L^p-ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE FRACTIONAL HEAT EQUATION ON RIEMANNIAN SYMMETRIC SPACES OF NONCOMPACT TYPE

MUNA NAIK, SWAGATO K. RAY, AND JAYANTA SARKAR

ABSTRACT. The main goal of this paper is to study the L^p-asymptotic behaviour of solutions to the heat equation and the fractional heat equations on a Riemannian symmetric space of non-compact type. For $\alpha \in (0,1]$, let h_t^α denotes the fractional heat kernel on a Riemannian symmetric space of non-compact type $X = G/K$ (with h_t^1 being the standard heat kernel of X). Suppose $f \in L^1(X)$ is a left K-invariant function. We show that if $p \in [1,2]$ then

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p = 0,$$

where $\gamma_p = 2/p - 1$, ρ is the half sum of positive roots and $\widehat{f}$ denotes the spherical Fourier transform of f. If $p \in (2,\infty]$ then

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - \widehat{f}(0)h_t^\alpha\|_p = 0.$$

In fact, we will show that the above results are valid for more general class of left K-invariant functions larger than the space of integrable left K-invariant functions on X. The above statements for symmetric spaces were earlier studied in the important special case $p = 1 = \alpha$, in [16, 1].

1. INTRODUCTION AND MAIN RESULT

For all unexplained notation of this section we refer to Section 2. This paper mainly deals with generalizations of two results, proved initially for $L^p(\mathbb{R}^n)$. The first such result was proved in [13, Proposition 3.10]: if for $r > 0$, $m_r(x) = r^{-n}\chi_{B(0,r)}(x)$ then for any complex Borel measure μ,

$$\lim_{r\to\infty} \|\mu * m_r - \mu(\mathbb{R}^n)m_r\| = 0, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 43A85; Secondary 22E30.

Key words and phrases. symmetric space, heat kernel, fractional heat kernel.

where $B(0,r)=\{x\in\mathbb{R}^n \mid \|x\|<r\}$, the n-dimensional Lebesgue measure of $B(0,1)$ is normalized and $\|\cdot\|$ denotes the total variation norm. The second result was proved in [14, Theorem 2.1] (see also [15]): if $f\in L^1(\mathbb{R}^n)$ then for all $p\in[1,\infty]$,

$$\lim_{t\to\infty}\frac{\|f*h_t-M(f)h_t\|_p}{\|h_t\|_p}=0, \tag{1.2}$$

where

$$M(f)=\int_{\mathbb{R}^n}f(x)dx,\quad h_t(x)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4t}},\ \ x\in\mathbb{R}^n,$$

is the standard heat kernel of $\mathbb{R}^n$. We note that using the Fourier transform on $\mathbb{R}^n$ we can write $M(f)=\widehat{f}(0)$ and $\mu(\mathbb{R}^n)=\widehat{\mu}(0)$. We consider a Riemannian symmetric spaces $X=G/K$ of noncompact type where G is a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup. We first show by a counter example that the exact analogue of (1.1) is false in this case (see Subsection 5.2). However, the main results of this paper pertains to analogues of (1.2). In recent times several generalizations and variants of (1.2) have been proved in [16, 1, 6, 7]. We will mostly concentrate on the papers by Vázquez [16] which proves analogue of (1.2) on real hyperbolic spaces and by Anker et al. [1] which proves analogue of (1.2) for all Riemannian symmetric spaces of noncompact type. Precisely, it was proved that if $f\in L^1(G/K)$ is left K-invariant and $M(f)$ denotes the integral of f then

$$\lim_{t\to\infty}\|f*h_t-M(f)h_t\|_1=0, \tag{1.3}$$

where h_t is the heat kernel for $X=G/K$. This is a deep result as the proof deviates completely from the Euclidean case. Vázquez's argument in [16] uses L^1 concentration of h_t for real hyperbolic spaces which were later generalized in [1]. But in [1] another proof of this result was given which uses the spherical Fourier transform in a fundamental way. This Fourier analytic proof from [1] is of fundamental importance for us. In [16, 1] it has been explained in great detail why analogue of (1.2) cannot be true for functions f defined on $X=G/K$ which are not left K-invariant. In this paper, we will therefore restrict our discussion mostly to functions $f:G\to\mathbb{C}$ which are K-bi-invariant. In the above mentioned papers it has also been shown that the exact analogue of (1.2) breaks down for $p=\infty$. However, we will see that by merely changing the constant $M(f)$ one can prove a precise analogue of (1.2) for $p=\infty$ also. Our main interest in this paper lies in understanding the exact analogue of (1.2) for $p\in(1,\infty]$. Both the papers [16, 1] contain a weak analogue of (1.2) for the case $p=\infty$ and then use standard convexity argument to obtain an L^p-version of the following kind: for all $p \in (1, \infty)$,

$$\lim_{t\to\infty} \frac{\|f * h_t - M(f)h_t\|_p}{v_p(t)} = 0, \tag{1.4}$$

where $v_p : (0, \infty) \to (0, \infty)$ (given explicitly in [16, 1]) satisfies the following property:

$$\lim_{t\to\infty} \frac{\|h_t\|_p}{v_p(t)} = 0. \tag{1.5}$$

As $\lim_{t\to\infty} \frac{\|h_t\|_p}{v_p(t)} = 0$, the relation (1.4) follows by using Young's inequality. In fact, the relation (1.4) holds true even if one replaces $M(f)$ by any $z \in \mathbb{C}$ and in this case the function f does not need to be left K-invariant. Therefore, we feel that for $p > 1$, the result (1.4) is not an exact analogue of (1.3). As in the Euclidean case, to obtain an L^p-version of (1.3) it is therefore desirable to be able to replace the quantity $v_p(t)$ by $\|h_t\|_p$. We show that this can be done for $p \in (1, \infty]$ but then one has to allow the number $M(f)$ to depend on p. In fact, for $p = 2$, simply by applying the Plancherel theorem one can see that for all left K-invariant $f \in L^1(G/K)$ (see Proposition 3.1)

$$\lim_{t\to\infty} \frac{\|f * h_t - \widehat{f}(0)h_t\|_2}{\|h_t\|_2} = 0. \tag{1.6}$$

It is also important for us to note that using the spherical Fourier transform we can write $M(f) = \widehat{f}(i\rho)$, where ρ is half sum of positive roots. The equations (1.6) and (1.3) together motivates us to try to prove an L^p-version of (1.3) with $\widehat{f}(i\rho)$ replaced by the complex number $\widehat{f}(i\gamma_p\rho)$, the spherical Fourier transform of f at $i\gamma_p\rho$, where $\gamma_p := \left(\frac{2}{p} - 1\right)$, $1 \le p \le 2$. We note that $\gamma_2 = 0$.

We will prove an L^p-version of (1.3) for $p \in (1, \infty]$ with $M(f)$ replaced by $\widehat{f}(i\gamma_p\rho)$ when $p \in (1, 2]$ and by $\widehat{f}(0)$ when $p \in (2, \infty]$. Indeed, we will prove an L^p-analogue of (1.3) for a larger class of left K-invariant functions containing the class of left K-invariant L^1-function on X. In this regard, for $p \in [1, 2]$, we define

$$\mathcal{L}^p(G//K) := \{f : G \to \mathbb{C} \text{ measurable and } K\text{-bi-invariant} \mid \widehat{|f|}(i\gamma_p\rho) < \infty\}.$$

The first main result of this paper is the following:

Theorem 1.1. *a) If $p \in [1, 2]$ and $f \in \mathcal{L}^p(G//K)$ then*

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(i\gamma_p\rho)h_t\|_{L^p(G)} = 0. \tag{1.7}$$

Moreover for any $z \in \mathbb{C}$,

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - z\, h_t\|_{L^p(G)} = 0,$$

if and only if $z = \widehat{f}(i\gamma_p\rho)$.

b) If $p \in (2, \infty]$ and $f \in \mathcal{L}^2(G//K)$ then

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(0) h_t\|_{L^p(G)} = 0. \tag{1.8}$$

Moreover for any $z \in \mathbb{C}$,

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - z\, h_t\|_{L^p(G)} = 0,$$

if and only if $z = \widehat{f}(0)$.

The proof of Theorem 1.1 for the case $p \in [1, 2)$ depends heavily on the first proof of [1, Theorem 1.2] along with the L^p-concentration of the heat kernel h_t on $X = G/K$ proved in [2, Theorem 4.1.2]. At this point, one may wonder why the constant $\widehat{f}(0)$ works for all $p \in [2, \infty]$ in Theorem 1.1. This can be attributed to the Kunze-Stein phenomenon and the fact that for all $p \in (2, \infty]$ and for all large t, $\|h_t\|_p$ is comparable to $\|h_{t/2}\|_2^2$ (see (2.24)). Using Theorem 1.1 we also prove an exact analogue of Theorem 1.1 for the fractional heat kernels h_t^α, $\alpha \in (0, 1)$. More precisely, we prove the following result.

Theorem 1.2. *a) If $p \in [1, 2]$ and $f \in \mathcal{L}^p(G//K)$ then*

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho) h_t^\alpha\|_{L^p(G)} = 0. \tag{1.9}$$

Moreover for any $z \in \mathbb{C}$,

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - z\, h_t^\alpha\|_{L^p(G)} = 0,$$

if and only if $z = \widehat{f}(i\gamma_p\rho)$.

b) If $p \in (2, \infty]$ and $f \in \mathcal{L}^2(G//K)$ then

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - \widehat{f}(0) h_t^\alpha\|_{L^p(G)} = 0. \tag{1.10}$$

Moreover for any $z \in \mathbb{C}$,

$$\lim_{t\to\infty} \|h_t^\alpha\|_p^{-1} \|f * h_t^\alpha - z\, h_t^\alpha\|_{L^p(G)} = 0,$$

if and only if $z = \widehat{f}(0)$.

The organization of this paper is as follows. In Section 2, we establish notations and all required preliminaries are gathered. Section 3 mainly deals with results related to the heat kernel where we prove Theorem 1.1. We give the proof of Theorem 1.2 in Section 4. In section 5, we discuss optimality of some of our results and produce the counter example for ball average for rank one symmetric space of non-compact type which we have mentioned in the beginning.

2. Preliminaries

In this section we shall establish notation and collect all the ingredients required for this paper.

2.1. **Basic Notation.** The letters $\mathbb{R}$, $\mathbb{R}^\times$, $\mathbb{R}^+$, $\mathbb{C}$ and $\mathbb{N}$ denote respectively the set of real numbers, nonzero real numbers, positive real numbers, complex numbers and natural numbers. For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote respectively the real and imaginary parts of z. For a real number a, let a_+ denote the non-negative part of a defined by $a_+ := \max\{a, 0\}$. For $1 < p < \infty$, let $p' := \frac{p}{p-1}$ be the conjugate exponent of p. For $p = 1$, we define $p' = \infty$ and for $p = \infty$, we define $p' = 1$. For $1 \leq p < \infty$, let $\gamma_p := \frac{2}{p} - 1$ and $\gamma_\infty := -1$. Everywhere in this article the symbol $f_1 \asymp f_2$ for two positive expressions f_1 and f_2 means that there are positive constants C_1, C_2 such that $C_1 f_1 \leq f_2 \leq C_2 f_1$. Similarly the symbol $f_1 \lesssim f_2$ means that there exists a positive constant C such that $f_1 \leq C f_2$. We write C_ε for a constant to indicate its dependency on the parameter ε. For a set A in a topological space, $\overline{A}$ is its closure and for a set A in a measure space, $|A|$ denotes its measure. For a topological space E, let $C(E)$ and $C_c(E)$ denote the space of all continuous functions and compactly supported continuous functions on E respectively.

2.2. **Symmetric space.** Let X be a Riemannian symmetric space of non-compact type. That is $X = G/K$ where G is a noncompact connected semisimple Lie group with finite center and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebra of G and K respectively. Let Θ be the corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let ℓ denote the dimension of $\mathfrak{a}$, called as the rank of X as well as the real rank of G. Let Σ denote the the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and W be its Weyl group. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ be the root space corresponding to the root α and m_α be the dimension of $\mathfrak{g}_\alpha$. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let Σ^+ denote the corresponding set of positive roots and and $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Sigma^+} m_\alpha \alpha$. Let Σ_r^+ and Σ_s^+ denote the set of positive indivisible roots and positive simple roots respectively. Let $n = \ell + \sum_{\alpha\in\Sigma^+} m_\alpha$ and $\nu = \ell + 2|\Sigma_r^+|$. The constant n is equal to dimension of the manifold X and the constant ν will be referred to the dimension at infinity.

Let $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ be the nilpotent Lie algebra spanned by the root subspaces for the positive restricted roots. Let A and N denote the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively, precisely $A = \exp(\mathfrak{a})$ and $N = \exp(\mathfrak{n})$. For $a \in A$, let $\log(a)$ be the element of $\mathfrak{a}$ such that $\exp(\log a) = a$. Let M be the centralizer of A in K. The groups M and A normalize N. We note that K/M is the Furstenberg boundary of X. For the group G, we have the Iwasawa decomposition $G = NAK$, that is, every $g \in G$ can be uniquely written as

$$g = n(g) \exp A(g) k(g), \quad k(g) \in K, A(g) \in \mathfrak{a}, n(g) \in N,$$

and the map $(n, a, k) \mapsto nak$ is a diffeomorphism of $N \times A \times K$ onto G. Via Cartan decomposition i.e. $G = K(\exp \overline{\mathfrak{a}^+})K$, each $g \in G$ can be written as

$$g = k_1(\exp H)k_2, \tag{2.1}$$

for some $k_1, k_2 \in K$ and unique $H \in \overline{\mathfrak{a}_+}$. Let g^+ denote the middle component of g in the Cartan decomposition i.e. H in the decomposition (2.1).

