Quantum dynamics of the effective field theory of the Calogero-Sutherland model

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Abstract

We consider the known effective field theory of the Calogero-Sutherland model in the thermodynamic limit of large number of particles, obtained from the standard procedure in conformal field theory: the Hilbert space is constructed *a priori* in terms of irreducible representations of the symmetry algebra, and not by diagonalization of the hamiltonian, which is given in terms of fields that carry representations of the $W_{1+\infty}$ algebra (representing the incompressibility of the Fermi sea). Nevertheless, the role of the effective hamiltonian of the theory is to establish a specific dynamics, which deserves further consideration. We show that the time evolution of the (chiral or antichiral) density field is given by the quantum Benjamin-Ono equation, in agreement with previous results obtained from the alternative description of the continuous limit of the model, based on quantum hydrodynamics. In this study, all calculations are performed at the quantum operator level, without making any assumption on the semiclassical limit of the fields and their equations of motion. This result may be considered as a reliable indication of the equivalence between the quantum field theoretic and quantum hydrodynamical formulations of the effective theories of the model. A one-dimensional quantum compressible fluid that includes both chiralities is the physical picture that emerges for the continuous limit of the Calogero-Sutherland model.

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1 Introduction

The Calogero-Sutherland (CS) model [1, 2] has a long and rich history in theoretical physics. It exhibits several interesting properties in its first quantized formulation, with a finite number of non-relativistic particles, as well as in its field theoretic formulation, which describes the thermodynamic limit of infinite number of particles. In this work, we shall consider the second alternative, after recalling that many of the interesting features of the first one are reviewed, *e.g.*, in [3]. The second quantized formulation of this model has been studied with at least two types of formulations: the quantum hydrodynamic approach of [4, 24] and the extended conformal field theory (CFT) [5] procedure of [6, 7, 8] (for alternative formulations, see also [9, 10]). Here we shall consider the second alternative with the scope of developing a better understanding of the physical picture that is implied by it, and to make contact with the most relevant results and conclusions of the first option, which involves a semi-classical and attractive picture of the dynamics. Quantum hydrodynamics and field theory are two ways of studying quantum matter and there relationship deserves to be more deeply understood (see, *e.g.*, [11]).

The extended CFT of the thermodynamic limit of the CS model naturally incorporates the idea of bosonization of the lowest energy fluctuations of the Fermi surface as the relevant semiclassical degrees of freedom [12] (see also [13]). The same theory describes general Luttinger systems as well [14]. The effective field theory (EFT) is constructed following the general prescription described in [15]: the low-lying deformations of the Fermi surface are parametrized in terms of the generators of the $W_{1+\infty}$ dynamical symmetry [16, 17, 18], which, therefore, describe the lowest energy ("gapless") fluctuations of the many-body states of the system in an algebraic formulation. Furthermore, the EFT allows for the inclusion of non perturbative effects due to the interaction among fermions by a change in the representation of the symmetry algebra, a result implied by Luttinger's theorem [19]. This is the main reason for writing the EFT in terms of the $W_{1+\infty}$ generators, which may be considered as favorable choice given the structure provided by this mathematical structure. In this work, we further investigate the physical consequences of this algebraic formulation of the effective theory, focusing on the time evolution of the quantum density field.

This paper is organized as follows: in section 2, we review the EFT of the CS model and the $W_{1+\infty}$ algebra. In section 3 we discuss the spectrum of the effective hamiltonian. In section 4 we study the equation of time evolution of the density field and discuss the relationship of our results to those obtained by the quantum hydrodynamical formulation of the CS effective theory. Finally, we provide some conclusions in section 5.

2 The EFT of the Calogero-Sutherland model

We start by briefly reviewing the CS model. Consider a system of N non-relativistic $(1+1)$-dimensional spinless interacting fermions on a circle of length L, with hamiltonian [2] (in units where $\hbar = 1$ and $2m = 1$, with m being the mass of the particles)

$$h_{CS} = \sum_{j=1}^{N} \left(\frac{1}{i} \frac{\partial}{\partial x_j} \right)^2 + g \frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j)/L)} , \tag{1}$$

where x_i $(i = 1, \ldots, N)$ is the coordinate of the i-th particle, and g is the dimensionless coupling constant. Ground state stability demands $g \geq -1/2$, with both attractive $(-1/2 \leq g < 0)$ and repulsive $(0 < g)$ regimes. A usual reparametrization of the coupling constant is given by $g = 2\xi(\xi - 1)$, so that $\xi \geq 0$ and $0 \leq \xi < 1$ is the attractive regime and $1 < \xi$ the repulsive one.

