Characteristic Currents on Cohesive Modules

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Abstract

Let \mathcal{F} be a coherent sheaf on a complex variety X that has a locally free resolution E^{\bullet} . In [19], the authors constructed a pseudomeromorphic current whose support is contained in $supp(E^{\bullet})$ that represents products of Chern classes of \mathcal{F} . In this paper, we show that their construction works for general de-Rham characteristic classes and then generalize it to represent products (in de-Rham cohomology) of characteristic forms of cohesive modules defined by Block[8]. Finally, we state a corollary to a transgression result in [16] that show that it is sufficient to only use the degree-0 and degree-1 parts of the superconnection to construct currents[6][5] that represent characteristic forms of cohesive modules in the Bott-Chern cohomology.

1 Introduction

While coherent sheaves on any quasi-projective scheme over a Noetherian affine scheme admits a locally free resolution [14, Example 6.5.1], this is not generally true. An example is certain coherent sheaves on $Spec(K[x]/(x^2)$ for any field K.[12, Example 4.18]. To circumvent such issue, in [8, Definition 2.3.2][9], Block introduced the differential-graded category $\mathcal{P}_{\mathcal{D}}$ of **Cohesive Modules** over the differential-graded algebra (dga) $\mathcal{D} = (\mathcal{A}^{\bullet}(X), d, 0) = (\mathcal{A}^{\bullet,0}(X), \overline{\partial}, 0)$ the Dolbeault dga of a complex manifold X (and also over general curved dga's) and studied their properties. It has the important property that

Theorem 1.1. [8, Theorem 4.1.3] Let X be a compact complex manifold, and $D^b_{coh}(X)$ be the bounded derived category of complexes \mathcal{O}_X -sheaves with coherent cohomology. Then the homotopy category $Ho(\mathcal{P}_{\mathcal{D}})$, whose objects are exactly those of $\mathcal{P}_{\mathcal{D}}$ and whose morphisms $Ho(\mathcal{P}_{\mathcal{D}})(x,y) = H^0(\mathcal{P}_{\mathcal{D}}(x,y))$, is equivalent to $D^b_{coh}(X)$.

Thus many statements about coherent sheaves admitting a locally free resolution can be translated into more general statements about cohesive modules. Our main result in this paper (Theorem 5.1) generalizes Theorem 5.1 in [19],

which constructs, for a coherent sheaf \mathcal{F} that admits a locally free resolution on a complex manifold X, a current having the same support as a resolution for \mathcal{F}

that represents products of Chern classes/Chern forms of \mathcal{F} , to a more general class of characteristic forms on cohesive modules which define classes in both the de-Rham cohomology and Bott-Chern cohomology of the manifold X. At the end, we ask some questions about transgression formulae for superconnections and superconnection currents that represent certain Bott-Chern characteristic forms.

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3 Cohesive Modules and Unitary Connections

3.1 The $\overline{\partial}$ -superconnection

Let X be a complex manifold, and $\mathcal{D} = (\mathcal{A}^{0,\bullet}(X), \overline{\partial})$ be its Dolbeault differential graded algebra, we can define the dg-category $\mathcal{P}_{\mathcal{D}}$ of \mathcal{D} -cohesive modules as follows[16]: the objects are $E = (E^{\bullet}, \mathbb{E}')$. Here, $E^{\bullet} = \bigoplus_{k=0}^{N} E_k$, with each E^k a finite dimensional complex vector bundle over X.

Proposition 3.1. Here are some basic facts about the sheaves of E^{\bullet} – and $End(E^{\bullet})$ – valued differential forms

- 1. $\mathcal{A}^{0,\bullet}(X, E^{\bullet}) \cong \mathcal{A}^{0,\bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}(X, E^{\bullet}) \text{ and } \mathcal{A}^{\bullet,0}(X, E^{\bullet}) \cong \mathcal{A}^{\bullet,0}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}(X, E^{\bullet}).$ Therefore $\mathcal{A}^{\bullet}(X, E^{\bullet}) \cong \mathcal{A}^{\bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}(X, E^{\bullet}).$
- 2. Same can be said if we replace E^{\bullet} by $End_{\mathbb{C}}(E^{\bullet})$ in (1).
- 3. Therefore $End_{\mathcal{O}_X}(\mathcal{A}^{\bullet}(X, E^{\bullet})) \cong \mathcal{A}^{\bullet}(X, End_{\mathbb{C}}(E^{\bullet}))$ as \mathcal{O}_X -modules.

Now let $\mathbb{E}'' : \mathcal{A}^{0,\bullet}(X, E^{\bullet}) \to \mathcal{A}^{0,\bullet}(X, E^{\bullet})$ be \mathcal{O}_X -linear of total degree-1 and satisfy the following:

- 1. $\mathbb{E}'' \circ \mathbb{E}'' = 0$; i.e. \mathbb{E}'' is flat.
- 2. The $\overline{\partial}$ -Leibniz formula $\forall s \in \mathcal{A}^0(X, E^{\bullet}), \forall \omega \in \mathcal{A}^{0, \bullet}(X),$

$$\mathbb{E}^{''}(s \otimes \omega) := \mathbb{E}^{''}(s) \otimes \omega + (-1)^{\deg(\omega)}s \otimes \overline{\partial}(\omega) \tag{1}$$

The meaning of total degree -1 is that $\forall p, q \in \mathbb{N}$

$$\mathbb{E}^{''}(\mathcal{A}^{0,p}(X,E^q)) \subseteq \bigoplus_{k \ge \max\{-p,-q+1\}} \mathcal{A}^{p+k}(X,E^{q-k+1}) \Rightarrow \mathbb{E}^{''} = \bigoplus_{k \in \mathbb{Z}} \mathbb{E}_k^{''},$$

with $\mathbb{E}_{k}^{''} = 0, \forall k < \max\{-p, -q+1\}$. Note that by definition of $\mathbb{E}^{''}$ and degree, we know that $\mathbb{E}_{k}^{''}$ is $\mathcal{A}^{\bullet}(X)$ -linear $\forall k \neq 1$. Now consider $\forall 0 \leq k \leq n$, we have $\mathbb{E}^{''}(\mathbb{E}^{''}|_{\mathcal{A}^{0}(X, E^{k})}) = 0$, so its projection onto $\mathcal{A}^{0}(X, E^{k+2})$ is also 0. Therefore $\mathbb{E}_{0}^{''} \circ \mathbb{E}_{0}^{''}(\mathcal{A}^{0}(X, E^{k})) = 0$, and we know that

$$0 \longrightarrow \mathcal{A}^{0}(X, E^{0}) \xrightarrow{\mathbb{E}_{0}^{''}} \mathcal{A}^{0}(X, E^{1}) \xrightarrow{\mathbb{E}_{0}^{''}} \cdots \xrightarrow{\mathbb{E}_{0}^{''}} \mathcal{A}^{0}(X, E^{N}) \longrightarrow 0$$

is a complex of coherent sheaves.

