

## ASYMPTOTICS IN INFINITE MONOIDAL CATEGORIES

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ABSTRACT. We discuss formulas for the asymptotic growth rate of the number of summands in tensor powers in certain (finite or infinite) monoidal categories. Our focus is on monoidal categories with infinitely many indecomposable objects, with our main tools being generalized Perron–Frobenius theory alongside techniques from random walks.

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## 1. INTRODUCTION

**1A. Growth problems.** Let  $\mathbf{C}$  be an additive Krull–Schmidt monoidal category. Let  $\mathbf{X} \in \mathbf{C}$  be an object of  $\mathbf{C}$ . We define

$$b_n = b_n^{\mathbf{C}, \mathbf{X}} := \#\text{indecomposable summands in } \mathbf{X}^{\otimes n} \text{ counted with multiplicities.}$$

Let  $b: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ ,  $b(n) = b_n$  denote the associated function.

**Notation 1A.1.** We use  $b_n$  to denote the entries of the sequence of numbers  $(b_n)_{n \in \mathbb{Z}_{\geq 0}}$ ,  $b$  for the associated function and  $b(n)$  for the evaluation of  $b$ . We use a similar notation for other sequences and functions.  $\diamond$

**Question 1A.2.** The main point of this paper is to address, and partially answer, the following questions:

- What is the *dominating growth* of  $b_n$ ?
- Can we get an *asymptotic formula*  $a: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  expressing  $b$ , e.g.

$$b(n) \sim a(n),$$

where  $a$  is “nice”,  $\sim$  denotes asymptotically equal?

- Say we have found  $a$  as in the previous point. Can we bound the *variance* or *mean absolute difference*  $|b_n - a_n|$ , or alternatively the convergence rate of  $\lim_{n \rightarrow \infty} b(n)/a(n) = 1$ ?

We refer to (a), (b), and (c) as the *growth problems* associated with  $(\mathbf{C}, \mathbf{X})$ . We also explore several statements along the same lines, which we will also call growth problems.  $\diamond$

**Remark 1A.3.** The order in [Question 1A.2](#) is roughly by difficulty. As an analogy: if  $b$  would be the prime counting function, then (a) could say that it grows essentially linearly, (b) could be the prime number theorem, and (c) could be (a consequence of) the Riemann hypothesis.  $\diamond$

**Example 1A.4.** If  $\mathbf{X}$  and the monoidal unit  $\mathbb{1}$  are indecomposable, and we have  $\mathbf{X} \otimes \mathbf{X} \cong \mathbb{1} \oplus \mathbf{X}$ , then  $(b_n)_{n \in \mathbb{Z}_{\geq 0}}$  is the Fibonacci sequence (with the first two terms equal to 1). In this case the dominating growth is  $\phi^n$  for  $\phi \approx 1.618$  the golden ratio. We have  $b(n) \sim \frac{\phi}{\sqrt{5}} \cdot \phi^n$  and  $|b_n - a_n| < (\phi - 1)^n$  for  $\phi - 1 \approx 0.618$ .

Note the following: the action matrix of  $-\otimes \mathbf{X}$  acting on  $\{\mathbb{1}, \mathbf{X}\}$  is  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$  has eigenvalues  $\phi$  and  $\phi - 1$ . The largest eigenvalue  $\phi$  gives the *dominating growth*, the second largest eigenvalue  $\phi - 1$  the variance.  $\diamond$

For  $\mathbf{C}$  with finitely many isomorphism classes of indecomposable objects there is a general answer to all questions in [Question 1A.2](#). The main player is the *Perron–Frobenius (PF for short) dimension* which is the largest eigenvalue of the action matrix associated with the given growth problem.

For arbitrary additive Krull–Schmidt monoidal categories there is not much we can say about growth problems. However, we will add some natural assumptions (specified below) and address the following:

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- (i) Below we will attach  $\text{PFdim}\mathbf{X} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$  to  $\mathbf{X}$  such that  $(\text{PFdim}\mathbf{X})^n$  is the dominating growth of  $b_n$ . We show that this works in two cases with **two different generalizations** of the PF dimension which agree for finite growth problems (and we write  $\text{PFdim}$  if the difference does not play a role):
- (1) Whenever a certain condition is satisfied which we call *PF admissible*. A key in this definition is that the associated PF dimension  $\text{PFdim}^f$  can be approximated using data from finite graphs.
  - (2) Whenever a certain random walk associated to  $\mathbf{X}$  is *sustainably positively recurrent*. This case uses a slightly different version of PF dimension  $\text{PFdim}^r$  which can also be computed from finite approximations.

In particular, in these cases we have the *exponential growth theorem*  $\beta = \lim_{n \rightarrow \infty} \sqrt[n]{b_n} = \text{PFdim}\mathbf{X}$ .

- (ii) If the category generated by  $\mathbf{X}$  is sustainably positively recurrent, then  $\text{PFdim}^r\mathbf{X} \neq \infty$  and we will give an explicit expression for  $a(n)$  with  $b(n) \sim a(n)$ . In this case we also determine the variance.

In the main body of the paper we will define the various generalizations of PF dimensions, (*positive and null*) *recurrent* and *transient categories*, following the classical notions in the theory of random walks, as well as PF admissible and sustainably positively recurrent growth problems.

**1B. Key examples.** All cases where  $\mathbf{C}$  has only finitely many indecomposable objects up to isomorphism are sustainably positively recurrent and PF admissible. We discuss two additional classes of examples with potentially infinitely many isomorphism classes of indecomposable objects:

- (a)  $(G, \mathbf{X}) = (\mathbf{Rep}(G), \mathbf{X})$ : Here  $\mathbf{Rep}(G)$  means finite dimensional  $G$ -representations over an arbitrary field, for  $G$  a finite group and  $\mathbf{X} \in \mathbf{Rep}(G)$  arbitrary. This case is always sustainably positively recurrent. More generally, all growth problems in finite tensor categories are sustainably positively recurrent, and we will explore these as well. (*Sustainably positively recurrent.*)
- (b) Again  $(G, \mathbf{X})$ : This time we consider  $\mathbf{Rep}(G)$  for  $G$  a simple reductive group over a field of characteristic zero with a representation  $\mathbf{X} \in \mathbf{Rep}(G)$  in defining characteristic. We show that this case is (almost always) transient, but PF admissible with  $\text{PFdim}^f\mathbf{X} \neq \infty$ . (*PF admissible.*)

Along the way we also discuss:

- (c)  $(\mathbf{SBim}, \mathbf{X})$ : We discuss the category of Soergel bimodules  $\mathbf{SBim}$  for an arbitrary finite Coxeter group  $W$ , and  $\mathbf{X}$  also arbitrary. This case is sustainably positive recurrent. We will see that  $\text{PFdim}^f\mathbf{X} \neq \infty$  by giving an explicit formula for  $\text{PFdim}^f\mathbf{X}$  and for  $a(n)$  with  $b(n) \sim a(n)$ . (*Both, sustainably positively recurrent and PF admissible.*)
- (d) We also discuss examples where  $\text{PFdim}^f\mathbf{X} = \infty$ . Our examples include  $(\text{SL}_{\mathbb{Z}_{\geq 0}}(\mathbb{C}), \mathbb{C}^{\mathbb{Z}_{\geq 0}})$  and (generalized) Deligne categories. This gives new proofs that these categories have superexponential growth.

**1C. Wrap up.** Finally, in the process, we classify for which  $(G, \mathbf{X})$  is (positive and null) recurrent.

*Remark 1C.1.* Our paper generalizes results that are known in the literature:

- (a) Our theorems applied to categories with finitely many indecomposable objects up to isomorphism recover the setting in [LTV23].
- (b) The exponential growth theorem partially generalizes [COT23, Theorems 1.4 and 1.9].
- (c) Our classification of the (positive and null) recurrent groups and representations is a version of Polya's random walk theorem [Pól21].

This will be explained in the main body of the paper. ◇

*Remark 1C.2.* All the above immediately generalize to certain algebras which have a basis with structure constants in  $\mathbb{R}_{\geq 0}$ . We will use this setting below. The connection to the above is that Grothendieck rings of additive Krull–Schmidt monoidal categories are  $\mathbb{R}_{\geq 0}$ -algebras. ◇

*Remark 1C.3.* Magma and Mathematica code for some of the calculations in this paper are available on GitHub: [LTV24]. ◇

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## 2. THE BASIC SETTING

The following mildly generalizes the usual notion of based  $\mathbb{R}_{\geq 0}$ -algebras as e.g. in [KM16, Section 2]. Let  $\mathbb{K}$  be a unital subring of  $\mathbb{C}$ , e.g.  $\mathbb{K} = \mathbb{Z}$ . A *based  $\mathbb{R}_{\geq 0}$ -algebra* is a pair  $(R, C)$  where  $C = (c_i)_{i \in I}$  with  $1 \in C$  is a  $\mathbb{K}$ -basis of a  $\mathbb{K}$ -algebra  $R$  such that all structure constants lie in  $\mathbb{R}_{\geq 0}$ , i.e.:

$$(2.1) \quad c_i c_j = \sum_{k \in I} m_{ij}^k c_k \quad \text{with} \quad m_{ij}^k \in \mathbb{R}_{\geq 0}.$$

Note that  $m_{ij}^k \in \mathbb{K} \cap \mathbb{R}_{\geq 0}$ , so that  $m_{ij}^k \in \mathbb{Z}_{\geq 0}$  when  $\mathbb{K} = \mathbb{Z}$ . We usually only write  $R$  for  $(R, C)$  if no confusion can arise. Note that the sum in (2.1) is finite since  $C$  is a  $\mathbb{K}$ -basis.

*Remark 2.2.* The reader who wants to consider  $\mathbb{R}_{\geq 0}$ -algebras without the condition  $1 \in C$  can split their growth problem along an idempotent decomposition of the identity, so  $1 \in C$  is not a restriction.

By taking a subring of  $R$  if necessary, we can, and will, always assume that  $I$  is countable and that the identity is the element  $c_0$  of the basis  $C$ . This is no restriction for growth problems as we will see below.  $\diamond$

We denote by  $\mathbb{R}_{\geq 0}C$  the subset of  $R$  of finite  $\mathbb{R}_{\geq 0}$ -linear combinations of elements of the basis  $C$  with nonnegative real coefficients. Given  $c \in \mathbb{R}_{\geq 0}C$ , we write  $cc_j = \sum_{k \in I} m_{c,j}^k c_k$  and we denote  $c^n = \sum_{i \in I} m_n^i(c) c_i$ . We define the function

$$b^{R,c}: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, n \mapsto b(n) = b^{R,c}(n) := \sum_{i \in I} m_n^i(c).$$

We are interested in the asymptotic behavior of this function. In particular, we use the terminology *growth problems* associated to  $(R, c)$  in the same way as in Question 1A.2.