The EFT of (1) has been derived in [6][7] by reformulating the system dynamics in terms of variables (fields) that directly describe that of the $1D$ Fermi surface (isolated points, actually) in the thermodynamic limit. This method amounts to defining initially suitable non-relativistic fermionic fields, taking then the thermodynamic limit $N \to \infty$ properly on the fields and hamiltonian to find a EFT that involves two sets of independent relativistic (in the sense of a linear dispersion relation) fermion fields that describe the low energy fluctuations around each of the two Fermi points of the $1D$ effective theory. The last step consists in writing down the EFT in terms of fields that display the $W_{1+\infty}$ symmetry. This step is crucial in our approach, as it allows to diagonalize the hamiltonian and find the Hilbert space of the EFT by exploiting the algebraic properties of the $W_{1+\infty}$ algebra. Moreover, the $W_{1+\infty}$ algebra found in the original fermionic basis may be realized as well by bosonic operators, that take into account interactions in the fermionic picture, as anticipated. Here we outline the procedure: in the free case $(g = 0)$, the single-particle wave functions are given by plane waves:

$$\phi_k(x) = \frac{1}{\sqrt{L}} \exp \left(i \frac{2\pi}{L} kx \right) , \tag{2}$$

where k is an integer (half integer) for periodic (anti periodic) boundary conditions. The new variables are derived from a second quantized non-relativistic fermion field which is defined in terms of the above wave functions as:

$$\Psi(x,t) \equiv \sum_{k=-\infty}^{\infty} c_k \, \phi_k(x,t) , \qquad \{ c_k, c_l^\dagger \} = \delta_{k,l} . \tag{3}$$

Here $\phi_k(x,t) = \phi_k(x) \exp(-i\epsilon_k t)$, $\epsilon_k = (2\pi/L)^2 k^2$ and c_k, $c_l^\dagger$ are Fock space operators. The ground state of the system is

$$| \, \Omega \, , N \, \rangle = c_{-M}^\dagger c_{-M+1}^\dagger \ldots c_{M-1}^\dagger c_M^\dagger | \, 0 \, \rangle , \tag{4}$$

where $|\ 0\ \rangle$ is the Fock vacuum and $M \equiv (N-1)/2$. Note that k is an integer (half integer) if N is odd (even). The ground state numerical density $n(x,t)$ is given by

$$n(x,t) \ \equiv\ \langle\ \Omega\ , N\ |\Psi^\dagger(x,t)\Psi(x,t)|\ \Omega\ , N\ \rangle \ =\ \sum_{k=-M}^{M} |\phi_k(x,t)|^2 \ =\ \frac{N}{L}\ . \tag{5}$$

Therefore the spatial density $n(x,t) = n_0 = N/L$ is uniform and stationary.

Next we consider the thermodynamic limit of large N and L, with n_0 finite. The Fermi momentum is $p_F = \pi(N-1)/L$ and the Fermi sea a segment in momentum space between the two Fermi points, located at $\pm p_F$. The Fermi surface consists of the two isolated Fermi points. We refer to these two Fermi points by their locations as R and L, for right and left, respectively (we hope that no confusion arises with the length variable). In this limit, the ground state (4) becomes a relativistic (Dirac) sea for each one of them. The new variables are defined as 'shifted' Fock operators around each Fermi point [20][19][12]: $a_r \ \equiv\ c_{M+r}$ for the R Fermi point and $\overline{a}_r \ \equiv\ c_{-M-r}$ for the L one , where $|r| \ll \sqrt{N}$ describes small fluctuations, *i.e.*, the lowest energy excitations of the system. In this limit, the fields become Weyl fermion ones [5], which describe the relevant degrees of freedom of the system in the vicinity of the Fermi surface (points in this case) [15]. The corresponding hamiltonian may be derived and rewritten in terms of new fermionic fields that satisfy the $W_{1+\infty}$ algebra. Once this form is found, a one-parameter deformation keeping the algebraic structure extends this expression to a bosonized form the theory, which incorporates also steps towards factorization of the hamiltonian into chiral and antichiral sectors.

Making use of the bosonization expressions derived in [6, 7], we may write the effective hamiltonian as the following operator:

$$\begin{aligned}\mathcal{H}_{CS} \ &= \ \left(2\pi n_0\sqrt{\xi}\right)^2 \left\{\left[\frac{\sqrt{\xi}}{4}\, W^{\,0}_{\,0} + \frac{1}{N}\, W^{\,1}_{\,0} + \frac{1}{N^2}\left(\frac{1}{\sqrt{\xi}}\, W^{\,2}_{\,0} - \frac{\sqrt{\xi}}{12}\, W^{\,0}_{\,0}\right.\right.\right.\\ &\left.\left.\left. - \ \ \frac{g}{2\xi^2}\sum_{\ell=1}^{\infty} \ell\, W^{\,0}_{\,-\ell} W^{\,0}_{\,\ell}\right)\right] + \left(W \ \leftrightarrow\ \overline{W}\right)\right\} \quad ,\end{aligned} \tag{6}$$

where $\xi \ =\ \left(1+\sqrt{1+2g}\right)/2$ is the parameter defined after (1). Note that $\xi\ =\ 1$ corresponds to the free fermion case. Nevertheless, the relationship between ξ and g is not relevant when discussing the properties of the EFT, which leaves behind all the small-scale details of the underlying dynamics. With this idea in mind, we shall think of ξ as a real positive *free independent* parameter for the rest of the discussion. The operators $W^{\,m}_{\,\ell}$ in (6) are the lowest (in $m = i+1$, where i is the conformal spin [5]) generators of the infinite dimensional algebra known as $W_{1+\infty}$ [16, 17]. The terms in the $W^{\,m}_{\,\ell}$ ($\overline{W}^{\,i}_{\,\ell}$) operators describe the dynamics at the right (R) (left (L)) Fermi point, respectively. We remark that the *complete factorization* of (6) into chiral and antichiral sectors is possible for the $CS\ EFT$ only after performing a Bogoliubov transformation that decouples both sectors, that are generically mixed by backward scattering terms in the first fermionic form of the hamiltonian obtained in the thermodynamic limit [7, 8].

