamma\}| < \mathfrak{c}$, and hence there exists $F \in p$ such that $F \subset^* F_x \cap G_f$ for any $x \in X_\gamma$ and $f \in \Phi_\gamma$.

For any $n \in \omega$ and $x \in X_\gamma$ set

$$b_n^x = \lim_{i \in F} f_{U_{n,\gamma}^x}(z_i)$$

and note that $b_n^x = 0$ for $n > m(x)$.

Let $d_\gamma = \{z_i : i \in F\}$. Condition (4) implies that $\text{cl}_{Y_\gamma}(U_{n+1,\gamma}^x) \subseteq U_{n,\gamma}^x$ for every $x \in X_\gamma$ and $n \in \omega$. Then for any $x \in X_\gamma$ we have

$$d_\gamma \subset^* U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x \subset U_{m(x),\gamma}^x \setminus \text{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x).$$

It follows that for every $x \in X_\gamma$ there exists a function $g_x \in \omega^F$ such that for all but finitely many $i \in F$ the following holds:

$$\text{cl}_{Y_\gamma}(U_{g_x(i),\gamma}^{z_i}) \subset U_{m(x),\gamma}^x \setminus \text{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x).$$

By the continuity, for any $f \in \Phi_\gamma$ there is a function $\pi_f \in \omega^F$ such that

$$(\circ_0) \quad f(U_{\pi_f(i),\gamma}^{z_i}) \subset (f(z_i) - \frac{1}{2^{i+1}}, f(z_i) + \frac{1}{2^{i+1}}) \cap [0, 1].$$

Let $h_\gamma \in \omega^F$ be a function such that $h_\gamma \geq^* g_x$ and $h_\gamma \geq^* \pi_f$ for each $x \in X_\gamma$ and $f \in \Phi_\gamma$. Such a function h_γ exists because $|\mathcal{B}_\gamma \cup \Phi_\gamma| < \mathfrak{b} = \mathfrak{c}$. Without loss of generality we can additionally assume that $U_{h_\gamma(i),\gamma}^{z_i} \cap U_{h_\gamma(j),\gamma}^{z_j} = \emptyset$ for each distinct $i, j \in \omega$.

We are in a position now to define the open neighborhood base $\mathcal{B}_\xi^\gamma$ at the point γ in the space Y_ξ : put

$$(\circ_1) \quad U_{0,\xi}^\gamma = X_\xi \quad \text{and} \quad U_{n,\xi}^\gamma = \bigcup_{i \in F \setminus n} U_{h_\gamma(i)+n,\gamma}^{z_i} \cup \{\gamma\}$$

for all $n \in \mathbb{N}$.

Next, we define $\mathcal{B}_\xi^x$ for $x \in X_\gamma$. Let $U_{0,\xi}^x = X_\xi$ for every $x \in X_\gamma$. For each $n \in \mathbb{N}$ let $U_{n,\xi}^x = U_{n,\gamma}^x$ if either $n \geq m(x) + 1$ or $n = m(x)$ and $b_n^x = 0$, otherwise, put $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$. It is easy to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ , and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ . Clearly, the first equality in condition (5) is satisfied for every $\beta < \alpha \leq \xi$ and $x \in X_\beta$. Fix an arbitrary $x \in X_\gamma$. The choice of g_x and the inequality $g_x \leq^* h_\gamma$ ensure the existence of large enough $n \in \omega$ such that $U_{m(x)+2,\xi}^x \cap U_{n,\xi}^\gamma = \emptyset$.

Thus Y_ξ is Hausdorff. The next claim proves the second equality in condition (5) for $\alpha = \gamma$, and therefore also for any $\alpha < \xi$.

