Point-Charge Models and Averages for Electromagnetic Quantities Considered in Two Relativistic Inertial Frames

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Abstract

Electromagnetic quantities at a spacetime point have tensor Lorentz transformations between relatively-moving inertial frames. However, since the Lorentz transformation of time between inertial frames depends upon both the time and space coordinates, *averages* of electrodynamic quantities at a single time will in general depend upon the inertial frame, and will differ between inertial frames. Here we illustrate how the use of continuous charge and current distributions rather than point-charge distributions can lead to physically mystifying and even inaccurate results for electromagnetic quantities and physical phenomena. The discrepancy noted between the average electric field values in different inertial frames is particularly striking because it is first order in the relatative velocity between the frames.

I. INTRODUCTION

A. Apparent Relativity Paradox

There is an apparent relativity paradox involving magnets which has puzzled some teachers of electromagnetism. A toroid is claimed to have negligible electric and magnetic fields outside the surface currents. Thus, using tensor transformations for the fields, we should expect that there should be negligible electromagnetic fields outside the magnet in an inertial frame in which the magnet is moving. However, there is a problem in Griffiths' junior-level electromagnetism text[1] and in the first edition of Jackson's graduate text[2] (and only the first edition) suggesting that in an inertial frame in which the magnet is moving perpendicular to the magnetic field inside its winding, the magnet develops a non-zero scalar potential outside the magnet. The existence of a scalar potential suggests the possibility of electric fields outside the magnet in the inertial frame in which the magnet is moving. Indeed, the electric fields outside a magnet in the inertial frame in which it is moving have been used in a classical electromagnetic analysis[3] of the interaction of a charge and a magnet, an interaction made famous by the claims of Aharonov and Bohm.[4] The following question arises. In an inertial frame in which a electrically-neutral magnet is moving, are there or are there not electric fields outside the magnet in the direction parallel to the velocity?

This puzzling situation for teachers of electromagnetism is related to the use of continuous charge and current distributions rather than point charges. Much of classical electrodynamics is taught in the historical sequence involving continuous charge and current distributions. Indeed, it has been suggested[5] that *classical* electrodynamics should deal *only* with continuous sources. However, if one deals with electrodynamics from a *relativistic* perspective, then suddenly point spacetime events and point charges become important. Now most physicists seem convinced that any limit may be taken and interchanged without error with any other limit. Indeed, most of the time, such nonchalant interchange of limits does not lead to errors. In this article, we point out an error in connection with the interchange of Lorentz transformation and the limit of a continuous current in a current loop. By extension, the same error occurs in the classical electromagnetic analysis of the interaction of a continuous-current magnet and a passing charge in different inertial frames.

B. Tensor Transformations for Fields, *Not* for Averages

The explanation of the apparent relativity paradox mentioned in the first paragraph above involves the following crucial understanding. The electromagnetic field tensor assigned to a spacetime point is a mathematical representation of a *physical object* at that point, and so undergoes tensor transformations[6][7] between inertial frames. On the other hand, *averages at a single time* of electromagnetic quantities do *not* represent *physical objects,* and need not undergo tensor transformations between inertial frames. Indeed, averages at a single time over physical quantities may vary between *relativistic* inertial frames. This variation in averages at a single time between *relativistic* inertial frames arises because Lorentz transformations applied to the mathematical representations of physical quantities at different spacetime points involve the *spatial coordinates in the time transformation.*

It should be emphasized that this *variation in averages at a single time* over physical quantities between *relativistic* inertial frame is something which does *not* occur in *nonrelativistic* physics, and so seems surprising to many physicists. In nonrelativistic physics, time does *not* vary between inertial frames. Therefore a sum over the values of some quantity at a single time in one inertial frame will correspond to the sum over the values at a single time of the transformed quantity in a new inertial frame. Like so many other *relativity* paradoxes, the relativistic mixing of space and time coordinates on Lorentz transformation produces unfamiliar and sometimes surprising results.

