# Characteristic Currents on Cohesive Modules 

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#### Abstract

Let $\mathcal{F}$ be a coherent sheaf on a complex variety $X$ that has a locally free resolution $E^{\bullet}$. In [19], the authors constructed a pseudomeromorphic current whose support is contained in $\operatorname{supp}\left(E^{\bullet}\right)$ that represents products of Chern classes of $\mathcal{F}$. In this paper, we show that their construction works for general de-Rham characteristic classes and then generalize it to represent products (in de-Rham cohomology) of characteristic forms of cohesive modules defined by Block [8]. Finally, we state a corollary to a transgression result in 16] that show that it is sufficient to only use the degree- 0 and degree- 1 parts of the superconnection to construct currents [6] [5] that represent characteristic forms of cohesive modules in the Bott-Chern cohomology.


## 1 Introduction

While coherent sheaves on any quasi-projective scheme over a Noetherian affine scheme admits a locally free resolution [14, Example 6.5.1], this is not generally true. An example is certain coherent sheaves on $\operatorname{Spec}\left(K[x] /\left(x^{2}\right)\right.$ for any field K. 12, Example 4.18]. To circumvent such issue, in [8, Definition 2.3.2] 9], Block introduced the differential-graded category $\mathcal{P}_{\mathcal{D}}$ of Cohesive Modules over the differential-graded algebra (dga) $\mathcal{D}=\left(\mathcal{A}^{\bullet}(X), d, 0\right)=\left(\mathcal{A}^{\bullet, 0}(X), \bar{\partial}, 0\right)$ the Dolbeault dga of a complex manifold $X$ (and also over general curved dga's) and studied their properties. It has the important property that

Theorem 1.1. [8, Theorem 4.1.3] Let $X$ be a compact complex manifold, and $D_{c o h}^{b}(X)$ be the bounded derived category of complexes $\mathcal{O}_{X}$-sheaves with coherent cohomology. Then the homotopy category $\operatorname{Ho}\left(\mathcal{P}_{\mathcal{D}}\right)$, whose objects are exactly those of $\mathcal{P}_{\mathcal{D}}$ and whose morphisms $\operatorname{Ho}\left(\mathcal{P}_{\mathcal{D}}\right)(x, y)=H^{0}\left(\mathcal{P}_{\mathcal{D}}(x, y)\right)$, is equivalent to $D_{\text {coh }}^{b}(X)$.

Thus many statements about coherent sheaves admitting a locally free resolution can be translated into more general statements about cohesive modules. Our main result in this paper (Theorem 5.1) generalizes Theorem 5.1 in [19],
which constructs, for a coherent sheaf $\mathcal{F}$ that admits a locally free resolution on a complex manifold $X$, a current having the same support as a resolution for $\mathcal{F}$
that represents products of Chern classes/Chern forms of $\mathcal{F}$, to a more general class of characteristic forms on cohesive modules which define classes in both the de-Rham cohomology and Bott-Chern cohomology of the manifold $X$. At the end, we ask some questions about transgression formulae for superconnections and superconnection currents that represent certain Bott-Chern characteristic forms.

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## 3 Cohesive Modules and Unitary Connections

### 3.1 The $\bar{\partial}$-superconnection

Let $X$ be a complex manifold, and $\mathcal{D}=\left(\mathcal{A}^{0, \bullet}(X), \bar{\partial}\right)$ be its Dolbeault differential graded algebra, we can define the dg-category $\mathcal{P}_{\mathcal{D}}$ of $\mathcal{D}$-cohesive modules as follows [16]: the objects are $E=\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)$. Here, $E^{\bullet}=\bigoplus_{k=0}^{N} E_{k}$, with each $E^{k}$ a finite dimensional complex vector bundle over $X$.

Proposition 3.1. Here are some basic facts about the sheaves of $E^{\bullet}-$ and $\operatorname{End}\left(E^{\bullet}\right)$-valued differential forms

1. $\mathcal{A}^{0, \bullet}\left(X, E^{\bullet}\right) \cong \mathcal{A}^{0, \bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}\left(X, E^{\bullet}\right)$ and $\mathcal{A}^{\bullet, 0}\left(X, E^{\bullet}\right) \cong$ $\mathcal{A}^{\bullet}, 0(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}\left(X, E^{\bullet}\right)$. Therefore $\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right) \cong \mathcal{A}^{\bullet}(X) \otimes_{\mathcal{A}^{0}(X)}$ $\mathcal{A}^{0}\left(X, E^{\bullet}\right)$.
2. Same can be said if we replace $E^{\bullet}$ by $E n d_{\mathbb{C}}\left(E^{\bullet}\right)$ in (1).
3. Therefore $\operatorname{End}_{\mathcal{O}_{X}}\left(\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right)\right) \cong \mathcal{A}^{\bullet}\left(X, \operatorname{End}_{\mathbb{C}}\left(E^{\bullet}\right)\right)$ as $\mathcal{O}_{X}$-modules.

Now let $\mathbb{E}^{\prime \prime}: \mathcal{A}^{0, \bullet}\left(X, E^{\bullet}\right) \rightarrow \mathcal{A}^{0, \bullet}\left(X, E^{\bullet}\right)$ be $\mathcal{O}_{X}$-linear of total degree-1 and satisfy the following:

1. $\mathbb{E}^{\prime \prime} \circ \mathbb{E}^{\prime \prime}=0$; i.e. $\mathbb{E}^{\prime \prime}$ is flat.
2. The $\bar{\partial}$-Leibniz formula $\forall s \in \mathcal{A}^{0}\left(X, E^{\bullet}\right), \forall \omega \in \mathcal{A}^{0, \bullet}(X)$,

$$
\begin{equation*}
\mathbb{E}^{\prime \prime}(s \otimes \omega):=\mathbb{E}^{\prime \prime}(s) \otimes \omega+(-1)^{\operatorname{deg}(\omega)} s \otimes \bar{\partial}(\omega) \tag{1}
\end{equation*}
$$

The meaning of total degree- 1 is that $\forall p, q \in \mathbb{N}$

$$
\mathbb{E}^{\prime \prime}\left(\mathcal{A}^{0, p}\left(X, E^{q}\right)\right) \subseteq \bigoplus_{k \geq \max \{-p,-q+1\}} \mathcal{A}^{p+k}\left(X, E^{q-k+1}\right) \Rightarrow \mathbb{E}^{\prime \prime}=\bigoplus_{k \in \mathbb{Z}} \mathbb{E}_{k}^{\prime \prime}
$$

with $\mathbb{E}_{k}^{\prime \prime}=0, \forall k<\max \{-p,-q+1\}$. Note that by definition of $\mathbb{E}^{\prime \prime}$ and degree, we know that $\mathbb{E}_{k}^{\prime \prime}$ is $\mathcal{A}^{\bullet}(X)$-linear $\forall k \neq 1$. Now consider $\forall 0 \leq k \leq n$, we have $\mathbb{E}^{\prime \prime}\left(\left.\mathbb{E}^{\prime \prime}\right|_{\mathcal{A}^{0}\left(X, E^{k}\right)}\right)=0$, so its projection onto $\mathcal{A}^{0}\left(X, E^{k+2}\right)$ is also 0 . Therefore $\mathbb{E}_{0}^{\prime \prime} \circ \mathbb{E}_{0}^{\prime \prime}\left(\mathcal{A}^{0}\left(X, E^{k}\right)\right)=0$, and we know that

$$
0 \longrightarrow \mathcal{A}^{0}\left(X, E^{0}\right) \xrightarrow{\mathbb{E}_{0}^{\prime \prime}} \mathcal{A}^{0}\left(X, E^{1}\right) \xrightarrow{\mathbb{E}_{0}^{\prime \prime}} \cdots \xrightarrow{\mathbb{E}_{0}^{\prime \prime}} \mathcal{A}^{0}\left(X, E^{N}\right) \longrightarrow 0
$$

is a complex of coherent sheaves.