The general form of the $W_{1+\infty}$ algebra is:

$$\left[W^{\,i}_{\,\ell}, W^{\,j}_{\,m} \right] = (j\ell - im) W^{\,i+j-1}_{\,\ell+m} + q(i,j,\ell,m) W^{\,i+j-3}_{\,\ell+m} + \cdots + \delta^{ij}\delta_{\ell+m,0}\; c\; d(i,\ell) \ , \quad (7)$$

where the structure constants $q(i,j,\ell,m)$ and $d(i,\ell)$ are polynomial in their arguments, c is the central charge, and the dots denote a *finite* number of terms involving the operators $W^{\,i+j-2k}_{\,\ell+m}$. The ground state $|\Omega\rangle$ is a highest-weight state with respect to the $W_{1+\infty}$ operators, namely $W^{\,i}_{\,\ell}|\Omega\rangle = 0,\ \ell > 0, i \geq 0$, that is to say that it is incompressible in momentum space. We remark here that the basis $W^{\,i}_{\,\ell}$ of $W_{1+\infty}$ operators in the hamiltonian (6) is not the original fermionic one, inherited from the CS model, which we denote as $V^{\,i}_{\,\ell}$ but rather a bosonic one, as we shall explain right away. One major advantage for choosing the basis of the $W_{1+\infty} \times \overline{W}_{1+\infty}$ operators is that, once the algebraic content of the theory has been established in the free fermionic picture, the bosonic realization (in terms of bosonic field) of the algebra can be used, and the free value of the compactification radius of the boson can be chosen so as to diagonalize the hamiltonian. This method is consequently termed as *algebraic bosonization* [8]. For the case of the CS EFT $c = 1$, $W_{1+\infty}$ is interpreted as the enveloping algebra of the fermion number $U(1)$ symmetry, and all the relevant commutation relations are:

$$\begin{aligned}
\left[W^{\,0}_{\,\ell}, W^{\,0}_{\,m} \right] &= c\,\xi\ell\,\delta_{\ell+m,0} \quad , \\
\left[W^{\,1}_{\,\ell}, W^{\,0}_{\,m} \right] &= -m\, W^{\,0}_{\,\ell+m} \quad , \\
\left[W^{\,1}_{\,\ell}, W^{\,1}_{\,m} \right] &= (\ell - m) W^{\,1}_{\,\ell+m} + \frac{c}{12}\ell(\ell^2-1)\delta_{\ell+m,0} \quad , \\
\left[W^{\,2}_{\,\ell}, W^{\,0}_{\,m} \right] &= -2m\, W^{\,1}_{\,\ell+m} \quad , \\
\left[W^{\,2}_{\,\ell}, W^{\,1}_{\,m} \right] &= (\ell - 2m)\, W^{\,2}_{\,\ell+m} - \frac{1}{6}\left(m^3 - m\right) W^{\,0}_{\,\ell+m} \quad , \\
\left[W^{\,2}_{\,n}, W^{\,2}_{\,m} \right] &= (2n - 2m)\, W^{\,3}_{\,n+m} + \frac{n-m}{15}\left(2n^2 + 2m^2 - nm - 8\right) W^{\,1}_{\,n+m} \\
&\quad + c\,\frac{n(n^2-1)(n^2-4)}{180}\,\delta_{n+m,0} \quad .
\end{aligned} \quad (8)$$

The first and third equations in (8) show that the generators $W^{\,0}_{\,\ell}$ satisfy the abelian Kac-Moody algebra $\widehat{U(1)}$, and the generators $W^{\,1}_{\,\ell}$ satisfy the Virasoro algebra, respectively. The operators $\overline{W}^{\,i}_{\,\ell}$ satisfy the same algebra (8) with central charge $\overline{c} = 1$ and commute with the all the operators $W^{\,i}_{\,\ell}$. For this reason, the complete EFT of the CS model is a $(c, \overline{c}) = (1, 1)$ CFT, but since both chiral (R) and antichiral (L) sectors are isomorphic, we will often consider one of them for the sake of simplicity.

The $c = 1$ $W_{1+\infty}$ algebra can be realized by either *fermionic* or *bosonic* operators.