Let B denote the Killing form of $\mathfrak{g}$. It is known that $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and hence induces an inner product and a norm $|\cdot|$ on $\mathfrak{p}$. Since the tangent space of the homogeneous space $X = G/K$ at the base point $o = eK$ can be identified with $\mathfrak{p}$, it has a G-invariant Riemannian structure induced from the form $B|_{\mathfrak{p}\times\mathfrak{p}}$. Let Δ be the Laplace–Beltrami operator on X associated to this Riemannian structure. Using nondegenracy of the positive bilinear form $B|_{\mathfrak{a}\times\mathfrak{a}}$, we identify $\mathfrak{a}$ with $\mathfrak{a}^*$ and using this identification we have a canonical action of the Weyl group W on $\mathfrak{a}^*$. We equip $\mathfrak{a}$ with the inner product induced from $\mathfrak{p}$ and naturally extend it to an inner product on $\mathfrak{a}^*$. The norm on $\mathfrak{a}^*$ induced from this inner product $\langle , \rangle$ will be again denoted by $|\cdot|$. Let

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \mid \langle\lambda, \alpha\rangle > 0 \text{ for } \alpha \in \Sigma^+\}.$$

Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the complexification of $\mathfrak{a}^*$, that is, the set of all complex-valued real linear functionals on $\mathfrak{a}$. If $\lambda : \mathfrak{a} \to \mathbb{C}$ is a real linear functional, then $\Re\lambda : \mathfrak{a} \to \mathbb{R}$ and $\Im\lambda : \mathfrak{a} \to \mathbb{R}$, given by

$$\Re\lambda(H) := \text{ Real part of } \lambda(H), \quad H \in \mathfrak{a},$$

$$\Im\lambda(H) := \text{ Imaginary part of } \lambda(H), \quad H \in \mathfrak{a},$$

are real-valued linear functionals on $\mathfrak{a}$ and $\lambda = \Re\lambda + i\Im\lambda$. Hence $\mathfrak{a}_{\mathbb{C}}^*$ can be naturally identified with $\mathbb{C}^\ell$ and the real inner product on $\mathfrak{a}^*$ can be extended to an inner product on $\mathfrak{a}_{\mathbb{C}}^*$ such that

$$|\lambda| = \left(|\Re\lambda| + |\Im\lambda|\right)^{\frac{1}{2}}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

We also extend the action of Weyl group W on $\mathfrak{a}_{\mathbb{C}}^*$ from $\mathfrak{a}^*$.

As usual on the compact group K, we fix the normalized Haar measure dk and let dn denotes a Haar measure on N. The following integral formula describes the Haar measure of G in term of the Iwasawa and Cartan decomposition. For any $f \in C_c(G)$,

$$\begin{aligned} \int_G f(g)\, dg &= \int_K \int_A \int_N f(nak) e^{-2\langle \rho, \log(a)\rangle}\, dn\, da\, dk & (2.2)\\ &= \int_K \int_{\overline{\mathfrak{a}+}} \int_K f(k_1(\exp g^+)k_2) J(g^+)\, dk_1\, dg^+\, dk_2, & (2.3)\end{aligned}$$

where dg^+ is the Lebesgue measure on $\mathbb{R}^\ell$ and

$$J(g^+) = |K/M| \prod_{\alpha\in\Sigma^+} \left(\sinh\langle \alpha, g^+\rangle\right)^{m_\alpha} \asymp \prod_{\alpha\in\Sigma^+} \left(\frac{\langle \alpha, g^+\rangle}{1 + \langle \alpha, g^+\rangle}\right)^{m_\alpha} e^{2\langle \rho, g^+\rangle}, \quad (g^+ \in \overline{\mathfrak{a}_+}). \tag{2.4}$$

We will identify a function on $X = G/K$ with a right K-invariant function on G. For an integrable function f on X,

$$\int_G f(g)\, dg = \int_X f(x)\, dx,$$

where in the left hand side f is considered as a right K-invariant function on G and dg is the Haar measure on G, while on the right hand side dx is the G-invariant measure on X. We shall slur over the difference between integrating over G and that on $X = G/K$ as we shall deal with functions on X.

For a function space $\mathcal{F}(X)$ on X, let $\mathcal{F}(G//K)$ denote the subspace of $F(X)$ consisting of K-bi-invariant functions in $F(X)$. Similarly let $\mathcal{F}(X)^W$ denote the space of W-invariant functions in $\mathcal{F}(X)$.

2.3. **Spherical Fourier analysis.** For any $\lambda \in \mathfrak{a}_{\mathbb{C}}$, we define the elementary spherical function φ_λ by

$$\varphi_\lambda(x) := \int_K e^{\langle i\lambda + \rho, A(kx)\rangle}\, dk, \quad x \in G. \tag{2.5}$$

The elementary spherical function φ_λ has the following properties ([9, 10]).

(a) For each fixed $\lambda \in \mathfrak{a}_{\mathbb{C}}$, $\Delta\varphi_\lambda = -(|\lambda|^2 + |\rho|^2)\varphi_\lambda$ and $\varphi_\lambda(e) = 1$.
(b) φ_λ is a smooth K-bi-invariant function on G.
(c) φ_λ is Weyl group invariant. More precisely φ_{λ_1} and φ_{λ_2} are identical if and only if $\lambda_1 = w\lambda_2$ for some $w \in W$.
(d) $\varphi_\lambda(x) = \varphi_{-\lambda}(x^{-1})$.
(e) For each $x \in G$, $\varphi_\lambda(x)$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}$.

For all $H \in \overline{\mathfrak{a}^+}$, the spherical function φ_0 satisfy the following estimate:

$$\varphi_0(\exp H) \asymp \left\{ \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, H \rangle) \right\} e^{-\langle \rho, H \rangle}, \quad (H \in \overline{\mathfrak{a}^+}). \tag{2.6}$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and a suitable function f on X, the spherical Fourier transform of f at λ is given by

$$\widehat{f}(\lambda) := \int_G f(x)\varphi_{-\lambda}(x)\, dx. \tag{2.7}$$

Let $\mathcal{S}(G//K)$ denote the space of all K-bi-invariant smooth functions f on G satisfying

$$\sup_{k_1, k_2 \in K, H \in \overline{\mathfrak{a}^+}} (1 + |H|)^N e^{\langle \rho, H \rangle} |f(D_1 : k_1(\exp H)k_2 : D_2)| < \infty$$

for every D_1, D_2 belonging to the universal enveloping algebra $U(\mathfrak{g})$ and $N \geq 0$, where $f(D_1 : x : D_2)$ denotes left and right differentiation of f at $x \in G$ with respect to D_1 and D_2 respectively. The space $\mathcal{S}(G//K)$ is called as L^2-Schwartz space or simply Schwartz space. Let $\mathcal{S}(\mathfrak{a})^W$ denote the subspace of W-invariant functions in the Schwartz space $\mathcal{S}(\mathfrak{a})$. It is known that the map $f \mapsto \widehat{f}$ is a topological isomorphism between the Schwartz spaces $\mathcal{S}(G//K)$ and $\mathcal{S}(\mathfrak{a})^W$ (see [10]). Moreover we have the following inversion formula:

$$f(x) = C_o \int_{\mathfrak{a}} \widehat{f}(\lambda)\varphi_\lambda(x)|\mathrm{c}(\lambda)|^{-2}\, d\lambda, \quad (x \in G, f \in \mathcal{S}(G//K)) \tag{2.8}$$

where $C_o := 2^{n-\ell}/(2\pi)^\ell |K/M||W|$ and $\mathrm{c}(\lambda)$ is the Harish–Chandra function (see (2.10) below). It is also known that the Spherical Fourier transform $f \mapsto \widehat{f}$ defined by (2.7) on $\mathcal{S}(G//K)$ extends to an isometry of $L^2(G//K)$ onto $L^2(\mathfrak{a})^W$ (with the measure $|\mathrm{c}(\lambda)|^{-2}\, d\lambda$ on $\mathfrak{a}$) such that the following Plancheral theorem holds:

$$\int_X |f(x)|^2\, dx = \int_{\mathfrak{a}} |\widehat{f}(\lambda)|^2 |\mathrm{c}(\lambda)|^{-2}\, d\lambda. \tag{2.9}$$

For $\alpha \in \Sigma_r^+$, $z \in \mathbb{C}$, let

$$\mathrm{c}_\alpha(z) := \frac{\Gamma\left(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{2}m_\alpha\right)}{\Gamma\left(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}\right)} \frac{\Gamma\left(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)}{\Gamma\left(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha\right)} \frac{\Gamma(iz)}{\Gamma\left(iz + \frac{1}{2}m_\alpha\right)} \frac{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha)}{\Gamma\left(\frac{i}{2}z + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)}.$$

The following formula for c-function is due to Gindikin and Karpelevič:

$$\mathrm{c}(\lambda) = \prod_{\alpha\in\Sigma_r^+} \mathrm{c}_\alpha\left(\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\right). \tag{2.10}$$

It is also known that

$$|\mathrm{c}(\lambda)|^{-2} = \mathrm{c}(w.\lambda)^{-1}\mathrm{c}(-w.\lambda)^{-1}, \quad (\lambda \in \mathfrak{a}, w \in W) \tag{2.11}$$

For $\alpha \in \Sigma_r^+$, $z \in \mathbb{C}$, let

$$\begin{aligned} b_\alpha(z) \;&:=\; |\alpha|^2 iz\mathrm{c}_\alpha(z) \\ &=\; |\alpha|^2 \frac{\Gamma\left(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{2}m_\alpha\right)}{\Gamma\left(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}\right)} \frac{\Gamma\left(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)}{\Gamma\left(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha\right)} \frac{\Gamma(iz+1)}{\Gamma\left(iz + \frac{1}{2}m_\alpha\right)} \frac{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha)}{\Gamma\left(\frac{i}{2}z + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)}. \end{aligned}$$

Let $\mathrm{b}(\lambda) := \prod_{\alpha\in\Sigma_r^+} \mathrm{b}_\alpha\left(\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\right)$. It is easy to see that

$$\mathrm{b}(\lambda) = \pi(i\lambda)\mathrm{c}(\lambda), \tag{2.12}$$

where

$$\pi(\lambda) := \prod_{\alpha\in\Sigma_r^+} \langle\alpha,\lambda\rangle. \tag{2.13}$$

It can be seen that the function $\lambda \mapsto \mathrm{b}(-\lambda)^{\pm 1}$ is a holomorphic function in a neighborhood of $\mathfrak{a} + i\overline{\mathfrak{a}^+}$ which has the following behaviour:

$$|\mathrm{b}(-\lambda)|^{\pm 1} \asymp \prod_{\alpha\in\Sigma_r^+} (1 + |\langle\alpha,\lambda\rangle|)^{\mp\frac{m_\alpha+m_{2\alpha}}{2}\pm 1}, \quad (\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}) \tag{2.14}$$

and whose derivatives satisfy the following estimate:

$$|p(\nabla_\lambda)\mathrm{b}(-\lambda)|^{\pm 1} = O(|\mathrm{b}(-\lambda)|^{\pm 1}), \quad (\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}) \tag{2.15}$$

where $\nabla_\lambda := (\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \cdots, \frac{\partial}{\partial\lambda_\ell})$ and $p(\nabla_\lambda)$ is any differential polynomial.

From (2.12) and (2.14), it is clear that c-function has the following global behaviour for $\Im\lambda \in -\overline{\mathfrak{a}^+}$:

$$|\mathrm{c}(\lambda)| \asymp \prod_{\alpha\in\Sigma_r^+} |\langle\alpha,\lambda\rangle|^{-1}(1 + |\langle\alpha,\lambda\rangle|)^{1-\frac{m_\alpha+m_{2\alpha}}{2}}, \quad (\Im\lambda \in -\overline{\mathfrak{a}^+}). \tag{2.16}$$

A vector $\lambda \in \mathfrak{a}$ is said to be regular if $\langle \lambda, \alpha \rangle \neq 0$ for every root $\alpha \in \Sigma$. For all $\lambda \in \mathfrak{a}$ regular, the following converging Harish–Chandra expansion is known to be true for all $H \in \mathfrak{a}^+$,

$$\varphi_\lambda(\exp H) = \sum_{w \in W} \mathrm{c}(w.\lambda)\Phi_{w.\lambda}(H), \tag{2.17}$$

$$\text{where } \Phi_\lambda(H) := e^{\langle i\lambda - \rho, H\rangle} \sum_{q \in 2Q} \gamma_q(\lambda) e^{-\langle q, H\rangle}. \tag{2.18}$$

In the above, $Q = \sum_{\alpha \in \Sigma_s^+} \mathbb{N}\alpha$ denotes the positive root lattice and the leading coefficient γ_0 is equal to one. The other coefficients $\gamma_q(\lambda)$ are rational functions in $\lambda \in \mathfrak{a}_{\mathbb{C}}$, which have no poles in $\mathfrak{a} + i\overline{\mathfrak{a}^+}$ and satisfy there

$$|\gamma_q(\lambda)| \leq C(1 + |q|)^d, \tag{2.19}$$

for some non-negative constants C and d (independent of $q \in Q$ and $\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}$). It is also known that derivatives of γ_q satisfy following estimate:

$$p(\nabla_\lambda)(\gamma_q(\lambda)) = O(\gamma_q(\lambda)), \tag{2.20}$$

where $p(\nabla_\lambda)$ is any differential polynomial.

2.4. **Heat kernel.** For $t > 0$, the heat kernel h_t is defined as the K-bi-invariant Schwartz function on X whose spherical Fourier transform is given by

$$\widehat{h_t}(\lambda) := e^{-t(|\lambda|^2 + |\rho|^2)}, \quad (\lambda \in \mathfrak{a}). \tag{2.21}$$

Recall that the heat kernel h_t on X is also the fundamental solution of the heat equation and solution of the heat equation:

$$\Delta u(x,t) = \frac{\partial u}{\partial t}(x,t), \quad u(x,0) = f(x), \quad (t > 0, x \in X). \tag{2.22}$$

is given by

$$u(x,t) = f * h_t(x).$$

It is known that the heat kernel h_t is a positive K-bi-invariant function and satisfies the global estimate [2]:

$$h_t(\exp H) \asymp t^{-\frac{n}{2}} \left\{ \prod_{\alpha \in \Sigma_r^+} (1 + t + \langle \alpha, H\rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \right\} \varphi_0(\exp(H)) e^{-|\rho|^2 t - \frac{|H|^2}{4t}} \tag{2.23}$$

$$\asymp \quad t^{-\frac{n}{2}} \left\{ \prod_{\alpha \in \Sigma_r^+} \left((1 + \langle \alpha, H \rangle)(1 + t + \langle \alpha, H \rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \right) \right\} e^{-|\rho|^2 t - \langle \rho, H \rangle - \frac{|H|^2}{4t}},$$

for all $t > 0$ and $H \in \overline{\mathfrak{a}^+}$.

