# SKEW TWO-SIDED BRACOIDS

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Abstract. Isabel Martin-Lyons and Paul J.Truman generalized the definition of a skew brace to give a new algebraic object, which they termed a skew bracoid. Their construction involves two groups interacting in a manner analogous to the compatibility condition found in the definition of a skew brace. They formulated tools for characterizing and classifying skew bracoids, and studied substructures and quotients of skew bracoids. In this paper we study two-sided bracoids. In [9] Rump showed that if a left brace $(B, \star, \cdot)$ is a two-sided brace and the operation $* : B \times B \longrightarrow B$ is defined by $a * b = a \cdot b \star \overline{a} \star \overline{b}$ for all $a, b \in B$ then $(B, \star, *)$ is a Jacobson radical ring. Lau showed that if $(B, \star, \cdot)$ is a left brace and the operation is asssociative, then $B$ is a two-sided brace. We will prove bracoid versions of this results.

## 1. Introduction and notation

Skew braces has been introduced by Guarnieri and Vedramin in [5] as a generalisation of the left braces introduced by Rump in [9].

**Definition 1.1.** A *(left) skew brace* is a triple $(B, \star, \cdot)$ such that $(B, \cdot)$ and $(B, \star)$ are groups such that

$$a \cdot (b \star c) = (a \cdot b) \star \overline{a} \star (a \cdot c) \tag{1.1}$$

for all $a, b, c \in B$.

Throughout, the identity element of a group $G$ is denoted $e_G$. We also denote by $\overline{\eta}$ the inverse element of $\eta$ in a group $(N, \star)$ and $g^{-1}$ the inverse element of $g$ in a group $(G, \cdot)$. Note that if $(B, \star)$ is an abelian group, we obtain the definition of left brace introduced by Rump, as formulated by Cedó, Jespers and Okniński in [3]. Skew left braces have been devised with the aim of attacking the problem of finding all set-theoretical non-degenerate solutions of the Yang-Baxter Equation, a consistency equation that plays a relevant role in quantum statistical mechanics, in the foundation of quantum groups, and that provides a multidisciplinary approach from a wide variety of areas such as Hopf algebras, knot theory and braid theory among others.

In [1] Brzeziński proposed to study a set with two binary operations connected by a rule which can be seen as the interpolation between the ring-type (i.e. the standard) and brace distributive laws. A *skew left truss* is a set $A$ with binary operations $\star$ and $\cdot$, such that $(A, \star)$ is a group, $(A, \cdot)$ is a semigroup and, for all $a, b, c \in A$, the following generalised distributive law holds

$$a \cdot (b \star c) = (a \cdot b) \star \overline{a \cdot e_\star} \star (a \cdot c),$$

where $e_\star$ is the neutral element of the group $(A, \star)$. This truss distributive law interpolates the ring (standard) and brace distributive laws: the former one is obtained by setting $a \cdot e_\star = e_\star$, the latter is obtained by setting $a \cdot e_\star = a$. A special case of skew left truss is a near brace, considered by Doikou and Rybołowicz in [6]. A near brace is a skew left truss $A$, where $(A, \cdot)$ is a group. Using the

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notion of the near braces they produced new multi-parametric, non-degenerate, non-involutive solutions of the set-theoretic Yang-Baxter equation. These solutions are generalisations of the known ones coming from braces and skew braces.

In [8] Martin-Lyons and Truman generalised the definition of a skew brace to give a new algebraic object, which they termed a skew bracoid.

**Definition 1.2.** [8, Definition 2.1.] A *(left) skew bracoid* is a 5-tuple $(G,\cdot,N,\star,\odot)$ such that $(G,\cdot)$ and $(N,\star)$ are groups and $\odot$ is a transitive action of $(G,\cdot)$ on $N$ such that

$$g\odot(\mu\star\eta)=(g\odot\mu)\star\overline{(g\odot e_N)}\star(g\odot\eta) \tag{1.2}$$

for all $g\in G$ and $\mu,\eta\in N$. If $(N,\star)$ is an abelian group, then we shall call $(G,\cdot,N,\star,\odot)$ a *(left) bracoid*.