3.2 Extended Hermitian Metric and the d-connection

Let h be a Hermitian metric on E^{\bullet} . Then, using Proposition 2.1.(1), we can extend h to $\mathcal{A}^{\bullet}(X, E^{\bullet})$ via

$$h(\alpha \otimes f, \beta \otimes g) = \overline{\alpha} \wedge h(f, g) \wedge \beta, \quad \forall \alpha, \beta \in \mathcal{A}^{\bullet}(X), \quad \forall f, g \in \mathcal{A}^{0}(X, E^{\bullet})$$

We will write a cohesive module as $(E^{\bullet}, \mathbb{E}'', h)$ to emphasize the dependence of various properties/constructions on the Hermitian metric. Now we state an important structure theorem:

Theorem 3.1. [16] For a Hermitian cohesive module $(E^{\bullet}, \mathbb{E}^{''}, h)$, there is a unique $\mathbb{E}' : \mathcal{A}^{\bullet,0}(X, E^{\bullet}) \to \bigcup_{q \in \mathbb{Z}} \mathcal{A}^{\bullet+q+1,0}(X, E^{\bullet-q})$ satisfying the following:

- 1. \mathbb{E}' is a ∂ -superconnection, i.e.¹ $\forall \alpha \otimes f$, with $\alpha \in \mathcal{A}^{\bullet}(X), f \in \mathcal{A}^{0}(X, E^{\bullet})$, we have $\mathbb{E}'(\alpha \otimes f) = \mathbb{E}'(\alpha) \otimes f + (-1)^{deg(\alpha)} \alpha \otimes (\partial f)$.
- 2. Extending \mathbb{E}'' and \mathbb{E}' to $\mathcal{A}^{\bullet,\bullet}(X, E^{\bullet})$ linearly (considering that $\mathcal{A}^{\bullet,\bullet}(X, E^{\bullet})$ $\cong \mathcal{A}^{0,\bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{\bullet,0}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}(X, E^{\bullet}))$, and we have $\mathbb{E} = \mathbb{E}'' + \mathbb{E}'$ is a *d*-superconnection.
- 3. \mathbb{E} is h-unitary, i.e. $\forall s, t \in \mathcal{A}^{\bullet}(X, E^{\bullet})$, we have $(-1)^{deg(s)}d(h(s, t)) = -h(\mathbb{E}(s), t) + h(s, \mathbb{E}(t))$.

¹Note that $\mathcal{A}^{\bullet,0}(X, E^{\bullet}) \cong \mathcal{A}^{\bullet,0}(X) \otimes_{\mathcal{A}^0(X)} \mathcal{A}^0(X, E^{\bullet})$

4. Writing $\mathbb{E}' = \bigoplus_{q \in \mathbb{Z}} \mathbb{E}'_q, \nabla = \mathbb{E}''_1 + \mathbb{E}'_1 : \mathcal{A}^{\bullet, \bullet}(X, E^{\bullet}) \to \mathcal{A}^{\bullet+1, \bullet+1}(X, E^{\bullet})$ is a unitary *d*-connection.

From the last statement we see that ∇ restricts to connections on each $E^k, 0 \leq k \leq N$. Therefore, writing $\nabla_k = \nabla|_{E_k}$ and $\nabla = \bigoplus_{k=0}^N \nabla_k$, we know that (E^k, ∇_k) is a vector bundle with a *d*-connection. Note that ∇ induces a connection ∇^{End} on $End(E^{\bullet})$ by

$$\nabla^{End}: \mathcal{A}^{\bullet}(X, End(E^{\bullet})) \to \mathcal{A}^{\bullet+1}(X, End(E^{\bullet})), \quad \phi \mapsto \nabla \circ \phi - (-1)^{deg(\phi)} \phi \circ \nabla.$$

3.3 Construction of a Compatible Unitary Connection

Definition 3.1. [19, Section 4] Connections $\{\Theta_k\}_{0 \le k \le N}$ on $\{E_k\}$ are compatible with the complex $0 \longrightarrow E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{N-1}} E_N \longrightarrow 0$ if $\Theta_{k+1} \circ \phi_k = -\phi_k \circ \Theta_k \Leftrightarrow \Theta_{k+1} \circ \phi_k + \phi_k \circ \Theta_k = 0.$

The reason why compatibility is important will be seen in the next section. In the case of cohesive modules, the ∇_k 's might not be compatible with $\mathbb{E}_0^{''}$. Let

 Z_i be the set where $\mathbb{E}_0^{''}$ is not exact, and let $Z = \bigcup_{i=0}^N Z_i$. We will call Z the support of the cohesive module. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth characteristic function such that $\chi \equiv 0$ on $(-\infty, 1-\delta)$ and $\chi \equiv 1$ on $(1+\delta, \infty)$, for some arbitrarily small positive δ .

Now for a positive $\epsilon > 0$, if $Z = \emptyset$, define $\chi_{\epsilon} \equiv 1$ on X. Otherwise, define $F = \boxtimes_{k=1}^{N} F_k = \boxtimes_{k=1}^{N} \det(\mathbb{E}_0^{''}|_{E_{k-1}})^{\bigwedge \operatorname{rank}(E_{k-1})}$ on $X[2, \operatorname{Section} 2]$, which is a section to the coherent sheaf $\mathcal{F} = \boxtimes_{k=1}^{N} \left(\bigwedge^{\operatorname{rank}(E_{k-1})} E_{k-1}^* \otimes \bigwedge^{\operatorname{rank}(E_{k-1})} E_k\right)$. Then it is clear that $Z = \{F = 0\} = \bigcup_{k=1}^{N} \{F_k = 0\}$. If it is impossible to find an F that is generically nonvanishing such that $Z \subseteq \{F = 0\}$, define $\chi_{\epsilon} \equiv 1$. Otherwise define $\chi_{\epsilon}(x) = \chi\left(\frac{|F(x)|^2}{\epsilon}\right), \forall x \in X$. In this case $\chi_{\epsilon} \equiv 1$ except on a small neighborhood of Z when ϵ is small. (Since F is generically nonvanishing, we can modify F such that $F \equiv 1$ except in a small neighborhood of Z.) Now we need another concept before constructing the compatible connections: the minimal inverse.

Definition 3.2. [19, 2] For $0 \le k \le N-1$, write $\mathcal{A}^0(X, E_{k+1}) = \mathbb{E}_0''(\mathcal{A}^0(X, E_k))$ $\oplus F_{k+1}$. Define the **minimal inverse** $\sigma_k : E_{k+1} \to E_k$, a morphism between vector bundles, by the following conditions: if $e \in \mathbb{E}_0''(E_k)$, then $\sigma_k(e) \equiv n$, where $\mathbb{E}_0''(n) = e$, and n has pointwise the minimal h-norm among all such vectors. If under $h, e \perp \mathbb{E}_0''(E_k)$, then $\sigma_k(e) \equiv 0$. It then follows that $\mathbb{E}_0'' \circ \sigma_k \circ \mathbb{E}_0'' = \mathbb{E}_0''$.

Remark. Here are some properties of σ_k

1. The minimality of $\sigma_k(e)$ is equivalent to stating that $n \perp Ker(\mathbb{E}_0''|_{E_k})$, since any complement of $Ker(\mathbb{E}_0''|_{E_k})$ injects onto $\mathbb{E}_0''(E_k)$ under $\mathbb{E}_0''.[1]$

- 2. From Remark (1), we know that $Im(\sigma_k) \perp Ker(\mathbb{E}_0''|_{E_k}) \Rightarrow Im(\sigma_k) \perp \mathbb{E}_0''(E_{k-1})$, since $(\mathbb{E}_0'')^2 = 0$. This means that $\sigma_{k-1}\sigma_k = 0.[19]$
- 3. σ_k is smooth on $X \setminus Z_k$. This is because on $X \setminus Z_k$, the rank of $\mathbb{E}''_0|_{E_k}$ is constant. It then suffices to show that $\sigma_k|_{\mathbb{E}''_0(E_k)}$ is smooth, since σ_k on the orthogonal complement is constant. This follows from a description of σ_k in [1, Section 3].

