THE METRIC FOR MATRIX DEGENERATE KATO SQUARE ROOT OPERATORS

GIANMARCO BROCCHI AND ANDREAS ROSÉN

Abstract. We prove a Kato square root estimate with anisotropically degenerate matrix coefficients. We do so by doing the harmonic analysis using an auxiliary Riemannian metric adapted to the operator. We also derive L^2-solvability estimates for boundary value problems for divergence form elliptic equations with matrix degenerate coefficients. Main tools are chain rules and Piola transformations for fields in matrix weighted L^2 spaces, under $W^{1,1}$ homeomorphism.

Contents

Introduction

Our point of departure is the celebrated Kato square root estimate

$$\|\sqrt{-\mathrm{div} A\nabla u}\|_{L^2(\mathbb{R}^d)} \eqsim \|\nabla u\|_{L^2(\mathbb{R}^d)} \tag{0.1}$$

proved by [Aus+02], where the complex valued coefficient matrix A is assumed only to be bounded, measurable and accretive. After its formulation by Tosio Kato in [Kat61], [Kat95, p. 332], already the one dimensional result, $d = 1$, was only solved 20 years later by Coifman, McIntosh, and Meyer [CMM82]. The higher dimensional result [Aus+02] in $d \geq 2$ took some additional 20 years, and a reason was that the non-surjectivity of ∇ requires a more elaborated stopping time argument in the Carleson measure estimate at the heart of the proof. That the estimate (0.1) is beyond the scope of classical Calderón–Zygmund theory for $d \geq 2$ is clear from the fact that, in general, the Kato square root estimate may hold in $L^p(\mathbb{R}^d)$ only for p in a small interval around $p = 2$, depending on the matrix A.

In this paper we consider the extension of (0.1) to weighted L^2 estimates. Cruz-Uribe and Rios [CR15] proved the weighted Kato square root estimate

$$\|\sqrt{-(1/w)\mathrm{div} A\nabla u}\|_{L^2(\mathbb{R}^d,w)} \eqsim \|\nabla u\|_{L^2(\mathbb{R}^d,w)} \tag{0.2}$$

2020 *Mathematics Subject Classification.* 42B37, 35J70, 58J32, 35J56, 35J57, 47B12.

Key words and phrases. Matrix weight, Riemannian metric, anisotropically degenerate coefficients, holomorphic functional calculus, weighted norm inequalities.

for Muckenhoupt weight $w \in A_2(\mathbb{R}^d)$ and degenerate coefficient matrices A satisfying

$$\mathfrak{Re}\langle A(x)v, v\rangle \gtrsim w(x)|v|^2\,, \quad |A(x)| \lesssim w(x) \quad \text{for all } x \in \mathbb{R}^d, v \in \mathbb{C}^d.$$

It should be noted that Rubio de Francia extrapolation is not applicable here, since the operator $-(1/w)\mathrm{div}A\nabla$ and the $L^2(w)$-norm are coupled. However, under additional assumption on w, Cruz-Uribe, Martell, and Rios [CMR18] proved (0.2) with degenerate coefficients also in the unweighted $L^2(\mathbb{R}^d)$-norm.

We shall however follow a different path, where we seek to decouple A from w in the operator $-(1/w)\mathrm{div}A\nabla$. To this end, we consider more general *anisotropically* degenerate elliptic operators $-(1/a)\mathrm{div}A\nabla$, where the complex-valued scalar function $a(x)$ is controlled by a scalar weight μ, as

$$\mathfrak{Re}\, a(x) \gtrsim \mu(x)\,, \quad |a(x)| \lesssim \mu(x) \tag{a}$$

and the complex matrix function $A(x)$ is controlled as

$$\mathfrak{Re}\langle A(x)v, v\rangle \gtrsim \langle W(x)v, v\rangle\,, \quad |W(x)^{-1/2}A(x)W(x)^{-1/2}| \lesssim 1 \tag{A}$$

by a matrix weight W, meaning that $W(x)$ is a positive definite matrix at almost every point $x \in \mathbb{R}^d$. The second condition in (A) is equivalent to

$$\langle A(x)v, v\rangle \lesssim \langle W(x)v, v\rangle \quad \text{for all } x \in \mathbb{R}^d, v \in \mathbb{C}^d.$$

Note carefully that for such degenerate elliptic operators $-(1/a)\mathrm{div}A\nabla$, not only the size of the two coefficients a and A can differ unboundedly, but the size of $A(x)v$ can vary unboundedly between different directions $v \in \mathbb{C}^d$, $|v| = 1$, at $x \in \mathbb{R}^d$.

The natural norms for the operator $-(1/a)\mathrm{div}A\nabla$ appear using the standard duality proof of the Kato square root estimate in the special case of self-adjoint coefficients $a = \mu$ and $A = W$:

$$\begin{aligned}&\|\sqrt{-(1/\mu)\mathrm{div}W\nabla}u\|^2_{L^2(\mu)}\\ &\qquad = \langle -(1/\mu)\mathrm{div}W\nabla u, u\rangle_{L^2(\mu)} = \langle W\nabla u, \nabla u\rangle_{L^2(\mathbb{R}^d)} = \|\nabla u\|^2_{L^2(\mathbb{R}^d, W)}.\end{aligned}$$

Our problem is thus to understand under what conditions on μ and W the matrix-weighted Kato square root estimate

$$\|\sqrt{-(1/a)\mathrm{div}A\nabla}u\|_{L^2(\mathbb{R}^d,\mu)} \eqsim \|\nabla u\|_{L^2(\mathbb{R}^d,W)} \tag{0.3}$$

holds for general a and A satisfying (a) and (A) respectively. We study (0.3) using a framework of first order differential operators, which goes back to [AMN97] and [AKM06]. The approach consists in proving boundedness of the H^∞ functional calculus for perturbations of a first order self-adjoint differential operator D, perturbed by a bounded and accretive multiplication operator B. In our context, we set

$$D = \begin{bmatrix} 0 & -(1/\mu)\mathrm{div}W \\ \nabla & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \mu/a & 0 \\ 0 & W^{-1}A \end{bmatrix}. \tag{0.4}$$

The operators D and B act on the Hilbert space $\mathscr{H} = L^2(\mu) \oplus L^2(W)$. The perturbed operator

$$BD = \begin{bmatrix} 0 & -(1/a)\mathrm{div}W \\ W^{-1}A\nabla & 0 \end{bmatrix} \tag{0.5}$$

has spectrum in a bisector around the real line, and we show the boundedness of the H^∞ functional calculus for BD, as defined in §0.2. The Kato square root estimate

(0.3) then follows from the boundedness of the sign function of BD, namely from the estimate

$$\left\|\sqrt{(BD)^2}\begin{bmatrix} u \\ 0 \end{bmatrix}\right\|_{\mathscr{H}} \eqsim \left\|BD\begin{bmatrix} u \\ 0 \end{bmatrix}\right\|_{\mathscr{H}} \tag{0.6}$$

since $\sqrt{(BD)^2} = \operatorname{sgn}(BD)BD$ and

$$\sqrt{(BD)^2} = \begin{bmatrix} \sqrt{-\frac{1}{a}\mathrm{div} A\nabla} & 0 \\ 0 & \sqrt{-W^{-1}A\nabla\frac{1}{a}\mathrm{div} W} \end{bmatrix},$$

while the right hand side of (0.6) is equivalent to $\|\nabla u\|_{L^2(W)}$ as desired.

In the isotropically degenerate case with $W = \mu I$, boundedness of the H^∞ functional calculus of BD, and in particular (0.6), was proved in [ARR15]. Important to note is that the proof in [ARR15] does not require B to be block diagonal, as compared to the one in [CR15], as [ARR15] uses a more elaborate double stopping argument for test function and weight. Our results in the present paper do not require B to be block diagonal either. Non-block diagonal B are important in applications to boundary value problems: we extend [AMR22, §4] to anisotropic degenerate elliptic equations in §3.

When trying to prove boundedness of the H^∞ functional calculus for our operator BD from (0.5), following the local Tb argument in [ARR15], one soon realises that the main obstacle when $W \neq \mu I$ is the L^2 off-diagonal estimates for the resolvents of BD. In all previous works, one has an estimate

$$\|(I + itBD)^{-1}u\|_{L^2(F,\mu)} \lesssim \eta\Big(\frac{\mathrm{dist}(E,F)}{t}\Big)\,\|u\|_{L^2(E,W)} \tag{0.7}$$

with $\eta(x)$ rapidly decaying to 0 as $x \to \infty$ and $\mathrm{dist}(E, F)$ being the distance between sets $E, F \subseteq \mathbb{R}^d$. So the resolvents are not only bounded, but act almost locally at scale t. When $W \neq \mu I$, this crucial estimate in the local Tb theorem may fail. Indeed, the commutator estimate used in the proof of (0.7) fails, as it uses boundedness of

$$[D, \eta] = \begin{bmatrix} 0 & -\frac{1}{\mu}[\mathrm{div}, \eta]W \\ [\nabla, \eta] & 0 \end{bmatrix}.$$

This is a bounded multiplier on $L^2(\mu)\oplus L^2(W)$, with norm $\|\nabla\eta\|_{L^\infty}$, only if $|W| \lesssim \mu$. But even assuming this latter bound, it is still unclear to us how to extend the remaining part of the euclidean proof from [ARR15] which seems to require nontrivial two-weight bounds.

The way we instead resolve this problem is to replace the euclidean metric by a Riemannian metric g adapted to the operator BD. We show in §2 that the euclidean operator BD on $L^2(\mathbb{R}^d, \mu) \oplus L^2(\mathbb{R}^d; \mathbb{C}^d, W)$ is in fact similar to an operator $B_M D_M$ acting on $L^2(M, \nu)\oplus L^2(TM, \nu I)$ for a auxiliary Riemannian manifold M with metric g and a single scalar weight ν associated with μ, W.

Note that the scalar weight ν determines the norms *both* on scalar *and* vector functions. Thus we have reduced to the situation in [ARR15], but with $\mathbb{R}^d$ replaced by a manifold M. The euclidean proof in [ARR15] has been generalised to a class of manifolds in [AMR22], notably those with positive injectivity radius and Ricci curvature bounded from below. Applying [AMR22] to $B_M D_M$ gives boundedness of its H^∞ functional calculus and, via similarity, also for our anisotropically degenerate operator BD on $\mathbb{R}^d$. This in particular shows the matrix-weighted Kato square root estimate (0.3) for a class of weights (μ, W) determined by properties

$$\begin{tikzcd}
\mathscr{H}_M := L^2(M,\nu) \oplus L^2(TM,\nu I) \arrow[r, "D_M B_M"] \arrow[d, "\mathsf{P}"] & \mathscr{H}_M \\
\mathscr{H} := L^2(\mathbb{R}^d,\mu) \oplus L^2(\mathbb{R}^d;\mathbb{C}^d, W) \arrow[r, "DB"] & \mathscr{H} \arrow[u, "\mathsf{P}^{-1}"']
\end{tikzcd}$$

Figure 1. We will use a unitary map P and its inverse, introduced in §1 and defined in (2.1).

of (g,ν). The examples at the end of §2 show that indeed this class covers weights beyond [ARR15]. In forthcoming papers, we shall relax further the hypotheses on the auxiliary manifold (M,g).

Preliminaries

Notations. For two quantities $X, Y \geq 0$, the expression $X \lesssim Y$ means that there exists a finite, positive constant C such that $X \leq CY$. The expression $X \gtrsim Y$ means that $Y \lesssim X$. When both expressions hold at the same time, with possibly different constants, we will write $X \eqsim Y$. Given a matrix W the quantities $|W|$ and $\|W\|_{\mathrm{op}}$ denote any of the equivalent matrix norms of W.

As discussed in the introduction, the Kato square root estimate follows from the boundedness of functional calculus for a bisectorial operator BD. Here we recall these concepts.

0.1. **Bisectorial operators.**

For an angle $\theta \in [0, \pi/2)$, consider the closed bisector

$$S_\theta := \{z \in \mathbb{C} \,:\, |\arg(z)| \leq \theta\} \cup \{0\} \cup \{z \in \mathbb{C} \,:\, |\arg(-z)| \leq \theta\}.$$

Definition 0.1 (Bisectorial operator)**.** A closed, densely defined operator D on a Hilbert space is bisectorial if there exists an angle $\theta \in [0, \pi/2)$ such that

- the spectrum $\sigma(D)$ is contained in the bisector S_θ;
- outside S_θ we have resolvent bounds: $\|(\lambda I - D)^{-1}\| \lesssim 1/\mathrm{dist}(\lambda, S_\theta)$.

Given a densely defined operator D, its domain will be denoted by $\mathsf{dom}(D)$. If D is bisectorial, we have the topological (not necessarily orthogonal) splitting [AAM10, Proposition 3.3 (ii)]

$$\mathscr{H} = \mathsf{ker}(D) \oplus \overline{\mathsf{im}(D)}$$

where $\mathsf{ker}(D) := \{u \in \mathsf{dom}(D), Du = 0\}$ is always closed and $\mathsf{im}(D) := \{Du \in \mathscr{H}, u \in \mathsf{dom}(D)\}$. In particular, restricting D to the closure of its range gives an injective bisectorial operator.

0.2. **Bounded holomorphic functional calculus.**

Given $\theta' > \theta$, with $\theta', \theta \in [0, \pi/2)$, let $\mathring{S}_{\theta'}$ be the interior of the bisector $S_{\theta'}$. Denote by $H^\infty(\mathring{S}_{\theta'})$ the space of bounded holomorphic functions on $\mathring{S}_{\theta'}$. Given an injective operator D which is bisectorial on S_θ, we say that D has bounded H^∞ functional calculus on $\mathring{S}_{\theta'}$ if any function $f \in H^\infty(\mathring{S}_{\theta'})$ defines a bounded operator $f(D)$ with norm bound

$$\|f(D)\|_{\mathscr{H} \to \mathscr{H}} \lesssim \|f\|_{L^\infty(\mathring{S}_{\theta'})}.$$

For a non-injective operator D, the H^∞ functional calculus can be extended to the whole space $\mathscr{H}$ by setting $f(D)\restriction_{\mathsf{ker}(D)} := f(0) I\restriction_{\mathsf{ker}(D)}$, for $f \colon \{0\} \cup \mathring{S}_{\theta'} \to \mathbb{C}$ such that $f\restriction_{\mathring{S}_{\theta'}} \in H^\infty(\mathring{S}_{\theta'})$.