**Example 1.1.** [8, Example 2.2.] A skew brace $(B,\star,\cdot)$ can be viewed as a skew bracoid with $(G,\cdot)=(B,\cdot)$, $(N,\star)=(B,\star)$ and $\odot=\cdot$.

Clearly a near brace is also a special case of a skew bracoid. Martin-Lyons and Truman showed that several important constructions and identities associated with skew braces (such as $\gamma$-functions) have natural skew bracoid counterparts. We recall that the $\gamma$-function of a skew brace $(B,\cdot,\star)$ is the function $\gamma: B\longrightarrow Perm(B)$ defined by ${}^{\gamma(b)}a=\overline{b}\star(b\cdot a)$. In fact, we have $\gamma(b)\in Aut(B,\star)$ for each $b\in B$, and the map $\gamma$ is a homomorphism from $(B,\cdot)$ to $Aut(B,\star)$. The $\gamma$-functions of skew braces (often called $\lambda$-functions) are used, among others, to define left ideals and ideals of skew braces (see for example [4, 5]). In [8] Martin-Lyons and Truman showed that certain skew bracoids arise from a natural quotienting procedure on skew braces. They also defined $\gamma$-functions, left ideals and ideals of skew braces.

**Definition 1.3.** [8, Definition 2.10.] Let $(G,\cdot,N,\star,\odot)$ be a skew bracoid. The homomorphism $\gamma: G\longrightarrow Aut(N)$ defined by

$${}^{\gamma(g)}\eta=\overline{(g\odot e_N)}\star(g\odot\eta)\,\text{for all}\, g\in G\,\text{and}\,\eta\in N$$

is called the $\gamma$*-function* of the skew bracoid.

As it was noticed in [8] when a skew brace is viewed as a skew bracoid, the $\gamma$-function of the skew bracoid coincides with that of the skew brace.

In Section 3 we consider two-sided bracoids. In [9] Rump showed that if a left brace $(B,\star,\cdot)$ is a two-sided brace and the operation $*: B\times B\longrightarrow B$ is defined by $a*b=a\cdot b\star\overline{a}\star\overline{b}$ for all $a,b\in B$ then $(B,\star,*)$ is a Jacobson radical ring (see also [2, Proposition 1]). In [7] Lau showed that if $(B,\star,\cdot)$ is a left brace and the operation $*$ is asssociative, then $B$ is a two-sided brace. We will prove bracoid versions of this results.

## 2. Preliminaries

Here are some elementary properties of skew bracoids.

*Remark.* (see [4, Remark 1.3.]) Let $(G,\cdot,N,\star,\odot)$ be a skew left bracoid. Then the following formulas hold:

$$(g\odot e_N)\star{}^{\gamma(g)}\,\eta=g\odot\eta,\;(g\odot e_N)\star\eta=g\odot({}^{\gamma(g)^{-1}}\eta),\;{}^{\gamma(g)}(g^{-1}\odot e_N)=\overline{g\odot e_N}$$

for all $g\in G$ and $\eta\in N$.

**Lemma 2.1.** [8, Proposition 2.13.] *Let $(G,\cdot,N,\star,\odot)$ be a skew bracoid. Then for all $g\in G$ and $\eta\in N$ we have*

$$\overline{(g\odot e_N)}\star(g\odot\overline{\eta})\star\overline{(g\odot e_N)}=\overline{g\odot\eta}.$$

Now we generalise the $*$-operation of a skew brace $(B, \star, \cdot)$. Recall that the operation $* : B \times B \longrightarrow B$ is defined by setting

$$a * b = \overline{a} \star (a \cdot b) \star \overline{b}$$

for all $a, b \in B$.

**Definition 2.1.** Let $(G, \cdot, N, \star, \odot)$ be a skew bracoid. For every $g \in G$ define the map $\alpha(g) : (N, \star) \longrightarrow (N, \star)$ by setting

$$ {}^{\alpha(g)}\eta = \overline{(g \odot e_N)} \star (g \odot \eta) \star \overline{\eta} = ({}^{\gamma(g)}\eta) \star \overline{\eta}, \tag{2.1}$$

for all $\eta \in N$.