**Notation 2.3.** If not explicitly stated otherwise, representations are always finite dimensional.  $\diamond$

**Example 2.4.** *Good examples* of growth problems (coming from categories) that we can treat nicely are:

- (a) Let  $\mathbf{C} = \langle \mathbf{X} \rangle$  be the monoidal category generated by  $\mathbf{X}$ . This example is good if  $\mathbf{C}$  is a finite tensor category over an arbitrary field  $\mathbb{F}$  in the sense of [EGNO15, Definition 1.8.5 and Definition 4.1.1] or finite with respect to isomorphism classes of indecomposable objects. This includes:
  - (i)  $(H, \mathbf{X}) = (\mathbf{Rep}(H), \mathbf{X})$  for  $H$  a finite dimensional Hopf  $\mathbb{F}$ -algebra and  $\mathbf{X}$  a finite dimensional  $H$ -representation over  $\mathbb{F}$  that generates  $\mathbf{Rep}(H)$ . Specifically,  $H$  could be the group ring of a finite group and  $\mathbf{X}$  a faithful  $H$ -representation.
  - (ii)  $(\mathbf{SBim}, \mathbf{X})$  as in Section 1, see for example [EW16] for details about  $\mathbf{SBim}$ .
- (b)  $(G, \mathbf{X}) = (\mathbf{Rep}(G), \mathbf{X})$  where  $G$  is a simple reductive group and  $\mathbf{X}$  is a  $G$ -representation, both in characteristic zero. (The assumption on  $G$  being simple can be relaxed, see Remark 3.17.)
- (c)  $(\mathbf{C}, \mathbf{X})$  for either of the following:  $\mathbf{C} = \mathbf{SL}_{\mathbb{Z}_{\geq 0}}(\mathbb{C})$  and  $\mathbf{X} = \mathbb{C}^{\mathbb{Z}_{\geq 0}}$  the defining representation;  $\mathbf{C} = \mathbf{Rep}(S_t, \mathbb{C})$  for  $t \in \mathbb{C} \setminus \mathbb{Z}$  and  $\mathbf{X}$  the generating object, following the conventions in [CO11], and  $\mathbf{C}$  a Delannoy category with  $\mathbf{X}$  the generating object following the conventions in [HSS22] (and [HS22, Theorem 13.2] shows that this is a good example).

We will revisit these several times below.  $\diamond$

*Remark 2.5.* In an additive Krull–Schmidt monoidal category  $\mathbf{C}$  the *fusion graph* of an object  $\mathbf{X}$  is the graph with vertices corresponding to the indecomposable objects in  $\mathbf{C}$  and outgoing edges of some vertex  $\mathbf{Y}$  going to the direct summands of  $\mathbf{X} \otimes \mathbf{Y}$ . The below is motivated by this terminology.  $\diamond$

Let us consider the (*oriented and weighted*) *fusion graph*  $\Gamma = \Gamma(c)$  associated to some growth problem  $(R, c)$ . This graph is defined as follows:

- (i) The vertices of  $\Gamma$  correspond to the basis elements  $c_i \in C$  appearing in  $c^n$  for some  $n$ , i.e. those  $c_i$  for which some  $m_n^i \neq 0$ .
- (ii) There is an edge of label  $m_{c,j}^i$  from the vertex  $c_j$  to the vertex  $c_i$ .

We identify the basis elements  $c_i \in C$  with the vertices of  $\Gamma$ . After fixing some ordering of the vertices, the matrix associated to a fusion graph  $\Gamma$  is called the *action matrix*  $M(\Gamma) = (m_{c,j}^k)_{k,j}$ .

**Example 2.6.** The key example the reader should keep in mind is the growth problem for  $(\mathbf{SL}_2(\mathbb{C}), \mathbb{C}^2)$ . In this case the fusion graph is  $\mathbb{Z}_{\geq 0}$ :

$$(2.7) \quad \Gamma(\mathbf{SL}_2) = \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \dots,$$

$$M(\mathbf{SL}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We will come back to this growth problem several times throughout the paper.  $\diamond$

**Example 2.8.** The Grothendieck group of an additive Krull–Schmidt monoidal category  $\mathbf{C}$  is an example of a based  $\mathbb{R}_{\geq 0}$ -algebra with  $\mathbb{K} = \mathbb{Z}$  with basis given by the classes of indecomposable objects. The growth problems as in [Question 1A.2](#) can then be reformulated in terms of growth problems for based  $\mathbb{R}_{\geq 0}$ -algebras, and we do this silently throughout. In this case the fusion graph  $\Gamma(\mathbf{X})$  for  $\mathbf{X} \in \mathbf{C}$  is the fusion graph on the subcategory generated by  $\mathbf{X}^{\otimes n}$  in the usual sense.  $\diamond$

**Lemma 2.9.** *For a fusion graph  $\Gamma$  for a growth problem we have:*

- (a)  $\Gamma$  has countably many vertices.
- (b) There is a path from  $c_0$  to any other vertex of  $\Gamma$ . In particular,  $\Gamma$  is connected in the unoriented sense.
- (c) Every vertex of  $\Gamma$  is of finite degree.

*Proof.* This follows because we consider only  $c^n$ , which, for fixed  $n$ , has only finitely many nonzero coefficients in terms of  $C$ .  $\square$

### 3. THE DOMINATING GROWTH VIA TRUNCATIONS

The *naive cutoff* of  $\Gamma$  is the sequence  $(\Gamma_k)_{k \in \mathbb{Z}_{\geq 0}}$  of subgraphs of  $\Gamma$  of the form

$$(\Gamma_k)_{k \in \mathbb{Z}_{\geq 0}} = (\Gamma_0 = \{c_0\} \subset \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma) \text{ with } \bigcup_{k \in \mathbb{Z}_{\geq 0}} \Gamma_k = \Gamma,$$

such that  $\Gamma_k$  is the induced subgraph of  $\Gamma$  that has precisely the vertices that can be reached with a path of length  $\leq k$  from  $c_0$ . Note that [Lemma 2.9.\(c\)](#) implies that  $\Gamma_k$  will be finite.

**Definition 3.1.** A *good filtration* of a growth problem  $(R, C)$  is a sequence of (not necessarily associative)  $\mathbb{R}_{\geq 0}$ -algebras  $((R_k, C_k))_{k \in \mathbb{Z}_{\geq 0}}$  over  $\mathbb{K}$  such that:

- (a)  $C_k \subset C$  is finite and

$$(C_k)_{k \in \mathbb{Z}_{\geq 0}} = (C_0 = \{c_0\} \subset C_1 \subset \dots \subset C_k \subset \dots \subset C) \text{ with } \bigcup_{k \in \mathbb{Z}_{\geq 0}} C_k = C.$$

- (b)  $c_0$  is the unit of all  $R_k$ .

- (c) The multiplication in  $R_k$  is given by taking the product in  $(R, C)$  followed by the projection to  $(R_k, C_k)$ .

The *naive filtration* of  $(R, C)$  is the good filtration where  $C_k$  are the vertices of  $\Gamma_k$  for the naive cutoff.  $\diamond$

**Example 3.2.** In [\(2.7\)](#) the naive cutoff consists of  $\Gamma_k$  being a line graph with  $k$  vertices. In [\(4.15\)](#) below, all the  $\Gamma_k$  for  $k > 1$  will include the vertex labeled  $P_4$ .  $\diamond$

For a good filtration, in  $(R_k, C_k)$  (this is of finite rank over  $\mathbb{K}$ ) we can define  $\text{PFdim}_k c_i$  to be the *PF eigenvalue* of the action matrix of  $c_i$  with respect to  $C_k$ .

**Lemma 3.3.** *We have  $\text{PFdim}_k c_i \leq \text{PFdim}_{k+1} c_i$  and  $\lim_{k \rightarrow \infty} \text{PFdim}_k c_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$  is well-defined.*

*Proof.* Immediate.  $\square$

**Definition 3.4.** Given  $c \in \mathbb{R}_{\geq 0}C$ , define the (good filtration) *PF dimension* as the limit  $\text{PFdim}^f c = \lim_{k \rightarrow \infty} \text{PFdim}_k c \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ .  $\diamond$

Note that [Definition 3.4](#) depends on the choice of a good filtration.

**Lemma 3.5.** *If  $|I| < \infty$ , then [Definition 3.4](#) agrees with the usual definition of PF dimension using the largest eigenvalue of the associated matrix (or, equivalently, graph).*

*Proof.* Immediate.  $\square$

**Definition 3.6.** A growth problem  $(R, C)$  is called *PF admissible* if there exists a good filtration such that:

- (a)  $\text{PFdim}^f$  is superadditive and subpowermultiplicative:

$$\begin{aligned} \text{PFdim}^f c + \text{PFdim}^f d &\leq \text{PFdim}^f (c + d), \\ \text{PFdim}^f c^n &\leq (\text{PFdim}^f c)^n, \end{aligned}$$

for all  $c, d \in \mathbb{R}_{\geq 0}C$ ,  $n \in \mathbb{Z}_{\geq 0}$ .

- (b)  $\text{PFdim}^f c_i \geq 1$  for all  $c_i \in C$ .

We always associate any such good filtration to a PF admissible growth problem. We use the same terminology for the fusion graphs and action matrices themselves.  $\diamond$

*Remark 3.7.* If  $R$  is of finite  $\mathbb{K}$ -rank, then two good filtrations will agree on all but finitely many pages of the filtrations. Abusing terminology a bit, we do not need to and will not specify a specific good filtration.  $\diamond$

**Example 3.8.** A growth problem coming from a transitive unital  $\mathbb{Z}_{\geq 0}$ -ring of finite  $\mathbb{Z}$ -rank is PF admissible. (This includes decategorifications of fusion categories.) The condition  $\text{PFdim}^f c_i \geq 1$  for all  $c_i \in C$  holds in this case, see e.g. [EGNO15, Proposition 3.3.4]. In fact, this assumption is motivated from these rings.  $\diamond$

**Example 3.9.** By a classical result of Kronecker, as summarized in e.g. [EGNO15, Lemma 3.3.14], every growth problem corresponding to a finite graph satisfies Definition 3.6.(b) unless the PF dimension is zero.  $\diamond$

**Notation 3.10.** We use the usual *Bachmann–Landau* (also called Landau–Bachmann or just Landau or even capital O) *notation*: A function  $f$  satisfies  $f \in \Theta(g)$  if there exist constants  $A, B \in \mathbb{R}_{>0}$  such that  $A \cdot g(n) \leq f(n) \leq B \cdot g(n)$  for all  $n_0 < n$  for some fixed  $n_0 \in \mathbb{Z}_{\geq 0}$ . If one has the upper bound, then we write  $f \in O(g)$ , and if the lower bound holds, then we write  $f \in \Omega(g)$ .

Moreover, a function  $f$  is of *superexponential growth* if  $f \in \Omega(\gamma^n)$  for all  $\gamma \in \mathbb{R}_{>1}$ , and is of *subexponential growth* if  $f \in O(\gamma^n)$  for all  $\gamma \in \mathbb{R}_{>1}$ .  $\diamond$

**Theorem 3.11.** *Let  $(R, c)$  be a growth problem.*

(a) *For a good filtration we have*

$$(3.12) \quad b(n) \in \Omega(\gamma^n) \text{ for all } \gamma < \text{PFdim}^f c.$$

*Assume now that  $(R, c)$  is PF admissible.*

(b) *If  $\text{PFdim}^f c \neq \infty$ , then*

$$(3.13) \quad b(n) \in O((\text{PFdim}^f c)^n).$$

(c) *We have the exponential growth theorem:*

$$\beta = \lim_{n \rightarrow \infty} \sqrt[n]{b_n} = \text{PFdim}^f c.$$

*In particular,  $b_n$  grows superexponentially if and only if  $\text{PFdim}^f c = \infty$ .*

In general the PF dimension depends on the choice of a good filtration. However, if one has two different good filtrations such that the growth problem is PF admissible, then the PF dimension is the same for both of them by Theorem 3.11.(c).

*Proof of Theorem 3.11.* As in the proof of [LTV23, Theorem 3B.2], the number  $b_n$  can be computed as the column sum of the matrix  $M(\Gamma)^n$  for the first column corresponding to the identity. (We will use this throughout.) To see this we consider the equation

$$M(\Gamma)c(n-1) = c(n) \xrightarrow{\text{iterate}} M(\Gamma)^n c(0) = c(n),$$

where  $c(k) = (c_0(k), c_1(k), \dots) \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}$  are vectors such that their  $i$ th entry is the multiplicity of  $c_i$  in  $c^k$ , and  $c(0) = (1, 0, \dots)^T$  with the one is in the slot of  $c_0 = 1$ . Observe  $M(\Gamma)^n c(0)$  is the first column of  $M(\Gamma)^n$ .