Now we construct the connections $\nabla_k^{\epsilon} = \nabla_k - \chi_{\epsilon}(\sigma_k \circ \nabla^{End} \circ \mathbb{E}_0'')$ on E_k 's, and we write $\nabla^{\epsilon} = \bigoplus_{k=0}^N \nabla_k^{\epsilon}$. Then we have

Theorem 3.2. [19, Lemma 4.4] For any $\epsilon > 0$, the connections $\{\nabla_k^{\epsilon}\}_{0 \le k \le N}$ are compatible with \mathbb{E}_0'' exactly where $\chi_{\epsilon} \equiv 1$.

4 Characteristic Class in de-Rham Cohomology

4.1 Characteristic Forms of $(E^{\bullet}, \mathbb{E}'', h)$

Let $(E^{\bullet}, \mathbb{E}'', h)$ be a Hermitian cohesive module. Define the curvature $R_h = \mathbb{E}^2 = \frac{1}{2}[\mathbb{E}, \mathbb{E}] = [\mathbb{E}', \mathbb{E}''].$

Remark. Noting that $\mathbb{E} \in End(\mathcal{A}^{\bullet}(X, E^{\bullet})) \cong \mathcal{A}^{\bullet}(X, End(E^{\bullet})) \cong \mathcal{A}^{\bullet}(X) \otimes_{\mathcal{A}^{0}(X)} \mathcal{A}^{0}(X, End(E^{\bullet}))$, so if we write $\mathbb{E} = \alpha \otimes f$, then we have $R_{h} = (\alpha \wedge \alpha) \otimes (f \circ f)$.

Then, following Quillen's notion of the supertrace[20], for a fixed convergent complex power series f(T), we define its **characteristic form** to be $Tr_s(f(R_h))$, where $Tr_s : \mathcal{A}^{\bullet}(X, End(E^{\bullet})) \to \mathcal{A}^{\bullet}(X)$ is defined as follows [20, 2]: letting $E^{\bullet} =$ $E^+ \oplus E^-$, with $E^+ = \bigoplus_{2|k} E_k, E^- = \bigoplus_{2|k} E_k$, we define $Tr_s : End(E^{\bullet}) \to$ $\mathbb{C}, X \mapsto tr(\epsilon X)$, where for $e \in E^+, \epsilon X(e) = X(e)$, and for $e \in E^-, \epsilon X(e) =$ -X(e). Now extend Tr_s linearly to $\mathcal{A}^{\bullet}(X, End(E^{\bullet}))$. Then we have the following facts

Theorem 4.1. [16][Corollary 2.26] Characteristic forms are closed, so they define classes in $H^{\bullet}_{dR}(X, \mathbb{C})$. These classes are well-defined by Serre's Vanishing Theorem. We then have $[Tr_s(f(R_h))] = [Tr_s(f(\Theta_{\nabla}))]^2$ in $H^{\bullet}_{dR}(X, \mathbb{C})$, where $\Theta_{\nabla} = \nabla^2$ is the curvature form associated to ∇ .

4.2 Characteristic Forms of Exact Chain Complexes

We first show the following claim establishing a more explicit relation between the characteristic form and the curvature form. Let (E, ∇) be a vector bundle on X. Denote $tr : \mathcal{A}^{\bullet}(X, End(E)) \to \mathcal{A}^{\bullet}(X)$ the extension of the trace function on $\mathcal{A}^{0}(X, End(E))$. Then we have

²This ∇ was defined in Theorem 2.1.(4).

Proposition 4.1. $[tr(f(\Theta_{\nabla}))] \in H^*_{dR}(X)$ is a polynomial in the Chern classes of E. Specifically, it is a symmetric polynomial in $[\Theta_{\nabla}]$. It is also a sum of homogeneous polynomials in $[\Theta_{\nabla}]$.

Proof. Let X, Y be two complex algebraic varieties, and (E, Δ) a vector bundle of rank k on X. We show that $(E, \Delta) \mapsto [tr(f(\Theta_{\Delta}))]$ is a natural transformation from $\mathsf{Vect}_k(-;\mathbb{C})$ to $H^*(-)$. For a morphism $\phi: Y \to X$, let $(\phi^*E, \phi^*\Delta)$ be the pullback vector bundle on Y with the pullback connection which is functorial (as defined in [21, Theorem 3.6(a)]), we know that $\Theta_{\phi^*\Delta} = \phi^*\Theta_{\Delta}$. Now it suffices to show that $\forall i \in \mathbb{N}, \phi^*[tr(\Theta_{\Delta}^i)] = [tr((\phi^*\Theta_{\Delta})^i)]$. Recall the splitting principle

Lemma 4.2. [10, Section 21] Let $E \to X$ a C^{∞} complex vector bundle and $p: \mathbb{P}(E) \to M$ be the projection map. Then $p^*(E) \to \mathbb{P}(E)$ splits into a direct sum of line bundles and $p^*: H^*(X) \to H^*(\mathbb{P}(E))$ is an embedding.

Using the lemma and the fact that $\phi^* \mathbb{P}(E) = \mathbb{P}(\phi^*(E))$, and considering the commutative diagram

and then noting that $\mathbb{P}(\phi^*(E)) = \phi^*(\mathbb{P}(E))$, we can reduce to when $E \to X$ is a line bundle, in which case ϕ^* amounts to multiplication by an element in \mathcal{O}_X on both sides.

Now recall that every natural transformation $\operatorname{Vect}_k(-,\mathbb{C}) \to H^*_{dR}(-)$ can be expressed as a polynomial in the Chern classes [10, Proposition 23.11], it remains to show that the Chern classes $c_n(E)$ is a polynomial of $[\Theta_{\Delta}]$. This follows directly from [21, Definition 3.4].

Remark. To show that $[tr(f(\Theta_{\nabla}))]$ is a polynomial, not a series, in the Chern classes, we implicitly used the fact that degrees $\geq 2dim_{\mathbb{C}}(X)$ vanish in $H^*_{dR}(X)$.

Now we recall a Whitney formula [4, Lemma 4.22]

Theorem 4.3. Let ϕ be a symmetric homogeneous polynomial of degree less than or equal to $\dim_{\mathbb{C}}(X)$, then let $\{D_k\}_{0 \le k \le N}$ be compatible connections on $(E^{\bullet}, \mathbb{E}'')$. If the complex $(E^{\bullet}, \mathbb{E}''_0)$ is exact, then in de-Rham cohomology, $[\phi(\Theta_{D_0})] = -\left(\frac{2\pi}{i}\right)^{deg(\phi)} \cdot [\phi(\sum_{i=1}^N (-1)^i E_i)]/4$, Equation 4.15], which is defined as follows: letting $\sigma_k(D_j) = \left(\frac{2\pi}{i}\right)^k \cdot c_k(E_j, D_j)^3/4$, Equation 3.34], and $\phi(D_j) =$ $\tilde{\phi}(\sigma_1(D_j), \cdots, \sigma_{deg(\phi)}(D_j))$, and writing $\sum_{i=1}^N (-1)^i E_i$ as $\widehat{E^{\bullet}}$, we define $\phi(\widehat{E^{\bullet}}) \equiv \tilde{\phi}(c_1(\widehat{E^{\bullet}}), \cdots, c_{deg(\phi)}(\widehat{E^{\bullet}}))$.

³Here σ_k denotes the elementary symmetric polynomial of degree k.