### 3.2 Extended Hermitian Metric and the d-connection

Let $h$ be a Hermitian metric on $E^{\bullet}$. Then, using Proposition 2.1.(1), we can extend $h$ to $\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right)$ via

$$
h(\alpha \otimes f, \beta \otimes g)=\bar{\alpha} \wedge h(f, g) \wedge \beta, \quad \forall \alpha, \beta \in \mathcal{A}^{\bullet}(X), \quad \forall f, g \in \mathcal{A}^{0}\left(X, E^{\bullet}\right)
$$

We will write a cohesive module as $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$ to emphasize the dependence of various properties/constructions on the Hermitian metric. Now we state an important structure theorem:

Theorem 3.1. [16] For a Hermitian cohesive module $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$, there is a unique $\mathbb{E}^{\prime}: \mathcal{A}^{\bullet}, 0\left(X, E^{\bullet}\right) \rightarrow \bigcup_{q \in \mathbb{Z}} \mathcal{A}^{\bullet+q+1,0}\left(X, E^{\bullet-q}\right)$ satisfying the following:

1. $\mathbb{E}^{\prime}$ is a $\partial$-superconnection, i.e $\forall \alpha \otimes f$, with $\alpha \in \mathcal{A}^{\bullet}(X), f \in \mathcal{A}^{0}\left(X, E^{\bullet}\right)$, we have $\mathbb{E}^{\prime}(\alpha \otimes f)=\mathbb{E}^{\prime}(\alpha) \otimes f+(-1)^{\operatorname{deg}(\alpha)} \alpha \otimes(\partial f)$.
2. Extending $\mathbb{E}^{\prime \prime}$ and $\mathbb{E}^{\prime}$ to $\mathcal{A}^{\bullet \bullet}\left(X, E^{\bullet}\right)$ linearly (considering that $\mathcal{A}^{\bullet \bullet}\left(X, E^{\bullet}\right)$ $\left.\cong \mathcal{A}^{0, \bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{\bullet, 0}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}\left(X, E^{\bullet}\right)\right)$, and we have $\mathbb{E}=\mathbb{E}^{\prime \prime}+\mathbb{E}^{\prime}$ is a $d$-superconnection.
3. $\mathbb{E}$ is $h$-unitary, i.e. $\forall s, t \in \mathcal{A}^{\bullet}\left(X, E^{\bullet}\right)$, we have $(-1)^{\operatorname{deg}(s)} d(h(s, t))=$ $-h(\mathbb{E}(s), t)+h(s, \mathbb{E}(t))$.

[^0]4. Writing $\mathbb{E}^{\prime}=\bigoplus_{q \in \mathbb{Z}} \mathbb{E}_{q}^{\prime}, \nabla=\mathbb{E}_{1}^{\prime \prime}+\mathbb{E}_{1}^{\prime}: \mathcal{A}^{\bullet \bullet}\left(X, E^{\bullet}\right) \rightarrow \mathcal{A}^{\bullet+1, \bullet+1}\left(X, E^{\bullet}\right)$ is a unitary d-connection.

From the last statement we see that $\nabla$ restricts to connections on each $E^{k}, 0 \leq k \leq N$. Therefore, writing $\nabla_{k}=\left.\nabla\right|_{E_{k}}$ and $\nabla=\bigoplus_{k=0}^{N} \nabla_{k}$, we know that $\left(E^{k}, \nabla_{k}\right)$ is a vector bundle with a $d$-connection. Note that $\nabla$ induces a connection $\nabla^{E n d}$ on $\operatorname{End}\left(E^{\bullet}\right)$ by

$$
\nabla^{E n d}: \mathcal{A}^{\bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right) \rightarrow \mathcal{A}^{\bullet+1}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right), \quad \phi \mapsto \nabla \circ \phi-(-1)^{\operatorname{deg}(\phi)} \phi \circ \nabla
$$

### 3.3 Construction of a Compatible Unitary Connection

Definition 3.1. [19, Section 4] Connections $\left\{\Theta_{k}\right\}_{0 \leq k \leq N}$ on $\left\{E_{k}\right\}$ are com-
patible with the complex $0 \longrightarrow E_{0} \xrightarrow{\phi_{0}} E_{1} \xrightarrow{\phi_{1}} \cdots \xrightarrow{\phi_{N-1}} E_{N} \longrightarrow 0$ if $\Theta_{k+1} \circ \phi_{k}=-\phi_{k} \circ \Theta_{k} \Leftrightarrow \Theta_{k+1} \circ \phi_{k}+\phi_{k} \circ \Theta_{k}=0$.

The reason why compatibility is important will be seen in the next section. In the case of cohesive modules, the $\nabla_{k}$ 's might not be compatible with $\mathbb{E}_{0}^{\prime \prime}$. Let
$Z_{i}$ be the set where $\mathbb{E}_{0}^{\prime \prime}$ is not exact, and let $Z=\bigcup_{i=0}^{N} Z_{i}$. We will call $Z$ the support of the cohesive module. Let $\chi: \mathbb{R} \rightarrow[0,1]$ be a smooth characteristic function such that $\chi \equiv 0$ on $(-\infty, 1-\delta)$ and $\chi \equiv 1$ on $(1+\delta, \infty)$, for some arbitrarily small positive $\delta$.

Now for a positive $\epsilon>0$, if $Z=\emptyset$, define $\chi_{\epsilon} \equiv 1$ on $X$. Otherwise, define $F=\boxtimes_{k=1}^{N} F_{k}=\boxtimes_{k=1}^{N} \operatorname{det}\left(\left.\mathbb{E}_{0}^{\prime \prime}\right|_{E_{k-1}}\right)^{\wedge \operatorname{rank}\left(E_{k-1}\right)}$ on $X[2$, Section 2], which is a section to the coherent sheaf $\mathcal{F}=\boxtimes_{k=1}^{N}\left(\bigwedge^{\operatorname{rank}\left(E_{k-1}\right)} E_{k-1}^{*} \otimes \bigwedge^{\operatorname{rank}\left(E_{k-1}\right)} E_{k}\right)$. Then it is clear that $Z=\{F=0\}=\bigcup_{k=1}^{N}\left\{F_{k}=0\right\}$. If it is impossible to find an $F$ that is generically nonvanishing such that $Z \subseteq\{F=0\}$, define $\chi_{\epsilon} \equiv 1$. Otherwise define $\chi_{\epsilon}(x)=\chi\left(\frac{|F(x)|^{2}}{\epsilon}\right), \forall x \in X$. In this case $\chi_{\epsilon} \equiv 1$ except on a small neighborhood of $Z$ when $\epsilon$ is small. (Since $F$ is generically nonvanishing, we can modify $F$ such that $F \equiv 1$ except in a small neighborhood of $Z$.) Now we need another concept before constructing the compatible connections: the minimal inverse.

Definition 3.2. [19, 2] For $0 \leq k \leq N-1$, write $\mathcal{A}^{0}\left(X, E_{k+1}\right)=\mathbb{E}_{0}^{\prime \prime}\left(\mathcal{A}^{0}\left(X, E_{k}\right)\right)$ $\oplus F_{k+1}$. Define the minimal inverse $\sigma_{k}: E_{k+1} \rightarrow E_{k}$, a morphism between vector bundles, by the following conditions: if $e \in \mathbb{E}_{0}^{\prime \prime}\left(E_{k}\right)$, then $\sigma_{k}(e) \equiv n$, where $\mathbb{E}_{0}^{\prime \prime}(n)=e$, and $n$ has pointwise the minimal $h-$ norm among all such vectors. If under $h, e \perp \mathbb{E}_{0}^{\prime \prime}\left(E_{k}\right)$, then $\sigma_{k}(e) \equiv 0$. It then follows that $\mathbb{E}_{0}^{\prime \prime} \circ \sigma_{k} \circ \mathbb{E}_{0}^{\prime \prime}=\mathbb{E}_{0}^{\prime \prime}$.

Remark. Here are some properties of $\sigma_{k}$

1. The minimality of $\sigma_{k}(e)$ is equivalent to stating that $n \perp \operatorname{Ker}\left(\left.\mathbb{E}_{0}^{\prime \prime}\right|_{E_{k}}\right)$, since any complement of $\operatorname{Ker}\left(\mathbb{E}_{0}^{\prime \prime} \mid E_{k}\right)$ injects onto $\mathbb{E}_{0}^{\prime \prime}\left(E_{k}\right)$ under $\mathbb{E}_{0}^{\prime \prime} \cdot[1]$
2. From Remark (1), we know that $\operatorname{Im}\left(\sigma_{k}\right) \perp \operatorname{Ker}\left(\left.\mathbb{E}_{0}^{\prime \prime}\right|_{E_{k}}\right) \Rightarrow \operatorname{Im}\left(\sigma_{k}\right) \perp$ $\mathbb{E}_{0}^{\prime \prime}\left(E_{k-1}\right)$, since $\left(\mathbb{E}_{0}^{\prime \prime}\right)^{2}=0$. This means that $\sigma_{k-1} \sigma_{k}=0$. [19]
3. $\sigma_{k}$ is smooth on $X \backslash Z_{k}$. This is because on $X \backslash Z_{k}$, the rank of $\left.\mathbb{E}_{0}^{\prime \prime}\right|_{E_{k}}$ is constant. It then suffices to show that $\left.\sigma_{k}\right|_{\mathbb{E}_{0}^{\prime \prime}\left(E_{k}\right)}$ is smooth, since $\sigma_{k}$ on the orthogonal complement is constant. This follows from a description of $\sigma_{k}$ in [1, Section 3].

