BIHOM-Ω-ASSOCIATIVE ALGEBRAS AND RELATED STRUCTURES

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Abstract. In this paper, we first propose the concepts of BiHom-Ω-associative algebras, BiHom-Ω-dendriform algebras, BiHom-Ω-pre-Lie algebras and BiHom-Ω-Lie algebras. We then obtain a new BiHom-Ω-associative (resp. Lie) algebra by defining a new multiplication on a BiHom-Ω-associative (resp. Lie) algebra with the Rota-Baxter family of weight λ. In addition, we generalize the classical relationships of associative algebras, pre-Lie algebras, dendriform algebras and Lie algebras to the BiHom-Ω version.

Contents

1. Introduction

BiHom-algebras. The origin of Hom-structures may be found in the physics literature around 1990, concerning q-deformations of algebras of vector fields, especially Witt and Virasoro algebras, see for instance [6, 10, 12, 18, 22]. In the last few years, many articles have generalized the classical algebraic structures to Hom-algebraic structure, see for instance [4, 5, 8, 9, 11, 17]. A generalization has been given in [15], where the construction of a Hom-category including a group action led to the concept of BiHom-type algebras. Yau posed the twisting principle, a main tool for constructing (Bi)Hom-type algebras. Up to now, many concepts of BiHom-type algebras have been proposed, such as BiHom-associative algebras [15], BiHom-Lie algebras [15], BiHom-(tri)dendriform algebras [20], BiHom-pre-Lie algebras [19] and BiHom-PostLie algebras [3].

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Ω-algebras. The notion of associative algebras relative to a commutative semigroup Ω first proposed by Aguiar in [1]. The proposal of this concept has received widespread attention. Later, Das and Zhang et.al referred to Ω-relative associative algebras as Ω-associative algebras in articles [13, Definition 4.1] and [27], respectively. In this paper, we will continue to refer to Ω-relative algebras as Ω-algebras. In fact, Ω-type algebras are a generalization of the original type algebras which correspond to the case in which the semigroup Ω is a single point set. Prior to this study, there was another generalization of algebraic structures, namely family algebra. Rota-Baxter family algebras [14] were the first example of family algebraic structures. In recent, the concepts of (tri)dendriform family algebras [25] and pre-Lie family algebras [26] were proposed by Zhang, Gao and Manchon. Let us take the dendriform algebra as an example to illustrate the relationship between Ω-algebras and family algebras. In the definition of Ω-algebraic structures, if the operation $\prec_{\alpha,\beta}$ is independent of α and the operation $\succ_{\alpha,\beta}$ is independent of β in the Ω-dendriform algebra $(D, \prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega}$, then D reduces to a dendriform family algebra. In the definition of morphisms, the family algebra is a special case in the Ω-algebra where the map f_α is independent of α.

BiHom-Ω-algebras. Inspired by BiHom-associative algebras and Ω-associative algebras, we propose the concept of BiHom-Ω-associative algebras. It's not only a promotion of BiHom-associative algebra, but also a generlization of Ω-associative algebra. On the one hand, when the semigroup Ω is a trivial semigroup with one single element, a BiHom-Ω-associative algebra is precisely a BiHom-associative algebra. On the other hand, when the structure maps of a BiHom-Ω-associative algebra are all identity maps, the BiHom-Ω-associative algebra reduces to an Ω-associative algebra. In this paper, we mainly introduce the second generalized method. In order to better observe the relationships among different BiHom-Ω-type algebras studied in this paper, we give the following commutative diagram.

$$\begin{tikzcd}
\text{BiHom-}\Omega\text{-dendriform algebras} \arrow[r, "\text{Prop3.7}"] \arrow[dd, shift left, "\text{Prop2.17}"] & \text{BiHom-}\Omega\text{-pre-Lie algebras} \arrow[dd, shift left, "\text{Prop3.15}"] & \\
& & \text{BiHom-}\Omega\text{-PostLie algebras} \arrow[ul, "\text{Remk4.5 (a)}"'] \arrow[dl, shift left, "\text{Prop4.7}"] \\
\text{BiHom-}\Omega\text{-associative algebras} \arrow[r, "\text{Prop 3.16}"] \arrow[uu, shift left, "\text{Prop2.18}"] \arrow[uur, "\text{Prop3.8}"] & \text{BiHom-}\Omega\text{-Lie algebras.} \arrow[uu, shift left, "\text{Theorem3.19}"] \arrow[ur, shift left, "\text{Prop4.8}"] &
\end{tikzcd}$$

The outline of this paper. In Section 2, we first propose the concepts of BiHom-Ω-associative algebras and BiHom-Ω-dendriform algebras, then we study the relationship between them (Proposition 2.17 and 2.18). Also we prove that a new BiHom-Ω-associative algebra can be induced by a Rota-Baxter family of weight λ on the BiHom-Ω-associative algebra (Theorem 2.9). In Section 3, we mainly introduce BiHom-Ω-pre-Lie algebras and BiHom-Ω-Lie algebras and we also give the links between them (Proposition 3.15 and Theorem 3.19). Moreover, similar to Theorem 2.9, we obtain a new BiHom-Ω-Lie algebra by defining a new multiplication on a BiHom-Ω-Lie algebra with the Rota-Baxter family of weight λ (Theorem 3.17). In Section 4, we first introduce the concept of BiHom-Ω-PostLie algebra, then by studying the relationship between BiHom-Ω-PostLie algebras and BiHom-Ω-Lie algebras, we get Proposition 4.7 and Proposition 4.8, which are the generalizations of Theorem 3.17 and Theorem 3.19, respectively. Finally, we simply introduce the concept of BiHom-Ω-pre-Possion algebras.

Notation. Throughout this paper, we fix a commutative unitary ring $\mathbf{k}$, which will be the base ring of all algebras as well as linear maps. By an algebra we mean a unitary associative noncommutative algebra, unless the contrary is specified. Denote by Ω a semigroup, unless otherwise specified. For the composition of two maps p and q, we will write either $p \circ q$ or simply pq.

2. BiHom-Ω-associative algebras and BiHom-Ω-dendriform algebras

In this section, we mainly introduce the definitions of BiHom-Ω-associative algebras and BiHom-Ω-dendriform algebras, then we give some results of them.

2.1. **BiHom-Ω-associative algebras.** In this subsection, we first introduce the BiHom version of Ω-associative algebras and its Yau twist property, then we prove that a new BiHom-Ω-associative algebra can be induced by a Rota-Baxter family of weight λ on the BiHom-Ω-associative algebra. Now let's recall the related concepts of Ω-associative algebras.

Definition 2.1. [1] An **Ω-associative algebra** $(A, \cdot_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space A equipped with a family of operations $(\cdot_{\alpha,\beta} : A \times A \to A)_{\alpha,\beta\in\Omega}$ such that

$$(x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z = x \cdot_{\alpha,\beta\gamma} (y \cdot_{\beta,\gamma} z), \tag{1}$$

for all $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$.

Definition 2.2. [1] Let $(A, \cdot_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(A', \cdot'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-associative algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : A \to A'$ is called an **Ω-associative algebra morphism** if

$$f_{\alpha\beta}(x \cdot_{\alpha,\beta} y) = f_\alpha(x) \cdot'_{\alpha,\beta} f_\beta(y), \tag{2}$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$.

Inspired by the concepts of BiHom-associative algebras [15] and Ω-associative algebras [1], now we introduce the BiHom-Ω-associative algebras.

Definition 2.3. A **BiHom-Ω-associative algebra** $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space A equipped with two commuting families of linear maps $p_\alpha, q_\alpha : A \to A$ and a family of bilinear maps $(\bullet_{\alpha,\beta} : A \otimes A \to A)_{\alpha,\beta\in\Omega}$ such that

$$p_{\alpha\beta}(x \bullet_{\alpha,\beta} y) = p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y)\ ,\ q_{\alpha\beta}(x \bullet_{\alpha,\beta} y) = q_\alpha(x) \bullet_{\alpha,\beta} q_\beta(y), \quad \text{(multiplicativity)} \tag{3}$$

$$p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z) = (x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z), \quad \text{(BiHom-Ω-associativity)} \tag{4}$$

for all $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of A.

Obviously, if the structure maps of BiHom-Ω-associative algebra $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ are the identity maps, then A reduces to an Ω-associative algebra.

Definition 2.4. Let $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ and $(A', \bullet'_{\alpha,\beta}, p'_\alpha, q'_\alpha)_{\alpha,\beta\in\Omega}$ be two BiHom-Ω-associative algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : A \to A'$ is called a **BiHom-Ω-associative algebra morphism** if $(f_\alpha)_{\alpha\in\Omega}$ is an Ω-associative algebra morphism and

$$p'_\alpha \circ f_\alpha = f_\alpha \circ p_\alpha, \quad q'_\alpha \circ f_\alpha = f_\alpha \circ q_\alpha, \quad \text{for all } \alpha \in \Omega.$$

We give an example for BiHom-Ω-associative algebras as follows.

Example 2.5. Let maps $c : \Omega\times\Omega \to \mathbf{k}$, $\leftthreetimes : \Omega\times\mathbf{k} \to \mathbf{k}$, and $\rightthreetimes : \mathbf{k}\times\Omega \to \mathbf{k}$ satisfy the following conditions

$$\alpha\beta \leftthreetimes 1_k = (\alpha \leftthreetimes 1_k)(\beta \leftthreetimes 1_k), \quad 1_k \rightthreetimes \alpha\beta = (1_k \rightthreetimes \alpha)(1_k \rightthreetimes \beta),$$

$$c(\alpha, \beta)(1_k \rightthreetimes \gamma)c(\alpha\beta, \gamma) = c(\alpha, \beta\gamma)(\alpha \leftthreetimes 1_k)c(\beta, \gamma),$$

for all $\alpha, \beta, \gamma \in \Omega$, and 1_k is the unit of $\mathbf{k}$. We define the operations on the 2-dimensional unital space $\mathbf{k}\{e_1, e_2\}$:

$$\begin{aligned} p_\alpha(e_1) &:= (\alpha \leftthreetimes 1_k)e_1, \quad & p_\alpha(e_2) &:= (\alpha \leftthreetimes 1_k)e_2, \\ q_\alpha(e_1) &:= (1_k \rightthreetimes \alpha)e_1, \quad & q_\alpha(e_2) &:= (1_k \rightthreetimes \alpha)e_1, \\ e_1 \bullet_{\alpha,\beta} e_1 &:= c(\alpha, \beta)e_1, \quad & e_1 \bullet_{\alpha,\beta} e_2 &:= c(\alpha, \beta)e_1, \\ e_2 \bullet_{\alpha,\beta} e_1 &:= c(\alpha, \beta)e_2, \quad & e_2 \bullet_{\alpha,\beta} e_2 &:= c(\alpha, \beta)e_2. \end{aligned}$$

Then $(\mathbf{k}\{e_1, e_2\}, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-associative algebra.

Now we show that BiHom-Ω-associative algebras can be obtained from the classical Ω-associative algebras as follows.

Proposition 2.6. *Let $(A, \cdot_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-associative algebra. If $p_\alpha, q_\alpha : A \to A$ are two commuting Ω-associative algebra morphisms and we define the multiplication on A by*

$$x \bullet_{\alpha,\beta} y := p_\alpha(x) \cdot_{\alpha,\beta} q_\beta(y), \quad \text{for all } x, y \in A,\ \alpha, \beta \in \Omega.$$

Then $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-associative algebra, called the Yau twist of $(A, \cdot_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Proof. First, we prove the multiplicativity property. For $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, we have

$$\begin{aligned} p_{\alpha\beta}(x \bullet_{\alpha,\beta} y) &= p_{\alpha\beta}(p_\alpha(x) \cdot_{\alpha,\beta} q_\beta(y)) \\ &= p_\alpha^2(x) \cdot_{\alpha,\beta} p_\beta q_\beta(y) && \text{(by Eq. (2))} \\ &= p_\alpha^2(x) \cdot_{\alpha,\beta} q_\beta p_\beta(y) && \text{(by } p_\beta \circ q_\beta = q_\beta \circ p_\beta) \\ &= p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y). \end{aligned}$$

Similarly, we get $q_{\alpha\beta}(x \bullet_{\alpha,\beta} y) = q_\alpha(x) \bullet_{\alpha,\beta} q_\beta(y)$. Next, we prove the BiHom-Ω-associativity, we have

$$\begin{aligned} p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z) &= p_\alpha(x) \bullet_{\alpha,\beta\gamma} (p_\beta(y) \cdot_{\beta,\gamma} q_\gamma(z)) \\ &= p_\alpha^2(x) \cdot_{\alpha,\beta\gamma} q_{\beta\gamma}(p_\beta(y) \cdot_{\beta,\gamma} q_\gamma(z)) \\ &= p_\alpha^2(x) \cdot_{\alpha,\beta\gamma} (q_\beta p_\beta(y) \cdot_{\beta,\gamma} q_\gamma^2(z)) && \text{(by } (q_\alpha)_{\alpha\in\Omega} \text{ satisfying Eq. (2))} \\ &= (p_\alpha^2(x) \cdot_{\alpha,\beta} q_\beta p_\beta(y)) \cdot_{\alpha\beta,\gamma} q_\gamma^2(z) && \text{(by Eq. (1))} \\ &= p_{\alpha\beta}(p_\alpha(x) \cdot_{\alpha,\beta} q_\beta(y)) \cdot_{\alpha\beta,\gamma} q_\gamma^2(z) \\ &= p_{\alpha\beta}(x \bullet_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} q_\gamma^2(z) \\ &= (x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z). \end{aligned}$$

This completes the proof. □

The Yau twisting procedure for BiHom-Ω-associative algebras admits a more general form, which we state in the next result.

Proposition 2.7. *Let $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-associative algebra. If $p'_\alpha, q'_\alpha : A \to A$ are two BiHom-Ω-associative algebra morphisms and any two families of the maps $p_\alpha, q_\alpha, p'_\alpha, q'_\alpha$ commute with each other. Define the multiplication on A by*

$$x \bullet'_{\alpha,\beta} y := p'_\alpha(x) \bullet_{\alpha,\beta} q'_\beta(y),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \bullet'_{\alpha,\beta}, p_\alpha \circ p'_\alpha, q_\alpha \circ q'_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-associative algebra.

Proof. For $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, the multiplicativity is obvious. Now we only need to prove the BiHom-Ω-associativity.

$$\begin{aligned}
p_\alpha \circ p'_\alpha(x) \bullet'_{\alpha,\beta\gamma} (y \bullet'_{\beta,\gamma} z) &= p'_\alpha p_\alpha p'_\alpha(x) \bullet_{\alpha,\beta\gamma} q'_{\beta\gamma}(y \bullet'_{\beta,\gamma} z)\\
&= p_\alpha(p'_\alpha)^2(x) \bullet_{\alpha,\beta\gamma} q'_{\beta\gamma}(p'_\beta(y) \bullet_{\beta,\gamma} q'_\gamma(z)) \qquad (\text{by } p_\alpha \circ p'_\alpha = p_\alpha \circ' p_\alpha)\\
&= p_\alpha(p'_\alpha)^2(x) \bullet_{\alpha,\beta\gamma} (q'_\beta p'_\beta(y) \bullet_{\beta,\gamma} (q'_\gamma)^2(z))\\
&\qquad (\text{by } q'_{\beta\gamma} \text{ being a BiHom-}\Omega\text{-associative algebra morphism}))\\
&= ((p'_\alpha)^2(x) \bullet_{\alpha,\beta} q'_\beta p'_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma(q'_\gamma)^2(z) \qquad (\text{by Eq. } (4))\\
&= (p'_\alpha p'_\alpha(x) \bullet_{\alpha,\beta} p'_\beta q'_\beta(y)) \bullet_{\alpha\beta,\gamma} q'_\gamma q_\gamma q'_\gamma(z)\\
&\qquad (\text{by } q_\alpha, p'_\alpha, q'_\alpha \text{ commuting with each other})\\
&= p'_{\alpha\beta}(p'_\alpha(x) \bullet_{\alpha,\beta} q'_\beta(y)) \bullet_{\alpha\beta,\gamma} q'_\gamma q_\gamma q'_\gamma(z)\\
&\qquad (\text{by } p'_{\alpha\beta} \text{ being a BiHom-}\Omega\text{-associative algebra morphism}))\\
&= p'_{\alpha\beta}(x \bullet'_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q'_\gamma q_\gamma q'_\gamma(z)\\
&= (x \bullet'_{\alpha,\beta} y) \bullet'_{\alpha\beta,\gamma} q_\gamma \circ q'_\gamma(z).
\end{aligned}$$

This completes the proof. □

Let $\lambda \in \mathbf{k}$, a Rota-Baxter family of weight λ on the BiHom-Ω-algebra is defined as follows.