Remark. Here, the Chern classes of $\widehat{E^{\bullet}}$ satisfy the following:

$$1 + \sum_{i=1}^{2dim_{\mathbb{C}}(X)} c_i(\widehat{E^{\bullet}}) = c(\widehat{E^{\bullet}}) \equiv \bigoplus_{k=0}^N c(E_i)^{(-1)^n}$$

Since ϕ is homogeneous, we know that $[\phi(\Theta_{D_0})] = \left(\frac{2\pi}{i}\right)^{deg(\phi)}$ $[\tilde{\phi}(c_1(E_0, D_0), \cdots, c_{deg(\phi)}(E_0, D_0)]$. Therefore From this and Proposition 3.1, we automatically have

Corollary 4.3.1. On $X \setminus Z$,

$$Tr_s(f(\Theta_{\nabla})) = \bigoplus_{k=0}^N Tr_s(f(\Theta_{\nabla_k})) = \sum_{k=0}^N (-1)^k tr(\Theta_{\nabla_k}) = 0.$$

5 Characteristic Currents

Now we define the characteristic currents on a cohesive module $(E^{\bullet}, \mathbb{E}'', h)$. For $a \in H^*_{dR}(X)$, denote a_k to be the degree-k component of a.

Definition 5.1. For $(p_1, \dots, p_k) \in [0, 2dim_{\mathbb{C}}(X)]^k$ define the **characteristic** current to be $Tr_{p_1}(E^{\bullet}, \nabla) \wedge \dots \wedge Tr_{p_k}(E^{\bullet}, \nabla) \equiv \lim_{\epsilon \to 0} [Tr_s(f(\nabla^{\epsilon}))]_{p_1} \wedge \dots \wedge [Tr_s(f(\nabla^{\epsilon}))]_{p_k}.$

Remark. We will explain the meaning of wedge product right after Definition 5.2 (assuming Theorem 5.1 a priori).

Definition 5.2. [3] Let f be a holomorphic function on X. For $a \in \mathbb{N}$, the current $\left[\frac{1}{f^{a}}\right]$ is defined as the functional on test forms $\xi \mapsto \lim_{\epsilon \to 0} \int_{|f| > \epsilon} \frac{\xi}{f^{a}}$ and $\overline{\partial}\left[\frac{1}{f^{a}}\right]$ sends test form ξ to $\lim_{\epsilon \to 0} \int_{|f| > \epsilon} \frac{\overline{\partial}(\xi)}{f^{a}}$. These are well-defined by[15, Theorem 7.1]. Let $\Pi = \Pi_{1} \circ \Pi_{2} \circ \cdots \circ \Pi_{r}$ be a sequence of resolutions of singularities, with $\Pi_{i}: Y_{i} \to Y_{i-1}$ with $Y_{0} = X$. Then a current on X is **pseudomeromorphic** if it can be written as $\sum_{\ell} \prod_{*} \tau_{\ell}$, where τ_{ℓ} is a current on some Y_{ℓ} of the form $\left(\prod_{i=1}^{k} \left[\frac{1}{f^{a_{i}}}\right]\right) \overline{\partial}\left[\frac{1}{f^{b_{1}}}\right] \land \cdots \land \overline{\partial}\left[\frac{1}{f^{b_{m}}}\right]$ for some holomorphic f on Y_{ℓ} .

Here, we need to define a notion of "wedge product" of currents. This is given by the **Coleff-Herrera product**[11, Theorem 1.7.2][18, Theorem 2]. Call $(\epsilon_1, \dots, \epsilon_p) \to (0, \dots, 0)$ along an *admissible path* if $\forall k \in \mathbb{N}$ and $\forall j \geq 2, \frac{\epsilon_{j-1}}{\epsilon_j^k} \to 0$. in this case we write $\epsilon_1 \ll \dots \ll \epsilon_p \to 0$. Then for f_1, \dots, f_p holomorphic, we define

$$\overline{\partial}[\frac{1}{f_1}] \wedge \dots \wedge \overline{\partial}[\frac{1}{f_p}] = \lim_{\epsilon_1 \ll \dots \ll \epsilon_p \to 0} \frac{\overline{\partial}\chi(|f_1|^2/\epsilon_1)}{f_1} \wedge \dots \wedge \frac{\overline{\partial}\chi(|f_p|^2/\epsilon_p)}{f_p},$$

where for each $(\epsilon_1, \dots, \epsilon_p)$ and for any test form ϕ of bi-degree $(\dim_{\mathbb{C}}(X), \dim_{\mathbb{C}}(X)-p)$, the expression on the right hand side denotes $\phi \mapsto \int_X \frac{\overline{\partial}\chi(|f_1|^2/\epsilon_1)}{f_1} \wedge \int_X \frac{\overline{\partial}\chi$

 $\cdots \wedge \frac{\overline{\partial}_{\chi(|f_p|^2/\epsilon_p)}}{f_p} \wedge \phi. \text{ Then for holomorphic } f_1, \cdots, f_k, g_1, \cdots, g_m, \text{ and for a} \\ (dim_{\mathbb{C}}(X), dim_{\mathbb{C}}(X) - p) - \text{test form } \phi, \text{ define } \left(\prod_{i=1}^k [\frac{1}{f_i}]\right) \overline{\partial}[\frac{1}{g_1}] \wedge \cdots \wedge \overline{\partial}[\frac{1}{g_m}](\phi) \\ \text{as} \\ f \qquad \qquad \wedge \frac{\int_{i=1}^m \overline{\partial}_{\chi}\left(\frac{|g_i|^2}{\epsilon_i}\right) \wedge \phi }{\int_{i=1}^m \overline{\partial}_{\chi}\left(\frac{|g_i|^2}{\epsilon_i}\right) \wedge \phi}$

$$\lim_{\substack{\delta_1, \cdots, \delta_k \to 0\\\epsilon_1 \ll \cdots \ll \epsilon_m \to 0}} \int_{|f_i| > \delta_i, \forall i} \frac{\bigwedge_{j=1} O\chi\left(\frac{m_j}{\epsilon_j}\right) \land q}{\prod_{i=1}^k f_i \prod_{j=1}^m g_j}$$

Remark. It follows from Definition 5.2 that pushforwards of pseudomeromorphic currents under resolutions of singularities are still pseudomeromorphic.

Theorem 5.1. The characteristic current $Tr_{p_1}(E^{\bullet}, \nabla) \wedge \cdots \wedge Tr_{p_k}(E^{\bullet}, \nabla)$ is a well-defined closed pseudomeromorphic current, with support contained in Z, an analytic subvariety of positive codimension, that represents $Tr_s(f(\Theta_{\nabla}))_{p_1} \wedge \cdots \wedge Tr_s(f(\Theta_{\nabla}))_{p_k}$ for any k-tuple (p_1, \cdots, p_k) .

Remark. Denote $(\mathbb{D}^*(X), d)$ the complex of currents on M. Here $\mathbb{D}^q(X)$ is the dual space to $\Omega_c^{2dim_{\mathbb{C}}(X)-q}(X)$, the vector space of compactly supported smooth forms on X, with the dual topology. Also, the chain map $d : \mathbb{D}^q(X) \to \mathbb{D}^{q+1}(X)$ is given by $(dT)(\phi) = (-1)^{q+1}T(d\phi)$. Then we have $H^*_{dR}(X) \xrightarrow{\cong} H^*(\mathbb{D}^*(X), d)$.[13, Chapter 3, Section 1]