Claim 6.4. *In the space Y_ξ , $U_{n,\gamma}^x$ is dense in $U_{n,\xi}^x$ for all $n \in \omega$ and $x \in X_\gamma$.*

Proof. It suffices to consider the case $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$. Thus either $n < m(x)$ or $n = m(x)$ and $b_n^x > 0$. In both of these cases $z_i \in U_{n,\gamma}^x$ for almost all $i \in F$, and hence $U_{n,\gamma}^x$ intersects all neighbourhoods of γ in Y_ξ . $\square$

Now we shall define a family $\Phi_\xi = \{f_{U_{n,\xi}^x} : U_{n,\xi}^x \in \mathcal{B}_\xi\}$, such that conditions (3), (4) and (6) are satisfied when α is replaced with ξ . Put $f_{U_{0,\xi}^x} \equiv 1$ for all $x \in X_\xi$. For each $x \in X_\gamma$ and $n \in \mathbb{N}$ set

$$f_{U_{n,\xi}^x} = f_{U_{n,\gamma}^x} \cup \{(\gamma, b_n^x)\}.$$

Conditions $(\circ_0)$, $(\circ_1)$ together with the choice of b_n^x as the limit of the sequence $\{f_{U_{n,\gamma}^x}(z_i) : i \in F\}$, ensure the continuity of functions $f_{U_{n,\xi}^x}$ for all $n \in \omega$ and $x \in X_\gamma$. Finally, we set

$$f_{U_{n,\xi}^\gamma}(y) = \begin{cases} f_{U_{h_\gamma(i)+n,\gamma}^{z_i}}(y) & \text{if } y \in U_{h_\gamma(i)+n,\gamma}^{z_i} \text{ and } i \in F \setminus n; \\ 1 & \text{if } y = \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Since the family $\{U_{h_\gamma(i),\gamma}^{z_i} : i \in \omega\}$ is locally finite in Y_γ , it remains to check that $f_{U_{n,\xi}^\gamma}$ is continuous at γ . For this it is enough to prove the first equality of condition (4) for $x = \gamma$ and $\xi = \alpha$. This follows from the following equation:

$$f_{U_{n,\xi}^\gamma}(U_{n+1,\xi}^\gamma) = \{1\} \cup \bigcup_{i \in F \setminus (n+1)} f_{U_{n,\xi}^\gamma}(U_{h_\gamma(i)+n+1,\gamma}^{z_i}) = \{1\} \cup \bigcup_{i \in F \setminus (n+1)} f_{U_{h_\gamma(i)+n,\gamma}^{z_i}}(U_{h_\gamma(i)+n+1,\gamma}^{z_i}) = \{1\}.$$

The fact that functions from Φ_ξ satisfy the first equation from (4) also for $x \in X_\gamma$ follows from (5) established in Claim 6.4 and from the continuity of functions in Φ_ξ . It is clear that any function from Φ_ξ satisfies the second equation of condition (4). Finally, it is a tedious routine to check that the families $\mathcal{B}_\xi$ and Φ_ξ satisfy conditions (1), (2), (3), and (6), which concludes case 2.

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$. For each $x \in X_\xi$ and $n \in \omega$ put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x \quad \text{and} \quad f_{U_{n,\xi}^x} = \bigcup_{\theta_x \leq \alpha < \xi} f_{U_{n,\alpha}^x}.$$

It is straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ on X_ξ , and the families $\Phi_\xi = \{f_{U_{n,\xi}^x} : U_{n,\xi}^x \in \mathcal{B}_\xi\}$ and $\mathcal{B}_\xi$ satisfy conditions (1)–(6). Thus, it remains to establish the continuity of functions in Φ_ξ , which we do next.

Assume to the contrary that there exists $f_{U_{n,\xi}^x} \in \Phi_\xi$ which is not continuous at a point $z \in X_\xi$. Then there exists a sequence $\{a_k\}_{k \in \omega}$ such that $\lim_{k \in \omega} a_k = z$ but $\lim_{k \in \omega} f_{U_{n,\xi}^x}(a_k) \neq f_{U_{n,\xi}^x}(z)$. For each $k \in \omega$ let $\mu_k \in \xi$ be such that $\{a_k, x, z\} \subset X_{\mu_k}$. Since Y_ξ is first-countable, X_0 is a dense subset of X_ξ and the functions f_{U_{n,μ_k}^x} , $k \in \omega$ are continuous on Y_{μ_k} , for each $k \in \omega$ we can pick $b_k \in X_0$ such that

$$\lim_{k \in \omega} b_k = \lim_{k \in \omega} a_k = z \quad \text{and} \quad |f_{U_{n,\mu_k}^x}(b_k) - f_{U_{n,\mu_k}^x}(a_k)| \leq 1/k.$$