Definition 2.8. Let Ω be a semigroup. A **Rota-Baxter family of weight** λ on the BiHom-Ω-algebra $(A, \mu_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a collection of linear operators $(R_\alpha)_{\alpha\in\Omega}$ on A such that

$$\mu_{\alpha,\beta}(R_\alpha(x), R_\beta(y)) = R_{\alpha\beta}(\mu_{\alpha,\beta}(R_\alpha(x), y) + \mu_{\alpha,\beta}(x, R_\beta(y)) + \lambda\mu_{\alpha,\beta}(x, y)), \tag{5}$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$.

If further $(R_\alpha)_{\alpha\in\Omega}$ commute with the structure maps, then $(A, \mu_{\alpha,\beta}, p_\alpha, q_\alpha, R_\alpha)_{\alpha,\beta\in\Omega}$ is called a **Rota-Baxter family BiHom-Ω-algebra of weight** λ.

The main purpose of the following result is to show that a new BiHom-Ω-associative algebra can be constructed by the Rota-Baxter family of weight λ on the BiHom-Ω-associative algebra.

Theorem 2.9. *Let $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-associative algebra. If $(R_\alpha)_{\alpha\in\Omega}$ is a Rota-Baxter family of weight λ on A satisfying*

$$R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha \text{ and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha.$$

Define a new operation on A by

$$x \star_{\alpha,\beta} y := x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y,$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \star_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-associative algebra.

Proof. For any $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, we have $p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha$ and

$$\begin{aligned}
p_{\alpha\beta}(x \star_{\alpha,\beta} y) &= p_{\alpha\beta}(x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y)\\
&= p_{\alpha\beta}(x \bullet_{\alpha,\beta} R_\beta(y)) + p_{\alpha\beta}(R_\alpha(x) \bullet_{\alpha,\beta} y) + \lambda p_{\alpha\beta}(x \bullet_{\alpha,\beta} y)\\
&= p_\alpha(x) \bullet_{\alpha,\beta} p_\beta R_\beta(y) + p_\alpha R_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y) + \lambda p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y)\\
&\qquad \text{(by Eq. (3))}\\
&= p_\alpha(x) \bullet_{\alpha,\beta} R_\beta(p_\beta(y)) + R_\alpha(p_\alpha(x)) \bullet_{\alpha,\beta} p_\beta(y) + \lambda p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y)\\
&= p_\alpha(x) \star_{\alpha,\beta} p_\beta(y).
\end{aligned}$$

Similarly, we get $q_{\alpha\beta}(x \star_{\alpha,\beta} y) = q_\alpha(x) \star_{\alpha,\beta} q_\beta(y)$. On the one hand, we have

$$\begin{aligned}
(x \star_{\alpha,\beta} y) \star_{\alpha\beta,\gamma} q_\gamma(z) =& (x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y) \star_{\alpha\beta,\gamma} q_\gamma(z)\\
=& (x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} R_\gamma q_\gamma(z)\\
&+ R_{\alpha\beta}(x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&+ \lambda(x \bullet_{\alpha,\beta} R_\beta(y) + R_\alpha(x) \bullet_{\alpha,\beta} y + \lambda x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
=& (x \bullet_{\alpha,\beta} R_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma R_\gamma(z) + (R_\alpha(x) \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma R_\gamma(z)\\
&+ \lambda(x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma R_\gamma(z) + (R_\alpha(x) \bullet_{\alpha,\beta} R_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&+ \lambda(x \bullet_{\alpha,\beta} R_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma(z) + \lambda(R_\alpha(x) \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&+ \lambda^2(x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z) \qquad \text{(by Eq. (5) and } R_\gamma \circ q_\gamma = q_\gamma \circ R_\gamma)\\
=& p_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} R_\gamma(z)) + p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z))\\
&+ \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z)) + p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} z)\\
&+ \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} z) + \lambda p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z)\\
&+ \lambda^2 p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z). \quad \text{(by Eq. (4))}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p_\alpha(x) \star_{\alpha,\beta\gamma} (y \star_{\beta,\gamma} z) =& p_\alpha(x) \star_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z) + R_\beta(y) \bullet_{\beta,\gamma} z + \lambda y \bullet_{\beta,\gamma} z)\\
=& p_\alpha(x) \bullet_{\alpha,\beta\gamma} R_{\beta\gamma}(y \bullet_{\beta,\gamma} R_\gamma(z) + R_\beta(y) \bullet_{\beta,\gamma} z + \lambda y \bullet_{\beta,\gamma} z)\\
&+ R_\alpha p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z) + R_\beta(y) \bullet_{\beta,\gamma} z + \lambda y \bullet_{\beta,\gamma} z)\\
&+ \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z) + R_\beta(y) \bullet_{\beta,\gamma} z + \lambda y \bullet_{\beta,\gamma} z)\\
=& p_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} R_\gamma(z)) + p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z))\\
&+ p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} z) + \lambda p_\alpha R_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z)\\
&+ \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z)) + \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} z)\\
&+ \lambda^2 p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z) \qquad \text{(by Eq. (5) and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha).
\end{aligned}$$

By comparing the items of both sides, we get $(x \star_{\alpha,\beta} y) \star_{\alpha\beta,\gamma} q_\gamma(z) = p_\alpha(x) \star_{\alpha,\beta\gamma} (y \star_{\beta,\gamma} z)$. □

Remark 2.10. In Theorem 2.9, we notice that $(R_\alpha)_{\alpha\in\Omega}$ is also a Rota-Baxter family of weight λ for the BiHom-Ω-associative algebra $(A, \star_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$.

2.2. **BiHom-Ω-dendriform algebras.** In this subsection, we mainly introduce the BiHom-Ω-dendriform algebra, which is a generalization of Ω-dendriform algebras [1]. Further, we give the relationship between BiHom-Ω-dendriform algebras and BiHom-Ω-associative algebras. For this, let us first briefly recall the definition of Ω-dendriform algebras.

Definition 2.11. [1] An **Ω-dendriform algebra** $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space A equipped with two families of bilinear maps $(\prec_{\alpha,\beta},\ \succ_{\alpha,\beta}\colon A\times A\to A)_{\alpha,\beta\in\Omega}$ such that

$$(x \prec_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z = x \prec_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z + y \succ_{\beta,\gamma} z), \tag{6}$$

$$(x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z = x \succ_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z), \tag{7}$$

$$(x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z = x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z) \tag{8}$$

for all $x, y, z \in A,\ \alpha,\ \beta,\ \gamma \in \Omega$.

Definition 2.12. Let $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(A', \prec'_{\alpha,\beta}, \succ'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-dendriform algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : A \to A'$ is called an **Ω-dendriform algebra morphism** if

$$f_{\alpha\beta}(x \prec_{\alpha,\beta} y) = f_\alpha(x) \prec'_{\alpha,\beta} f_\beta(y), \quad f_{\alpha\beta}(x \succ_{\alpha,\beta} y) = f_\alpha(x) \succ'_{\alpha,\beta} f_\beta(y), \tag{9}$$

for all $x, y \in A,\ \alpha,\ \beta \in \Omega$.

Now, we generalize the above definitions of the Ω-dendriform algebra to BiHom-verision.

Definition 2.13. A **BiHom-Ω-dendriform algebra** $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space A equipped with two families of bilinear maps $(\prec_{\alpha,\beta}, \succ_{\alpha,\beta}\colon A\otimes A\to A)_{\alpha,\beta\in\Omega}$ and two commuting families of linear maps $p_\alpha, q_\alpha : A \to A$ such that

$$p_{\alpha\beta}(x \prec_{\alpha,\beta} y) = p_\alpha(x) \prec_{\alpha,\beta} p_\beta(y),\ p_{\alpha\beta}(x \succ_{\alpha,\beta} y) = p_\alpha(x) \succ_{\alpha,\beta} p_\beta(y), \tag{10}$$

$$q_{\alpha\beta}(x \prec_{\alpha,\beta} y) = q_\alpha(x) \prec_{\alpha,\beta} q_\beta(y),\ q_{\alpha\beta}(x \succ_{\alpha,\beta} y) = q_\alpha(x) \succ_{\alpha,\beta} q_\beta(y), \tag{11}$$

$$(x \prec_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q_\gamma(z) = p_\alpha(x) \prec_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z + y \succ_{\beta,\gamma} z), \tag{12}$$

$$(x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q_\gamma(z) = p_\alpha(x) \succ_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z), \tag{13}$$

$$p_\alpha(x) \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z) = (x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} q_\gamma(z), \tag{14}$$

for all $x,\ y,\ z \in A,\ \alpha,\ \beta \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of A.

Definition 2.14. Let $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ and $(A', \prec'_{\alpha,\beta}, \succ'_{\alpha,\beta}, p'_\alpha, q'_\alpha)_{\alpha,\beta\in\Omega}$ be two BiHom-Ω-dendriform algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : A \to A'$ is called a **BiHom-Ω-dendriform algebra morphism** if $(f_\alpha)_{\alpha\in\Omega}$ is an Ω-dendriform algebra morphism and

$$f_\alpha \circ p_\alpha = p'_\alpha \circ f_\alpha, \quad f_\alpha \circ q_\alpha = q'_\alpha \circ f_\alpha, \qquad \text{for all } \alpha \in \Omega.$$

Inspired by [20, Proposition 2.2], here we characterize the Yau twisting procedure for BiHom-Ω-dendriform algebras as follows.

Proposition 2.15. *Let $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-dendriform algebra. If $p_\alpha, q_\alpha : A \to A$ are two commuting Ω-dendriform algebra morphisms. Define the operations by*

$$x \prec'_{\alpha,\beta} y := p_\alpha(x) \prec_{\alpha,\beta} q_\beta(y), \quad x \succ'_{\alpha,\beta} y := p_\alpha(x) \succ_{\alpha,\beta} q_\beta(y),$$

for all $x, y \in A,\ \alpha,\ \beta \in \Omega$, then $(A, \prec'_{\alpha,\beta}, \succ'_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-dendriform algebra, called the Yau twist of $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Proof. For $x, y, z \in A,\ \alpha,\ \beta,\ \gamma \in \Omega$, we need to prove the structure maps satisfying Eqs. (10)-(11). Here we prove

$$\begin{aligned} p_{\alpha\beta}(x \prec'_{\alpha,\beta} y) =& p_{\alpha\beta}(p_\alpha(x) \prec_{\alpha,\beta} q_\beta(y)) \\ =& p_\alpha^2(x) \prec_{\alpha,\beta} p_\beta q_\beta(y) \qquad \text{(by } p_{\alpha\beta} \text{ satisfying Eq. (9))} \\ =& p_\alpha p_\alpha(x) \prec_{\alpha,\beta} q_\beta p_\beta(y) \qquad \text{(by } p_\beta \circ q_\beta = q_\beta \circ p_\beta) \end{aligned}$$

$$=p_\alpha(x) \prec'_{\alpha,\beta} p_\beta(y).$$

Other relations in Eqs. (10)-(11) are similar to prove. Next, we only need to prove Eq. (12) and Eqs. (13)-(14) are similar to prove by using Eqs. (7)-(8).

$$\begin{aligned}
(x \prec'_{\alpha,\beta} y) \prec'_{\alpha\beta,\gamma} q_\gamma(z) =& p_{\alpha\beta}(x \prec'_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q^2_\gamma(z)\\
=& p_{\alpha\beta}(p_\alpha(x) \prec_{\alpha,\beta} q_\beta(y)) \prec_{\alpha\beta,\gamma} q^2_\gamma(z)\\
=& (p^2_\alpha(x) \prec_{\alpha,\beta} p_\beta q_\beta(y)) \prec_{\alpha\beta,\gamma} q^2_\gamma(z) \qquad (\text{by } p_{\alpha\beta} \text{ satisfying Eq. } (9))\\
=& p^2_\alpha(x) \prec_{\alpha,\beta\gamma} (p_\beta q_\beta(y) \prec_{\beta,\gamma} q^2_\gamma(z) + p_\beta q_\beta(y) \succ_{\beta,\gamma} q^2_\gamma(z)) \quad (\text{by Eq. } (6))\\
=& p^2_\alpha(x) \prec_{\alpha,\beta\gamma} (q_\beta p_\beta(y) \prec_{\beta,\gamma} q^2_\gamma(z) + q_\beta p_\beta(y) \succ_{\beta,\gamma} q^2_\gamma(z))\\
&\qquad (\text{by } p_\beta \circ q_\beta = q_\beta \circ p_\beta)\\
=& p^2_\alpha(x) \prec_{\alpha,\beta\gamma} q_{\beta\gamma}(p_\beta(y) \prec_{\beta,\gamma} q_\gamma(z) + p_\beta(y) \succ_{\beta,\gamma} q_\gamma(z))\\
&\qquad (\text{by } q_{\beta\gamma} \text{ satisfying Eq. } (9))\\
=& p^2_\alpha(x) \prec_{\alpha,\beta\gamma} q_{\beta\gamma}(y \prec'_{\beta,\gamma} z + y \succ'_{\beta,\gamma} z)\\
=& p_\alpha(x) \prec'_{\alpha,\beta\gamma} (y \prec'_{\beta,\gamma} z + y \succ'_{\beta,\gamma} z).
\end{aligned}$$

This completes the proof. □

The following result is a more general of the Yau twisting procedure for BiHom-Ω-dendriform algebras and the proof is similar to Proposition 2.15.

Proposition 2.16. *If $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-dendriform algebra with $p'_\alpha, q'_\alpha : A \to A$ are two BiHom-Ω-dendriform algebra morphisms and any two families of the maps $p_\alpha, q_\alpha, p'_\alpha, q'_\alpha$ commute with each other. Define the multiplications on A by*

$$x \prec'_{\alpha,\beta} y := p'_\alpha(x) \prec_{\alpha,\beta} q'_\beta(y), \quad x \succ'_{\alpha,\beta} y := p'_\alpha(x) \succ_{\alpha,\beta} q'_\beta(y),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \prec'_{\alpha,\beta}, \succ'_{\alpha,\beta}, p_\alpha \circ p'_\alpha, q_\alpha \circ q'_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-dendriform algebra.

Next, we will introduce the relationship between BiHom-Ω-associative algebras and BiHom-Ω-dendriform algebras.

Proposition 2.17. *Let $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-dendriform algebra. If we define the multiplication by*

$$x \bullet_{\alpha,\beta} y := x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y,$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-associative algebra.

Proof. For any $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, owing to the commutativity, we get $p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha$ easily. First, we prove the multiplicativity in Definition 2.3, we have

$$\begin{aligned}
p_{\alpha\beta}(x \bullet_{\alpha,\beta} y) =& p_{\alpha\beta}(x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y)\\
=& p_{\alpha\beta}(x \prec_{\alpha,\beta} y) + p_{\alpha\beta}(x \succ_{\alpha,\beta} y) \qquad (\text{by } (p_\alpha)_{\alpha\in\Omega} \text{ being a family of linear maps})\\
=& p_\alpha(x) \prec_{\alpha,\beta} p_\beta(y) + p_\alpha(x) \succ_{\alpha,\beta} p_\beta(y) \qquad (\text{by Eq. } (10))\\
=& p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y).
\end{aligned}$$

Similarly, we have $q_{\alpha\beta}(x \bullet_{\alpha,\beta} y) = q_\alpha(x) \bullet_{\alpha,\beta} q_\beta(y)$. Next, we prove the BiHom-Ω-associativity in Definition 2.3, we have

$$p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} z) = p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z + y \succ_{\beta,\gamma} z)$$

$$
\begin{aligned}
&= p_\alpha(x) \prec_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z + y \succ_{\beta,\gamma} z) + p_\alpha(x) \succ_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z + y \succ_{\beta,\gamma} z)\\
&= (x \prec_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q_\gamma(z) + (x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q_\gamma(z)\\
&\quad + (x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} q_\gamma(z) \qquad \text{(by Eqs. (12)-(14))}\\
&= (x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} q_\gamma(z) + (x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} q_\gamma(z)\\
&= (x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&= (x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z).
\end{aligned}
$$

This completes the proof. □

It's well known that Rota-Baxter algebras induce dendriform family algebras, now we generalize this property to BiHom-Ω version.