(a). By an appropriate version of Feteke's subadditive lemma, the sequence  $(\sqrt[n]{b_n})_{n \in \mathbb{Z}_{\geq 0}}$  has a limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{b_n} = \beta \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Similarly, if we denote by  $(b_n^{(k)})_{n \in \mathbb{Z}_{\geq 0}}$  the respective growth problem in the cutoff  $R_k$  for  $c \in R_k$  and set  $b_n^{(k)} = 0$  whenever  $c \notin R_k$  (this happens for only finitely many  $k \in \mathbb{Z}_{\geq 0}$ ), then

$$\lim_{n \rightarrow \infty} \sqrt[n]{b_n^{(k)}} = \beta_k \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

exists. Note hereby that  $\beta_k = \text{PFdim}_k c$ , by [LTV23, Theorem 1] (which uses PF theory of finite graphs). Moreover, the definition of a good filtration Definition 3.1 implies that  $b_n^{(k)} \leq b_n$  which in turn implies that  $\beta_k \leq \beta$  for all  $k, n \in \mathbb{Z}_{\geq 0}$ . Therefore,  $\text{PFdim}^f c \leq \beta$  by Definition 3.4. Hence,  $b(n) \in \Omega(\gamma^n)$  for all  $\gamma < \text{PFdim}^f c$ .

(b). We go back to the definition of  $b_n$ . We have  $c^n = \sum_{i \in I} m_n^i(c) c_i$  and we obtain:

$$(\text{PFdim}^f c)^n \geq \text{PFdim}^f c^n \geq \sum_{i \in I} m_n^i(c) \text{PFdim}^f c_i \geq \sum_{i \in I} m_n^i(c) = b_n,$$

the inequalities following from the PF admissibility condition.

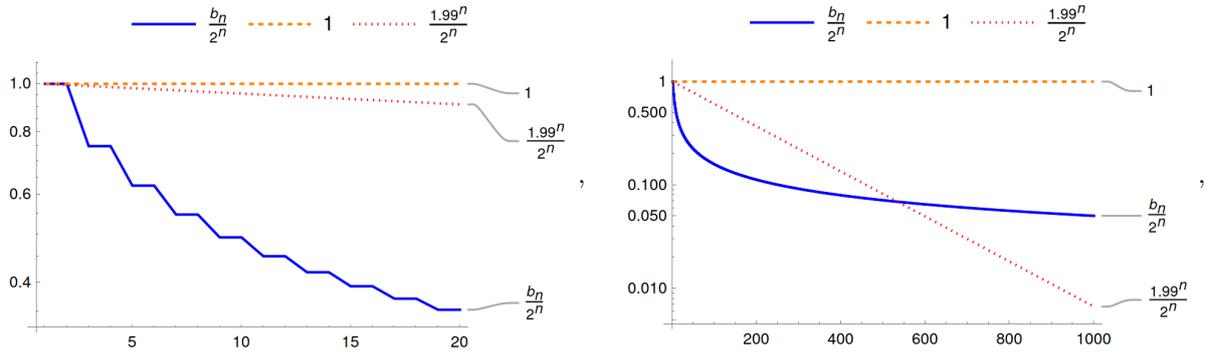
(c). This follows from the just established (3.12) and (3.13).  $\square$

**Example 3.14.** Coming back to Example 2.6, the cutoff  $(\Gamma_i)_{i \in \mathbb{Z}_{\geq 0}}$  of  $\Gamma(\text{SL}_2)$  is such that  $\Gamma_i$  is the finite subgraph on the first  $i$  vertices exemplified by:

$$\Gamma_7 = \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \cdot$$

We will prove later that this growth problem with the naive truncation is PF admissible, see Proposition 3.16. It is easy to see that  $\text{PFdim}^f \Gamma_i = 2 \cos(\pi/(i+1))$ . Then  $\text{PFdim}^f \Gamma(\text{SL}_2) = \lim_{i \rightarrow \infty} 2 \cos(\pi/(i+1)) = 2$ . Thus,

**Theorem 3.11** implies that the dominating growth factor of  $b_n$  is  $2^n$ . Moreover, dividing by the dominating growth factor, the log plots



shows (3.12) for  $\gamma = 1.99$  and (3.13).  $\diamond$

Finally, a “nonexample”:

**Example 3.15.** We give an example where the PF admissibility condition is necessary for **Theorem 3.11**. Suppose that our  $\mathbb{R}_{\geq 0}$ -algebra has a basis  $(c_i)_{i \in \mathbb{Z}_{\geq 0}}$  with multiplication  $c_i c_j = c_{i+j}$  and that  $c = \alpha c_0 + c_1$  for some  $\alpha \in \mathbb{R}_{\geq 0}$ . Then  $c$  satisfies  $cc_i = \alpha \cdot c_i + c_{i+1}$  for all  $i \in \mathbb{Z}_{\geq 0}$ . The action matrix is an infinite Jordan block with  $\alpha$ -labeled loops and we have

$$\Gamma(Jo) = \begin{array}{c} \alpha \quad \alpha \quad \alpha \quad \alpha \quad \alpha \quad \alpha \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots \end{array},$$

$$M(Jo) = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & \dots \\ 1 & \alpha & 0 & 0 & 0 & \dots \\ 0 & 1 & \alpha & 0 & 0 & \dots \\ 0 & 0 & 1 & \alpha & 0 & \dots \\ 0 & 0 & 0 & 1 & \alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

as the fusion graph. The cutoff is the sequence of finite Jordan blocks, in particular,  $\text{PFdim}^f c_i = 0$  for  $i > 0$  and  $\alpha = 0$ , so this example does not satisfy **Definition 3.6.(b)**. And in fact we get:

$$c^n = \sum_{i=0}^n \binom{n}{i} \alpha^{n-i} \cdot c_i,$$

and therefore  $b_n = (1 + \alpha)^n$  but  $\text{PFdim}^f c = \alpha$ .  $\diamond$

Here are key examples of categories with infinitely many indecomposable objects up to isomorphism that are PF admissible:

**Proposition 3.16.** *Any growth problem in **Example 2.4.(b)** is PF admissible.*

*Proof.* We will make use of the *Verlinde categories*  $\mathbf{V}_k(G)$  of level  $k$ , see for example [**Saw06**] for a nice summary of the main properties we need (when we refer to it we use the arXiv numbering). For example, that the PF dimension is the categorical dimension.

In these semisimple categories the indecomposable (= simple) objects  $\bar{L}(\lambda, k)$  are indexed by  $\Lambda^k$ , which are cutoffs of the dominant integral Weyl chamber, see [**Saw06**, Section 4] for details. Moreover, we have

$$\text{PFdim}_k \bar{L}(\lambda, k) = \dim_{\mathbf{V}_k(G)} \bar{L}(\lambda, k) = \prod_{\beta > 0} \frac{q^{\langle \lambda + \rho, \beta \rangle} - q^{-\langle \lambda + \rho, \beta \rangle}}{q^{\langle \rho, \beta \rangle} - q^{-\langle \rho, \beta \rangle}}$$

(the notation is as in [**Saw06**, Section 1]) the so-called *quantum Weyl dimension formula*. Here  $q = q(k)$  is a certain complex root of unity.

The only thing we need to know about this formula is that

$$\lim_{k \rightarrow \infty} \text{PFdim}_k \bar{L}(\lambda, k) = \prod_{\beta > 0} \frac{\langle \lambda + \rho, \beta \rangle}{\langle \rho, \beta \rangle},$$

which is now the *classical* Weyl dimension formula. This is well-known, since  $k \rightarrow \infty$  corresponds to  $q \rightarrow 1$ .

Let us choose the cutoff such that the vertices of  $\Gamma_k$  can be matched with  $\Lambda^k$ . Using this choice the induced product matches the product in the Verlinde category, by the fusion rules as recalled in [**Saw06**, Section 5]. Hence, this shows that **Definition 3.6.(b)** holds.

Moreover, since the Verlinde categories are braided, it follows that the action matrices  $M(c, k)$  of any  $\mathbb{R}_{\geq 0}$ -linear combination of the  $[\bar{L}(\lambda, k)]$  commute with one another, and these matrices can be simultaneously diagonalized. It is easily checked that the vector

$$\sum_{\lambda \in \Lambda^k} \text{PFdim}_k \bar{L}(\lambda, k) \cdot [\bar{L}(\lambda, k)]$$

is an eigenvector of  $M(c, k)$ . The classical PF theorem implies that the associated eigenvalue of  $M(c, k)$  is  $\text{PFdim}_k M(c, k)$ . It follows that we have subpowermultiplicativity, or more precisely even:

$$\text{PFdim}_k (M(c, k)M(c', k)) = \text{PFdim}_k (M(c, k))\text{PFdim}_k (M(c', k)).$$

and superadditivity, ore more precisely even:

$$\text{PFdim}_k (M(c, k) + M(c', k)) = \text{PFdim}_k (M(c, k)) + \text{PFdim}_k (M(c', k)).$$

This then implies [Definition 3.6](#).(a). □

*Remark 3.17.* The argument in the proof of [Proposition 3.16](#) also works for, for example, the nonsimple group  $\text{GL}_n(\mathbb{C})$  and its defining representation. However, the setting we use needs to be adjusted since the induced product is not the product in the Verlinde cutoff. ◇

*Remark 3.18.* [Proposition 3.16](#) and its proof also generalize [[LSYZ22](#), Theorem 1.2]. ◇

**Definition 3.19.** We say an  $\mathbb{R}_{\geq 0}$ -algebra is of *superexponential growth* if there exists some  $c$  such that  $b_n$  grows superexponentially.

We say an additive Krull–Schmidt monoidal category is of *superexponential growth* if its Grothendieck ring is of superexponential growth in the above sense. ◇

*Remark 3.20.* [Definition 3.19](#) runs in parallel to the definition of superexponential growth for abelian monoidal categories using the length of objects. However, note that:

- (a) An abelian growth problem could be of superexponential growth while its version defined using  $b_n$  might not grow superexponentially.
- (b) If  $b_n$  grows superexponentially, then the abelian one does so as well.

Hence, the notion from [Definition 3.19](#) “grows slower” than the abelian one. ◇

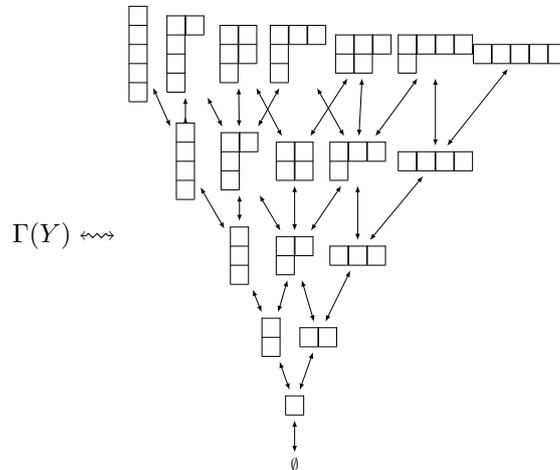
The next example is simple but crucial to determine categories of superexponential growth.

**Example 3.21.** Consider the following graph:

$$\Gamma(\star_N) = \begin{array}{c} \textcircled{2} \\ \updownarrow \\ \textcircled{0} \\ \updownarrow \\ \textcircled{N} \end{array} \quad \dots, \quad M(\star_N) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This is called the *star graph* with  $N + 1$  vertices. It is easy to see that  $\text{PFdim}^f \Gamma(\star_N) = \sqrt{N}$ .