Now we construct the connections $\nabla_{k}^{\epsilon}=\nabla_{k}-\chi_{\epsilon}\left(\sigma_{k} \circ \nabla^{E n d} \circ \mathbb{E}_{0}^{\prime \prime}\right)$ on $E_{k}$ 's, and we write $\nabla^{\epsilon}=\bigoplus_{k=0}^{N} \nabla_{k}^{\epsilon}$. Then we have

Theorem 3.2. [19, Lemma 4.4] For any $\epsilon>0$, the connections $\left\{\nabla_{k}^{\epsilon}\right\}_{0 \leq k \leq N}$ are compatible with $\mathbb{E}_{0}^{\prime \prime}$ exactly where $\chi_{\epsilon} \equiv 1$.

## 4 Characteristic Class in de-Rham Cohomology

### 4.1 Characteristic Forms of $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$

Let $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$ be a Hermitian cohesive module. Define the curvature $R_{h}=$ $\mathbb{E}^{2}=\frac{1}{2}[\mathbb{E}, \mathbb{E}]=\left[\mathbb{E}^{\prime}, \mathbb{E}^{\prime \prime}\right]$.

Remark. Noting that $\mathbb{E} \in \operatorname{End}\left(\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right)\right) \cong \mathcal{A}^{\bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right) \cong$ $\mathcal{A}^{\bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right.$, so if we write $\mathbb{E}=\alpha \otimes f$, then we have $R_{h}=$ $(\alpha \wedge \alpha) \otimes(f \circ f)$.

Then, following Quillen's notion of the supertrace 20, for a fixed convergent complex power series $f(T)$, we define its characteristic form to be $\operatorname{Tr}_{s}\left(f\left(R_{h}\right)\right)$, where $\operatorname{Tr}_{s}: \mathcal{A}^{\bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right) \rightarrow \mathcal{A}^{\bullet}(X)$ is defined as follows 20, 2]: letting $E^{\bullet}=$ $E^{+} \oplus E^{-}$, with $E^{+}=\bigoplus_{2 \mid k} E_{k}, E^{-}=\bigoplus_{2 \mid k} E_{k}$, we define $\operatorname{Tr}_{s}: \operatorname{End}\left(E^{\bullet}\right) \rightarrow$ $\mathbb{C}, X \mapsto \operatorname{tr}(\epsilon X)$, where for $e \in E^{+}, \epsilon X(e)=X(e)$, and for $e \in E^{-}, \epsilon X(e)=$ $-X(e)$. Now extend $\operatorname{Tr}_{s}$ linearly to $\mathcal{A}^{\bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$. Then we have the following facts

Theorem 4.1. [16]/[Corollary 2.26] Characteristic forms are closed, so they define classes in $H_{d R}^{\bullet}(X, \mathbb{C})$. These classes are well-defined by Serre's Vanishing Theorem. We then have $\left[\operatorname{Tr}_{s}\left(f\left(R_{h}\right)\right)\right]=\left[\operatorname{Tr}_{s}\left(f\left(\Theta_{\nabla}\right)\right)\right]^{2}$ in $H_{d R}^{\bullet}(X, \mathbb{C})$, where $\Theta_{\nabla}=\nabla^{2}$ is the curvature form associated to $\nabla$.

### 4.2 Characteristic Forms of Exact Chain Complexes

We first show the following claim establishing a more explicit relation between the characteristic form and the curvature form. Let $(E, \nabla)$ be a vector bundle on $X$. Denote $\operatorname{tr}: \mathcal{A}^{\bullet}(X, \operatorname{End}(E)) \rightarrow \mathcal{A}^{\bullet}(X)$ the extension of the trace function on $\mathcal{A}^{0}(X, \operatorname{End}(E))$. Then we have

[^1]Proposition 4.1. $\left[\operatorname{tr}\left(f\left(\Theta_{\nabla}\right)\right)\right] \in H_{d R}^{*}(X)$ is a polynomial in the Chern classes of $E$. Specifically, it is a symmetric polynomial in $\left[\Theta_{\nabla}\right]$. It is also a sum of homogeneous polynomimals in $\left[\Theta_{\nabla}\right]$.

Proof. Let $X, Y$ be two complex algebraic varieties, and $(E, \Delta)$ a vector bundle of rank $k$ on $X$. We show that $(E, \Delta) \mapsto\left[\operatorname{tr}\left(f\left(\Theta_{\Delta}\right)\right)\right]$ is a natural transformation from $\operatorname{Vect}_{k}(-; \mathbb{C})$ to $H^{*}(-)$. For a morphism $\phi: Y \rightarrow X$, let $\left(\phi^{*} E, \phi^{*} \Delta\right)$ be the pullback vector bundle on $Y$ with the pullback connection which is functorial (as defined in [21, Theorem 3.6(a)]), we know that $\Theta_{\phi^{*} \Delta}=\phi^{*} \Theta_{\Delta}$. Now it suffices to show that $\forall i \in \mathbb{N}, \phi^{*}\left[\operatorname{tr}\left(\Theta_{\Delta}^{i}\right)\right]=\left[\operatorname{tr}\left(\left(\phi^{*} \Theta_{\Delta}\right)^{i}\right)\right]$. Recall the splitting principle
Lemma 4.2. [10, Section 21] Let $E \rightarrow X$ a $C^{\infty}$ complex vector bundle and $p: \mathbb{P}(E) \rightarrow M$ be the projection map. Then $p^{*}(E) \rightarrow \mathbb{P}(E)$ splits into a direct sum of line bundles and $p^{*}: H^{*}(X) \rightarrow H^{*}(\mathbb{P}(E))$ is an embedding.

Using the lemma and the fact that $\phi^{*} \mathbb{P}(E)=\mathbb{P}\left(\phi^{*}(E)\right)$, and considering the commutative diagram

and then noting that $\mathbb{P}\left(\phi^{*}(E)\right)=\phi^{*}(\mathbb{P}(E))$, we can reduce to when $E \rightarrow X$ is a line bundle, in which case $\phi^{*}$ amounts to multiplication by an element in $\mathcal{O}_{X}$ on both sides.

Now recall that every natural transformation $\operatorname{Vect}_{k}(-, \mathbb{C}) \rightarrow H_{d R}^{*}(-)$ can be expressed as a polynomial in the Chern classes [10, Proposition 23.11], it remains to show that the Chern classes $c_{n}(E)$ is a polynomial of $\left[\Theta_{\Delta}\right]$. This follows directly from [21, Definition 3.4].

Remark. To show that $\left[\operatorname{tr}\left(f\left(\Theta_{\nabla}\right)\right)\right]$ is a polynomial, not a series, in the Chern classes, we implicitly used the fact that degrees $\geq 2 \operatorname{dim}_{\mathbb{C}}(X)$ vanish in $H_{d R}^{*}(X)$.

Now we recall a Whitney formula [4, Lemma 4.22]
Theorem 4.3. Let $\phi$ be a symmetric homogeneous polynomial of degree less than or equal to $\operatorname{dim}_{\mathbb{C}}(X)$, then let $\left\{D_{k}\right\}_{0 \leq k \leq N}$ be compatible connections on $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)$. If the complex $\left(E^{\bullet}, \mathbb{E}_{0}^{\prime \prime}\right)$ is exact, then in de-Rham cohomology, $\left[\phi\left(\Theta_{D_{0}}\right)\right]=-\left(\frac{2 \pi}{i}\right)^{\operatorname{deg}(\phi)} \cdot\left[\phi\left(\sum_{i=1}^{N}(-1)^{i} E_{i}\right)\right]$ [4, Equation 4.15], which is defined as follows: letting $\sigma_{k}\left(D_{j}\right)=\left(\frac{2 \pi}{i}\right)^{k} \cdot c_{k}\left(E_{j}, D_{j}\right.$ 3 4, Equation 3.34], and $\phi\left(D_{j}\right)=$ $\tilde{\phi}\left(\sigma_{1}\left(D_{j}\right), \cdots, \sigma_{\operatorname{deg}(\phi)}\left(D_{j}\right)\right)$, and writing $\sum_{i=1}^{N}(-1)^{i} E_{i}$ as $\widehat{E^{\bullet}}$, we define $\phi\left(\widehat{E^{\bullet}}\right) \equiv \tilde{\phi}\left(c_{1}\left(\widehat{E^{\bullet}}\right), \cdots, c_{\text {deg }(\phi)}\left(\widehat{E^{\bullet}}\right)\right)$.