Proposition 2.18. *Let $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-associative algebra.*

(a) *If $(R_\alpha)_{\alpha\in\Omega}$ is a Rota-Baxter family of weight 0 on A and*

$$p_\alpha \circ R_\alpha = R_\alpha \circ p_\alpha, \quad q_\alpha \circ R_\alpha = R_\alpha \circ q_\alpha.$$

Define the operations by

$$x \prec_{\alpha,\beta} y := x \bullet_{\alpha,\beta} R_\beta(y), \quad x \succ_{\alpha,\beta} y := R_\alpha(x) \bullet_{\alpha,\beta} y,$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-dendriform algebra.

(b) *If $(R_\alpha)_{\alpha\in\Omega}$ is a Rota-Baxter family of weight λ on A and*

$$p_\alpha \circ R_\alpha = R_\alpha \circ p_\alpha, \quad q_\alpha \circ R_\alpha = R_\alpha \circ q_\alpha.$$

Define the operations by

$$x \prec'_{\alpha,\beta} y := x \bullet_{\alpha,\beta} R_\beta(y) + \lambda x \bullet_{\alpha,\beta} y, \quad x \succ'_{\alpha,\beta} y := R_\alpha(x) \bullet_{\alpha,\beta} y,$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \prec'_{\alpha,\beta}, \succ'_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-dendriform algebra.

Proof. We just prove Item (b). Item (a) can be proved in the same way. To verify the relations in Eqs. (10)-(14), for any $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, first of all, we have

$$
\begin{aligned}
p_{\alpha\beta}(x \prec'_{\alpha,\beta} y) &= p_{\alpha\beta}(x \bullet_{\alpha,\beta} R_\beta(y) + \lambda x \bullet_{\alpha,\beta} y)\\
&= p_\alpha(x) \bullet_{\alpha,\beta} p_\beta R_\beta(y) + \lambda p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y) \qquad \text{(by Eq. (3))}\\
&= p_\alpha(x) \bullet_{\alpha,\beta} R_\beta p_\beta(y) + \lambda p_\alpha(x) \bullet_{\alpha,\beta} p_\beta(y) \qquad (\text{by } p_\beta \circ R_\beta = R_\beta \circ p_\beta)\\
&= p_\alpha(x) \prec'_{\alpha,\beta} p_\beta(y),
\end{aligned}
$$

Other relations in Eqs. (10)-(11) can be similarly proved. Next, for Eqs. (12)-(14), we compute

$$
\begin{aligned}
(x \prec'_{\alpha,\beta} y) \prec'_{\alpha\beta,\gamma} q_\gamma(z) &= (x \bullet_{\alpha,\beta} R_\beta(y) + \lambda x \bullet_{\alpha,\beta} y) \prec'_{\alpha\beta,\gamma} q_\gamma(z)\\
&= (x \bullet_{\alpha,\beta} R_\beta(y) + \lambda x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} R_\gamma q_\gamma(z) + \lambda(x \bullet_{\alpha,\beta} R_\beta(y)\\
&\quad + \lambda x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&= (x \bullet_{\alpha,\beta} R_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma R_\gamma(z) + \lambda(x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma R_\gamma(z)\\
&\quad + \lambda(x \bullet_{\alpha,\beta} R_\beta(y)) \bullet_{\alpha\beta,\gamma} q_\gamma(z) + \lambda^2(x \bullet_{\alpha,\beta} y) \bullet_{\alpha\beta,\gamma} q_\gamma(z)\\
&\qquad (\text{by } R_\gamma \circ q_\gamma = q_\gamma \circ R_\gamma)\\
&= p_\alpha(x) \bullet_{\alpha,\beta\gamma} (R_\beta(y) \bullet_{\beta,\gamma} R_\gamma(z)) + \lambda p_\alpha(x) \bullet_{\alpha,\beta\gamma} (y \bullet_{\beta,\gamma} R_\gamma(z))
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda p_\alpha(x)\bullet_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z)+\lambda^2 p_\alpha(x)\bullet_{\alpha,\beta\gamma}(y\bullet_{\beta,\gamma}z) \qquad \text{(by Eq. (4))}\\
=&p_\alpha(x)\bullet_{\alpha,\beta\gamma}R_{\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z+y\bullet_{\beta,\gamma}R_\gamma(z)+\lambda y\bullet_{\beta,\gamma}z)\\
&+\lambda p_\alpha(x)\bullet_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z+y\bullet_{\beta,\gamma}R_\gamma(z)+\lambda y\bullet_{\beta,\gamma}z) \qquad \text{(by Eq. (5))}\\
=&p_\alpha(x)\prec'_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z+y\bullet_{\beta,\gamma}R_\gamma(z)+\lambda y\bullet_{\beta,\gamma}z)\\
=&p_\alpha(x)\prec'_{\alpha,\beta\gamma}(y\succ'_{\beta,\gamma}z+y\prec'_{\beta,\gamma}z),
\end{aligned}
$$

$$
\begin{aligned}
(x\succ'_{\alpha,\beta}y)\prec'_{\alpha\beta,\gamma}q_\gamma(z)=&(R_\alpha(x)\bullet_{\alpha,\beta}y)\bullet_{\alpha\beta,\gamma}R_\gamma q_\gamma(z)+\lambda(R_\alpha(x)\bullet_{\alpha,\beta}y)\bullet_{\alpha\beta,\gamma}q_\gamma(z)\\
=&(R_\alpha(x)\bullet_{\alpha,\beta}y)\bullet_{\alpha\beta,\gamma}q_\gamma R_\gamma(z)+\lambda(R_\alpha(x)\bullet_{\alpha,\beta}y)\bullet_{\alpha\beta,\gamma}q_\gamma(z)\\
&\qquad\text{(by } R_\gamma\circ q_\gamma=q_\gamma\circ R_\gamma)\\
=&p_\alpha R_\alpha(x)\bullet_{\alpha,\beta\gamma}(y\bullet_{\beta,\gamma}R_\gamma(z))+\lambda p_\alpha R_\alpha(x)\bullet_{\alpha,\beta\gamma}(y\bullet_{\beta,\gamma}z) \qquad \text{(by Eq. (4))}\\
=&R_\alpha p_\alpha(x)\bullet_{\alpha,\beta\gamma}(y\bullet_{\beta,\gamma}R_\gamma(z)+\lambda y\bullet_{\beta,\gamma}z) \qquad \text{(by } p_\alpha\circ R_\alpha=R_\alpha\circ p_\alpha)\\
=&p_\alpha(x)\succ'_{\alpha,\beta\gamma}(y\prec'_{\beta,\gamma}z).
\end{aligned}
$$

$$
\begin{aligned}
p_\alpha(x)\succ'_{\alpha,\beta\gamma}(y\succ'_{\beta,\gamma}z)=&p_\alpha(x)\succ'_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z)=R_\alpha p_\alpha(x)\bullet_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z)\\
=&p_\alpha R_\alpha(x)\bullet_{\alpha,\beta\gamma}(R_\beta(y)\bullet_{\beta,\gamma}z) \qquad (\text{ by } R_\alpha\circ p_\alpha=p_\alpha\circ R_\alpha)\\
=&(R_\alpha(x)\bullet_{\alpha,\beta}R_\beta(y))\bullet_{\alpha\beta,\gamma}q_\gamma(z) \qquad \text{(by Eq. (4))}\\
=&R_{\alpha\beta}(x\bullet_{\alpha,\beta}R_\beta(y)+R_\alpha(x)\bullet_{\alpha,\beta}y+\lambda x\bullet_{\alpha,\beta}y)\bullet_{\alpha\beta,\gamma}q_\gamma(z) \qquad \text{(by Eq. (5))}\\
=&(x\bullet_{\alpha,\beta}R_\beta(y)+\lambda x\bullet_{\alpha,\beta}y+R_\alpha(x)\bullet_{\alpha,\beta}y)\succ'_{\alpha\beta,\gamma}q_\gamma(z)\\
=&(x\prec'_{\alpha,\beta}y+x\succ'_{\alpha,\beta}y)\succ'_{\alpha\beta,\gamma}q_\gamma(z).
\end{aligned}
$$

This completes the proof. □

3. BiHom-Ω-pre-Lie algebras and BiHom-Ω-Lie algebras

In this section, we assume that Ω is a commutative semigroup. First, we introduce the definitions of BiHom-Ω-pre-Lie algebras and BiHom-Ω-Lie algebras, then we obtain a BiHom-Ω analogue of the classical result of Aguiar [2].

3.1. **BiHom-Ω-pre-Lie algebras.** The concept of Ω-pre-Lie algebras was proposed in [1], as a generalization of pre-Lie algebras invented by Gerstenhaber and Vinberg [16, 24].

Definition 3.1. [1] An **Ω-pre-Lie algebra** $(A, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space A equipped with a family of bilinear maps $(\rhd_{\alpha,\beta}: A\otimes A\to A)_{\alpha,\beta\in\Omega}$ such that

$$x\rhd_{\alpha,\beta\gamma}(y\rhd_{\beta,\gamma}z)-(x\rhd_{\alpha,\beta}y)\rhd_{\alpha\beta,\gamma}z=y\rhd_{\beta,\alpha\gamma}(x\rhd_{\alpha,\gamma}z)-(y\rhd_{\beta,\alpha}x)\rhd_{\beta\alpha,\gamma}z, \tag{15}$$

for all $x, y, z\in A$, $\alpha, \beta, \gamma\in\Omega$.

Definition 3.2. Let $(A, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(A', \rhd'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-pre-Lie algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega}: A\to A'$ is called an **Ω-pre-Lie algebra morphism** if

$$f_{\alpha\beta}(x\rhd_{\alpha,\beta}y)=f_\alpha(x)\rhd'_{\alpha,\beta}f_\beta(y), \tag{16}$$

for all $x, y\in A$, $\alpha, \beta\in\Omega$.

BiHom-pre-Lie algebras, as the BiHom version of classical pre-Lie algebras, has been proposed in [19]. Similarly, we give the BiHom version of Ω-pre-Lie algebras.

Definition 3.3. A **BiHom-Ω-pre-Lie algebra** $(A, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space A equipped with a family of bilinear maps $(\blacktriangleright_{\alpha,\beta}\colon A\otimes A \to A)_{\alpha,\beta\in\Omega}$ and two commuting Ω-pre-Lie algebra morphisms $p_\alpha, q_\alpha : A \to A$ such that

$$\begin{aligned}
&p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\
&\qquad = p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z),
\end{aligned} \tag{17}$$

for all $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of A.

Definition 3.4. Let $(A, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ and $(A', \blacktriangleright'_{\alpha,\beta}, p'_\alpha, q'_\alpha)_{\alpha,\beta\in\Omega}$ be two BiHom-Ω-pre-Lie algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : A \to A'$ is called a **BiHom-Ω-pre-Lie algebra morphism** if $(f_\alpha)_{\alpha\in\Omega}$ is an Ω-pre-Lie algebra morphism and

$$f_\alpha \circ p_\alpha = p'_\alpha \circ f_\alpha, \quad f_\alpha \circ q_\alpha = q'_\alpha \circ f_\alpha, \quad \text{for all } \alpha \in \Omega.$$

Inspired by [19], we have the similar property of Ω-pre-Lie algebras as follows.

Proposition 3.5. *Let $(A, \triangleright_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-pre-Lie algebra. If $p_\alpha, q_\alpha : A \to A$ are two commuting Ω-pre-Lie algebra morphisms. Define the multiplications on A by*

$$x \blacktriangleright_{\alpha,\beta} y := p_\alpha(x) \triangleright_{\alpha,\beta} q_\beta(y),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra, called the Yau twist of $(A, \triangleright_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Proof. For any $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, we only need to prove Eq. (17).

$$\begin{aligned}
&p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\
=&p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta^2(y) \triangleright_{\beta,\gamma} q_\gamma(z)) - (p_\alpha q_\alpha(x) \triangleright_{\alpha,\beta} q_\beta p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\
=&p_\alpha^2 q_\alpha(x) \triangleright_{\alpha,\beta\gamma} q_{\beta\gamma}(p_\beta^2(y) \triangleright_{\beta,\gamma} q_\gamma(z)) - p_{\alpha\beta}(p_\alpha q_\alpha(x) \triangleright_{\alpha,\beta} q_\beta p_\beta(y)) \triangleright_{\alpha\beta,\gamma} q_\gamma^2(z)\\
=&p_\alpha^2 q_\alpha(x) \triangleright_{\alpha,\beta\gamma} (q_\beta p_\beta^2(y) \triangleright_{\beta,\gamma} q_\gamma^2(z)) - (p_\alpha^2 q_\alpha(x) \triangleright_{\alpha,\beta} p_\beta q_\beta p_\beta(y)) \triangleright_{\alpha\beta,\gamma} q_\gamma^2(z)\\
&\qquad (\text{by } q_{\beta\gamma}, p_{\alpha\beta} \text{ satisfying Eq. (16)})\\
=&q_\alpha p_\alpha^2(x) \triangleright_{\alpha,\beta\gamma} (p_\beta^2 q_\beta(y) \triangleright_{\beta,\gamma} q_\gamma^2(z)) - (q_\alpha p_\alpha^2(x) \triangleright_{\alpha,\beta} p_\beta^2 q_\beta(y)) \triangleright_{\alpha\beta,\gamma} q_\gamma^2(z)\\
&\qquad (\text{by } p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha)\\
=&p_\beta^2 q_\beta(y) \triangleright_{\beta,\alpha\gamma} (q_\alpha p_\alpha^2(x) \triangleright_{\alpha,\gamma} q_\gamma^2(z)) - (p_\beta^2 q_\beta(y) \triangleright_{\beta,\alpha} q_\alpha p_\alpha^2(x)) \triangleright_{\beta\alpha,\gamma} q_\gamma^2(z)\\
&\qquad (\text{by } (A, \triangleright_{\alpha,\beta})_{\alpha,\beta\in\Omega} \text{ satisfying Eq. (15)})\\
=&p_\beta^2 q_\beta(y) \triangleright_{\beta,\alpha\gamma} (q_\alpha p_\alpha^2(x) \triangleright_{\alpha,\gamma} q_\gamma^2(z)) - (p_\beta^2 q_\beta(y) \triangleright_{\beta,\alpha} p_\alpha q_\alpha p_\alpha(x)) \triangleright_{\beta\alpha,\gamma} q_\gamma^2(z)\\
&\qquad (\text{by } q_\alpha \circ p_\alpha = p_\alpha \circ q_\alpha)\\
=&p_\beta^2 q_\beta(y) \triangleright_{\beta,\alpha\gamma} q_{\alpha\gamma}(p_\alpha^2(x) \triangleright_{\alpha,\gamma} q_\gamma(z)) - p_{\beta\alpha}(p_\beta q_\beta(y) \triangleright_{\beta,\alpha} q_\alpha p_\alpha(x)) \triangleright_{\beta\alpha,\gamma} q_\gamma^2(z)\\
&\qquad (\text{by } p_\alpha, q_\alpha \text{ satisfying Eq. (16)})\\
=&p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha^2(x) \triangleright_{\alpha,\gamma} q_\gamma(z)) - (p_\beta q_\beta(y) \triangleright_{\beta,\alpha} q_\alpha p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z)\\
=&p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z).
\end{aligned}$$

This completes the proof. □

The following result is a more general of the Yau twisting procedure for BiHom-Ω-pre-Lie algebras and the proof is similar to Proposition 3.5.

Proposition 3.6. *Let $(A, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-pre-Lie algebra. If $p'_\alpha, q'_\alpha : A \to A$ are two BiHom-Ω-pre-Lie algebra morphisms and any two families of the maps $p_\alpha, q_\alpha, p'_\alpha, q'_\alpha$ commute with each other. Define the multiplication on A by*

$$x \blacktriangleright'_{\alpha,\beta} y := p'_\alpha(x) \blacktriangleright_{\alpha,\beta} q'_\beta(y),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \blacktriangleright'_{\alpha,\beta}, p_\alpha \circ p'_\alpha, q_\alpha \circ q'_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.

The concept of BiHom-Ω-dendriform algebras has been given in Definition 2.13. When the structure maps are bijective, we get a BiHom-Ω-pre-Lie algebra from the BiHom-Ω-dendriform algebra as follows.