Let  $\Gamma(Y)$  be the Young lattice, considered as an oriented graph by putting an orientation in both directions. The first few layers of this graph are:



We get  $\text{PFdim}^f \Gamma(Y) = \infty$  since  $\Gamma(Y)$  contains star graphs for arbitrarily large  $N$ . To see this take the vertex 0 of the star graph to be a staircase partition  $(t, t - 1, t - 2, \dots, 1)$  for some large enough  $t$ . ◇

**Proposition 3.22.** *The following categories are of superexponential growth.*

- (a) *The category  $\mathbf{Rep}(\mathrm{SL}_{\mathbb{Z}_{\geq 0}}(\mathbb{C}), \mathbb{C}^{\mathbb{Z}_{\geq 0}})$ .*
- (b) *The Deligne category  $\mathbf{Rep}(S_t, \mathbb{C})$ , where  $t \in \mathbb{C} \setminus \mathbb{Z}$ .*
- (c) *The so-called Delannoy categories from [HSS22].*

*Proof.* We can apply Example 3.21 three times: For  $\mathbf{Rep}(\mathrm{SL}_{\mathbb{Z}_{\geq 0}}(\mathbb{C}), \mathbb{C}^{\mathbb{Z}_{\geq 0}})$  this follows directly from Theorem 3.11.(a) and Example 3.21.

The following argument is based on the description of the fusion graph of  $\mathbf{Rep}(S_t, \mathbb{C})$  for  $t \in \mathbb{C} \setminus \mathbb{Z}$ , which was provided to us by Victor Ostrik via email. We thank Victor for providing us with this description. For  $\mathbf{Rep}(S_t, \mathbb{C})$  we recall that the combinatorics of tensoring in is very similar to the combinatorics of tensoring with  $\star_N$ -representations if  $N \gg 0$ . The correspondence is as follows: if you have a partition  $(p_1, p_2, \dots)$  labeling object in the Deligne category, it corresponds to a partition  $(N - (p_1 + p_2 + \dots), p_1, p_2, \dots)$  for  $\star_N$ . Now tensoring by a 1-node Young diagram object is given by removing one node whenever possible and add it back. This description implies the existence of star graphs, so we are done in the same way as for  $\mathbf{Rep}(\mathrm{SL}_{\mathbb{Z}_{\geq 0}}(\mathbb{C}), \mathbb{C}^{\mathbb{Z}_{\geq 0}})$ .

For the Delannoy categories from [HSS22] we can again use star graphs for the fusion graphs computed in [HSS22, Section 7].  $\square$

*Remark 3.23.* We suspect that “most” of the categories discussed in [KOK22], as well as the generalizations of the Delannoy categories from [HSS22], exhibit superexponential growth  $\diamond$

#### 4. RECURRENT AND TRANSIENT CATEGORIES

We retain the setup in Section 2. Following the classical theory of random walks, see e.g. [Kit98] where the reader will also find some standard terminology, we now define *recurrent and transient growth problems*.

To this end, recall that a vertex  $v \in \Gamma$  in a graph with countably many vertices (not just a fusion graph) is *recurrent* if the probability of returning to  $v$  in the simple random walk on  $\Gamma$  is 1, and *transient* if its not recurrent. If  $v \in \Gamma$  is recurrent, then we call it *positive recurrent* if the expected amount of time between recurrences is finite, and *null recurrent* otherwise.

**Lemma 4.1.** *For a fusion graph  $\Gamma$  we have:*

- (a) *Every vertex  $v$  of  $\Gamma$  is either recurrent or transient.*
- (b) *Every recurrent vertex of  $\Gamma$  is either positively recurrent or null recurrent.*
- (c) *Let  $C(\Gamma)$  be a connected component. If  $v \in C(\Gamma)$  is positively recurrent (or null recurrent or transient), then so is any other  $w \in C(\Gamma)$ .*

*Proof.* Classical.  $\square$

Note that Lemma 4.1 allows us to consider (positively/null) recurrence and transience for one fixed connected component. We will use this below.

To state the main definition, we recall the notion of PF dimension of irreducible (not necessary finite)  $\mathbb{R}_{\geq 0}$ -matrices à la Vere–Jones [VJ67]. For such a matrix  $M = (m_{ij})_{i,j \in I}$  for a countable set  $I$ , let

$$(4.2) \quad \mathrm{PFdim}^{vj} M = \lim_{n \rightarrow \infty} \sqrt[h \cdot n]{m_{ij}^{(h \cdot n)}},$$

where  $m_{ij}^{(n)}$  is the  $(i, j)$ -entry of  $M^n$ , and  $h \in \mathbb{Z}_{\geq 1}$  is the period. By [VJ67, Theorem A]  $\mathrm{PFdim}^{vj} M$  is independent of  $i$  and  $j$ .

Recalling that irreducible matrices correspond to strongly connected graphs, we can define:

**Definition 4.3.** For a strongly connected graph  $\Gamma$  with countably many vertices define the *PF dimension*  $\mathrm{PFdim}^{vj} \Gamma$  using (4.2) with  $M$  being the adjacency matrix of  $\Gamma$ .  $\diamond$

*Remark 4.4.* A generalization of  $\mathrm{PFdim}^{vj} \Gamma$  to graphs that are not necessarily strongly connected was worked out in [Twe71]. It however turns out that this generalization is not useful for our purposes.  $\diamond$

The following is immediate:

**Lemma 4.5.** *If  $|I| < \infty$ , then Definition 4.3 agrees with the usual definition of PF dimension using the largest eigenvalue of the associated matrix (or, equivalently, graph).*  $\square$

*Remark 4.6.* We use the notation  $\mathrm{PFdim}^{vj}$  to indicate that this PF dimension is different from the one we introduced in Section 3. They however agree if  $|I| < \infty$  by Lemma 3.5 and Lemma 4.5.  $\diamond$

**Example 4.7.** We continue with Example 2.6. In this case we get  $\mathrm{PFdim}^{vj} \Gamma(\mathrm{SL}_2) = 2$  since

$$m_{11}^{(2n)}(\mathrm{SL}_2) = \frac{1}{n+1} \binom{2n}{n} \sim \pi^{-1/2} \cdot n^{-3/2} \cdot 2^{2n}.$$

This equation follows from a standard argument. Note that the period is two, which is why we consider every second value of  $m_{11}^{(n)}(\mathrm{SL}_2)$  only.  $\diamond$

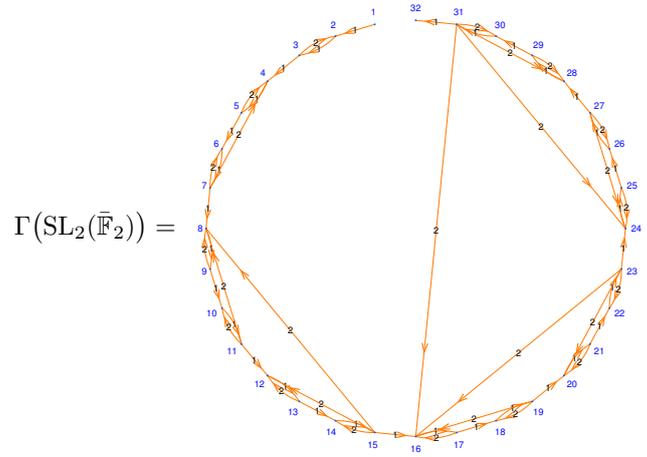
We also recall the notion of *final basic classes* (FBC). Firstly, a *class*  $C(\Gamma)$  is a strongly connected component of  $\Gamma$  and such a class is *basic* if:

$$\mathrm{PFdim}^{vj}C(\Gamma) \geq \mathrm{PFdim}^{vj}C'(\Gamma) \text{ for all strongly connected components } C'(\Gamma).$$

Finally, a basic class is *final* if there is no path to any other basic class. (Note that a FBC does not need to be final in  $\Gamma$  itself.)

**Example 4.8.** For a strongly connected graph  $\Gamma$  is the only class, so it's a FBC. In particular, what we will see below is a generalization of the strongly connected situation.  $\diamond$

**Example 4.9.** Consider  $G = \mathrm{SL}_2(\bar{\mathbb{F}}_2)$  and take the defining  $G$ -representation  $X = \bar{\mathbb{F}}_2^2$  over the field  $\bar{\mathbb{F}}_2$ . The growth problem  $(G, X)$  has the associated fusion graph



The pattern is as follows: There is always one forward arrow. The backward arrows are of weight 2 and start at  $2^k - 1$  and end at  $2^{k-2}$  for all  $k \in \mathbb{Z}_{\geq 2}$ . See, for example, [STWZ23] for details.

The strongly connected components are indexed by  $2^k$  for  $k \in \mathbb{Z}_{\geq 0}$ , with  $2^k$  being their minimal vertex. In this case there is no FBC since all strongly connected components have PF dimension  $< 2$ , but the limit  $k \rightarrow \infty$  of their PF dimensions is approaching 2.  $\diamond$

The following is a key definition:

**Definition 4.10.** Let  $(R, c)$  be a growth problem, and let FBC refer to its fusion graph  $\Gamma$ .

- The growth problem  $(R, c)$  is *recurrent* if all of its FBCs are recurrent and there exists at least one FBC. The growth problem is *transient* otherwise.
- If  $(R, c)$  is recurrent, then we say  $(R, c)$  is *positively recurrent* if all of its FBCs are positively recurrent. The growth problem is *null recurrent* otherwise.

We use the same terminology for graphs in general.  $\diamond$

Note that transient dominates recurrent, and null recurrent dominates positively recurrent.

**Lemma 4.11.** *Every growth problem is either recurrent or transient. Moreover, every recurrent growth problem is either positively or null recurrent.*

*Proof.* Immediate from Lemma 4.1.  $\square$

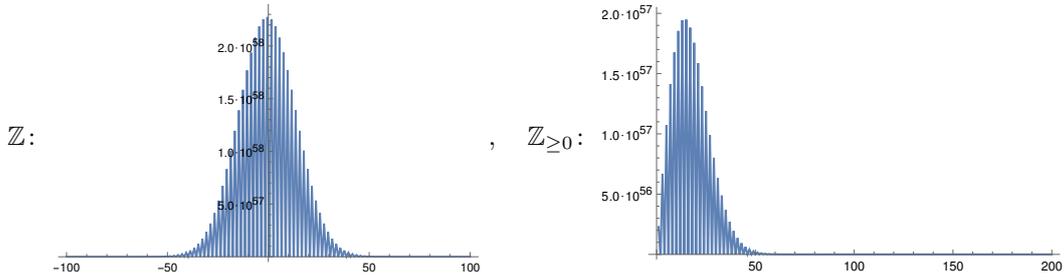
**Example 4.12.** Take the graph

$$\Gamma(\mathbb{Z}) = \dots \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \dots$$

The associated random walk is the classical random walk on  $\mathbb{Z}$ . This is Polya's first example of a recurrent graph [P6121]. Additionally, this example is null recurrent.

In contrast, the random walk on  $\Gamma(\mathrm{SL}_2)$  from (2.7) is transient. Below we will see that this is no coincidence, cf. Proposition 4.22.

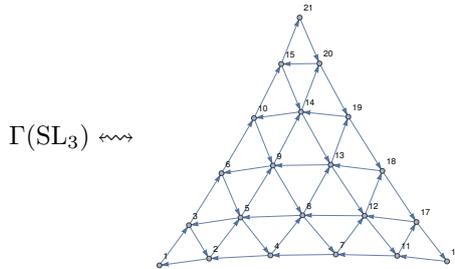
The difference becomes evident when one looks at the number of path of length  $n$  starting at the origin, and ending at vertex  $v$ . For  $n = 200$ , plotting this in  $\mathbb{R}^2$  with (end vertex, number of paths) gives:



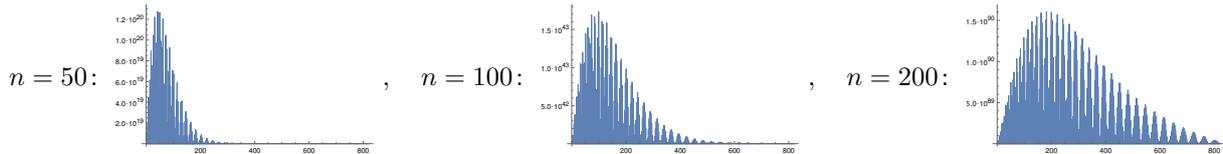
Note that for  $\Gamma(\text{SL}_2) \cong \mathbb{Z}_{\geq 0}$  the peak of the binomial distribution is roughly at  $\sqrt{200} \approx 14.1$ . In fact, the endpoints of paths in this case move to infinity, while they stay at the origin for  $\mathbb{Z}$ .