It is known that $\int_X h_t(x)\, dx = 1$ for every $t > 0$, and

$$\|h_t\|_p \asymp \begin{cases} t^{-\frac{\ell}{2p'}} e^{-\frac{4}{pp'}|\rho|^2 t}, & 1 \leq p < 2, \\ t^{-\frac{\ell}{4} - \frac{|\Sigma_r^+|}{2}} e^{-|\rho|^2 t}, & p = 2, \\ t^{-\frac{\ell}{2} - |\Sigma_r^+|} e^{-|\rho|^2 t}, & 2 < p \leq \infty, \end{cases} \tag{2.24}$$

for all sufficiently large t (see [2, Proposition 4.1.1]).

2.5. **Herz's principle.** Recall for $p \in [1, 2]$, we have defined the following function space in the introduction:

$$\mathcal{L}^p(G//K) := \{ f : G \to \mathbb{C} \text{ measurable and } K\text{-bi-invariant} \mid \widehat{|f|}(i\gamma_p \rho) < \infty \}.$$

It is clear that $\mathcal{L}^p(G//K)$ is a Banach space and for any $f \in \mathcal{L}^p(G//K)$ norm of f is equal to $\widehat{|f|}(i\gamma_p\rho)$. We note that $\mathcal{L}^1(G//K) = L^1(G//K)$. For $p \in (1, 2]$, using estimate of $\varphi_{-i\gamma_p\rho}$, it can be seen that $L^q(G//K) \subset \mathcal{L}^p(G//K)$ for $q \in [1, p)$. When real rank of G is one, it can also be seen that the Lorentz space $L^{p,1}(G//K) \subset \mathcal{L}^p(G//K)$. For a function $f : G \to \mathbb{C}$, we define $f^\# : G \to \mathbb{C}$ by

$$f^\#(x) := f(x^{-1}), \quad (x \in G).$$

Let $f \in \mathcal{L}^p(G//K)$, $g \in L^p(G//K)$ and $h \in L^{p'}(G//K)$.

$$\begin{aligned} & |\langle f * g, h \rangle| = |\langle (f * g)^\#, h^\# \rangle| = |\langle g^\# * f^\#, h^\# \rangle| \\ = \quad & \left| \int_G \left[\int_{N \times A} g^\#(nax) h^\#(na) e^{-2\langle \rho, \log a \rangle}\, dn\, da \right] f(x)\, dx \right| \\ = \quad & \left| \int_{K \times A} \left[\int_{N \times A} g^\#(naka_1) h^\#(na) e^{-2\langle \rho, \log a \rangle}\, dn\, da \right] f(a_1)\, J(a_1) da_1 dk \right| \\ \leq \quad & \int_{K \times A} \left[\int_{N \times A} |g|^\#(naka_1) |h|^\#(na) e^{-2\langle \rho, \log a \rangle}\, dn\, da \right] |f|(a_1)\, J(a_1) da_1 dk \\ \leq \quad & \|h\|_{p'} \int_{K \times A} \left[\int_{N \times A} |g|^\#(naka_1)|^p e^{-2\langle \rho, \log a \rangle}\, dn\, da \right]^{1/p} |f|(a_1)\, J(a_1) da_1 dk. \end{aligned} \tag{2.25}$$

Let $ka_1 = n_2a_2k_2$. Then $\log(a_2) = A(ka_1)$. Since A normalizes N, $naka_1 = nan_2a_2k_2 = n_3a_3k_3$, where $n_3 = nan_2a^{-1}$ and $a_3 = aa_2$. Thus we get,

$$\begin{aligned}\int_{N\times A} |g|^{\#}(naka_1)|^p e^{-2\langle\rho,\log a\rangle}\, dn\, da &= e^{2\langle\rho,\log a_2\rangle}\int_{N\times A} |g|^{\#}(n_3a_3)|^p e^{-2\langle\rho,\log a_3\rangle}\, dn_3\, da_3 \\ &= \|g\|_p^p e^{2\langle\rho,A(ka_1)\rangle}. \end{aligned} \tag{2.26}$$

From (2.25) and (2.26) we get

$$|\langle f*g,h\rangle| \leq \|h\|_{p'}\|g\|_p \int_{KA} |f|(a_1)e^{\frac{2}{p}\langle\rho,A(ka_1)\rangle}J(a_1)da_1dk = \|g\|_p\|h\|_{p'}\widehat{|f|}(i\gamma_p\rho). \tag{2.27}$$

Owing to (2.27) we conclude that

$$\|f*g\|_p \leq \widehat{|f|}(i\gamma_p\rho)\|g\|_p, \tag{2.28}$$

and the convolution operator T_f given by $T_f(g) := f*g$ is a well defined bounded operator from $L^p(G//K)$ to $L^p(G//K)$ with its operator norm $|T_f| \leq \widehat{|f|}(i\gamma_p\rho)$. We will see later that if $f \in \mathcal{L}^p(G//K)$ for $1 \leq p \leq 2$ and f is non-negative, our results will imply that $|T_f| = \widehat{f}(i\gamma_p\rho)$ (see Remark 4.2), which is already a known result due to the following version of the Herz principle [3].

Theorem 2.1 (Herz principle)**.** *Let f be a non-negative locally integrable K-bi-invariant function. For $p \in [1,2]$, the convolution operator given by $T_f(g) := f*g$, $g \in C_c(G//K)$, extends to a bounded operator from $L^p(G//K)$ to $L^p(G//K)$ if and only if $\widehat{f}(i\gamma_p\rho) < \infty$. Moreover, we have $|T_f| = \widehat{f}(i\gamma_p\rho)$.*

We will also use the following convolution inequality which follows from the Kunze-Stein phenomenon (see [4, p.122]: For all $f_j \in L^2(G)$, $j = 1,2$ we have

$$\|f_1 * f_2\|_{L^p(G)} \lesssim \|f_1\|_{L^2(G)}\|f_2\|_{L^2(G)}, \tag{2.29}$$

for all $p \in (2,\infty]$.

3. Results for heat kernel

We first consider a result for the L^2-case.

Proposition 3.1. *Let $f \in L^1(G//K)$. Then*

$$\lim_{t\to\infty} \|h_t\|_2^{-1}\|f*h_t - \widehat{f}(0)h_t\|_2 = 0.$$

Moreover, for any $z \in \mathbb{C}$,

$$\lim_{t\to\infty} \|h_t\|_2^{-1}\|f*h_t - z\,h_t\|_2 = 0 \text{ if and only if } z = \widehat{f}(0).$$

Proof. From (2.9), (2.24) and (2.16) it follows that

$$\begin{aligned}
&\|h_t\|_2^{-2}\|f*h_t - z\,h_t\|_2^2\\
\asymp\;& t^{\frac{\nu}{2}}e^{2t|\rho|^2}\int_{\mathfrak{a}}|\widehat{f}(\lambda)-z|^2e^{-2t(|\lambda|^2+|\rho|^2)}|\mathrm{c}(\lambda)|^{-2}\,d\lambda\\
\asymp\;& t^{\frac{\nu}{2}}\int_{\mathfrak{a}}|\widehat{f}(\lambda)-z|^2e^{-2t|\lambda|^2}\left[\prod_{\alpha\in\Sigma_r^+}|\langle\alpha,\lambda\rangle|^2(1+|\langle\alpha,\lambda\rangle|)^{m_\alpha+m_{2\alpha}-2}\right]d\lambda\\
\asymp\;& \int_{\mathfrak{a}}\left|\widehat{f}\left(\frac{\lambda}{\sqrt{t}}\right)-z\right|^2e^{-2|\lambda|^2}\left[\prod_{\alpha\in\Sigma_r^+}|\langle\alpha,\lambda\rangle|^2\left(1+\frac{|\langle\alpha,\lambda\rangle|}{\sqrt{t}}\right)^{(m_\alpha+m_{2\alpha}-2)}\right]d\lambda \quad (3.1)
\end{aligned}$$

Since $f\in L^1(G//K)$ observe that,

$$\begin{aligned}
&\left|\widehat{f}\left(\frac{\lambda}{\sqrt{t}}\right)-z\right|^2\left[\prod_{\alpha\in\Sigma_r^+}|\langle\alpha,\lambda\rangle|^2\left(1+\frac{|\langle\alpha,\lambda\rangle|}{\sqrt{t}}\right)^{(m_\alpha+m_{2\alpha}-2)}\right]\\
\lesssim\;& \prod_{\alpha\in\Sigma_r^+}|\langle\alpha,\lambda\rangle|^2\,(1+|\langle\alpha,\lambda\rangle|)^{(m_\alpha+m_{2\alpha}-2)_+}, \qquad (3.2)
\end{aligned}$$

for all $t>1$. Owing to (3.1), (3.2) and the dominated convergence theorem we get

$$\lim_{t\to\infty}\|h_t\|_2^{-2}\|f*h_t-z\,h_t\|_2^2\asymp\int_{\mathfrak{a}}\left|\widehat{f}(0)-z\right|^2e^{-2|\lambda|^2}\left(\prod_{\alpha\in\Sigma_r^+}|\langle\alpha,\lambda\rangle|^2\right)d\lambda. \qquad (3.3)$$

Hence from (3.3) it is clear that

$$\lim_{t\to\infty}\|h_t\|_2^{-1}\|f*h_t-z\,h_t\|_2=0 \text{ if and only if } z=\widehat{f}(0).$$

Owing to (3.3) it also follows that

$$\lim_{t\to\infty}\|h_t\|_2^{-1}\|f*h_t-\widehat{f}(0)h_t\|_2=0.$$

□

As we have said in the introduction, previous result and L^1-analogue of previous result serves as motivation for formulation of the statements of our main results. Let $r(t)$ be a positive function on $(0,\infty)$ such that

$$\lim_{t\to\infty}\frac{r(t)}{\sqrt{t}}=\infty \quad\text{and}\quad \lim_{t\to\infty}\frac{r(t)}{t}=0. \qquad (3.4)$$

Let $\Omega_t^p:=B(2t\gamma_p\rho,r(t))$ denotes the ball in $\mathfrak{a}$ with radius $r(t)$ and center $2t\gamma_p\rho$. We will use the following proposition later for proving our result for the case $p\in[1,2)$, which says that for $1 \leq p < 2$, the L^p-norm of h_t concentrates asymptotically in the bi-K-orbit of $\exp \Omega_t^p$. Compare with [2, Theorem 4.1.2] and [1, Lemma 2.1].

Proposition 3.2. *Let $1 \leq p < 2$. Then for any $N \geq 0$ the following estimate holds:*

$$\|h_t\|_p^{-1} \|h_t\|_{L^p(G \setminus K(\exp \Omega_t^p)K)} \lesssim \left(\frac{r(t)}{\sqrt{t}} \right)^{-N}, \tag{3.5}$$

for all sufficiently large t. In particular

$$\lim_{t \to \infty} \|h_t\|_p^{-1} \|h_t\|_{L^p(G \setminus K(\exp \Omega_t^p)K)} = 0. \tag{3.6}$$

Proof. Idea of the proof is similar to that of [1, Lemma 2.1]. Let $\eta > 0$ be a fixed constant. As $\lim_{t \to \infty} \dfrac{r(t)}{t} = 0$, there exists $t_o > 0$ such that $r(t) < \eta t$, for all $t > t_o$.

Let $H \in \overline{\mathfrak{a}^+}$ and $g_t(H) := e^{-|\rho|^2 t - \langle \rho, H \rangle - \frac{|H|^2}{4t}}$. From (2.23) it follows that

$$h_t(\exp H) \ \lesssim g_t(H) \times \begin{cases} t^{-\frac{\ell}{2}} & \text{if } H \in B(2t\gamma_p \rho, \eta t), \\ t^L, & \text{if } H \in B(0, 3t\gamma_p |\rho|) \setminus B(2t\gamma_p\rho, \eta t), \\ |H|^L, & \text{if } |H| \geq 3t\gamma_p|\rho|, \end{cases} \tag{3.7}$$

for some $L > 0$ and all sufficiently large t. Thus, owing to (3.7) we obtain

$$|h_t(\exp H)|^p \lesssim |g_t(H)|^p \times \begin{cases} t^{-\frac{p\ell}{2}} & \text{if } H \in B(2t\gamma_p \rho, \eta t), \\ t^{pL}, & \text{if } H \in B(0, 3t\gamma_p |\rho|) \setminus B(2t\gamma_p\rho, \eta t), \\ |H|^{pL}, & \text{if } |H| \geq 3t\gamma_p|\rho|, \end{cases} \tag{3.8}$$

for some $L > 0$ and all sufficiently large t.