Proposition 3.7. *Let $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-dendriform algebra. If p_α, q_α are bijective and we define the operation by*

$$x \blacktriangleright_{\alpha,\beta} y := x \succ_{\alpha,\beta} y - (p_\beta^{-1} q_\beta(y)) \prec_{\beta,\alpha} (p_\alpha q_\alpha^{-1}(x)),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$. Then $(A, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.

Proof. For $x, y, z \in A$, $\alpha, \beta, \gamma \in \Omega$, we only need to check Eq. (17). On the one hand, we have

$$\begin{aligned}
&p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\
=&p_\alpha q_\alpha(x) \succ_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (p_{\beta\gamma}^{-1} q_{\beta\gamma}(p_\beta(y) \blacktriangleright_{\beta,\gamma} z)) \prec_{\beta\gamma,\alpha} (p_\alpha q_\alpha^{-1} p_\alpha q_\alpha(x))\\
&- (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \succ_{\alpha\beta,\gamma} q_\gamma(z) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} p_{\alpha\beta} q_{\alpha\beta}^{-1}(q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y))\\
=&p_\alpha q_\alpha(x) \succ_{\alpha,\beta\gamma} (p_\beta(y) \succ_{\beta,\gamma} z) - p_{\beta\gamma}^{-1} q_{\beta\gamma}(p_\beta(y) \succ_{\beta,\gamma} z) \prec_{\beta\gamma,\alpha} p_\alpha^2(x)\\
&- p_\alpha q_\alpha(x) \succ_{\alpha,\beta\gamma} (p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\beta} p_\beta^2 q_\beta^{-1}(y)) + p_{\beta\gamma}^{-1} q_{\beta\gamma}(p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\beta} p_\beta^2 q_\beta^{-1}(y)) \prec_{\beta\gamma,\alpha} p_\alpha^2(x)\\
&- (q_\alpha(x) \succ_{\alpha,\beta} p_\beta(y)) \succ_{\alpha\beta,\gamma} q_\gamma(z) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} p_{\alpha\beta} q_{\alpha\beta}^{-1}(q_\alpha(x) \succ_{\alpha,\beta} p_\beta(y))\\
&+ (q_\beta(y) \prec_{\beta,\alpha} p_\alpha(x)) \succ_{\alpha\beta,\gamma} q_\gamma(z) - p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} p_{\alpha\beta} q_{\alpha\beta}^{-1}(q_\beta(y) \prec_{\beta,\alpha} p_\alpha(x))\\
=&p_\alpha q_\alpha(x) \succ_{\alpha,\beta\gamma} (p_\beta(y) \succ_{\beta,\gamma} z) - (q_\beta(y) \succ_{\beta,\gamma} p_\gamma^{-1} q_\gamma(z)) \prec_{\beta\gamma,\alpha} p_\alpha^2(x)\\
&- p_\alpha q_\alpha(x) \succ_{\alpha,\beta\gamma} (p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\beta} p_\beta^2 q_\beta^{-1}(y)) + (p_\gamma^{-2} q_\gamma^2(z) \prec_{\gamma,\beta} p_\beta(y)) \prec_{\beta\gamma,\alpha} p_\alpha^2(x)\\
&- (q_\alpha(x) \succ_{\alpha,\beta} p_\beta(y)) \prec_{\alpha\beta,\gamma} q_\gamma(z) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} (p_\alpha(x) \succ_{\alpha,\beta} p_\beta^2 q_\beta^{-1}(y))\\
&+ (q_\beta(y) \prec_{\beta,\alpha} p_\alpha(x)) \succ_{\alpha\beta,\gamma} q_\gamma(z) - p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} (p_\beta(y) \prec_{\beta,\alpha} p_\alpha^2 q_\alpha^{-1}(x))\\
&\qquad\qquad \text{(by } p_{\beta\gamma}^{-1}, q_{\beta\gamma}, p_{\alpha\beta}, q_{\alpha\beta}^{-1} \text{ satisfying Eq. (9))}\\
=&(q_\alpha(x) \prec_{\alpha,\beta} p_\beta(y)) \succ_{\alpha\beta,\gamma} q_\gamma(z) - p_\beta q_\beta(y) \succ_{\beta,\gamma\alpha} (p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\alpha} q_\alpha^{-1} p_\alpha^2(x))\\
&- (q_\alpha(x) \succ_{\alpha,\gamma} p_\gamma^{-1} q_\gamma(z)) \prec_{\alpha\gamma,\beta} p_\beta^2(y) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} (p_\beta(y) \succ_{\beta,\alpha} q_\alpha^{-1} p_\alpha^2(x))\\
&+ p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\alpha\beta} (p_\alpha(x) \succ_{\alpha,\beta} p_\beta^2 q_\beta^{-1}(y)) + (q_\beta(y) \prec_{\beta,\alpha} p_\alpha(x)) \succ_{\alpha\beta,\gamma} q_\gamma(z). \quad \text{(by Eqs. (12)-(14))}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z)\\
=&p_\beta q_\beta(y) \succ_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - p_{\alpha\gamma}^{-1} q_{\alpha\gamma}(p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) \prec_{\alpha\gamma,\beta} p_\beta q_\beta^{-1} p_\beta q_\beta(y)\\
&- (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \succ_{\beta\alpha,\gamma} q_\gamma(z) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\beta\alpha} p_{\beta\alpha} q_{\beta\alpha}^{-1}(q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x))\\
=&p_\beta q_\beta(y) \succ_{\beta,\alpha\gamma} (p_\alpha(x) \succ_{\alpha,\gamma} z) - p_{\alpha\gamma}^{-1} q_{\alpha\gamma}(p_\alpha(x) \succ_{\alpha,\gamma} z) \prec_{\alpha\gamma,\beta} p_\beta^2(y)\\
&- p_\beta q_\beta(y) \succ_{\beta,\alpha\gamma} (p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\alpha} p_\alpha^2 q_\alpha^{-1}(x)) + p_{\alpha\gamma}^{-1} q_{\alpha\gamma}(p_\gamma^{-1} q_\gamma(z) \prec_{\gamma,\alpha} p_\alpha^2 q_\alpha^{-1}(x)) \prec_{\alpha\gamma,\beta} p_\beta^2(y)\\
&- (q_\beta(y) \succ_{\beta,\alpha} p_\alpha(x)) \succ_{\beta\alpha,\gamma} q_\gamma(z) + p_\gamma^{-1} q_\gamma^2(z) \prec_{\gamma,\beta\alpha} p_{\beta\alpha} q_{\beta\alpha}^{-1}(q_\beta(y) \succ_{\beta,\alpha} p_\alpha(x))
\end{aligned}$$

$$+\,(q_\alpha(x)\prec_{\alpha,\beta} p_\beta(y))\succ_{\alpha\beta,\gamma} q_\gamma(z)-p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\alpha\beta} p_{\alpha\beta}q_{\alpha\beta}^{-1}(q_\alpha(x)\prec_{\alpha,\beta} p_\beta(y))$$
$$=p_\beta q_\beta(y)\succ_{\beta,\alpha\gamma}(p_\alpha(x)\succ_{\alpha,\gamma} z)-(q_\alpha(x)\succ_{\alpha,\gamma} p_\gamma^{-1}q_\gamma(z))\prec_{\alpha\gamma,\beta} p_\beta^2(y)$$
$$-\,p_\beta q_\beta(y)\succ_{\beta,\alpha\gamma}(p_\gamma^{-1}q_\gamma(z)\prec_{\gamma,\alpha} p_\alpha^2q_\alpha^{-1}(x))+(p_\gamma^{-2}q_\gamma^2(z)\prec_{\gamma,\alpha} p_\alpha(x))\prec_{\alpha\gamma,\beta} p_\beta^2(y)$$
$$-\,(q_\beta(y)\succ_{\beta,\alpha} p_\alpha(x))\succ_{\beta\alpha,\gamma} q_\gamma(z)+p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\beta\alpha}(p_\beta(y)\succ_{\beta,\alpha} p_\alpha^2q_\alpha^{-1}(x))$$
$$+\,(q_\alpha(x)\prec_{\alpha,\beta} p_\beta(y))\succ_{\alpha\beta,\gamma} q_\gamma(z)-p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\alpha\beta}(p_\alpha(x)\prec_{\alpha,\beta} p_\beta^2q_\beta^{-1}(y))$$
$$\quad(\text{by } p_{\alpha\gamma}^{-1}, q_{\alpha\gamma}, p_{\alpha\beta}, q_{\alpha\beta}^{-1} \text{ satisfying Eqs. (10)-(11)})$$
$$=(q_\beta(y)\prec_{\beta,\alpha} p_\alpha(x))\succ_{\beta\alpha,\gamma} q_\gamma(z)-(q_\alpha(x)\succ_{\alpha,\gamma} p_\gamma^{-1}q_\gamma(z))\prec_{\alpha\gamma,\beta} p_\beta^2(y)$$
$$-\,p_\beta q_\beta(y)\succ_{\beta,\alpha\gamma}(p_\gamma^{-1}q_\gamma(z)\prec_{\gamma,\alpha} p_\alpha^2q_\alpha^{-1}(x))+p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\alpha\beta}(p_\alpha(x)\succ_{\alpha,\beta} q_\beta^{-1}p_\beta^2(y))$$
$$+\,p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\beta\alpha}(p_\beta(y)\succ_{\beta,\alpha} p_\alpha^2q_\alpha^{-1}(x))+(q_\alpha(x)\prec_{\alpha,\beta} p_\beta(y))\succ_{\alpha\beta,\gamma} q_\gamma(z)\quad(\text{by Eq. (12) and Eq. (14)})$$
$$=(q_\beta(y)\prec_{\beta,\alpha} p_\alpha(x))\succ_{\beta\alpha,\gamma} q_\gamma(z)-(q_\alpha(x)\succ_{\alpha,\gamma} p_\gamma^{-1}q_\gamma(z))\prec_{\alpha\gamma,\beta} p_\beta^2(y)$$
$$-\,p_\beta q_\beta(y)\succ_{\beta,\alpha\gamma}(p_\gamma^{-1}q_\gamma(z)\prec_{\gamma,\alpha} q_\alpha^{-1}p_\alpha^2(x))+p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\alpha\beta}(p_\alpha(x)\succ_{\alpha,\beta} p_\beta^2q_\beta^{-1}(y))$$
$$+\,p_\gamma^{-1}q_\gamma^2(z)\prec_{\gamma,\beta\alpha}(p_\beta(y)\succ_{\beta,\alpha} q_\alpha^{-1}p_\alpha^2(x))+(q_\alpha(x)\prec_{\alpha,\beta} p_\beta(y))\succ_{\alpha\beta,\gamma} q_\gamma(z).\quad(\text{by } p_\alpha q_\alpha^{-1}=q_\alpha^{-1}p_\alpha)$$

By comparing the items of both sides, we get

$$\begin{aligned}&p_\alpha q_\alpha(x)\blacktriangleright_{\alpha,\beta\gamma}(p_\beta(y)\blacktriangleright_{\beta,\gamma} z)-(q_\alpha(x)\blacktriangleright_{\alpha,\beta} p_\beta(y))\blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\=&p_\beta q_\beta(y)\blacktriangleright_{\beta,\alpha\gamma}(p_\alpha(x)\blacktriangleright_{\alpha,\gamma} z)-(q_\beta(y)\blacktriangleright_{\beta,\alpha} p_\alpha(x))\blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z).\end{aligned}$$

□

In classical cases, any associative algebra is also a pre-Lie algebra. We generlize this result to BiHom-Ω version.

Proposition 3.8. *If $(A,\bullet_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-assocative algebra, then $(A,\bullet_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.*

Proof. For any $x,y,z\in A$, $\alpha,\beta,\gamma\in\Omega$, we only need to prove Eq. (17). By BiHom-Ω-associativity, we have

$$p_\alpha(x)\bullet_{\alpha,\beta\gamma}(y\bullet_{\beta,\gamma} z)=(x\bullet_{\alpha,\beta} y)\bullet_{\alpha\beta,\gamma} q_\gamma(z),$$

by replacing x with $q_\alpha(x)$ and y with $p_\beta(y)$, we get

$$p_\alpha q_\alpha(x)\bullet_{\alpha,\beta\gamma}(p_\beta(y)\bullet_{\beta,\gamma} z)-(q_\alpha(x)\bullet_{\alpha,\beta} p_\beta(y))\bullet_{\alpha\beta,\gamma} q_\gamma(z)=0.$$

Similarly, we get

$$p_\beta q_\beta(y)\bullet_{\beta,\alpha\gamma}(p_\alpha(x)\bullet_{\alpha,\gamma} z)-(q_\beta(y)\bullet_{\beta,\alpha} p_\alpha(x))\bullet_{\beta\alpha,\gamma} q_\gamma(z)=0.$$

Thus, we get Eq. (17). This completes the proof. □

3.2. **BiHom-Ω-Lie algebras.** In this subsection, we first introduce the concept of BiHom-Ω-Lie algebras, then we prove that a new BiHom-Ω-Lie algebra can be induced by a Rota-Baxter family of weight λ on the BiHom-Ω-Lie algebra. Finally, we generalize the classical relationships of associative algebras, pre-Lie algebras and Lie algebras to the BiHom-Ω version.

Definition 3.9. [1] An **Ω-Lie algebra** $(L,[\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space L equipped with a family of binary operations $([\cdot,\cdot]_{\alpha,\beta}:L\times L\to L)_{\alpha,\beta\in\Omega}$ such that

$$[x,y]_{\alpha,\beta}=-[y,x]_{\beta,\alpha},\tag{18}$$
$$[x,[y,z]_{\beta,\gamma}]_{\alpha,\beta\gamma}+[y,[z,x]_{\gamma,\alpha}]_{\beta,\gamma\alpha}+[z,[x,y]_{\alpha,\beta}]_{\gamma,\alpha\beta}=0,\tag{19}$$

for all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$.

Definition 3.10. Let $(L, [\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(L', [\cdot,\cdot]'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-Lie algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : L \to L'$ is called an **Ω-Lie algebra morphism** if

$$f_{\alpha\beta}([x, y]_{\alpha,\beta}) = [f_\alpha(x), f_\beta(y)]'_{\alpha,\beta}, \tag{20}$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$.

Now, we generlize the above definitions of the Ω-Lie algebra to BiHom version.

Definition 3.11. A **BiHom-Ω-Lie algebra** $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space L equipped with a family of bilinear maps $(\{\cdot,\cdot\}_{\alpha,\beta} : L \times L \to L)_{\alpha,\beta\in\Omega}$ and two commuting Ω-Lie algebra morphisms $p_\alpha, q_\alpha : L \to L$ such that

$$\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} = -\{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}, \qquad \text{(BiHom-}\Omega\text{-skew-symmetry)} \tag{21}$$

$$\{q^2_\alpha(x), \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q^2_\beta(y), \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{q^2_\gamma(z), \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} = 0,$$

$$\text{(BiHom-}\Omega\text{-Jacobi condition)} \tag{22}$$

for all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of L.

Definition 3.12. Let $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ and $(L', \{\cdot,\cdot\}'_{\alpha,\beta}, p'_\alpha, q'_\alpha)_{\alpha,\beta\in\Omega}$ be two BiHom-Ω-Lie algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : L \to L'$ is called a **BiHom-Ω-Lie algebra morphism** if f_α is an Ω-Lie algebra morphism and

$$p'_\alpha \circ f_\alpha = f_\alpha \circ p_\alpha, \quad q'_\alpha \circ f_\alpha = f_\alpha \circ q_\alpha, \quad \text{for all } \alpha \in \Omega.$$

Similar to [15, Proposition 3.16], here we get the similar property of Ω-Lie algebras as follows.