Furthermore, an understanding of this apparent relativity paradox suggests again the importance of using point charges when discussing relativity and electrodynamics. The limit of an electrically neutral *continuous* current loop of zero-spatial extent (an "ideal" magnetic moment) produces a magnetic field only perpendicular to the plane of the current loop and so precludes an electric field parallel to the direction of relative velocity between inertial frames. In contrast, a current loop based upon point charges leads to a *time-varying* electric field component parallel to the relative velocity between inertial frames. The average values of the time-varying electric fields are different in different inertial frames. Toroids (or long solenoids) can be regarded as stacks of current loops. A stack of "ideal" magnetic moments leads to different electric fields from the time-varying electric fields outside a stack of current loops based on point charges. The contrast in the results of these two different models leads to different understandings of natural phenomena. Indeed, the point-charge model allows a classical electromagnetic understanding[3] of the Aharonov-Bohm situation whereas the "ideal"-magnetic-moment model does not.

C. Outline of the Article

Our analysis deals almost exclusively with the electromagnetic quantities associated with a neutral current loop formed by a charge q moving with constant angular velocity ω in a circle of radius R with an opposite charge $-q$ at rest at the center of the loop. Our aim is to point out that a loop with a *continuous* current leads to a different and restricted electric field outside the current loop compared to the *point-charge* model of the current loop. First we treat the charge densities, potential functions, and electromagnetic fields and their averages for this point-charge magnetic moment model in the inertial frame S in which the circular loop is at rest. Secondly, we consider these same quantities and their averages when seen in a second S' inertial frame in which the circular loop is moving with uniform velocity $\mathbf{V} = -\widehat{x}V$ in the x-direction parallel to the plane of the loop. For simplicity in the analysis, we use the Darwin Lagrangian approximation[8][9] and the associated potentials and fields. Also, we assume that the orbital circle has a small radius compared to the distance to the field point. The analysis suggests the behavior of a current loop with many charges, and also the behavior of the electromagnetic potentials and fields outside magnetic toroids and long solenoids. In addition, we show that a relativity problem in Griffiths' text involving an "ideal magnetic dipole" gives a different result from the analysis using point charges. We trace the discrepancy to the use of tensor transformations for *averages* of electromagnetic quantities rather than using tensor transformations for the actual *time-varying* fields at spacetime points. Thus sometimes the use of continuous charge and current densities leads to results different from those obtained using point charges. Finally, we note that the relativistic effect is unusual in involving *first* order in V/c, (where c is the speed of light in vacuum) like both the Fizeau experiment,[12] and the interaction of a magnet and a passing charge. We point out that the classical electromagnetic analysis suggests a classical lag basis[3] for the Aharonov-Bohm situation when a charged particle passes through a magnetic toroid. Gaussian units are used throughout the calculations.

II. CIRCULAR ORBIT FOR A POINT CHARGE MOVING WITH CONSTANT SPEED

A. Exact Expressions for the Trajectory of a Point Charge

We consider a point-charge model for a magnetic moment consisting of a point charge $+q$ moving with constant angular velocity ω in a circle of radius R in the xy-plane, centered on the origin. A negative charge $-q$ is at rest at the origin, the center of the circle. Then we have for the displacement of the moving charge q

$$\mathbf{r}_q(t) = \widehat{x}R\cos(\omega t + \phi_q) + \widehat{y}R\sin(\omega t + \phi_q)\,. \tag{1}$$

The velocity of the charge q follows as

$$\mathbf{u}_q(t) = -\widehat{x}R\omega\sin(\omega t + \phi_q) + \widehat{y}R\omega\cos(\omega t + \phi_q)\,, \tag{2}$$

and the acceleration as

$$\mathbf{a}_q(t) = -\omega^2\mathbf{r}_q(t) = -\widehat{x}\omega^2R\cos(\omega t + \phi_q) - \widehat{y}\omega^2R\sin(\omega t + \phi_q)\,. \tag{3}$$

The Liénard-Wiechert expressions for the retarded potentials and fields are given in standard textbooks of classical electrodynamics[10][11] All of the field quantities must be evaluated at the *retarded* time t_{q-ret} depending on the time, the source point, and the field point. If the charge is moving and the field point is not one of high symmetry, the calculation of the retarded time, and hence the evaluation of the electromagnetic field, may require numerical calculation.