0.3. **Quadratic estimates.** A bisectorial operator D acting on a Hilbert space $\mathscr{H}$ satisfies quadratic estimates if

$$\left(\int_0^\infty \|\psi_t(D)u\|^2_{\mathscr{H}} \frac{\mathrm{d}t}{t}\right)^{1/2} \lesssim \|u\|_{\mathscr{H}} \qquad \text{for all } u \in \mathscr{H}, \tag{0.8}$$

where $\psi_t(\zeta) := \psi(t\zeta)$ and ψ is any function in $H^\infty(\mathring{S}_{\theta'})$ which is non-vanishing on both sectors and decaying $|\psi(\zeta)| \lesssim |\zeta|^s(1+|\zeta|^{2s})^{-1}$ for some $s > 0$. Since quadratic estimates for one such ψ implies quadratic estimates for all such ψ, for simplicity we take $\psi(\zeta) = \zeta/(1+\zeta^2)$. Bisectorial operators D, for which both D and $D^\star$ satisfy the quadratic estimates (0.8) have H^∞ functional calculus. See [ADM96], where this is shown for sectorial operators. The extension to bisectorial operators is straightforward.

0.4. **Weights.** A scalar weight is a function $x \mapsto \mu(x)$ which is positive almost everywhere, while a matrix weight is a matrix-valued function $x \mapsto W(x)$ such that $W(x)$ is symmetric, positive definite matrix at almost every x. We will consider weights on $\mathbb{R}^d$ and on a complete Riemannian manifold M with Riemannian measure $\mathrm{d}y$.

Definition 0.2. Let W be a matrix weight. A multiplication operator B is said to be W-bounded if

$$|W^{1/2}BW^{-1/2}| \lesssim 1 \quad \text{a.e.}$$

and it is said to be W-accretive if

$$\mathfrak{Re}\langle W^{1/2}BW^{-1/2}v, v\rangle \gtrsim |v|^2 \quad \text{a.e. and } \forall v \in \mathbb{C}^d.$$

Note that

- B is W-bounded if and only if the map $v \mapsto Bv$ is bounded in the norm $v \mapsto |W^{1/2}v|$
- B is W-accretive if and only if the map $v \mapsto Bv$ is accretive with respect to the inner product $\langle Wv, v\rangle$ associated to the norm $|W^{1/2}v|$.
- For scalar weights $W = w$ this reduces to standard unweighted notions of boundedness and accretivity.

When W is a block diagonal matrix $\left[\begin{smallmatrix}\mu & 0\\ 0 & w\end{smallmatrix}\right]$, we will use the notation $(\mu \oplus w)$, and say that a multiplication operator is $(\mu \oplus w)$-bounded and $(\mu \oplus w)$-accretive.

A special class of weights are Muckenhoupt weights, which are defined in terms of averages. Let $B = B(x,r)$ be a geodesic ball of radius $r > 0$ centred at x. If $|B|$ denotes the Riemannian measure of a ball B, the average of a scalar weight ν over B is $⨍_B \nu \,\mathrm{d}y := |B|^{-1}\int_B \nu \,\mathrm{d}y$.

Definition 0.3 (Muckenhoupt A_2^R weights)**.** Let $R > 0$ be fixed. A scalar weight $\nu\colon M \to [0,\infty]$ belongs to the Muckenhoupt class $A_2^R(M)$, with respect to the Riemannian measure $\mathrm{d}y$, if

$$[\nu]_{A_2^R} := \sup_{\substack{y_0 \in M\\ r<R}} \left(⨍_{B(y_0,r)} \nu(y)\,\mathrm{d}y\right)\left(⨍_{B(y_0,r)} \frac{1}{\nu(y)}\,\mathrm{d}y\right) < \infty.$$

We say that a weight $\nu \in A_2(M)$ if $[\nu]_{A_2} := \sup_{R>0}[\nu]_{A_2^R}$ is finite.

We also introduce local Muckenhoupt weights, as these are used to apply dominated convergence locally, for example in proving the density of smooth functions in matrix-weighted Sobolev spaces. Note that we do not use the A_2^{loc} property quantitatively.

Definition 0.4 (Local Muckenhoupt weights)**.** Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let μ and W be a scalar and a matrix weight, respectively. We say that μ is in $A_2^{\mathrm{loc}}(\Omega)$ if for any compact $K \subset \Omega$

$$\sup_{B \subset K} \Big(\fint_B \mu(x) \mathrm{d}x \Big) \Big(\fint_B \frac{1}{\mu(x)} \mathrm{d}x \Big) < \infty,$$

where the supremum is over balls B. Similarly, W is in $A_2^{\mathrm{loc}}(\Omega)$ if for any compact $K \subset \Omega$

$$\sup_{B \subset K} \Big\| \Big(\fint_B W(x) \mathrm{d}x \Big)^{1/2} \Big(\fint_B W^{-1}(x) \mathrm{d}x \Big)^{1/2} \Big\|_{\mathrm{op}}^2 < \infty$$

where $\| \cdot \|_{\mathrm{op}}$ is the operator norm on the space of linear operators acting on $\mathbb{C}^d$.

As in Definition 0.4 we define $A_2^{\mathrm{loc}}(M)$ on a manifold M for scalar weights. One can show that for scalar weights it holds that $A_2^R \subset A_2^{\mathrm{loc}}$ for any $R > 0$.

Defining matrix weights on a Riemannian manifold M is more subtle. At any $y \in M$, $W(y)$ should be a positive definite map of T_yM, and in a chart $\varphi \colon \mathbb{R}^d \to M$, it should be represented by $W_\varphi := (\mathrm{d}\varphi)^{-1} W (\mathrm{d}\varphi^\star)^{-1}$. However, the following example indicates that the matrix A_2 condition on W_φ is not in general invariant under transition maps between different smooth charts φ.

Example 0.5. Let $W \colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ be the matrix weight

$$W(x) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1+2r \end{bmatrix} \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}.$$

The constant diagonal matrix $W(0) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1+2r \end{smallmatrix}\right]$ is trivially a matrix A_2 weight with $[W(0)]_{A_2} = 1$ for any $r \geq 0$. A direct computation shows that

$$\lim_{r \to +\infty} \Big\| \Big(\fint_0^\pi W(x) \mathrm{d}x \Big)^{1/2} \Big(\fint_0^\pi W^{-1}(x) \mathrm{d}x \Big)^{1/2} \Big\|_{\mathrm{op}}^2 = \infty.$$

See also [Bow01, Proposition 5.3] and [BLM17, Example 4.3].

Therefore, we make the following auxiliary definition:

Definition 0.6. A matrix weight $W \in \mathsf{End}(TM)$ belongs to $A_2^{\mathrm{loc}}(M)$ if at each $y \in M$ there exists a chart φ such that $(\mathrm{d}\varphi)^{-1} W (\mathrm{d}\varphi^\star)^{-1}$ is a weight in $A_2^{\mathrm{loc}}(\mathbb{R}^d)$.

1. Two scalar weights in one dimension

Following the historical tradition of the Kato square root problem, we first consider the one dimensional problem. We treat this case separately since all one-dimensional manifolds are locally isometric, so no hypothesis on the Riemannian metric g is needed, only hypothesis on the weight ν.

In dimension $d = 1$ the matrix weight $W(x)$ reduces to a scalar weight $w(x)$, and $\nabla = \mathrm{div} = \partial_x$ is the derivative. Consider the differential operator

$$D = \begin{bmatrix} 0 & -(1/\mu)\partial_x w \\ \partial_x & 0 \end{bmatrix}. \tag{1.1}$$

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a "rubber band" parametrisation, a map stretching the real line, with $y = \rho(x)$ for $x \in \mathbb{R}$. To see g, ν appear, we consider the pullback

$$\mathsf{P}\colon \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix} \mapsto \begin{bmatrix} v_1(\rho(x)) \\ v_2(\rho(x))\rho'(x) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}. \tag{1.2}$$

The basic observation is the following.

Lemma 1.1. *Let μ, w be two weights that are smooth on an interval $I \subset \mathbb{R}$. Let $\rho\colon I \to \mathbb{R}$ be such that $\rho'(x) = \sqrt{\mu(x)/w(x)}$. Set $M := \rho(I) \subset \mathbb{R}$. Let $\nu(\rho(x)) := \sqrt{\mu(x)w(x)}$ and*

$$D_M := \begin{bmatrix} 0 & -(1/\nu)\partial_y \nu \\ \partial_y & 0 \end{bmatrix}. \tag{1.3}$$

Then the map P is an isometry between the Hilbert spaces $\mathscr{H} = L^2(I, \mu) \oplus L^2(I, w)$ and $\mathscr{H}_M := L^2(M, \nu) \oplus L^2(M, \nu)$, and $\mathsf{P}^{-1} D \mathsf{P} = D_M$.

Proof. We verify that $\mathsf{P} D_M = D\mathsf{P}$. This amounts to check the equality (!) in

$$\mathsf{P} D_M \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \Big((-1/\nu)\partial_y \nu v_2\Big) \circ \rho \\ \rho'(\partial_y v_1) \circ \rho \end{bmatrix} \overset{(!)}{=} \begin{bmatrix} -(1/\mu)\partial_x w(v_2 \circ \rho)\,\rho' \\ \partial_x(v_1 \circ \rho) \end{bmatrix} = D\mathsf{P} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The identity for the second component is the chain rule in Theorem A.2 in one dimension. The identity for the first component is seen by multiplying and dividing by ρ';

$$\frac{1}{\nu(\rho(x))\rho'(x)} \cdot \rho'(x)\partial_y(\nu\, v_2)(\rho(x)) \overset{(!)}{=} \frac{1}{\mu(x)}\partial_x(w(x)\rho'(x)(v_2 \circ \rho)(x)),$$

and noting that

$$\begin{cases} \mu(x) = \nu(\rho(x))\rho'(x), \\ w(x)\rho'(x) = \nu(\rho(x)). \end{cases} \tag{1.4}$$

Using the identities in (1.4) and the definition of P, the weighted norms $\|u_1\|_{L^2(\mu)}$ and $\|u_2\|_{L^2(w)}$ become

$$\int |u_1(x)|^2 \mu(x)\,\mathrm{d}x = \int |v_1(\rho(x))|^2 \nu(\rho(x))\rho'(x)\,\mathrm{d}x = \int |v_1(y)|^2 \nu(y)\,\mathrm{d}y,$$
$$\int |u_2(x)|^2 w(x)\,\mathrm{d}x = \int |v_2(\rho(x))\rho'(x)|^2 w(x)\,\mathrm{d}x = \int |v_2(y)|^2 \nu(y)\,\mathrm{d}y.$$

This shows that P is an isometry and concludes the proof. $\square$

Lemma 1.1 shows that formally, D in $L^2(I, \mu) \oplus L^2(I, w)$ is similar to D_M defined in (1.3) acting on $L^2(M, \nu) \oplus L^2(M, \nu)$, to which [ARR15] applies. To this end, for non-smooth μ and w, we need that $\nu \in A_2(\mathbb{R}, \mathrm{d}y)$ and the map ρ to be absolutely continuous, which amounts to $\rho' = \sqrt{\mu/w} \in L^1_{\mathrm{loc}}$. This holds in particular if $\mu, w \in A_2^{\mathrm{loc}}$ which we need in order to apply Theorem A.2. Somewhat more subtle, to ensure that we obtain a complete manifold, we must also take into account the completeness of the y-axis, that is, M. This corresponds to the problem of defining D as self-adjoint operator in $L^2(\mu) \oplus L^2(w)$. Indeed, if ρ maps onto an interval $M \subsetneq \mathbb{R}$, boundary conditions need to be imposed for D_M to be self-adjoint in $\mathscr{H}_M$, and hence for $D = \mathsf{P} D_M \mathsf{P}^{-1}$ to be self-adjoint. Although this can be done, here we limit our study to the case in which M is a complete manifold. See also Example 1.6 below.

Theorem 1.2. *Consider a possibly unbounded interval $I = (c_1, c_2) \subseteq \mathbb{R}$. Let μ, w be weights in $A_2^{\mathrm{loc}}(I)$ and assume that*

$$\int_{c_1}^{c} \sqrt{\frac{\mu}{w}}\,\mathrm{d}t = \int_{c}^{c_2} \sqrt{\frac{\mu}{w}}\,\mathrm{d}t = \infty \quad \text{for } c_1 < c < c_2.$$

For some fixed $c \in (c_1, c_2)$, let

$$\rho(x) = \int_c^x \sqrt{\frac{\mu}{w}}\,\mathrm{d}t \quad \text{and} \quad \nu(y) := \sqrt{\mu(\rho^{-1}(y))w(\rho^{-1}(y))}.$$

Assume that $\nu \in A_2(\mathbb{R}, \mathrm{d}y)$. Let D be the operator defined in (1.1) and let B be a $(\mu\oplus w)$-bounded and $(\mu\oplus w)$-accretive multiplication operator on $L^2(I,\mu)\oplus L^2(I,w)$ as in Definition 0.2. Then BD and DB are bisectorial operators satisfying quadratic estimates and have bounded H^∞ functional calculus in $L^2(I,\mu)\oplus L^2(I,w)$.

Proof. The operator D_M in (1.3) has domain $\mathcal{H}^1_\nu \oplus (\mathcal{H}^1_\nu)^\star$ where

$$\mathcal{H}^1_\nu := \{v \in L^2(\nu) \,:\, \partial_y v \in L^2(\nu)\}$$

and the adjoint space $(\mathcal{H}^1_\nu)^\star = \{v \in L^2(\nu) \,:\, (1/\nu)\partial_y \nu v \in L^2(\nu)\}$. The operator D has domain $\mathcal{H}^1_{\mu,w} \oplus (\mathcal{H}^1_{\mu,w})^\star$ where

$$\mathcal{H}^1_{\mu,w} := \{u \in L^2(\mu) \,:\, \partial_x u \in L^2(w)\}$$

and the adjoint space $(\mathcal{H}^1_{\mu,w})^\star = \{u \in L^2(w) \,:\, (1/\mu)\partial_x w u \in L^2(\mu)\}$. Note that the operator $(1/\mu)\partial_x w \colon L^2(w) \to L^2(\mu)$ is unitary equivalent to $\partial_x \colon L^2(w^{-1}) \to L^2(\mu^{-1})$, since the multiplication by w is a unitary map from $L^2(w) \to L^2(w^{-1})$.