Proof. By Proposition 4.1 and linearity, it suffices to consider the case where $Tr_s(f(\Theta_{\nabla}))$ is a monomial in the Chern classes. Then from [19, Lemma 2.1] and [19, Theorem 5.1], it suffices to show that $\forall \ell_1, \ell_2$ we have

$$\lim_{\epsilon \to 0} c_{\ell_1}(E^{\bullet}, \nabla^{\epsilon}) \wedge \lim_{\delta \to 0} c_{\ell_2}(E^{\bullet}, \nabla^{\delta}) = \lim_{\epsilon \to 0} c_{\ell_1}(E^{\bullet}, \nabla^{\epsilon}) \wedge c_{\ell_2}(E^{\bullet}, \nabla^{\epsilon})$$

and then proceed inductively (for wedge products of more Chern classes). Since $[tr(f(\Theta_{\nabla}))]$ is a characteristic class, it does not depend on the choice of connection. As in the proof of [19, Theorem 5.1], for any ϵ and δ , we can write $c_{\ell_1}(E^{\bullet}, \epsilon)$ as $A_1 + \sum_{j\geq 1} \chi_{\epsilon}^j B_j + \sum_{j\geq 1} \chi_{\epsilon}^{j-1} \wedge d\chi_{\epsilon} \wedge B'_j$ and $c_{\ell_2}(E^{\bullet}, \delta)$ as $A_2 + \sum_{j\geq 1} \chi_{\delta}^j C_j + \sum_{j\geq 1} \chi_{\delta}^{j-1} \wedge d\chi_{\delta} \wedge C'_j$, where $A_1, A_2, B_j, C_j, B'_j, C'_j$ are independent of ϵ and δ , with A_1, A_2 smooth and B_j, C_j, B'_j, C'_j polynomials in the entries of the minimal inverses σ_k (cf. Definition 2.2), $D_{End(E^{\bullet})}\mathbb{E}_0''$, and θ_k , which are the matrices representing ∇_k ($0 \leq k \leq N$). Also, $\chi_{\epsilon} = \chi(\frac{|F|^2}{\epsilon})$ as defined in Section 3.3. Therefore it suffices to consider where $\chi_{\epsilon} \neq 1$ for ϵ small enough and show that for any i, j and any s, t products of entries of $\sigma_k, D_{End(E^{\bullet})}\mathbb{E}_0''$ and θ_k , we have $\chi_{\epsilon}, \lim_{\epsilon\to 0} \chi_{\epsilon}^i s \wedge \lim_{\delta\to 0} \chi_{\delta}^j t = \lim_{\epsilon\to 0} \chi_{\epsilon}^{i+j} d\chi_{\epsilon} \wedge s \wedge d\chi_{\epsilon} \wedge t$.

We use similar ideas to [19, Lemma 2.1]. By resolution of singularities [17, Theorem 3.36], since in Section 2.3 we found a section $F = \boxtimes_{j=0}^{N} F_j$ to a coherent sheaf (actually a vector bundle) \mathcal{F} such that $Z \subseteq Z(F)$, which implies that $(F) \cdot \mathcal{O}_X^4$ is not invertible on Z(F), we know that there exists a (composition of) birational and projective modification(s) $\pi : Y \to X$ such that

⁴Here (F) is the ideal of \mathcal{O}_X generated by (F).

 $\pi|_{Y-\pi^{-1}(Z(F))}: Y-\pi^{-1}(Z(F)) \to X-Z(F)$ is an isomorphism, and the coherent sheaf of ideals on Y generated by pullbacks of local sections to $(F) \cdot \mathcal{O}_X$, which we write as $\pi^{-1}((F) \cdot \mathcal{O}_X)$, is a monomonial sheaf of ideals.⁵ Equivalently, this is the subsheaf of \mathcal{O}_Y generated by $\pi^*F = \boxtimes_{j=0}^N \pi^*F_j$. This means that $\pi^*F_j = F_{j0}F_{j1}$, where $F_{j0} = \prod_{i=1}^n z_i^{c_i}$ is a monomial in local coordinates $\{z_1, \cdots, z_n\}$ and $Z(F_{j0}) \subseteq \pi^{-1}(Z(F))$ and F_{j1} is holomorphic and nonvanishing. Then on Y, we have the local formula

$$\pi^* \sigma_k = \frac{1}{F_{k0}} \phi_k, \forall 0 \le k \le N \tag{2}$$

with ϕ_k smooth everywhere on Y. (cf.the definition of σ_k in Definition 2.2. This can also be found in [2, Section 2]) Note that $D_{End(E^{\bullet})}\mathbb{E}_0^{''}$ and θ_k are everywhere smooth. Now by [19, Equation 2.2], writing $\pi^*s = \frac{1}{\psi_1}\tilde{s}, \pi^*t = \frac{1}{\psi_2}\tilde{t}$, with ψ_1, ψ_2 products of monomials (thus also monomials) of local coordinates on Y and \tilde{s}, \tilde{t} smooth.⁶ In view of Equation (2), it suffices to show that for any test $2dim_{\mathbb{C}}(X)$ -form ξ on Y (here we note that $\chi \sim \chi^i, \forall i \in \mathbb{N}$, and also $\{\psi_1 = 0\} \bigcup \{\psi_2 = 0\} \subseteq \pi^{-1}(Z) = \{\pi^*F = 0\}$, using the fact that π is an isomorphism on X - Z(F) and s, t are smooth outside of Z)

$$\begin{bmatrix} \lim_{\epsilon \to 0} \frac{\chi \left(\frac{|\pi^* F|^2}{\epsilon}\right)^{i+j}}{\psi_1 \psi_2} \end{bmatrix} (\xi) = \begin{bmatrix} \lim_{\epsilon \to 0} \frac{\chi \left(\frac{|\pi^* F|^2}{\epsilon}\right)^i}{\psi_1} \end{bmatrix} \begin{bmatrix} \lim_{\delta \to 0} \frac{\chi \left(\frac{|\pi^* F|^2}{\delta}\right)^j}{\psi_2} \end{bmatrix} (\xi) \iff \lim_{\epsilon \to 0} \int_{|\psi_1 \psi_2| > \epsilon} \frac{\xi}{\psi_1 \psi_2} = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{|\psi_1| > \epsilon} \frac{\xi}{\psi_1 \psi_2}$$
(3)

since this current does not depend on the choice of characteristic function.[19, Lemma 2.1] Then the difference between the two is (noting that ξ is a test form so $||\xi||_{L^{\infty}(X)}$ exists)

$$\begin{split} \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} & \int_{|\psi_1| > \epsilon} \frac{\left(\mathbbm{1}_{|\psi_2| > \epsilon/|\psi_1|} - \mathbbm{1}_{|\psi_2| > \delta}\right) \xi}{\psi_1 \psi_2} + \lim_{\epsilon \to 0} \int_{|\psi_1| < \epsilon} \frac{\mathbbm{1}_{|\psi_2| > \epsilon/|\psi_1|} \xi}{\psi_1 \psi_2} \\ & \leq \left(\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\substack{\delta > \epsilon/|\psi_1| < \epsilon \\ \epsilon/|\psi_1| < |\psi_2| < \delta}} \left|\frac{\xi}{\psi_1 \psi_2}\right| + \int_{|\psi_1 \psi_2| < \epsilon} \left|\frac{\xi}{\psi_1 \psi_2}\right|\right) + \lim_{\epsilon \to 0} \int_{|\psi_1| < \epsilon} \left|\frac{\xi}{\psi_1}\right| \le \\ & ||\xi||_{L^{\infty}(X)} \left(\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\substack{\delta > \epsilon/|\psi_1| \\ \delta > \epsilon/|\psi_1| < |\psi_2| < \delta}} \left|\frac{\mathbbm{1}_{Supp(\xi)}}{\psi_1 \psi_2}\right| + \lim_{\epsilon \to 0} \int_{|\psi_1 \psi_2| < \epsilon} \left|\frac{\mathbbm{1}_{Supp(\xi)}}{\psi_1 \psi_2}\right| + \int_{|\psi_1| < \epsilon} \left|\frac{\mathbbm{1}_{Supp(\xi)}}{\psi_1}\right| \right) \end{split}$$

 $^5\mathrm{Also}$ reference [17, Note 3.16] for the equivalent characterizations of monomial ideal sheaves.