[^2]Remark. Here, the Chern classes of $\widehat{E^{\bullet}}$ satisfy the following:

$$
1+\sum_{i=1}^{2 \operatorname{dim}_{\mathbb{C}}(X)} c_{i}\left(\widehat{E_{\bullet}}\right)=c\left(\widehat{E_{\bullet}}\right) \equiv \bigoplus_{k=0}^{N} c\left(E_{i}\right)^{(-1)^{i}}
$$

Since $\phi$ is homogeneous, we know that $\left[\phi\left(\Theta_{D_{0}}\right)\right]=\left(\frac{2 \pi}{i}\right)^{\operatorname{deg}(\phi)}$
$\left[\tilde{\phi}\left(c_{1}\left(E_{0}, D_{0}\right), \cdots, c_{d e g(\phi)}\left(E_{0}, D_{0}\right)\right]\right.$. Therefore From this and Proposition 3.1, we automatically have

Corollary 4.3.1. On $X \backslash Z$,

$$
\operatorname{Tr}_{s}\left(f\left(\Theta_{\nabla}\right)\right)=\bigoplus_{k=0}^{N} \operatorname{Tr}_{s}\left(f\left(\Theta_{\nabla_{k}}\right)\right)=\sum_{k=0}^{N}(-1)^{k} \operatorname{tr}\left(\Theta_{\nabla_{k}}\right)=0
$$

## 5 Characteristic Currents

Now we define the characteristic currents on a cohesive module $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$. For $a \in H_{d R}^{*}(X)$, denote $a_{k}$ to be the degree- $k$ component of $a$.

Definition 5.1. For $\left(p_{1}, \cdots, p_{k}\right) \in\left[0,2 \operatorname{dim}_{\mathbb{C}}(X)\right]^{k}$ define the characteristic current to be $\operatorname{Tr}_{p_{1}}\left(E^{\bullet}, \nabla\right) \wedge \cdots \wedge \operatorname{Tr}_{p_{k}}\left(E^{\bullet}, \nabla\right) \equiv \lim _{\epsilon \rightarrow 0}\left[\operatorname{Tr}_{s}\left(f\left(\nabla^{\epsilon}\right)\right)\right]_{p_{1}} \wedge \cdots \wedge$ $\left[\operatorname{Tr}_{s}\left(f\left(\nabla^{\epsilon}\right)\right)\right]_{p_{k}}$.

Remark. We will explain the meaning of wedge product right after Definition 5.2 (assuming Theorem 5.1 a priori).

Definition 5.2. [3] Let $f$ be a holomorphic function on $X$. For $a \in \mathbb{N}$, the current $\left[\frac{1}{f^{a}}\right]$ is defined as the functional on test forms $\xi \mapsto \lim _{\epsilon \rightarrow 0} \int_{|f|>\epsilon} \frac{\xi}{f^{a}}$ and $\bar{\partial}\left[\frac{1}{f^{a}}\right]$ sends test form $\xi$ to $\lim _{\epsilon \rightarrow 0} \int_{|f|>\epsilon} \frac{\bar{\partial}(\xi)}{f^{a}}$. These are well-defined by 15, Theorem 7.1]. Let $\Pi=\Pi_{1} \circ \Pi_{2} \circ \cdots \circ \Pi_{r}$ be a sequence of resolutions of singularities, with $\Pi_{i}: Y_{i} \rightarrow Y_{i-1}$ with $Y_{0}=X$. Then a current on $X$ is pseudomeromorphic if it can be written as $\sum_{\ell} \Pi_{*} \tau_{\ell}$, where $\tau_{\ell}$ is a current on some $Y_{\ell}$ of the form $\left(\prod_{i=1}^{k}\left[\frac{1}{f^{a_{i}}}\right]\right) \bar{\partial}\left[\frac{1}{f^{b_{1}}}\right] \wedge \cdots \wedge \bar{\partial}\left[\frac{1}{f^{b_{m}}}\right]$ for some holomorphic $f$ on $Y_{\ell}$.

Here, we need to define a notion of "wedge product" of currents. This is given by the Coleff-Herrera product[11, Theorem 1.7.2][18, Theorem 2]. Call $\left(\epsilon_{1}, \cdots, \epsilon_{p}\right) \rightarrow(0, \cdots, 0)$ along an admissible path if $\forall k \in \mathbb{N}$ and $\forall j \geq 2, \frac{\epsilon_{j-1}}{\epsilon_{j}^{k}} \rightarrow$ 0 . in this case we write $\epsilon_{1} \ll \cdots \ll \epsilon_{p} \rightarrow 0$. Then for $f_{1}, \cdots, f_{p}$ holomorphic, we define

$$
\bar{\partial}\left[\frac{1}{f_{1}}\right] \wedge \cdots \wedge \bar{\partial}\left[\frac{1}{f_{p}}\right]=\lim _{\epsilon_{1} \ll \cdots<\epsilon_{p} \rightarrow 0} \frac{\bar{\partial} \chi\left(\left|f_{1}\right|^{2} / \epsilon_{1}\right)}{f_{1}} \wedge \cdots \wedge \frac{\bar{\partial} \chi\left(\left|f_{p}\right|^{2} / \epsilon_{p}\right)}{f_{p}}
$$

where for each $\left(\epsilon_{1}, \cdots, \epsilon_{p}\right)$ and for any test form $\phi$ of bi-degree $\left(\operatorname{dim}_{\mathbb{C}}(X)\right.$, $\left.\operatorname{dim}_{\mathbb{C}}(X)-p\right)$, the expression on the right hand side denotes $\phi \mapsto \int_{X} \frac{\bar{\partial} \chi\left(\left|f_{1}\right|^{2} / \epsilon_{1}\right)}{f_{1}} \wedge$
$\cdots \wedge \frac{\bar{\partial} \chi\left(\left|f_{p}\right|^{2} / \epsilon_{p}\right)}{f_{p}} \wedge \phi$. Then for holomorphic $f_{1}, \cdots, f_{k}, g_{1}, \cdots, g_{m}$, and for a $\left(\operatorname{dim}_{\mathbb{C}}(X), \operatorname{dim}_{\mathbb{C}}(X)-p\right)-$ test form $\phi$, define $\left(\prod_{i=1}^{k}\left[\frac{1}{f_{i}}\right]\right) \bar{\partial}\left[\frac{1}{g_{1}}\right] \wedge \cdots \wedge \bar{\partial}\left[\frac{1}{g_{m}}\right](\phi)$ as

$$
\lim _{\substack{\delta_{1}, \cdots, \delta_{k} \rightarrow 0 \\ \epsilon_{1}<\cdots<\epsilon_{m} \rightarrow 0}} \int_{\left|f_{i}\right|>\delta_{i}, \forall i} \frac{\bigwedge_{j=1}^{m} \bar{\partial} \chi\left(\frac{\left|g_{j}\right|^{2}}{\epsilon_{j}}\right) \wedge \phi}{\prod_{i=1}^{k} f_{i} \prod_{j=1}^{m} g_{j}}
$$

Remark. It follows from Definition 5.2 that pushforwards of pseudomeromorphic currents under resolutions of singularities are still pseudomeromorphic.
Theorem 5.1. The characteristic current $\operatorname{Tr}_{p_{1}}\left(E^{\bullet}, \nabla\right) \wedge \cdots \wedge \operatorname{Tr}_{p_{k}}\left(E^{\bullet}, \nabla\right)$ is a well-defined closed pseudomeromorphic current, with support contained in $Z$, an analytic subvariety of positive codimension, that represents $\operatorname{Tr}_{s}\left(f\left(\Theta_{\nabla}\right)\right)_{p_{1}} \wedge$ $\cdots \wedge \operatorname{Tr}_{s}\left(f\left(\Theta_{\nabla}\right)\right)_{p_{k}}$ for any $k-$ tuple $\left(p_{1}, \cdots, p_{k}\right)$.