The pullback transformation P maps between the domains of D_M and D. Indeed, if $v \in \mathcal{H}^1_\nu$, then by Theorem A.2 applied with $v = \nu$ and $V = \nu$, we have that $u := \rho^* v \in L^2(\mu)$ and

$$\partial_x u = \partial_x(\rho^* v) = \rho^*(\partial_y v) = \rho'\,(\partial_y v)\circ\rho \in L^2(w)$$

since $v_\rho = \mu$ and $V_\rho = w$. Similarly, we see that the L^2-adjoint of ρ^*, ρ_*/ρ', maps

$$\{u \in L^2(w^{-1}) \,:\, \partial_x u \in L^2(\mu^{-1})\} \to \mathcal{H}^1_{\nu^{-1}}.$$

By applying Theorem A.3 with $v = w^{-1}$ and $V = \mu^{-1}$, we see that both v^ρ and V^ρ equals $1/\nu$, so we have that $(\rho')^{-1}\rho_* u \in L^2(\nu^{-1})$ and

$$\partial_y\Big(\frac{\rho_*}{\rho'}u\Big) = \frac{\rho_*}{\rho'}(\partial_x u) \in L^2(\nu^{-1}).$$

Let $B_M := \mathsf{P}^{-1}B\mathsf{P}$. We show that B is $(\mu\oplus w)$-bounded and $(\mu\oplus w)$-accretive if and only if the operator B_M is $(\nu\oplus\nu)$-bounded and $(\nu\oplus\nu)$-accretive. The $(\nu\oplus\nu)$-boundedness of B_M means that

$$\int \big(\left[\begin{smallmatrix}\nu & 0\\ 0 & \nu\end{smallmatrix}\right]\mathsf{P}^{-1}B\mathsf{P}v, \mathsf{P}^{-1}B\mathsf{P}v\big)\,\mathrm{d}y \lesssim \int \big|\left[\begin{smallmatrix}\nu & 0\\ 0 & \nu\end{smallmatrix}\right]^{1/2} v\big|^2\,\mathrm{d}y. \tag{1.5}$$

Let $u = \mathsf{P}v$, then the left hand side of (1.5) equals

$$\langle \mathsf{P}^{-1}Bu, \mathsf{P}^{-1}Bu\rangle_{L^2\left(\left[\begin{smallmatrix}\nu & 0\\ 0 & \nu\end{smallmatrix}\right]\right)} = \langle \mathsf{P}\mathsf{P}^{-1}Bu, Bu\rangle_{L^2\left(\left[\begin{smallmatrix}\mu & 0\\ 0 & w\end{smallmatrix}\right]\right)}$$

where we used that $\mathsf{P}^{-1} = \mathsf{P}^\star$, since P is an isometry, as shown in Lemma 1.1. The same applies to show that B_M is $(\nu\oplus\nu)$-accretive if and only if B is $(\mu\oplus w)$-accretive.

Now, to prove the theorem, apply [ARR15, Theorem 3.3] to $D_M B_M$, where $D_M := \mathsf{P}^{-1}D\mathsf{P}$. It follows that $D_M B_M$ satisfies quadratic estimates. The same holds for the operator DB via the isometry P, and for $BD = B(DB)B^{-1}$. $\square$

Remark 1.3. Since the Riemannian measure of $\rho(J)$ for any subinterval $J \subseteq I$ is $\int_J \rho'(x)\,\mathrm{d}x$, the condition $\nu \in A_2(\mathbb{R}, \mathrm{d}y)$ explicitly means that for all intervals J, we have

$$\Big(\int_J \mu(x)\,\mathrm{d}x\Big)\Big(\int_J \frac{1}{w(x)}\,\mathrm{d}x\Big) \lesssim \Big(\int_J \sqrt{\frac{\mu}{w}}\,\mathrm{d}x\Big)^2. \tag{1.6}$$

Note that the hypothesis $\mu,\mu^{-1},w,w^{-1} \in L^1_{\mathrm{loc}}$ and more precisely $\mu, w \in A_2^{\mathrm{loc}}$, is not used quantitatively, but only to ensure that:

(1) $L^2(I,\mu)$ and $L^2(M,\nu)$ are contained in $L^1_{\mathrm{loc}}(\mathrm{d}x)$, so that the derivatives in the operator D can be, and are, interpreted in the sense of distributions;
(2) the isometry P maps $\mathsf{dom}(D_M)$ bijectively onto $\mathsf{dom}(D)$.

A way to extend Theorem 1.2 to more rough weights would be to define the domain $\mathsf{dom}(D)$ as the image of $\mathsf{dom}(D_M)$ under the isometry P. In this way, one only requires that $\sqrt{\mu/w} \in L^1_{\mathrm{loc}}$ and (1.6) uniformly for all $J \subseteq I$, but, in this generality, the derivatives in D do not have the standard distributional definition.

Curiously, in one dimension we have the following

Proposition 1.4. *If $\mu, w \in A_2(I, \mathrm{d}x)$, then $\nu \in A_2(\mathbb{R}, \mathrm{d}y)$.*

Proof. The weight ν is in $A_2(\mathbb{R}, \mathrm{d}y)$ if (1.6) holds for all $J \subset \mathbb{R}$. The A_2 condition on an interval J for μ and w means

$$\int_J \mu(x)\,\mathrm{d}x \lesssim \frac{|J|^2}{\int_J 1/\mu(x)\,\mathrm{d}x} \quad , \quad \int_J \frac{1}{w(x)}\,\mathrm{d}x \lesssim \frac{|J|^2}{\int_J w(x)\,\mathrm{d}x}.$$

Applying Cauchy–Schwarz twice gives as claimed

$$\Big(\int_J \mu\,\mathrm{d}x\Big)\Big(\int_J \frac{1}{w}\,\mathrm{d}x\Big) \lesssim \frac{|J|^4}{\big(\int_J 1/\mu\big)\big(\int_J w\big)} \le \left(\frac{|J|^2}{\int_J \sqrt{w/\mu}\,\mathrm{d}x}\right)^2 \le \Big(\int_J \sqrt{\frac{\mu}{w}}\,\mathrm{d}x\Big)^2.$$

□

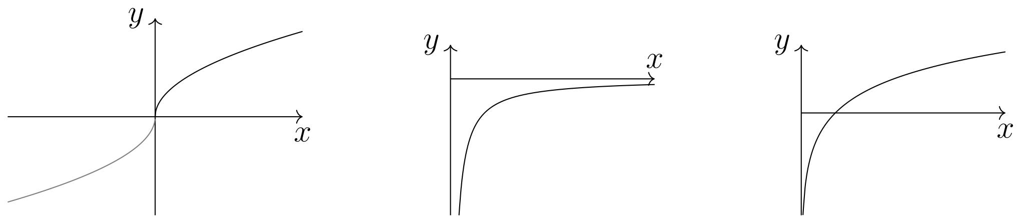
Remark 1.5. It is not clear to us if such relation between (μ, w) and ν exists in higher dimension. Moreover note that, since the Jacobian $|\rho'|$ is not necessarily bounded, but only locally integrable, the composition $\mu \circ \rho^{-1}$ is not guaranteed to be in L^1_{loc}, and so it is not a Muckenhoupt weight. Still ν can be in A_2, as Case 2 in the next example shows.

Example 1.6. Consider the power weights $\mu(x) = x^\alpha$ and $w(x) = x^{-\beta}$ for $x > 0$. Then $\rho'(x) = \sqrt{x}^{\alpha+\beta}$ and $\nu(\rho(x)) = \sqrt{x}^{\alpha-\beta}$. In computing ρ^{-1}, we distinguish three cases.

Case 1: $\alpha + \beta + 2 > 0$. In this case $\rho(x) = \frac{2}{\alpha+\beta+2}\sqrt{x}^{\alpha+\beta+2}$ is strictly positive and increasing. Thus $\nu(y) = (\frac{\alpha+\beta+2}{2}y)^{\frac{\alpha-\beta}{\alpha+\beta+2}}$. The weight $\nu \in A_2(\mathrm{d}y)$ if and only if $-1 < \frac{\alpha-\beta}{\alpha+\beta+2} < 1$, or equivalently if $\alpha > -1$ and $\beta > -1$.

Case 2: $\alpha + \beta + 2 < 0$. In this case ρ is negative and equals $\frac{1}{c}x^c$ where $c = \frac{\alpha+\beta+2}{2} < 0$ and $\nu(y) = (-cy)^{\frac{\alpha-\beta}{\alpha+\beta+2}} > 0$. The weight $\nu \in A_2(\mathrm{d}y)$ if and only if $-1 < \frac{\alpha-\beta}{\alpha+\beta+2} < 1$, or equivalently if $\alpha < -1$ and $\beta < -1$.

Case 3: $\alpha + \beta = -2$. In this case $\rho'(x) = 1/x$ and so $\rho(x) = \ln x$. Then $\rho^{-1}(y) = e^y$ and $\nu(y) = (e^y)^{(\alpha-\beta)/2}$ is in $A_2(\mathrm{d}y)$ if and only if $\alpha = \beta = -1$.



(a) Case 1. **(b)** Case 2. **(c)** Case 3.

Figure 2. Completeness of the y-axes. In Case 1, $\rho(x) = \sqrt{x}$ on $\mathbb{R}_+$ can be extended to an odd bijection $\mathbb{R} \to \mathbb{R}$. In Case 2, $\rho(x) = -1/x$ is not surjective onto $\mathbb{R}$. In Case 3, $\rho(x) = \ln(x)$ is a bijection from $\mathbb{R}_+$ to $\mathbb{R}$.

In either case $\nu \in A_2$ if and only if $\operatorname{sgn}(\alpha+1) = \operatorname{sgn}(\beta+1)$. Case 2 shows that it is possible that $\nu \in A_2$ even if μ and w are not. Note that in the extension of Case 1 to an odd bijection, and in Case 3, the map ρ is a bijection and maps onto a complete manifold, while in Case 2 the map ρ is not surjective. See Figure 2.

Assuming that $|\alpha|, |\beta| < 1$ and extending to power weights $\mu(x) = |x|^\alpha$ and $w(x) = |x|^{-\beta}$, Theorem 1.2 applies and gives quadratic estimates for the operator BD, where

$$D = \begin{bmatrix} 0 & -|x|^{-\alpha}\partial_x |x|^{-\beta} \\ \partial_x & 0 \end{bmatrix},$$

on the weighted space $L^2(\mathbb{R}, |x|^\alpha) \oplus L^2(\mathbb{R}, |x|^{-\beta})$.

Corollary 1.7. *Let $I \subseteq \mathbb{R}$ and let $\mu, w \in A_2^{\mathrm{loc}}(\mathbb{R})$ satisfy the assumptions of Theorem 1.2. In particular $\nu \in A_2(\mathbb{R}, \mathrm{d}y)$. Let a, b be two complex-valued functions on I such that*

$$\begin{aligned} \mu(x) &\lesssim \mathfrak{Re}\, a(x)\,,\ |a(x)| \lesssim \mu(x)\,, \\ w(x) &\lesssim \mathfrak{Re}\, b(x)\,,\ |b(x)| \lesssim w(x) \end{aligned} \tag{1.7}$$

for a.e. $x \in I$. Then the following Kato square root estimate holds:

$$\|\sqrt{-(1/a)\partial_x b \partial_x} u\|_{L^2(I,\mu)} \eqsim \|\partial_x u\|_{L^2(I,w)}.$$

Proof. Consider the multiplication operator $B = \left[\begin{smallmatrix} \mu/a & 0 \\ 0 & b/w \end{smallmatrix}\right]$. The hypothesis in (1.7) implies that B is bounded and accretive. Since

$$B = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & \sqrt{w} \end{bmatrix} B \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & \sqrt{w} \end{bmatrix}^{-1}$$

holds for any diagonal matrix B, then B is $\left[\begin{smallmatrix} \mu & \\ & w \end{smallmatrix}\right]$-bounded and $\left[\begin{smallmatrix} \mu & \\ & w \end{smallmatrix}\right]$-accretive. The desired estimate follows by applying Theorem 1.2 to B and D as defined in (1.1). Indeed, the perturbed operator BD equals

$$BD = \begin{bmatrix} 0 & -(1/a)\partial_x w \\ \frac{b}{w}\partial_x & 0 \end{bmatrix}$$

and so

$$\|\sqrt{-(1/a)\partial_x b \partial_x} u\|_{L^2(I,\mu)} = \left\|\sqrt{(BD)^2}\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\right\|_{\mathscr{H}}.$$

The boundedness of the H^∞ functional calculus for BD on $\mathscr{H} = L^2(I, \mu) \oplus L^2(I, w)$ implies that $\operatorname{sgn}(BD)$ is a bounded and invertible operator on $\mathscr{H}$. Since $\sqrt{(BD)^2} =$

$\operatorname{sgn}(BD)BD$, we have

$$\big\|\sqrt{(BD)^2}\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\big\|_{\mathscr{H}} \eqsim \big\|BD\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\big\|_{\mathscr{H}} \eqsim \big\|D\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\big\|_{\mathscr{H}} \eqsim \|\partial_x u\|_{L^2(I,w)}.$$

□

Example 1.8 (Cauchy integral on rectifiable graphs). Consider a curve $\gamma := (t, \varphi(t))$ as the graph of a function $\varphi\colon \mathbb{R} \to \mathbb{R}$. The curve γ is Lipschitz if and only if $\varphi' \in L^\infty$.

The Cauchy singular integral

$$\mathscr{C}_\gamma(x) := \frac{i}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{u(y)}{y + i\varphi(y) - (x + i\varphi(x))}(1 + i\varphi'(y))\, \mathrm{d}y$$

and its boundedness on $L^2(\gamma)$ for Lipschitz curves is a classical and famous problem in analysis. It was first showed by Calderón [Cal77] that $\mathscr{C}_\gamma\colon L^2(\gamma) \to L^2(\gamma)$ for a curve $\gamma \subseteq \mathbb{C}$ with small Lipschitz constant $\|\varphi'\|_{L^\infty}$. This smallness assumption was removed by Coifman–M$^{\text{c}}$Intosh–Meyer in [CMM82], where only $\|\varphi'\|_{L^\infty} < \infty$ was assumed. and finally David [Dav84] showed that $\mathscr{C}_\gamma$ is bounded on $L^2(\gamma)$ if and only if the curve γ is Ahlfors–David-regular, meaning that the 1-dimensional Hausdorff measure $\mathscr{H}^1$ restricted on the curve satisfies

$$\mathscr{H}^1(\gamma \cap B(x, r)) \eqsim r$$

for any ball $B(x, r)$ centred at $x \in \gamma$. A crucial observation due to Alan M$^{\text{c}}$Intosh which led to the seminal work [CMM82] is that the Kato estimate

$$\|\sqrt{-(1/a)\partial_x b \partial_x} u\|_{L^2(\mathbb{R})} \eqsim \|\partial_x u\|_{L^2(\mathbb{R})},$$

for $b = 1/a$, implies the L^2-estimate for $\mathscr{C}_\gamma$ on Lipschitz curves. See also Kenig and Meyer [KM85].

