# NONTRIVIAL MASSEY PRODUCTS ON COMPACT KÄHLER MANIFOLDS 

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#### Abstract

We show that the bigraded quasi-isomorphism type of the bigraded, bidifferential algebra of forms on a compact Kähler manifold generally contains more information than the de Rham cohomology algebra with its real Hodge structure. More precisely, on any closed Riemann surface of genus at least two, there is a nontrivial $A B C$-Massey product. Furthermore, starting from dimension three, there are simply connected projective manifolds with a nonzero $A B C$-Massey product of three divisor classes. In particular, compact Kähler manifolds are generally not formal in the sense of pluripotential homotopy theory.


## 1. Introduction

For manifolds, the quasi-isomorphism class of the commutative differential graded algebra (cdga) of differential forms provides a much finer algebraic invariant than the de Rham cohomology algebra. For instance, one may compute from it certain topological higher operations called Massey products and, under assumptions of finite type and nilpotent $\pi_{1}$, the ranks of the higher homotopy groups [16].

One of the early highlights of Sullivan's approach to rational homotopy theory is the theorem of [4] that any compact Kähler manifold is (de Rham) formal: The cdga of forms can be connected by a chain of quasi-isomorphisms to the de Rham cohomology algebra, considered as a cdga with trivial differential. In other words, the quasi-isomorphism class of the differential forms is encoded in the de Rham cohomology and hence the former does not contain more information than the latter. In particular, on compact Kähler manifolds all Massey products need to vanish.

For complex manifolds, the splitting $d=\partial+\bar{\partial}$ equips the complex-valued differential forms with the enriched structure of a commutative bigraded, bidifferential algebra (cbba). We consider cbbas up to bigraded, or pluripotential, quasi-isomorphisms, cf. [15], meaning maps of cbbas which induce an isomorphism in the Bott-Chern and Aeppli cohomologies

$$
H_{B C}=\frac{\operatorname{ker} \partial \cap \operatorname{ker} \bar{\partial}}{\operatorname{im} \partial \bar{\partial}} \quad \text { and } \quad H_{A}=\frac{\operatorname{ker} \partial \bar{\partial}}{\operatorname{im} \partial+\operatorname{im} \bar{\partial}} .
$$

This notion of quasi-isomorphism is in a certain sense universal [15, Thm C]. In particular, it is stronger than requiring the map to induce an isomorphism in de Rham or Dolbeault cohomology. A complex manifold is called strongly formal if its cbba of forms is connected by a chain of bigraded quasi-isomorphisms to a cbba with trivial differentials [10]. Hermitian symmetric spaces, all compact Kähler solvmanifolds, and all compact Kähler manifolds of dimension $\geq 2$ with cohomology of complete intersection type (the Kähler class generates all cohomology below middle degree) are strongly formal [10], [14], [15].

[^0]The goal of this paper is to show that, nevertheless, compact Kähler manifolds are generally not strongly formal. I.e., the bigraded quasi-isomorphism class of the cbba of forms contains more information than the de Rham cohomology algebra with its real Hodge structure. To prove this, we consider a secondary variant of Massey products, called Aeppli-Bott-Chern-Massey products (ABC-Massey products), introduced in [1]. They are defined like ordinary Massey products, replacing $d$ by $i \partial \bar{\partial}$. We recall their precise definition in Section 2. Schematically, they yield a partially defined map of bidegree $(-1,-1)$

$$
\langle\cdot, \cdot, \cdot\rangle_{A B C}: H_{B C}^{\otimes 3}(X)--H_{A}(X)
$$

For compact Kähler manifolds, the natural maps $H_{B C}(X) \rightarrow H_{d R}(X) \rightarrow H_{A}(X)$ are isomorphisms. Therefore, one might view $\langle\cdot, \cdot, \cdot\rangle_{A B C}$ as a partially defined operation of degree -2 in de Rham cohomology. As with de Rham formality and ordinary Massey products, on any strongly formal manifold, all $A B C$-Massey products are trivial [10].
Theorem A. On any closed Riemann surface of genus at least two there is a nontrivial ABCMassey product.

As a consequence, on any compact Kähler manifold with a surjective map to a curve of genus at least two there is a nontrivial $A B C$-Massey product, Corollary 3. On the other hand, simply connectedness and nilpotency play a large role in rational homotopy theory. It is thus natural to ask whether simply connected compact Kähler manifolds which are not strongly formal exist as well. Simply connected compact complex surfaces have cohomology of complete intersection type and hence are strongly formal. Therefore, the minimal dimension where this could occur is 3 . In that regard our main result is:
Theorem B. There is a finite sequence of blow-ups in points and lines of $\mathbb{C P}^{3}$ such that on the resulting space there is a nontrivial ABC-Massey product.