For completeness: The graph  $\Gamma(\mathbb{Z})$  is also the fusion graph for a monoidal category: take  $(\mathbf{Rep}(\mathbb{S}^1), \mathbf{X}_1 \oplus \mathbf{X}_{-1})$  with  $\mathbf{X}_{\pm 1}$  being the rotation of an irrational angle  $\pm\theta$ .  $\diamond$

**Example 4.13.** Similarly as for  $\text{SL}_3(\mathbb{C})$ , the growth problem for  $G = \text{SL}_3(\mathbb{C})$  and  $\mathbf{X} = \mathbb{C}^3$  with the defining  $G$ -action is also transient. In this case, the fusion graph is (here a cutoff):

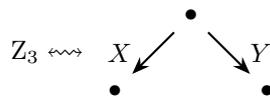


And similarly as in Example 4.12, the number of paths of length  $n$  starting at 1 and ending at  $v$  moves outwards (illustrated using a large enough cutoff):



In Proposition 4.22 we will see that this is not a coincidence.  $\diamond$

**Example 4.14.** Every finite growth problem is positively recurrent. More generally, every growth problem where the FBCs are finite is also positively recurrent. Explicitly, let  $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  be the Klein four group, and we take the ground field  $\mathbb{K} = \mathbb{F}_2$ . Then one has  $\mathbb{K}[G] \cong \mathbb{K}[X, Y]/(X^2, Y^2)$ , and using this presentation one can check that



defines an indecomposable  $G$ -representation of dimension three, call it  $V = Z_3$ . Moreover, it is also easy to see that  $G$  has indecomposable  $G$ -representations  $Z_k$  for  $k \in \mathbb{Z}_{\geq 1}$  odd determined by the property that it is minimal with respect to  $Z_k$  being a direct summand of  $V^{\otimes (k-1)/2}$ . Note that  $Z_1 \cong L_1$  is the trivial  $G$ -representation. One can check that  $\dim_{\mathbb{K}} Z_k = k$  and that all odd numbers appear.

Let  $P_4$  denote the regular  $G$ -representation. An easy calculation gives us then the following fusion graph.

$$(4.15) \quad \Gamma(Kl) = \begin{array}{c} \begin{array}{ccccccccc} \textcircled{L_1} & \longrightarrow & \textcircled{Z_3} & \longrightarrow & \textcircled{Z_5} & \longrightarrow & \textcircled{Z_7} & \longrightarrow & \textcircled{Z_9} & \longrightarrow & \textcircled{Z_{11}} & \longrightarrow & \dots \end{array} \\ \begin{array}{c} \textcircled{Z_3} \xrightarrow{1} \textcircled{P_4} \\ \textcircled{Z_5} \xrightarrow{2} \textcircled{P_4} \\ \textcircled{Z_7} \xrightarrow{3} \textcircled{P_4} \\ \textcircled{Z_9} \xrightarrow{4} \textcircled{P_4} \\ \textcircled{Z_{11}} \xrightarrow{5} \textcircled{P_4} \\ \dots \end{array} \\ \begin{array}{c} \textcircled{P_4} \xrightarrow{3} \textcircled{P_4} \end{array} \end{array},$$

$$M(Kl) = \begin{pmatrix} 3 & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \ddots \end{pmatrix}.$$

The labeled loop at the vertex with label 3 represents three loops.

Let us consider the growth rate problem for  $(G, V)$ . Then this is positively recurrent since the only FBC is  $\{P_4\}$  and therefore a finite graph.  $\diamond$

For a more general statement see [Proposition 4.22](#) below.

**Example 4.16.** The growth problem in [Example 4.9](#) is transient since there is no FBC.  $\diamond$

Recall that a group  $G$  is *virtually*  $H$  if  $H$  is isomorphic to a finite index subgroup of  $G$ .

**Theorem 4.17** (Polya's random walk theorem for representations). *Let  $\mathbb{K}$  be a field of characteristic zero. For any (finitely presentable) group  $G$  and finite dimensional faithful completely reducible  $G$ -representation  $\mathbf{X}$  the following are equivalent:*

- (a) *The growth problem  $(G, \mathbf{X})$  is recurrent.*
- (b)  *$G$  is virtually a torus of rank 0, 1 or 2.*

Moreover, the following are also equivalent:

- (i) *The growth problem  $(G, \mathbf{X})$  is positively recurrent.*
- (ii)  *$G$  is virtually a torus of rank 0, i.e.  $G$  is finite.*

If  $\mathbf{X}$  is not faithful, then go to the biggest quotient group that acts faithfully and repeat.

*Proof.* We will make use of the following. We will formulate it for bialgebra, but we use the same notation as for group.

**Lemma 4.18.** *Let  $G$  be a bialgebra, and  $H \subset G$  be a subbialgebra. Let  $\mathbf{X}$  be a  $G$ -representation such that:*

- (a) *The growth problem  $(G, \mathbf{X})$  has a strongly connected fusion graph with  $\text{PFdim}^{vj} \mathbf{X} < \infty$ .*
- (b) *The growth problem  $(H, \mathbf{X})$  obtained via restriction has a strongly connected fusion graph with the same PF dimension  $\text{PFdim}^{vj} \mathbf{X}$ .*

*Let  $\mathbf{X} \cong \bigoplus_{i=1}^j \mathbf{X}_i$  as an  $H$ -representation obtained by restriction. If the growth problem  $(H, \mathbf{X}_i)$  is transient for some  $i \in \{1, \dots, j\}$ , then  $(G, \mathbf{X})$  is also transient.*

*Proof.* We only prove the statement in the aperiodic case, to keep the formulas simple. First note that the transience of  $(H, \mathbf{X}_i)$  implies that  $(H, \mathbf{X})$  is also transient as there is a subgraph corresponding to  $(H, \mathbf{X}_i)$ . Now, since  $(H, \mathbf{X})$  is transient with strongly connected fusion graph we have

$$\sum_{n \in \mathbb{Z}_{\geq 0}} h_{11}^{(n)} \cdot (\text{PFdim}^{vj} \mathbf{X})^{-n} < \infty,$$

where  $h_{11}^{(n)}$  is the  $(1, 1)$ -entry of the  $n$ th power of the action matrix corresponding to the fusion graph of  $(H, \mathbf{X})$ . Using a similar notation for  $(G, \mathbf{X})$ , we also have

$$\sum_{n \in \mathbb{Z}_{\geq 0}} g_{11}^{(n)} \cdot (\text{PFdim}^{vj} \mathbf{X})^{-n} \leq \sum_{n \in \mathbb{Z}_{\geq 0}} h_{11}^{(n)} \cdot (\text{PFdim}^{vj} \mathbf{X})^{-n},$$

by assumption. Taking both together, the claim follows.  $\square$

The following is then key:

**Lemma 4.19.** *Let  $G = \text{SL}_2(\mathbb{C})$  and  $\mathbf{X}$  any simple  $G$ -representation of dimension  $\dim_{\mathbb{C}} \mathbf{X} > 1$ . Then the growth problem  $(G, \mathbf{X})$  is transient. The same is true when replacing  $\mathbb{C}$  by any characteristic zero field.*

*Proof.* Recall that the fusion rules for  $G = \mathrm{SL}_2(\mathbb{C})$  are as follows. We index the simple  $G$ -representations  $L(\lambda)$  by their highest weight  $\lambda \in \mathbb{Z}_{\geq 0}$ . Then, using the convention that  $L(\nu) = 0$  if  $\nu \notin \mathbb{Z}_{\geq 0}$ , we have

$$L(\lambda) \otimes L(\mu) \cong L(\lambda - \mu) \oplus L(\lambda - \mu + 2) \oplus \dots \oplus L(\lambda + \mu - 2) \oplus L(\lambda + \mu)$$

for  $\lambda \geq \mu$  and similarly for  $\lambda < \mu$ .

The fusion graphs are now easily computed. We give a few examples of how their action matrices look like:

$$(4.20) \quad M(\lambda = 1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad M(\lambda = 3) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and the number of offdiagonals with 1s keep on increasing with steps of two for  $\lambda = 5$ ,  $\lambda = 7$  etc. The case where  $\lambda$  is even is similar, but with 1s on the main diagonal except in the top left entry which is still zero. For example:

$$(4.21) \quad M(\lambda = 2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad M(\lambda = 4) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the matrices for  $\lambda \in 2\mathbb{Z}_{\geq 0}$  are on a different basis on the Grothendieck level (namely on the Grothendieck classes of  $L(0)$ ,  $L(2)$ ,  $L(4)$ , etc.)

Let us now prove the case where  $\lambda = 2$  as an example. In this case  $m_{11}^n$  is given by A005043 on [OEI23]. In particular,  $m_{11}^{(n)} \sim \frac{3^{3/2}}{8\sqrt{\pi}} \cdot n^{-3/2} \cdot 3^n$  and this implies that

$$\sum_{n \in \mathbb{Z}_{\geq 0}} m_{11}^{(n)} \cdot 3^{-n} = \frac{3^{3/2}}{8\sqrt{\pi}} \sum_{n \in \mathbb{Z}_{\geq 0}} n^{-3/2} < \infty,$$

which is the usual calculation one need to do in order to verify that a problem is transient. For general even  $\lambda > 0$  we have  $m_{11}^{(n)} \sim \text{some scalar} \cdot n^{-3/2} \cdot \lambda^n$ , as follows from e.g. [Bia93, Theorem 2.2]. So the same calculations as for  $\lambda = 2$  works. Moreover, again by e.g. [Bia93, Theorem 2.2], when  $\lambda$  is odd, then  $m_{11}^{(2n)} \sim \text{some scalar} \cdot n^{-3/2} \cdot \lambda^{2n}$ , and again the same calculation works.  $\square$

We can assume that  $G$  is an algebraic group since we can replace it with the Zariski closure of its image in  $\mathrm{GL}(\mathbf{X})$ . So assume that  $G$  is an algebraic group.

If  $G$  is not a torus, then we can apply the Jacobson–Morozov theorem to find a copy of  $\mathrm{SL}_2(\mathbb{C})$  in  $G$ . Then Lemma 4.19 (see also Example 4.12) and Lemma 4.18, whose assumptions are satisfied by [COT23, Theorem 1.4], imply that the growth problem  $(G, \mathbf{X})$  is transient.

On the other hand, if  $G$  is torus, then Polya’s classification of recurrent random walks applies, and we get a recurrent growth problem if and only if the rank of the torus is 0, 1 or 2.

The claim about positively recurrent versus null recurrent follows then from Example 4.12.  $\square$

**Proposition 4.22.** *The following growth problems are positively recurrent.*

- (a) For an arbitrary field,  $(G, \mathbf{X})$  for a finite group  $G$  and  $\mathbf{X}$  any  $G$ -representation.
- (b) For an arbitrary field,  $(\mathbf{C}, \mathbf{X})$  for a finite tensor category  $\mathbf{C}$  and  $\mathbf{X} \in \mathbf{C}$  any object.
- (c) For an arbitrary field,  $(\mathbf{SBim}, \mathbf{X})$  for Soergel bimodules  $\mathbf{SBim}$  attached to a finite Coxeter group and  $\mathbf{X} \in \mathbf{SBim}$  any object.

Moreover, the following growth problems are transient.