One can ask if the weighted estimates in Corollary 1.7 can be used to prove that $\mathscr{C}_\gamma$ is bounded on Ahlfors–David-regular graphs more general than Lipschitz graphs. This is still unclear to us. The natural strategy is as follows. As in [MQ91], the Cauchy singular integral can be written as $\operatorname{sgn}((1/a(x))i\partial_x)$, for multiplier $a(x) = 1 + i\varphi'(x)$, see also [AKM06, Consequence 3.2]. Note that the arclength measure on γ is $\mathrm{d}s := \sqrt{1 + (\varphi')^2}\, \mathrm{d}x = \mu\, \mathrm{d}x$. Boundedness of $\mathscr{C}_\gamma$ in $L^2(\gamma, \mathrm{d}s)$ thus amounts to

$$\|\operatorname{sgn}\big((1/a)i\partial_x\big)u\|_{L^2(\mathbb{R},\mu)} \lesssim \|u\|_{L^2(\mathbb{R},\mu)}.$$

By functional calculus, this is equivalent to

$$\|\sqrt{-(1/a)\partial_x (1/a) \partial_x} u\|_{L^2(\mathbb{R},\mu)} \lesssim \|(1/a)\partial_x u\|_{L^2(\mathbb{R},\mu)} = \|\partial_x u\|_{L^2(1/\mu)}.$$

The latter estimate would follow from Corollary 1.7 with $b = 1/a$, $w = 1/\mu$, if the hypotheses were satisfied, since in this case $\sqrt{\mu/w} = \mu = \sqrt{1 + (\varphi')^2}$ and $\nu(y) = \sqrt{\mu w} = 1$. However Corollary 1.7 does not apply here, since the accretivity condition $\mathfrak{Re}\, a(x) = 1 \gtrsim \mu(x)$ is not satisfied, unless φ' is bounded.

We end this section by noting that the matrix-weighted Kato square root estimate (0.3) which we consider in this paper, despite looking like a two-weight estimate, should be seen as a one-weight estimate, as the proof of Theorem 1.2 clearly shows. In the following example we see that our results apply only when the weights in the square root operator correctly match the weights in the norms.

Example 1.9 (Two-weight Hilbert transform). Consider the two-weight estimate

$$\|Hu\|_{L^2(\mu)} \lesssim \|u\|_{L^2(w)} \tag{1.8}$$

for the Hilbert transform

$$Hu(x) := \frac{i}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{u(y)}{y-x}\,\mathrm{d}y.$$

The problem of characterising for which weights μ, w the estimate (1.8) holds was solved in [Lac+14]. If we use functional calculus to write H as $\sqrt{-\partial_x^2}(i\partial_x)^{-1}$, then (1.8) amounts to

$$\|\sqrt{-\partial_x^2}u\|_{L^2(\mu)} \lesssim \|\partial_x u\|_{L^2(w)}. \tag{1.9}$$

Changing variables $y = \rho(x)$ and $u(x) = v(\rho(x))$ as in Lemma 1.1, and using the chain rule: $\partial_x = \rho'\partial_y$, the two-weight estimate (1.8) becomes

$$\int \left|\sqrt{-(\rho'\partial_y)^2}v(\rho(x))\right|^2 \mu(x)\,\mathrm{d}x \lesssim \int |(\rho'(x)\partial_y v)(\rho(x))|^2 w(x)\,\mathrm{d}x.$$

Choosing $\rho'(x) = \sqrt{\mu(x)/w(x)}$ gives $(\rho')^2 w = \mu$ in the right hand side. Changing variables and using $\nu \circ \rho = \sqrt{w\mu}$ yields

$$\mu(x)\,\mathrm{d}x = \sqrt{\mu(x)w(x)} \cdot \sqrt{\mu(x)/w(x)}\,\mathrm{d}x = (\nu\circ\rho)(x)\cdot\rho'(x)\,\mathrm{d}x = \nu(y)\,\mathrm{d}y.$$

Thus estimate (1.8) holds if and only if the one-weight estimate

$$\left\|\sqrt{-(\lambda\partial_y)^2}v\right\|_{L^2(\nu)} \lesssim \|\partial_y v\|_{L^2(\nu)} \tag{1.10}$$

holds with the weight $\lambda(y) := \rho'(\rho^{-1}(y)) = \sqrt{\mu(\rho^{-1}(y))/w(\rho^{-1}(y))}$ in the Kato square root operator. Corollary 1.7 does not apply directly to (1.10), nor to (1.9), since it requires that the weights in the Kato square root operator correctly match the weights in the norms.

2. The (μ, W) manifold M

We now seek to generalise the results in §1 to higher dimension $d \geq 2$, starting with Lemma 1.1. To cover general matrix weights W, we need to allow for more general diffeomorphisms $\rho\colon \mathbb{R}^d \to M$, where now M is some auxiliary smooth d-dimensional Riemannian manifold. The metric g for M will be determined by μ and W, but not the differential structure on M. In general, smooth weights (μ, W) will define a metric g for a manifold with non-zero curvature. For this reason we need to allow for curved manifolds. The natural pullback generalising (1.2) for the differential operator D in (0.4) is now

$$\mathsf{P}\colon \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix} \mapsto \begin{bmatrix} v_1(\rho(x)) \\ (\mathrm{d}\rho_x)^\star v_2(\rho(x)) \end{bmatrix} =: \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}. \tag{2.1}$$

Here $v_1\colon M \to \mathbb{C}$ is a scalar function on M and v_2 is a section of the cotangent bundle $T^\star M$, which we identify with TM using the metric g. This is important because, although we can view v_2 as a vector on M, it is actually a 1-form, so its pullback is obtained by multiplying $v_2 \circ \rho$ by the transpose $(\mathrm{d}\rho)^\star$ of the Jacobian matrix $\mathrm{d}\rho$. Below J_ρ denotes the determinant of the Jacobian matrix $\det(\mathrm{d}\rho) := \det(g)^{1/2}$, where $g = (\mathrm{d}\rho)^\star\mathrm{d}\rho$ is the Riemannian metric on M pulled back to $\mathbb{R}^d$.

Here and below, to ease notation, we shall identify maps defined on $\mathbb{R}^d$ and on M through ρ, writing for example $\nabla_M v_1$ for $(\nabla_M v_1)\circ\rho$. We use $v(y)$ for functions defined on M and $u(x)$ for functions defined on $\mathbb{R}^d$. With a slight abuse of notation, we use the abbreviations $J_\rho(y), \mathrm{d}\rho_y$ and $u(y)$ for $J_\rho(\rho^{-1}(y)),\ \mathrm{d}\rho_{\rho^{-1}(y)},\ u(\rho^{-1}(y))$. The differential operators ∇ and div are always defined on $\mathbb{R}^d$.

In order to write the operator D_M similar to D we need the chain rule:

$$\nabla u_1 = (\mathrm{d}\rho)^\star \nabla_M v_1,$$

which holds in the weak sense by Theorem A.2. We also require the L^2-adjoint result for vector fields $u_2\colon \mathbb{R}^d \to \mathbb{C}^d$ in Theorem A.3. We compute

$$\begin{aligned}
\mathsf{P}^{-1} D \mathsf{P} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \mathsf{P}^{-1} D \begin{bmatrix} v_1 \circ \rho \\ (\mathrm{d}\rho_x^\star v_2)\circ\rho \end{bmatrix} = \mathsf{P}^{-1} \begin{bmatrix} -(1/\mu)\mathrm{div} W[(\mathrm{d}\rho_x^\star v_2)\circ\rho] \\ \nabla(v_1\circ\rho) \end{bmatrix} \\
&= \mathsf{P}^{-1} \begin{bmatrix} -(1/\mu)\mathrm{div}_M \Big\{ J_\rho^{-1}\, \mathrm{d}\rho_x \big(W (\mathrm{d}\rho_x)^\star v_2 \big) \Big\} J_\rho \\ \nabla u_1 \end{bmatrix} \\
&= \begin{bmatrix} -(1/\mu) J_\rho \mathrm{div}_M \Big\{ J_\rho^{-1}\, \mathrm{d}\rho_y \big(W (\mathrm{d}\rho_y)^\star v_2 \big) \Big\} \\ \nabla_M v_1 \end{bmatrix}.
\end{aligned} \tag{2.2}$$

We obtain the following generalisation of Lemma 1.1.

Lemma 2.1. *Assume that μ is a scalar weight on $\mathbb{R}^d$ and that W is a matrix weight on $\mathbb{R}^d$. Assume that μ and W are smooth around $\rho(x_0) \in \mathbb{R}^d$. Set*

$$g := \mu W^{-1} \quad \text{and} \quad \nu := \mu/\sqrt{\det g}.$$

Let M be a Riemannian manifold with chart (U, ρ) around x_0 and metric g in this chart. Let

$$D_M := \begin{bmatrix} 0 & -(1/\nu)\mathrm{div}_M \nu \\ \nabla_M & 0 \end{bmatrix}. \tag{2.3}$$

Then $\mathsf{P}\colon L^2(U,\nu) \oplus L^2(TU, \nu I) \to L^2(\rho^{-1}(U), \mu) \oplus L^2(\rho^{-1}(U); \mathbb{C}^d, W)$ defined in (2.1), is an isometry, and $\mathsf{P}^{-1} D \mathsf{P} = D_M$.

Remark 2.2. There is one-to-one correspondence between the pairs of weights (μ, W) and the pairs (g, ν) of Riemannian metric and weight, since inversely $\mu = \nu\sqrt{\det g}$, and $W = (\nu\sqrt{\det g})g^{-1}$.

Proof of Lemma 2.1. To obtain the operator D_M with a single scalar weight ν on a manifold, in (2.2) we require that

$$(1/\mu) J_\rho = 1/\nu \quad \text{and} \quad J_\rho^{-1}\, \mathrm{d}\rho\, W\, \mathrm{d}\rho^\star = \nu I,$$

where I is the identity matrix. The first condition yields $\mu = J_\rho \nu$. Since the volume change is $J_\rho = \sqrt{\det g}$, we have $\nu = \mu/\sqrt{\det g}$ as stated. For the second one, since the metric in a chart ρ is $g = \mathrm{d}\rho^\star \mathrm{d}\rho$, and the matrices $\mathrm{d}\rho$ and $\mathrm{d}\rho^\star$ commute with the scalars ν and J_ρ, we have

$$\frac{W}{J_\rho \nu} = \mathrm{d}\rho^{-1}(\,\mathrm{d}\rho^\star)^{-1} = \big(\mathrm{d}\rho^\star \mathrm{d}\rho\big)^{-1} = g^{-1},$$

and so $g = \mu W^{-1}$. To see that the map P in (2.1) is an isometry, it is enough to compute

$$\int_{\mathbb{R}^d} |u_1(x)|^2 \mu(x)\, \mathrm{d}x = \int_M |v_1(y)|^2 \underbrace{\frac{\mu}{\sqrt{\det g}}(y)}_{=\nu(y)}\, \mathrm{d}y \tag{2.4}$$

where $\mathrm{d}y$ is the Riemannian measure on M. Also

$$\begin{aligned}
\int_{\mathbb{R}^d} \langle W(x)u_2(x), u_2(x)\rangle \,\mathrm{d}x &= \int_{\mathbb{R}^d} \langle W(x)(\mathrm{d}\rho_x)^\star v_2(\rho(x)), (\mathrm{d}\rho_x)^\star v_2(\rho(x))\rangle \,\mathrm{d}x \\
&= \int_M \langle W \mathrm{d}\rho^\star v_2(y), \mathrm{d}\rho^\star v_2(y)\rangle \frac{\mathrm{d}y}{\sqrt{\det g}} \\
&= \int_M \langle \frac{1}{\sqrt{\det g}} \mathrm{d}\rho W \mathrm{d}\rho^\star v_2(y), v_2(y)\rangle \,\mathrm{d}y \\
&= \int_M |v_2(y)|^2 \nu(y)\,\mathrm{d}y.
\end{aligned} \tag{2.5}$$

This concludes the proof. □

We aim to prove a matrix weighted Kato square root estimate on $\Omega \subseteq \mathbb{R}^d$, by applying [AMR22, Theorem 1.1] to the one-scalar-weight operator D_M on M in (2.3) and pulling back the result to $\mathbb{R}^d$. However, this requires a modification of Lemma 2.1 since [AMR22, Theorem 1.1 and Theorem 1.2] only apply to prove inhomogeneous Kato square root estimates, since only local square function estimates can be proved on M without further hypothesis on its geometry at infinity. As in [AMR22, Eq. (2.4)] we introduce inhomogeneous first order differential operators

$$\widetilde{D} = \begin{bmatrix} 0 & I & -(1/\mu)\mathrm{div}W \\ I & 0 & 0 \\ \nabla & 0 & 0 \end{bmatrix} \quad \text{acting on } \widetilde{\mathscr{H}} := \begin{bmatrix} L^2(\Omega,\mu) \\ L^2(\Omega,\mu) \\ L^2(\Omega;\mathbb{C}^d, W) \end{bmatrix}, \tag{2.6}$$

$$\widetilde{D}_M = \begin{bmatrix} 0 & I & -(1/\nu)\mathrm{div}_M\nu \\ I & 0 & 0 \\ \nabla_M & 0 & 0 \end{bmatrix} \quad \text{acting on } \widetilde{\mathscr{H}}_M := \begin{bmatrix} L^2(M,\nu) \\ L^2(M,\nu) \\ L^2(TM,\nu) \end{bmatrix}, \tag{2.7}$$

where divergence and ∇ in (2.6) are on $\mathbb{R}^d$. The domains of the operators ∇ and ∇_M are weighted Sobolev spaces

$$\begin{aligned}
\mathcal{H}^1_{\mu,W}(\Omega) &:= \big\{f \in W^{1,1}_{\mathrm{loc}}(\Omega), f \in L^2_{\mathrm{loc}}(\Omega,\mu) \text{ with } \nabla f \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^d, W)\big\}, \\
\mathcal{H}^1_{\nu}(M) &:= \big\{f \in W^{1,1}_{\mathrm{loc}}(M), f \in L^2_{\mathrm{loc}}(M,\nu) \text{ with } \nabla_M f \in L^2_{\mathrm{loc}}(TM,\nu I)\big\}
\end{aligned}$$

respectively, so $\mathsf{dom}(\nabla) = \mathcal{H}^1_{\mu,W}(\Omega)$ and $\mathsf{dom}(\nabla_M) = \mathcal{H}^1_\nu(M)$. The closed operator $-\mathrm{div}$ with domain

$$\mathsf{dom}(\mathrm{div}) = \big\{h \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^d, W^{-1}), \mathrm{div}h \in L^2_{\mathrm{loc}}(\Omega,\mu^{-1})\big\}$$

is the adjoint of ∇ with respect the unweighted L^2 pairing. In the same way, $-\mathrm{div}_M$ is the closed operator with domain

$$\mathsf{dom}(\mathrm{div}_M) = \big\{h \in L^2_{\mathrm{loc}}(TM,\nu^{-1}), \mathrm{div}_M h \in L^2_{\mathrm{loc}}(M,\nu^{-1})\big\}$$

and it is the adjoint of ∇_M with respect the unweighted L^2 pairing on M. Consider the pullback $\widetilde{\mathsf{P}} \colon \widetilde{\mathscr{H}}_M \to \widetilde{\mathscr{H}}$ given by

$$\widetilde{\mathsf{P}} \colon \begin{bmatrix} v_1(y) \\ v_0(y) \\ v_2(y) \end{bmatrix} \mapsto \begin{bmatrix} v_1 \circ \rho \\ v_0 \circ \rho \\ \mathrm{d}\rho^\star v_2 \circ \rho \end{bmatrix} =: \begin{bmatrix} u_1(x) \\ u_0(x) \\ u_2(x) \end{bmatrix}.$$

The map $\widetilde{\mathsf{P}}$ preserves the domains of the operators $\widetilde{D}$ and $\widetilde{D}_M$.