Proposition 3.13. *Let $(L, [\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-Lie algebra and let $p_\alpha, q_\alpha : L \to L$ be two families of commuting Ω-Lie algebra morphisms. We define the bilinear maps $\{\cdot,\cdot\}_{\alpha,\beta} : L \times L \to L$ by*

$$\{x, y\}_{\alpha,\beta} := [p_\alpha(x), q_\beta(y)]_{\alpha,\beta}, \quad \text{for all } x, y \in L,\ \alpha, \beta \in \Omega.$$

Then $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra, called the Yau twist of $(L, [\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Proof. For $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$, owing to the commutativity, we get $p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha$. First, we prove p_α, q_α are the Ω-Lie algebra endomorphisms on $(L, \{\cdot,\cdot\}_{\alpha,\beta})_{\alpha,\beta\in\Omega}$, we have

$$\begin{aligned}
p_{\alpha\beta}(\{x, y\}_{\alpha,\beta}) &= p_{\alpha\beta}([p_\alpha(x), q_\beta(y)]_{\alpha,\beta}) \\
&= [p^2_\alpha(x), p_\beta q_\beta(y)]_{\alpha,\beta} \qquad \text{(by } p_{\alpha\beta} \text{ satisfying Eq. (20))} \\
&= [p^2_\alpha(x), q_\beta p_\beta(y)]_{\alpha,\beta} \qquad \text{(by } p_\beta \circ q_\beta = q_\beta \circ p_\beta \text{)} \\
&= \{p_\alpha(x), p_\beta(y)\}_{\alpha,\beta}.
\end{aligned}$$

Similarly, we get $q_{\alpha\beta}(\{x, y\}_{\alpha,\beta}) = \{q_\alpha(x), q_\beta(y)\}_{\alpha,\beta}$. Next, we prove the BiHom-Ω-skew-symmetry, we have

$$\begin{aligned}
\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} &= [p_\alpha q_\alpha(x), q_\beta p_\beta(y)]_{\alpha,\beta} \\
&= -[q_\beta p_\beta(y), p_\alpha q_\alpha(x)]_{\beta,\alpha} \qquad \text{(by Eq. (18))} \\
&= -[p_\beta q_\beta(y), q_\alpha p_\alpha(x)]_{\beta,\alpha} \qquad \text{(by } p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha) \\
&= -\{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}.
\end{aligned}$$

Finally, we prove the BiHom-Ω-Jcaobi condition, we have

$$\begin{aligned}
&\{q_\alpha^2(x),\{q_\beta(y),p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\{q_\beta^2(y),\{q_\gamma(z),p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}+\{q_\gamma^2(z),\{q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
=&\{q_\alpha^2(x),[p_\beta q_\beta(y),q_\gamma p_\gamma(z)]_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\{q_\beta^2(y),[p_\gamma q_\gamma(z),q_\alpha p_\alpha(x)]_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\{q_\gamma^2(z),[p_\alpha q_\alpha(x),q_\beta p_\beta(y)]_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
=&\{q_\alpha^2(x),[p_\beta q_\beta(y),p_\gamma q_\gamma(z)]_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\{q_\beta^2(y),[p_\gamma q_\gamma(z),p_\alpha q_\alpha(x)]_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\{q_\gamma^2(z),[p_\alpha q_\alpha(x),p_\beta q_\beta(y)]_{\alpha,\beta}\}_{\gamma,\alpha\beta}\qquad(\text{by } p_\alpha\circ q_\alpha=q_\alpha\circ p_\alpha)\\
=&[p_\alpha q_\alpha^2(x),q_{\beta\gamma}([p_\beta q_\beta(y),p_\gamma q_\gamma(z)]_{\beta,\gamma})]_{\alpha,\beta\gamma}+[p_\beta q_\beta^2(y),q_{\gamma\alpha}([p_\gamma q_\gamma(z),p_\alpha q_\alpha(x)]_{\gamma,\alpha})]_{\beta,\gamma\alpha}\\
&+[p_\gamma q_\gamma^2(z),q_{\alpha\beta}([p_\alpha q_\alpha(x),q_\beta p_\beta(y)]_{\alpha,\beta})]_{\gamma,\alpha\beta}\\
=&[p_\alpha q_\alpha^2(x),[p_\beta q_\beta^2(y),p_\gamma q_\gamma^2(z)]_{\beta,\gamma}]_{\alpha,\beta\gamma}+[p_\beta q_\beta^2(y),[p_\gamma q_\gamma^2(z),p_\alpha q_\alpha^2(x)]_{\gamma,\alpha}]_{\beta,\gamma\alpha}\\
&+[p_\gamma q_\gamma^2(z),[p_\alpha q_\alpha^2(x),p_\beta q_\beta^2(y)]_{\alpha,\beta}]_{\gamma,\alpha\beta}\qquad(\text{by } q_{\beta\gamma},q_{\gamma\alpha},q_{\alpha\beta} \text{ satisfying Eq. (20)})\\
=&0.\qquad(\text{by Eq. (19)})
\end{aligned}$$

□

Inspired by [19, Claim 3.17], we have the following result and the proof is similar to Proposition 3.13.

Proposition 3.14. *Let $(L,\{\cdot,\cdot\}_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-Lie algebra. If $p'_\alpha,\ q'_\alpha:L\to L$ are two BiHom-Ω-Lie algebra morphisms and the maps $p_\alpha,q_\alpha,p'_\alpha,q'_\alpha$ commute with each other. Define the bilinear maps $\langle\cdot,\cdot\rangle_{\alpha,\beta}:L\times L\to L$ by*

$$\langle x,y\rangle_{\alpha,\beta}:=\{p'_\alpha(x),q'_\beta(y)\}_{\alpha,\beta},$$

for all $x,y\in L,\ \alpha,\ \beta\in\Omega$, then $(L,\langle\cdot,\cdot\rangle_{\alpha,\beta},p_\alpha\circ p'_\alpha,q_\alpha\circ q'_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

The following result precises the link between BiHom-Ω-pre-Lie algebras and BiHom-Ω-Lie algebras.

Proposition 3.15. *Let $(A,\blacktriangleright_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-pre-Lie algebra. If $p_\alpha,\ q_\alpha$ are bijective and we define $\{\cdot,\cdot\}_{\alpha,\beta}:A\times A\to A$ by*

$$\{x,y\}_{\alpha,\beta}:=x\blacktriangleright_{\alpha,\beta}y-(p_\beta^{-1}q_\beta(y))\blacktriangleright_{\beta,\alpha}(p_\alpha q_\alpha^{-1}(x)),$$

for all $x,y\in A,\ \alpha,\ \beta\in\Omega$, then $(A,\{\cdot,\cdot\}_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

Proof. For any $x,y,z\in A,\ \alpha,\ \beta,\ \gamma\in\Omega$, we have

$$\begin{aligned}
p_{\alpha\beta}(\{x,y\}_{\alpha,\beta})=&p_{\alpha\beta}(x\blacktriangleright_{\alpha,\beta}y-p_\beta^{-1}q_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha q_\alpha^{-1}(x))\\
=&p_{\alpha\beta}(x\blacktriangleright_{\alpha,\beta}y)-p_{\alpha\beta}(p_\beta^{-1}q_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha q_\alpha^{-1}(x))\\
=&p_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y)-p_\beta p_\beta^{-1}q_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha^2 q_\alpha^{-1}(x)\qquad(\text{by } p_{\alpha\beta} \text{ satisfying Eq. (16)})\\
=&p_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y)-p_\beta^{-1}q_\beta p_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha q_\alpha^{-1}p_\alpha(x)\qquad(\text{by } p_\alpha\circ q_\alpha=q_\alpha\circ p_\alpha)\\
=&\{p_\alpha(x),p_\beta(y)\}_{\alpha,\beta}.
\end{aligned}$$

Similarly, we get $q_{\alpha\beta}(\{x,y\}_{\alpha,\beta})=\{q_\alpha(x),q_\beta(y)\}_{\alpha,\beta}$. Now we prove the BiHom-Ω-skew-symmetry, we have

$$\begin{aligned}
\{q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}=&q_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y)-p_\beta^{-1}q_\beta p_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha q_\alpha^{-1}q_\alpha(x)\\
=&q_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y)-q_\beta(y)\blacktriangleright_{\beta,\alpha}p_\alpha(x)
\end{aligned}$$

$$
\begin{aligned}
&= -\left(q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x) - p_\alpha^{-1} q_\alpha p_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta q_\beta^{-1} q_\beta(y)\right)\\
&= -\{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}.
\end{aligned}
$$

Next, we are going to prove the BiHom-Ω-Jacobi condition, we have

$$
\begin{aligned}
&\{q_\alpha^2(x), \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\beta^2(y), \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{q_\gamma^2(z), \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
=&\{q_\alpha^2(x), q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z) - p_\gamma^{-1} q_\gamma p_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta q_\beta^{-1} q_\beta(y)\}_{\alpha,\beta\gamma}\\
&+ \{q_\beta^2(y), q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x) - p_\alpha^{-1} q_\alpha p_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma q_\gamma^{-1} q_\gamma(z)\}_{\beta,\gamma\alpha}\\
&+ \{q_\gamma^2(z), q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - p_\beta^{-1} q_\beta p_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha q_\alpha^{-1} q_\alpha(x)\}_{\gamma,\alpha\beta}\\
=&\{q_\alpha^2(x), q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z) - q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)\}_{\alpha,\beta\gamma} + \{q_\beta^2(y), q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x) - q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)\}_{\beta,\gamma\alpha}\\
&+ \{q_\gamma^2(z), q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)\}_{\gamma,\alpha\beta}\\
=&q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z) - q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)) - p_{\beta\gamma}^{-1} q_{\beta\gamma}(q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)\\
&- q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)) \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha^{-1} q_\alpha^2(x) + q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x) - q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z))\\
&- p_{\gamma\alpha}^{-1} q_{\gamma\alpha}(q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x) - q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)) \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta^{-1} q_\beta^2(y)\\
&+ q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) - p_{\alpha\beta}^{-1} q_{\alpha\beta}(q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)\\
&- q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma^{-1} q_\gamma^2(z)\\
=&q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)) - q_\alpha^2(x) \blacktriangleright_{\alpha,\gamma\beta} (q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y))\\
&- (p_\beta^{-1} q_\beta^2(y) \blacktriangleright_{\beta,\gamma} q_\gamma(z)) \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha(x) + (p_\gamma^{-1} q_\gamma^2(z) \blacktriangleright_{\gamma,\beta} q_\beta(y)) \blacktriangleright_{\gamma\beta,\alpha} p_\alpha q_\alpha(x)\\
&+ q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)) - q_\beta^2(y) \blacktriangleright_{\beta,\alpha\gamma} (q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z))\\
&- (p_\gamma^{-1} q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha} q_\alpha(x)) \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta(y) + (p_\alpha^{-1} q_\alpha^2(x) \blacktriangleright_{\alpha,\gamma} q_\gamma(z)) \blacktriangleright_{\alpha\gamma,\beta} p_\beta q_\beta(y)\\
&+ q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) - q_\gamma^2(z) \blacktriangleright_{\gamma,\beta\alpha} (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x))\\
&- (p_\alpha^{-1} q_\alpha^2(x) \blacktriangleright_{\alpha,\beta} q_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma(z) + (p_\beta^{-1} q_\beta^2(y) \blacktriangleright_{\beta,\alpha} q_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} p_\gamma q_\gamma(z)\\
&\qquad\qquad \text{(by } p_{\beta\gamma}^{-1}, q_{\beta\gamma}, p_{\gamma\alpha}^{-1}, q_{\gamma\alpha}, p_{\alpha\beta}^{-1}, q_{\alpha\beta} \text{ satisfying Eq. (16))}\\
=&p_\alpha q_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)) - (q_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta p_\beta^{-1} q_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma p_\gamma(z)\\
&- p_\beta q_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)) + (q_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha p_\alpha^{-1} q_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma p_\gamma(z)\\
&- p_\alpha q_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\gamma\beta} (p_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)) + (q_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma p_\gamma^{-1} q_\gamma(z)) \blacktriangleright_{\alpha\gamma,\beta} q_\beta p_\beta(y)\\
&+ p_\gamma q_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\alpha\beta} (p_\alpha p_\alpha^{-1} q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) - (q_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha p_\alpha^{-1} q_\alpha(x)) \blacktriangleright_{\gamma\alpha,\beta} q_\beta p_\beta(y)\\
&+ p_\beta q_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\gamma\alpha} (p_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)) - (q_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma p_\gamma^{-1} q_\gamma(z)) \blacktriangleright_{\beta\gamma,\alpha} q_\alpha p_\alpha(x)\\
&- p_\gamma q_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\beta\alpha} (p_\beta p_\beta^{-1} q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) + (q_\gamma p_\gamma^{-1} q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta p_\beta^{-1} q_\beta(y)) \blacktriangleright_{\gamma\beta,\alpha} q_\alpha p_\alpha(x)\\
&\qquad\qquad \text{(by } p_\alpha, p'_\alpha, q_\alpha, q'_\alpha \text{ commuting with each other)}\\
=&0. \qquad\quad \text{(by Eq. (17))}
\end{aligned}
$$

This completes the proof. □

Similar to associative algebras induce Lie algebras, now we generalize this classical result to BiHom-Ω version.

Proposition 3.16. *Let $(A, \bullet_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-associative algebra. If p_α, q_α are bijective and we define*

$$\{x, y\}_{\alpha,\beta} := x \bullet_{\alpha,\beta} y - p_\beta^{-1} q_\beta(y) \bullet_{\beta,\alpha} p_\alpha q_\alpha^{-1}(x),$$

for all $x, y \in A$, $\alpha, \beta \in \Omega$, then $(A, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

Proof. It follows from Proposition 3.8 and 3.15. □

Similar to Theorem 2.9, we obtain a new BiHom-Ω-Lie algebra by defining a new binary operation by the Rota-Baxter family of weight λ on the BiHom-Ω-Lie algebra.

Theorem 3.17. *Let $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-Lie algebra. If $(R_\alpha)_{\alpha\in\Omega}$ is a Rota-Baxter family of weight λ on L satisfying*

$$R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha \text{ and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha.$$

Define a new operation on L by

$$\langle x, y\rangle_{\alpha,\beta} := \{R_\alpha(x), y\}_{\alpha,\beta} + \{x, R_\beta(y)\}_{\alpha,\beta} + \lambda\{x, y\}_{\alpha,\beta},$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$, then $(L, \langle\cdot,\cdot\rangle_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

Proof. For all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$, we have

$$\begin{aligned}
\langle p_\alpha(x), p_\beta(y)\rangle_{\alpha,\beta} =& \{R_\alpha(p_\alpha(x)), p_\beta(y)\}_{\alpha,\beta} + \{p_\alpha(x), R_\beta(p_\beta(y))\}_{\alpha,\beta} + \lambda\{p_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
=& \{p_\alpha(R_\alpha(x)), p_\beta(y)\}_{\alpha,\beta} + \{p_\alpha(x), p_\beta(R_\beta(y))\}_{\alpha,\beta} + \lambda\{p_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
&\quad (\text{by } R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha)\\
=& p_{\alpha\beta}(\{R_\alpha(x), y\}_{\alpha,\beta}) + p_{\alpha\beta}(\{x, R_\beta(y)\}_{\alpha,\beta}) + \lambda p_{\alpha\beta}(\{x, y\}_{\alpha,\beta})\\
&\quad (\text{by } p_{\alpha\beta} \text{ being a BiHom-}\Omega\text{-Lie algebra morphism}))\\
=& p_{\alpha\beta}(\{R_\alpha(x), y\}_{\alpha,\beta} + \{x, R_\beta(y)\}_{\alpha,\beta} + \lambda\{x, y\}_{\alpha,\beta})\\
=& p_{\alpha\beta}(\langle x, y\rangle_{\alpha,\beta}).
\end{aligned}$$

Similarly, we get $q_{\alpha\beta}(\langle x, y\rangle_{\alpha,\beta}) = \langle q_\alpha(x), q_\beta(y)\rangle_{\alpha,\beta}$. For the BiHom-Ω-skew-symmetry, we have

$$\begin{aligned}
\langle q_\alpha(x), p_\beta(y)\rangle_{\alpha,\beta} =& \{R_\alpha(q_\alpha(x)), p_\beta(y)\}_{\alpha,\beta} + \{q_\alpha(x), R_\beta(p_\beta(y))\}_{\alpha,\beta} + \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
=& \{q_\alpha(R_\alpha(x)), p_\beta(y)\}_{\alpha,\beta} + \{q_\alpha(x), p_\beta(R_\beta(y))\}_{\alpha,\beta} + \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
&\quad (\text{by } R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha \text{ and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha)\\
=& -\{q_\beta(y), p_\alpha(R_\alpha(x))\}_{\beta,\alpha} - \{q_\beta(R_\beta(y)), p_\alpha(x)\}_{\beta,\alpha} - \lambda\{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}\\
&\quad (\text{by Eq. } (21))\\
=& -\{q_\beta(y), R_\alpha(p_\alpha(x))\}_{\beta,\alpha} - \{R_\beta(q_\beta(y)), p_\alpha(x)\}_{\beta,\alpha} - \lambda\{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}\\
&\quad (\text{by } R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha \text{ and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha)\\
=& -\langle q_\beta(y), p_\alpha(x)\rangle_{\beta,\alpha}.
\end{aligned}$$