We observe that

$$\begin{aligned} |g_t(H)|^p e^{2\langle \rho, H \rangle} &= e^{-p|\rho|^2 t + (2-p)\langle \rho, H \rangle - \frac{p|H|^2}{4t}} \\ &= e^{-p\left(|\rho|^2 t - \gamma_p \langle \rho, H \rangle - \frac{|H|^2}{4t} \right)} \\ &= e^{-\frac{p}{4t}\left(|H|^2 - 4t\gamma_p \langle \rho, H \rangle + 4|\rho|^2 t^2 \right)} \\ &= e^{-\frac{p}{4t}|H - 2t\gamma_p \rho|^2} e^{-p(1-\gamma_p^2)|\rho|^2 t} \\ &= e^{-\frac{p}{4t}|H - 2t\gamma_p \rho|^2} e^{-\frac{4|\rho|^2 t}{p'}}. \end{aligned} \tag{3.9}$$

Owing to (2.24) and (3.9) we get

$$\|h_t\|_p^{-p} |g_t(H)|^p e^{2\langle \rho, H \rangle} \asymp t^{\frac{p\ell}{2p'}} e^{-\frac{p}{4t}|H - 2t\gamma_p\rho|^2} \tag{3.10}$$

for all sufficiently large t. Since the Jacobian $J(H)$ of the Cartan decomposition satisfies $J(H) \lesssim e^{2\langle\rho,H\rangle}$ for $H \in \overline{\mathfrak{a}^+}$, using (3.8), (3.10) we get

$$\begin{aligned}
& \|h_t\|_p^{-p} \int\limits_{K(\exp B(2t\gamma_p\rho,\eta t))K\setminus K(\exp \Omega_t^p)K} |h_t(x)|^p\,dx \\
\lesssim\ & \|h_t\|_p^{-p} \int\limits_{r(t)\leq|H-2t\gamma_p\rho|\leq\eta t} |g_t(H)|^p t^{-\frac{p\ell}{2}} J(H)\,dH \\
\lesssim\ & t^{-\frac{p\ell}{2}} \int\limits_{r(t)\leq|H-2t\gamma_p\rho|\leq\eta t} \|h_t\|_p^{-p}|g_t(H)|^p e^{2\langle\rho,H\rangle}\,dH \\
\lesssim\ & t^{-\frac{\ell}{2}} \int\limits_{r(t)\leq|H-2t\gamma_p\rho|\leq\eta t} e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH \\
\lesssim\ & t^{-\frac{\ell}{2}} \int\limits_{r(t)}^{\eta t} e^{-\frac{ps^2}{4t}} s^{\ell-1}\,ds \\
\lesssim\ & \int\limits_{\frac{r(t)}{\sqrt{t}}}^{\eta\sqrt{t}} e^{-\frac{ps^2}{4}} s^{\ell-1}\,ds \\
\lesssim\ & \int\limits_{\frac{r(t)}{\sqrt{t}}}^{\infty} e^{-\frac{ps^2}{4}} s^{\ell-1}\,ds \\
\lesssim\ & \left(\frac{r(t)}{\sqrt{t}}\right)^{-N}, \quad \text{for any } N>0. \qquad (3.11)
\end{aligned}$$

Owing to (2.24) and (3.9) we obtain

$$\begin{aligned}
& \|h_t\|_p^{-p} \int\limits_{K(\exp B(0,3t\gamma_p|\rho|))K\setminus K(\exp B(2t\gamma_p\rho,\eta t))K} |h_t(x)|^p\,dx \\
\lesssim\ & \|h_t\|_p^{-p} \int\limits_{[\overline{\mathfrak{a}^+}\cap B(0,3t\gamma_p|\rho|)]\setminus B(2t\gamma_p\rho,\eta t)} |g_t(H)|^p t^{pL} J(H)\,dH \\
\lesssim\ & t^{pL} \int\limits_{[\overline{\mathfrak{a}^+}\cap B(0,3t\gamma_p|\rho|)]\setminus B(2t\gamma_p\rho,\eta t)} \|h_t\|_p^{-p}|g_t(H)|^p e^{2\langle\rho,H\rangle}\,dH
\end{aligned}$$

$$
\begin{aligned}
&\lesssim t^{p(L+\frac{\ell}{2p'})} \int\limits_{[\overline{\mathfrak{a}^+}\cap B(0,3t\gamma_p|\rho|)]\setminus B(2t\gamma_p\rho,\eta t)} e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH \\
&\lesssim t^{p(L+\frac{\ell}{2p'})} \int\limits_{\eta t\le|H-2t\gamma_p\rho|\le 5\gamma_p|\rho|t} e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH \\
&\lesssim t^{p(L+\frac{\ell}{2p'})} \int\limits_{\eta t}^{5\gamma_p|\rho|t} e^{-\frac{ps^2}{4t}} s^{\ell-1}\,ds \\
&\lesssim t^{p(L+\frac{\ell}{2p'})+\frac{\ell}{2}} \int\limits_{\eta\sqrt{t}}^{5\gamma_p|\rho|\sqrt{t}} e^{-\frac{ps^2}{4}} s^{\ell-1}\,ds \\
&\lesssim t^{p(L+\frac{\ell}{2p'})+\frac{\ell}{2}} \int\limits_{\eta\sqrt{t}}^{\infty} e^{-\frac{ps^2}{4}} s^{\ell-1}\,ds \\
&\lesssim \left(\frac{r(t)}{\sqrt{t}}\right)^{-N}, \quad \text{for any } N>0. \qquad (3.12)
\end{aligned}
$$

Also, we take note that

$$
\begin{aligned}
&\|h_t\|_p^{-p} \int\limits_{G\setminus K(\exp B(0,3t\gamma_p|\rho|))K} |h_t(x)|^p\,dx \\
\lesssim\ &\|h_t\|_p^{-p} \int\limits_{\overline{\mathfrak{a}^+}\setminus B(0,3t\gamma_p|\rho|)} |g_t(H)|^p|H|^{pL}J(H)\,dH \\
\lesssim\ &\int\limits_{\overline{\mathfrak{a}^+}\setminus B(0,3t\gamma_p|\rho|)} |H|^{pL}\|h_t\|_p^{-p}|g_t(H)|^p e^{2\langle\rho,H\rangle}\,dH \\
\lesssim\ &t^{\frac{p\ell}{2p'}} \int\limits_{\overline{\mathfrak{a}^+}\setminus B(0,3t\gamma_p|\rho|)} |H|^{pL}e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH \\
\lesssim\ &t^{\frac{p\ell}{2p'}} \int\limits_{|H|\ge 3t\gamma_p|\rho|} |H|^{pL}e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH \\
\lesssim\ &t^{\frac{p\ell}{2p'}} \int\limits_{|H-2t\gamma_p\rho|\ge\gamma_p|\rho|t} |H|^{pL}e^{-\frac{p}{4t}|H-2t\gamma_p\rho|^2}\,dH
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \; t^{\frac{p\ell}{2p'}} \int\limits_{|H|\geq \gamma_p|\rho|t} |H+2t\gamma_p\rho|^{pL} e^{-\frac{p}{4t}|H|^2}\, dH \\
&\lesssim \; t^{\frac{p\ell}{2p'}} \int\limits_{|H|\geq \gamma_p|\rho|t} |H|^{pL} e^{-\frac{p}{4t}|H|^2}\, dH \quad (\text{As } |H+2t\gamma_p\rho| \lesssim |H|) \\
&\lesssim \; t^{\frac{p\ell}{2p'}} \int\limits_{\gamma_p|\rho|t}^{\infty} s^{pL} e^{-\frac{ps^2}{4t}} s^{\ell-1}\, ds \\
&\lesssim \; t^{\frac{p\ell}{2p'}+\frac{\ell}{2}+\frac{Lp}{2}} \int\limits_{\gamma_p|\rho|\sqrt{t}}^{\infty} e^{-\frac{ps^2}{4}} s^{pL+\ell-1}\, ds \\
&\lesssim \; \left(\frac{r(t)}{\sqrt{t}}\right)^{-N}, \quad \text{for any } N>0. \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
\text{As } \; G\setminus K(\exp \Omega_t^p)K \subset\; & [K(\exp B(2t\gamma_p\rho,\eta t))K \setminus K(\exp\Omega_t^p)K] \\
& \cup [K(\exp B(0,3t\gamma_p|\rho|))K \setminus K(\exp B(2t\gamma_p\rho,\eta t))K] \\
& \cup [G\setminus K(\exp B(0,3t\gamma_p|\rho|))K],
\end{aligned}$$

the result follows from (3.11), (3.12) and (3.13).

□

Proposition 3.3. *Let $1 \leq p \leq 2$ and $\frac{1}{2} < a < 1$. Then*

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|h_t\|_{L^p(G\setminus K(\exp B(2t\gamma_p\rho, t^a))K)} = 0. \qquad (3.14)$$

Proof. For $1 \leq p < 2$, proof follows from Proposition 3.2 if we take $r(t) = t^a$. For $p=2$, exactly same argument as in Proposition 3.2 will give the result. □

Proposition 3.4. *Let $1 \leq p < 2$ be fixed. Then for any $f \in C_c^\infty(G//K)$, we have*

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(i\gamma_p\rho)h_t\|_{L^p(K(\exp\Omega_t^p)K)} = 0. \qquad (3.15)$$

Proof. Let $H \in \Omega_t^p$. By inversion formula (2.8) we have

$$\begin{aligned}
& f*h_t(\exp H) - \widehat{f}(i\gamma_p\rho)h_t(\exp H) \\
=\;& C_o \int_{\mathfrak{a}} e^{-t(|\lambda|^2+|\rho|^2)} (\widehat{f}(\lambda) - \widehat{f}(i\gamma_p\rho))\varphi_\lambda(\exp H)|\mathrm{c}(\lambda)|^{-2}\, d\lambda \\
=\;& C_o e^{-t|\rho|^2} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda)\varphi_\lambda(\exp H)|\mathrm{c}(\lambda)|^{-2}\, d\lambda \qquad (\text{Here } U(\lambda) := \widehat{f}(\lambda) - \widehat{f}(i\gamma_p\rho))
\end{aligned}$$

$$= C_o e^{-t|\rho|^2} \sum_{w\in W} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) \mathrm{c}(w.\lambda) \Phi_{w.\lambda}(H) |\mathrm{c}(\lambda)|^{-2}\, d\lambda \quad \text{(see (2.17))}. \tag{3.16}$$

As the function $\lambda \mapsto e^{-t|\lambda|^2} U(\lambda) |\mathrm{c}(\lambda)|^{-2}$ is invariant under the action of Weyl group W, because of (3.16), we get

$$\begin{aligned}
& f * h_t(\exp H) - \widehat{f}(i\gamma_p\rho) h_t(\exp H) \\
= \;& C_o e^{-t|\rho|^2} |W| \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) \Phi_\lambda(H) \mathrm{c}(\lambda) |\mathrm{c}(\lambda)|^{-2}\, d\lambda \\
= \;& C_o |W| e^{-t|\rho|^2} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) \Phi_\lambda(H) \mathrm{c}(-\lambda)^{-1} d\lambda \qquad (\text{since } |\mathrm{c}(\lambda)|^{-2} = \mathrm{c}(\lambda)^{-1}\mathrm{c}(-\lambda)^{-1}) \\
= \;& C_o |W| e^{-t|\rho|^2} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) \left(e^{\langle i\lambda - \rho, H\rangle} \sum_{q\in 2Q} \gamma_q(\lambda) e^{-\langle q, H\rangle} \right) \mathrm{c}(-\lambda)^{-1} d\lambda \quad \text{(see (2.18))} \\
= \;& C_o |W| e^{-|\rho|^2 t - \langle \rho, H\rangle} \sum_{q\in 2Q} e^{-\langle q,H\rangle} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) e^{i\langle\lambda, H\rangle} \gamma_q(\lambda) \mathrm{c}(-\lambda)^{-1}\, d\lambda \\
= \;& C_o |W| e^{-|\rho|^2 t - \langle \rho, H\rangle} \sum_{q\in 2Q} e^{-\langle q,H\rangle} \int_{\mathfrak{a}} e^{-t|\lambda|^2} U(\lambda) e^{i\langle\lambda, H\rangle} \gamma_q(\lambda) \pi(-i\lambda) \mathrm{b}(-\lambda)^{-1}\, d\lambda \quad \text{(see (2.12))} \\
= \;& C_o |W| e^{-|\rho|^2 t - \langle \rho, H\rangle} \sum_{q\in 2Q} e^{-\langle q,H\rangle} E_q(t, H)
\end{aligned} \tag{3.17}$$

where $E_q(t,H) := \int_{\mathfrak{a}} \pi(-i\lambda) e^{-t|\lambda|^2} e^{i\langle\lambda,H\rangle} \mathrm{b}(-\lambda)^{-1} \gamma_q(\lambda) U(\lambda)\, d\lambda$.

For a vector $v \in \mathfrak{a}$ and a complex valued function F on $\mathfrak{a}$, let $\partial_v F$ denote the directional derivative of F in the direction of v. We observe that $\left(\prod_{\alpha\in\Sigma_r^+} \partial_\alpha\right) e^{-t|\lambda|^2} = (2t)^{|\Sigma_r^+|} \pi(-\lambda) e^{-t|\lambda|^2}$. Hence

$$\pi(-i\lambda) e^{-t|\lambda|^2} = (2t)^{-|\Sigma_r^+|} \left(\prod_{\alpha\in\Sigma_r^+} (i\partial_\alpha) \right) e^{-t|\lambda|^2}. \tag{3.18}$$

Owing to (3.18) and integration by parts, we get

$$\begin{aligned}
E_q(t,H) &= \int_{\mathfrak{a}} \pi(-i\lambda) e^{-t|\lambda|^2} e^{i\langle\lambda,H\rangle} \mathrm{b}(-\lambda)^{-1} \gamma_q(\lambda) U(\lambda)\, d\lambda \\
&= (2t)^{-|\Sigma_r^+|} \int_{\mathfrak{a}} \left[\left(\prod_{\alpha\in\Sigma_r^+} (i\partial_\alpha) \right) e^{-t|\lambda|^2} \right] e^{i\langle\lambda,H\rangle} \mathrm{b}(-\lambda)^{-1} \gamma_q(\lambda) U(\lambda)\, d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= (2t)^{-|\Sigma_r^+|}\int_{\mathfrak a} e^{-t|\lambda|^2}\left(\prod_{\alpha\in\Sigma_r^+}(-i\partial_\alpha)\right)\left(e^{i\langle\lambda,H\rangle}\mathrm{b}(-\lambda)^{-1}\gamma_q(\lambda)U(\lambda)\right)\,d\lambda\\
&= (2t)^{-|\Sigma_r^+|}\sum_{\Sigma_r^+=\Sigma'\sqcup\Sigma''}\left\{\prod_{\alpha'\in\Sigma'}\langle\alpha',H\rangle\right\}\int_{\mathfrak a} e^{-t|\lambda|^2}e^{i\langle\lambda,H\rangle}\\
&\qquad\times\left(\prod_{\alpha''\in\Sigma''}(-i\partial_{\alpha''})\right)\left(\mathrm{b}(-\lambda)^{-1}\gamma_q(\lambda)U(\lambda)\right)\,d\lambda\\
&= (2t)^{-|\Sigma_r^+|}e^{-\frac{|H|^2}{4t}}\sum_{\Sigma_r^+=\Sigma'\sqcup\Sigma''}\left\{\prod_{\alpha'\in\Sigma'}\langle\alpha',H\rangle\right\}\int_{\mathfrak a} e^{-t\left|\lambda-i\frac{H}{2t}\right|^2} \qquad (3.19)\\
&\qquad\times\left(\prod_{\alpha''\in\Sigma''}(-i\partial_{\alpha''})\right)\left(\mathrm{b}(-\lambda)^{-1}\gamma_q(\lambda)U(\lambda)\right)\,d\lambda
\end{aligned}
$$