When a skew brace $(B, \star, \cdot)$ is viewed as a skew bracoid we have ${}^{\alpha(a)}b = a * b$ for all $a, b \in B$ (see [4]).

We begin with some essential properties of the map $\alpha(g)$.

**Lemma 2.2.** *Let $(G, \cdot, N, \star, \odot)$ be a skew bracoid. Then for every $g, h \in G$ and $\mu, \eta \in N$, we have:*

(1) ${}^{\alpha(g)}(\eta \star \mu) = ({}^{\alpha(g)}\eta) \star \eta \star ({}^{\alpha(g)}\mu) \star \overline{\eta}$.
(2) ${}^{\alpha(g)}e_N = {}^{\alpha(e_G)}\ \eta = e_N$.
(3) ${}^{\alpha(g)}\overline{\eta} = \overline{\eta} \star \overline{({}^{\alpha(g)}\eta)} \star \eta$.
(4) ${}^{\alpha(gh)}\eta = ({}^{\alpha(g)}({}^{\alpha(h)}\eta)) \star ({}^{\alpha(h)}\eta) \star ({}^{\alpha(g)}\eta)$.

*Proof.* Only Statement (4) is in doubt.

$$({}^{\alpha(g)}({}^{\alpha(h)}\eta)) \star ({}^{\alpha(h)}\eta) \star ({}^{\alpha(g)}\eta) = ({}^{\gamma(g)}({}^{\alpha(h)}\eta)) \star \overline{({}^{\alpha(h)}\eta)} \star ({}^{\alpha(h)}\eta) \star ({}^{\alpha(g)}\eta) =$$
$$({}^{\gamma(g)}({}^{\gamma(h)}\eta \star \overline{\eta})) \star ({}^{\gamma(g)}\eta) \star \overline{\eta} = ({}^{\gamma(g)}({}^{\gamma(h)}\eta \star \overline{\eta} \star \eta)) \star \overline{\eta} = ({}^{\gamma(gh)}\eta) \star \overline{\eta} = {}^{\alpha(gh)}\ \eta.$$

□

**Corollary 2.3.** *Let $(G, \cdot, N, \star, \odot)$ be a bracoid (i.e. $(N, \star)$ is an abelian group). Then for every $g \in G$ the map $\alpha(g)$ is an endomorhism of the group $(N, \star)$.*

A right skrew bracoid is defined analogously to a left skew bracoid.

**Definition 2.2.** A *right skew bracoid* is a 5-tuple $(H, \circ, N, \star, \boxdot)$ such that $(H, \circ)$ and $(N, \star)$ are groups and $\boxdot$ is a transitive right action of $(H, \circ)$ on $(N, \star)$ such that

$$(\eta \star \mu) \boxdot g = (\eta \boxdot g) \star \overline{(e_N \boxdot g)} \star (\mu \boxdot g) \tag{2.2}$$

for all $g \in H$ and $\eta, \mu \in N$. If $(N, \star)$ is an abelian group, then we shall call $(H, \circ, N, \star, \boxdot)$ a *right bracoid.*

Let $(H, \circ, N, \star, \boxdot)$ be a right skew bracoid. We can define the homomorphism $\delta : H \longrightarrow Aut(N)$ as

$$\eta^{\delta(h)} = (\eta \boxdot h) \star \overline{(e_N \boxdot h)} \text{ for all } h \in H \text{ and } \eta \in N.$$

We call it the *$\delta$-function* of the right skew bracoid.

**Definition 2.3.** Let $(H, \circ)$ and $(N, \star)$ be groups and $\boxdot$ be a transitive right action of $(H, \circ)$ on $(N, \star)$. We define for every $h \in H$ the map $\beta(h) : (N, \star) \longrightarrow (N, \star)$ by setting

$$\eta^{\beta(h)} = \overline{\eta} \star (\eta \boxdot h) \star \overline{(e_N \boxdot h)}, \tag{2.3}$$

for all $\eta \in N$.