Next, we are going to prove that $\langle\cdot,\cdot\rangle_{\alpha,\beta}$ satisfies the BiHom-Ω-Jacobi condition, we have

$$\begin{aligned}
&\langle q_\alpha^2(x), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\rangle_{\alpha,\beta\gamma}\\
=& \{R_\alpha(q_\alpha^2(x)), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(x), R_{\beta\gamma}(\langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma})\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
=& \{R_\alpha(q_\alpha^2(x)), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \lambda\{q_\alpha^2(x), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{q_\alpha^2(x), R_{\beta\gamma}(\{R_\beta q_\beta(y), p_\gamma(z)\}_{\beta,\gamma} + \{q_\beta(y), R_\gamma p_\gamma(z)\}_{\beta,\gamma} + \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma})\}_{\alpha,\beta\gamma}
\end{aligned}$$

$$
\begin{aligned}
=&\{R_\alpha(q_\alpha^2(x)), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(x), \{R_\beta(q_\beta(y)), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\}_{\alpha,\beta\gamma} \qquad \text{(by Eq. (5))}\\
=&\{R_\alpha(q_\alpha^2(x)), \{R_\beta(q_\beta(y)), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{R_\alpha(q_\alpha^2(x)), \{q_\beta(y), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{R_\alpha(q_\alpha^2(x)), \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(x), \{R_\beta(q_\beta(y)), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \{R_\beta(q_\beta(y)), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \lambda\{q_\alpha^2(x), \{q_\beta(y), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\langle q_\beta^2(y), \langle q_\gamma(z), p_\alpha(x)\rangle_{\gamma,\alpha}\rangle_{\beta,\gamma\alpha}\\
=&\{R_\beta(q_\beta^2(y)), \{R_\gamma(q_\gamma(z)), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{R_\beta(q_\beta^2(y)), \{q_\gamma(z), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \{R_\beta(q_\beta^2(y)), \lambda\{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{q_\beta^2(y), \{R_\gamma(q_\gamma(z)), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \lambda\{q_\beta^2(y), \{R_\gamma(q_\gamma(z)), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \lambda\{q_\beta^2(y), \{q_\gamma(z), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \lambda\{q_\beta^2(y), \lambda\{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha},
\end{aligned}
$$

$$
\begin{aligned}
&\langle q_\gamma^2(z), \langle q_\alpha(x), p_\beta(y)\rangle_{\alpha,\beta}\rangle_{\gamma,\alpha\beta}\\
=&\{R_\gamma(q_\gamma^2(z)), \{R_\alpha(q_\alpha(x)), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \{R_\gamma(q_\gamma^2(z)), \{q_\gamma(x), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \{R_\gamma(q_\gamma^2(z)), \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \{q_\gamma^2(z), \{R_\alpha(q_\alpha(x)), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \lambda\{q_\gamma^2(z), \{R_\alpha(q_\alpha(x)), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \lambda\{q_\gamma^2(z), \{q_\alpha(x), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \lambda\{q_\gamma^2(z), \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}.
\end{aligned}
$$

By adding the items, we obtain

$$
\begin{aligned}
&\langle q_\alpha^2(x), \langle q_\beta(y), p_\gamma(z)\rangle_{\beta,\gamma}\rangle_{\alpha,\beta\gamma} + \langle q_\beta^2(y), \langle q_\gamma(z), p_\alpha(x)\rangle_{\gamma,\alpha}\rangle_{\beta,\gamma\alpha} + \langle q_\gamma^2(z), \langle q_\alpha(x), p_\beta(y)\rangle_{\alpha,\beta}\rangle_{\gamma,\alpha\beta}\\
=&\{R_\alpha(q_\alpha^2(x)), \{R_\beta(q_\beta(y)), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{R_\alpha(q_\alpha^2(x)), \{q_\beta(y), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{R_\alpha(q_\alpha^2(x)), \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(x), \{R_\beta(q_\beta(y)), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \{R_\beta(q_\beta(y)), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \lambda\{q_\alpha^2(x), \{q_\beta(y), R_\gamma(p_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \lambda\{q_\alpha^2(x), \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{R_\beta(q_\beta^2(y)), \{R_\gamma(q_\gamma(z)), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{R_\beta(q_\beta^2(y)), \{q_\gamma(z), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \{R_\beta(q_\beta^2(y)), \lambda\{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{q_\beta^2(y), \{R_\gamma(q_\gamma(z)), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \lambda\{q_\beta^2(y), \{R_\gamma(q_\gamma(z)), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \lambda\{q_\beta^2(y), \{q_\gamma(z), R_\alpha(p_\alpha(x))\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \lambda\{q_\beta^2(y), \lambda\{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ \{R_\gamma(q_\gamma^2(z)), \{R_\alpha(q_\alpha(x)), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \{R_\gamma(q_\gamma^2(z)), \{q_\alpha(x), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \{R_\gamma(q_\gamma^2(z)), \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \{q_\gamma^2(z), \{R_\alpha(q_\alpha(x)), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \lambda\{q_\gamma^2(z), \{R_\alpha(q_\alpha(x)), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} + \lambda\{q_\gamma^2(z), \{q_\alpha(x), R_\beta(p_\beta(y))\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \lambda\{q_\gamma^2(z), \lambda\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
=&\{q_\alpha^2(R_\alpha(x)), \{q_\beta(R_\beta(y)), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(R_\alpha(x)), \{q_\beta(y), p_\gamma(R_\gamma(z))\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{q_\alpha^2 R_\alpha(x), \lambda\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{q_\alpha^2(x), \{q_\beta R_\beta(y), p_\gamma R_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\{q_\alpha^2(x),\{q_\beta R_\beta(y),p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\lambda\{q_\alpha^2(x),\{q_\beta(y),p_\gamma R_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+\lambda\{q_\alpha^2(x),\lambda\{q_\beta(y),p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\{q_\beta^2R_\beta(y),\{q_\gamma R_\gamma(z),p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\{q_\beta^2R_\beta(y),\{q_\gamma(z),p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}+\{q_\beta^2R_\beta(y),\lambda\{q_\gamma(z),p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\{q_\beta^2(y),\{q_\gamma R_\gamma(z),p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}+\{q_\beta^2(y),\{q_\gamma R_\gamma(z),p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\lambda\{q_\beta^2(y),\{q_\gamma(z),p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}+\lambda\{q_\beta^2(y),\lambda\{q_\gamma(z),p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+\{q_\gamma^2R_\gamma(z),\{q_\alpha R_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}+\{q_\gamma^2R_\gamma(z),\{q_\alpha(x),p_\beta R_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+\{q_\gamma^2R_\gamma(z),\lambda\{q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}+\{q_\gamma^2(z),\{q_\alpha R_\alpha(x),p_\beta R_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+\lambda\{q_\gamma^2(z),\{q_\alpha R_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}+\lambda\{q_\gamma^2(z),\{q_\alpha(x),p_\beta R_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+\lambda\{q_\gamma^2(z),\lambda\{q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\qquad(\text{by } R_\alpha\circ p_\alpha=p_\alpha\circ R_\alpha,\ R_\alpha\circ q_\alpha=q_\alpha\circ R_\alpha)\\
=&0.\qquad(\text{by Eq. (22)})
\end{aligned}
$$

□

Corollary 3.18. *Let $(L,[\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-Lie algebra. If $(R_\alpha)_{\alpha\in\Omega}$ is a Rota-Baxter family of weight λ on L. Define a new multiplication on L by*

$$[x,y]'_{\alpha,\beta}:=[R_\alpha(x),y]_{\alpha,\beta}+[x,R_\beta(y)]_{\alpha,\beta}+\lambda[x,y]_{\alpha,\beta},$$

for all $x,y\in L$, $\alpha,\beta\in\Omega$, then $(L,[\cdot,\cdot]'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-Lie algebra.

Proof. It follows from Theorem 3.17 by taking $p_\alpha=q_\alpha=\mathrm{id}_L$ for $\alpha\in\Omega$. □

In [2], Aguiar has proved that a pre-Lie algebra induced by the Rota-Baxter family of weight zero on a Lie algebra, and the BiHom and Hom analogue of Aguiar's result were studied in [19, 21], now we generalize the classical results to BiHom-Ω version.

Theorem 3.19. *Let $(L,\{\cdot,\cdot\}_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-Lie algebra and $(R_\alpha)_{\alpha\in\Omega}$ be a Rota-Baxter family of weight 0 on L such that*

$$R_\alpha\circ p_\alpha=p_\alpha\circ R_\alpha \text{ and } R_\alpha\circ q_\alpha=q_\alpha\circ R_\alpha.$$

Define the operation on L by

$$x\blacktriangleright_{\alpha,\beta}y:=\{R_\alpha(x),y\}_{\alpha,\beta},$$

for all $x,y\in L$, $\alpha,\beta\in\Omega$. Then $(L,\blacktriangleright_{\alpha,\beta},p_\alpha,q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.

Proof. For any $x,y,z\in L$, $\alpha,\beta,\gamma\in\Omega$, we have

$$
\begin{aligned}
p_{\alpha\beta}(x\blacktriangleright_{\alpha,\beta}y)=&p_{\alpha\beta}(\{R_\alpha(x),y\}_{\alpha,\beta})\\
=&\{p_\alpha R_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\qquad(\text{by } p_{\alpha\beta} \text{ satisfying Eq. (20)})\\
=&\{R_\alpha p_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\qquad(\text{by } R_\alpha\circ p_\alpha=p_\alpha\circ R_\alpha)\\
=&p_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y).
\end{aligned}
$$

Similarly, we get $q_{\alpha\beta}(x\blacktriangleright_{\alpha,\beta}y)=q_\alpha(x)\blacktriangleright_{\alpha,\beta}q_\beta(y)$. Next, we only need to prove Eq. (17). On the one hand, we have

$$
\begin{aligned}
&p_\alpha q_\alpha(x)\blacktriangleright_{\alpha,\beta\gamma}(p_\beta(y)\blacktriangleright_{\beta,\gamma}z)-(q_\alpha(x)\blacktriangleright_{\alpha,\beta}p_\beta(y))\blacktriangleright_{\alpha\beta,\gamma}q_\gamma(z)\\
=&p_\alpha q_\alpha(x)\blacktriangleright_{\alpha,\beta\gamma}\{R_\beta p_\beta(y),z\}_{\beta,\gamma}-\{R_\alpha q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}\blacktriangleright_{\alpha\beta,\gamma}q_\gamma(z)\\
=&\{R_\alpha p_\alpha q_\alpha(x),\{R_\beta p_\beta(y),z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}-\{R_{\alpha\beta}(\{R_\alpha q_\alpha(x),p_\beta(y)\}_{\alpha,\beta}),q_\gamma(z)\}_{\alpha\beta,\gamma}\\
=&\{R_\alpha p_\alpha q_\alpha(x),\{R_\beta p_\beta(y),z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}+\{R_{\alpha\beta}(\{q_\beta(y),p_\alpha R_\alpha(x)\}_{\beta,\alpha}),q_\gamma(z)\}_{\beta\alpha,\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\text{(by Eq. (21))}\\
&=\{q_\alpha p_\alpha R_\alpha(x), \{p_\beta R_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}.\\
&\qquad\text{(by } p_\alpha, q_\alpha, R_\alpha \text{ commuting with each other)}
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z)\\
=& p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma} - \{R_\beta q_\beta(y), p_\alpha(x)\}_{\beta,\alpha} \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z)\\
=& \{R_\beta p_\beta q_\beta(y), \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{R_{\beta\alpha}(\{R_\beta q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
=& \{R_\beta p_\beta q_\beta(y), \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{\{R_\beta q_\beta(y), R_\alpha p_\alpha(x)\}_{\beta,\alpha}, q_\gamma(z)\}_{\beta\alpha,\gamma}\\
&+ \{R_{\beta\alpha}(\{q_\beta(y), R_\alpha p_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma} \qquad \text{(by Eq. (5))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{\{q_\beta R_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}, q_\gamma(z)\}_{\beta\alpha,\gamma}\\
&+ \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma} \qquad \text{(by } p_\alpha, q_\alpha, R_\alpha \text{ commuting with each other)}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{\{q_\alpha R_\alpha(x), p_\beta R_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma}\\
&+ \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma} \qquad \text{(by Eq. (21))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{q_{\alpha\beta} q_{\alpha\beta}^{-1}(\{q_\alpha R_\alpha(x), p_\beta R_\beta(y)\}_{\alpha,\beta}), p_\gamma q_\gamma p_\gamma^{-1}(z)\}_{\alpha\beta,\gamma}\\
&+ \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{q_\gamma^2 p_\gamma^{-1}(z), p_{\alpha\beta} q_{\alpha\beta}^{-1}(\{q_\alpha R_\alpha(x), p_\beta R_\beta(y)\}_{\alpha,\beta})\}_{\gamma,\alpha\beta}\\
&+ \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma} \qquad \text{(by Eq. (21))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{q_\gamma^2 p_\gamma^{-1}(z), \{p_\alpha R_\alpha(x), p_\beta^2 q_\beta^{-1} R_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&+ \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma} \qquad \text{(by } p_{\alpha\beta}, q_{\alpha\beta}^{-1} \text{ satisfying Eq. (20))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{q_\alpha^2 q_\alpha^{-1} p_\alpha R_\alpha(x), \{q_\beta p_\beta q_\beta^{-1} R_\beta(y), p_\gamma p_\gamma^{-1}(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{q_\beta^2 p_\beta q_\beta^{-1} R_\beta(y), \{q_\gamma p_\gamma^{-1}(z), p_\alpha q_\alpha^{-1} p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
&\qquad \text{(by Eq. (22))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{q_\alpha p_\alpha R_\alpha(x), \{p_\beta R_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ \{q_\beta p_\beta R_\beta(y), \{q_\gamma p_\gamma^{-1}(z), p_\alpha^2 q_\alpha^{-1} R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{q_\alpha p_\alpha R_\alpha(x), \{p_\beta R_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&- \{q_\beta p_\beta R_\beta(y), \{q_\alpha p_\alpha q_\alpha^{-1} R_\alpha(x), p_\gamma p_\gamma^{-1}(z)\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
&\qquad \text{(by Eq. (21))}\\
=& \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{q_\alpha p_\alpha R_\alpha(x), \{p_\beta R_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&- \{q_\beta p_\beta R_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}\\
=& \{q_\alpha p_\alpha R_\alpha(x), \{p_\beta R_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} + \{R_{\alpha\beta}(\{q_\beta(y), p_\alpha R_\alpha(x)\}_{\beta,\alpha}), q_\gamma(z)\}_{\beta\alpha,\gamma}.
\end{aligned}
$$

By comparing the items of both sides, we get

$$
\begin{aligned}
& p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\
=& p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z).
\end{aligned}
$$

This completes the proof. $\square$

4. BiHom-Ω-PostLie algebras and BiHom-Ω-pre-Possion algebras

In this section, we continue to assume that Ω is a commutative semigroup.

4.1. **BiHom-Ω-PostLie algebras.** In this subsection, we mainly study the relationship between BiHom-Ω-PostLie algebras and BiHom-Ω-Lie algebras. First, we generlize the concept of PostLie algebras [7, 23] to Ω version.

Definition 4.1. An **Ω-PostLie algebra** $(L, [\cdot,\cdot]_{\alpha,\beta}, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space L equipped with two families of bilinear operations $[\cdot,\cdot]_{\alpha,\beta}$, $\rhd_{\alpha,\beta} : L\times L \to L$, such that $(L, [\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-Lie algebra and

$$[x, y]_{\alpha,\beta} \rhd_{\alpha\beta,\gamma} z = x \rhd_{\alpha,\beta\gamma} (y \rhd_{\beta,\gamma} z) - (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\beta,\gamma} z - y \rhd_{\beta,\alpha\gamma} (x \rhd_{\alpha,\gamma} z) + (y \rhd_{\beta,\alpha} x) \rhd_{\beta\alpha,\gamma} z,$$
$$x \rhd_{\alpha,\beta\gamma} [y, z]_{\beta,\gamma} = [x \rhd_{\alpha,\beta} y, z]_{\alpha\beta,\gamma} + [y, x \rhd_{\alpha,\gamma} z]_{\beta,\alpha\gamma},$$

for all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$.