As a consequence of Theorem $B$, every complex manifold of dimension $\geq 4$ is bimeromorphic to one on which there is a nontrivial $A B C$-Massey product, Corollary 8.

The $A B C$-Massey product in Theorem A takes combinations of holomorphic and antiholomorphic 1-forms as input. In Theorem B, one takes divisor classes as inputs and the output is related to the cross ratio of four points on a line in $\mathbb{C P}^{3}$. In particular, in the latter example varying the positions of the centers of the blow-ups yields diffeomorphic manifolds which are, as complex manifolds, distinguished by their nontrivial $A B C$-Massey product, Corollary 9.

Related work. The nonformality of Kähler manifolds in pluripotential homotopy theory should be compared with the fact that in Dolbeault homotopy theory [11], where one considers forms up to Dolbeault quasi-isomorphism, compact Kähler manifolds are still formal, with the same proof as in [4]. The Dolbeault homotopy type can be obtained from the pluripotential homotopy type. On general compact complex manifolds there are many higher operations encoded in the pluripotential homotopy type which are, generally speaking, not encoded by Dolbeault or de Rham homotopy theory [6], [10], [17]. An important indication that such operations may also be interesting on compact Kähler manifolds was given by [13] who exhibited a (non Kähler) manifold satisfying the $\partial \bar{\partial}$-Lemma with a nontrivial $A B C$-Massey product. This is relevant as the proof of formality of compact Kähler manifolds in the de Rham and Dolbeault settings depends only on the $\partial \bar{\partial}$-Lemma. On the other hand, taking into account rational structures, [2] exhibit an example which shows that the rational Mixed Hodge Structure on homotopy groups is generally not a formal consequence of the rational Hodge Structure on the cohomology ring.

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## 2. Preliminaries and notation

2.1. Gradings. Given any bigraded vector space $H^{\bullet \bullet}$, a single superscript denotes the associated singly graded space with total grading $H^{\bullet}=\bigoplus_{p+q=\bullet} H^{p, q}$. When we talk about the total space, we omit the grading completely $H=\bigoplus_{p, q} H^{p, q}$ and for an element $h \in H$ we denote its (bi)degree by $|h|$.
2.2. Forms, currents and fundamental classes. For any manifold $X$, we write $A^{\bullet}(X)$ for its graded-commutative differential graded algebra of $\mathbb{C}$-valued smooth differential forms and $\mathcal{D}(X)$ for the space of currents, i.e. $\mathcal{D}^{k}(X)$ is the topological dual of the space of compactly supported ( $\operatorname{dim}(X)-k)$-forms. We write $d$ for the exterior differential and we will usually omit the wedge product $\alpha \wedge \beta=\alpha \beta$.

Let us now assume $X$ is a complex manifold. Then $A(X)$ (and $\mathcal{D}(X)$ ) carry a bigrading by type, refining the single grading and making it into a graded-commutative, bigraded, bidifferential algebra (resp. a bigraded, bidifferential module over $A_{X}$ ). We write $\partial$ and $\bar{\partial}$ for the components of bidegree $(1,0)$, respectively $(0,1)$, of the exterior derivative $d=\partial+\bar{\partial}$. For $X$ compact, the map $A(X) \rightarrow \mathcal{D}(X), \omega \mapsto \int(-1)^{|\omega|-1} \omega \wedge_{\text {_ }}$ is a bigraded quasi-isomorphism, see [12], [15, Lem. 1.27]. In particular one can (and we will) represent cohomology classes both by forms and by currents. As the most relevant example for us, any complex submanifold $Z \subseteq X$ of complex codimension $p$, or any formal linear combination of such, gives rise to a current of integration $[Z]:=\left(\left.\omega \mapsto \int_{Z} \omega\right|_{Z}\right) \in \mathcal{D}^{p, p}(X)$. This current is $\partial$ and $\bar{\partial}$-closed and therefore defines a class in $H_{B C}^{p, p}(X)$ which, by abuse of notation, we also denote by $[Z]$. It is called the fundamental class, or Thom class, of $Z$. If $Z$ is of complex codimension 1, i.e. a divisor, the Lelong-Poincaré formula yields form representatives for $[Z] \in H_{B C}(X)$ as follows: Let $\mathcal{O}(Z)$ be the associated line bundle, $s$ a meromorphic section of $\mathcal{O}(Z)$ with divisor of zero and poles $(s)=Z$ and $h$ a Hermitian metric on $\mathcal{O}(Z)$. Further, let $c_{1}^{h}(Z)$ be the form representative for $c_{1}(\mathcal{O}(Z)) \in H_{B C}^{1,1}(X)$ built as the curvature of the Chern connection with respect to $h$. Concretely, if locally $h=e^{-\varphi}\| \|^{2}$, then $c_{1}^{h}(Z)=\frac{i}{2 \pi} \partial \bar{\partial} \varphi$. Now $\log |s|_{h}$ is a locally integrable function and hence integration against it defines a current in $D^{0,0}(X)$. The Lelong-Poincaré formula is

$$
\begin{equation*}
\frac{i}{\pi} \partial \bar{\partial} \log |s|_{h}=[Z]-c_{1}^{h}(Z) \tag{1}
\end{equation*}
$$

We refer to [5] for further details.