⁶More accurately π^*s might be a sum of such terms, but we can certainly apply a normalization argument to achieve a fraction whose denominator is a monomial in local coordinates.

We have

$$\lim_{\substack{\epsilon \to 0\\\delta \to 0}} \int_{\substack{|\psi_1| > \epsilon\\\epsilon/|\psi_1| < |\psi_2| < \delta}} \left| \frac{\mathbbm{I}_{Supp(\xi)}}{\psi_1 \psi_2} \right| \le \lim_{\substack{\epsilon \to 0\\\delta \to 0}} \int_{\substack{|\psi_1| > \epsilon\\|\psi_2| < \delta}} \left| \frac{\mathbbm{I}_{Supp(\xi)}}{\psi_1 \psi_2} \right| \\
\le \lim_{\epsilon \to 0} \sqrt{\int_{|\psi_1| > \epsilon} \frac{\mathbbm{I}_{Supp(\xi)}}{|\psi_1|^2} \cdot \lim_{\delta \to 0} \sqrt{\int_{|\psi_2| < \delta} \frac{\mathbbm{I}_{Supp(\xi)}}{|\psi_2|^2}} = O\left(\lim_{\delta \to 0} \int_{|\psi_2| < \delta} \frac{\mathbbm{I}_{Supp(\xi)}}{|\psi_2|}\right),$$

which will follow from Lemma 5.3. To prove that this is 0, and also the second and third terms vanish, it suffices to prove the following two lemmas:

Lemma 5.2. Let ϕ be a holomorphic function on M whose vanishing locus is a measure-zero set (specifically, a subvariety of positive codimension), and such that $\frac{1}{\phi} \in L^1(Supp(\xi))$, then $\lim_{\epsilon \to 0} \int_{|\phi| < \epsilon} \frac{1}{|\phi|} = 0$. Here, $|\cdot|$ is the complex norm.

Proof. Note that $codim(Z(\phi)) > 0 \Rightarrow \mu(Z(\phi)) = 0$, where μ is the pullback (under the coordinate maps) of the Lebesgue measure on $\mathbb{C}^{\dim_{\mathbb{C}}(X)}$. Define

$$f_n: X \to [0, +\infty], \quad x \mapsto \begin{cases} \frac{1}{|\phi(x)|} & |\phi(x)| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$
(4)

Then note that $f_i(x) \ge f_j(x), \forall i < j$, and $\lim_{n\to\infty} f_n(x) \ne 0 \Leftrightarrow \phi(x) = 0$. Now since $\int_{Supp(\xi)} |f_1| \le \int_{Supp(\xi)} \frac{1}{|\phi|} < \infty$, then by monotone convergence, we have

$$\lim_{\epsilon \to 0} \int_{|\phi| < \epsilon} \frac{\mathbbm{1}_{Supp(\xi)}}{|\phi|} = \lim_{n \to \infty} \int_{|\phi| < \frac{1}{n}} \frac{\mathbbm{1}_{Supp(\xi)}}{|\phi|} = \lim_{n \to \infty} \int_{Supp(\xi)} f_n$$
$$= \int_{Supp(\xi)} \lim_{n \to \infty} f_n = \int_{Supp(\xi) \cap Z(\phi)} \lim_{n \to \infty} f_n,$$
$$0 \text{ since } \mu(Supp(\xi) \cap Z(\phi)) < \mu(Z(\phi)) = 0.$$

which is 0 since $\mu(Supp(\xi) \cap Z(\phi)) \leq \mu(Z(\phi)) = 0.$

Lemma 5.3. We can apply a further sequence of blow-ups $\Pi: \tilde{Y} \to Y$ such that $\frac{1}{\Pi^*\psi_1}$ and $\frac{1}{\Pi^*\psi_2} \in L^1(Supp(\xi))$. Therefore we can assume WLOG that $\frac{1}{\psi_1}, \frac{1}{\psi_2} \in L^1(Supp(\xi))$. $L^1(Supp(\xi)).$

Proof. We will show this for ψ_1 only. In local coordinates, write $\psi_1 = \prod_{i=1}^n z_i^{c_i}$. Then by Fubini, and writing $r = diam(Supp(\xi)) < \infty$ we have $||\psi_1||_{L^1(Supp(\xi))} \leq \infty$ $\prod_{i=1}^n \int_{|z_i| < r} \frac{1}{|z_i|^{c_i}}$, which is finite if c_i is smaller than $2dim_{\mathbb{C}}(Y), \forall i$. Now since $Supp(\xi) \cap Z(\pi^*F)$ is compact, we can cover it by finitely many coordinate neighborhoods, and $|\psi_1\psi_2|$ will have a strictly positive lower bound outside these neighborhoods. Thus the problem reduces to finding $\Pi: \tilde{Y} \to Y$ such that $\Pi^* \psi_1$ and $\Pi^* \psi_2$ are monomials in local coordinates covering $Supp(\xi) \cap Z(\pi^* F)$ in which the degree of every coordinate does not exceed $2dim_{\mathbb{C}}(Y) - 1$. This will follow directly from [17, Theorem 3.68], which states that if $I \subseteq \mathcal{O}_X$ is an ideal sheaf with $\mathsf{Maxord}_{Supp(\xi)\cap Z(\pi^*F)}(I) \leq m$ for some $m \in \mathbb{N}$, then there is a composition of blow-ups $\Pi = \Pi_r \circ \cdots \circ \Pi_1$ such that $\mathsf{Maxord}_{\Pi^{-1}(Supp(\xi) \cap Z(\pi^*F))} \Pi^{-1}(I) =$ $\mathsf{Maxord}_{\Pi^{-1}(Supp(\xi))\cap Z(\Pi^*\pi^*F)} < m.$ Here, for a point $y \in Y, \mathsf{ord}_y(I) \stackrel{def}{=} \max\{r:$ $\mathfrak{m}_{y}^{r}\mathcal{O}_{Y,y} \supset I_{x}$, where \mathfrak{m}_{y} is the maximal ideal of $\mathcal{O}_{Y,y}$; for a subset $Z \subseteq Y$, define $\mathsf{Maxord}_{Z}(I) = \sup_{y \in Z} \mathsf{ord}_{y}(I)$.[17, Definition 3.47] In our case, we let $I = (\psi_{1}, \psi_{2})\mathcal{O}_{Y}$. Then $\mathsf{ord}_{y}(I)$ is just the maximal order of the two monomials. \square

(Proof of Theorem 5.1–Continued)

Now we show that $\lim_{\epsilon \to 0} \pi^* (\chi_{\epsilon})^i d\pi^* \chi_{\epsilon} \wedge \pi^* s \wedge \lim_{\delta \to 0} (\pi^* \chi_{\delta})^j d\pi^* \chi_{\delta} \wedge \pi^* t = \lim_{\epsilon \to 0} (\pi^* \chi_{\epsilon})^{i+j} d\pi^* \chi_{\epsilon} \wedge \pi^* s \wedge d\pi^* \chi_{\epsilon} \wedge \pi^* t$. Since $\pi^* s, \pi^* t$ have nice expressions, for a test form ξ of matching degree, it suffices to consider

$$\begin{bmatrix} \lim_{\epsilon \to 0} \chi \left(\frac{|\pi^* F|^2}{\epsilon} \right)^i \frac{d\chi \left(\frac{|\pi^* F|^2}{\epsilon} \right)}{\psi_1} \bigwedge \lim_{\delta \to 0} \chi \left(\frac{|\pi^* F|^2}{\delta} \right)^j \frac{d\chi \left(\frac{|\pi^* F|^2}{\delta} \right)}{\psi_2} \end{bmatrix} (\xi)$$
$$= \lim_{\substack{\epsilon \to 0\\\delta \to 0}} \int_X \frac{d\chi \left(\frac{|\pi^* F|^2}{\epsilon} \right) \wedge d\chi \left(\frac{|\pi^* F|^2}{\delta} \right) \wedge \xi}{\psi_1 \psi_2}$$

and

$$\left[\lim_{\epsilon \to 0} \chi\left(\frac{|\pi^*F|^2}{\epsilon}\right)^{i+j} \frac{\bigwedge^2 d\chi\left(\frac{|\pi^*F|^2}{\epsilon}\right)}{\psi_1\psi_2}\right](\xi) = \lim_{\epsilon \to 0} \int_X \frac{\bigwedge^2 d\chi\left(\frac{|\pi^*F|^2}{\epsilon}\right) \wedge \xi}{\psi_1\psi_2}.$$