Definition 4.2. Let $(L, [\cdot,\cdot]_{\alpha,\beta}, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(L', [\cdot,\cdot]'_{\alpha,\beta}, \rhd'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-PostLie algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : L \to L'$ is called an **Ω-PostLie algebra morphism** if

$$f_{\alpha\beta}[x, y]_{\alpha,\beta} = [f_\alpha(x), f_\beta(y)]'_{\alpha,\beta},$$
$$f_{\alpha\beta}(x \rhd_{\alpha,\beta} y) = f_\alpha(x) \rhd'_{\alpha,\beta} f_\beta(y),$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$.

Remark 4.3. Let $(L, [\cdot,\cdot]_{\alpha,\beta}, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-PostLie algebra.

(a) If $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in L$, $\alpha, \beta \in \Omega$, then we get
$$x \rhd_{\alpha,\beta\gamma} (y \rhd_{\beta,\gamma} z) - (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\beta,\gamma} z = y \rhd_{\beta,\alpha\gamma} (x \rhd_{\alpha,\gamma} z) - (y \rhd_{\beta,\alpha} x) \rhd_{\beta\alpha,\gamma} z,$$
for all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$, that is $(L, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-pre-Lie algebra.

(b) If $x \rhd_{\alpha,\beta} y = 0$, for all $x, y \in L$, $\alpha, \beta \in \Omega$, then $(L, [\cdot,\cdot]_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-Lie algebra.

The concept of BiHom-PostLie algebras was given in [3]. Now we introduce a more general version of Definition 4.1.

Definition 4.4. A **BiHom-Ω-PostLie algebra** $(L, \{\cdot,\cdot\}_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space L equipped with two families of bilinear maps $\{\cdot,\cdot\}_{\alpha,\beta}$, $\blacktriangleright_{\alpha,\beta}$: $L\times L \to L$, and two commuting families of linear maps $(p_\alpha)_{\alpha\in\Omega}, (q_\alpha)_{\alpha\in\Omega} : L \to L$ such that $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra and

$$p_{\alpha\beta}(x \blacktriangleright_{\alpha,\beta} y) = p_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y), \quad q_{\alpha\beta}(x \blacktriangleright_{\alpha,\beta} y) = q_\alpha(x) \blacktriangleright_{\alpha,\beta} q_\beta(y),$$

$$\begin{aligned}\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z) =& p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)\\ &- p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z) + (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z),\end{aligned}$$

$$p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} \{y, z\}_{\beta,\gamma} = \{q_\alpha(x) \blacktriangleright_{\alpha,\beta} y, q_\gamma(z)\}_{\alpha\beta,\gamma} + \{q_\beta(y), p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z\}_{\beta,\alpha\gamma},$$

for all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α, q_α (in this order) are called the structure maps of L.

Remark 4.5. (a) If $\{x, y\}_{\alpha,\beta} = 0$, for all $x, y \in L$, $\alpha, \beta \in \Omega$, then $(L, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.

(b) If $x \blacktriangleright_{\alpha,\beta} y = 0$, for all $x, y \in L$, $\alpha, \beta \in \Omega$, then $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

Now we introduce the Yau twisting procedure for BiHom-Ω-PostLie algebras and the proof is similar to Proposition 3.5 and 3.13.

Proposition 4.6. *Let $(L, [\cdot,\cdot]_{\alpha,\beta}, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-PostLie algebra and let $p_\alpha, q_\alpha : L \to L$ be two commuting Ω-PostLie algebra morphisms. Define two operations on L by*

$$\{x,y\}_{\alpha,\beta} := [p_\alpha(x), q_\beta(y)]_{\alpha,\beta} \text{ and } x \blacktriangleright_{\alpha,\beta} y := p_\alpha(x) \rhd_{\alpha,\beta} q_\beta(y),$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$. Then $(L, \{\cdot,\cdot\}_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-PostLie algebra, called the Yau twist of $(L, [\cdot,\cdot]_{\alpha,\beta}, \rhd_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Now we give a way to construct a BiHom-Ω-Lie algebra by defining a new operation on the BiHom-Ω-PostLie algebra.

Proposition 4.7. *Let $(L, \{\cdot,\cdot\}_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-PostLie algebra. If p_α, q_α are bijective. Define a new multiplication on L by*

$$< x, y >_{\alpha,\beta} := x \blacktriangleright_{\alpha,\beta} y - (p_\beta^{-1} q_\beta(y)) \blacktriangleright_{\beta,\alpha} (p_\alpha q_\alpha^{-1}(x)) + \{x, y\}_{\alpha,\beta},$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$, then $(L, < \cdot,\cdot >_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra.

Proof. For all $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$, owing to the commutativity, we get $p_\alpha \circ q_\alpha = q_\alpha \circ p_\alpha$ and we have

$$p_{\alpha\beta} < x, y >_{\alpha,\beta} = < p_\alpha(x), p_\beta(y) >_{\alpha,\beta}, \quad q_{\alpha\beta} < x, y >_{\alpha,\beta} = < q_\alpha(x), q_\beta(y) >_{\alpha,\beta} .$$

First, we prove the BiHom-Ω-skew-symmetry, we have

$$\begin{aligned}
< q_\alpha(x), p_\beta(y) >_{\alpha,\beta} &= q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - p_\beta^{-1} q_\beta(p_\beta(y)) \blacktriangleright_{\beta,\alpha} p_\alpha q_\alpha^{-1} q_\alpha(x) + \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
&= q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x) + \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
&= -(q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x) - q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) + \{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha})\\
&\qquad \text{(by Eq. (21))}\\
&= -(q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x) - p_\alpha^{-1} q_\alpha p_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta q_\beta^{-1} q_\beta(y) + \{q_\beta(y), p_\alpha(x)\}_{\beta,\alpha})\\
&= - < q_\beta(y), p_\alpha(x) >_{\beta,\alpha} .
\end{aligned}$$

Next, we prove the BiHom-Ω-Jacobi condition, we have

$$\begin{aligned}
&< q_\alpha^2(x), < q_\beta(y), p_\gamma(z) >_{\beta,\gamma} >_{\alpha,\beta\gamma} + < q_\beta^2(y), < q_\gamma(z), p_\alpha(x) >_{\gamma,\alpha} >_{\beta,\gamma\alpha}\\
&\quad + < q_\gamma^2(z), < q_\alpha(x), p_\beta(y) >_{\alpha,\beta} >_{\gamma,\alpha\beta}\\
&= < q_\alpha^2(x), q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z) >_{\alpha,\beta\gamma} - < q_\alpha^2(x), q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y) >_{\alpha,\beta\gamma}\\
&\quad + < q_\alpha^2(x), \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma} >_{\alpha,\beta\gamma} + < q_\beta^2(y), q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x) >_{\beta,\gamma\alpha}\\
&\quad - < q_\beta^2(y), q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z) >_{\beta,\gamma\alpha} + < q_\beta^2(y), \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha} >_{\beta,\gamma\alpha}\\
&\quad + < q_\gamma^2(z), q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) >_{\gamma,\alpha\beta} - < q_\gamma^2(z), q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x) >_{\gamma,\alpha\beta}\\
&\quad + < q_\gamma^2(z), \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} >_{\gamma,\alpha\beta}\\
&= q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)) + \{q_\alpha^2(x), q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)\}_{\alpha,\beta\gamma} - q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y))\\
&\quad - \{q_\alpha^2(x), q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\gamma(z)\}_{\alpha,\beta\gamma} + q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma} + \{q_\alpha^2(x), \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&\quad + q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)) + \{q_\beta^2(y), q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)\}_{\beta,\gamma\alpha} - q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z))\\
&\quad - \{q_\beta^2(y), q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)\}_{\beta,\gamma\alpha} + q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha} + \{q_\beta^2(y), \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+ q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) + \{q_\gamma^2(z), q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)\}_{\gamma,\alpha\beta} - q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x))\\
&- \{q_\gamma^2(z), q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)\}_{\gamma,\beta\alpha} + q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} + \{q_\gamma^2(z), \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
&- p_{\beta\gamma}^{-1} q_{\beta\gamma}(q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)) \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha^{-1} q_\alpha^2(x) + p_{\gamma\beta}^{-1} q_{\gamma\beta}(q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)) \blacktriangleright_{\gamma\beta,\alpha} p_\alpha q_\alpha(x)\\
&- p_{\beta\gamma}^{-1} q_{\beta\gamma}\{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma} \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha(x) - p_{\gamma\alpha}^{-1} q_{\gamma\alpha}(q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)) \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta(y)\\
&+ p_{\alpha\gamma}^{-1} q_{\alpha\gamma}(q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)) \blacktriangleright_{\alpha\gamma,\beta} p_\beta q_\beta(y) - p_{\gamma\alpha}^{-1} q_{\gamma\alpha}\{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha} \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta(y)\\
&- p_{\alpha\beta}^{-1} q_{\alpha\beta}(q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma(z) + p_{\beta\alpha}^{-1} q_{\beta\alpha}(q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} p_\gamma q_\gamma(z)\\
&- p_{\alpha\beta}^{-1} q_{\alpha\beta}\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma(z)\\
=&q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)) - (p_\beta^{-1} q_\beta^2(y) \blacktriangleright_{\beta,\gamma} q_\gamma(z)) \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha(x) + \{q_\alpha^2(x), q_\beta(y) \blacktriangleright_{\beta,\gamma} p_\gamma(z)\}_{\alpha,\beta\gamma}\\
&- q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} (q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)) + (p_\gamma^{-1} q_\gamma^2(z) \blacktriangleright_{\gamma,\beta} q_\beta(y)) \blacktriangleright_{\gamma\beta,\alpha} p_\alpha q_\alpha(x) - \{q_\alpha^2(x), q_\gamma(z) \blacktriangleright_{\gamma,\beta} p_\beta(y)\}_{\alpha,\beta\gamma}\\
&+ q_\alpha^2(x) \blacktriangleright_{\alpha,\beta\gamma} \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma} - \{p_\beta^{-1} q_\beta^2(y), q_\gamma(z)\}_{\beta,\gamma} \blacktriangleright_{\beta\gamma,\alpha} p_\alpha q_\alpha(x) + \{q_\alpha^2(x), \{q_\beta(y), p_\gamma(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
&+ q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)) - (p_\gamma^{-1} q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha} q_\alpha(x)) \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta(y) + \{q_\beta^2(y), q_\gamma(z) \blacktriangleright_{\gamma,\alpha} p_\alpha(x)\}_{\beta,\gamma\alpha}\\
&- q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} (q_\beta(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)) + (p_\alpha^{-1} q_\alpha^2(x) \blacktriangleright_{\alpha,\gamma} q_\gamma(z)) \blacktriangleright_{\alpha\gamma,\beta} p_\beta q_\beta(y) - \{q_\beta^2(y), q_\alpha(x) \blacktriangleright_{\alpha,\gamma} p_\gamma(z)\}_{\beta,\gamma\alpha}\\
&+ q_\beta^2(y) \blacktriangleright_{\beta,\gamma\alpha} \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha} - \{p_\gamma^{-1} q_\gamma^2(z), q_\alpha(x)\}_{\gamma,\alpha} \blacktriangleright_{\gamma\alpha,\beta} p_\beta q_\beta(y) + \{q_\beta^2(y), \{q_\gamma(z), p_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&+ q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) - (p_\alpha^{-1} q_\alpha^2(x) \blacktriangleright_{\alpha,\beta} q_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma(z) + \{q_\gamma^2(z), q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)\}_{\gamma,\alpha\beta}\\
&- q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) + (p_\beta^{-1} q_\beta^2(y) \blacktriangleright_{\beta,\alpha} q_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} p_\gamma q_\gamma(z) - \{q_\gamma^2(z), q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)\}_{\gamma,\beta\alpha}\\
&+ q_\gamma^2(z) \blacktriangleright_{\gamma,\alpha\beta} \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} - \{p_\alpha^{-1} q_\alpha^2(x), q_\beta(y)\}_{\alpha,\beta} \blacktriangleright_{\alpha\beta,\gamma} p_\gamma q_\gamma(z) + \{q_\gamma^2(z), \{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta}\\
=&0. \qquad \text{(by } (L, \{\cdot,\cdot\}_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega} \text{ being a BiHom-}\Omega\text{-PostLie algebra)}
\end{aligned}
$$

□

Motivated by [7, Corollary 5.6], we have the following result.

Proposition 4.8. *Let $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-Lie algebra and let $(R_\alpha)_{\alpha\in\Omega}$ be a Rota-Baxter family of weight λ on L such that*

$$R_\alpha \circ p_\alpha = p_\alpha \circ R_\alpha \text{ and } R_\alpha \circ q_\alpha = q_\alpha \circ R_\alpha.$$

We define two operations on L by

$$\langle x, y\rangle_{\alpha,\beta} := \lambda\{x, y\}_{\alpha,\beta} \text{ and } x \blacktriangleright_{\alpha,\beta} y := \{R_\alpha(x), y\}_{\alpha,\beta},$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$. Then $(L, \langle\cdot,\cdot\rangle_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-PostLie algebra.

Proof. By $(L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ being a BiHom-Ω-Lie algebra, we get $(L, \langle\cdot,\cdot\rangle_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-Lie algebra, where $\langle x, y\rangle_{\alpha,\beta} := \lambda\{x, y\}_{\alpha,\beta}$. For $x, y, z \in L$, $\alpha, \beta, \gamma \in \Omega$, we have

$$
\begin{aligned}
p_{\alpha\beta}(x \blacktriangleright_{\alpha,\beta} y) =& p_{\alpha\beta}(\{R_\alpha(x), y\}_{\alpha,\beta}) = \{p_\alpha R_\alpha(x), p_\beta(y)\}_{\alpha,\beta}\\
=&\{R_\alpha p_\alpha(x), p_\beta(y)\}_{\alpha,\beta} \qquad \text{(by } p_\alpha \circ R_\alpha = R_\alpha \circ p_\alpha)\\
=&p_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y).
\end{aligned}
$$

Similarly, we get $q_{\alpha\beta}(x \blacktriangleright_{\alpha,\beta} y) = q_\alpha(x) \blacktriangleright_{\alpha,\beta} q_\beta(y)$. Next, we have

$$
\begin{aligned}
&\langle q_\alpha(x), p_\beta(y)\rangle_{\alpha,\beta} \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z) + (q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z) + p_\beta q_\beta(y) \blacktriangleright_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z)\\
&- p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) - (q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\beta\alpha,\gamma} q_\gamma(z)\\
=&\lambda\{R_{\alpha\beta}\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma} + \{R_{\alpha\beta}\{R_\alpha q_\alpha(x), p_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma}\\
&+ \{R_\beta p_\beta q_\beta(y), \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{R_\alpha p_\alpha q_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&-\{R_{\beta\alpha}\{R_\beta q_\beta(y), p_\alpha(x)\}_{\beta,\alpha}, q_\gamma(z)\}_{\beta\alpha,\gamma}\\
=&\{\lambda R_{\alpha\beta}\{q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} + R_{\alpha\beta}\{R_\alpha q_\alpha(x), p_\beta(y)\}_{\alpha,\beta} + R_{\alpha\beta}\{q_\alpha(x), R_\beta p_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma}\\
&+\{R_\beta p_\beta q_\beta(y), \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma} - \{R_\alpha p_\alpha q_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} \qquad \text{(by Eq. (21))}\\
=&\{\{R_\alpha q_\alpha(x), R_\beta p_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma} + \{R_\beta p_\beta q_\beta(y), \{R_\alpha p_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma}\\
&-\{R_\alpha p_\alpha q_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} \qquad \text{(by Eq. (5))}\\
=&\{\{q_\alpha R_\alpha(x), R_\beta p_\beta(y)\}_{\alpha,\beta}, q_\gamma(z)\}_{\alpha\beta,\gamma} + \{q_\beta R_\beta p_\beta(y), \{p_\alpha R_\alpha(x), z\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma}\\
&-\{p_\alpha q_\alpha R_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
=&\{q_{\alpha\beta}q_{\alpha\beta}^{-1}\{q_\alpha R_\alpha(x), R_\beta p_\beta(y)\}_{\alpha,\beta}, p_\gamma p_\gamma^{-1}q_\gamma(z)\}_{\alpha\beta,\gamma} + \{q_\beta R_\beta p_\beta(y), \{q_\alpha q_\alpha^{-1}p_\alpha R_\alpha(x), p_\gamma p_\gamma^{-1}(z)\}_{\alpha,\gamma}\}_{\beta,\alpha\gamma}\\
&-\{p_\alpha q_\alpha R_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma}\\
=&-\{q_\gamma p_\gamma^{-1}q_\gamma(z), p_{\alpha\beta}q_{\alpha\beta}^{-1}\{q_\alpha R_\alpha(x), R_\beta p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} - \{q_\beta R_\beta p_\beta(y), \{q_\gamma p_\gamma^{-1}(z), p_\alpha q_\alpha^{-1}p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&-\{p_\alpha q_\alpha R_\alpha(x), \{R_\beta p_\beta(y), z\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} \qquad \text{(by Eq. (21))}\\
=&-\{q_\alpha^2 q_\alpha^{-1}p_\alpha R_\alpha(x), \{q_\beta q_\beta^{-1}R_\beta p_\beta(y), p_\gamma p_\gamma^{-1}(z)\}_{\beta,\gamma}\}_{\alpha,\beta\gamma} - \{q_\beta^2 q_\beta^{-1}R_\beta p_\beta(y), \{q_\gamma p_\gamma^{-1}(z), p_\alpha q_\alpha^{-1}p_\alpha R_\alpha(x)\}_{\gamma,\alpha}\}_{\beta,\gamma\alpha}\\
&-\{q_\gamma^2 p_\gamma^{-1}(z), \{q_\alpha q_\alpha^{-1}p_\alpha R_\alpha(x), p_\beta q_\beta^{-1}R_\beta p_\beta(y)\}_{\alpha,\beta}\}_{\gamma,\alpha\beta} \qquad \text{(by } p_{\alpha\beta}, q_{\alpha\beta}^{-1} \text{ satisfying Eq. (20))}\\
=&0. \qquad \text{(by Eq. (22))}
\end{aligned}
$$