In our methodology for constructing EFTs, the first realization is useful for identifying the correct hamiltonian terms if one starts from fermionic systems, like the CS model, as it was mentioned above. Its explicit form is give by:

$$
\begin{aligned}
V^{\,0}_{\,n} &= \sum_{r=-\infty}^{\infty} :a^{\dagger}_{r-n}\, a_r: \quad , \\
V^{\,1}_{\,n} &= \sum_{r=-\infty}^{\infty} \left(r - \frac{n+1}{2} \right) :a^{\dagger}_{r-n}\, a_r: \quad , \\
V^{\,2}_{\,n} &= \sum_{r=-\infty}^{\infty} \left(r^2 - (n+1)\, r + \frac{(n+1)(n+2)}{6} \right) :a^{\dagger}_{r-n}\, a_r: \quad ,
\end{aligned}
\tag{9}
$$

with $\{\, a_k, a^{\dagger}_l \,\} \;=\; \delta_{k,l}$ and all other anticommutators vanishing. These are explicit expressions for the R Fermi point with analogous ones for the L one. The $W_{1+\infty}$ algebra satisfied by these operators is isomorphic to (8) with $\xi = 1$.

The second realization may be derived through a generalized Sugawara construction [17] in terms of a chiral bosonic field. In fact, if one introduces the right and left moving modes, α_ℓ and $\overline{\alpha}_\ell$, of a free compactified boson ($[\alpha_n, \alpha_m] = \xi n \delta_{n+m,0}$ and similarly for the $\overline{\alpha}_\ell$ operators), one can check that the commutation relations (8) are satisfied by defining $W^{\,i}_{\,\ell}$ (we only write the expressions for $i = 0, 1, 2$) as

$$
\begin{aligned}
W^{\,0}_{\,\ell} &= \alpha_\ell \quad , \\
W^{\,1}_{\,\ell} &= \frac{1}{2} \sum_{r=-\infty}^{\infty} :\alpha_r\, \alpha_{\ell-r}: \quad , \\
W^{\,2}_{\,\ell} &= \frac{1}{3} \sum_{r,s=-\infty}^{\infty} :\alpha_r\, \alpha_s\, \alpha_{\ell-r-s}: \quad ,
\end{aligned}
\tag{10}
$$

and analogously for the operators $\overline{W}^{\,i}_{\,\ell}$ in terms of $\overline{\alpha}_\ell$. We warn the reader that the naive generalization of (10) to higher values of conformal spin is incorrect (for example, see [8]). Finally, the relationship between these two realizations is given by the correspondence $W^{\,0}_{\,\ell} \leftrightarrow \sqrt{\xi} V^{\,0}_{\,\ell}$ and $W^{\,i}_{\,\ell} \leftrightarrow V^{\,i}_{\,\ell}$ $(i = 1, 2, \ldots)$. It follows that the constant density and the electric charge unit get renormalized by a factor $1/\xi$ with respect to the free fermion case.

A remark about the hamiltonian (6) is that it is given as a power series expansion in the small parameter $1/N$, but a *finite* one: the expansion stops at $1/N^2$. This result agrees with [4] and reflects the fact that the dispersion relation at the Fermi points is not linear, but still polynomial (quadratic in this case): on the contrary, the same method applied to the Heisenberg model yields an infinite power series [8]. For the CS EFT it is just a consequence of the dimensions of the interaction (1) that decays as $1/L^2$. The terms of $O(1)$ in (6) correspond to global operators (zero modes), the terms of $O(1/N)$ are the finite-size universal corrections given by CFT and the terms of $O(1/N^2)$ are beyond the usual scope of CFT.

3 Spectrum of the effective hamiltonian

A key remark regarding the derivation of (6) in [6, 7], is that the hamiltonian may be written as a sum of decoupled chiral sectors only after a Bogoliubov transformation has been performed on the fermionic $W_{1+\infty}$ operators. Indeed, the effective hamiltonian is naturally written in terms of the fermionic basis of the $W_{1+\infty}$ operators (9), and has backward scattering terms (that mix both chiralities). The Bogoliubov transformation decouples both Fermi points and defines a new $W_{1+\infty}$ basis that, moreover, is rewritten in the bosonic form (10), is given by:

$$\begin{aligned} W^{\,0}_{\,\ell} &= V^{\,0}_{\,\ell}\,\cosh\beta + \overline{V}^{\,0}_{\,-\ell}\,\sinh\beta \quad , \\ \overline{W}^{\,0}_{\,\ell} &= V^{\,0}_{\,-\ell}\,\sinh\beta + \overline{V}^{\,0}_{\,\ell}\,\cosh\beta \end{aligned} \tag{11}$$

for all ℓ, with

$$\tanh 2\beta = \frac{g}{2+g} \quad . \tag{12}$$

We now discuss the spectrum of (6). In the fermionic description $V^{\,i}_{\,\ell}$ it is easy to see that the highest weight states of the $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra are attained by the addition of ΔN particles to the ground state $|\Omega\rangle$, and by moving ΔD particles from the left to the right Fermi point; they are denoted by $|\Delta N, \Delta D\rangle_0$. The descendant states,

$$|\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 \; = V^{\,0}_{\,-k_1} \dots V^{\,0}_{\,-k_r} \overline{V}^{\,0}_{\,-\overline{k}_1} \dots \overline{V}^{\,0}_{\,-\overline{k}_s} |\Delta N, \Delta D\rangle_0 \quad ,$$

with $k_1 \geq k_2 \geq \ldots \geq k_r > 0$, and $\overline{k}_1 \geq \overline{k}_2 \geq \ldots \geq \overline{k}_s > 0$, coincide with the particle-hole excitations arising from $|\Delta N, \Delta D\rangle_0$. Using the expressions of $V^{\,0}_{\,0}$ and $\overline{V}^{\,0}_{\,0}$ given in (9), one finds that the charges associated to these states are

$$\begin{aligned} V^{\,0}_{\,0}\, |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} + \Delta D\right) |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 \quad , \\ \overline{V}^{\,0}_{\,0}\, |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} - \Delta D\right) |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 \quad . \end{aligned} \tag{13}$$