On changing the above integral $\lambda\mapsto\lambda+i\frac{H}{2t}$ by contour integration first and then by the change of variable $\lambda\mapsto\frac{\lambda}{\sqrt t}$, we get

$$
\begin{aligned}
E_q(t,H) &= 2^{-|\Sigma_r^+|}t^{-\frac{\nu}{2}}e^{-\frac{|H|^2}{4t}}\sum_{\Sigma_r^+=\Sigma'\sqcup\Sigma''}\left\{\prod_{\alpha'\in\Sigma'}\langle\alpha',H\rangle\right\} \qquad (3.20)\\
&\times\int_{\mathfrak a} e^{-|\lambda|^2}\left(\prod_{\alpha''\in\Sigma''}(-i\sqrt t\partial_{\alpha''})\right)\mathrm{b}\left(-\frac{\lambda}{\sqrt t}-i\frac{H}{2t}\right)^{-1}\gamma_q\left(\frac{\lambda}{\sqrt t}+i\frac{H}{2t}\right)\\
&\times U\left(\frac{\lambda}{\sqrt t}+i\frac{H}{2t}\right)\,d\lambda.
\end{aligned}
$$

For $j\in\mathbb N$, let $\nabla_\lambda^j := (\frac{\partial^j}{\partial\lambda_1^j},\frac{\partial^j}{\partial\lambda_2^j},\cdots,\frac{\partial^j}{\partial\lambda_\ell^j})$. Since $f\in C_c^\infty(G//K)$, by Paley–Weiner theorem $\widehat f(\lambda)$ is a W-invariant holomorphic function on $\mathfrak a_{\mathbb C}$ and there exists a constant $C'\geq 0$ such that for every $j\in\mathbb N$ and for every $k\in\mathbb N$,

$$
|\nabla_\lambda^j\widehat f(\lambda)|\leq C_{k,j}(1+|\lambda|)^{-k}e^{C'|\Im\lambda|} \qquad (3.21)
$$

for some constant $C_{k,j}\geq 0$ and for all $\lambda\in\mathfrak a_{\mathbb C}$. Owing to (3.21) and the mean value theorem, for all large t and $\lambda\in\mathfrak a$ we get

$$
\begin{aligned}
\left|U\left(\frac{\lambda}{\sqrt t}+i\frac{H}{2t}\right)\right| &\lesssim \left|\frac{\lambda}{\sqrt t}+i\frac{H}{2t}-i\gamma_p\rho\right|e^{C'\frac{|H|}{2t}} \qquad (3.22)\\
&\leq \left(\frac{|\lambda|}{\sqrt t}+\frac{|H-2t\gamma_p\rho|}{2t}\right)e^{C'\frac{|H|}{2t}}
\end{aligned}
$$

$$\leq \left(\frac{|\lambda|}{\sqrt{t}} + \frac{r(t)}{2t}\right) e^{C'\frac{|H|}{2t}}$$
$$\lesssim (|\lambda|+1)\,\frac{r(t)}{t}, \quad \left(\text{as } H \in \Omega_t^p \text{ and } 1 < \frac{r(t)}{\sqrt{t}}\right).$$

We observe that for $\alpha \in \Sigma_r^+$,

$$1 + \left|\left\langle \alpha, -\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right\rangle\right| \lesssim \left(1 + \frac{|\langle \alpha, \lambda\rangle|}{\sqrt{t}}\right)\left(1 + \frac{\langle \alpha, H\rangle}{2t}\right) \lesssim (1+|\lambda|). \tag{3.23}$$

Hence from (3.23) and (2.14) it follows that

$$\left|\mathrm{b}\left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right)\right|^{-1} \lesssim \prod_{\alpha\in\Sigma_r^+}\left(1 + \left|\left\langle \alpha, -\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right\rangle\right|\right)^{(\frac{m_\alpha+m_{2\alpha}}{2}-1)_+} \lesssim (1+|\lambda|)^m, \tag{3.24}$$

where $m = \sum_{\alpha\in\Sigma_r^+}(\frac{m_\alpha + m_{2\alpha}}{2} - 1)_+$.

Owing to (3.24) and (2.15) for all large t we get

$$\left|(\sqrt{t}\nabla_\lambda)^j\left(\mathrm{b}\left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right)^{-1}\right)\right| \lesssim \left|\mathrm{b}\left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right)\right|^{-1} \lesssim (1+|\lambda|)^m \tag{3.25}$$

In view of (2.19) and (2.20) we also have

$$\left|(\sqrt{t}\nabla_\lambda)^j\gamma_q\left(\frac{\lambda}{\sqrt{t}} + i\frac{H}{2t}\right)\right| \lesssim (1+|q|)^d. \tag{3.26}$$

After applying the Leibniz rule in (3.20), we notice that $E_q(t,H)$ is the sum of two types of terms. The first type consists of terms that do not contain any derivative of U, while the second type comprises terms that contains at least one derivative of U. Let T_1 be a term of $E_q(t,H)$ which is of first kind. Then from (3.22), (3.25) and (3.26) it follows that

$$|T_1| \lesssim t^{-\frac{\nu}{2}}e^{-\frac{|H|^2}{4t}}\left\{\prod_{\alpha'\in\Sigma'}\langle\alpha', H\rangle\right\}(1+|q|)^d\frac{r(t)}{t} \tag{3.27}$$
$$\lesssim t^{-\frac{\nu}{2}}e^{-\frac{|H|^2}{4t}}\left(\prod_{\Sigma_r^+}(1+|H|)\right)(1+|q|)^d\frac{r(t)}{t}$$
$$\lesssim t^{-\frac{\ell}{2}}e^{-\frac{|H|^2}{4t}}(1+|q|)^d\frac{r(t)}{t}. \quad (\text{since } 1+|H| \lesssim t).$$

Similarly if T_2 is a term of second kind, from (3.21), (3.25) and (3.26) we get

$$\begin{aligned} |T_2| &\lesssim t^{-\frac{\nu}{2}} e^{-\frac{|H|^2}{4t}} \left\{ \prod_{\alpha' \in \Sigma'} \langle \alpha', H \rangle \right\} (1+|q|)^d \\ &\lesssim t^{-\frac{\nu}{2}} e^{-\frac{|H|^2}{4t}} (1+|H|)^{|\Sigma_r^+|-1} (1+|q|)^d \quad (\text{Since } |\Sigma'| < |\Sigma_r^+|) \\ &\lesssim t^{-\frac{\ell}{2}-1} e^{-\frac{|H|^2}{4t}} (1+|q|)^d \quad (\text{since } 1+|H| \lesssim t) \\ &\lesssim t^{-\frac{\ell}{2}} e^{-\frac{|H|^2}{4t}} (1+|q|)^d \frac{r(t)}{t} \quad (\text{since } 1 \leq r(t) \text{ for all large } t). \end{aligned} \tag{3.28}$$

Hence owing to (3.20), (3.27) and (3.28) we get

$$|E_q(t,H)| \lesssim t^{-\frac{\ell}{2}} e^{-\frac{|H|^2}{4t}} (1+|q|)^d \frac{r(t)}{t}. \tag{3.29}$$

Thus for all sufficiently large t and $H \in \Omega_t^p$, from (3.17) and (3.29) we get

$$\begin{aligned} |f * h_t(\exp H) - \widehat{f}(i\gamma_p\rho) h_t(\exp H)| &\lesssim \frac{r(t)}{t} t^{-\frac{\ell}{2}} e^{-|\rho|^2 t - \langle \rho, H\rangle - \frac{|H|^2}{4t}} \overbrace{\sum_{q \in 2Q} (1+|q|)^d e^{-\langle q, H\rangle}}^{\lesssim 1} \\ &\lesssim \frac{r(t)}{t} t^{-\frac{\ell}{2}} g_t(H), \end{aligned} \tag{3.30}$$

where $g_t(H) := e^{-|\rho|^2 t - \langle \rho, H\rangle - \frac{|H|^2}{4t}}$.

Owing to (3.10) we have

$$\|h_t\|_p^{-p} |g_t(H)|^p e^{2\langle \rho, H\rangle} \asymp t^{\frac{\ell p}{2p'}} e^{-\frac{p}{4t}|H - 2t\gamma_p\rho|^2} \tag{3.31}$$

for all sufficiently large t. On integrating according to the Cartan decomposition and using (3.31) we get

$$\begin{aligned} & \|h_t\|_p^{-p} \int_{K(\exp \Omega_t^p)K} |f * h_t(x) - \widehat{f}(i\gamma_p\rho) h_t(x)|^p \, dx \\ = \ & C_0 \|h_t\|_p^{-p} \int_{\Omega_t^p} |f * h_t(\exp H) - \widehat{f}(i\gamma_p\rho) h_t(\exp H)|^p J(H) \, dH \\ \lesssim \ & \left(\frac{r(t)}{t}\right)^p t^{-\frac{\ell p}{2}} \int_{\Omega_t^p} \|h_t\|_p^{-p} |g_t(H)|^p e^{2\langle \rho, H\rangle} \, dH \\ \lesssim \ & \left(\frac{r(t)}{t}\right)^p t^{-\frac{\ell p}{2}} \int_{\Omega_t^p} t^{\frac{\ell p}{2p'}} e^{-\frac{p}{4t}|H - 2t\gamma_p\rho|^2} \, dH \\ \lesssim \ & \left(\frac{r(t)}{t}\right)^p t^{-\frac{\ell}{2}} \int_{B(2t\gamma_p\rho, r(t))} e^{-\frac{p}{4t}|H - 2t\gamma_p\rho|^2} \, dH \end{aligned}$$

$$\begin{aligned}
&\lesssim \left(\frac{r(t)}{t}\right)^p t^{-\frac{\ell}{2}} \int_0^{r(t)} e^{-\frac{ps^2}{4t}} s^{l-1}\, ds \\
&\lesssim \left(\frac{r(t)}{t}\right)^p \int_0^{\frac{r(t)}{\sqrt{t}}} e^{-\frac{ps^2}{4}} s^{l-1}\, ds \\
&\leq \left(\frac{r(t)}{t}\right)^p \int_0^{\infty} e^{-\frac{ps^2}{4}} s^{l-1}\, ds \\
&\lesssim \left(\frac{r(t)}{t}\right)^p . \qquad (3.32)
\end{aligned}$$

Hence the result follows from (3.32). □

Proposition 3.5. *Let $1 \leq p < 2$ and $f \in C_c^\infty(G//K)$. Then we have*

$$\lim_{t\to\infty} \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(i\gamma_p\rho)h_t\|_{L^p(G\setminus K(\exp \Omega_t^p)K)} = 0. \qquad (3.33)$$

Proof. Let support of f be contained in $K(\exp B(0,a))K$ for some $a > 0$. By using Minkowski integral inequality we get

$$\begin{aligned}
&\|f * h_t\|_{L^p(G\setminus K(\exp \Omega_t^p)K)} \qquad (3.34)\\
&= \left(\int_{G\setminus K(\exp \Omega_t^p)K} |f * h_t(x)|^p\, dx \right)^{\frac{1}{p}} \\
&= \left(\int_{G\setminus K(\exp \Omega_t^p)K} \left| \int_{K(\exp B(0,a))K} f(y) h_t(xy^{-1})\, dy \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \int_{K(\exp B(0,a))K} \left(\int_{G\setminus K(\exp \Omega_t^p)K} |f(y)|^p h_t(xy^{-1})^p\, dx \right)^{\frac{1}{p}} dy \\
&\leq \int_{K(\exp B(0,a))K} |f(y)| \left(\int_{G\setminus K(\exp \Omega_t^p)K} h_t(xy^{-1})^p\, dx \right)^{\frac{1}{p}} dy
\end{aligned}$$

It can be shown that if $x \in G \setminus K(\exp \Omega_t^p)K$ and $y \in K(\exp B(0,a))K$, then $xy^{-1} \in G \setminus K(\exp B(2t\gamma_p\rho, r(t) - a))K$ (see [1, Proposition 3.2]). Hence it follows that

$$\int_{G\setminus K(\exp \Omega_t^p)K} h_t(xy^{-1})^p\, dx \leq \int_{G\setminus K(\exp B(2t\gamma_p\rho, r(t)-a))K} h_t(z)^p\, dz. \qquad (3.35)$$

From (3.34) and (3.35) we get

$$\|h_t\|_p^{-1}\|f*h_t\|_{L^p(G\setminus K(\exp\Omega_t^p)K)} \leq \|f\|_1\|h_t\|_p^{-1}\|h_t\|_{L^p(G\setminus K(\exp B(2t\gamma_p\rho,r(t)-a))K)}. \tag{3.36}$$

From Proposition 3.2 and (3.36) it is clear that

$$\lim_{t\to\infty}\|h_t\|_p^{-1}\|f*h_t\|_{L^p(G\setminus K(\exp\Omega_t^p)K)}=0. \tag{3.37}$$

Since

$$\begin{aligned} & \|h_t\|_p^{-1}\|f*h_t-\widehat{f}(i\gamma_p\rho)h_t\|_{L^p(G\setminus K(\exp\Omega_t^p)K)} \\ \lesssim\ & \|h_t\|_p^{-1}\|f*h_t\|_{L^p(G\setminus K(\exp\Omega_t^p)K)}+\|h_t\|_p^{-1}\|h_t\|_{L^p(G\setminus K(\exp\Omega_t^p)K)}, \end{aligned} \tag{3.38}$$

the result follows from Proposition 3.2, (3.37) and (3.38). □

Lemma 3.6. *Let $1\leq p\leq 2$ and $f\in C_c^\infty(G//K)$. Then we have*

$$\lim_{t\to\infty}\|h_t\|_p^{-1}\|f*h_t-\widehat{f}(i\gamma_p\rho)h_t\|_{L^p(G)}=0. \tag{3.39}$$

Proof. For $1\leq p<2$, the Lemma follows from Proposition 3.4 and Proposition 3.5. For $p=2$, we have already proved the result in a more general set up (see Proposition 3.1). □

Completion of proof of Theorem 1.1.