Remark. Denote $\left(\mathbb{D}^{*}(X), d\right)$ the complex of currents on $M$. Here $\mathbb{D}^{q}(X)$ is the dual space to $\Omega_{c}^{2 d^{2} m_{\mathbb{C}}(X)-q}(X)$, the vector space of compactly supported smooth forms on $X$, with the dual topology. Also, the chain map $d: \mathbb{D}^{q}(X) \rightarrow \mathbb{D}^{q+1}(X)$ is given by $(d T)(\phi)=(-1)^{q+1} T(d \phi)$. Then we have $H_{d R}^{*}(X) \xrightarrow{\cong} H^{*}\left(\mathbb{D}^{*}(X), d\right) .[13$, Chapter 3, Section 1]
Proof. By Proposition 4.1 and linearity, it suffices to consider the case where $T r_{s}\left(f\left(\Theta_{\nabla}\right)\right)$ is a monomial in the Chern classes. Then from [19, Lemma 2.1] and [19, Theorem 5.1], it suffices to show that $\forall \ell_{1}, \ell_{2}$ we have

$$
\lim _{\epsilon \rightarrow 0} c_{\ell_{1}}\left(E^{\bullet}, \nabla^{\epsilon}\right) \wedge \lim _{\delta \rightarrow 0} c_{\ell_{2}}\left(E^{\bullet}, \nabla^{\delta}\right)=\lim _{\epsilon \rightarrow 0} c_{\ell_{1}}\left(E^{\bullet}, \nabla^{\epsilon}\right) \wedge c_{\ell_{2}}\left(E^{\bullet}, \nabla^{\epsilon}\right)
$$

and then proceed inductively (for wedge products of more Chern classes). Since $\left[\operatorname{tr}\left(f\left(\Theta_{\nabla}\right)\right)\right]$ is a characteristic class, it does not depend on the choice of connection. As in the proof of [19, Theorem 5.1], for any $\epsilon$ and $\delta$, we can write $c_{\ell_{1}}\left(E^{\bullet}, \epsilon\right)$ as $A_{1}+\sum_{j \geq 1} \chi_{\epsilon}^{j} B_{j}+\sum_{j \geq 1} \chi_{\epsilon}^{j-1} \wedge d \chi_{\epsilon} \wedge B_{j}^{\prime}$ and $c_{\ell_{2}}\left(E^{\bullet}, \delta\right)$ as $A_{2}+$ $\sum_{j \geq 1} \chi_{\delta}^{j} C_{j}+\sum_{j \geq 1} \chi_{\delta}^{j-1} \wedge d \chi_{\delta} \wedge C_{j}^{\prime}$, where $A_{1}, A_{2}, B_{j}, C_{j}, B_{j}^{\prime}, C_{j}^{\prime}$ are independent of $\epsilon$ and $\delta$, with $A_{1}, A_{2}$ smooth and $B_{j}, C_{j}, B_{j}^{\prime}, C_{j}^{\prime}$ polynomials in the entries of the minimal inverses $\sigma_{k}\left(c f . \quad\right.$ Definition 2.2), $D_{\operatorname{End}(E \bullet)} \mathbb{E}_{0}^{\prime \prime}$, and $\theta_{k}$, which are the matrices representing $\nabla_{k}(0 \leq k \leq N)$. Also, $\chi_{\epsilon}=\chi\left(\frac{|F|^{2}}{\epsilon}\right)$ as defined in Section 3.3. Therefore it suffices to consider where $\chi_{\epsilon} \not \equiv 1$ for $\epsilon$ small enough and show that for any $i, j$ and any $s, t$ products of entries of $\sigma_{k}, D_{E n d(E \bullet)} \mathbb{E}_{0}^{\prime \prime}$ and $\theta_{k}$, we have $\chi_{\epsilon}, \lim _{\epsilon \rightarrow 0} \chi_{\epsilon}^{i} s \wedge \lim _{\delta \rightarrow 0} \chi_{\delta}^{j} t=\lim _{\epsilon} \chi_{\epsilon}^{i+j} s \wedge t$, and also $\lim _{\epsilon \rightarrow 0} \chi_{\epsilon}^{i} d \chi_{\epsilon} \wedge s \wedge \lim _{\delta \rightarrow 0} \chi_{\delta}^{j} d \chi_{\delta} \wedge t=\lim _{\epsilon \rightarrow 0} \chi_{\epsilon}^{i+j} d \chi_{\epsilon} \wedge s \wedge d \chi_{\epsilon} \wedge t$.

We use similar ideas to [19, Lemma 2.1]. By resolution of singularities [17, Theorem 3.36], since in Section 2.3 we found a section $F=\boxtimes_{j=0}^{N} F_{j}$ to a coherent sheaf (actually a vector bundle) $\mathcal{F}$ such that $Z \subseteq Z(F)$, which implies that $(F) \cdot \mathcal{O}_{X}{ }^{4}$ is not invertible on $Z(F)$, we know that there exists a (composition of) birational and projective modification(s) $\pi: Y \rightarrow X$ such that

[^3]$\left.\pi\right|_{Y-\pi^{-1}(Z(F))}: Y-\pi^{-1}(Z(F)) \rightarrow X-Z(F)$ is an isomorphism, and the coherent sheaf of ideals on $Y$ generated by pullbacks of local sections to $(F) \cdot \mathcal{O}_{X}$, which we write as $\pi^{-1}\left((F) \cdot \mathcal{O}_{X}\right)$, is a monomonial sheaf of ideals 5 Equivalently, this is the subsheaf of $\mathcal{O}_{Y}$ generated by $\pi^{*} F=\boxtimes_{j=0}^{N} \pi^{*} F_{j}$. This means that $\pi^{*} F_{j}=F_{j 0} F_{j 1}$, where $F_{j 0}=\prod_{i=1}^{n} z_{i}^{c_{i}}$ is a monomial in local coordinates $\left\{z_{1}, \cdots, z_{n}\right\}$ and $Z\left(F_{j 0}\right) \subseteq \pi^{-1}(Z(F))$ and $F_{j 1}$ is holomorphic and nonvanishing. Then on $Y$, we have the local formula
\[

$$
\begin{equation*}
\pi^{*} \sigma_{k}=\frac{1}{F_{k 0}} \phi_{k}, \forall 0 \leq k \leq N \tag{2}
\end{equation*}
$$

\]

with $\phi_{k}$ smooth everywhere on $Y$. (cf.the definition of $\sigma_{k}$ in Definition 2.2. This can also be found in [2, Section 2]) Note that $D_{\operatorname{End}(E \bullet)} \mathbb{E}_{0}^{\prime \prime}$ and $\theta_{k}$ are everywhere smooth. Now by [19, Equation 2.2], writing $\pi^{*} s=\frac{1}{\psi_{1}} \tilde{s}, \pi^{*} t=\frac{1}{\psi_{2}} \tilde{t}$, with $\psi_{1}, \psi_{2}$ products of monomials (thus also monomials) of local coordinates on $Y$ and $\tilde{s}, \tilde{t}$ smooth $\sqrt[6]{ }$ In view of Equation (2), it suffices to show that for any test $2 \operatorname{dim}_{\mathbb{C}}(X)$-form $\xi$ on $Y$ (here we note that $\chi \sim \chi^{i}, \forall i \in \mathbb{N}$, and also $\left\{\psi_{1}=0\right\} \bigcup\left\{\psi_{2}=0\right\} \subseteq \pi^{-1}(Z)=\left\{\pi^{*} F=0\right\}$, using the fact that $\pi$ is an isomorphism on $X-Z(F)$ and $s, t$ are smooth outside of $Z$ )

$$
\begin{gather*}
{\left[\lim _{\epsilon \rightarrow 0} \frac{\chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)^{i+j}}{\psi_{1} \psi_{2}}\right](\xi)=\left[\lim _{\epsilon \rightarrow 0} \frac{\chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)^{i}}{\psi_{1}}\right]\left[\lim _{\delta \rightarrow 0} \frac{\chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\delta}\right)^{j}}{\psi_{2}}\right](\xi) \Longleftrightarrow} \\
\lim _{\epsilon \rightarrow 0} \int_{\left|\psi_{1} \psi_{2}\right|>\epsilon} \frac{\xi}{\psi_{1} \psi_{2}}=\lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{\left|\psi_{1}\right|>\epsilon} \frac{\xi}{\left|\psi_{2}\right|>\delta}< \tag{3}
\end{gather*}
$$

since this current does not depend on the choice of characteristic function. 19 , Lemma 2.1] Then the difference between the two is (noting that $\xi$ is a test form so $\|\xi\|_{L^{\infty}(X)}$ exists)

$$
\begin{gathered}
\lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{\left|\psi_{1}\right|>\epsilon} \frac{\left(\mathbb{1}_{\left|\psi_{2}\right|>\epsilon /\left|\psi_{1}\right|}-\mathbb{1}_{\left|\psi_{2}\right|>\delta}\right) \xi}{\psi_{1} \psi_{2}}+\lim _{\epsilon \rightarrow 0} \int_{\left|\psi_{1}\right|<\epsilon} \frac{\mathbb{1}_{\left|\psi_{2}\right|>\epsilon /\left|\psi_{1}\right|} \xi}{\psi_{1} \psi_{2}} \\
\leq\left(\lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{\substack{\delta>\epsilon /\left|\psi_{1}\right| \\
\epsilon /\left|\psi_{1}\right|<\left|\psi_{2}\right|<\delta}}^{\left|\psi_{1}\right|}\left|\frac{\xi}{\psi_{1} \psi_{2}}\right|+\int_{\left|\psi_{1} \psi_{2}\right|<\epsilon}\left|\frac{\xi}{\psi_{1} \psi_{2}}\right|\right)+\lim _{\epsilon \rightarrow 0} \int_{\left|\psi_{1}\right|<\epsilon}\left|\frac{\xi}{\psi_{1}}\right| \leq \\
\|\xi\|_{L^{\infty}(X)}\left(\lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{\substack{\left|\psi_{1}\right|>\epsilon \\
\delta /\left|\psi_{1}\right|<\left|\left|\psi_{2}\right|<\delta\right.}}^{\substack{\left|\psi_{1}\right|<\delta}}\left|\frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{\psi_{1} \psi_{2}}\right|+\lim _{\epsilon \rightarrow 0} \int_{\left|\psi_{1} \psi_{2}\right|<\epsilon}\left|\frac{\mathbb{1}_{\text {Supp }(\xi)}}{\psi_{1} \psi_{2}}\right|+\int_{\left|\psi_{1}\right|<\epsilon}\left|\frac{\mathbb{1}_{\text {Supp }(\xi)}}{\psi_{1}}\right|\right)
\end{gathered}
$$