Let $(H, \circ, N, \star, \boxdot)$ be a right skew bracoid. Then

$$\eta^{\beta(h)} = \overline{\eta} \star \eta^{\delta(h)}$$

for all $h \in H$ and $\eta \in N$. We can also prove some essential properties of the map $\beta(h)$ for every $h \in H$.

**Lemma 2.4.** *Let $(H, \circ, N, \star, \boxdot)$ be a right skew bracoid. Then for every $h \in H$ and $\mu, \eta \in N$, we have:*
(1) $(\mu \star \eta)^{\beta(h)} = \overline{\eta} \star (\mu^{\beta(h)}) \star \eta \star (\eta^{\beta(h)})$.
(2) $e_N^{\beta(h)} = \eta^{\beta(e_H)} = e_N$.
(3) $\overline{\eta}^{\beta(h)} = \eta \star (\overline{\eta^{\beta(h)}}) \star \overline{\eta}$.

**Corollary 2.5.** *Let $(H, \circ, N, \star, \boxdot)$ be a right bracoid. Then for every $h \in H$ the map $\beta(h)$ is an endomorhism of the group $(N, \star)$.*

When a skew brace $(B, \star, \cdot)$ is viewed as a skew bracoid we have $a^{\beta(b)} = a * b$ for all $a, b \in B$.

## 3. Two-sided bracoids

**Definition 3.1.** Let $(G, \cdot)$, $(H, \circ)$ and $(N, \star)$ be groups. If $(G, \cdot, N, \star, \odot)$ is a left skrew bracoid and $(H, \circ, N, \star, \boxdot)$ is a right skew bracoid such that

$$g \odot (\eta \boxdot h) = (g \odot \eta) \boxdot h \tag{3.1}$$

for all $g \in G$, $h \in H$ and $\eta \in N$, then we shall call $(G, \cdot, \odot, H, \circ, \boxdot, N, \star)$ a *two-sided skew bracoid*. If $(N, \star)$ is an abelian group, then we shall call $(G, \cdot, \odot, H, \circ, \boxdot, N, \star)$ a *two-sided bracoid*.

**Example 3.1.** In this example we denote by $g^{-1}$ the inverse element of g of a group. Let $t, w \in \mathbb{N}$, let $d$ be a positive divisor of $\gcd(t, w)$, let

$$G = \langle x, y \mid x^t = y^4 = 1, x^y = x^{-1} \rangle, \quad H = \langle a, b \mid a^{2w} = 1, a^w = b^2, a^b = a^{-1} \rangle$$

and let

$$N = \langle \mu, \eta \mid \mu^d = \eta^2 = 1, \mu^\eta = \mu^{-1} \rangle = D_d.$$

Then the rule

$$x^i y^j \odot \mu^r \eta^s = \mu^{i+(-1)^j r} \eta^{j+s}$$

defines a transitive action of $G$ on $N$, and we have

$$(x^i y^j \odot \mu^{r_1} \eta^{s_1}) \star (x^i y^j \odot e_N)^{-1} \star (x^i y^j \odot \mu^{r_2} \eta^{s_2}) =$$
$$\mu^{i+(-1)^j r_1} \eta^{j+s_1} \star (\mu^i \eta^j)^{-1} \star \mu^{i+(-1)^j r_2} \eta^{j+s_2} = \mu^{i+(-1)^j r_1 + (-1)^{j+s_1} r_2} \eta^{j+s_1+s_2},$$

$$(\mu^{r_1} \eta^{s_1}) \star (\mu^{r_2} \eta^{s_2}) = \mu^{r_1 + (-1)^{s_1} r_2} \eta^{s_1+s_2}$$

$$x^i y^j \odot ((\mu^{r_1} \eta^{s_1}) \star (\mu^{r_2} \eta^{s_2})) =$$
$$x^i y^j \odot (\mu^{r_1 + (-1)^{s_1} r_2} \eta^{s_1+s_2}) = \mu^{i+(-1)^j r_1 + (-1)^{j+s_1} r_2} \eta^{j+s_1+s_2},$$

Therefore $(G, \cdot, N, \star, \odot)$ is a left skew bracoid.