In terms of the bosonized operators basis $W^{\,i}_{\,\ell}$ the highest weight vectors, $|\Delta N; \Delta D\rangle_W$, are still characterized by the numbers ΔN and ΔD with the same meaning as before, but their charges are different. More precisely

$$\begin{aligned} W^{\,0}_{\,0}\, |\Delta N; \Delta D\rangle_W &= \left(\sqrt{\xi}\,\frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\xi}}\right) |\Delta N; \Delta D\rangle_W \\ \overline{W}^{\,0}_{\,0}\, |\Delta N; \Delta D\rangle_W &= \left(\sqrt{\xi}\,\frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\xi}}\right) |\Delta N; \Delta D\rangle_W \quad . \end{aligned} \tag{14}$$

The highest weight states $|\Delta N, \Delta D\rangle_W$ together with their descendants, denoted by $|\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_W$, form a *new* bosonic basis for the theory that has no simple expression in terms of the original free fermionic degrees of freedom.

The exact energies of these excitations in this basis are given by:

$$\mathcal{E} = \left(2\pi n_0\sqrt{\xi}\right)^2 \left\{ \left[\frac{\sqrt{\xi}}{4}\, Q + \frac{1}{N}\left(\frac{1}{2}\, Q^2 + k\right) + \frac{1}{N^2}\left(\frac{1}{3\sqrt{\xi}}\, Q^3 - \frac{\sqrt{\xi}}{12}\, Q \right.\right.\right. \tag{15}$$
$$\left.\left. + \frac{2k}{\sqrt{\xi}}\, Q + \frac{\sum_j k_j^2}{\xi} - \sum_j \left(2j-1\right) k_j \right)\right] + \left(Q \;\leftrightarrow\; \overline{Q}\;,\; \{k_j\} \;\leftrightarrow\; \{\overline{k}_j\}\right) \Bigg\} \quad ,$$

where

$$k = \sum_j k_j \quad , \quad \overline{k} = \sum_j \overline{k}_j$$

and the eigenvalues of $W^{\,0}_{\,0}$ and $\overline{W}^{\,0}_{\,0}$, respectively, are

$$Q = \sqrt{\xi}\,\frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\xi}} \quad , \quad \overline{Q} = \sqrt{\xi}\,\frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\xi}} \quad . \tag{16}$$

Moreover, the integers k_j are ordered according to $k_1 \geq k_2 \geq \ldots \geq 0$, and are different from zero only if $j << \sqrt{N}$, *i.e.*, within the range of validity of the EFT, and analogously for $\overline{k}_j$. Notice that Q and $\overline{Q}$ have the structure of the zero mode charges of a non-chiral bosonic field (which is the sum of chiral and antichiral bosons) compactified on a circle of radius

$$r = \frac{1}{\sqrt{\xi}} \quad . \tag{17}$$

Indeed, this is the exact value of the compactification radius of the bosonic field describing the density fluctuations of the fermions in the CS model [13]. The partition function of this $c = 1$ CFT is known to be invariant under the duality symmetry $r \leftrightarrow 1/(2r)$, which in our language is equivalent to the mapping $\sqrt{\xi} \leftrightarrow 2/\sqrt{\xi}$. The action of this mapping on the charges (16) is to interchange ΔN with ΔD, such that Q remains unchanged and $\overline{Q}$ maps onto minus itself. Some known identifications are: $\xi = 2$ is the self-dual point, $\xi = 1$ the free fermion point and $\xi = 1/2$ the Kosterlitz-Thouless point [5]. Notice that the zero modes are the only links between the L and R Fermi points within the framework of the EFT. From (15) we can also see that the exact value of the Fermi velocity is

$$v = 2\pi n_0\, \xi \quad , \tag{18}$$

as a consequence of Luttinger's theorem , that demands that the product vr^2 remains constant for any value of ξ [19]. In more physical terms, the spectrum (16) corresponds to both charged and neutral low-lying excitations. The charged (with respect to the $U(1)$ symmetry of fermion number) ones are labeled by Q, which is interpreted as a *soliton number*, represented in CFT by local vertex operators [5]. The neutral excitations are given by the integers k_j and correspond to *particle-hole*-like excitations. This analysis stems from the fermionic picture and generalizes to the bosonic one [6].

The structure of the spectrum is familiar in CFT: the charged excitations are highest weight states and the neutral fluctuations correspond to the Verma modules on top of each one of them [5]. That is to say that the EFT describes a uniform density ground state that may have solitons (located lumps or valleys) and fluctuations around them as the complete set of low-lying excitations. Unusually for a CFT, we have higher order corrections in Q beyond Q^2 in the hamiltonian, because the universal finite-size corrections are of order $1/N$.