- (i) For a field of characteristic zero,  $(G, \mathbf{X})$  for a group  $G$  and  $\mathbf{X}$  any  $G$ -representation such that the connected component of the Zariski closure of the image of  $G$  in  $\mathrm{GL}(\mathbf{X})$  is not a torus.
- (ii) In the below specified cases,  $(U_q, \mathbf{X})$  for a quantum enveloping algebra  $U_q = U_q(\mathfrak{g})$  (in the sense of [Lus90] or [APW91]) and  $\mathbf{X}$  any nontrivial tilting  $U_q(\mathfrak{g})$ -representation (meaning not a direct sum of one dimensional  $U_q$ -representations). The cases are:
  - $\mathbb{K}$  is arbitrary, and  $q$  is not a root of unity.
  - $\mathbb{K}$  is an algebraically closed field of characteristic zero, and  $q$  is a root of unity.
- (iii) For a field of characteristic zero,  $(\mathbf{SBim}, \mathbf{X})$  for Soergel bimodules  $\mathbf{SBim}$  attached to an affine Weyl group and  $\mathbf{X} \in \mathbf{SBim}$  nontrivial (meaning not a direct sum of the monoidal unit).

*Proof.* (a). A special case of (b).

(b). We can and will assume that  $\mathbf{X}$  is a tensor generator: if that is not the case then we would go to a smaller tensor category. By [CEOT24, Proposition A.1], every projective indecomposable object of  $\mathbf{C}$  is a direct summand of  $\mathbf{X}^{\otimes d}$  for some  $d \in \mathbb{Z}_{\geq 0}$ .

Let us first analyze the subgraph  $\Gamma^p$  of the fusion graph  $\Gamma$  that contains only vertices for the projective indecomposable objects. We call this the *projective cell*, and we claim it is strongly connected and basic. To see this we first note that [CEOT24, Proposition A.1] also gives us that the regular object appears in some tensor power of  $\mathbf{X}$ . Then [EGNO15, Section 3.3] implies that the projective cell is of maximal PF dimension since going from using Grothendieck classes of simple objects to compute PF dimensions, as in [EGNO15], to indecomposable objects can only decrease the PF dimension. That the projective cell is strongly connected also follows from the existence of the regular objects in some tensor power of  $\mathbf{X}$ .

Moreover, recall that projectives form a  $\otimes$ -ideal, see e.g. [EGNO15, Proposition 4.2.12]. This also implies by [CEOT24, Proposition A.1] that no strongly connected component without projective indecomposable objects is final since there is always a path to the projective cell.

Since every finite tensor category has only finitely many indecomposable projective objects, this problem is positively recurrent.

(c). In this case  $\Gamma$  is finite.

(i). As in the proof of Theorem 4.17, the Jacobson–Morozov theorem implies that we will find  $\mathrm{SL}_2 \subset \mathrm{GL}(\mathbf{X})$ , and Lemma 4.18 and Lemma 4.19 then imply the claim.

(ii). If  $q$  is not a root of unity, then this problem has the same combinatorics as (i), see for example [APW91, Corollary 7.7], so we are done. In the other case there are projective  $U_q$ -representations and they appear in some tensor power of  $\mathbf{X}$ :

**Lemma 4.23.** *Every projective indecomposable tilting  $U_q$ -representation is a direct summand of  $\mathbf{X}^{\otimes d}$  for some  $d \in \mathbb{Z}_{\geq 0}$ .*

*Proof.* We use two results from [APW91, Theorem 9.12]: there are enough projective tilting objects, and every projective tilting object is also injective. With these two facts, the proof of the lemma is, mutatis mutandis, the same as [CEOT24, Proof of Proposition A.1].  $\square$

Using Lemma 4.23, we get the the only final class can be the projective cell, and it remains to argue that this cell is transient. The results in [And18, Theorem 3.1 and Remark 2.(2)] imply that the projective cell (called the Steinberg component in [And18]) is a copy of the whole category in the semisimple case upon factoring the Steinberg  $U_q$ -representation. This implies that the projective cell is transient: The majority of paths in the starting category will eventually be in the projective cell, so they move away from the origin. Then the equivalence in [And18, Theorem 3.1 and Remark 2.(2)] jumps back to the original problem for which we already know that the majority of paths eventually leave the origin.

(iii). Using the correspondence between the cells from [Ost97], very similar as for the quantum group at a complex root of unity. Details are omitted.  $\square$

*Remark 4.24.* One could guess that the analogs of Proposition 4.22.(i) is also true in positive characteristic, e.g. for  $G = \mathrm{SL}_2(\overline{\mathbb{F}}_p)$  this follows from Example 4.9. Similarly, Proposition 4.22.(ii) might also be true for quantum groups in the mixed case, and Proposition 4.22.(iii) might be true for all infinite Coxeter types and regardless of the characteristic of the underlying field.  $\diamond$

## 5. ASYMPTOTICS FOR POSITIVELY RECURRENT CATEGORIES

Positively recurrent categories are essentially finite with respect to growth problems as justified by the main result of this section. However, we need a special case of positively recurrent growth problems, see Definition 5.5.

**Notation 5.1.** All growth problems in this section are positively recurrent in the sense of Definition 4.10.  $\diamond$

First, let us consider the fusion graph  $\Gamma$  associated to some growth problem  $(R, C)$ . Take the naive cutoff  $(\Gamma_k)_{k \in \mathbb{Z}_{\geq 0}}$  and let  $\lambda_k = \mathrm{PFdim}^f \Gamma_k$  (since  $\Gamma_k$  is finite, this is the classical PF dimension.) Let  $\lambda_k^{sec}$  be any second largest eigenvalue of  $\Gamma_k$ .

**Lemma 5.2.** *We have  $\lambda = \lim_{k \rightarrow \infty} \lambda_k$  and  $\lambda^{sec} = \lim_{k \rightarrow \infty} \lambda_k^{sec}$  exist, and  $|\lambda^{sec}| \in \mathbb{R}_{\geq 0}$ .*

*Proof.* Note that  $0 \leq \lambda_k \leq \lambda_{k+1}$  so that it remains to argue why  $\lambda = \lim_{k \rightarrow \infty} \lambda_k \neq \infty$ . As usual in the theory, see e.g. [Kit98, Section 7],  $1/\lambda$  is the radius of convergence of a certain power series, which in turn has a positive radius of convergence on any FBC. Thus, we get  $\lambda < \infty$ . The second largest eigenvalue exists in  $\mathbb{R}_{\geq 0}$  by the same reasoning and  $\lambda < \infty$ .  $\square$

Using Lemma 5.2, we will write  $\lambda = \lim_{k \rightarrow \infty} \lambda_k$  and  $\lambda^{sec} = \lim_{k \rightarrow \infty} \lambda_k^{sec}$  for a given positively recurrent growth problem.

**Definition 5.3.** We call  $\lambda$  the (positively recurrent) *PF dimension* of  $(R, c)$ , and write  $\mathrm{PFdim}^r c = \lambda$ .  $\diamond$

We think of  $\text{PFdim}^r c$  as another generalizations of the PF dimension. However, this PF dimension is not quite the same as in [Definition 3.4](#) or the one of Vere-Jones. We therefore write  $\text{PFdim}^r c$  to avoid confusion. They do agree in the finite case; the proof is immediate:

**Lemma 5.4.**  $\text{PFdim}^f c = \text{PFdim}^{vj} c = \text{PFdim}^r c$  for growth problems with  $|I| < \infty$ .  $\square$

**Definition 5.5.** Let  $(R, c)$  be a positively recurrent growth problem as defined in [Definition 4.10](#). We say  $(R, c)$  is *sustainably positively recurrent* if:

- (a) There is only one basic class.
- (b) There exists  $\varepsilon > 0$  such that the PF dimensions of the nonbasic classes are smaller than  $\lambda - \varepsilon$ .
- (c) We have  $(n \mapsto m_{ij}^{(n)}) \in o((\lambda + \varepsilon)^n)$  for all  $\varepsilon \in \mathbb{R}_{>0}$ .

We also say positively recurrent and sustainable instead of sustainably positively recurrent.  $\diamond$

We will denote the basic class from [Definition 5.5.\(a\)](#) by  $C^{FBC}(\Gamma)$ , which necessarily positive recurrent and a final basic class. However,  $C^{FBC}(\Gamma)$  is not assumed to be finite or final in  $\Gamma$ .

**Example 5.6.** All finite growth problems are sustainably positively recurrent. This includes all positively recurrent growth problems in [Theorem 4.17](#).  $\diamond$

Here are more examples of positively recurrent and sustainable growth problems:

**Proposition 5.7.** *All the positively recurrent examples in [Proposition 4.22](#) are sustainable. In all these cases we have  $\text{PFdim}^r \mathbf{X} \in \mathbb{R}_{\geq 1}$ , and the  $C^{FBC}(\Gamma)$  is finite and final in  $\Gamma$ .*

*Proof.* We have  $\text{PFdim}^r \mathbf{X} \in \mathbb{R}_{\geq 1}$  since the fusion graphs have edge weights in  $\mathbb{Z}_{\geq 1}$ . This follows from the argument in [Example 3.9](#) together with the observation that every vertex admits a path to the projective cell.

Second, the growth problem in (a) is a special case of the one in (b), while the growth problem in (c) has a finite fusion graph. So it remains to go through in [Definition 5.5](#) (a)-(c) for case (b), which we do after the following lemma.

**Lemma 5.8.** *For any growth problem as in [Proposition 4.22.\(b\)](#) we have*

$$b(n) \sim f(n) \cdot (\text{PFdim}^r \mathbf{X})^n \text{ with } f: \mathbb{Z}_{\geq 0} \rightarrow (0, 1] \text{ periodic with finite period.}$$

*Proof.* In this case let  $l_n = \ell(\mathbf{X}^{\otimes n})$  where  $\ell$  is the length. The fusion graph of the growth problem associated to the length is finite and it follows (e.g. by copying [\[LTV23\]](#)) that

$$l(n) \sim \tilde{f}(n) \cdot (\text{PFdim} \mathbf{X})^n \text{ with } \tilde{f}: \mathbb{Z}_{\geq 0} \rightarrow (0, 1] \text{ periodic with finite period,}$$

where  $\text{PFdim} \mathbf{X}$  is the usual PF dimension of this finite growth problem. Note that

$$b_n \leq l_n$$

since  $b_n$  is defined to count indecomposable summands, while  $l_n$  counts simple objects in the Jordan–Hölder filtration. Then [\[CEOT24, Proposition A.1\]](#) implies that this  $\text{PFdim}^r \mathbf{X}$  is the one from [Lemma 5.2](#).