B. Darwin-Lagrangian Approximations for a Point Charge

In this article, our interest is simply understanding the ideas associated with the apparent paradox noted in the opening paragraph and the validity of going to a continuous charge or current density. Thus we will restrict our discussion to small particle velocities $u_q = \omega R << c$ for the moving point charge $+q$ in the S inertial frame, and to small relative velocities $V << c$ between two inertial frames S and the S'. The relative velocity between the inertial frames is taken in the x-direction $\mathbf{V} = \widehat{x}V$. We will retain terms only through order Vu_q/c^2 in any expression for electromagnetic quantities. In this low-speed approximation and for field points which are not too distant from the source point, we may use the Darwin Lagrangian[8] and the associated electromagnetic potentials and fields. The Darwin approximation is vastly easier to work with because there are no retarded times involved. The electromagnetic potentials following from the Darwin Lagrangian are[8][9]

$$\Phi(\mathbf{r},t)=\sum\nolimits_q \frac{q}{|\mathbf{r}-\mathbf{r}_q(t)|} \tag{4}$$

and

$$\mathbf{A}\left(\mathbf{r},t\right)=\sum\nolimits_q \frac{q}{2c}\left[\frac{\mathbf{u}_q\left(t\right)}{|\mathbf{r}-\mathbf{r}_q(t)|}+\frac{\left[\mathbf{u}_q\left(t\right)\cdot\left(\mathbf{r}-\mathbf{r}_q(t)\right)\right]\left(\mathbf{r}-\mathbf{r}_q(t)\right)}{|\mathbf{r}-\mathbf{r}_q(t)|^3}\right], \tag{5}$$

and the electromagnetic fields are

$$\begin{aligned}\mathbf{E}\left(\mathbf{r},t\right)=&\sum\nolimits_q q\frac{\left(\mathbf{r}-\mathbf{r}_q(t)\right)}{|\mathbf{r}-\mathbf{r}_q(t)|^3}\left[1+\frac{1}{2}\left(\frac{u_q\left(t\right)}{c}\right)^2-\frac{3}{2}\left(\frac{\left(\mathbf{r}-\mathbf{r}_q(t)\right)\cdot\mathbf{u}_q\left(t\right)}{|\mathbf{r}-\mathbf{r}_q(t)|\,c}\right)^2\right]\\&-\frac{q}{2c^2\,|\mathbf{r}-\mathbf{r}_q(t)|}\left[\mathbf{a}_q\left(t\right)+\frac{\left[\mathbf{a}_q\left(t\right)\cdot\left(\mathbf{r}-\mathbf{r}_q(t)\right)\right]\left(\mathbf{r}-\mathbf{r}_q(t)\right)}{|\mathbf{r}-\mathbf{r}_q(t)|^2}\right]\end{aligned} \tag{6}$$

and

$$\mathbf{B}\left(\mathbf{r},\mathbf{t}\right)=\sum\nolimits_q q\frac{\mathbf{u}_q\left(t\right)\times\left(\mathbf{r}-\mathbf{r}_q(t)\right)}{c\,|\mathbf{r}-\mathbf{r}_q(t)|^3}. \tag{7}$$

If for our two-charge neutral current loop we choose the field point ($\mathbf{r}$,t) along the z-axis perpendicular to the plane of the loop, then the retarded-time correction is a fixed quantity depending on R and z, $t_{ret}=t-\sqrt{R^2+z^2}/c$, and it is easy to show that the result given by the Darwin approximation agrees with the exact result through order $\beta_q^2=u_q^2/c^2$, as indeed it should.

1. *First-Order in u_q/c and V/c*

We are interested in an effect which is first order in the particle speed u_x/c and first order in the relative velocity V/c. In this first-order approximation, we may drop the terms in the expression for the electric field in Eq. (6) which are already of order u_q/c^2 or ω^2R/c^2 without involving any term in V/c. Thus we may take the electric field as simply

$$\mathbf{E}\left(\mathbf{r},t\right)=\sum\nolimits_q q\frac{\left(\mathbf{r}-\mathbf{r}_q(t)\right)}{|\mathbf{r}-\mathbf{r}_q(t)|^3}. \tag{8}$$