### 2.3. Massey products.

Definition 1 ([8],[9]). Let $X$ be a manifold and $[\alpha] \in H_{d R}^{k_{1}}(X),[\beta] \in H_{d R}^{k_{2}}(X),[\gamma] \in H_{d R}^{k_{3}}(X)$ s.t. $[\alpha][\beta]=[\beta][\gamma]=0$. The triple Massey product of these classes is the class

$$
\langle\alpha, \beta, \gamma\rangle:=\left\langle(-1)^{|\alpha|} \alpha y-x \gamma\right\rangle \in H_{d R}^{k_{1}+k_{2}+k_{3}-1}(X) /\left([\alpha] H_{d R}(X)+H_{d R}(X)[\gamma]\right),
$$

where $\alpha \beta=d x$ and $\beta \gamma=d y$ for some $x \in A^{k_{1}+k_{2}-1}(X)$ and $y \in A^{k_{2}+k_{3}-1}(X)$.

Definition 2 ([1]). Let $X$ be a complex manifold and $[\alpha] \in H_{B C}^{k_{1}}(X),[\beta] \in H_{B C}^{k_{2}}(X),[\gamma] \in$ $H_{B C}^{k_{3}}(X)$ be s.t. $[\alpha][\beta]=[\beta][\gamma]=0$. The triple Aeppli-Bott-Chern Massey product of these classes is the class

$$
\langle\alpha, \beta, \gamma\rangle_{A B C}:=[\alpha y-x \gamma] \in H_{A}^{k_{1}+k_{2}+k_{3}-2}(X) /\left([\alpha] H_{A}(X)+H_{A}(X)[\gamma]\right),
$$

where $\alpha \beta=i \partial \bar{\partial} x$ and $\beta \gamma=i \partial \bar{\partial} y$ for some $x \in A^{k_{1}+k_{2}-2}$ and $y \in A^{k_{2}+k_{3}-2}$.
These definitions depend only on the cohomology classes and not on the choices of representatives or primitives made. Our conventions differ slightly from the references given in order to have less signs and be defined over the reals in the sense that if all three inputs are conjugation invariant classes, then so is the output. The $A B C$-Massey products are invariants of the (bigraded) quasi-isomorphism type of $A(X)$, [10, Proposition 4.4].

There are natural maps $H_{A} \rightarrow H_{B C}$ induced by $d^{c}=\frac{1}{2}(\bar{\partial}-\partial)$ as well as natural maps

$$
\begin{equation*}
H_{B C}(X) \rightarrow H_{d R}(X) \rightarrow H_{A}(X) \tag{2}
\end{equation*}
$$

induced by the identity. With these maps understood, the $A B C$-Massey products are secondary variants of the ordinary Massey products in the sense that one has the following relation in $H_{d R}(X)$ :

$$
\begin{equation*}
d^{c}\langle\alpha, \beta, \gamma\rangle_{A B C}=\langle\alpha, \beta, \gamma\rangle, \tag{3}
\end{equation*}
$$

up to a universal sign depending on one's convention for the ordinary triple Massey products.
One says that $X$ satisfies the $\partial \bar{\partial}$-Lemma if the natural maps (2) are isomorphisms. As is wellknown [4], compact Kähler (and hence projective) manifolds satisfy the $\partial \bar{\partial}$-Lemma. For this reason, we will mostly not distinguish these three cohomologies and instead simply write $H(X)$.