4 Dynamics of the density field operator

In (1+1) systems with quantum dynamics, such as the CS model, fields and operators are naturally defined on the boundary circle ($0 \leq \theta < 2\pi$), *i.e.*, on a compact space. In the mathematical literature, however, they are conventionally considered in an unbounded space. There is a conformal mapping between these two spaces, which are the *cylinder* ($u = \tau/R - i\theta$) and the *conformal plane* (z), $x = L\theta/(2\pi)$

$$z = \exp\left(\frac{u}{R}\right) = \exp\left(\frac{\tau}{R} - i\theta\right) \ . \tag{19}$$

where $R = L/(2\pi)$, $x = R\theta$, that is x is periodic with period L.For example, the operators (8) define the $W_{1+\infty}$ currents $W^i(z)$ on the conformal plane as follows,

$$W^i(z) \ \equiv \ \sum_n \ W^i_n \ z^{-n-i-1} \ . \tag{20}$$

When studying the dynamics of a physical system like the CS model, the hamiltonian of edge excitations should be expressed in terms of the $W_{1+\infty}$ generators $(W_R)^i_n$ on the cylinder.

$$\begin{aligned} W^0(u) &= \frac{dz}{du} \, W^0(z) \ , \\ W^1(u) &= \left(\frac{dz}{du}\right)^2 \, W^1(z) \ + \frac{c}{12} \, S(z,u) \ , \\ W^2(u) &= \left(\frac{dz}{du}\right)^3 \, W^2(z) \ + \frac{1}{6} \, \frac{dz}{du} \, S(z,u) \, W^0(z) \ . \end{aligned} \tag{21}$$

The $W_{1+\infty}$ currents on the cylinder are thus found by using the mapping ,

$$\begin{aligned} W^0_R(u) &= \frac{z}{R} W^0(z) \ , \\ W^1_R(u) &= \frac{z}{R^2}\left(z^2 W^1(z) - \frac{1}{24}\right) \ , \\ W^2_R(u) &= \frac{z}{R^3}\left(z^3 W^2(z) - \frac{z}{12} V^0(z)\right) \ . \end{aligned} \tag{22}$$

Using the definition

$$(W_R)^j_0 \equiv \int_0^{2\pi i R} \frac{du}{(-2\pi i)} \, W^j_R(u) \ , \tag{23}$$

we find the relation between the zero modes in the two geometries

$$(W_R)_0^0 = W_0^0 \ , \qquad (W_R)_0^1 = \frac{1}{R}\left(W_0^1 - \frac{c}{24}\right) \ , \qquad (W_R)_0^2 = \frac{1}{R^2}\left(W_0^2 - \frac{1}{12}W_0^0\right) \ . \tag{24}$$

We now consider the transformation of the effective hamiltonian (6) from the plane to the cylinder geometry. There is a natural unit of energy, with $\hbar = c = 1$ and restoring the mass m of the CS particles, given by:

$$E_0 \ = \ \frac{2\pi^2 n_0^2 \sqrt{\xi}}{m} \ = \ \frac{\pi n_0 \xi N}{mR} \ . \tag{25}$$

The exact value of the dimensionless Fermi velocity is (recall R$n_0 = N/L$ and $R = L/(2\pi)$)

$$v \ = \ \frac{\pi n_0 \ \xi}{m} \ = \ v_0 \ \xi \ , \tag{26}$$

The effective hamiltonian on the (τ,θ) cylinder is, therefore, the following operator:

$$\begin{aligned} H_{CS} \ &= \ E_0 \ \left\{ \left[\frac{\sqrt{\xi}}{4} W\,_0^0 + \frac{1}{N} W\,_0^1 + \frac{1}{N^2} \left(\frac{1}{\sqrt{\xi}} W\,_0^2 - \frac{\sqrt{\xi}}{12} W\,_0^0 \right.\right.\right. \\ &- \left.\left.\left. \frac{(\xi-1)}{\xi} \sum_{\ell=1}^{\infty} \ell \ W\,_{-\ell}^0 W\,_\ell^0 \right) \right] + \left(W \ \leftrightarrow \ \overline{W} \right) \right\} \ . \end{aligned} \tag{27}$$

This is indeed defined on the cylinder because the hamiltonian does not involve terms with dimensional factors of powers of $1/R$. However, the Laurent modes are those defined by the commutator relations of the $W_{1+\infty}$ algebra (7) on the plane, as usual.