The rule

$$\mu^r \eta^s \boxdot a^k b^l = \mu^{r+(-1)^s k} \eta^{s+l}$$

defines a transitive right action of $H$ on $N$, and we have

$$((\mu^{r_1} \eta^{s_1}) \star (\mu^{r_2} \eta^{s_2})) \boxdot a^k b^l =$$
$$(\mu^{r_1 + (-1)^{s_1} r_2} \eta^{s_1+s_2}) \boxdot a^k b^l = \mu^{r_1 + (-1)^{s_1} r_2 + (-1)^{s_1+s_2} k} \eta^{l+s_1+s_2},$$

$$((\mu^{r_1}\eta^{s_1}) \boxdot a^k b^l) \star (e_N \boxdot a^k b^l)^{-1} \star ((\mu^{r_2}\eta^{s_2}) \boxdot a^k b^l) =$$
$$\mu^{r_1+(-1)^{s_1}k}\eta^{s_1+l} \star (\mu^k \eta^l)^{-1} \star \mu^{r_2+(-1)^{s_2}k}\eta^{s_2+l} = \mu^{r_1+(-1)^{s_1}r_2+(-1)^{s_1+s_2}k}\eta^{l+s_1+s_2}.$$

Hence $(H, \circ, N, \star, \boxdot)$ is a right skew bracoid. Finally,

$$(x^i y^j \odot \mu^r \eta^s) \boxdot a^k b^l = \mu^{i+(-1)^j r}\eta^{j+s} \boxdot a^k b^l = \mu^{i+(-1)^j r+(-1)^{j+s}k}\eta^{j+s+l}$$

and

$$x^i y^j \odot (\mu^r \eta^s \boxdot a^k b^l) = x^i y^j \odot (\mu^{r+(-1)^s k}\eta^{s+l}) = \mu^{i+(-1)^j r+(-1)^{j+s}k}\eta^{j+s+l}$$

Therefore $(G, \cdot, \odot, H, \circ, \boxdot, N, \star)$ is a skew two-sided bracoid.

In [9] Rump showed that if a left brace $(B, \star, \cdot)$ is a two-sided brace and the operation $* : B \times B \longrightarrow B$ is defined by $a * b = a \cdot b \star \overline{a} \star \overline{b}$ for all $a, b \in B$ then $(B, \star, *)$ is a Jacobson radical ring (see also [2, Proposition 1]). Now we will prove a bracoid version of this result.

**Theorem 3.1.** *If $(G, \cdot, \odot, H, \circ, \boxdot, N, \star)$ is a two-sided bracoid (i.e. $(N, \star)$ is an abelian group), then for all $g \in G$, $h \in H$ and $\mu, \eta \in N$ we have*
(1) *$\alpha(g)$ is an endomorphism of $(N, \star)$.*
(2) *$\beta(h)$ is an endomorphism of $(N, \star)$.*
(3) *$({}^{\gamma(g)}\eta)^{\delta(h)} = {}^{\gamma(g)}(\eta^{\delta(h)})$.*
(4) *$({}^{\alpha(g)}\eta)^{\beta(h)} = {}^{\alpha(g)}(\eta^{\beta(h)})$.*

*Proof.* Since $(N, \star)$ is abelian, by Corollaries 2.3 and 2.5 we obtain (1)-(2).
Now we prove (3). Let $g \in G$, $h \in H$ and $\eta \in N$.

$$
\begin{aligned}
({}^{\gamma(g)}\eta)^{\delta(h)} &= (\overline{(g \odot e_N)} \star (g \odot \eta))^{\delta(h)} = \overline{(g \odot e_N)^{\delta(h)}} \star (g \odot \eta)^{\delta(h)} = \\
&\overline{((g \odot e_N) \boxdot h) \star \overline{(e_N \boxdot h)}} \star ((g \odot \eta) \boxdot h) \star \overline{(e_N \boxdot h)} = \\
&\overline{((g \odot e_N) \boxdot h)} \star (e_N \boxdot h) \star ((g \odot \eta) \boxdot h) \star \overline{(e_N \boxdot h)} = \\
&\overline{((g \odot e_N) \boxdot h)} \star ((g \odot \eta) \boxdot h) = \\
&\overline{(g \odot (e_N \boxdot h))} \star (g \odot e_N) \star (g \odot (\eta \boxdot h)) \star \overline{(g \odot e_N)} = \\
&\overline{{}^{\gamma(g)}(e_N \boxdot h)} \star {}^{\gamma(g)}(\eta \boxdot h) = {}^{\gamma(g)}(\overline{(e_N \boxdot h)} \star (\eta \boxdot h)) = {}^{\gamma(g)}(\eta^{\delta(h)}).
\end{aligned}
$$