Proof. We first prove a), that is, we assume that $p\in[1,2]$. Let $\epsilon>0$ be given. As $C_c^\infty(G//K)$ is dense in $\mathcal{L}^p(G//K)$, we choose $g\in C_c^\infty(G//K)$ be such that $\widehat{|f-g|}(i\gamma_p\rho)<\epsilon$. Clearly, we have

$$|\widehat{g}(i\gamma_p\rho)-\widehat{f}(i\gamma_p\rho)|\leq\widehat{|f-g|}(i\gamma_p\rho)<\epsilon. \tag{3.40}$$

From (2.28) we have

$$\|(f-g)*h_t\|_p\leq\widehat{|f-g|}(i\gamma_p\rho)\|h_t\|_p<\epsilon\|h_t\|_p. \tag{3.41}$$

As $g\in C_c^\infty(G//K)$, applying Lemma 3.6 we obtain $t_o>0$ such that

$$\|h_t\|_p^{-1}\|g*h_t-\widehat{g}(i\gamma_p\rho)h_t\|_{L^p(G)}<\epsilon, \tag{3.42}$$

whenever $t>t_o$. Owing to (3.41), (3.40) and triangle inequality we get

$$\begin{aligned} & \|f*h_t-\widehat{f}(i\gamma_p\rho)h_t\|_p \\ \leq\ & \|(f-g)*h_t\|_p+\|g*h_t-\widehat{g}(i\gamma_p\rho)h_t\|_p+|\widehat{g}(i\gamma_p\rho)-\widehat{f}(i\gamma_p\rho)|\|h_t\|_p \\ <\ & 2\epsilon\|h_t\|_p+\|g*h_t-\widehat{g}(i\gamma_p\rho)h_t\|_p. \end{aligned} \tag{3.43}$$

Thus, it follows from (3.42) and (3.43) that

$$\|h_t\|_p^{-1}\|f*h_t-\widehat{f}(i\gamma_p\rho)h_t\|_{L^p(G)}<3\epsilon,$$

whenever $t>t_o$. Now suppose

$$\lim_{t\to\infty}\|h_t\|_p^{-1}\|f*h_t-z\,h_t\|_{L^p(G)}=0.$$

Since

$$\begin{aligned}|z-\widehat{f}(i\gamma_p\rho)| &= \|h_t\|_p^{-1}\|zh_t-\widehat{f}(i\gamma_p\rho)h_t\|_p\\ &\leq \|h_t\|_p^{-1}\|f*h_t-z\,h_t\|_p+\|h_t\|_p^{-1}\|f*h_t-\widehat{f}(i\gamma_p\rho)\,h_t\|_p,\end{aligned}$$

by taking $t\to\infty$, we get $z=\widehat{f}(i\gamma_p\rho)$. This proves a).

To prove b), we first show that if $f\in\mathcal{L}^2(G//K)$ then $f*h_t\in L^\infty(G//K)$ and hence $f*h_t\in L^p(G//K)$, for all $p\in(2,\infty]$. From the estimate (2.23) it follows that for all $t\geq 1$ there exists a positive constant C_t such that

$$\frac{h_t(\exp H)}{\varphi_0(\exp H)}\leq C_t,\qquad \text{for all } H\in\overline{\mathfrak{a}^+}.$$

It follows that for all $x\in G$

$$\begin{aligned}|f*h_t(x)| &\leq \int_G|f(y^{-1}x)|h_t(y)\,dy\\ &= \int_G|f(y^{-1}x)|\varphi_0(y)\frac{h_t(y)}{\varphi_0(y)}\,dy\\ &\leq C_t|f|*\varphi_0(x)=C_t\widehat{|f|}(0)\varphi_0(x),\end{aligned}$$

implying that $f*h_t\in L^\infty(G//K)$. The convolution inequality (2.29) now implies that for any $p\in(2,\infty]$

$$\begin{aligned}\|f*h_t-\widehat{f}(0)h_t\|_p &= \|f*h_{t/2}*h_{t/2}-\widehat{f}(0)h_{t/2}*h_{t/2}\|_p\\ &\leq \|f*h_{t/2}-\widehat{f}(0)h_{t/2}\|_2\|h_{t/2}\|_2.\end{aligned}$$

Hence,

$$\frac{\|f*h_t-\widehat{f}(0)h_t\|_p}{\|h_t\|_p}\leq\frac{\|f*h_{t/2}-\widehat{f}(0)h_{t/2}\|_2}{\|h_{t/2}\|_2}\times\frac{\|h_{t/2}\|_2^2}{\|h_t\|_p}. \tag{3.44}$$

The estimates in (2.24) shows that

$$\frac{\|h_{t/2}\|_2^2}{\|h_t\|_p}\asymp\left(\frac{t}{2}\right)^{-\frac{2\nu}{4}}e^{-2|\rho|^2t/2}t^{\frac{\nu}{2}}e^{|\rho|^2t}\asymp 1,$$

where $\nu = l + 2|\Sigma_r^+|$, as mentioned in Subsection 2.2. Using this estimate in (3.44) we obtain

$$\frac{\|f * h_t - \widehat{f}(0)h_t\|_p}{\|h_t\|_p} \leq C\frac{\|f * h_{t/2} - \widehat{f}(0)h_{t/2}\|_2}{\|h_{t/2}\|_2}.$$

Thus from the proof of part a), it follws that the ration in the left hand side goes to zero as t goes to infinity. As before, for $2 < p \leq \infty$, we can write

$$\begin{aligned} |z - \widehat{f}(0)| &= \|h_t\|_p^{-1}\|zh_t - \widehat{f}(0)h_t\|_p \\ &\leq \|h_t\|_p^{-1}\|f * h_t - z\,h_t\|_p + \|h_t\|_p^{-1}\|f * h_t - \widehat{f}(0)\,h_t\|_p, \end{aligned}$$

by taking $t \to \infty$, we get $z = \widehat{f}(0)$. This completes the proof. ☐

3.1. **Counter example for the case when f is not K-bi-invariant.** Now, we show that Theorem 1.1 is not true when we assume f is not K-bi-invariant.

Proposition 3.7. *Let $1 \leq p \leq 2$ and $y \in G \setminus K$. Let $r(t) = t^a$ for some $\frac{1}{2} < a < 1$ and $\Omega_t^p := B(2t\gamma_p\rho, r(t))$. Then for every $x \in K(\exp\Omega_t^p)K$, we have*

$$\frac{h_t(y^{-1}x)}{h_t(x)} = e^{\langle 2\rho, A(k^{-1}y)\rangle} + O\left(\frac{r(t)}{t}\right), \tag{3.45}$$

where $x = k\exp(x^+)k'$ in the Cartan decomposition.

Proof. Proof is exactly similar to that of [1, Proposition 3.7]. Hence we leave the proof. ☐

Fix $1 \leq p \leq 2$. Let $r(t) = t^a$ for some $\frac{1}{2} < a < 1$ and $\Omega_t^p := B(2t\gamma_p\rho, r(t))$. Let $y \in G \setminus K$ be fixed and $x = k\exp(x^+)k' \in K(\exp\Omega_t^p)K$. From (3.45) we get

$$\begin{aligned} \left|\frac{h_t(y^{-1}x)}{h_t(x)} - \varphi_{i\gamma_p\rho}(y)\right|^p &\leq \left|e^{\langle 2\rho, A(k^{-1}y)\rangle} + O\left(\frac{r(t)}{t}\right) - \varphi_{i\gamma_p\rho}(y)\right|^p \\ &\leq \left(\left|e^{\langle 2\rho, A(k^{-1}y)\rangle} - \varphi_{i\gamma_p\rho}(y)\right| + O\left(\frac{r(t)}{t}\right)\right)^p \\ &\leq \left|e^{\langle 2\rho, A(k^{-1}y)\rangle} - \varphi_{i\gamma_p\rho}(y)\right|^p + O\left(\frac{r(t)}{t}\right). \end{aligned} \tag{3.46}$$

We observe that

$$\begin{aligned} &\|h_t\|_p^{-1}\|\delta_y * h_t - \widehat{\delta}_y(i\gamma_p\rho)h_t\|_p \\ =\ &\|h_t\|_p^{-1}\|h_t(y^{-1}\cdot) - \varphi_{i\gamma_p\rho}(y)h_t\|_p \\ =\ &\|h_t\|_p^{-1}\|h_t\left(\frac{h_t(y^{-1}\cdot)}{h_t} - \varphi_{i\gamma_p\rho}(y)\right)\|_p \end{aligned}$$

$$\geq \quad \|h_t\|_p^{-1} \|h_t \left(\frac{h_t(y^{-1}\cdot)}{h_t} - \varphi_{i\gamma_p\rho}(y) \right) \|_{L^p(K(\exp \Omega_t^p)K)}.$$

Using (3.46), we get

$$\|h_t\|_p^{-p} \|h_t \left(\frac{h_t(y^{-1}\cdot)}{h_t} - \varphi_{i\gamma_p\rho}(y) \right) \|_{L^p(K(\exp \Omega_t^p)K)}^p \tag{3.47}$$

$$= \quad \|h_t\|_p^{-p} \int\limits_{K(\exp \Omega_t^p)K} |h_t(x)|^p \left| \frac{h_t(y^{-1}x)}{h_t(x)} - \varphi_{i\gamma_p\rho}(y) \right|^p \, dx$$

$$= \quad \|h_t\|_p^{-p} \int\limits_{\Omega_t^p} |h_t(x^+)|^p \left[\int_K \left| e^{\langle 2\rho, A(k^{-1}y) \rangle} - \varphi_{i\gamma_p\rho}(y) \right|^p \, dk + O\left(\frac{r(t)}{t} \right) \right] J(x^+) \, dx^+$$

From Proposition 3.3 it follows that

$$\lim_{t\to\infty} \|h_t\|_p^{-p} \int\limits_{\Omega_t^p} |h_t(x^+)|^p J(x^+) \, dx^+ = 1. \tag{3.48}$$

Also observe that

$$\int_K \left| e^{\langle 2\rho, A(k^{-1}y) \rangle} - \varphi_{i\gamma_p\rho}(y) \right|^p \, dk > 0. \tag{3.49}$$

For, if $\int_K \left| e^{\langle 2\rho, A(k^{-1}y) \rangle} - \varphi_{i\gamma_p\rho}(y) \right|^p \, dk = 0$, then

$$e^{\langle 2\rho, A(k^{-1}y) \rangle} = \varphi_{i\gamma_p\rho}(y), \tag{3.50}$$

for every $k \in K$. As $y \notin K \Rightarrow y = k_3 \exp(y^+) k_4$ with $y^+ \neq 0$. Hence $\langle \rho, y^+ \rangle > 0$. Putting $k = k_3$ in (3.50) we get

$$1 < e^{\langle 2\rho, y^+ \rangle} = e^{\langle 2\rho, A(k_3^{-1}y) \rangle} = \varphi_{i\gamma_p\rho}(y) \leq 1 \tag{3.51}$$

which is a contradiction. Hence (3.49) is true. Thus it follows from (3.47), (3.48) and (3.49) that

$$\limsup_{t\to\infty} \|h_t\|_p^{-p} \|\delta_y * h_t - \widehat{\delta}_y(i\gamma_p\rho) h_t\|_p^p \geq \int_K \left| e^{\langle 2\rho, A(k^{-1}y) \rangle} - \varphi_{i\gamma_p\rho}(y) \right|^p \, dk > 0, \tag{3.52}$$

Hence we get

$$\limsup_{t\to\infty} \|h_t\|_p^{-1} \|\delta_y * h_t - \widehat{\delta}_y(i\gamma_p\rho) h_t\|_p > \alpha, \tag{3.53}$$

for some $\alpha > 0$. Let $f = \delta_y * h_1$ i.e.

$$f(x) = h_1(y^{-1}x) \quad (x \in G).$$

Clearly, $f \in L^q(G/K)$ for all $1 \leq q \leq \infty$. As $y \notin K$, it is also clear that f is not K-bi-invariant.

$$\begin{aligned}
& \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(i\gamma_p\rho) h_t\|_p \qquad (3.54)\\
= \;& \|h_t\|_p^{-1} \|\delta_y * h_{1+t} - \varphi_{i\gamma_p\rho}(y) \widehat{h}_1(i\gamma_p\rho) h_t\|_p \\
= \;& \|h_t\|_p^{-1} \|\delta_y * h_{1+t} - \varphi_{i\gamma_p\rho}(y) h_{1+t} + \varphi_{i\gamma_p\rho}(y) h_{1+t} - \varphi_{i\gamma_p\rho}(y) \widehat{h}_1(i\gamma_p\rho) h_t\|_p \\
\geq \;& \|h_t\|_p^{-1} \|\delta_y * h_{1+t} - \varphi_{i\gamma_p\rho}(y) h_{1+t}\|_p - \varphi_{i\gamma_p\rho}(y) \|h_t\|_p^{-1} \|h_{1+t} - \widehat{h}_1(i\gamma_p\rho) h_t\|_p \\
= \;& \frac{\|h_{t+1}\|_p}{\|h_t\|_p} \|h_{t+1}\|_p^{-1} \|\delta_y * h_{1+t} - \widehat{\delta}_y(i\gamma_p\rho) h_{1+t}\|_p - \varphi_{i\gamma_p\rho}(y) \|h_t\|_p^{-1} \|\|h_{1+t} - \widehat{h}_1(i\gamma_p\rho) h_t\|_p.
\end{aligned}$$

Owing to Theorem 1.1 we have

$$\lim_{t \to \infty} \|h_t\|_p^{-1} \|h_{1+t} - \widehat{h}_1(i\gamma_p\rho) h_t\|_p = 0. \tag{3.55}$$

Owing to (2.24), (3.53), (3.54) and (3.55) we get

$$\limsup_{t \to \infty} \|h_t\|_p^{-1} \|f * h_t - \widehat{f}(i\gamma_p\rho) h_t\|_p \geq e^{-\frac{4\rho^2}{pp'}} \alpha > 0.$$

Hence, Theorem 1.1 is not true in general for non K-bi-invariant functions.