We are now in a suitable position to investigate the dynamics induced by the effective hamiltonian (27). We choose to study the time evolution of the chiral component the *density field*, as it the most obviously related to the semiclassical hydrodynamical approach to the CS model. Note, however, that the hamiltonian is not needed in order to determine the Hilbert space of the EFT, which is known since the beginning by the ‘kinematical construction of CFTs. The antichiral sector of the theory could be considered as well along similar lines. The density field equation of motion is related to that of the operator W_R^0:

$$\frac{\partial W_R^0(u)}{\partial t} \ = \ -i \left[W_R^0(u), H_{CS} \right] = \ -i \frac{z}{R} \left[W^0(z), H_{CS} \right] \tag{28}$$

We use the Sugawara construction with the normal ordering defined on the z plane:

$$W\,^1(z) = \frac{1}{2} : \left(W\,^0(z) \right)^2 : \ , \tag{29}$$

Furthermore, we define the density field as

$$n(x,t) \ = \ \frac{1}{\pi \xi^{3/2}} \ \left(W\,_R^0(x,t) + \pi n_0 \sqrt{\xi} \ \right) \ , \tag{30}$$

where $W^{\,0}_{R}(x,t)$ is the fluctuation field over the uniform value $\langle\Omega|n(x,t)|\Omega\rangle = n_0/\xi$, with $\langle\Omega|W^{\,0}_{R}(x,t)|\Omega\rangle = 0$. We find:

$$\begin{aligned}\frac{\partial n}{\partial t} \;&=\; \left(\frac{\pi\xi^2}{2m}\right)\,\frac{\partial}{\partial x}\left(n^2\right)\;+\\ &-\;\frac{iz(\xi-1)n_0}{m\sqrt{\xi}NR^2}\sum_{\ell=1}^{\infty}\left[\ell^2\;z^{-\ell-1}W^{\,0}_{\ell}\;-\ell^2\;z^{\ell-1}W^{\,0}_{-\ell}\,\right]\quad ,\end{aligned}\tag{31}$$

Note that the free fermion part coincides exactly with the *quantum Hopf equation* [4].

Next we focus on the interaction term in (31), which may be rewritten as:

$$\sum_{\ell=1}^{\infty}\left[\ell^2\;z^{-\ell-1}W^{\,0}_{\ell}\;-\ell^2\;z^{\ell-1}W^{\,0}_{-\ell}\,\right]\;=\;-\frac{\partial}{\partial z}\left[z\frac{\partial}{\partial z}\left(zW^0_+(z)\right)\;-\;z\frac{\partial}{\partial z}\left(zW^0_-(z)\right)\,\right]\;,\tag{32}$$

where

$$\begin{aligned}W^0_+(z)\;&=\;\sum_{n=1}^{\infty}\;W^0_n\;z^{-n-1}\\ W^0_-(z)\;&=\;\sum_{n=-\infty}^{-1}\;W^0_n\;z^{-n-1}\;.\end{aligned}\tag{33}$$

are the positive and negative Laurent mode fields. The equation of motion becomes:

$$\begin{aligned}\frac{\partial n}{\partial t}\;&=\;\left(\frac{\pi\xi^2}{2m}\right)\,\frac{\partial}{\partial x}\left(n^2\right)\;+\\ &-\;\frac{i(\xi-1)}{2\pi m\sqrt{\xi}}\,\frac{\partial^2}{\partial x^2}\left(\left(W^0_R\right)_+\;-\;\left(W^0_R\right)_-\right)\;.\end{aligned}\tag{34}$$

We would like to understand better the non familiar field that appears in the interaction term. It corresponds to an interaction that in terms of the density field is local in space but non-local in the time domain. Indeed, according to [4]:

$$\left(W^0_R\right)_H(x,t)\;=\;\left(W^0_+(u)\;-\;W^0_-(u)\,\right)\tag{35}$$

where $(W^0_R)_H\,(x,t)$ is the Hilbert transform of the field $W^0_R(x,t)$:

$$\left(W^0_R\right)_H(x,t)\;=\;\frac{1}{\pi}PV\;\int_{-\infty}^{\infty}\frac{W^0_R(x,t')}{(t-t')}\,dt'\;,\tag{36}$$

where PV denotes the Principal Value (there is a potential singularity at $t'=t$). We now show that (35) may be derived within the framework of CFT as well. We have

$$W^0_R(u)\;=\frac{z}{R}W^0(z)\;=\;\frac{1}{R}\sum_{n=-\infty}^{\infty}\;z^{-n}\;W^0_n\;,\tag{37}$$

where $z = \exp\left[(\tau - ix)/R\right]$, $t = i\tau$ and $(\tau - ix) \ = \ R \ln z$. The integral in the z plane is along the radial direction ('time evolution' on the plane).

$$\left(W_R^0\right)_H (x,\tau) \ = \ \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \ W_n^0 \ PV \int_0^{\infty} \frac{(z')^{-n-1}}{(\ln z - \ln z')} \, dz' \ , \tag{38}$$

where we assume $\theta' = \theta$, that is, spatial locality, and setting $|z| = r$ and $|z'| = r'$, such that the potential singularity is now at $z' = z$, we have:

$$\left(W_R^0\right)_H (x,\tau) \ = \ -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \ W_n^0 \ PV \int_0^{\infty} \frac{(r')^{-n-1}}{\ln r'/r} \, dr' \ , \tag{39}$$

setting $s = r'/r$, the potential singularity is now at $s = 1$. Therefore, we arrive to the expansion