So only (4) is in doubt. Let $g \in G$, $h \in H$ and $\eta \in N$. Then by using (3) we have

$$
\begin{aligned}
({}^{\alpha(g)}\eta)^{\beta(h)} &= ({}^{\gamma(g)}\eta \star \overline{\eta})^{\beta(h)} = \overline{({}^{\gamma(g)}\eta \star \overline{\eta})} \star ({}^{\gamma(g)}\eta \star \overline{\eta})^{\delta(h)} = \\
&({}^{\gamma(g)}\overline{\eta} \star \eta) \star ({}^{\gamma(g)}\eta)^{\delta(h)} \star \overline{\eta}^{\delta(h)} = {}^{\gamma(g)}\overline{\eta} \star \eta \star {}^{\gamma(g)}(\eta^{\delta(h)}) \star \overline{\eta}^{\delta(h)} = \\
&{}^{\gamma(g)}(\overline{\eta} \star \eta^{\delta(h)}) \star \overline{(\overline{\eta} \star \eta^{\delta(h)})} = {}^{\alpha(g)}(\overline{\eta} \star \eta^{\delta(h)}) = {}^{\alpha(g)}(\eta^{\beta(h)}).
\end{aligned}
$$

□

When a two-sided brace $(B, \star, \cdot)$ is viewed as a two-sided bracoid we have

$$a^{\beta(b)} = a * b = {}^{\alpha(a)}b$$

for all $a, b \in B$. Hence Theorem 3.1 is a generalisation of the result of Rump.

In [7] Lau showed that if $(B, \star, \cdot)$ is a left brace and the operation $* : B \times B \longrightarrow B$ defined by $a * b = a \cdot b \star \overline{a} \star \overline{b}$ for all $a, b \in B$ is asssociative, then $B$ is a two-sided brace. In the next part of the section we will prove a bracoid version of this result.

**Proposition 3.2.** *Let $(G, \cdot, N, \star, \odot)$ be a left skew bracoid and $(H, \circ)$ be a group. Assume that $\boxdot$ is a transitive right action of $(H, \circ)$ on $N$ such that*

$$g \odot (\eta \boxdot h) = (g \odot \eta) \boxdot h$$

*for all $g \in G, h \in H$, $\eta \in N$. Then for all $g \in G$, $\eta \in N$ and $h \in H$ we have*

$$(g \odot \eta)^{\beta(h)} = ({}^{\alpha(g)}(\eta^{\beta(h)})) \star (\eta^{\beta(h)}) \star ({}^{\alpha(g)}(e_N \boxdot h)).$$

*Proof.* For all $g \in G$, $\eta \in N$ and $h \in H$ we obtain

$$(g \odot \eta)^{\beta(h)} = (\overline{g \odot \eta}) \star ((g \odot \eta) \boxdot h) \star \overline{(e_N \boxdot h)} = \overline{(g \odot \eta)} \star (g \odot (\eta \boxdot h)) \star \overline{(e_N \boxdot h)} =$$
$$\overline{(g \odot \eta)} \star (g \odot (\eta \star (\eta^{\beta(h)}) \star (e_N \boxdot h))) \star \overline{(e_N \boxdot h)} =$$
$$\overline{(g \odot \eta)} \star (g \odot \eta) \star \overline{(g \odot e_N)} \star (g \odot ((\eta^{\beta(h)}) \star (e_N \boxdot h))) \star \overline{(e_N \boxdot h)} =$$
$$\overline{(g \odot e_N)} \star (g \odot (\eta^{\beta(h)})) \star \overline{(g \odot e_N)} \star (g \odot (e_N \boxdot h)) \star \overline{(e_N \boxdot h)} =$$
$$({}^{\alpha(g)}(\eta^{\beta(h)})) \star (\eta^{\beta(h)}) \star ({}^{\alpha(g)}(e_N \boxdot h)).$$