[^4]We have

$$
\begin{aligned}
& \lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int \underset{\substack{\delta>\epsilon\left|>\epsilon \\
\epsilon /\left|\psi_{1}\right|<\left|\psi_{2}\right|<\delta\right.}}{\left|\psi_{1}\right|>\epsilon}\left|\frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{\psi_{1} \psi_{2}}\right| \leq \lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{\substack{\left|\psi_{1}\right|>\epsilon \\
\left|\psi_{2}\right|<\delta}}\left|\frac{\mathbb{1}_{\operatorname{Supp}(\xi)} \mid}{\psi_{1} \psi_{2}}\right| \\
& \leq \lim _{\epsilon \rightarrow 0} \sqrt{\int_{\left|\psi_{1}\right|>\epsilon} \frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{\left|\psi_{1}\right|^{2}}} \cdot \lim _{\delta \rightarrow 0} \sqrt{\int_{\left|\psi_{2}\right|<\delta} \frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{\left|\psi_{2}\right|^{2}}}=O\left(\lim _{\delta \rightarrow 0} \int_{\left|\psi_{2}\right|<\delta} \frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{\left|\psi_{2}\right|}\right),
\end{aligned}
$$

which will follow from Lemma 5.3. To prove that this is 0 , and also the second and third terms vanish, it suffices to prove the following two lemmas:

Lemma 5.2. Let $\phi$ be a holomorphic function on $M$ whose vanishing locus is a measure-zero set (specifically, a subvariety of positive codimension), and such that $\frac{1}{\phi} \in L^{1}(\operatorname{Supp}(\xi))$, then $\lim _{\epsilon \rightarrow 0} \int_{|\phi|<\epsilon} \frac{1}{|\phi|}=0$. Here, $|\cdot|$ is the complex norm.

Proof. Note that $\operatorname{codim}(Z(\phi))>0 \Rightarrow \mu(Z(\phi))=0$, where $\mu$ is the pullback (under the coordinate maps) of the Lebesgue measure on $\mathbb{C}^{\operatorname{dim}_{\mathbb{C}}(X)}$. Define

$$
f_{n}: X \rightarrow[0,+\infty], \quad x \mapsto \begin{cases}\frac{1}{|\phi(x)|} & |\phi(x)|<\frac{1}{n}  \tag{4}\\ 0 & \text { otherwise }\end{cases}
$$

Then note that $f_{i}(x) \geq f_{j}(x), \forall i<j$, and $\lim _{n \rightarrow \infty} f_{n}(x) \neq 0 \Leftrightarrow \phi(x)=0$. Now since $\int_{\text {Supp }(\xi)}\left|f_{1}\right| \leq \int_{\operatorname{Supp}(\xi)} \frac{1}{|\phi|}<\infty$, then by monotone convergence, we have

$$
\begin{gathered}
\lim _{\epsilon \rightarrow 0} \int_{|\phi|<\epsilon} \frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{|\phi|}=\lim _{n \rightarrow \infty} \int_{|\phi|<\frac{1}{n}} \frac{\mathbb{1}_{\operatorname{Supp}(\xi)}}{|\phi|}=\lim _{n \rightarrow \infty} \int_{\operatorname{Supp}(\xi)} f_{n} \\
=\int_{\operatorname{Supp}(\xi)} \lim _{n \rightarrow \infty} f_{n}=\int_{\operatorname{Supp}(\xi) \cap Z(\phi)} \lim _{n \rightarrow \infty} f_{n}
\end{gathered}
$$

which is 0 since $\mu(\operatorname{Supp}(\xi) \cap Z(\phi)) \leq \mu(Z(\phi))=0$.
Lemma 5.3. We can apply a further sequence of blow-ups $\Pi: \tilde{Y} \rightarrow Y$ such that $\frac{1}{\Pi^{*} \psi_{1}}$ and $\frac{1}{\Pi^{*} \psi_{2}} \in L^{1}(\operatorname{Supp}(\xi))$. Therefore we can assume $W L O G$ that $\frac{1}{\psi_{1}}, \frac{1}{\psi_{2}} \in$ $L^{1}(\operatorname{Supp}(\xi))$.

Proof. We will show this for $\psi_{1}$ only. In local coordinates, write $\psi_{1}=\prod_{i=1}^{n} z_{i}^{c_{i}}$. Then by Fubini, and writing $r=\operatorname{diam}(\operatorname{Supp}(\xi))<\infty$ we have $\left\|\psi_{1}\right\|_{L^{1}(\operatorname{Supp}(\xi))} \leq$ $\prod_{i=1}^{n} \int_{\left|z_{i}\right| \leq r} \frac{1}{\left|z_{i}\right|^{c_{i}}}$, which is finite if $c_{i}$ is smaller than $2 \operatorname{dim}_{\mathbb{C}}(Y), \forall i$. Now since $\operatorname{Supp}(\xi) \cap Z\left(\pi^{*} F\right)$ is compact, we can cover it by finitely many coordinate neighborhoods, and $\left|\psi_{1} \psi_{2}\right|$ will have a strictly positive lower bound outside these neighborhoods. Thus the problem reduces to finding $\Pi: \tilde{Y} \rightarrow Y$ such that $\Pi^{*} \psi_{1}$ and $\Pi^{*} \psi_{2}$ are monomials in local coordinates covering $S u p p(\xi) \cap Z\left(\pi^{*} F\right)$ in which the degree of every coordinate does not exceed $2 \operatorname{dim}_{\mathbb{C}}(Y)-1$. This will follow directly from [17, Theorem 3.68], which states that if $I \subseteq \mathcal{O}_{X}$ is an ideal sheaf with Maxord $\operatorname{Supp}(\xi) \cap Z\left(\pi^{*} F\right)(I) \leq m$ for some $m \in \mathbb{N}$, then there is a composition of blow-ups $\Pi=\Pi_{r} \circ \cdots \circ \Pi_{1}$ such that $\operatorname{Maxord}_{\Pi^{-1}\left(\operatorname{Supp}(\xi) \cap Z\left(\pi^{*} F\right)\right)} \Pi^{-1}(I)=$ $\operatorname{Maxord}_{\Pi^{-1}}(\operatorname{Supp}(\xi)) \cap Z\left(\Pi^{*} \pi^{*} F\right)<m$. Here, for a point $y \in Y, \operatorname{ord}_{y}(I) \stackrel{\text { def }}{=} \max \{r$ :
$\left.\mathfrak{m}_{y}^{r} \mathcal{O}_{Y, y} \supset I_{x}\right\}$, where $\mathfrak{m}_{y}$ is the maximal ideal of $\mathcal{O}_{Y, y}$; for a subset $Z \subseteq Y$, define $\operatorname{Maxord}_{Z}(I)=\sup _{y \in Z} \operatorname{ord}_{y}(I)$. [17, Definition 3.47] In our case, we let $I=$ $\left(\psi_{1}, \psi_{2}\right) \mathcal{O}_{Y}$. Then $\operatorname{ord}_{y}(I)$ is just the maximal order of the two monomials.
(Proof of Theorem 5.1-Continued)
Now we show that $\lim _{\epsilon \rightarrow 0} \pi^{*}\left(\chi_{\epsilon}\right)^{i} d \pi^{*} \chi_{\epsilon} \wedge \pi^{*} s \wedge \lim _{\delta \rightarrow 0}\left(\pi^{*} \chi_{\delta}\right)^{j} d \pi^{*} \chi_{\delta} \wedge \pi^{*} t=$ $\lim _{\epsilon \rightarrow 0}\left(\pi^{*} \chi_{\epsilon}\right)^{i+j} d \pi^{*} \chi_{\epsilon} \wedge \pi^{*} s \wedge d \pi^{*} \chi_{\epsilon} \wedge \pi^{*} t$. Since $\pi^{*} s, \pi^{*} t$ have nice expressions, for a test form $\xi$ of matching degree, it suffices to consider

$$
\begin{gathered}
{\left[\lim _{\epsilon \rightarrow 0} \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)^{i} \frac{d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)}{\psi_{1}} \bigwedge \lim _{\delta \rightarrow 0} \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\delta}\right)^{j} \frac{d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\delta}\right)}{\psi_{2}}\right](\xi)} \\
=\lim _{\substack{\epsilon \rightarrow 0 \\
\delta \rightarrow 0}} \int_{X} \frac{d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right) \wedge d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\delta}\right) \wedge \xi}{\psi_{1} \psi_{2}}
\end{gathered}
$$

and

$$
\left[\lim _{\epsilon \rightarrow 0} \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)^{i+j} \frac{\bigwedge^{2} d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)}{\psi_{1} \psi_{2}}\right](\xi)=\lim _{\epsilon \rightarrow 0} \int_{X} \frac{\bigwedge^{2} d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right) \wedge \xi}{\psi_{1} \psi_{2}}
$$