## 3. RiEmANN SURFACES

Let $\Sigma$ be any compact Riemann surface. Recall (see e.g. [7]) there is a positive definite sesquilinear form

$$
\begin{aligned}
\langle,\rangle_{D}: A^{1,0} \times A^{1,0} & \longrightarrow \mathbb{C} \\
\quad(\alpha, \beta) & \longmapsto\langle\alpha, \beta\rangle_{D}:=i \cdot \int_{\Sigma} \alpha \wedge \bar{\beta} .
\end{aligned}
$$

Proof of Theorem A. Let $\Sigma$ be a Riemann surface of genus at least two and let us fix some point $x \in \Sigma$. Let $\omega_{1}, \omega_{2} \in A^{1,0}(\Sigma)$ be two nonzero holomorphic one-forms such that $\left\langle\omega_{1}, \omega_{2}\right\rangle_{D}=$ $i \int_{\Sigma} \omega_{1} \wedge \bar{\omega}_{2}=0$. Then, there exists a unique function $F$ s.t. $F(x)=0$ and $i \partial \bar{\partial} F=\omega_{1} \wedge \bar{\omega}_{2}$. Note that locally $\omega_{i}$ has the form $f_{i} d z$ for some holomorphic function $f_{i}$ and therefore a local expression for $\omega_{1} \wedge \bar{\omega}_{2}$ is given by $f_{1} \bar{f}_{2} d z d \bar{z}$ which can not vanish identically by the identity principle. Now we distinguish two cases: First, let us assume that $\omega_{1} \wedge \bar{\omega}_{2}$ is real. In this case, also $F$ will be a real function, since $i \partial \bar{\partial}$ is a real operator. Then we claim that

$$
0 \neq\left\langle\omega_{1}, \omega_{1}, \bar{\omega}_{2}\right\rangle_{A B C}=\left[F \omega_{1}\right] \in H^{1,0}(\Sigma) /\left(\left[\omega_{1}\right]\right)
$$

To see this, we pair the representative with $\bar{\omega}_{2}$, which pairs trivially with the indeterminacy and therefore gives a well-defined element in top degree. Then we compute

$$
\int_{\Sigma} F \omega_{1} \wedge \bar{\omega}_{2}=i \int_{\Sigma} F \wedge \partial \bar{\partial} F=-i \int_{\Sigma} \partial F \wedge \bar{\partial} F=-i \int_{\Sigma} \partial F \wedge \overline{\partial F}=-\langle\partial F, \partial F\rangle_{D} \neq 0
$$

because $F$ is real and not constant.

Next, if $\omega_{1} \wedge \bar{\omega}_{2}$ has nontrivial imaginary part, set $\alpha_{i}=\omega_{i}+\bar{\omega}_{i}$. In this case, $\alpha_{1} \wedge \alpha_{2}$ has the local expression $2 i \operatorname{Im}\left(f_{1} \bar{f}_{2}\right) d z d \bar{z}$, which is nonzero by assumption and therefore there exists a nonconstant real function $u$ with $i \partial \bar{\partial} u=\alpha_{1} \wedge \alpha_{2}$ (in fact, up to a constant, we may take $\operatorname{Re}(F)$ ). Then, we have

$$
0 \neq\left\langle\alpha_{1}, \alpha_{1}, \alpha_{2}\right\rangle_{A B C}=\left[u \alpha_{1}\right] \in H^{1}(\Sigma) /\left(\left[\alpha_{1}\right],\left[\alpha_{2}\right]\right)
$$

In fact, since $\left[\alpha_{1} \wedge \alpha_{2}\right]=0=\left[\alpha_{2} \wedge \alpha_{2}\right]$ this follows as before from

$$
\int_{\Sigma} u \alpha_{1} \wedge \alpha_{2}=-\langle\partial u, \partial u\rangle_{D} \neq 0
$$

Corollary 3. Any compact Kähler manifold which admits a nonconstant holomorphic map to a closed Riemann surface of genus $g \geq 2$ admits a nontrivial ABC-Massey product.

Proof. Given such $\pi: Y \rightarrow X=\Sigma_{g \geq 2}$, the induced map in cohomology is injective [19]. Let $\mathfrak{m}=\langle\alpha, \beta, \gamma\rangle_{A B C} \in H^{1}(X) /([\alpha],[\gamma])$ be a nonzero Massey product. Then, we have $\pi^{*} \mathfrak{m}=$ $\left\langle\pi^{*} \alpha, \pi^{*} \beta, \pi^{*} \gamma\right\rangle_{A B C} \neq 0 \in H^{1}(Y) /\left(\pi^{*}[\alpha], \pi^{*}[\gamma]\right)$, where we use that in degree 1 the ideal generated by $\left[\pi^{*} \alpha\right],\left[\pi^{*} \gamma\right]$ equals the image under $\pi^{*}$ of the ideal generated by $[\alpha]$ and $[\gamma]$, i.e. the relevant indeterminacy in $Y$ is not bigger than in $X$.

Remark 4. The property of admitting a fibration over a curve of genus $g \geq 2$ depends only on the de Rham cohomology ring [3]. So Corollary 3 can be rephrased as follows: No strongly formal Kähler structure can exist on a manifold $X$ which admits an isotropic subspace $U \subset H^{1}(X)$ with $\operatorname{dim}(U) \geq 2$.