□

**Proposition 3.3.** *Let $(G, \cdot, N, \star, \odot)$ be a left bracoid and $(H, \circ)$ be a group. Assume that*

(a) *$\boxdot$ is a transitive right action of $(H, \circ)$ on $N$ such that*

$$g \odot (\eta \boxdot h) = (g \odot \eta) \boxdot h$$

*for all $g \in G, h \in H$, $\eta \in N$,*

(b) ${}^{\alpha(g)}(\eta^{\beta(h)}) = ({}^{\alpha(g)}\eta)^{\beta(h)}$ *for all $g \in G, h \in H$ and $\eta \in N$.*

*Then for all $\eta \in N$ and $h \in H$ we have*

(1) $\overline{\eta}^{\beta(h)} = \overline{\eta^{\beta(h)}}$.

(2) $\overline{\eta} \boxdot h = (e_N \boxdot h)^2 \star \overline{\eta \boxdot h}$.

*Proof.* (1) Let $\eta \in N$ and $h \in H$. Since $\odot$ is a transitive action of $(G, \cdot)$ on $N$, there exists $g \in G$ such that $g \odot e_N = \eta$. Hence by Proposition 3.2 we have

$$({}^{\alpha(g)}\overline{\eta})^{\beta(h)} = (({}^{\alpha(g)}\overline{\eta}) \star \eta \star \overline{\eta})^{\beta(h)} = (({}^{\alpha(g)}\overline{\eta}) \star (g \odot e_N) \star \overline{\eta})^{\beta(h)} =$$
$$(g \odot \overline{\eta})^{\beta(h)} = ({}^{\alpha(g)}(\overline{\eta}^{\beta(h)})) \star (\overline{\eta}^{\beta(h)}) \star ({}^{\alpha(g)}(e_N \boxdot h)) =$$
$$({}^{\alpha(g)}(\overline{\eta}^{\beta(h)})) \star (\overline{\eta}^{\beta(h)}) \star (\overline{g \odot e_N}) \star (g \odot (e_N \boxdot h)) \star (\overline{e_N \boxdot h}) =$$
$$({}^{\alpha(g)}(\overline{\eta}^{\beta(h)})) \star (\overline{\eta}^{\beta(h)}) \star \overline{\eta} \star (\eta \boxdot h) \star (\overline{e_N \boxdot h}) = ({}^{\alpha(g)}(\overline{\eta}^{\beta(h)})) \star (\overline{\eta}^{\beta(h)}) \star (\eta^{\beta(h)}).$$

By $(b)$ we have

$$\overline{\eta}^{\beta(h)} = \overline{\eta^{\beta(h)}}$$

and this implies (2). □

**Theorem 3.4.** *Let $(G, \cdot, N, \star, \odot)$ be a left bracoid and $(H, \circ)$ be a group. Assume that*

(a) *$\boxdot$ is a transitive right action of $(H, \circ)$ on $N$ such that*

$$g \odot (\eta \boxdot h) = (g \odot \eta) \boxdot h$$

*for all $g \in G, h \in H$, $\eta \in N$,*

(b) ${}^{\alpha(g)}(\eta^{\beta(h)}) = ({}^{\alpha(g)}\eta)^{\beta(h)}$ *for all $g \in G, h \in H$ and $\eta \in N$.*