The difference is

$$
\lim _{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0 \\ \tau \rightarrow 0}} \int_{X} \frac{\xi}{\psi_{1} \psi_{2}} \wedge\left(d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right) \bigwedge d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\delta}\right)-\bigwedge^{2} d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\tau}\right)\right)
$$

Note that the term in the parenthesis is nonzero only when $\left|\pi^{*} F\right|^{2}<(1+\nu)$. $\min \{\epsilon, \delta, \tau\}$, where $\operatorname{supp}(\chi) \subseteq[0,1+\nu)$. By definition of $\chi,\|\nabla \chi\|_{L^{\infty}(\mathbb{R})}<\infty$, and

$$
\left\|d \chi\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)\right\| \leq\|\nabla \chi\|_{L^{\infty}(\mathbb{R})} \cdot\left\|\nabla\left(\frac{\left|\pi^{*} F\right|^{2}}{\epsilon}\right)\right\|,
$$

so the difference does not exceed (writing $k=\|\xi\|_{L^{\infty}(X)} \cdot\|\nabla \chi\|_{L^{\infty}(\mathbb{R})}$ $\cdot\left\|\nabla\left(\left|\pi^{*} F\right|^{2}\right)\right\|_{L^{\infty}(\operatorname{Supp}(\xi))}$, and denoting the region $\left\{\left|\pi^{*} F\right|^{2}<(1+\nu) \cdot \min \{\epsilon, \delta, \tau\}\right\}$ by $C_{\epsilon, \delta, \tau}$ ),

$$
k \cdot \lim _{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0 \\ \tau \rightarrow 0}} \int_{C_{\epsilon, \delta, \tau} \cap \operatorname{Supp}(\xi)} \frac{1}{\left|\psi_{1} \psi_{2}\right|}\left(\frac{1}{\epsilon \delta}-\frac{1}{\tau^{2}}\right) .
$$

Note that by Hölder's Inequality, the integrand does not exceed

$$
\begin{aligned}
& \sqrt{\int_{\operatorname{Supp}(\xi) \cap C_{\epsilon, \delta, \tau}} \frac{1}{\left|\psi_{1} \psi_{2}\right|^{2}}} \cdot \sqrt{\int_{\operatorname{Supp}(\xi) \cap C_{\epsilon, \delta, \tau}}\left|\frac{1}{\epsilon \delta}-\frac{1}{\tau^{2}}\right|^{2}} \\
& \quad \leq \sqrt{\int_{\operatorname{Supp}(\xi)} \frac{1}{\left|\psi_{1} \psi_{2}\right|^{2}}} \cdot \sqrt{\int_{\operatorname{Supp}(\xi) \cap C_{\epsilon, \delta, \tau}} \frac{2}{\left|\pi^{*} F\right|^{2}}} .
\end{aligned}
$$

Since $\operatorname{Supp}(\xi)$ is compact, from what we have shown before the first term is finite. Also, the second term goes to 0 as $\epsilon, \delta$ and $\tau$ go to 0 simultaneously, since $\mu\left(Z\left(\left|\pi^{*} F\right|^{2}\right)\right)=0$ and after applying a further sequence of resolutions making the degrees of local coordinates low enough in $\left|\pi^{*} F\right|^{2}$, just as in the proof of Lemma 5.3, we can follow essentially the same proof as for Equation (3)). Then the statement will then follow from Proposition 5.1.

Proposition 5.1. $\pi_{*}\left(\lim _{\epsilon \rightarrow 0} \pi^{*}\left(\chi_{\epsilon}\right)^{i} d \pi^{*} \chi_{\epsilon} \wedge \pi^{*} s \wedge \lim _{\delta \rightarrow 0}\left(\pi^{*} \chi_{\delta}\right)^{j} d \pi^{*} \chi_{\delta} \wedge \pi^{*} t\right)=$ $\lim _{\epsilon \rightarrow 0} \chi_{\epsilon}^{i} d \chi_{\epsilon} \wedge s \wedge \lim _{\delta \rightarrow 0} \chi_{\delta}^{j} d \chi_{\delta} \wedge t$, and we can say the same about the other three currents we are considering. 7

Proof. This is clear from the definition.
Remark. It is clear from our method that [19, Theorem 5.1] also applies to any other characteristic class in de-Rham cohomology.

## 6 Chern Currents in Bott-Chern Cohomology

### 6.1 The Bott-Chern Character

We first define the double complex of Bott-Chern cohomology classes of the cohesive module $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$. 7] Letting $d=\partial+\bar{\partial}$ be the de-Rham differential, we have

Definition 6.1. $H_{B C}^{p, q}(X) \equiv\left(\mathcal{A}^{p, q}(X) \cap \operatorname{Ker}(d)\right) / \bar{\partial} \partial \mathcal{A}^{p-1, q-1}(X)$.
The Bott-Chern character of $\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)$ is defined by $c h_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}, h\right)=$ $\operatorname{Trs}\left(\exp \left(-\mathcal{R}_{h}\right)\right)$. By [16, Lemma 2.20], this defines a class in $H_{B C}(X)$. The Bott-Chern character is also independent of the Hermitian metric $h$ by [16, Corollary 3.14][7], Theorem 8.2]. Therefore it makes sense to write it as $\operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)$. If the complex $\left(\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right), \mathbb{E}_{0}^{\prime \prime}\right)$ is exact, then we have

Theorem 6.1. [16, Theorem 4.21] Let $\mathbb{E}_{t}^{\prime \prime}=\left.\sum_{k=0}^{N} t^{\frac{1-k}{2}} \mathbb{E}^{\prime \prime}\right|_{E_{k}}$ for $t>0$ and $\mathcal{R}_{t}$ denote the corresponding curvatures. Let $N_{H}$ be the number operator $h \mapsto k h$ if $h \in \mathcal{A}^{\bullet}\left(X, E_{k}\right)[16$, Definition 4.18][6, Definition 2.3], and we have

$$
\operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)=\partial \bar{\partial} \int_{1}^{\infty} \operatorname{Trs}\left(\frac{N_{H} \cdot \exp \left(-\mathcal{R}_{t}\right)}{t}\right) d t \Rightarrow \operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)=0
$$

Now we consider the case where $\mathcal{A}^{\bullet}\left(X, E^{\bullet}\right)$ is not exact. Again, denote the support of this complex by $Z$.

[^5]
### 6.2 Transgression Formulae and Superconnection Currents

There are Bott-Chern currents representing $\operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)$. Denote $Z(F)=X^{\prime}$ and $\phi: \bigwedge T^{*} X \rightarrow \bigwedge T^{*} X, a \mapsto(2 \pi i)^{-|a| / 2} a$. Now write $\delta_{X^{\prime}} \in \mathbb{D}^{*}(X)$ be the current of integration along $X^{\prime}$. Let $\mathscr{H} \mathscr{E}$ be the sheaf of cohomology groups of $E^{\bullet}$. Note that $\forall x \in X$ there is a canonical isomorphism $\mathscr{H} \mathscr{E}_{x} \cong\left\{y \in E^{\bullet}\right.$ : $\left.\mathbb{E}_{0}^{\prime \prime}(y)=0, \mathbb{E}_{0}^{\prime}(y)=0\right\}$, so by [6, Theorem 1.2] it inherits a Hermitian metric from $h$. Let $\nabla^{\mathscr{H} \mathscr{E}}$ be the connection compatible with the inherited Hermitian metric. Now we define the following superconnections:

1. Fix a $y \in\left(N_{X / X^{\prime}}\right)_{\mathbb{R}}$ with $\bar{y} \in \overline{\left(N_{X / X^{\prime}}\right)_{\mathbb{R}}}$, define $B=\nabla^{\mathscr{H} \mathscr{E}}+\partial_{y} \mathbb{E}_{0}^{\prime \prime}+\partial_{\bar{y}} \mathbb{E}_{0}^{\prime}$.
2. For $t>0$, let $A_{t}=\nabla^{E^{\bullet}}+\sqrt{t} \mathbb{E}_{0}$.