2. *First-Order in R/r*

Finally, we assume that the radius R of the current loop is small compared to the distance to a field point $\mathbf{r} = \widehat{x}X + \widehat{y}Y$ in the xy-plane, $R << \sqrt{X^2 + Y^2}$, so that we can use the familiar approximations involving the electric dipole moment $\mathbf{p} = q\mathbf{r}_q$ and the magnetic dipole moment $\mathbf{m} = q\mathbf{r}_q \times \mathbf{u}_q/(2c)$. Then we find the potentials

$$\Phi(\mathbf{r}, t) = q\frac{\mathbf{r}_q(t) \cdot (\mathbf{r} - \mathbf{r_p})}{|\mathbf{r} - \mathbf{r_p}|^3} \tag{9}$$

and

$$\mathbf{A}(\mathbf{r}, t) = q\left[\frac{\mathbf{r}_q(t) \times \mathbf{u}_q(t)}{2c}\right]\frac{\times(\mathbf{r} - \mathbf{r_p})}{|\mathbf{r} - \mathbf{r_p}|^3} \tag{10}$$

where $\mathbf{r_p}$ is the location of the center of the current loop. In this approximation, the fields are given by dipole approximations as

$$\mathbf{E}(\mathbf{r},t) = q\left(\frac{3[\mathbf{r}_q(t) \cdot (\mathbf{r} - \mathbf{r_p})](\mathbf{r} - \mathbf{r_p})}{|\mathbf{r} - \mathbf{r_p}|^5} - \frac{\mathbf{r}_q(t)}{|\mathbf{r} - \mathbf{r_p}|^3}\right) \tag{11}$$

and

$$\mathbf{B}(\mathbf{r}, t) = q\left\{3\left[\frac{(\mathbf{r}_q \times \mathbf{u}_q)}{2c}\right] \cdot \frac{(\mathbf{r} - \mathbf{r_p})}{|\mathbf{r} - \mathbf{r_p}|^5} - \left[\frac{(\mathbf{r}_q \times \mathbf{u}_q)}{2c}\right]\frac{1}{|\mathbf{r} - \mathbf{r_p}|^3}\right\}. \tag{12}$$

C. Time-Varying Electric Quantities in the S Inertial Frame

The position $\mathbf{r}_q(t)$ and the velocity $\mathbf{u}_q(t)$ of the moving point charge are varying in time and lead to time-variations for the electric potential in Eq. (9) and the electric field in Eq. (11) with the center of the current loop at the origin, $\mathbf{r_p} = 0$. On the other hand the velocity $\mathbf{u}_q(t)$ of the charge q is always perpendicular to the displacement $\mathbf{r}_q(t)$ so that $\mathbf{r}_q(t) \times \mathbf{u}_q(t) = \widehat{z}Ru_q = \widehat{z}\omega R^2$ is constant in time. Accordingly the vector potential and the magnetic field are both constant in time as

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3} \tag{13}$$

and

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}) = \left\{\frac{3(\mathbf{m} \cdot \mathbf{r}_q)\mathbf{r}_q}{r^5} - \frac{\mathbf{m}}{r^3}\right\} \tag{14}$$

where the magnetic moment $\mathbf{m}$ is constant in time

$$\mathbf{m}(t) = q\left[\frac{(\mathbf{r}_q \times \mathbf{u}_q)}{2c}\right] = \widehat{z}q\frac{Ru_q}{2c} = \widehat{z}q\frac{\omega R^2}{2c} = \widehat{z}m. \tag{15}$$

D. Averages and the "Ideal" Magnetic Dipole Moment

1. *Averages at time t for the Point-Charge Model*

Although the electric potential and electric field of our *point-charge* magnetic moment model are *varying* in time, these electric quantities vanish when averaged over the phase ϕ_q at a single time t in the S inertial frame

$$\langle \mathbf{r}_q(t) \rangle_{\phi_q} = 0, \quad \langle \Phi(\mathbf{r}, t) \rangle_{\phi_q} = 0, \quad \text{and} \quad \langle \mathbf{E}(\mathbf{r}, t) \rangle_{\phi_q} = 0. \tag{16}$$