The difference is

$$\lim_{\substack{\epsilon \to 0 \\ \tau \to 0}} \int_X \frac{\xi}{\psi_1 \psi_2} \wedge \left(d\chi\left(\frac{|\pi^* F|^2}{\epsilon}\right) \bigwedge d\chi\left(\frac{|\pi^* F|^2}{\delta}\right) - \bigwedge^2 d\chi\left(\frac{|\pi^* F|^2}{\tau}\right) \right)$$

Note that the term in the parenthesis is nonzero only when $|\pi^*F|^2 < (1+\nu) \cdot \min\{\epsilon, \delta, \tau\}$, where $supp(\chi) \subseteq [0, 1+\nu)$. By definition of $\chi, ||\nabla \chi||_{L^{\infty}(\mathbb{R})} < \infty$, and

$$\left\| \left| d\chi\left(\frac{|\pi^*F|^2}{\epsilon}\right) \right\| \le ||\nabla\chi||_{L^{\infty}(\mathbb{R})} \cdot \left\| \nabla\left(\frac{|\pi^*F|^2}{\epsilon}\right) \right\|,$$

so the difference does not exceed (writing $k = ||\xi||_{L^{\infty}(X)} \cdot ||\nabla\chi||_{L^{\infty}(\mathbb{R})}$ $\cdot ||\nabla(|\pi^*F|^2)||_{L^{\infty}(Supp(\xi))}$, and denoting the region $\{|\pi^*F|^2 < (1+\nu)\cdot\min\{\epsilon,\delta,\tau\}\}$ by $C_{\epsilon,\delta,\tau}$),

$$k \cdot \lim_{\substack{\epsilon \to 0 \\ \delta \to 0 \\ \tau \to 0}} \int_{C_{\epsilon,\delta,\tau} \cap Supp(\xi)} \frac{1}{|\psi_1 \psi_2|} \left(\frac{1}{\epsilon \delta} - \frac{1}{\tau^2}\right).$$

Note that by Hölder's Inequality, the integrand does not exceed

$$\begin{split} \sqrt{\int_{Supp(\xi)\cap C_{\epsilon,\delta,\tau}} \frac{1}{|\psi_1\psi_2|^2}} \cdot \sqrt{\int_{Supp(\xi)\cap C_{\epsilon,\delta,\tau}} \left|\frac{1}{\epsilon\delta} - \frac{1}{\tau^2}\right|^2} \\ &\leq \sqrt{\int_{Supp(\xi)} \frac{1}{|\psi_1\psi_2|^2}} \cdot \sqrt{\int_{Supp(\xi)\cap C_{\epsilon,\delta,\tau}} \frac{2}{|\pi^*F|^2}}. \end{split}$$

Since $Supp(\xi)$ is compact, from what we have shown before the first term is finite. Also, the second term goes to 0 as ϵ , δ and τ go to 0 simultaneously, since $\mu(Z(|\pi^*F|^2)) = 0$ and after applying a further sequence of resolutions making the degrees of local coordinates low enough in $|\pi^*F|^2$, just as in the proof of Lemma 5.3, we can follow essentially the same proof as for Equation (3)). Then the statement will then follow from Proposition 5.1.

Proposition 5.1. $\pi_* \left(\lim_{\epsilon \to 0} \pi^* (\chi_{\epsilon})^i d\pi^* \chi_{\epsilon} \wedge \pi^* s \wedge \lim_{\delta \to 0} (\pi^* \chi_{\delta})^j d\pi^* \chi_{\delta} \wedge \pi^* t \right) = \lim_{\epsilon \to 0} \chi_{\epsilon}^i d\chi_{\epsilon} \wedge s \wedge \lim_{\delta \to 0} \chi_{\delta}^j d\chi_{\delta} \wedge t$, and we can say the same about the other three currents we are considering.⁷

Proof. This is clear from the definition.

Remark. It is clear from our method that [19, Theorem 5.1] also applies to any other characteristic class in de-Rham cohomology.

6 Chern Currents in Bott-Chern Cohomology

6.1 The Bott-Chern Character

We first define the double complex of Bott-Chern cohomology classes of the cohesive module $(E^{\bullet}, \mathbb{E}^{''}, h)$.[7] Letting $d = \partial + \overline{\partial}$ be the de-Rham differential, we have

Definition 6.1. $H^{p,q}_{BC}(X) \equiv \left(\mathcal{A}^{p,q}(X) \cap Ker(d)\right) / \overline{\partial} \partial \mathcal{A}^{p-1,q-1}(X).$

The **Bott-Chern character** of $(E^{\bullet}, \mathbb{E}^{''}, h)$ is defined by $ch_{BC}(E^{\bullet}, \mathbb{E}^{''}, h) = Trs(exp(-\mathcal{R}_h))$. By [16, Lemma 2.20], this defines a class in $H_{BC}(X)$. The Bott-Chern character is also independent of the Hermitian metric h by [16, Corollary 3.14][7, Theorem 8.2]. Therefore it makes sense to write it as $ch_{BC}(E^{\bullet}, \mathbb{E}^{''})$. If the complex $(\mathcal{A}^{\bullet}(X, E^{\bullet}), \mathbb{E}_0^{''})$ is exact, then we have

Theorem 6.1. [16, Theorem 4.21] Let $\mathbb{E}''_t = \sum_{k=0}^N t^{\frac{1-k}{2}} \mathbb{E}''|_{E_k}$ for t > 0 and \mathcal{R}_t denote the corresponding curvatures. Let N_H be the number operator $h \mapsto kh$ if $h \in \mathcal{A}^{\bullet}(X, E_k)$ [16, Definition 4.18][6, Definition 2.3], and we have

$$ch_{BC}(E^{\bullet}, \mathbb{E}^{''}) = \partial \overline{\partial} \int_{1}^{\infty} Trs\left(\frac{N_{H} \cdot exp(-\mathcal{R}_{t})}{t}\right) dt \Rightarrow ch_{BC}(E^{\bullet}, \mathbb{E}^{''}) = 0.$$

Now we consider the case where $\mathcal{A}^{\bullet}(X, E^{\bullet})$ is not exact. Again, denote the support of this complex by Z.

⁷i.e. Pushing forward by π is the same as taking away all the π^* in the expressions.