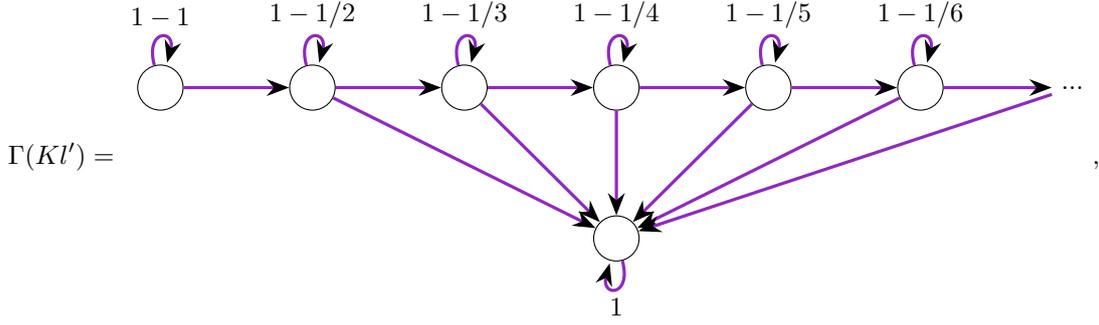
As the next step, using the universal grading group of  $\mathbf{C}$ , one can see that  $b(n)/(\text{PFdim}^r \mathbf{X})^n$  has only finitely many limiting points. This implies that the generating function of  $b_n$  satisfies the properties necessary to run e.g. [\[Mis20, Section 7.7\]](#), and we get  $b(hn + s) \sim t_{h,s} \cdot (\text{PFdim}^r \mathbf{X})^{hn}$  for some  $t_{h,s} \in (0, 1]$ , where  $h$  is the number of limiting points and  $s \in \{0, \dots, h - 1\}$ . Collecting the scalars  $t_{h,s} \in (0, 1]$  into a piecewise constant function  $f: \mathbb{Z}_{\geq 0} \rightarrow [0, 1)$  shows the claim.  $\square$

*Property (a).* As in the proof of [Proposition 4.22](#), by [\[CEOT24, Proposition A.1\]](#) the unique FBC is the projective cell, and its PF dimension is  $\text{PFdim}^r \mathbf{X}$ . We additionally argue now that all other strongly connected components have PF dimension  $< \text{PFdim}^r \mathbf{X}$ . Indeed, assuming the contrary leads to a contradiction with [Lemma 5.8](#). This is easy to see, for example in the toy case

$$\Gamma = \text{PFdim}^r \mathbf{X} \begin{array}{c} \curvearrowright \\ \circ \\ \curvearrowleft \end{array} \longrightarrow \begin{array}{c} \circ \\ \curvearrowright \\ \curvearrowleft \end{array} \text{PFdim}^r \mathbf{X}, \quad M(\Gamma) = \begin{pmatrix} \text{PFdim}^r \mathbf{X} & 0 \\ 1 & \text{PFdim}^r \mathbf{X} \end{pmatrix},$$

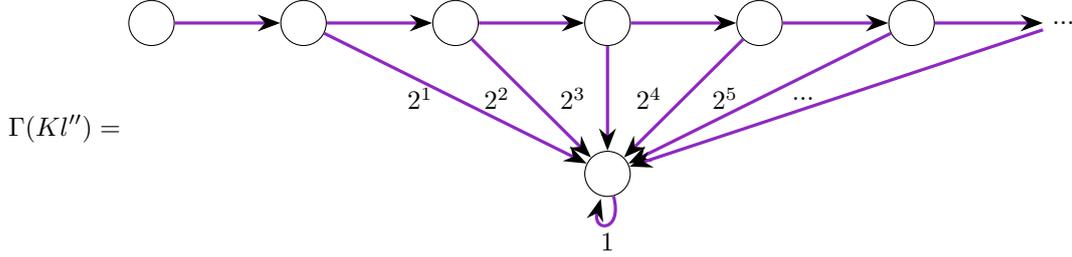
one already gets  $b(n) \sim \frac{n}{\text{PFdim}^r \mathbf{X}} \cdot (\text{PFdim}^r \mathbf{X})^n$ . Recalling that all strongly connected components have some path to the projective cell, the general case then reduces to at least this toy case.

*Property (b).* Similarly as in (a), having basic classes  $B_n$  with  $\lim_{n \rightarrow \infty} \text{PFdim}^{v^j} B_n = \text{PFdim}^r \mathbf{X}$  contradicts [Lemma 5.8](#). For example,



has  $b(n) \in \Omega(1.5^n)$  while  $\text{PFdim}^r \mathbf{X} = 1$ .

*Property (c).* As in the points above, nonsustainable would contradict [Lemma 5.8](#) so we are done. Roughly, consider the following modification of the graph from (4.15):



The growth of  $b_n$  in this case overshoots  $(\text{PFdim}^r \mathbf{X})^n = 1^n$  and we have  $b(n) \in \Omega(1.5^n)$ .

*Final and finite in  $\Gamma$ .* We already discussed this above.  $\square$

Fix  $\Gamma_k$ . Let  $h_k$  be the period and  $\zeta_k = \exp(2\pi i/h_k)$ . Similarly as in [[LTV23](#), Section 1], let us denote the right (the one for the left action) and left (the one for the right action) eigenvectors for  $\zeta_k^i \lambda_k$  by  $v_i^k$  and  $w_i^k$ , normalized such that  $(w_i^k)^T v_i^k = 1$ . Let  $v_i^k (w_i^k)^T [1]$  denote taking the sum of the first column of the matrix  $v_i^k (w_i^k)^T$ . Using this, we define

$$a_k(n) = (v_0^k (w_0^k)^T [1] \cdot 1 + v_1^k (w_1^k)^T [1] \cdot \zeta_k^n + v_2^k (w_2^k)^T [1] \cdot (\zeta_k^2)^n + \dots + v_{h_k-1}^k (w_{h_k-1}^k)^T [1] \cdot (\zeta_k^{h_k-1})^n) \cdot \lambda^n \in \overline{\mathbb{Z}}.$$

Similarly, but directly for  $\Gamma$ , we can define

$$(5.9) \quad a(n) = (v_0(w_0)^T [1] \cdot 1 + v_1(w_1)^T [1] \cdot \zeta^n + v_2(w_2)^T [1] \cdot (\zeta^2)^n + \dots + v_{h-1}(w_{h-1})^T [1] \cdot (\zeta^{h-1})^n) \cdot \lambda^n,$$

where  $h$  is the period of the FBC of  $\Gamma$  (for strongly connected graphs like the FBC this is defined in e.g. [[Kit98](#), Begin of Section 7.1]). Note that, a priori,  $a(n)$  might not be a finite expression.

**Theorem 5.10.** *Assume that the growth problem  $(R, C)$  is sustainably positively recurrent. Then:*

(a) *We have  $a(n) \in \overline{\mathbb{Z}}$  and*

$$b(n) \sim a(n).$$

(b) *If the FBC is final in  $\Gamma$ , the limit  $\lim_{k \rightarrow \infty} a_k(n)$  exists and is equal to  $a(n)$ .*

(c) *If the FBC is finite and final in  $\Gamma$ , then the convergence of  $\lim_{n \rightarrow \infty} b_n/a_n = 1$  is geometric with ratio  $|\lambda^{sec}/\lambda|$  and  $|b_n - a_n| \leq C \cdot (\lambda^{sec})^n$  for  $C \in \mathbb{R}_{>0}$ .*

*Remark 5.11.* [[LTV23](#)] states an analog of [Theorem 5.10](#) for arbitrary finite graphs, but the statement itself is a bit nasty. We decided not to include its positively recurrent analog in this paper.  $\diamond$

*Proof of [Theorem 5.10](#).* We start with the statement about the asymptotic and then proof the other claims that we call ‘‘Finite approximation’’ and ‘‘Variance’’.

*Asymptotic.* Condition (a) of [Definition 5.5](#) implies that there is a unique basic class  $C^{FBC}(\Gamma)$  (which is a final basic class, of course). Note that  $\lambda < \infty$  by [Lemma 5.2](#). Moreover,  $\lambda = 0$  if and only if the growth problem is zero, and hence we can and will assume that  $\lambda \in \mathbb{R}_{>0}$ .

We borrow from and adjust [[Kit98](#), Section 7.1]. There are two differences between our setting and [[Kit98](#), Section 7.1] to keep in mind: Firstly, and crucial, we consider graphs that are potentially not strongly connected. Secondly, we can have a period, but that is a rather harmless generalization. We postpone this discussion to the end of the proof and assume first that we are in the aperiodic situation.

Unless stated otherwise, our indexing set below will be the vertices of the fusion graph  $\Gamma$  associated to  $(R, C)$ , and these vertices are often called  $i, j$ .

**Notation 5.12.** To have the same conventions as [Kit98] in this proof, let  $T$  be the transpose of  $M$ .  $\diamond$

We define three generating functions  $T$ ,  $L$  and  $R$ :

$$T_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} t_{ij}^{(n)} z^n, \quad L_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} l_{ij}(n) z^n, \quad R_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} r_{ij}(n) z^n,$$

$$L'_{ii} = \sum_{n \in \mathbb{Z}_{\geq 0}} n l_{ii}(n) z^n, \quad \mu(i) = \lambda^{-1} L'_{ii}(\lambda^{-1}) \text{ (with } \mu(i) = \infty \text{ allowed),}$$

where the coefficients of  $L$  and  $R$  are defined as follows. The coefficient  $l_{ij}(n)$  is the sum of the labels of the paths  $i \rightarrow j$  of length  $n$  that do not return to  $i$  with  $< n$  steps. Similarly,  $r_{ij}(n)$  is the sum of the labels of the paths  $i \rightarrow j$  of length  $n$  that do not return to  $j$  with  $< n$  steps.

*Step 1.* We start with two crucial lemmas. We define  $\lambda_{ij} = \limsup_{n \rightarrow \infty} \sqrt[n]{t_{ij}^{(n)}}$ .

**Lemma 5.13.** *We have  $\lambda_{ij} \leq \lambda$  for all  $i, j \in \Gamma$ .*

*Proof.* Immediate from  $(n \mapsto t_{ij}^{(n)}) \in o((\lambda + \varepsilon)^n)$  for all  $\varepsilon \in \mathbb{R}_{>0}$ .  $\square$

**Lemma 5.14.** *For all  $i, j \in \Gamma$ , the radii of convergences of  $T_{ij}(z)$ ,  $L_{ij}(z)$  and  $R_{ij}(z)$  are  $\geq \lambda^{-1}$ .*

*Proof.* The radius of convergence of  $T_{ij}(z)$  is  $\lambda_{ij}^{-1}$  by the Cauchy–Hadamard theorem, and the result follows from Lemma 5.13 and the inequalities  $l_{ij}(n) \leq t_{ij}^{(n)}$  and  $r_{ij}(n) \leq t_{ij}^{(n)}$ .  $\square$

We are now ready to go through statements in [Kit98, Section 7.1]. The references below are all with respect to [Kit98].

*Step 2.* Most parts of Lemma 7.1.6 are formal and work verbatim. Some of the arguments need Lemma 5.14, but then can be proven mutatis mutandis with it.

*Step 3.* Next, we see a mild modification. Namely, Lemma 7.1.7 for  $i, j \in \Gamma$  in the same component works mutatis mutandis since  $\lambda$  is the leading eigenvalue so  $\lambda^{-1}$  is the minimal radius of convergence, see Lemma 5.14. Moreover,  $L_{ij}(\lambda^{-1})$  is finite in general if there is a path from  $j$  to  $i$ , and vice versa for  $R_{ij}(\lambda^{-1})$ , by the same arguments as in [Kit98, Section 7.1].

*Step 4.* Lemma 7.1.8 remains true with the following changes. If  $i \in C^{FBC}(\Gamma)$ , then  $T_{ij}(\lambda^{-1}) = \infty$  and  $L_{ii}(\lambda^{-1}) = 1$  since the FBC is recurrent. We do not need the case where  $i \notin C^{FBC}(\Gamma)$ .

*Step 5.* Choose some enumeration of the vertices of  $\Gamma$ . For each  $i \in C^{FBC}(\Gamma)$  define a row and a column vector

$$\ell^{(i)} = (L_{i1}(\lambda^{-1}) \quad L_{i2}(\lambda^{-1}) \quad \dots \quad L_{ij}(\lambda^{-1}) \quad \dots),$$

$$r^{(i)} = (R_{1i}(\lambda^{-1}) \quad R_{2i}(\lambda^{-1}) \quad \dots \quad R_{ji}(\lambda^{-1}) \quad \dots)^T.$$

These are well-defined by Step 3.

*Step 6.* Lemma 7.1.9.(i) remains true, that is, the  $\ell^{(i)}$  and the  $r^{(i)}$  are left and right  $\lambda$ -eigenvectors of the action matrix  $T$ :

$$\ell^{(i)} T = \lambda \cdot \ell^{(i)}, \quad T r^{(i)} = \lambda \cdot r^{(i)}.$$

(For the meticulous reader who want to double check that  $r^{(i)}$  is an eigenvector we point out that the  $R_{ii}$  in Lemma 7.1.6.(viii) should be a  $R_{jj}$ .)