*Then $(G, \cdot, \odot, H, \circ, \boxdot, N, \star)$ is a two-sided bracoid.*

*Proof.* Let $g \in G, h \in H$ and $\eta \in N$. We have

$${}^{\alpha(g)}(\eta^{\beta(h)}) = {}^{\alpha(g)} (\overline{\eta} \star (\eta \boxdot h) \star \overline{(e_N \boxdot h)}) =$$
$$\overline{(g \odot e_N)} \star (g \odot (\overline{\eta} \star (\eta \boxdot h) \star \overline{(e_N \boxdot h)})) \star \overline{(\overline{\eta} \star (\eta \boxdot h) \star \overline{(e_N \boxdot h)})} =$$
$$\overline{(g \odot e_N)} \star (g \odot (\overline{\eta} \star (\eta \boxdot h) \star \overline{(e_N \boxdot h)})) \star \eta \star \overline{(\eta \boxdot h)} \star (e_N \boxdot h)$$

and

$$\begin{aligned}({}^{\alpha(g)}\eta)^{\beta(h)} &= (\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})^{\beta(h)} = \\ &\overline{(\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})}\star((\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})\boxdot h)\star\overline{(e_N\boxdot h)} = \\ &(g\odot e_N)\star\overline{(g\odot\eta)}\star\eta\star((\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})\boxdot h)\star\overline{(e_N\boxdot h)}\end{aligned}$$

By $(b)$ it follows that

$$\begin{aligned}&((\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})\boxdot h)\star\overline{(g\odot\eta)}\star(g\odot e_N)^2 = \\ &\qquad (g\odot(\overline{\eta}\star(\eta\boxdot h)\star\overline{(e_N\boxdot h)}))\star\overline{(\eta\boxdot h)}\star(e_N\boxdot h)^2.\end{aligned}$$

Hence by Lemma 2.1 and Proposition 3.3(2) it follows that

$$\begin{aligned}&((\overline{(g\odot e_N)}\star(g\odot\eta)\star\overline{\eta})\boxdot h)\star(g\odot\overline{\eta}) = \\ &\qquad (g\odot(\overline{\eta}\star(\eta\boxdot h)\star\overline{(e_N\boxdot h)}))\star(\overline{\eta}\boxdot h).\end{aligned}$$

Thus by (1.2)

$$\begin{aligned}&((g\odot(\eta\star(g^{-1}\odot\overline{\eta})))\boxdot h)\star(g\odot\overline{\eta}) = \\ &\qquad (g\odot(\overline{\eta}\star(\eta\boxdot h)\star\overline{(e_N\boxdot h)}))\star(\overline{\eta}\boxdot h)\end{aligned}$$

and by $(a)$ we have

$$\begin{aligned}&g^{-1}\odot((g\odot((\eta\star(g^{-1}\odot\overline{\eta}))\boxdot h))\star(g\odot\overline{\eta})) = \\ &\qquad g^{-1}\odot((g\odot(\overline{\eta}\star(\eta\boxdot h)\star\overline{(e_N\boxdot h)})))\star(\overline{\eta}\boxdot h)).\end{aligned}$$

By applying the action $\odot$ of the group $(G,\cdot)$ on $(N,\star)$ we have

$$\begin{aligned}&((\eta\star(g^{-1}\odot\overline{\eta}))\boxdot h)\star\overline{\eta}\star\overline{(g^{-1}\odot e_N)} = \\ &\qquad \overline{\eta}\star(\eta\boxdot h)\star\overline{(e_N\boxdot h)}\star(g^{-1}\odot(\overline{\eta}\boxdot h))\star\overline{(g^{-1}\odot e_N)}.\end{aligned}$$

Hence by $(a)$ we obtain

$$(\eta\star(g^{-1}\odot\overline{\eta}))\boxdot h = (\eta\boxdot h)\star\overline{(e_N\boxdot h)}\star((g^{-1}\odot\overline{\eta})\boxdot h).$$

Since $\odot$ is a transitive action of $(G,\cdot)$ on $(N,\star)$ for every $\mu\in N$ there exists $g\in G$ such that $g^{-1}\odot\overline{\eta}=\mu$. Therefore for every $\mu,\eta\in N$ and $h\in H$ we have

$$(\eta\star\mu)\boxdot h = (\eta\boxdot h)\star\overline{(e_N\boxdot h)}\star(\mu\boxdot h).$$

□

When a two-sided brace $(B,\star,\cdot)$ is viewed as a left bracoid we have

$$a^{\beta(b)} = a * b = {}^{\alpha(a)}b$$

for all $a,b\in B$. Hence Theorem 3.4 is a generalisation of the result of Lau.

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