Lemma 2.3. *The map $\widetilde{\mathsf{P}}$ is an isometry and $\widetilde{\mathsf{P}}(\mathsf{dom}(\widetilde{D}_M)) = \mathsf{dom}(\widetilde{D})$.*

Proof. For scalar-valued functions, apply Theorem A.2 with $v = \nu$ and $V = \nu I$. Note that since $\nu \circ \rho = \mu / J_\rho$, then $v_\rho = \mu$. Also, since the metric $\mathrm{d}\rho^{-1}(\mathrm{d}\rho^{-1})^\star = g^{-1} = \mu^{-1} W$, it follows that $V_\rho = W$. For vector fields, if $\vec{u} \in L^2(\Omega; \mathbb{R}^d, W^{-1})$ with $\mathrm{div}\vec{u}$ in $L^2(\Omega, \mu^{-1})$, apply Theorem A.3 with $V = W^{-1}$ and $v = \mu^{-1}$. Indeed, $V^\rho = \nu^{-1} I$ and $v^\rho = \nu^{-1}$, so $J_\rho^{-1} \rho_* \vec{u} = J_\rho^{-1} \mathrm{d}\rho \vec{u} \circ \rho^{-1} \in L^2(TM, \nu^{-1} I)$ and

$$\mathrm{div}\left(\frac{\rho_*}{J_\rho}\vec{u}\right) = \frac{\rho_*}{J_\rho}(\mathrm{div}\vec{u}) \in L^2(M, \nu^{-1}).$$

□

As in the proof of Lemma 2.1, one sees that $\widetilde{\mathsf{P}}$ is an isometry. A calculation as in (2.2) shows that

$$\widetilde{\mathsf{P}}^{-1} \widetilde{D} \widetilde{\mathsf{P}} = \widetilde{D}_M.$$

We have the following generalisation of Theorem 1.2.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $\rho \colon \Omega \to M$ be a $W^{1,1}_{\mathrm{loc}}$ homeomorphism onto a complete, smooth Riemannian manifold M. Let μ, W be scalar and matrix weights in $A_2^{\mathrm{loc}}(\Omega)$. Assume that the metric on M pulled back via ρ is*

$$g = \mu W^{-1}$$

and define the scalar weight $\nu = \mu / \sqrt{\det g}$ on M. Let $\widetilde{D}$ be the differential operator in (2.6) and let $\widetilde{B}$ be a $(\mu \oplus \mu \oplus W)$-bounded, $(\mu \oplus \mu \oplus W)$-accretive multiplication operator on $\widetilde{\mathscr{H}}$ as in Definition 0.2. If the manifold M has Ricci curvature and injectivity radius bounded from below, and if $\nu \in A_2^R(M)$, for some $R > 0$, then $\widetilde{B}\widetilde{D}$ and $\widetilde{D}\widetilde{B}$ are bisectorial operators that satisfy quadratic estimates and have bounded H^∞ functional calculus in $\widetilde{\mathscr{H}}$.

Remark 2.5. The Riemannian manifold M is assumed to be smooth with smooth metric. But since the map ρ is not smooth in general, the pullback g of the smooth metric of M on Ω may be non-smooth. See Figure 3.

$$\begin{tikzcd} & M \\ \Omega \subseteq \mathbb{R}^d \arrow[ur, "\rho"] \arrow[r, "\varphi^{-1}\circ\rho"] & \mathbb{R}^d \arrow[u, "\varphi"'] \end{tikzcd}$$

Figure 3. The Riemannian manifold M with a chart φ from its smooth atlas. A function f on M is smooth if $f \circ \varphi$ is smooth. But $f \circ \rho$ is not in general smooth since the map $\varphi^{-1} \circ \rho$ is only in $W^{1,1}$.

Proof of Theorem 2.4. Given the differential operator $\widetilde{D}$ as in (2.6), consider the operators $\widetilde{D}_M := \widetilde{\mathsf{P}}^{-1} \widetilde{D} \widetilde{\mathsf{P}}$ given in (2.7) and the operator $\widetilde{B}_M := \widetilde{\mathsf{P}}^{-1} \widetilde{B} \widetilde{\mathsf{P}}$.

Lemma 2.1 shows that the extended pullback transformation $\widetilde{\mathsf{P}}$ is an isometry between the weighted spaces $\widetilde{\mathscr{H}}_M$ and $\widetilde{\mathscr{H}}$. Indeed, let $u = \widetilde{\mathsf{P}} v$, then

$$\langle \widetilde{\mathsf{P}}^{-1}(\widetilde{B}u), \widetilde{\mathsf{P}}^{-1}(\widetilde{B}u)\rangle_{\widetilde{\mathscr{H}}_M} = \langle \widetilde{\mathsf{P}}\widetilde{\mathsf{P}}^{-1}(\widetilde{B}u), \widetilde{B}u\rangle_{\widetilde{\mathscr{H}}}$$

from which follows that $\widetilde{B}_M$ is $(\nu \oplus \nu \oplus \nu I)$-bounded, and $(\nu \oplus \nu \oplus \nu I)$-accretive if and only if $\widetilde{B}$ is $(\mu \oplus \mu \oplus W)$-bounded, $(\mu \oplus \mu \oplus W)$-accretive.

[AMR22, Lemma 2.3] implies that $\widetilde{D}_M$ is self-adjoint, and so is the operator $\widetilde{D} = \widetilde{\mathsf{P}}\widetilde{D}_M\widetilde{\mathsf{P}}^{-1}$, since $\widetilde{\mathsf{P}}$ is unitary. By [AMR22, Theorem 1.1], the operator $\widetilde{B}_M\widetilde{D}_M$ has bounded H^∞ functional calculus in $L^2(M; \mathbb{C}^d \oplus TM, \nu I)$. The same holds for the operator $\widetilde{B}\widetilde{D}$ via the isometry $\widetilde{\mathsf{P}}$, and for $\widetilde{D}\widetilde{B} = \widetilde{B}^{-1}(\widetilde{B}\widetilde{D})\widetilde{B}$. $\square$

Analogous to Corollary 1.7, we derive from Theorem 2.4 the following Kato square root estimate.

Corollary 2.6. *Assume that $\rho\colon \Omega \to M$, μ, W, g, ν satisfy the hypotheses of Theorem 2.4. Consider the operator*

$$Lu := -\frac{1}{\mu}\mathrm{div} A\nabla u - \frac{1}{\mu}\mathrm{div}(\vec{b}u) + \frac{1}{\mu}\langle \vec{c}, \nabla u\rangle + d \cdot u,$$

where the coefficient matrix

$$\begin{bmatrix} d & \mu^{-1/2}\vec{c}W^{-1/2} \\ W^{-1/2}\vec{b}\mu^{-1/2} & W^{-1/2}AW^{-1/2} \end{bmatrix}$$

is bounded and accretive. Then the following Kato square root estimate

$$\|\sqrt{aL}u\|_{L^2(\Omega,\mu)} \eqsim \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d,W)} + \|u\|_{L^2(\Omega,\mu)}$$

holds for any complex-valued function $a \in L^\infty(\Omega)$ such that $\inf_\Omega \mathfrak{Re}(a) \gtrsim 1$.

Proof. Apply Theorem 2.4 to $\widetilde{D}$ defined in (2.6) and coefficients

$$\widetilde{B} = \begin{bmatrix} a & 0 & 0 \\ 0 & d & \mu^{-1}\vec{c} \\ 0 & W^{-1}\vec{b} & W^{-1}A \end{bmatrix}.$$

By the hypothesis on the coefficient, and the property of a, the matrix $\widetilde{B}$ is $(\mu \oplus \mu \oplus W)$-bounded and accretive. By Theorem 2.4 the operator $\widetilde{B}\widetilde{D}$ has bounded H^∞ functional calculus on $\widetilde{\mathscr{H}} = L^2(\Omega, \mu)^2 \oplus L^2(\Omega; \mathbb{C}^d, W)$. This implies the boundedness and invertibility of the operator $\mathrm{sgn}(\widetilde{B}\widetilde{D})$, and so by writing $\sqrt{(\widetilde{B}\widetilde{D})^2} = \mathrm{sgn}(\widetilde{B}\widetilde{D})\widetilde{B}\widetilde{D}$ we have

$$\left\|\sqrt{(\widetilde{B}\widetilde{D})^2}\left[\begin{smallmatrix} u \\ 0 \\ 0 \end{smallmatrix}\right]\right\|_{\widetilde{\mathscr{H}}} \eqsim \left\|\widetilde{B}\widetilde{D}\left[\begin{smallmatrix} u \\ 0 \\ 0 \end{smallmatrix}\right]\right\|_{\widetilde{\mathscr{H}}} \eqsim \left\|\widetilde{D}\left[\begin{smallmatrix} u \\ 0 \\ 0 \end{smallmatrix}\right]\right\|_{\widetilde{\mathscr{H}}} \eqsim \|\nabla u\|_{L^2(\Omega,W)} + \|u\|_{L^2(\Omega,\mu)}.$$

This concludes the proof, since $\sqrt{(\widetilde{B}\widetilde{D})^2}$ applied to $[u\ 0\ 0]^\intercal$ equals $[\sqrt{aL}u\ 0\ 0]^\intercal$. $\square$

We end this section with some examples of matrix weights, and discuss when the hypotheses on the manifold M associated with μ, W are met. To obtain examples of μ, W, we consider manifolds M embedded in $\mathbb{R}^N$ obtained as graphs of functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}^m$, with $N = d + m$. In Theorem 2.4 we thus have

$$\begin{aligned} \rho\colon \mathbb{R}^d &\to M \\ x &\mapsto (x, \varphi(x)) = (x, y) \end{aligned}$$

with Jacobian matrix $\mathrm{d}\rho_x = (I, \mathrm{d}\varphi_x)^\intercal$. By reverse engineering, we get from φ an example of a Riemannian metric on $\mathbb{R}^d$

$$g = \mathrm{d}\rho_x^\star \mathrm{d}\rho_x = I + \mathrm{d}\varphi_x^\star \mathrm{d}\varphi_x.$$

For any choice of scalar weight μ, this yields an example of a matrix weight $W = \mu g^{-1}$.

Example 2.7. Consider the graph of

$$\varphi(x_1, x_2) = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}\right) = (y_1, y_2), \tag{2.8}$$

for $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Here $\rho(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2))$ and $M \subseteq \mathbb{R}^4$ is complete and asymptotically isometric to $\mathbb{R}^2$ both when $|x|^2 = x_1^2 + x_2^2 \to +\infty$ and when $|x|^2 \to 0$. Therefore Ricci curvature and injectivity radius is bounded from below by a compactness argument. In this case

$$g_\varphi = I + \mathrm{d}\varphi_x^\star \mathrm{d}\varphi_x = \left(1 + \frac{1}{|x|^4}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a conformal metric. Therefore this only gives scalar weighted examples of W to which Theorem 2.4 applies. To see a more general matrix weight W appear, we can tweak (2.8) by composing φ with a non-conformal diffeomorphism. Consider

$$\phi(x_1, x_2) = \left(h\left(\frac{x_1}{x_1^2 + x_2^2}\right), \frac{x_2}{x_1^2 + x_2^2}\right)$$

where $h(t) = t\sqrt{1 + t^2}$, for $t \in \mathbb{R}$. Again M is asymptotically isometric to $\mathbb{R}^2$ both as $|x|^2 \to \infty$ and when $|x|^2 \to 0$, so the geometric hypotheses on M are satisfied. To see that the metric g_ϕ obtained from ϕ, and hence the matrix W, is not equivalent to a scalar weight, we verify that the singular values of $\mathrm{d}\phi_x$ do not have bounded quotient. We calculate

$$\partial_{x_1}\phi(t, 0) = \Big(h'(1/t) \cdot (-1/t^2), 0\Big)$$
$$\partial_{x_2}\phi(t, 0) = \Big(0, 1/t^2\Big)$$

so the ratio $|\partial_{x_1}\phi|/|\partial_{x_2}\phi|(t, 0) = |h'(1/t)| \eqsim 1/t \to +\infty$ as $t \to 0^+$.

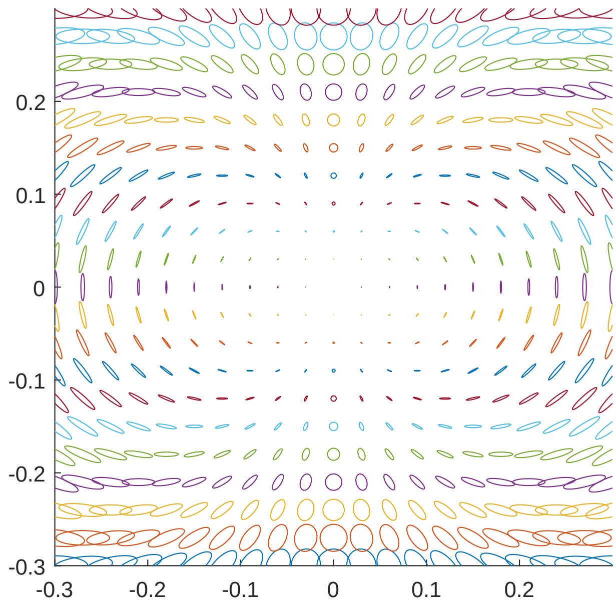


Figure 4. Geodesic discs in the metric g_ϕ of Example 2.7 are ellipses shrinking anisotropically towards the origin.