Now for any $t>0$, define currents $\zeta_{E} \bullet(t), \zeta_{E}^{\prime} \bullet(0)$ and $T(h)$ by

$$
\begin{gathered}
\zeta_{E} \bullet(t)=\frac{1}{\Gamma(t)} \int_{0}^{\infty} u^{t-1}\left(\operatorname{Tr} s\left(N_{H} \cdot \exp \left(-A_{u}^{2}\right)\right)-\int_{X^{\prime}} \operatorname{Tr}\left(N_{H} \cdot \exp \left(-B^{2}\right)\right) \cdot \delta_{X^{\prime}}\right) d u \\
\int_{X} \mu \zeta_{E}^{\prime} \bullet(0)=\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{X} \mu \zeta_{E} \bullet(t), \forall \mu \in \mathcal{A}^{\bullet}(X) \\
T(h)=\phi\left(\zeta_{E}^{\prime} \bullet(0)\right)
\end{gathered}
$$

Then we have the following representation formula
Theorem 6.2. [5, Theorem 2.5]

$$
\operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)=\left(\int_{N_{X / X^{\prime}}} \phi\left(\operatorname{Tr} s\left(\exp \left(-B^{2}\right)\right)\right)\right) \delta_{X^{\prime}}-\frac{\bar{\partial} \partial}{2 \pi i} T(h)
$$

By [6. Theorem 3.2], We know that the wave-front set of $\left(\int_{N_{X / X^{\prime}}} \phi\left(\operatorname{Trs}\left(\exp \left(-B^{2}\right)\right)\right)\right) \delta_{X^{\prime}}$ is contained in $\left(N_{X / X^{\prime}}\right)_{\mathbb{R}^{*}}$, and there is the following convergence resembling our previous construction:

Theorem 6.3. As $t \rightarrow \infty$, we have

$$
\operatorname{Trs}\left(\exp \left(-A_{t}^{2}\right)\right) \rightarrow\left(\int_{N_{X / X^{\prime}}} \operatorname{Trs}\left(\exp \left(-B^{2}\right)\right)\right) \delta_{X^{\prime}}
$$

and also (abusing the notation a bit and writing $A_{t}$ also as the current $\xi \mapsto$ $\int_{X} \operatorname{Trs}\left(\exp \left(-A_{t}^{2}\right)\right) \wedge \xi$ and $B$ by the current $\left.\xi \mapsto\left(\int_{N_{X / X^{\prime}}} \operatorname{Trs}\left(\exp \left(-B^{2}\right)\right)\right) \delta_{X^{\prime}}(\xi).\right)$

$$
\lim _{t \rightarrow \infty} \sup _{\xi \in \Gamma}|\xi|^{m} \cdot\left|\phi \cdot \widehat{\left(A_{t}-B\right)}(\xi)\right|=\lim _{t \rightarrow \infty} \sup _{\xi \in \Gamma}|\xi|^{m} \cdot\left|\left(A_{t}-B\right)(\phi \widehat{\xi})\right|=0
$$

with the following nice convergence: $\exists C^{\prime}>0$ such that for $t \ll 1$, we have

$$
\lim _{t \rightarrow \infty} \sup _{\xi \in \Gamma}|\xi|^{m} \cdot\left|\phi \cdot \widehat{\left(A_{t}-B\right)}(\xi)\right| \leq \frac{C^{\prime}}{\sqrt{t}}
$$

for any fixed $m \geq 1$, for any open $U \subseteq X$ biholomorphic to a ball and contained in a trivializing neighborhood of $T_{\mathbb{R}}^{*} X$ and any smooth function $\phi$ supported on $U$, and any $\Gamma$ a closed cone such that on $U \cap X^{\prime}, \Gamma \cap\left(N_{X / X^{\prime}}\right)_{\mathbb{R}}^{*}=\{0\}$.

Observe that all the above constructions involve only the degree-0 and degree- 1 terms of $\mathbb{E}$. It is then natural to ask the following question:

Question 6.1. What effects do the $\mathbb{E}_{k}^{\prime \prime}$ terms $(k \geq 2)$ have on $\operatorname{ch}_{B C}\left(E^{\bullet}, \mathbb{E}^{\prime \prime}\right)$ ?
The answer is that they have no effects. This follows from Qiang's transgression formula with respect to superconnections [16.

### 6.2.1 Known Transgression Formulae

There are two main types of transgression formulae. The first type is with respect to the moduli space (with the topology of uniform $C^{\infty}$ convergence on compact sets) $\mathscr{M}$ of Hemitian metrics on $E^{\bullet}$. [7, Theorem 8.1.2] [6, Theorems 2.1, 2.2, 2.4] [16, Proposition 3.10, Corollary 3.13, Theorem 3.19]. The second type is with respect to the moduli space of superconnections. We will need the following (combining Corollary 4.8 and Proposition 4.15 of [16]):

Theorem 6.4. Let $f$ be a convergent power series. Let $\mathcal{E}$ be the space of all $\bar{\partial}$-superconnections of degree- 1 on $\left(E^{\bullet}, h\right)$. Then $\exists \delta_{1}, \delta_{2}$, which are 1 -forms on the subspace of $\mathcal{A}^{\bullet}, 0\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$ of exotic degre $\underbrace{8}-1$ and the subspace of $\mathcal{A}^{0, \bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$ of exotic degree 1 respectively, and $\gamma_{1}, \gamma_{2}$ which are sections to the subspaces of exotic degree 0 of $\mathcal{A}^{0, \bullet}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$ and $\mathcal{A}^{\bullet, 0}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$ respectively such that

$$
-d^{\mathcal{E}} \operatorname{Tr} s\left(f\left(\mathcal{R}_{h}\right)\right)=\partial \operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h}\right) \cdot \delta_{1}\right)+\bar{\partial} \operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h}\right) \cdot \delta_{2}\right)
$$

and

$$
\begin{aligned}
& \bar{\partial} \operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h} \cdot \gamma_{1}\right)\right)=\operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h}\right) \cdot \delta_{1}\right) \\
& \partial \operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h} \cdot \gamma_{2}\right)\right)=\operatorname{Tr} s\left(f^{\prime}\left(\mathcal{R}_{h}\right) \cdot \delta_{2}\right)
\end{aligned}
$$

For the construction of $\gamma_{1}, \gamma_{2}, \delta_{1}, \delta_{2}$ refer to Definition 4.11 and Definition 4.5 in 16. It then directly follows that in $H_{B C}^{\bullet}(X)$,

Corollary 6.4.1. Let $\psi_{t}\left(\mathbb{E}^{\prime \prime}\right) \in \mathcal{E}$ be the superconnection $\mathbb{E}_{0}^{\prime \prime}+\mathbb{E}_{1}^{\prime \prime}+t \sum_{t>2} \mathbb{E}_{k}^{\prime \prime}$, and let the associated curvature forms be $\phi_{t}\left(\mathcal{R}_{h}\right)$. Then $\operatorname{Trs}\left(f\left(\phi_{1}\left(\mathcal{R}_{h}\right)\right)\right) \simeq$ $\operatorname{Trs}\left(f\left(\phi_{0}\left(\mathcal{R}_{h}\right)\right)\right)=\operatorname{Trs}\left(f\left(\mathcal{R}_{h}\right)\right)$ in $H_{B C}^{\bullet}(X)$.

[^6]
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[^0]:    ${ }^{1}$ Note that $\mathcal{A}^{\bullet, 0}\left(X, E^{\bullet}\right) \cong \mathcal{A}^{\bullet, 0}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}\left(X, E^{\bullet}\right)$

[^1]:    ${ }^{2}$ This $\nabla$ was defined in Theorem 2.1.(4).

[^2]:    ${ }^{3}$ Here $\sigma_{k}$ denotes the elementary symmetric polynomial of degree $k$.

[^3]:    ${ }^{4}$ Here $(F)$ is the ideal of $\mathcal{O}_{X}$ generated by $(F)$.

[^4]:    ${ }^{5}$ Also reference [17, Note 3.16] for the equivalent characterizations of monomial ideal sheaves.
    ${ }^{6}$ More accurately $\pi^{*} s$ might be a sum of such terms, but we can certainly apply a normalization argument to achieve a fraction whose denominator is a monomial in local coordinates.

[^5]:    ${ }^{7}$ i.e. Pushing forward by $\pi$ is the same as taking away all the $\pi^{*}$ in the expressions.

[^6]:    ${ }^{8}$ From Proposition 3.1, we can write an element $A \in \mathcal{A}^{p, q}\left(X, \operatorname{End}\left(E^{\bullet}\right)\right)$ as $\phi \otimes \tau$, with $\phi \in \mathcal{A}^{p, q}(X)$ and $\tau \in E n d^{d}\left(E^{\bullet}\right)$ for some $d$. Then we define the exotic degre to be $d+q-p$.