## 4. A SIMPLY-CONNECTED EXAMPLE

Let $P, Q, R \in \mathbb{C P}^{3}$ and let $C$ be the line through $P, Q$. Choose distinct points $S_{i} \in C \backslash\{P, Q\}$ for $i=1,2$ and let $L_{i}$ be the line through $R$ and $S_{i}$. Now let $X$ be the space obtained from $\mathbb{C P}^{3}$ by fist blowing up $P, Q, R$, then blowing up the strict transform of $C$, and finally blowing up the strict transforms of the $L_{i}$. We denote by $E_{P}, E_{Q}, \ldots$ the (strict transforms of the) respective exceptional divisors in $X$. Let furthermore $A_{i} \subset X$ be the strict transform of a hyperplane containing $L_{i}$ and intersecting $C$ transversally.


Recall the definition of the cross-ratio of four distinct points $P_{i} \in \mathbb{C P}^{1}=\mathbb{C} \cup\{\infty\}$ :

$$
\begin{equation*}
\chi\left(P_{1}, P_{2}, P_{3}, P_{4}\right):=\frac{\left(P_{3}-P_{1}\right)\left(P_{4}-P_{2}\right)}{\left(P_{3}-P_{2}\right)\left(P_{4}-P_{1}\right)} \tag{4}
\end{equation*}
$$

where one applies the naive calculation rules in case one of the points is $\infty$, e.g. $0=\frac{1}{\infty}$. This quantity is invariant under projective transformations and hence well defined for any 4 -tuple of distinct points on a line. The main objective of this section is to prove
Theorem 5. The map

$$
\begin{aligned}
H^{2,2}(X) & \longrightarrow \mathbb{C} \\
x & \longmapsto \int_{A_{1}-A_{2}} x
\end{aligned}
$$

vanishes on the indeterminacy of the Massey product

$$
\mathfrak{m}=\left\langle\left[E_{L_{1}}+E_{L_{2}}+E_{R}\right],\left[E_{P}-E_{Q}\right],\left[E_{C}-E_{R}\right]\right\rangle_{A B C}
$$

and

$$
\int_{A_{1}-A_{2}} \mathfrak{m}=\frac{\log \left|\chi\left(P, Q, S_{1}, S_{2}\right)\right|}{\pi}
$$

Let us first check that the Massey product $\mathfrak{m}$ is indeed defined. The divisors $E_{L_{1}}+E_{L_{2}}+E_{R}$ and $E_{P}-E_{Q}$ are disjoint so the product on the left hand side of the Massey product is indeed 0 . On the right hand side $\left[E_{P}-E_{Q}\right] \cdot\left[E_{C}-E_{R}\right]=\left[E_{C} \cap E_{P}-E_{C} \cap E_{Q}\right]$, which is the trivial class since all fibres of $E_{C} \rightarrow C$ are rationally, and hence homologically, equivalent. To exhibit an explicit primitive, let us identify $C \cong \mathbb{C P}^{1}=\mathbb{C} \cup\{\infty\}$ in such a way that $P=0$ and $Q=\infty$. We may then view the projection $f: E_{C} \rightarrow C$ as a meromorphic function. Thus, by (1) applied to $f$ and the trivial line bundle, we have

$$
\left[E_{C} \cap E_{P}-E_{C} \cap E_{Q}\right]=\frac{i}{\pi} \partial \bar{\partial} \log |f|
$$

as currents on $E_{C}$ and the push-forward along the inclusion $j: E_{C} \rightarrow X$ yields the desired $i \partial \bar{\partial}$-primitive $p:=j_{*}\left(\frac{1}{\pi} \log |f|\right) \in \mathcal{D}^{1,1}(X)$ certifying triviality of the product of the right hand cohomology classes in $\mathfrak{m}$.

Lemma 6. Let $F_{i}=E_{C} \cap E_{L_{i}} \subset X$. With respect to the above coordinates on $C$ one has $\frac{1}{\pi}\left(\log \left|S_{1}\right|\left[F_{1}\right]+\log \left|S_{2}\right|\left[F_{2}\right]\right) \in \mathfrak{m}$.

Proof. We set $D_{1}=E_{L_{1}}+E_{L_{2}}+E_{R}, D_{2}=E_{P}-E_{Q}$, and $D_{3}=E_{C}-E_{R}$. In order to compute $\mathfrak{m}$ we choose a defining system of $\mathfrak{m}$ in $A(X)$. Let $\mathcal{O}\left(D_{i}\right)$ be the line bundle with section $s_{i}$ corresponding to the divisor $D_{i}$. Then for any Hermitian metric $h_{i}$ on $\mathcal{O}\left(D_{i}\right)$ we obtain the Chern form $\gamma_{i}:=c_{1}^{h_{i}}\left(D_{i}\right)$.