Choose any  $i \in C^{FBC}(\Gamma)$  and define

$$\ell = \ell^{(i)}, \quad r = r^{(i)}.$$

It is easy to see that the choice of the defining  $i \in C^{FBC}(\Gamma)$  is not important for what we will do below.

*Step 7.* Lemma 7.1.10 remains true, with the same proof, but needs some adjustments. Firstly, the vectors  $x$  and  $y$  are assumed to have a nonzero entry corresponding to some  $k \in C^{FBC}(\Gamma)$ . Then  $x$  and  $y$  are strictly positive on the entries corresponding to all  $k \in C^{FBC}(\Gamma)$ . This uses irreducibility of the matrix supported on  $C^{FBC}(\Gamma)$ . (ii) and (iii) then remain true without further change.

*Step 8.* For Lemma 7.1.11 we replace  $xT \leq \lambda x$  and  $Ty \leq \lambda y$  by  $xT = \lambda x$  and  $Ty = \lambda y$  and we keep the same assumptions on  $x$  and  $y$  from the previous step. We renormalize  $x$  and  $y$  such that  $x_i = 1$  and  $y_i = 1$ , and the previous step ensure that  $x - l^{(i)}$  and  $y - r^{(i)}$  are nonnegative and have a zero entry on  $C^{FBC}(\Gamma)$ . We then obtain that  $x - l^{(i)}$  and  $y - r^{(i)}$  are supported on the nonbasic classes. We can extract from  $x - l^{(i)}$  and  $y - r^{(i)}$  eigenvectors of a matrix whose spectrum does not contain  $\lambda$ , because of the condition (b) in Definition 5.5. We finally obtain that  $x = l^{(i)}$  and  $y = r^{(i)}$ .

*Step 9.* Since the FBC is positively recurrent, we get  $L'_{ii}(\lambda^{-1}) < \infty$  for  $i \in C^{FBC}(\Gamma)$ . It follows that  $l \cdot r = \mu(i) < \infty$  (dot product) by copying the calculation in the proof of Lemma 7.1.14.

*Step 10.* Lemma 7.1.15 remains true (it is independent of the growth problem).

*Step 11.* Theorem 7.1.18 has two cases: for  $i \notin C^{FBC}(\Gamma)$  we are in case (i), and for  $i \in C^{FBC}(\Gamma)$  we are in case (ii). The justification for  $i \notin C^{FBC}(\Gamma)$  is clear, and the proof for  $i \in C^{FBC}(\Gamma)$  works mutatis mutandis as in [Kit98], using the adjustments made in the steps above.

*Step 12.* Lemma 7.1.19.(i) works verbatim. For Lemma 7.1.19.(ii) there are three cases. If  $i \in C^{FBC}(\Gamma)$ , then the first equality holds. Moreover, for  $j \in C^{FBC}(\Gamma)$  the equality between the first and the last term holds. Thus, for  $i, j \in C^{FBC}(\Gamma)$  Lemma 7.1.19.(ii) works in the same way as in [Kit98].

*Step 13.* Lemma 7.1.20 works with the extra assumption that  $i, k \in C^{FBC}(\Gamma)$  in part (a) of (i),  $j, k \in C^{FBC}(\Gamma)$  in part (b) of (i) and  $i, j \in C^{FBC}(\Gamma)$  in (ii).

*Step 14.* As in [Kit98], steps 1-13 prove [Kit98, Theorem 7.1.3.(f)] for sustainably positively recurrent growth problems.

*Step 15.* Now, we can mimic the proof of [LTV23, Theorem 7] to obtain the precise asymptotic on  $b(n)$ .

For periodic matrices evaluate the power series at  $\zeta^k \lambda$  for  $\zeta$  a root of unity. With this change all the above work mutatis mutandis. See also the final part of [Kit98, Section 7.1].

*Finite approximation.* Since the FBC is final, the vector  $\ell$  is supported by the FBC. The first column of the matrices in (5.9) are then supported by the FBC. Hence, [Kit98, Theorem 7.1.4] applies and we are done.

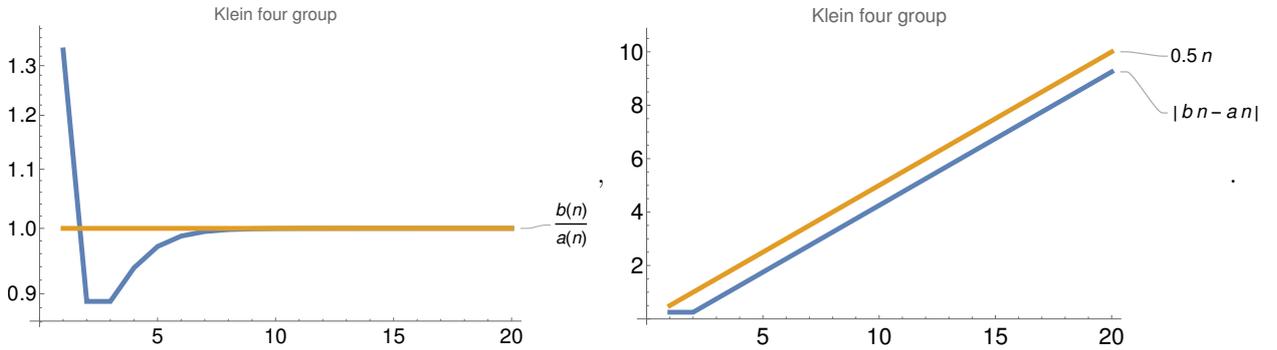
*Variance.* As in the previous point, but since the FBC is now assumed to be finite, we can use [LTV23, Theorem 1].  $\square$

*Remark 5.15.* For Krull–Schmidt monoidal categories with finitely many indecomposable objects Theorem 5.10 recovers the main results of [LTV23] (up to the point made in Remark 5.11).  $\diamond$

**Example 5.16.** Let us come back to Example 4.14. The random walk on the fusion graph  $\Gamma$  in (4.15) is positively recurrent. Thus, Theorem 5.10 applies and we get

$$b_n \sim \frac{1}{4} \cdot 3^n$$

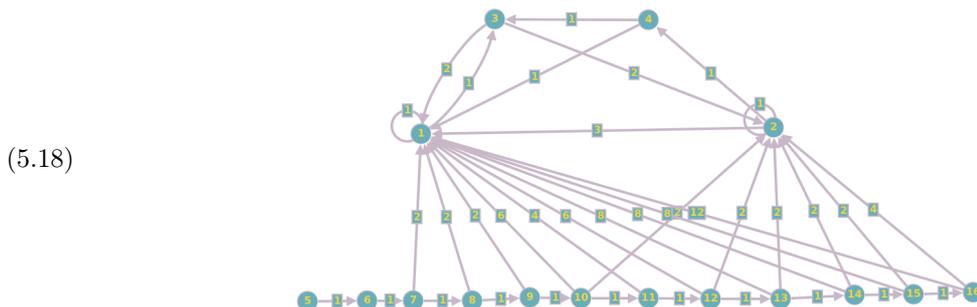
(note that  $|G| = 4$  and  $\sum_{L \text{ simples}} \dim_{\mathbb{F}_2} L = 1$ ) using the finite cutoffs as indicated in (4.15). We get:



The left plot is a logplot, the right is a standard plot.  $\diamond$

**Example 5.17.** Another example from the realm of finite groups is the following. Consider  $G = \text{PSL}_2(\mathbb{F}_7)$ . Over  $\mathbb{F}_2$  the representation theory of  $G$  is not semisimple. In this case there are two three dimensional simple representations, and let  $X$  be any of these two. The choice of this example is motivated by [Cra13, Theorem 1.4], and is the smallest example on that list.

The fusion graph of the associated growth problem  $(G, X)$  can be seen to be (illustrated below with a cutoff)



$$M(PSL2) = \begin{pmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 2 & 2 & 2 & 6 & 4 & 6 & 8 & 8 & 8 & 12 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 4 & 4 & 4 & 4 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The projective cell is illustrated at the top and has the adjacency matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The dimensions of the projective indecomposables, in the order as in (5.18), are 8, 16, 16, 8. The eigenvalues of  $M$  are  $\{3, \frac{1}{2}(-1 + i\sqrt{7}), \frac{1}{2}(-1 - i\sqrt{7}), 0\}$ , and our dominating growth rate is 3.

The dimensions of the indecomposables along the bottom in (5.18) is

$$1, 3, 9, 11, 17, 35, 25, 43, 49, 51, 57, 75, 65, 83, 89, 91, 97, 115, 105, 123, 129, \dots,$$

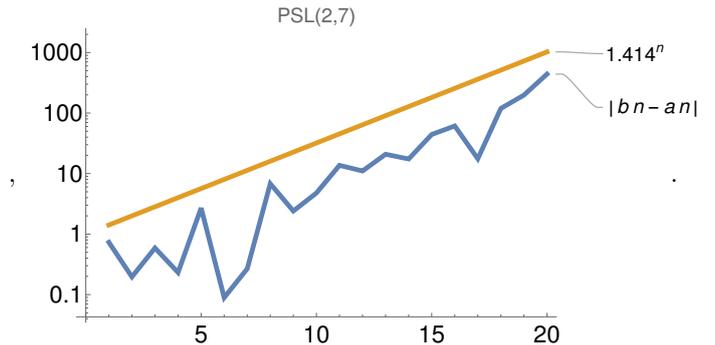
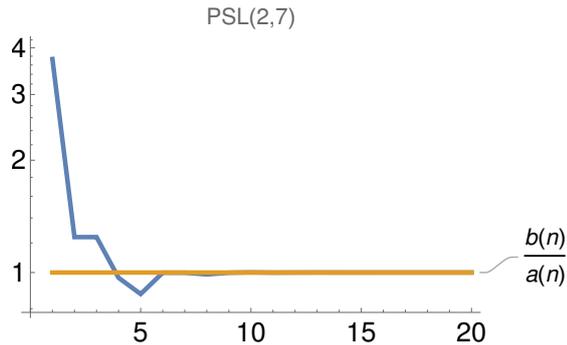
$$\text{pattern: } +6, +2, +6, +18, -10, +18, \text{repeat,}$$

$$\text{generating function: } \frac{16x^5 + 5x^4 + 7x^3 + 8x^2 + 3x + 1}{(x-1)^2(x^3 + 2x^2 + 2x + 1)},$$

and this pattern continues. From this pattern one gets the whole graph.

Moreover, one can check that  $v_0^k(w_0^k)^T[1]$  for the leading eigenvectors converges to  $\frac{15}{168} \approx 0.089$  as  $k \rightarrow \infty$ . (Note that  $|G| = 168$  and  $\sum_{L \text{ simples}} \dim_{\mathbb{F}_2} L = 15$ .) Thus, Theorem 5.10 implies that:

$$b_n \sim \frac{15}{168} \cdot 3^n.$$



Note that  $|\lambda^{sec}| = |\frac{1}{2}(-1 + i\sqrt{7})| = \sqrt{2} \approx 1.414$ . The convergence is geometric with ratio  $\sqrt{2}/3 = |\frac{1}{2}(-1 + i\sqrt{7})|/3 \approx 0.4714$ .  $\diamond$

**Example 5.19.** Let  $W$  be a finite Coxeter group and  $\mathbf{SBim} = \mathbf{SBim}(W)$  the associated monoidal category of Soergel bimodules. Let  $X \in \mathbf{SBim}$  be some object. The growth problem  $(\mathbf{SBim}, X)$  is positively recurrent and has the following asymptotics.

In the Grothendieck ring, the group ring  $\mathbb{Z}W$ , write  $[X] = \sum_{s \in W} m_s \cdot s$  where  $m_s \in \mathbb{Z}_{\geq 0}$ . One gets

$$b(n) \sim \frac{1}{|W|} \cdot \left( \sum_{s \in W} m_s \right)^n.$$

This follows directly from Theorem 5.10 since one only has one FBC corresponding to the indecomposable Soergel bimodule associated to the longest word.  $\diamond$

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