4. Fractional heat Equation

For $0 < \alpha \leq 1$, the fractional heat kernel h_t^α is a bi-K-invariant L^2-function on X defined by its spherical Fourier transform which is given by

$$h_t^\alpha(\lambda) := e^{-t(|\lambda|^2 + |\rho|^2)^\alpha}, \quad (\lambda \in \mathfrak{a}). \tag{4.1}$$

Observe that for $\alpha = 1$, h_t^α is the heat kernel h_t while $h_t^{\frac{1}{2}}$ is the Poisson kernel on X. Let u be an L^2-tempered distribution on X. We define action of $(-\Delta)^\alpha$ on u by

$$\widehat{(-\Delta)^\alpha u} := -\left(|\cdot|^2 + |\rho|^2\right)^\alpha \widehat{u}.$$

One knows that h_t^α is the fundamental solution of the fractional heat equation on X:

$$\partial_t u + (-\Delta)^\alpha u = 0, \;\; t > 0.$$

It is known that $\int_X h_t^\alpha(x)\, dx = 1$ for every $t > 0$ and

$$h_t^\alpha * h_s^\alpha = h_{t+s}^\alpha, \tag{4.2}$$

for every $t > 0$ and $s > 0$.

For $t>1$, we also have [5, Lemma 3 (i)–(iii)]

$$\|h_t^\alpha\|_p \asymp \begin{cases} t^{-\frac{\ell}{2p'}} e^{-t\left(\frac{4|\rho|^2}{pp'}\right)^\alpha}, & 1\le p<2,\\ t^{-\frac{\ell}{4}-\frac{|\Sigma_r^+|}{2}} e^{-|\rho|^{2\alpha}t}, & p=2,\\ t^{-\frac{\ell}{2}-|\Sigma_r^+|} e^{-|\rho|^{2\alpha}t}, & p=\infty. \end{cases} \tag{4.3}$$

For $\alpha\in(0,1)$, there exists a positive function η_t^α (known as the Bochner's subordinator function) on $(0,\infty)$ such that the following subordination formula holds:

$$h_t^\alpha(x)=\int_0^\infty h_u(x)\eta_t^\alpha(u)\,du, \quad (u>0, x\in X). \tag{4.4}$$

We also have

$$e^{-t\gamma^\alpha}=\int_0^\infty e^{-u\gamma}\eta_t^\alpha(u)\,du, \quad (\gamma\ge 0). \tag{4.5}$$

We have the following global estimate for the subordinator function η_t^α [8, (8)–(9)]:

$$\eta_t^\alpha(u)\asymp \begin{cases} t^{\frac{1}{2(1-\alpha)}} u^{-\frac{(2-\alpha)}{2(1-\alpha)}} e^{-c_\alpha t^{\frac{1}{1-\alpha}} u^{-\frac{\alpha}{1-\alpha}}}, & u\le t^{\frac{1}{\alpha}},\\ t u^{-(1+\alpha)} e^{-c_\alpha t^{\frac{1}{1-\alpha}} u^{-\frac{\alpha}{1-\alpha}}}, & u>t^{\frac{1}{\alpha}}, \end{cases} \tag{4.6}$$

where $c_\alpha=(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$. From (4.6) it follows that there exist real numbers a_1 and a_2 such that

$$\eta_t^\alpha(u)\lesssim t^{a_1}(1+u^{a_2})e^{-c_\alpha t^{\frac{1}{1-\alpha}} u^{-\frac{\alpha}{1-\alpha}}}, \quad (t>1, u>0) \tag{4.7}$$

4.1. **Completion of proof of Theorem 1.2.**

Proof. We first prove part a), that is, we assume that $p\in(1,2]$. Without loss of generality we may assume that $f\in C_c^\infty(G//K)$, for, using standard density arguments as in the proof of Theorem 1.1, we can always extend the result to $\mathcal{L}^p(G//K)$. From (2.24) and (4.3) it is clear that for all $t>1$, there exists $a\ge 0$, $b\ge 0$ such that

$$\|h_t\|_p\asymp t^{-a}e^{-tb} \text{ and } \|h_t^\alpha\|_p\asymp t^{-a}e^{-tb^\alpha}. \tag{4.8}$$

Let $\epsilon>0$ be given. Owing to Theorem 1.1 we get $M_1>0$ such that

$$\|h_u\|_p^{-1}\|f*h_u-\widehat{f}(i\gamma_p\rho)h_u\|_p<\epsilon, \tag{4.9}$$

for all $u>M_1$.

From (4.4) it follows that

$$f*h_t^\alpha(x)-\widehat{f}(i\gamma_p\rho)h_t^\alpha(x)=\int_0^\infty \Big(f*h_u(x)-\widehat{f}(i\gamma_p\rho)h_u(x)\Big)\,\eta_t^\alpha(u)\,du.$$

Using Minkowski integral inequality we get

$$\|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p \leq \int_0^\infty \|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \, \eta_t^\alpha(u)\, du. \tag{4.10}$$

We choose a positive number $L > 0$ such that

$$c_\alpha L^{-\frac{\alpha}{(1-\alpha)}} - b^\alpha > \delta,$$

for some $\delta > 0$. From (4.10) we get

$$\|h_t^\alpha\|_p^{-1}\|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p \leq A_t + B_t, \tag{4.11}$$

where

$$A_t := \|h_t^\alpha\|_p^{-1} \int_0^{Lt} \|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \, \eta_t^\alpha(u)\, du,$$
$$B_t := \|h_t^\alpha\|_p^{-1} \int_{Lt}^\infty \|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \, \eta_t^\alpha(u)\, du.$$

For all large t with $t > \frac{M_1}{L}$, we have

$$\begin{aligned} B_t &= \|h_t^\alpha\|_p^{-1} \int_{Lt}^\infty \|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \, \eta_t^\alpha(u)\, du \\ &< \epsilon\|h_t^\alpha\|_p^{-1} \int_{Lt}^\infty \|h_u\|_p \, \eta_t^\alpha(u)\, du \\ &\lesssim \epsilon e^{tb^\alpha} \int_{Lt}^\infty e^{-ub}\, \eta_t^\alpha(u)\, du \quad (\text{as } u^{-a} \lesssim t^{-a}) \\ &\lesssim \epsilon e^{tb^\alpha} \int_0^\infty e^{-ub}\, \eta_t^\alpha(u)\, du \\ &\lesssim \epsilon \quad (\text{see } (4.5)). \end{aligned} \tag{4.12}$$

Since $\|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \lesssim \|h_u\|_p \leq u^{-a}$, from (4.7) we get

$$\begin{aligned} A_t &= \|h_t^\alpha\|_p^{-1} \int_0^{Lt} \|f * h_u - \widehat{f}(i\gamma_p\rho)h_u\|_p \, \eta_t^\alpha(u)\, du \\ &\lesssim \|h_t^\alpha\|_p^{-1} \int_0^{Lt} u^{-a}\, t^{a_1}(1 + u^{a_2}) e^{-c_\alpha t^{\frac{1}{1-\alpha}} u^{-\frac{\alpha}{1-\alpha}}}\, du. \end{aligned} \tag{4.13}$$

On putting $v = t^{\frac{\alpha}{1-\alpha}} u^{-\frac{\alpha}{1-\alpha}}$ in (4.13), we get

$$A_t \lesssim \|h_t^\alpha\|_p^{-1} t^{a_3} \int_{L^{-\frac{\alpha}{1-\alpha}}}^{\infty} v^{a_4}\, (1 + t^{a_5} v^{a_6}) e^{-c_\alpha t v}\, dv, \quad \text{for some } a_3, a_4, a_5, a_6 \in \mathbb{R}$$

$$
\begin{aligned}
&\lesssim t^{a_7}\int\limits_{L^{-\frac{\alpha}{1-\alpha}}}^{\infty} v^{a_4}\,(1+t^{a_5}v^{a_6})e^{-t(c_\alpha v-b^\alpha)}\,dv, \quad \text{(see (4.6), here } a_7:=a+a_3)\\
&\lesssim t^{a_7}\int\limits_{c_\alpha L^{-\frac{\alpha}{(1-\alpha)}}-b^\alpha}^{\infty} (u+b^\alpha)^{a_4}\,(1+t^{a_5}(u+b^\alpha)^{a_6})e^{-tu}\,du\\
&\lesssim t^{a_7}\int\limits_{\delta}^{\infty}(u+b^\alpha)^{a_4}\,(1+t^{a_5}(u+b^\alpha)^{a_6})e^{-tu}\,du\\
&\lesssim t^{a_7}\int\limits_{\delta}^{\infty}u^{a_4}\,(1+t^{a_5}u^{a_6})e^{-tu}\,du, \quad (\text{as } u+b^\alpha\asymp u),\\
&\lesssim t^{a_8}\int\limits_{\delta t}^{\infty}u^{a_4}\,(1+t^{a_9}u^{a_6})e^{-u}\,du, \quad \text{for some } a_8,a_9\in\mathbb{R}, \qquad (4.14)
\end{aligned}
$$

From (4.11), (4.12) and (4.14) it follows that

$$\|h_t^\alpha\|_p^{-1}\|f*h_t^\alpha-\widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p\lesssim\epsilon, \tag{4.15}$$

for all sufficiently large t. Proof of remaining part of the Theorem 1.2, a) is similar to that of Theorem 1.1, a).

The proof of Theorem 1.2, b) is analogous to that of Theorem 1.1, b) due to the convolution inequality (2.29) once we show that for all $p\in(2,\infty]$

$$\|h_{t/2}^\alpha\|_2^2\asymp\|h_t^\alpha\|_p, \quad \text{for all large } t.$$

Applying the Hölder's inequality, we obtain

$$\|h_{t+1}^\alpha\|_\infty=\|h_1^\alpha*h_t^\alpha\|_\infty\le\|h_1^\alpha\|_{p'}\|h_t^\alpha\|_p\lesssim\|h_t^\alpha\|_p.$$

Using the estimate (4.3) for $p=\infty$, and $p=2$, we get

$$\|h_t^\alpha\|_p\gtrsim(t+1)^{-\frac{\nu}{2}}e^{-(t+1)|\rho|^\alpha}\gtrsim t^{-\frac{\nu}{2}}e^{-t|\rho|^\alpha}\gtrsim\|h_{t/2}^\alpha\|_2^2.$$

The reverse inequality follows straight away from (2.29)

$$\|h_t^\alpha\|_p=\|h_{t/2}^\alpha*h_{t/2}^\alpha\|_p\lesssim\|h_{t/2}^\alpha\|_2^2.$$

This completes the proof. ∎

Remark 4.1. Theorem 1.2 for the case $\alpha = 1/2$ (which corresponds the Poisson kernel case) and $p = 1$ was proved in [6, 7], where the author also provides a counter example for a non K-bi-invariant initial data.

Remark 4.2. Let $p \in [1, 2]$ and $\alpha \in (0, 1]$ be fixed. Let $f \in \mathcal{L}^p(G//K)$ be a non-negative measurable function. We know that convolution by f from left defines a bounded linear operator $T_f : L^p(G//K) \to L^p(G//K)$ with $|T_f| \leq \widehat{f}(i\gamma_p\rho)$. We observe that

$$\begin{aligned} \widehat{f}(i\gamma_p\rho) &= \frac{\|\widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p}{\|h_t^\alpha\|_p} \leq \frac{\|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p}{\|h_t^\alpha\|_p} + \frac{\|f * h_t^\alpha\|_p}{\|h_t^\alpha\|_p} \\ &\leq \frac{\|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p}{\|h_t^\alpha\|_p} + |T_f| \\ &\leq \frac{\|f * h_t^\alpha - \widehat{f}(i\gamma_p\rho)h_t^\alpha\|_p}{\|h_t^\alpha\|_p} + \widehat{f}(i\gamma_p\rho). \end{aligned} \tag{4.16}$$

From (4.16) it is clear that

$$\widehat{f}(i\gamma_p\rho) = |T_f| = \lim_{t\to\infty} \frac{\|f * h_t^\alpha\|_p}{\|h_t^\alpha\|_p}. \tag{4.17}$$

(4.17) precisely says that in order to get the operator norm of T_f it is enough to look at the asymptotic limit of action of T_f on the family of unit vectors $\{\|h_t^\alpha\|_p^{-1}h_t^\alpha \mid t > 0\}$ instead of whole unit sphere of $L^p(G//K)$. In other words, the family $\{\|h_t^\alpha\|_p^{-1}h_t^\alpha \mid t > 0\}$ is an extremizer for the operator T_f.

5. Rank one results: Optimality and counterexamples

In this section we will discuss about sharpness of Theorem 1.1 for L^2-case in rank one symmetric space of noncompact type. For general rank symmetric spaces and for any $p \in [1, 2]$, we still don't know whether Theorem 1.1 is sharp or not. However we show that if $p = 2$ and $f \in L^q(G//K)$ for $1 \leq q < 2$, it is possible to get a better version of Theorem 1.1 (see Prosition 5.1). Since Theorem 1.1 is true for a much bigger class of functions, at this moment we can not conclude anything about its sharpness.

Now onwards we assume that X is a rank one symmetric space of non-compact type i.e. G is of real rank one. Thus the Lie group A is isomorphic to $\mathbb{R}$ and by fixing a suitable isomorphism, we write $A = \{a_t \mid t \in \mathbb{R}\}$.

5.1. **Abel transform.** For a suitable bi-K-invariant function f on G, Abel transform of f is defined by

$$\mathscr{A}f(t) := e^{\rho t} \int_N f(a_t n)\, dn, \quad (t \in \mathbb{R}).$$

For a nice function g on $\mathbb{R}$, let

$$\mathscr{F}g(\lambda) := \int_{\mathbb{R}} g(s) e^{-i\lambda s}\, ds, \ \ (\lambda \in \mathbb{R}).$$

denote the classical Fourier transform of g.