6.2 Transgression Formulae and Superconnection Currents

There are Bott-Chern currents representing $ch_{BC}(E^{\bullet}, \mathbb{E}'')$. Denote Z(F) = X'and $\phi : \bigwedge T^*X \to \bigwedge T^*X, a \mapsto (2\pi i)^{-|a|/2}a$. Now write $\delta_{X'} \in \mathbb{D}^*(X)$ be the current of integration along X'. Let $\mathscr{H}\mathscr{E}$ be the sheaf of cohomology groups of E^{\bullet} . Note that $\forall x \in X$ there is a canonical isomorphism $\mathscr{H}\mathscr{E}_x \cong \{y \in E^{\bullet} :$ $\mathbb{E}_0''(y) = 0, \mathbb{E}_0'(y) = 0\}$, so by [6, Theorem 1.2] it inherits a Hermitian metric from h. Let $\nabla^{\mathscr{H}\mathscr{E}}$ be the connection compatible with the inherited Hermitian metric. Now we define the following superconnections:

- 1. Fix a $y \in (N_{X/X'})_{\mathbb{R}}$ with $\overline{y} \in \overline{(N_{X/X'})_{\mathbb{R}}}$, define $B = \nabla^{\mathscr{H}\mathscr{E}} + \partial_y \mathbb{E}_0^{''} + \partial_{\overline{y}} \mathbb{E}_0^{'}$.
- 2. For t > 0, let $A_t = \nabla^{E^{\bullet}} + \sqrt{t} \mathbb{E}_0$.

Now for any t > 0, define currents $\zeta_{E^{\bullet}}(t), \zeta_{E^{\bullet}}'(0)$ and T(h) by

$$\begin{aligned} \zeta_{E^{\bullet}}(t) &= \frac{1}{\Gamma(t)} \int_{0}^{\infty} u^{t-1} \left(Trs(N_{H} \cdot exp(-A_{u}^{2})) - \int_{X'} Trs(N_{H} \cdot exp(-B^{2})) \cdot \delta_{X'} \right) du \\ &\int_{X} \mu \zeta_{E^{\bullet}}'(0) = \frac{\partial}{\partial t}|_{t=0} \int_{X} \mu \zeta_{E^{\bullet}}(t), \ \forall \mu \in \mathcal{A}^{\bullet}(X). \\ &T(h) = \phi(\zeta_{E^{\bullet}}'(0)). \end{aligned}$$

Then we have the following representation formula

Theorem 6.2. [5, Theorem 2.5]

$$ch_{BC}(E^{\bullet}, \mathbb{E}^{''}) = \left(\int_{N_{X/X'}} \phi(Trs(exp(-B^2)))\right) \delta_{X'} - \frac{\overline{\partial}\partial}{2\pi i} T(h).$$

By [6, Theorem 3.2], We know that the wave-front set of $\left(\int_{N_{X/X'}} \phi(Trs(exp(-B^2)))\right) \delta_{X'}$ is contained in $(N_{X/X'})_{\mathbb{R}^*}$, and there is the following convergence resembling our previous construction:

Theorem 6.3. As $t \to \infty$, we have

$$Trs(exp(-A_t^2)) \to \left(\int_{N_{X/X'}} Trs(exp(-B^2))\right) \delta_{X'}$$

and also (abusing the notation a bit and writing A_t also as the current $\xi \mapsto \int_X Trs(exp(-A_t^2)) \wedge \xi$ and B by the current $\xi \mapsto \left(\int_{N_{X/X'}} Trs(exp(-B^2))\right) \delta_{X'}(\xi)$.)

$$\lim_{t \to \infty} \sup_{\xi \in \Gamma} |\xi|^m \cdot \left| \phi \cdot \widehat{(A_t - B)}(\xi) \right| = \lim_{t \to \infty} \sup_{\xi \in \Gamma} |\xi|^m \cdot \left| (A_t - B)(\phi \widehat{\xi}) \right| = 0.$$

with the following nice convergence: $\exists C' > 0$ such that for $t \ll 1$, we have

$$\lim_{t \to \infty} \sup_{\xi \in \Gamma} |\xi|^m \cdot \left| \phi \cdot \widehat{(A_t - B)}(\xi) \right| \le \frac{C'}{\sqrt{t}},$$

for any fixed $m \geq 1$, for any open $U \subseteq X$ biholomorphic to a ball and contained in a trivializing neighborhood of $T^*_{\mathbb{R}}X$ and any smooth function ϕ supported on U, and any Γ a closed cone such that on $U \cap X', \Gamma \cap (N_{X/X'})^*_{\mathbb{R}} = \{0\}$.

Observe that all the above constructions involve only the degree-0 and degree-1 terms of \mathbb{E} . It is then natural to ask the following question:

Question 6.1. What effects do the $\mathbb{E}_{k}^{''}$ terms $(k \geq 2)$ have on $ch_{BC}(E^{\bullet}, \mathbb{E}^{''})$?

The answer is that they have no effects. This follows from Qiang's transgression formula with respect to superconnections[16].

6.2.1 Known Transgression Formulae

There are two main types of transgression formulae. The first type is with respect to the moduli space (with the topology of uniform C^{∞} convergence on compact sets) \mathscr{M} of Hemitian metrics on E^{\bullet} .[7, Theorem 8.1.2][6, Theorems 2.1, 2.2, 2.4][16, Proposition 3.10, Corollary 3.13, Theorem 3.19]. The second type is with respect to the moduli space of superconnections. We will need the following (combining Corollary 4.8 and Proposition 4.15 of [16]):

Theorem 6.4. Let f be a convergent power series. Let \mathcal{E} be the space of all $\overline{\partial}$ -superconnections of degree-1 on (E^{\bullet}, h) . Then $\exists \delta_1, \delta_2$, which are 1-forms on the subspace of $\mathcal{A}^{\bullet,0}(X, End(E^{\bullet}))$ of exotic degree⁸ -1 and the subspace of $\mathcal{A}^{0,\bullet}(X, End(E^{\bullet}))$ of exotic degree 1 respectively, and γ_1, γ_2 which are sections to the subspaces of exotic degree 0 of $\mathcal{A}^{0,\bullet}(X, End(E^{\bullet}))$ and $\mathcal{A}^{\bullet,0}(X, End(E^{\bullet}))$ respectively such that

$$-d^{\mathcal{E}}Trs(f(\mathcal{R}_h)) = \partial Trs(f'(\mathcal{R}_h) \cdot \delta_1) + \overline{\partial}Trs(f'(\mathcal{R}_h) \cdot \delta_2)$$

and

$$\overline{\partial}Trs(f'(\mathcal{R}_h \cdot \gamma_1)) = Trs(f'(\mathcal{R}_h) \cdot \delta_1)$$

$$\partial Trs(f'(\mathcal{R}_h \cdot \gamma_2)) = Trs(f'(\mathcal{R}_h) \cdot \delta_2).$$

For the construction of $\gamma_1, \gamma_2, \delta_1, \delta_2$ refer to Definition 4.11 and Definition 4.5 in [16]. It then directly follows that in $H^{\bullet}_{BC}(X)$,

Corollary 6.4.1. Let $\psi_t(\mathbb{E}'') \in \mathcal{E}$ be the superconnection $\mathbb{E}_0'' + \mathbb{E}_1'' + t \sum_{t \geq 2} \mathbb{E}_k''$, and let the associated curvature forms be $\phi_t(\mathcal{R}_h)$. Then $Trs(f(\phi_1(\mathcal{R}_h))) \simeq Trs(f(\phi_0(\mathcal{R}_h))) = Trs(f(\mathcal{R}_h))$ in $H^{\bullet}_{\mathbf{B}C}(X)$.

⁸From Proposition 3.1, we can write an element $A \in \mathcal{A}^{p,q}(X, End(E^{\bullet}))$ as $\phi \otimes \tau$, with $\phi \in \mathcal{A}^{p,q}(X)$ and $\tau \in End^d(E^{\bullet})$ for some d. Then we define the exotic degre to be d + q - p.

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