Recall from Equation (1) that

$$
\frac{i}{\pi} \partial \bar{\partial} \log \left|s_{i}\right|_{h_{i}}=\left[D_{i}\right]-\gamma_{i}
$$

so we may take $\gamma_{i}$ as a form representative for $\left[D_{i}\right]$. Since $\mathcal{O}\left(D_{i}\right)$ is trivial outside $D_{i}$, one may always pick $h_{i}$ such that $\left|s_{i}\right| \equiv 1$ except on some small neighborhood of $D_{i}$. Hence we may choose $\log \left|s_{i}\right|_{h_{i}}$ and in particular $\gamma_{i}$ to be supported near $D_{i}$.

Due to the supports we have $\gamma_{1} \gamma_{2}=0$ and may choose 0 as a $i \partial \bar{\partial}$-primitive for the left hand side of the defining system of $\mathfrak{m}$. Let $p$ be the previously constructed $i \partial \bar{\partial}$-primitive for $\left[D_{2} \cap D_{3}\right]$. Then we define

$$
q:=p-\frac{1}{\pi} \log \left|s_{2}\right|_{h_{2}}\left[D_{3}\right]-\frac{1}{\pi} \log \left|s_{3}\right|_{h_{3}} \gamma_{2} \in \mathcal{D}^{1,1}(X)
$$

where we note that the middle term is a well-defined current since $\log \left|s_{2}\right|_{h_{2}}$ restricts to a locally integrable function on $D_{3}$. We have $i \partial \bar{\partial} q=\gamma_{2} \gamma_{3}$. Now we choose another $i \partial \bar{\partial}$-primitive $r \in$ $A^{1,1}(X)$ for $\gamma_{2} \gamma_{3}$ in such a way that $r-q$ defines a trivial class in $H_{A}(X)$. This completes the construction of a defining system of $\mathfrak{m}$ which is represented by $\left[\gamma_{1} r\right] \in H_{A}^{2,2}(X)$. If one allows current representatives of cohomology classes, one in fact has

$$
\left[\gamma_{1} r\right]=\left[\gamma_{1} q\right]=\left[\gamma_{1} p\right]
$$

where the first equality is due to $[r-q]=0$ and for the second equality we use that $p-q$ is supported away from $D_{1}$. Now by the definition of $p$ we have $\gamma_{1} p=j_{*}\left(\left.\frac{1}{\pi} \log |f| \cdot \gamma_{1}\right|_{E_{C}}\right)$. Note that $\left.\log |f| \cdot \gamma_{1}\right|_{E_{C}}$ is supported inside a small open neighbourhood $U \subset E_{C}$ of $D_{1} \cap E_{C}$ for which $\left[\left.\gamma_{1}\right|_{U}\right]$ and $\left[D_{1} \cap E_{C}\right]$ agree in the Bott-Chern cohomology $H_{B C, c p t}(U)$ with compact support. Furthermore the restriction of $\log |f|$ to $U$ defines an element in $H_{A}(U)$ and, using that the pairing

$$
H_{A}(U) \otimes H_{B C, c p t}(U) \rightarrow H_{A, c p t}(U)
$$

is well-defined, we find that $\left[\left.\log |f| \gamma_{1}\right|_{E_{C}}\right]$ and $\left[\log |f|\left[D_{1} \cap E_{C}\right]\right]$ agree in $H_{A, c p t}(U)$. But then so do their images in $H_{A}\left(E_{C}\right)$ under the pushforward of the inclusion $U \rightarrow E_{C}$. This shows that $\mathfrak{m}$ is represented by $\frac{1}{\pi} \log |f|\left[D_{1} \cap E_{C}\right]=\frac{1}{\pi} \log |f|\left[F_{1}+F_{2}\right]$. By its definition $f$ is constant on $F_{i}$ and takes the value $S_{i}$ with respect to the initial identification $C=\mathbb{C} \cup\{\infty\}$.