$$\left(W_R^0\right)_H (x,\tau) \ = \ -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \ C_n \ \exp(-n\tau/R) \ W_n^0 \ , \tag{40}$$

with the coefficients

$$C_n \ = PV \int_0^{\infty} \frac{s^{-n-1}}{\ln s} \, ds \ . \tag{41}$$

These coefficients may be evaluated, taking care of subtracting the UV divergences properly, so that:

$$C_n = \begin{cases} -i\pi & (n > 0) \\ +i\pi & (n < 0) \end{cases} \tag{42}$$

Therefore, the coefficients perform the projection on the positive and negative modes of the Laurent expansion of the field W^0. Analogously, for the 0-mode we find $C_0 = -i\pi$, so that we can add it to the definition of C_n with $n > 0$. We have, therefore, shown that:

$$\left(W_R^0\right)_H \ = \ i \left[\left(W_R^0\right)_+ \ - \ \left(W_R^0\right)_- \right] \ . \tag{43}$$

Replacing this result in (34) we finally obtain:

$$\frac{\partial n}{\partial t} \ = \ \left(\frac{\pi \xi^2}{2m} \right) \ \frac{\partial}{\partial x} \left(n^2 \right) \ + \ \frac{(\xi - 1)\xi}{2m} \frac{\partial^2 n_H}{\partial x^2} \ , \tag{44}$$

where n_H is the Hilbert transform of the density, defined by (30) with W_R^0 replaced by $(W_R^0)_H$. This is the quantum Benjamin-Ono (BO) equation [4] , written with two coupling constants. It is identical to the classical BO equation but it is satisfied by a quantum field rather than by a real function. The classical Benjamin-Ono equation is a nonlinear partial integro-differential equation that describes one-dimensional internal waves in deep water [21, 22] . Each term has dimension 2 (in natural units), therefore the couplings $\alpha = \pi\xi^2/(2m)$ and $\beta = \xi(\xi - 1)/(2m)$ have both dimension 1. Their quotient is, therefore, dimensionless: $\beta/\alpha = (\xi - 1)/(\pi\xi)$. In [4] , the corresponding quantum BO equation has terms with different dimensions to (44) and one coupling constant, which is given by $(\xi-1)/(2\sqrt{\xi})$ and that should be put in correspondence with $(\xi-1)/(\pi\xi)$. Note, however, that in [4] , the Kac-Moody algebra has no ξ factor as in (8), so that one needs to introduce it for a correct comparison. The coupling α may be eliminated by a redefinition of the time coordinate. We therefore find a complete agreement between the dynamics of the density field predicted by quantum hydrodynamics and that of quantum field theory. A related study on the classical hydrodynamics of the CS model and the (classical) Benjamin-One equation [23] is available in [24] .

Finally, we discuss a physical interpretation of (35): the Hilbert transform associates to a quantum field decomposed in Laurent modes another one. This mapping is non-local in time, and involves both the distant past and future with respect to the field at present time. The field is a function of the coordinate z. In radial quantization, a growing $|z|$ amounts to time evolution, from the distant past ($|z| \to 0$) to the distant future ($|z| \to \infty$). Laurent expansion can be viewed as a partial wave decomposition. In this setting, the Hilbert transform may be viewed as transformation that takes an instantaneous function (potential) to a retarded or advanced one, as in the picture of electromagnetic charge wave emission. This idea was developed by Wheeler and Feynman [25] and it means that the terms with negative sign in (35) are interpreted as circular waves moving backwards in time from the distant future, 'the advanced waves' . Note that this interpretation is also consistent with the classical interpretation of the surface waves described by the BO equation, involving deep water and therefore long time reflected waves in the bottom of the sea.

We have explicitly shown how the Hilbert transform acting on the density field performs the separation of positive and negative Laurent modes, reversing the sign of half of them. Notice that the decomposition (35) arises as an effect on the density field by the CS interaction term in the hamiltonian (27).

5 Conclusions

The EFT that describes the thermodynamic limit of the CS model may be cast as a $(c,\overline{c}) = (1,1)$ CFTs with extended symmetry $W_{1+\infty} \times \overline{W}_{1+\infty}$ and chiral and antichiral sectors that are isomorphic. The Hilbert space and partition function for these theories are well known [5, 20]. Nonetheless, what singles out this theory is the specific dynamics and time evolution induced by its effective hamiltonian. In particular, the $1/r^2$ interaction in the CS model implies the finiteness of the expansion of the effective hamiltonian in terms of the $W_{1+\infty}$ generators (6) (see also [26]).

Regarding the equivalence of the descriptions based on quantum hydrodynamics and effective field theories, we mention here that besides the agreement on the

Benjamin-Ono equation obeyed by the density field, both the precise form of the hamiltonian (6) and the bosonic operator content (10) match exactly as well. This equivalence may help to gain further insights in both formulations.

We conclude by remarking that the consistent physical portrait arising from the continuous limit of the CS model is that of a one-dimensional quantum compressible fluid (with non-linear waves of the Benjamin-Ono type) that involves the two chiralities. This view stresses similarities and differences with the quantum incompressible fluids that appear in the quantum Hall effect. We believe that this qualitative picture may be useful in future developments.

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