Example 2.8. Let M be the graph of the scalar function $\varphi(x,y) = (x^2+y^2)^{-a}$, for $a > 0$. One checks that Gaussian/Ricci curvature $\mathrm{Ric}(M) \sim -(x^2+y^2)^{2a}$ when $x^2+y^2 \to 0^+$, so the Ricci curvature is bounded below, but the injectivity radius is not bounded away from zero. Indeed, as discussed in [AMR22, §2.1], the geometric hypothesis in [AMR22, Theorem 1.1] implies in particular that geodesic balls of radius 1 are Lipschitz diffeomorphic to Euclidean balls. But this is not true in this example, so [AMR22] does not apply to this manifold.

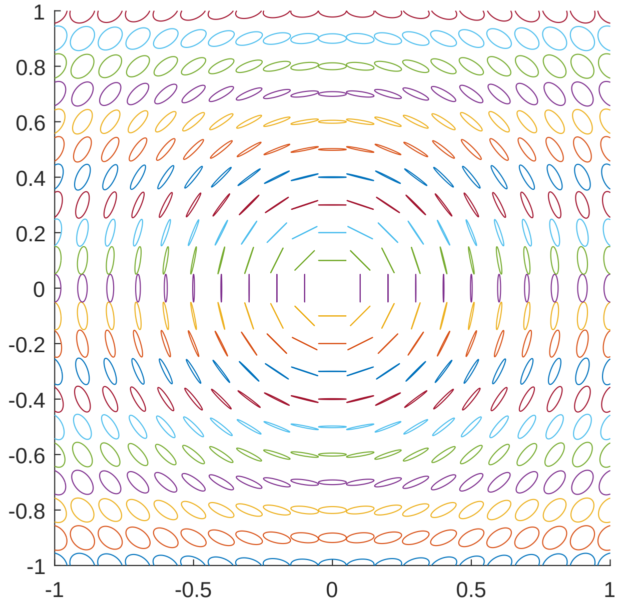


Figure 5. Geodesic discs in the metric of Example 2.8 for $a = 1$.

3. Matrix degenerate Boundary Value Problems

We show in this final section how the methods in this paper yield solvability estimates of elliptic Boundary Value Problems (BVPs) for matrix-degenerate divergence form equations

$$\mathrm{div} A\nabla u = 0 \tag{3.1}$$

on a compact manifold Ω with Lipschitz boundary $\partial\Omega$. We assume that there exists a matrix weight V that describe the degeneracy of the coefficients A, in the following way.

Lemma 3.1. *Let V be a matrix weight and let A be a multiplication operator. The following are equivalent:*

- $V^{-1/2}AV^{-1/2}$ *is uniformly bounded and accretive;*
- $V^{-1}A$ *is V-bounded and V-accretive;*
- *for all vectors $v, w \in \mathbb{C}^{d+1}$ we have*

$$\mathfrak{Re}\langle Av, v\rangle \gtrsim \langle Vv, v\rangle \quad \text{and} \quad |\langle Av, w\rangle| \lesssim \langle Vv, w\rangle. \tag{3.2}$$

A weak solution u to (3.1) is a function such that $\nabla u \in L^2_{\mathrm{loc}}(T\Omega, V)$, where $T\Omega$ is the tangent bundle on Ω. Since the weighted space $L^2_{\mathrm{loc}}(T\Omega, V) \hookrightarrow L^1_{\mathrm{loc}}(T\Omega)$, then $A\nabla u \in L^1_{\mathrm{loc}}(T\Omega)$ and $\nabla u \in L^1_{\mathrm{loc}}(T\Omega)$, so $u \in W^{1,1}_{\mathrm{loc}}(T\Omega)$ by Poincaré inequality.

Further we assume given a closed Riemannian manifold M_0 and, for $\delta > 0$, a bi-Lipschitz map

$$\begin{aligned} \rho_0\colon [0,\delta) \times M_0 &\to U \subseteq \Omega\,, \\ (t,x) &\mapsto \rho_0(t,x) \end{aligned}$$

between a finite part of the cylinder $\mathbb{R} \times M_0$ and a neighbourhood U of the boundary $\partial\Omega$, so that $\rho_0(\{0\} \times M_0) = \partial\Omega$. See Figure 6.

To analyse a weak solution u of (3.1) near $\partial\Omega$, we define the pullback $u_0 := u \circ \rho_0$ on the cylinder $\mathcal{C}_0 := [0,\delta) \times M_0$. Then u_0 satisfies

$$\mathrm{div}_{\mathcal{C}_0} A_0 \nabla_{\mathcal{C}_0} u_0 = 0, \tag{3.3}$$

with coefficients $A_0 := J_{\rho_0}(\rho_0)_*^{-1} A (\rho_0^*)^{-1}$ where $(\rho_0)_*$ denotes the pushforward via ρ_0, so $J_{\rho_0}^{-1}(\rho_0)_*(v) := J_{\rho_0}^{-1} \mathrm{d}\rho_0(v \circ \rho_0^{-1})$ is the Piola transformation, and $\rho_0^* v = (\mathrm{d}\rho_0)^\star v \circ \rho_0$ denotes the pullback via ρ_0. See [Ros19, §7.2 and Example 7.2.12] for more details on this transformation. The differential operators in (3.3) are

$$\begin{aligned} \nabla_{\mathcal{C}_0} u_0 &:= [\partial_t u_0, \nabla_{M_0} u_0]^\intercal, \\ \mathrm{div}_{\mathcal{C}_0} \vec{v}_0 &:= \partial_t (e_0 \cdot \vec{v}_0) + \mathrm{div}_{M_0} (\vec{v}_0)_\parallel, \end{aligned} \tag{3.4}$$

where e_0 denotes the vertical unit vector along the cylinder, and $(\vec{v}_0)_\parallel$ is the tangetial part of $\vec{v}_0$. Define the pulled back matrix weight $V_0 := J_{\rho_0}(\rho_0)_*^{-1} V (\rho_0^*)^{-1}$.

Lemma 3.2. *The matrix $V^{-1/2} A V^{-1/2}$ is uniformly bounded and accretive on a neighbourhood U of the boundary $\partial\Omega$ if and only if $V_0^{-1/2} A_0 V_0^{-1/2}$ is uniformly bounded and accretive on $[0,\delta) \times M_0$.*

Indeed, the condition (3.2) for A and V is seen to be equivalent to (3.2) for A_0 and V_0. To obtain solvability estimates, we require that the matrix weight V_0 has the structure

$$V_0(t,x) = \begin{bmatrix} \mu(x) & 0 \\ 0 & W(x) \end{bmatrix},$$

meaning that V_0 is constant along the cylinder $\mathcal{C}_0$ and that the vertical direction is a principal direction of V_0. The functions μ and W are assumed to be scalar and matrix weights on M_0, respectively. Using a transformation of coefficients $A \mapsto B$ from [AAM10], the divergence form equation (3.3) can be turned into an evolution equation

$$(\partial_t + DB) f_0 = 0 \tag{3.5}$$

for the conormal gradient $f_0 := [(1/\mu)\partial_{\nu_{A_0}} u_0, \nabla_{M_0} u_0]^\intercal$ of u_0 on the cylinder $[0,\delta) \times M_0$. Here $\partial_{\nu_{A_0}} u_0 := e_0 \cdot A_0 \nabla_{\mathcal{C}_0} u_0$ is the conormal derivative. We make this correspondence precise in the following lemma.

Lemma 3.3. *A function u_0 is a weak solution to the divergence form equation*

$$\mathrm{div}_{\mathcal{C}_0} A_0 \nabla_{\mathcal{C}_0} u_0 = 0\,, \qquad \textit{with } A_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

if and only if its conormal gradient f_0 solves the Cauchy–Riemann system (3.5) with

$$D = \begin{bmatrix} 0 & -(1/\mu)\mathrm{div}_{M_0} W \\ \nabla_{M_0} & 0 \end{bmatrix} \quad \textit{and } B = \begin{bmatrix} \mu a^{-1} & -a^{-1} b \\ W^{-1} c a^{-1} \mu & W^{-1}(d - c a^{-1} b) \end{bmatrix}.$$

The operator D is self-adjoint on $L^2(M_0,\mu) \oplus L^2(TM_0, W)$ and B is $(\mu \oplus W)$-bounded and $(\mu \oplus W)$-accretive.

Proof. Consider the transformation of the coefficient $A_0 \mapsto \mathcal{J}(A_0)$ given by

$$\mathcal{J}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}.$$

This map is an involution and preserves accretivity and boundedness [AAM10, Proposition 3.2]. Following [AAM10; AMR22] the divergence form equation (3.3) is equivalent to

$$\left(\partial_t + \begin{bmatrix} 0 & -\mathrm{div}_{M_0} \\ \nabla_{M_0} & 0 \end{bmatrix} \mathcal{J}(A_0)\right) \begin{bmatrix} \partial_{\nu_{A_0}} u_0 \\ \nabla_{M_0} u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.6}$$

Then a computation shows that

$$\mathcal{J}\left(\begin{bmatrix} v_1 & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_2 & 0 \\ 0 & W_2 \end{bmatrix}\right) = \begin{bmatrix} v_2^{-1} & 0 \\ 0 & W_1 \end{bmatrix} \mathcal{J}(A_0) \begin{bmatrix} v_1^{-1} & 0 \\ 0 & W_2 \end{bmatrix}. \tag{3.7}$$

We introduce weights into the system (3.6) as following:

$$\begin{aligned} &\begin{bmatrix} 1/\mu & 0 \\ 0 & I \end{bmatrix} \left(\partial_t + \begin{bmatrix} 0 & -\mathrm{div}_{M_0} \\ \nabla_{M_0} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix} \mathcal{J}(A_0) \begin{bmatrix} \mu & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 1/\mu & 0 \\ 0 & I \end{bmatrix}\right) \\ &\qquad = \left(\partial_t + D \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix} \mathcal{J}(A_0) \begin{bmatrix} \mu & 0 \\ 0 & I \end{bmatrix}\right) \begin{bmatrix} 1/\mu & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

where we used that multiplication by $(1/\mu)$ and ∂_t commute since μ is independent of t. Using (3.7) we define

$$B := \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix} \mathcal{J}(A_0) \begin{bmatrix} \mu & 0 \\ 0 & I \end{bmatrix} = \mathcal{J}\left(\begin{bmatrix} \mu^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} A_0 \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}\right).$$

The argument of $\mathcal{J}$ on the right hand side is $(\mu \oplus W)$-bounded and $(\mu \oplus W)$-accretive. Since $\mathcal{J}$ preserves accretivity and boundedness, B is uniformly bounded and accretive. The reader can check that B coincides with the expression in the statement of the lemma. $\square$

We note that DB, with D and B from Lemma 3.3, has the same structure as the operators considered in §2, if we replace $\mathbb{R}^d$ by a compact manifold M_0. As in §2, we use a metric on M_0 adapted to the weights μ, W: we assume the existence of a smooth, closed Riemannian manifold (M_1, g_1) and a $W^{1,1}_{\mathrm{loc}}$-homeomorphism $\rho\colon M_0 \to M_1$ such that the pullback of the metric g_1 on M_1 via ρ is

$$g_0 := \rho^* g_1 = \mu W^{-1},$$

and we defined the scalar weight $\nu := \rho_* \mu / \sqrt{\det g_1}$ on M_1, where $\rho_* \mu = \mu \circ \rho^{-1}$ denotes the pushforward via ρ. We extend the map ρ to a map between the corresponding cylinders by setting

$$\begin{aligned} \rho_1\colon [0,\delta) \times M_0 &\to [0,\delta) \times M_1 \\ (t,x) &\mapsto (t, \rho(x)). \end{aligned}$$

The extension of the Riemannian metric on the cylinder and its pullback via ρ_1 are

$$\widetilde{g}_1 := \begin{bmatrix} 1 & 0 \\ 0 & g_1 \end{bmatrix}, \quad \widetilde{g}_0 = \rho_1^* \widetilde{g}_1 := \begin{bmatrix} 1 & 0 \\ 0 & \mu W^{-1} \end{bmatrix}. \tag{3.8}$$

In the following, the variable x is in M_0, while $y = \rho(x) \in M_1$. We denote by $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ the Riemannian measures on M_0, M_1 and on Ω, respectively. See Figure 6.

We also denote by dist_0 and dist_1 the distance functions on M_0 and M_1 induced by g_0 and g_1.

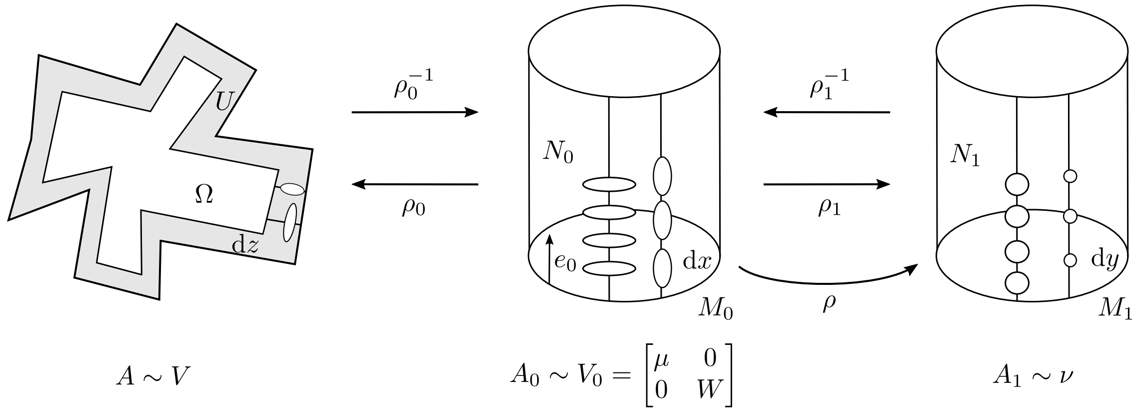


Figure 6. The neighborhood U of $\partial\Omega$ in Ω is transformed by the bi-Lipschitz map ρ_0^{-1} into the cylinder $[0, \delta) \times M_0$ with anisotropic degenerate coefficients A_0. The coefficients A_1 on the cylinder $[0, \delta) \times M_1$ are isotropically degenerate.

Note that A_1 is isotropically degenerate, meaning that $V_1 = \nu I$ is a scalar weight in each component. Weak solutions to the anisotropically degenerate equation (3.3) correspond to weak solutions to an isotropically degenerate equation on $[0, \delta) \times M_1$.