To deal with the indeterminacy of $\mathfrak{m}$, we will now compute the multiplicative structure of $H(X)$. Recall the formulae for the blow-up of a submanifold $Z \subseteq Y$ of codimension $r$ :

$$
H\left(B l_{Z} Y\right) \cong H(Y) \oplus \bigoplus_{i=1}^{r-1} H(Z) E_{Z}^{i}
$$

The algebra structure on the right is given as follows (see e.g. [18, Thm. 7.31]): $H(Y)$ is a subring, $H(Z)$ is an $H(Y)$-module via restriction, and $E_{Z}^{r}=(-1)^{r-1}[Z]+\sum_{i=1}^{r-1}(-1)^{r-i+1} c_{i}\left(\mathcal{N}_{Z}\right) E_{Z}^{r-i}$, where $\mathcal{N}_{Z}$ is the normal bundle. The isomorphism from right to left is given by pullback on $H(Y)$, sends the symbol $E_{Z}$ to $\left[E_{Z}\right]$, and is a map of $H(Y)$-modules on $H(Z)$.

It follows that $H^{1,1}(X)$ is generated by the pullbacks of the classes of a generic plane $G$ in $\mathbb{C P}^{3}$ as well as the exceptional divisors $E_{P}, E_{Q}, E_{R}, E_{C}, E_{L_{1}}$, and $E_{L_{2}}$. Note that the pullback of the Thom class of a divisor along a blow down is the Thom class of its strict transform. For ease of notation, in the following we omit square brackets around these divisors when indicating the corresponding cohomology class in $H(X)$. One obtains that a basis for $H^{2,2}(X)$ is given by $G^{2}, E_{P}^{2}, E_{Q}^{2}, E_{R}^{2}$, $G E_{C}, G E_{L_{1}}$, and $G E_{L_{2}}$, where we have used that $G$ restricts to a generator of $H^{1,1}(C), H^{1,1}\left(L_{1}\right)$, and $H^{1,1}\left(L_{2}\right)$. The following table displays the map $H^{1,1}(X) \otimes H^{1,1}(X) \rightarrow H^{2,2}(X)$ in the chosen bases.

The off-diagonal entries can be computed via the transverse intersections of the divisors and the fact that all fibers in the blowup of a line are rationally equivalent. Regarding the diagonal entries, only $E_{C}^{2}, E_{L_{1}}^{2}$, and $E_{L_{2}}^{2}$ need justification. We provide the computation for $E_{C}^{2}$. The computations for $E_{L_{i}}$ are similar and left to the reader. In the above formula for the blowup cohomology we get $E_{C}^{2}=-C+c_{1}\left(\mathcal{N}_{C}\right) E_{C}$, where the right hand side is to be understood in the blow-up of

|  | $G$ | $E_{P}$ | $E_{Q}$ | $E_{R}$ | $E_{C}$ | $E_{L_{1}}$ | $E_{L_{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $G$ | $G^{2}$ | 0 | 0 | 0 | $G E_{C}$ | $G E_{L_{1}}$ | $G E_{L_{2}}$ |
| $E_{P}$ | 0 | $E_{P}^{2}$ | 0 | 0 | $G E_{C}$ | 0 | 0 |
| $E_{Q}$ | 0 | 0 | $E_{Q}^{2}$ | 0 | $G E_{C}$ | 0 | 0 |
| $E_{R}$ | 0 | 0 | 0 | $E_{R}^{2}$ | 0 | $G E_{L_{1}}$ | $G E_{L_{2}}$ |
| $E_{C}$ | $G E_{C}$ | $G E_{C}$ | $G E_{C}$ | 0 | $-G^{2}-E_{P}^{2}-E_{Q}^{2}$ | $G E_{L_{1}}$ | $G E_{L_{2}}$ |
| $E_{L_{1}}$ | $G E_{L_{1}}$ | 0 | 0 | $G E_{L_{1}}$ | $G E_{L_{1}}$ | $-G E_{L_{1}}$ | 0 |
| $E_{L_{2}}$ | $G E_{L_{2}}$ | 0 | 0 | $G E_{L_{2}}$ | $G E_{L_{2}}$ | $E_{R}^{2}+G E_{C}$ | 0 |

Figure 1. Multiplication table for $H(X)$.
$\mathbb{C P}^{3}$ at the points $P, Q, R$, say $X_{1}$, where we are about to blow up $C$. We determine the class $C$ in $H^{2,2}\left(X_{1}\right)$ by pairing it geometrically with the basis $G, E_{P}, E_{Q}, E_{R}$ of $H^{1,1}\left(X_{1}\right)$. We find that $C=G^{2}+E_{P}^{2}+E_{Q}^{2}$. We further compute

$$
\int_{C} c_{1}\left(\mathcal{N}_{C}\right)=\int_{C}\left(\left.c_{1}\left(X_{1}\right)\right|_{C}-c_{1}(C)\right)=\left.\int_{C}\left(4 G-2 E_{P}-2 E_{Q}-2 E_{R}\right)\right|_{C}-2=-2 .
$$

Lemma 7. Let $I \subset H^{*, *}(X)$ be the submodule generated by $E_{C}-E_{R}$ and $E_{R}+E_{L_{1}}+E_{L_{2}}$. Then for any representative of an element of $I^{2,2}$ the integrals over $A_{1}$ and $A_{2}$ agree.