2. *The "Ideal" Magnetic Dipole Moment*

This vanishing of the averages of the time-varying electric quantities arising in the point-charge model for a magnetic moment has suggested the idea of an "ideal" magnetic dipole moment. The average over the phase angle ϕ_q has the same effect as considering many non-interacting point charges in a current loop and then taking the continuous current limit of many point charges subdivided repeatedly while maintaining the same magnetic moment. Indeed, the time-independent magnetic dipole moment $\mathbf{m}$ is the only non-vanishing quantity for our small-radius magnetic moment. If we imagine the radius R of the current loop as negligible in size, then we have an "ideal" point magnetic dipole of magnetic moment $\mathbf{m}$ and magnetization $\mathbf{M}(\mathbf{r}) = \mathbf{m}\delta^3(\mathbf{r})$. The associated current density is $\mathbf{J_m}(\mathbf{r}) = c\nabla \times \mathbf{M}(\mathbf{r}) = c\mathbf{m} \times \nabla\delta^3(\mathbf{r})$, while the charge density vanishes $\rho_{\mathbf{m}}(\mathbf{r}) = 0$, which is consistent with the charge continuity equation $\partial\rho_{\mathbf{m}}/\partial t = -\nabla \cdot \mathbf{J_m}$. In the xy-plane containing the "ideal" magnetic moment, the only magnetic field will be in the $\widehat{\mathbf{z}}$-direction.

3. *"Ideal" Magnetic Toroids and Solenoids*

We can imagine forming magnetic toroids and magnetic solenoids as stacks of "ideal" magnetic dipoles. From this perspective, there are no electric or magnetic field outside a *toroid* formed from such "ideal" magnetic moments. Indeed, one can consider a field point along the axis of a *magnetic toroid* formed from such "ideal" magnetic and can conclude by symmetry alone, that the magnetic field must vanish. What we wish to point out is that this "ideal" magnetic dipole moment involves different results compared to the point-charge model for the electric field in an inertial frame S' in which the magnetic moment is moving.

III. ANALYSIS OF THE POINT-CHARGE MAGNETIC MOMENT IN THE S' INERTIAL FRAME

A. Trajectory Equations for the Moving Charge q in the Inertial Frame S'

We will be considering relativistic transformations of the electromagnetic quantities between the inertial frames labeled by S and S'. We will take the S' inertial frame as moving with constant velocity $\mathbf{V} = \widehat{x}V$ relative to the S inertial frame. The displacement of the charge q in the S' inertial frame is obtained by Lorentz transformations $x' = \Gamma(x - Vt)$ and $y' = y$, $z' = z$, and $t' = \Gamma(t - Vx/c^2)$ where $\Gamma = \left[1 - (V/c)^2\right]^{-1/2}$ giving

$$\mathbf{r}'_q(t') = \widehat{x}\{\Gamma R\cos[\omega t + \phi_q] - Vt\} + \widehat{y}R\sin[\omega t + \phi_q]. \tag{17}$$

We note that $x_q = \Gamma\left(x'_q + Vt'\right)$ and use the Lorentz transformation for the time $t = \Gamma(t' + Vx'/c^2)$ to eliminate the unprimed time t in favor of primed quantities,

$$x'_q = -Vt' + \frac{1}{\Gamma}R\cos\left[\omega\Gamma(t' + Vx'_q/c^2) + \phi_q\right] \tag{18}$$

and

$$y'_q = R\sin\left[\omega\Gamma(t' + Vx'_q/c^2) + \phi_q\right]. \tag{19}$$

These equations give the exact trajectory for the charge in the S' inertial frame. Thus, in the S' inertial frame, we have *implicit* functions for the coordinates x'_q and y'_q in terms of the time t'. However, there seems to be no closed-form explicit solution for these equations. Thus, the situation here is quite different from that for a charged particle moving with constant velocity where the trajectory in a different relativistic inertial frame can be given in closed form.[11]

B. Trajectory of the Moving Charge in the Darwin-Lagrangian Approximation

Although there is no exact closed-form expression for the trajectory of the charge q in the S' inertial frame, we can find an approximate expression within the approximations we have introduced. Thus if we keep only terms through first order in V/c and first order in u_q/c, and through first order in r_q/r, then we find for the trajectory of the moving charge,

$$x'_q(t') = \Gamma(x_q - Vt) \approx x_q(t) - Vt = x_q(t) - Vt', \quad y'_q(t') = y_q(t), \quad z'_q = z_q = 0 \tag{20}$$

and

$$t'_q = \Gamma\left(t - Vx_q\left(t\right)/c^2\right) \approx t - \left(V/c\right)x_q\left(t\right)/c = t - \left(V/c\right)x_q\left(t'\right)/c, \tag{21}$$

and where the displacement $x_q\left(t\right)$ in the unprimed S inertial frame is given in Eq. (1). We notice that the approximate transformation involving first-order terms for the relative speed V gives a transformation for the displacement x'_q in Eq. (20) which is the same as is involved for nonrelativistic physics. It is the time transformation which is completely different. The time transformation is first-order in the relative velocity V between the S and S' inertial frames but is of order $1/c^2$. Since we are interested in averages over the phase ϕ_q (or equivalently averaging over many charged particles) at a single time t in the S inertial frame or at a single time t' in the S' inertial frame, the time transformation may lead to quite different results for averages in different inertial frames.