Lemma 3.4. *Define the coefficients A_1 on the cylinder $[0, \delta) \times M_1$ by*

$$A_1 \coloneqq \frac{1}{J_{\rho_1}} (\rho_1)_* A_0 \rho_1^* = \frac{1}{J_{\rho_1}} \mathrm{d}\rho_1 (A_0 \circ \rho_1^{-1}) \mathrm{d}\rho_1^\star.$$

Then A_1/ν is uniformly bounded and accretive. Moreover, the function $u_1 = u_0 \circ \rho_1^{-1}$ on $\mathcal{C}_1 = (0, \delta) \times M_1$ is a weak solution to

$$\mathrm{div}_{\mathcal{C}_1} A_1 \nabla_{\mathcal{C}_1} u_1 = 0 \tag{3.9}$$

if and only if u_0 is a weak solution to

$$\mathrm{div}_{\mathcal{C}_0} A_0 \nabla_{\mathcal{C}_0} u_0 = 0 \tag{3.10}$$

on $\mathcal{C}_0 = (0, \delta) \times M_0$.

Proof. Define the matrix weight $V_1 \coloneqq \frac{1}{J_{\rho_1}} (\rho_1)_* V_0 (\rho_1)^*$ on $[0, \delta) \times M_1$. Replacing ρ_0^{-1} by ρ_1 in Lemma 3.2 shows that $V_1^{-1/2} A_1 V_1^{-1/2}$ is uniformly bounded and accretive. We have

$$V_1 = \frac{1}{\sqrt{\det g_1}} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{d}\rho \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{d}\rho^\star \end{bmatrix} = \begin{bmatrix} \nu & 0 \\ 0 & \nu I \end{bmatrix},$$

since $J_\rho = \sqrt{\det g_1}$ and $J_\rho^{-1} \,\mathrm{d}\rho W \,\mathrm{d}\rho^\star = \nu I$. Thus $V_1^{-1/2} A_1 V_1^{-1/2} = A_1/\nu$. If $\nabla_{C_0} u_0 \in L^2(V_0)$, then $(\rho_1^{-1})^* \nabla_{C_0} u_0 = \nabla_{C_1}(u_0 \circ \rho_1^{-1})$ is in $L^2(T\mathcal{C}_1, \nu I)$ Moreover $A_0 \nabla_{C_0} u_0 \in L^2(T\mathcal{C}_0, V_0^{-1})$, so the non-smooth Piola transformation in Theorem A.3 shows that

$$\mathrm{div}_{\mathcal{C}_1} \frac{(\rho_1)_*}{J_{\rho_1}} \big(A_0 \nabla_{\mathcal{C}_0} u_0 \big) = \frac{(\rho_1)_*}{J_{\rho_1}} \big(\mathrm{div}_{\mathcal{C}_0} A_0 \nabla_{\mathcal{C}_0} u_0 \big) = 0$$

in $L^2(\mathcal{C}_1, \nu^{-1})$. This completes the proof. $\square$

Since A_1 is isotropically degenerate, we can apply results from [AMR22, §4] to obtain solvability estimates of BVPs for $\mathrm{div}_{\mathcal{C}_1} A_1 \nabla_{\mathcal{C}_1} u_1 = 0$. One can then translate to matrix-weighted norms on the cylinder $\mathcal{C}_0$ and in Ω to obtain the corresponding results for our BVPs for matrix-degenerate equations. To illustrate this, we consider the L^2 non-tangential maximal Neumann solvability estimate

$$\|\nabla u\|_{\mathcal{X}} \lesssim \|\partial_{\nu_{A_0}} u_\rho \restriction_M\|_{L^2(M,\omega_0^{-1})} \tag{3.11}$$

proved in [AMR22, Theorem 1.4]. In the notation of the present paper, the right hand side of (3.11) is

$$\Big(\int_{M_1} |e_0 \cdot A_1 \nabla_{\mathcal{C}_1} u_1|^2 \frac{1}{\nu} \, \mathrm{d}y \Big)^{1/2}$$

where $\nabla_{\mathcal{C}_1} u_1$ is the full gradient of u_1 as defined in (3.4). Note that

$$\begin{aligned} \nabla_{\mathcal{C}_1} u_1 &= (\rho_1^*)^{-1} \nabla_{\mathcal{C}_0} u_0, \\ \frac{1}{\nu} \mathrm{d}y &= \Big(\frac{J_\rho}{\mu}\Big)(J_\rho \mathrm{d}x) = \frac{J_\rho^2}{\mu} \mathrm{d}x. \end{aligned} \tag{3.12}$$

Since $A_1 = J_{\rho_1}^{-1} (\rho_1)_* \, A_0 \, \rho_1^*$, we get

$$e_0 \cdot A_1 \nabla_{\mathcal{C}_1} u_1 = J_{\rho_1}^{-1} (\rho_1^* e_0) \cdot A_0 \nabla_{\mathcal{C}_0} u_0 = J_{\rho_1}^{-1} e_0 \cdot A_0 \nabla_{\mathcal{C}_0} u_0$$

and since $J_{\rho_1} \restriction_{M_0} = J_\rho$, by using (3.12) we have

$$\int_{M_1} |e_0 \cdot A_1 \nabla_{\mathcal{C}_1} u_1|^2 \frac{1}{\nu} \, \mathrm{d}y = \int_{M_0} |e_0 \cdot A_0 \nabla_{\mathcal{C}_0} u_0|^2 \frac{1}{\mu} \, \mathrm{d}x.$$

As for the left hand side in (3.11), translating the Banach norm in [AMR22, eq. (4.13)] to our present notation gives

$$\|\nabla u\|_{\mathcal{X}}^2 = \int_{M_1} |\widetilde{N}_*(\eta \nabla_{\mathcal{C}_1} u_1)|^2 \nu \mathrm{d}y + \int_\Omega \langle V \nabla u, \nabla u \rangle (1-\eta)^2 \mathrm{d}z,$$

where $\eta(t)$ is a smooth cut-off towards the top of the cylinder, for example $\eta(t) = \max\{0, \min(1, 2 - 2t/\delta)\}$. Note that in the second term, with abuse of notation, we denoted again by η the pullback $\eta \circ \rho_0^{-1}$ on Ω. We recall the definition of the modified non-tangential maximal function $\widetilde{N}_*$ used on the cylinder $[0,\delta) \times M_1$.

Definition 3.5 (Modified non-tangential maximal function)**.** Let $c_0 > 1$, $c_1 > 0$ be fixed constants. For a point $(t, y) \in [0, \delta) \times M_1$, we define the Whitney region

$$W_1(t, y) := (t/c_0, c_0 t) \times B_1(y, c_1 t)$$

where B_1 denotes the geodesic ball of M_1 with respect to the metric dist_1. Then the non-tangential maximal function at a point $y_1 \in M_1$ is

$$\widetilde{N}_* f(y_1) := \sup_{t \in (0, c_0 \delta)} \left(\frac{1}{\nu\big(W_1(t, y_1)\big)} \iint_{W_1(t,y_1)} |f(s,y)|^2 \nu(y) \, \mathrm{d}s \, \mathrm{d}y \right)^{1/2},$$

where the measure $\nu\big(W_1(t, y_1)\big)$ is taken with respect to the weighted measure $\nu \, \mathrm{d}s \, \mathrm{d}y$ and equals $t(c_0 - c_0^{-1}) \, \nu(B_1)$.

Consider on M_0 the distance $\mathrm{dist}_0(x, \xi) := \mathrm{dist}_1(\rho(x), \rho(\xi))$ which is the geodesic distance on M_1 pulled back to M_0. The Whitney regions on $[0, \delta) \times M_0$ are

$$W_0(t, x) := (t/c_0, c_0 t) \times \{\xi \in M_0 : \ \mathrm{dist}_0(x, \xi) < c_1 t\}.$$

Changing variables with $y_1 = \rho(x_1)$, since $W_0(t,x_1) = \rho_1\big(W_1(t,y_1)\big)$, we get

$$\iint_{W_1(t,y_1)} \nu(y)\,\mathrm{d}s\,\mathrm{d}y = \iint_{W_0(t,x_1)} \mu(x)\,\mathrm{d}s\,\mathrm{d}x =: \mu\big(W_0(t,x_1)\big) \tag{3.13}$$

Changing variables using ρ_1 and the expression of the metric $\widetilde{g}_1$ in (3.8), we also get

$$\begin{aligned}\iint_{W_1(t,y_1)} &|\eta(s)\nabla_{\mathcal{C}_1} u_1|^2 \nu(y)\,\mathrm{d}s\,\mathrm{d}y \\ &= \iint_{W_0(t,x_1)} \eta(s)^2 \langle \mathrm{d}\rho_1^{-1}(\mathrm{d}\rho_1^{-1})^\star \nabla_{\mathcal{C}_0}(\rho_1^* u_1), \nabla_{\mathcal{C}_0}(\rho_1^* u_1)\rangle \mu(x)\,\mathrm{d}s\,\mathrm{d}x \\ &= \iint_{W_0(t,x_1)} \eta(s)^2 \langle \left[\begin{smallmatrix}\mu & 0\\ 0 & W\end{smallmatrix}\right] \nabla_{\mathcal{C}_0} u_0, \nabla_{\mathcal{C}_0} u_0\rangle\,\mathrm{d}s\,\mathrm{d}x\end{aligned}$$

since $\mathrm{d}\rho_1^{-1}(\mathrm{d}\rho_1^{-1})^\star = \widetilde{g}_1^{-1}$ and $\rho_1^* u_1 = u_0$. We also have

$$\int_{M_1} |\widetilde{N}_*(\eta\nabla_{\mathcal{C}_1} u_1)|^2 \nu(y)\,\mathrm{d}y = \int_{M_0} |\widetilde{N}_0(\eta\nabla_{\mathcal{C}_0} u_0)|^2 \mu(x)\,\mathrm{d}x,$$

where the new modified non-tangential maximal function is

$$\widetilde{N}_0 f(x_1) := \sup_{t\in(0,c_0\delta)} \left(\frac{1}{\mu\big(W_0(t,x_1)\big)} \iint_{W_0(t,x_1)} \langle \left[\begin{smallmatrix}\mu & 0\\ 0 & W\end{smallmatrix}\right] f(s,x), f(s,x)\rangle\,\mathrm{d}s\,\mathrm{d}x \right)^{1/2},$$

and $\mu\big(W_0(t,x_1)\big)$ is as in (3.13).

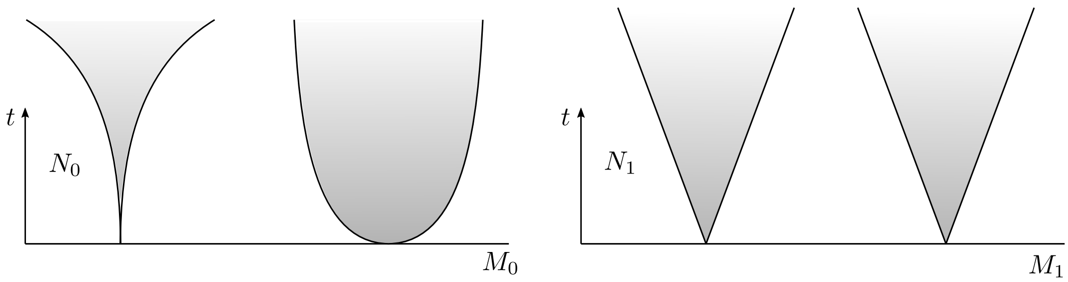


Figure 7. Non-tangential approach regions. On the left the μ, W-adapted approach regions: in the first $\mu W^{-1} \to \infty$ at M_0, in the second region $\mu W^{-1} \to 0$. On the right hand side, the corresponding non-tangential conical approach regions to M_1.

Note that the approach regions for $\widetilde{N}_0$ shown in Figure 7 left are intimately connected to the failure of standard off-diagonal estimates for the resolvent of the operator DB from Lemma 3.3. On the other hand, such off-diagonal estimates do hold for the corresponding operator associated to $\mathrm{div}_{\mathcal{C}_1} A_1 \nabla_{\mathcal{C}_1} u_1 = 0$, from [AMR22, Proposition 4.2]. And indeed on M_1 we have standard non-tangential approach regions on the right in Figure 7, and in [AMR22, Theorem 1.4].

For our solvability result, we also need the analogue of the *Carleson discrepancy* $\|\cdot\|_*$ from [AMR22, Eq. (4.10)] for a multiplier $\mathcal{E}$ on the cylinder $[0,\delta)\times M_0$ with Whitney regions W_0 and balls $B_0 \subseteq M_0$ taken with respect to the distance $\mathrm{dist}_0(\cdot,\cdot)$. The quantity $\|\mathcal{E}\|_*^2$ is given by

$$\sup_{\substack{\zeta\in M_0\\ r<\delta}} \iint_{\left\{\substack{0<t<r\\ x\in B_0(\zeta,r)}\right\}} \Big(\sup_{(s,\xi)\in W_0(t,x)} \big|V_0(\xi)^{-1/2}\mathcal{E}(s,\xi)V_0(\xi)^{-1/2}\big| \Big)^2 \frac{\mathrm{d}t}{t} \frac{\mu(x)\,\mathrm{d}x}{\mu(B_0(\zeta,r))},$$

where $\mu(B_0(\zeta,r)) = \int_{B_0(\zeta,r)} \mu(x)\,\mathrm{d}x$.

Summarising, we have obtained the following solvability result for the Neumann BVP for anisotropically degenerate divergence form equations (3.3).