Proof. From the multiplication table above one computes that

$$
I^{2,2}=\left\langle G E_{C}, E_{R}^{2}, E_{P}^{2}+E_{Q}^{2}, G\left(E_{L_{1}}+E_{L_{2}}\right), G^{2}\right\rangle
$$

All intersection numbers between the $A_{i}$ and the generating cycles of $I^{2,2}$ can be computed directly via transverse intersections and agree for $i=1,2$.

Proof of Theorem 5. By Lemma 7, multiplying with $\left[A_{1}-A_{2}\right]$ sends the indeterminacy ideal $I$ to 0 . From the computation in Lemma 6 and since $A_{i} \cap F_{i}=\{p t\}$, we deduce that

$$
\pi \cdot \int_{A_{1}-A_{2}} \mathfrak{m}=\log \left|S_{1}\right|-\log \left|S_{2}\right|=\log \left|\frac{S_{1}}{S_{2}}\right| .
$$

As $P=0$ and $Q=\infty$ in the chosen coordinate on $C$, this is indeed $\log \left|\chi\left(P, Q, S_{1}, S_{2}\right)\right|$.
As an immediate consequence of Theorem 5 we obtain
Corollary 8. For any complex manifold $Y$ of dimension at least 4 there is a finite sequence of blowups of points and lines such that the resulting manifold $\widetilde{Y}$ admits a non trivial triple $A B C$-Massey product and is thus in particular not strongly formal.

Proof. Let $Y$ be a complex manifold of dimension $n \geq 4$. After possibly blowing up a point in $Y$, we can assume the existence of an embedding $\mathbb{C P}^{3} \rightarrow Y$. Now choose a configuration of points $P, Q, R$ and lines $C, L_{i}$ inside $\mathbb{C P}^{3}$ and blow up $Y$ along these submanifolds as in the construction of $X$. The Massey product

$$
\left\langle\left[E_{L_{1}}+E_{L_{2}}+E_{R}\right],\left[E_{P}-E_{Q}\right],\left[E_{C}-E_{R}\right]\right\rangle_{A B C}
$$

is defined in the resulting space $\widetilde{Y}$. Furthermore there is an embedding $X \rightarrow \widetilde{Y}$ given by the strict transform of $\mathbb{C P}^{3}$. Cohomologically it maps the class of the (strict transform of the) exceptional divisors to their respective intersections with $X$, which are the (strict transforms of the) exceptional divisors used to define $\mathfrak{m}$ in Theorem 5. Hence there is an $A B C$-Massey product $\widetilde{\mathfrak{m}} \subset H_{A}(\widetilde{Y})$ which pulls back to a subset of $\mathfrak{m} \subset H_{A}(X)$ and is thus nontrivial for suitable choices in the construction.

Corollary 9. For uncountably many choices of $P, Q, R, S_{1}, S_{2}$, the resulting manifolds $X$ as in Theorem 5 are pairwise non-biholomorphic, and distinguished by the ABC-Massey product.

Proof. Consider a biholomorphism $\varphi: X^{\prime} \rightarrow X$ between two manifolds corresponding to different choices, say $X=X\left(P, Q, R, S_{1}, S_{2}\right)$ and $X^{\prime}=X\left(P^{\prime}, Q^{\prime}, R^{\prime}, S_{1}^{\prime}, S_{2}^{\prime}\right)$, and the corresponding pullback $H(X) \rightarrow H\left(X^{\prime}\right)$. In $X^{\prime}$, there is at most a countable collection of Massey products with integral $(1,1)$-classes as entries and correspondingly only a countable collection of values when integrating these Massey products over integral cycles annihilating the indeterminacy. Thus, for any such $\varphi, \int_{A_{1}-A_{2}} \mathfrak{m}=\int_{\varphi^{-1}\left(A_{1}-A_{2}\right)} \varphi^{*} \mathfrak{m}$ must be among these values.

Remark 10. One may be more precise and show that the automorphism group of $H(X)$ is generated by the involutions exchanging $E_{P}$ and $E_{Q}$, resp. $E_{L_{1}}$ and $E_{L_{2}}$. Thus, whenever the values $|\log | \chi\left(P, Q, S_{1}, S_{2}\right)|\mid$ and $| \log \left|\chi\left(P^{\prime}, Q^{\prime}, S_{1}^{\prime}, S_{2}^{\prime}\right)\right| \mid$ differ, the manifolds $X$ and $X^{\prime}$ are not biholomorphic.

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