C. Electric Dipole Moment for the Charges in S'

In the S' inertial frame the charge q has the location $x'_q\left(t'\right) = x_q\left(t\right) - Vt = x_q\left(t\right) - Vt', y'_q\left(t'\right) = y_q\left(t\right)$ at time $t' = t - \left(V/c\right)x_q\left(t\right)/c = t - \left(V/c\right)x_q\left(t'\right)/c$. Then $x'_q(t') + Vt' = x_q(t)$ or keeping only first-order terms in V/c, we have

$$\begin{aligned}
x'_q(t') + Vt' &= x_q\left[t' + \frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&\approx R\cos\left\{\omega_q\left[t' + \frac{(V)}{c}\frac{x_q(t)}{c}\right] + \phi_q\right\} \\
&= R\cos\left(\omega_q t' + \phi_q\right)\cos\left[\omega_q\frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&- R\sin\left(\omega_q t' + \phi_q\right)\sin\left[\omega_q\frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&= R\cos\left(\omega_q t' + \phi_q\right) - \frac{(V)}{c}\frac{\omega_q R}{c}R\cos\left(\omega_q t'\right)\sin\left(\omega_q t' + \phi_q\right)
\end{aligned} \tag{22}$$

and

$$
\begin{aligned}
y_q'(t') &= y_q(t) = y\left[t' + \frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&= R\sin\left\{\omega_q\left[t' + \frac{(V)}{c}\frac{x_q(t)}{c}\right] + \phi_q\right\} \\
&= R\sin\left(\omega_q t' + \phi_q\right)\cos\left[\omega_q\frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&+ R\cos\left(\omega_q t' + \phi_q\right)\sin\left[\omega_q\frac{(V)}{c}\frac{x_q(t)}{c}\right] \\
&= R\sin\left(\omega_q t' + \phi_q\right) + \frac{(V)}{c}\frac{\omega_q R}{c}R\cos^2\left(\omega_q t' + \phi_q\right),
\end{aligned} \tag{23}
$$

where we have noted that $\sin(a + bx/c^2) \approx \sin a + (bx/c^2)\cos a$ and $\cos(a + bx/c^2) \approx \cos a - (bx/c^2)\sin a$ through order $1/c^2$. Our results are first order in V/c and first order in $u_q/c = \omega R/c$.

If we calculate the ϕ_q-average electric dipole moment at time t', we find $\mathbf{p}'(t') = q\mathbf{r}_q'(t') = \widehat{x}qx_q'(t') + \widehat{y}qy_q'(t')$ gives an average at time t'

$$
\langle\mathbf{p}'(t')\rangle = \left\langle q\mathbf{r}_q'(t')\right\rangle = \widehat{x}q\left\langle x_q'(t')\right\rangle + \widehat{y}q\left\langle y_q'(t')\right\rangle = \widehat{y}\frac{V\omega_q R^2}{2c^2}, \tag{24}
$$

which does not vanish in S'. The electric dipole moment of the $\pm q$ current loop has vanishing average value in S but non-zero average in S'. Thus, indeed averages over extended electromagnetic systems at a fixed time t or t' can vary with the choice of inertial frame.