Let $f \in L^p(G//K)$ for $1 \leq p < 2$. Since $f \in L^p(G//K)$, we know that $\widehat{f}(\lambda)$ exists for all $\lambda \in \mathbb{C}$ with $|\Im\lambda| < \gamma_p \rho$ and the function $\lambda \mapsto \widehat{f}(\lambda)$ is an even analytic function. For such a function f, $\mathscr{A}f$ exists as an even function on $\mathbb{R}$ with

$$\int_{\mathbb{R}} |\mathscr{A}f(t)| e^{\alpha|t|}\, dt \lesssim \|f\|_p, \tag{5.1}$$

for any $0 \leq \alpha < \gamma_p \rho$ and

$$\mathscr{F}(\mathscr{A}f)(\lambda) = \widehat{f}(\lambda), \quad (\lambda \in \mathbb{C},\ |\Im\lambda| < \gamma_p \rho). \tag{5.2}$$

We observe that the c-function has the following behaviour in rank one symmetric space of non-compact type(see (2.16)):

$$|\mathrm{c}(\lambda)| \asymp |\lambda|^{-1}(1+|\lambda|)^{-\frac{n-3}{2}}, \quad (\Im\lambda \leq 0). \tag{5.3}$$

Proposition 5.1. *Let $p \in [1,2)$ and $\theta : (0,\infty) \to [0,\infty)$ be a non-negative function .*

(a) Suppose $\lim_{t\to\infty} \frac{\theta(t)}{t} = 0$. Then for any $f \in L^p(G//K)$,

$$\lim_{t\to\infty} \theta(t) \|h_t\|_2^{-1} \|f * h_t - \widehat{f}(0) h_t\|_2 = 0.$$

(b) Suppose $\limsup_{t\to\infty} \frac{\theta(t)}{t} > 0$. Then there exists $f \in L^p(G//K)$ such that

$$\limsup_{t\to\infty} \theta(t) \|h_t\|_2^{-1} \|f * h_t - \widehat{f}(0) h_t\|_2 > 0.$$

Proof. (a) From Plancheral theorem, (2.24) and (5.3) it follows that,

$$\begin{aligned} &\|h_t\|_2^{-2} \|f * h_t - \widehat{f}(0) h_t\|_2^2 \\ \asymp\ & t^{\frac{3}{2}} e^{2t\rho^2} \int_0^\infty |\widehat{f}(\lambda) - \widehat{f}(0)|^2 e^{-2t(\lambda^2+\rho^2)} |\mathrm{c}(\lambda)|^{-2}\, d\lambda \\ \asymp\ & t^{\frac{3}{2}} \int_0^\infty |\widehat{f}(\lambda) - \widehat{f}(0)|^2 e^{-2t\lambda^2} \lambda^2 (1+\lambda)^{n-3}\, d\lambda \end{aligned} \tag{5.4}$$

$$\asymp \int_0^\infty \left|\widehat{f}\left(\frac{\lambda}{\sqrt{t}}\right)-\widehat{f}(0)\right|^2 e^{-2\lambda^2}\lambda^2\left(1+\frac{\lambda}{\sqrt{t}}\right)^{n-3} d\lambda. \tag{5.5}$$

Since $\widehat{f}$ is an even real analytic function, by expanding $\widehat{f}$ around zero, we get

$$\widehat{f}\left(\frac{\lambda}{\sqrt{t}}\right)-\widehat{f}(0)=\frac{\lambda^2}{t}\,\widehat{f}^{(2)}(s_{\lambda,t}), \tag{5.6}$$

for some $s_{\lambda,t}\in\mathbb{R}$ depending on λ, t. We define a function $h:\mathbb{R}\to\mathbb{C}$ by

$$h(s)=s^2\mathscr{A}f(s),\quad s\in\mathbb{R}.$$

Owing to (5.1) and (5.2) we obtain

$$\begin{aligned}|\widehat{f}^{(2)}(u)| &= |(\mathscr{F}(\mathscr{A}f))^{(2)}(u)|=|\mathscr{F}(h)(u)|\le\|\mathscr{F}(h)\|_{L^\infty(\mathbb{R})}\\ &\le \|h\|_{L^1(\mathbb{R})}=\int_{\mathbb{R}}|\mathscr{A}f(s)|s^2\,ds\lesssim\int_{\mathbb{R}}|\mathscr{A}f(s)|e^{\frac{\gamma_p\rho}{2}s}\,ds\lesssim\|f\|_p. \end{aligned}\tag{5.7}$$

Hence from (5.5), (5.7) and (5.6) we get

$$\begin{aligned}&\theta(t)^2\|h_t\|_2^{-2}\|f*h_t-\widehat{f}(0)h_t\|_2^2\\ &\lesssim \left(\frac{\theta(t)}{t}\right)^2\int_0^\infty e^{-2\lambda^2}\lambda^6\left(1+\frac{\lambda}{\sqrt{t}}\right)^{n-3} d\lambda\end{aligned}\tag{5.8}$$

Since $\lim\limits_{t\to\infty}\dfrac{\theta(t)}{t}=0$ and $\left(1+\frac{\lambda}{\sqrt{t}}\right)^{n-3}\le(1+\lambda)^{(n-3)_+}$, by applying dominated convergence theorem in (5.8) we get the desired result.

(b) Let $f=h_1$ where h_1 is the heat kernel. Owing to (5.4) we get

$$\begin{aligned}&\|h_t\|_2^{-2}\|f*h_t-\widehat{f}(0)h_t\|_2^2\\ &\asymp t^{\frac{3}{2}}\int_0^\infty|e^{-(\lambda^2+\rho^2)}-e^{-\rho^2}|^2e^{-2t\lambda^2}\lambda^2(1+\lambda)^{n-3}\,d\lambda\\ &\asymp t^{\frac{3}{2}}\int_0^\infty|e^{-\lambda^2}-1|^2e^{-2t\lambda^2}\lambda^2(1+\lambda)^{n-3}\,d\lambda.\end{aligned}\tag{5.9}$$

Using the fact $|e^{-x}-1|\asymp\frac{x}{1+x}$ for $x\ge0$ in (5.9) we get

$$\begin{aligned}&\|h_t\|_2^{-2}\|f*h_t-\widehat{f}(0)h_t\|_2^2\\ &\asymp t^{\frac{3}{2}}\int_0^\infty\frac{\lambda^4}{(1+\lambda^2)^2}e^{-2t\lambda^2}\lambda^2(1+\lambda)^{n-3}\,d\lambda\\ &\asymp \frac{1}{t^2}\int_0^\infty\frac{\lambda^4}{\left(1+\frac{\lambda^2}{t}\right)^2}e^{-2\lambda^2}\lambda^2\left(1+\frac{\lambda}{\sqrt{t}}\right)^{n-3} d\lambda.\end{aligned}\tag{5.10}$$

Hence, from (5.10) it follows that

$$\limsup_{t\to\infty} \theta(t)^2 \|h_t\|_2^{-2} \|f * h_t - \widehat{f}(0) h_t\|_2^2$$
$$\asymp \limsup_{t\to\infty} \left(\frac{\theta(t)}{t}\right)^2 \int_0^\infty \lambda^6 e^{-2\lambda^2}\, d\lambda > 0.$$

This completes the proof.

□

5.2. **Counter example for ball average in L^1-case.** Let $m_r(x) = \frac{\chi_{B(o,r)}(x)}{|B(o,r)|}$. We know that for any $f \in L^1(\mathbb{R}^n)$ (see 1.1),

$$\lim_{r\to\infty} \|f * m_r - M(f) m_r\|_{L^1(\mathbb{R}^n)} = 0.$$

However this result is not true in rank one symmetric space of non-compact type. Indeed, we will show that

$$\lim_{r\to\infty} \|f * m_r - M(f) m_r\|_{L^1(X)} \nrightarrow 0$$

for any K-biinvarinat non-zero non-negative $f \in L^1(X)$. To see this let $0 \neq f \in L^1(G//K)$ be a non-negative measurable function. Suppose that

$$\lim_{r\to\infty} \|f * m_r - M(f) m_r\|_{L^1(X)} = 0.$$

Since $|\langle f * m_r - M(f) m_r, g\rangle| \leq \|f * m_r - M(f) m_r\|_1 \|g\|_\infty$, it follows that

$$\lim_{r\to\infty} \langle f * m_r - M(f) m_r, g\rangle = 0 \text{ for any } g \in L^\infty(G//K). \tag{5.11}$$

Let $g = \varphi_{\alpha+i\rho}$ with $\alpha \neq 0$. Clearly $g \in L^\infty(G//K)$. Let $\psi_\lambda(r) := \widehat{m_r}(\lambda) = \int_X m_r(x) \varphi_\lambda(x)\, dx$.

$$\begin{aligned} \langle f * m_r - M(f) m_r, g\rangle &= (f * (m_r * g))(e) - M(f)(m_r * g)(e) \qquad (5.12)\\ &= (f * (m_r * \varphi_{\alpha+i\rho}))(e) - M(f)(m_r * \varphi_{\alpha+i\rho})(e) \\ &= \psi_{\alpha+i\rho}(r)((f * \varphi_{\alpha+i\rho})(e) - M(f)) \\ &= C_f\, \psi_{\alpha+i\rho}(r) \end{aligned}$$

where $C_f := (f * \varphi_{\alpha+i\rho})(e) - M(f) = \int_X f(x)(\varphi_{\alpha+i\rho}(x) - 1)\, dx$.

As $0 \neq f \in L^1(G//K)$ is non-negative, from [12, Proposition 2.5.2(a)] it follows that $C_f < 0$. It is also known that there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals with $0 < r_n \uparrow \infty$ and

$$\lim_{n\to\infty} \psi_{\alpha+i\rho}(r_n) = C_\alpha \neq 0, \text{ (see [11, (4.0.3), p. 19]).} \tag{5.13}$$

Hence from (5.13) and (5.12) it is clear that

$$\lim_{n\to\infty}\langle f*m_{r_n}-M(f)m_{r_n},g\rangle=C_\alpha C_f\neq 0, \tag{5.14}$$

which clearly contradicts (5.11). Hence $\limsup_{r\to\infty}\|f*m_r-M(f)m_r\|_{L^1(X)}>0$ for any K-bi-invarinat non-zero non-negative $f\in L^1(X)$.

Acknowledgments

The authors would like to thank Mithun Bhowmik, Manjunath Krishnapur, and Sundaram Thangavelu for several useful discussions. Muna Naik and Jayanta Sarkar are supported by the INSPIRE faculty fellowship (Faculty Registration No.: IFA20-MA151, Faculty Registration No.: IFA22-MA-172) from the Department of Science and Technology, Government of India.

References

[1] Anker, J., Papageorgiou, E. & Zhang, H. Asymptotic behavior of solutions to the heat equation on noncompact symmetric spaces. *J. Funct. Anal.*. **284**, Paper No. 109828, 43 (2023), `https://doi.org/10.1016/j.jfa.2022.109828`

[2] Anker, J. & Ji, L. Heat kernel and Green function estimates on noncompact symmetric spaces. *Geom. Funct. Anal.*. **9**, 1035-1091 (1999), `https://doi.org/10.1007/s000390050107`

[3] Cowling, M., Herz's "principe de majoration" and the Kunze-Stein phenomenon, Harmonic analysis and Number Theory, CMS Conf. Proc. 21, A. M. S., Providence, RI, 1997, pp. 73–88.

[4] Cowling, M., Giulini, S., Meda, S. L^p-L^q-Estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces, I*. *Duke Math. J.* 72 (1993), no. 1, 109–150.

[5] Cowling, M., Giulini, S., Meda, S. L^p-L^q-Estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces, II*. *Journal of Lie Theory* . Volume 5 (1995), 1-14.

[6] Papageorgiou, E. Large-Time Behavior of Two Families of Operators Related to the Fractional Laplacian on Certain Riemannian Manifolds. *Potential Anal.* (2023), `https://doi.org/10.1007/s11118-023-10109-1`

[7] Papageorgiou, E. Asymptotic behavior of solutions to the extension problem for the fractional Laplacian on noncompact symmetric spaces. *J. Evol. Equ.* 24, 34 (2024). `https://doi.org/10.1007/s00028-024-00959-6`

[8] Graczyk, Piotr. Transition Density Estimates for stable processes on symmetric spaces. *Pacific Journal of Mathematics*, Vol. 217, No. 1, 2004

[9] Helgason, S. Geometric analysis on symmetric spaces. Second edition. *Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2008.*

[10] Helgason, S. Helgason, Sigurdur Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions. Corrected reprint of the 1984 original. *Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000.*

[11] Naik, M., Ray, S. & Sarkar, R. Mean value property in limit for eigenfunctions of the Laplace-Beltrami operator. *Trans. Amer. Math. Soc.*. **373**, 4735-4756 (2020), `https://doi.org/10.1090/tran/8078`

[12] Naik, M. & Sarkar, R. Characterization of eigenfunctions of the Laplace-Beltrami operator using Fourier multipliers. *J. Funct. Anal.*. **279**, 108737, 43 (2020), `https://doi.org/10.1016/j.jfa.2020.108737`

[13] Hansen, W. & Netuka, I. Successive averages and harmonic functions. *J. Anal. Math.*. **71** pp. 159-171 (1997), `https://doi.org/10.1007/BF02788028`

[14] Vázquez, J. Asymptotic behaviour methods for the heat equation. Convergence to the Gaussian, preprint, arXiv :1706.10034, 2018.

[15] Vázquez, J. Asymptotic behaviour for the fractional heat equation in the Euclidean space. *Complex Var. Elliptic Equ.*. **63**, 1216-1231 (2018).

[16] Vázquez, J. Asymptotic behaviour for the heat equation in hyperbolic space. *Comm. Anal. Geom.*. **30**, 2123-2156 (2022).

(Muna Naik) Indian Institute of Science, CV Raman Rd, Bengaluru, Karnataka, 560012 India

Email address: `mnaik41@gmail.com, muna@iisc.ac.in`

(Swagato K. Ray) Stat-Math Unit, Indian Statistical Institute, 203 B. T. Rd., Calcutta 700108, India

Email address: `swagato@isical.ac.in`

(Jayanta Sarkar) Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur-741246, Nadia, West Bengal, India

Email address: `jayantasarkarmath@gmail.com, jayantasarkar@iiserkol.ac.in`