For each $k \in \omega$, by condition (6), $f_{U_{n,\xi}^x} \upharpoonright_{X_{\mu_k}} = f_{U_{n,\mu_k}^x}$ implying that

$$(\bullet) \quad |f_{U_{n,\xi}^x}(b_k) - f_{U_{n,\xi}^x}(a_k)| = |f_{U_{n,\mu_k}^x}(b_k) - f_{U_{n,\mu_k}^x}(a_k)| \leq 1/k.$$

Since the function f_{U_{n,μ_0}^x} is continuous on Y_{μ_0} condition (6) implies the following:

$$\lim_{k \in \omega} f_{U_{n,\xi}^x}(b_k) = \lim_{k \in \omega} f_{U_{n,\mu_0}^x}(b_k) = f_{U_{n,\mu_0}^x}(\lim_{k \in \omega} b_k) = f_{U_{n,\mu_0}^x}(z) = f_{U_{n,\xi}^x}(z).$$

But then

$$0 \neq f_{U_{n,\xi}^x}(z) - \lim_{k \in \omega} f_{U_{n,\xi}^x}(a_k) = \lim_{k \in \omega} f_{U_{n,\xi}^x}(b_k) - \lim_{k \in \omega} f_{U_{n,\xi}^x}(a_k) = \lim_{k \in \omega} (f_{U_{n,\xi}^x}(b_k) - f_{U_{n,\xi}^x}(a_k)) = 0,$$

where the latter equality follows from equation $(\bullet)$. The obtained contradiction ensures that all functions in Φ_ξ are continuous, and thus concludes case 3.

By the construction, the space $Y_\mathfrak{c}$ is Tychonoff, first-countable and contains X as a dense subspace. Let A be any countable subset of $Y_\mathfrak{c}$. If the set $B = A \cap \mathfrak{c}$ is infinite, then consider $h(B) \in \mathfrak{c}$. By the construction, either B has an accumulation point in $Y_{h(B)}$ or $h(B)$ is an accumulation point of B in $Y_{h(B)+1}$. In both cases B has an accumulation point in $Y_\mathfrak{c}$. If $A \subset^* X$, then either it has an accumulation point in X , or A is closed and discrete in X . In the latter case either A has an accumulation point in $Y_{h(A)}$ or $h(A)$ is an accumulation point of A in $Y_{h(A)+1}$. Thus $Y_\mathfrak{c}$ is countably compact, which completes our proof. $\square$

Proof of Theorem 3.9. We need to show that the following assertions are equivalent:

- (1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
- (2) Every first-countable zero-dimensional Hausdorff space of weight $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.
- (3) Every first-countable zero-dimensional Hausdorff space of cardinality $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 4.11. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1). Corollary 5.7 implies that $\mathfrak{b} = \mathfrak{c}$. Assuming that $\mathfrak{s} < \mathfrak{c}$, we get that the normal first-countable space Y constructed in Proposition 5.2 has cardinality $\mathfrak{s} < \mathfrak{c}$ and, thus, is zero-dimensional. Then, by the assumption, Y embeds densely into a first-countable zero-dimensional pseudocompact space. But this contradicts Proposition 5.2. Hence $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$. $\square$

Proof of Theorem 3.10. We need to show that a subspace X of the Cantor space is a λ -set if and only if the Pixley-Roy hyperspace $\text{PR}(X)$ embeds densely into a first-countable pseudocompact space.

Proof. The implication $(\Rightarrow)$ follows from [5, Theorem 4.2].

The implication $(\Leftarrow)$ follows from Proposition 5.5. $\square$

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