Similarly, we have

$$\langle q_\alpha(x) \blacktriangleright_{\alpha,\beta} y, q_\gamma(z)\rangle_{\alpha\beta,\gamma} + \langle q_\beta(y), p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z\rangle_{\beta,\alpha\gamma} = p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} \langle y, z\rangle_{\beta,\gamma}.$$

This completes the proof. □

Remark 4.9. If $\lambda = 0$, then Proposition 4.8 reduces to Theorem 3.19.

Remark 4.10. By the link among Theorem 3.17, Proposition 4.7 and Proposition 4.8, we have the following commutative diagram.

$$
\begin{tikzcd}
\begin{array}{c}\text{BiHom-}\Omega\text{-Lie algebra}\\ (L, \{\cdot,\cdot\}_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}\end{array} \arrow[rr, "\text{Proposition 4.8}", "{\langle x,y\rangle_{\alpha,\beta}:=\lambda\{x,y\}_{\alpha,\beta} \text{ and } x\blacktriangleright_{\alpha,\beta}y:=\{R_\alpha(x),y\}_{\alpha,\beta}}"'] \arrow[dr, "\text{Theorem 3.17}", "{<x,y>_{\alpha,\beta}:=\{R_\alpha(x),y\}_{\alpha,\beta}+\{x,R_\beta(y)\}_{\alpha,\beta}+\lambda\{x,y\}_{\alpha,\beta}}"'] & & \begin{array}{c}\text{BiHom-}\Omega\text{-PostLie algebra}\\ (L, \langle\cdot,\cdot\rangle_{\alpha,\beta}, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}\end{array} \arrow[dl, "\text{Proposition 4.7}"', "{<x,y>_{\alpha,\beta}:=x\blacktriangleright_{\alpha,\beta}y-(p_\beta^{-1}q_\beta(y))\blacktriangleright_{\beta,\alpha}(p_\alpha q_\alpha^{-1}(x))+\langle x,y\rangle_{\alpha,\beta}}"] \\
& \begin{array}{c}\text{BiHom-}\Omega\text{-Lie algebra}\\ (L, <\cdot,\cdot>_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}.\end{array} &
\end{tikzcd}
$$

More precisely, we have

$$
\begin{aligned}
<x, y>_{\alpha,\beta} =& x \blacktriangleright_{\alpha,\beta} y - (p_\beta^{-1}q_\beta(y)) \blacktriangleright_{\beta,\alpha} (p_\alpha q_\alpha^{-1}(x)) + \langle x, y\rangle_{\alpha,\beta}\\
=&\{R_\alpha(x), y\}_{\alpha,\beta} - \{R_\beta p_\beta^{-1}q_\beta(y), p_\alpha q_\alpha^{-1}(x)\}_{\beta,\alpha} + \lambda\{x, y\}_{\alpha,\beta}\\
=&\{R_\alpha(x), y\}_{\alpha,\beta} - \{q_\beta p_\beta^{-1}R_\beta(y), p_\alpha q_\alpha^{-1}(x)\}_{\beta,\alpha} + \lambda\{x, y\}_{\alpha,\beta}\\
=&\{R_\alpha(x), y\}_{\alpha,\beta} + \{q_\alpha q_\alpha^{-1}(x), p_\beta p_\beta^{-1}R_\beta(y)\}_{\alpha,\beta} + \lambda\{x, y\}_{\alpha,\beta}\\
&\qquad \text{(by Eq. (21))}\\
=&\{R_\alpha(x), y\}_{\alpha,\beta} + \{x, R_\beta(y)\}_{\alpha,\beta} + \lambda\{x, y\}_{\alpha,\beta},
\end{aligned}
$$

for all $x, y \in L$, $\alpha, \beta \in \Omega$.

4.2. **BiHom-Ω-pre-Possion algebras.** In this subsection, we generlize the relationship between pre-Lie algebras and pre-Possion algebras to BiHom-Ω version. First of all, let's recall the concepts of Ω-zinbiel algebras and Ω-pre-Possion algebras.

Definition 4.11. [1]

(a) An **Ω-zinbiel algebra** $(Z, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space Z equipped with a family of binary operations $(*_{\alpha,\beta} : Z\times Z \to Z)_{\alpha,\beta\in\Omega}$ such that

$$x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) = (x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z + (y *_{\beta,\alpha} x) *_{\beta\alpha,\gamma} z, \tag{23}$$

for all $x, y, z \in Z$, α, β, $\gamma \in \Omega$.

(b) An **Ω-pre-Possion algebra** $(B, \triangleright_{\alpha,\beta}, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is a vector space B equipped with two families of bilinear operations $\triangleright_{\alpha,\beta}, *_{\alpha,\beta} : B\times B \to B$ such that $(B, \triangleright_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-pre-Lie algebra, $(B, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ is an Ω-zinbiel algebra and

$$(x \triangleright_{\alpha,\beta} y - y \triangleright_{\beta,\alpha} x) *_{\alpha\beta,\gamma} z = x \triangleright_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) - y *_{\beta,\alpha\gamma} (x \triangleright_{\alpha,\gamma} z),$$
$$(x *_{\alpha,\beta} y + y *_{\beta,\alpha} x) \triangleright_{\alpha\beta,\gamma} z = x *_{\alpha,\beta\gamma} (y \triangleright_{\beta,\gamma} z) + y *_{\beta,\alpha\gamma} (x \triangleright_{\alpha,\gamma} z),$$

for all $x, y, z \in B$, α, β, $\gamma \in \Omega$.

Definition 4.12. (a) Let $(Z, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(Z', *'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-zinbiel algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : Z \to Z'$ is called an **Ω-zinbiel algebra morphism** if

$$f_{\alpha\beta}(x *_{\alpha,\beta} y) = f_\alpha(x) *'_{\alpha,\beta} f_\beta(y), \tag{24}$$

for all $x, y \in Z$, α, $\beta \in \Omega$.

(b) Let $(B, \triangleright_{\alpha,\beta}, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ and $(B', \triangleright'_{\alpha,\beta}, *'_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be two Ω-pre-Possion algebras. A family of linear maps $(f_\alpha)_{\alpha\in\Omega} : B \to B'$ is called an **Ω-pre-Possion algebra morphism** if

$$f_{\alpha\beta}(x \triangleright_{\alpha,\beta} y) = f_\alpha(x) \triangleright'_{\alpha,\beta} f_\beta(y),$$
$$f_{\alpha\beta}(x *_{\alpha,\beta} y) = f_\alpha(x) *'_{\alpha,\beta} f_\beta(y),$$

for all $x, y \in B$, α, $\beta \in \Omega$.

Below, we will generlize the above definitions to the BiHom version.

Definition 4.13. A **BiHom-Ω-zinbiel algebra** $(Z, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space Z equipped with a family of binary operations $(\circledast_{\alpha,\beta} : Z\times Z \to Z)_{\alpha,\beta\in\Omega}$ and two commuting Ω-zinbiel algebra morphisms $p_\alpha, q_\alpha : Z \to Z$ such that

$$p_\alpha q_\alpha(x) \circledast_{\alpha,\beta\gamma} (p_\beta(y) \circledast_{\beta,\gamma} z) = (q_\alpha(x) \circledast_{\alpha,\beta} p_\beta(y)) \circledast_{\alpha\beta,\gamma} q_\gamma(z) + (q_\beta(y) \circledast_{\beta,\alpha} p_\alpha(x)) \circledast_{\beta\alpha,\gamma} q_\gamma(z),$$

for all $x, y, z \in Z$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of Z.

Combining BiHom-Ω-zinbiel algebras and BiHom-Ω-pre-Lie algebras, we propose the following definition.

Definition 4.14. A **BiHom-Ω-pre-Possion algebra** $(B, \blacktriangleright_{\alpha,\beta}, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a vector space B equipped with two families of bilinear maps $\blacktriangleright_{\alpha,\beta}, \circledast_{\alpha,\beta} : B \times B \to B$ and two commuting families of linear maps $(p_\alpha)_{\alpha\in\Omega}, (q_\alpha)_{\alpha\in\Omega} : B \to B$ such that $(B, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra, $(B, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-zinbiel algebra and

$$\begin{aligned}&(q_\alpha(x) \blacktriangleright_{\alpha,\beta} p_\beta(y) - q_\beta(y) \blacktriangleright_{\beta,\alpha} p_\alpha(x)) \circledast_{\alpha\beta,\gamma} q_\gamma(z)\\ &\qquad = p_\alpha q_\alpha(x) \blacktriangleright_{\alpha,\beta\gamma} (p_\beta(y) \circledast_{\beta,\gamma} z) - p_\beta q_\beta(y) \circledast_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z),\end{aligned}$$

$$(q_\alpha(x) \circledast_{\alpha,\beta} p_\beta(y) + q_\beta(y) \circledast_{\beta,\alpha} p_\alpha(x)) \blacktriangleright_{\alpha\beta,\gamma} q_\gamma(z)$$
$$= p_\alpha q_\alpha(x) \circledast_{\alpha,\beta\gamma} (p_\beta(y) \blacktriangleright_{\beta,\gamma} z) + p_\beta q_\beta(y) \circledast_{\beta,\alpha\gamma} (p_\alpha(x) \blacktriangleright_{\alpha,\gamma} z),$$

for all $x, y, z \in B$, $\alpha, \beta, \gamma \in \Omega$. The maps p_α and q_α (in this order) are called the structure maps of B.

Next, we introduce the relationship between BiHom-Ω-pre-Possion algebras and BiHom-Ω-pre-Lie algebras.

Remark 4.15. Let $(B, \blacktriangleright_{\alpha,\beta}, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ be a BiHom-Ω-pre-Possion algebra.

(a) If $x \circledast_{\alpha,\beta} y = 0$, for all $x, y \in B$, $\alpha, \beta \in \Omega$, then $(B, \blacktriangleright_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Lie algebra.

(b) If $x \blacktriangleright_{\alpha,\beta} y = 0$, for all $x, y \in B$, $\alpha, \beta \in \Omega$, then $(B, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-zinbiel algebra.

As usual, we characterize the Yau twisting procedure for BiHom-Ω-zinbiel algebras as follows.

Proposition 4.16. *Let $(Z, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-zinbiel algebra. If $p_\alpha, q_\alpha : Z \to Z$ are two commuting Ω-zinbiel algebra morphisms and we define the multiplication on Z by*

$$x \circledast_{\alpha,\beta} y := p_\alpha(x) *_{\alpha,\beta} q_\beta(y), \quad \textit{for all } x, y \in Z,\ \alpha, \beta \in \Omega.$$

Then $(Z, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-zinbiel algebra, called the Yau twist of $(Z, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

Proof. For $x, y, z \in Z$, $\alpha, \beta, \gamma \in \Omega$, we have

$$\begin{aligned}
& p_\alpha q_\alpha(x) \circledast_{\alpha,\beta\gamma} (p_\beta(y) \circledast_{\beta,\gamma} z) \\
=& p_\alpha^2 q_\alpha(x) *_{\alpha,\beta\gamma} q_{\beta\gamma}(p_\beta(y) \circledast_{\beta,\gamma} z) \\
=& p_\alpha^2 q_\alpha(x) *_{\alpha,\beta\gamma} q_{\beta\gamma}(p_\beta^2(y) *_{\beta,\gamma} q_\gamma(z)) \\
=& p_\alpha^2 q_\alpha(x) *_{\alpha,\beta\gamma} (q_\beta p_\beta^2(y) *_{\beta,\gamma} q_\gamma^2(z)) \qquad \text{(by } q_{\beta\gamma} \text{ satisfying Eq. (24))} \\
=& (p_\alpha^2 q_\alpha(x) *_{\alpha,\beta} q_\beta p_\beta^2(y)) *_{\alpha\beta,\gamma} q_\gamma^2(z) + (q_\beta q_\beta^2(y) *_{\beta,\alpha} p_\alpha^2 q_\alpha(x)) *_{\beta\alpha,\gamma} q_\gamma^2(z) \\
& \qquad \text{(by Eq. (23))} \\
=& (p_\alpha^2 q_\alpha(x) *_{\alpha,\beta} p_\beta q_\beta p_\beta(y)) *_{\alpha\beta,\gamma} q_\gamma^2(z) + (p_\beta^2 q_\beta(y) *_{\beta,\alpha} p_\alpha q_\alpha p_\alpha(x)) *_{\beta\alpha,\gamma} q_\gamma^2(z) \\
& \qquad \text{(by } p_\alpha, q_\alpha \text{ commuting with each other)} \\
=& p_{\beta\alpha}(p_\alpha q_\alpha(x) *_{\alpha,\beta} q_\beta p_\beta(y)) *_{\alpha\beta,\gamma} q_\gamma^2(z) + p_{\beta\alpha}(p_\beta q_\beta(y) *_{\beta,\alpha} q_\alpha p_\alpha(x)) *_{\beta\alpha,\gamma} q_\gamma^2(z) \\
& \qquad \text{(by } p_{\alpha\beta} \text{ satisfying Eq. (24))} \\
=& (q_\alpha(x) \circledast_{\alpha,\beta} p_\beta(y)) \circledast_{\alpha\beta,\gamma} q_\gamma(z) + (q_\beta(y) \circledast_{\beta,\alpha} p_\alpha(x)) \circledast_{\beta\alpha,\gamma} q_\gamma(z).
\end{aligned}$$

□

The following result is the Yau twisting procedure for BiHom-Ω-pre-Possion algebras and the proof is similar to Proposition 4.16.

Proposition 4.17. *Let $(B, \triangleright_{\alpha,\beta}, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$ be an Ω-pre-Possion algebra and let $p_\alpha, q_\alpha : B \to B$ be two commuting Ω-pre-Possion algebra morphisms. Define two operations on B by*

$$x \blacktriangleright_{\alpha,\beta} y := p_\alpha(x) \triangleright_{\alpha,\beta} q_\beta(y), \qquad x \circledast_{\alpha,\beta} y := p_\alpha(x) *_{\alpha,\beta} q_\beta(y),$$

for all $x, y \in B$, $\alpha, \beta \in \Omega$. Then $(B, \blacktriangleright_{\alpha,\beta}, \circledast_{\alpha,\beta}, p_\alpha, q_\alpha)_{\alpha,\beta\in\Omega}$ is a BiHom-Ω-pre-Possion algebra, called the Yau twist of $(B, \triangleright_{\alpha,\beta}, *_{\alpha,\beta})_{\alpha,\beta\in\Omega}$.

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