Theorem 3.6. *Let* Ω,V,A,ρ_0,M_0,$V_0 = \left[\begin{smallmatrix}\mu & 0\\ 0 & W\end{smallmatrix}\right]$, A_0, ρ, M_1, *and* ν *be as above and summarized in Figure 6. Assume that:*

- $\nu \in A_2(M_1)$ *and that* μ *and* W *are scalar and matrix weights,*
- *the matrix degenerate coefficients* $A_0(t,x)$ *has trace*

$$\underline{A}_0(x) := A_0(0,x) = \lim_{t\to 0} A_0(t,x)$$

such that

(1) the Carleson discrepancy $\|A_0 - \underline{A}_0\|_* < \varepsilon_1$,

(2) $\underline{A}_0$ *is close to its adjoint as operator on* $L^2(\mathcal{C}_0, V_0)$,

$$\sup_{x\in M_0} \left|V_0(x)^{-1/2}\big(\underline{A}_0^\star(x) - A_0(x)\big)V_0(x)^{-1/2}\right| < \varepsilon_2,$$

with $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ *small enough.*

Then the Neumann solvability estimate

$$\int_{M_0} |\widetilde{N}_0(\eta\nabla_{\mathcal{C}_0} u_0)|^2 \mu\,\mathrm{d}x + \int_\Omega \langle V\nabla u, \nabla u\rangle (1-\eta)^2\,\mathrm{d}z \lesssim \int_{M_0} |\partial_{\nu_{A_0}} u_0|^2 \frac{1}{\mu}\,\mathrm{d}x$$

holds for all weak solutions u *to* $\mathrm{div} A\nabla u = 0$ *in* Ω, *with near boundary values* u_0 *of* u, *in* $\mathcal{C}_0$, *as above.*

Moreover ε *depends only on* $[\nu]_{A_2(M_1)}$, $\|V_0^{-1/2}A_0V_0^{-1/2}\|_{L^\infty}$ *and the accretivity constant of* $V_0^{-1/2}A_0V_0^{-1/2}$, *other than the structural geometric constants of* M_1*: dimension, injectivity radius and lower bound on the Ricci curvature.*

Proof. Apply [AMR22, Theorem 1.4] to the isotropically degenerate equation (3.9) on $[0,\delta)\times M_1$ (see Figure 6). Translation of this result to the anisotropically degenerate equation $\mathrm{div} A\nabla u = 0$ in Ω (and the Lipschitz equivalent equation $\mathrm{div}_{\mathcal{C}_0} A_0 \nabla_{\mathcal{C}_0} u_0 = 0$ on the cylinder $[0,\delta)\times M_0$, near $\partial\Omega$) gives the stated result. We have seen above the translation of the solvability estimate. The translation of the Carleson discrepancy and almost self-adjointness hypothesis is done similarly using Lemma 3.2 with A, A_0 replaced by A_1, A_0 and a change of variables in the integrals. □

The solvability estimates for the L^2 Dirichlet and Dirichlet regularity BVPs from [AMR22, Theorem 1.4] and the Atiyah–Patodi–Singer BVPs from [AMR22, Theorems 4.5, 4.6] can similarly be extended to anisotropically degenerate equations. We leave the details to the interested reader.

Appendix A. $W^{1,1}$ pullbacks and Piola transformations

We generalise the commutation theorem [Ros19, Theorem 7.2.9, Lemma 10.2.4] for external derivatives and pullbacks to $W^{1,1}_{\mathrm{loc}}$ homeomorphisms and weighted L^2 fields. (We only deal with the scalar and vector case which we need).

Throughout this section, $\rho\colon \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be a $W^{1,1}_{\mathrm{loc}}$ homeomorphism, meaning that ρ,ρ^{-1} are continuous with weak Jacobian matrix $\mathrm{d}\rho$,$\mathrm{d}\rho^{-1}$ in L^1_{loc}.

Theorem A.1 (Change of variables)**.** *If ρ is a $W^{1,1}_{\mathrm{loc}}$ homeomorphism then*

$$\int_\Omega f(\rho(x))J_\rho(x)\,\mathrm{d}x = \int_{\rho(\Omega)} f(y)\,\mathrm{d}y$$

holds for all integrable, compactly supported functions f.

See [Haj93, Thorem 2 and §3].

For $f \in C^\infty_c(\mathbb{R}^d)$ and $h \in C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$, the chain rule in the weak sense reads

$$-\int f(\rho(x))\mathrm{div}h(x)\,\mathrm{d}x = \int (\mathrm{d}\rho_x)^\star(\nabla f)(\rho(x))h(x)\,\mathrm{d}x. \tag{A.1}$$

This holds for $W^{1,1}_{\mathrm{loc}}$ homeomorphism ρ, as readily seen by mollifying ρ and passing to the limit. We first extend to non-smooth f:

Theorem A.2 (Non-smooth chain rule)**.** *Assume $v, V \in A^{\mathrm{loc}}_2$ and $f \in L^2(v)$ is compactly supported with weak gradient $\nabla f \in L^2(V)$. Let ρ be a $W^{1,1}_{\mathrm{loc}}$ homeomorphism. Define the weights*

$$v_\rho(x) := J_\rho(x)v(\rho(x)), \quad V_\rho(x) := J_\rho(x)\mathrm{d}\rho_x^{-1}V(\rho(x))(\mathrm{d}\rho_x)^\star)^{-1}$$

and assume $v_\rho, V_\rho \in A^{\mathrm{loc}}_2$. Then $\rho^* f = f\circ\rho \in L^2(v_\rho)$ has weak gradient

$$\nabla(\rho^* f) = \rho^*\nabla f = \mathrm{d}\rho^\star(\nabla f\circ\rho) \in L^2(V_\rho).$$

Proof. Mollify $f_t := \eta_t * f$, so that $\nabla f_t = \eta_t * \nabla f$. It follows that $f_t \to f$ in $L^2(v)$ and $\nabla f_t \to \nabla f$ in $L^2(V)$ using dominated convergence and bounds for the vector Hardy–Littlewood maximal operator introduced by Christ and Goldberg [CG01], see [Gol03, Theorem 3.2]. Note that $\|\rho^* f\|_{L^2(v_\rho)} = \|f\|_{L^2(v)}$ and $\|\rho^*(\nabla f)\|_{L^2(V_\rho)} = \|\nabla f\|_{L^2(V)}$. Apply the chain rule (A.1) to f_t and ρ for a fixed test function h. We can pass to the limit in t and conclude since the left hand side of (A.1) is bounded as

$$\begin{aligned}\int |f_t(\rho(x)) - f(\rho(x))|v_\rho(x)\,\mathrm{d}x &\lesssim \Big(\int |f_t(\rho(x)) - f(\rho(x))|^2 v_\rho(x)\,\mathrm{d}x\Big)^{1/2}\\ &= \Big(\int |f_t(y) - f(y)|^2 v(y)\,\mathrm{d}y\Big)^{1/2} \to 0\end{aligned}$$

where the first integral is on the compact support of h and we used Theorem A.1 when changing variables. For the right hand side in (A.1), using that $|V_\rho^{-1}| \in L^1_{\mathrm{loc}}$, we bound

$$\begin{aligned}\int |\langle V_\rho^{1/2}\big(\rho^*(\nabla f_t) - \rho^*(\nabla f)\big), V_\rho^{-1/2}h\rangle|\,\mathrm{d}x &\lesssim \Big(\int |V_\rho^{1/2}\big(\rho^*(\nabla f_t) - \rho^*(\nabla f)\big)|^2\,\mathrm{d}x\Big)^{1/2}\\ &= \Big(\int |V^{1/2}(\nabla f_t - \nabla f)|^2\,\mathrm{d}y\Big)^{1/2} \to 0.\end{aligned}$$

This concludes the proof. □

Changing variables in (A.1) gives

$$-\int f(y)\frac{1}{J_\rho(\rho^{-1}(y))}(\mathrm{div}h)(\rho^{-1}(y))\,\mathrm{d}y = \int \nabla f(y)\cdot\Big(\frac{1}{J_\rho}\mathrm{d}\rho\, h\Big)(\rho^{-1}(y))\,\mathrm{d}y. \tag{A.2}$$

We refer to the transformation applied to h on the right hand side of (A.2) as the Piola transformation $J_\rho^{-1}\rho_*$, where ρ_* denotes the pushforward via ρ. This transformation is the adjoint of the pullback ρ^* with respect to the unweighted L^2 pairing.

We extend identity (A.2) to non-smooth vector fields h.

Theorem A.3 (Non-smooth Piola transformation). *Assume that* $v, V \in A_2^{\mathrm{loc}}$ *and* $h \in L^2(\mathbb{R}^d;\mathbb{R}^d, V)$ *is compactly supported with weak divergence* $\mathrm{div} h \in L^2(\mathbb{R}^d, v)$. *Let* ρ *be a* $W^{1,1}_{\mathrm{loc}}$ *homeomorphism. Define the weights*

$$v^\rho(y) := J_\rho(\rho^{-1}(y))v(\rho^{-1}(y)), \quad V^\rho(y) := \big(J_\rho(\mathrm{d}\rho^\star)^{-1} V \mathrm{d}\rho^{-1}\big) \circ \rho^{-1}(y)$$

and assume $v^\rho, V^\rho \in A_2^{\mathrm{loc}}$. *Then* $J_\rho^{-1}\rho_* h = \big(\frac{1}{J_\rho}\mathrm{d}\rho h\big) \circ \rho^{-1} \in L^2(V^\rho)$ *and has weak divergence*

$$\mathrm{div}(J_\rho^{-1}\rho_* h) = \Big(\frac{1}{J_\rho}\mathrm{div} h\Big) \circ \rho^{-1} \in L^2(v^\rho).$$

The proof is analogous to the one of Theorem A.2, where we pass to the limit $h_t \to h$ in (A.2).

Acknowledgements

G.B. is supported by the Knut and Alice Wallenberg foundation, KAW grant 2020.0262 postdoctoral program in Mathematics for researchers from outside Sweden. A.R. was supported by Grant 2022-03996 from the Swedish research council, VR.

References

[AAM10] P. Auscher, A. Axelsson, and A. McIntosh. "Solvability of elliptic systems with square integrable boundary data". In: *Ark. Mat.* 48.2 (2010), pp. 253–287 (cit. on pp. 4, 19, 20).

[ADM96] D. Albrecht, X. Duong, and A. McIntosh. "Operator theory and harmonic analysis". In: *Instructional workshop on analysis and geometry, Canberra, Australia, January 23 - February 10, 1995. Part III: Operator theory and nonlinear analysis*. Canberra: Australian National University, Centre for Mathematics and its Applications, 1996, pp. 77–136 (cit. on p. 5).

[AKM06] A. Axelsson, S. Keith, and A. McIntosh. "Quadratic estimates and functional calculi of perturbed Dirac operators". In: *Invent. Math.* 163.3 (2006), pp. 455–497 (cit. on pp. 2, 11).

[AMN97] P. Auscher, A. McIntosh, and A. Nahmod. "The square root problem of Kato in one dimension, and first order elliptic systems". In: *Indiana Univ. Math. J.* 46.3 (1997), pp. 659–695 (cit. on p. 2).

[AMR22] P. Auscher, A. J. Morris, and A. Rosén. "Quadratic estimates for degenerate elliptic systems on manifolds with lower Ricci curvature bounds and boundary value problems". In: *arXiv:2209.11529* (2022) (cit. on pp. 3, 14, 16, 18, 20, 22–24).

[ARR15] P. Auscher, A. Rosén, and D. Rule. "Boundary value problems for degenerate elliptic equations and systems". In: *Ann. Sci. Éc. Norm. Supér. (4)* 48.4 (2015), pp. 951–1000 (cit. on pp. 3, 4, 7, 8).

[Aus+02] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian. "The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$." In: *Ann. Math. (2)* 156.2 (2002), pp. 633–654 (cit. on p. 1).

[BLM17] K. Bickel, K. Lunceford, and N. Mukhtar. "Characterizations of A_2 matrix power weights". In: *J. Math. Anal. Appl.* 453.2 (2017), pp. 985–999 (cit. on p. 6).

[Bow01] M. Bownik. "Inverse volume inequalities for matrix weights". In: *Indiana Univ. Math. J.* 50.1 (2001), pp. 383–410 (cit. on p. 6).

[Cal77] A. P. Calderon. "Cauchy integrals on Lipschitz curves and related operators". In: *Proc. Natl. Acad. Sci. USA* 74 (1977), pp. 1324–1327 (cit. on p. 11).

[CG01] M. Christ and M. Goldberg. "Vector A_2 weights and a Hardy-Littlewood maximal function". In: *Trans. Am. Math. Soc.* 353.5 (2001), pp. 1995–2002 (cit. on p. 25).

[CMM82] R. R. Coifman, A. McIntosh, and Y. Meyer. "L'intégrale de Cauchy définit un opératuer borne sur L^2pour les courbes lipschitziennes". In: *Ann. Math. (2)* 116 (1982), pp. 361–387 (cit. on pp. 1, 11).

[CMR18] D. Cruz-Uribe, J. M. Martell, and C. Rios. "On the Kato problem and extensions for degenerate elliptic operators". In: *Anal. PDE* 11.3 (2018), pp. 609–660 (cit. on p. 2).

[CR15] D. Cruz-Uribe and C. Rios. "The Kato problem for operators with weighted ellipticity". In: *Trans. Am. Math. Soc.* 367.7 (2015), pp. 4727–4756 (cit. on pp. 1, 3).

[Dav84] G. David. "Opérateurs intégraux singuliers sur certaines courbes du plan complexe". In: *Ann. Sci. Éc. Norm. Supér. (4)* 17 (1984), pp. 157–189 (cit. on p. 11).

[Gol03] M. Goldberg. "Matrix A_p weights via maximal functions." In: *Pac. J. Math.* 211.2 (2003), pp. 201–220 (cit. on p. 25).

[Haj93] P. Hajłasz. "Change of variables formula under minimal assumptions". In: *Colloq. Math.* 64.1 (1993), pp. 93–101 (cit. on p. 25).

[Kat61] T. Kato. "Fractional powers of dissipative operators". In: *J. Math. Soc. Japan* 13 (1961), pp. 246–274 (cit. on p. 1).

[Kat95] T. Kato. *Perturbation theory for linear operators.* Reprint of the corr. print. of the 2nd ed. 1980. Class. Math. Berlin: Springer-Verlag, 1995 (cit. on p. 1).

[KM85] C. Kenig and Y. Meyer. "Kato's square roots of accretive operators and Cauchy kernels on Lipschitz curves are the same". Recent progress in Fourier analysis, Proc. Semin., El Escorial/Spain 1983, North-Holland Math. Stud. 111, 123-143 (1985). 1985 (cit. on p. 11).

[Lac+14] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero. "Two-weight inequality for the Hilbert transform: a real variable characterization. I". In: *Duke Math. J.* 163.15 (2014), pp. 2795–2820 (cit. on p. 12).

[MQ91] A. McIntosh and T. Qian. "Convolution singular integral operators on Lipschitz curves". In: *Harmonic analysis. Proceedings of the special program at the Nankai Institute of Mathematics, Tianjin, PR China, March-July, 1988.* Berlin etc.: Springer-Verlag, 1991, pp. 142–162 (cit. on p. 11).

[Ros19] A. Rosén. *Geometric multivector analysis. From Grassmann to Dirac.* Birkhäuser Adv. Texts, Basler Lehrbüch. Cham: Birkhäuser, 2019 (cit. on pp. 19, 24).

Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Göteborg, Sweden

Email address: brocchi@chalmers.se

Email address: andreas.rosen@chalmers.se