D. Scalar Potential and Electric Field E_x in the S' Inertial Frame

All the approximations which held in the S inertial frame are also valid in the S' inertial frame. Thus, in the S' inertial frame, the scalar potential due to the charge $\pm q$ is given by

$$
\Phi'(\mathbf{r}', t') = \frac{\mathbf{p}'(t')\cdot\left(\mathbf{r}' - \mathbf{r}_\mathbf{p}'\right)}{\left|\mathbf{r}' - \mathbf{r}_\mathbf{p}'\right|^3} = \frac{q\mathbf{r}'(t')\cdot\left(\mathbf{r}' - \mathbf{r}_\mathbf{p}'\right)}{\left|\mathbf{r}' - \mathbf{r}_\mathbf{p}'\right|^3} \tag{25}
$$

with the center of the dipole located at $\mathbf{r}_\mathbf{p}' = -\mathbf{V}T'$. If we take the field point in the xy-plane at $X', Y', 0, T'$, this becomes

$$
\Phi'(X', Y', 0, T') = q\frac{(X' + VT')\,x_a'(T') + Y'y_q'(T')}{\left[(X' + VT') + Y'^2\right]^{3/2}}, \tag{26}
$$

where x'_q and y'_q are given in Eqs. (22) and (23). The ϕ_q-average value at time T' is

$$\begin{aligned}\left\langle \Phi'\left(X',Y',0,T'\right)\right\rangle_{\phi_q} &= q\frac{\left(X'+VT'\right)\left\langle x'_a\left(T'\right)\right\rangle + Y'\left\langle y'_q\left(T'\right)\right\rangle}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{3/2}} \\ &= q\frac{V\omega_q R^2}{2c^2}\frac{Y'}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{3/2}} \end{aligned} \tag{27}$$

from Eq. (24).

The electric field in S' is given by

$$\mathbf{E}'\left(\mathbf{r}',t'\right) = q\left(\frac{3\left[\mathbf{r}'_q\left(t'\right)\cdot\left(\mathbf{r}'-\mathbf{r}'_{\mathbf{p}}\right)\right]\left(\mathbf{r}'-\mathbf{r}'_{\mathbf{p}}\right)}{\left|\mathbf{r}'-\mathbf{r}'_{\mathbf{p}}\right|^5} - \frac{\mathbf{r}'_q\left(t'\right)}{\left|\mathbf{r}'-\mathbf{r}'_{\mathbf{p}}\right|^3}\right) \tag{28}$$

or

$$\begin{aligned}\mathbf{E}'\left(X',Y',0,T'\right) = q&\left(\frac{3\left[x'_q\left(t'\right)\left(X'+VT'\right)+y'_q\left(t'\right)Y'\right]\left[\widehat{x}\left(X'+VT'\right)+\widehat{y}Y'\right]}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{5/2}}\right. \\ &\left.-\frac{\widehat{x}x'_q\left(t'\right)+\widehat{y}y'_q\left(t'\right)}{\left[\left(X'-VT'\right)^2+Y'^2\right]^{3/2}}\right)\end{aligned} \tag{29}$$

where the moving dipole has its center at $\mathbf{r}'_{\mathbf{p}}\left(t'\right) = -\mathbf{V}t'$. The ϕ_q-average at time T' is

$$\begin{aligned}\left\langle \mathbf{E}'\left(X',Y',0,T'\right)\right\rangle_{\phi_q} &= q\left(\frac{3\left[\left\langle x'_q\left(t'\right)\right\rangle\left(X'+VT'\right)+\left\langle y'_q\left(t'\right)\right\rangle Y'\right]\left[\widehat{x}\left(X'+VT'\right)+\widehat{y}Y'\right]}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{5/2}}\right. \\ &\quad\left.-\frac{\widehat{x}\left\langle x'_q\left(t'\right)\right\rangle+\widehat{y}\left\langle y'_q\left(t'\right)\right\rangle}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{3/2}}\right) \\ &= \left[\frac{V\omega_q R^2}{2c^2}\right]\left(3\frac{Y'\left[\widehat{x}\left(X'+VT'\right)'+\widehat{y}Y'\right]}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{5/2}} - \frac{\widehat{y}}{\left[\left(X'+VT'\right)^2+Y'^2\right]^{3/2}}\right) \\ &= \left[\frac{V\omega_q R^2}{2c^2}\right]\left(3\frac{Y'\left[\widehat{x}X'+\widehat{y}Y'\right]}{\left[\left(X'\right)^2+Y'^2\right]^{5/2}} - \frac{\widehat{y}}{\left[\left(X'\right)^2+Y'^2\right]^{3/2}}\right)\end{aligned} \tag{30}$$

again from Eq. (24).

E. The Vector Potential and Magnetic Field in S'

Since the vector potential is already first order in $1/c$, we may use nonrelativistic transformations between S and S' when dealing with the vector potential. The vector potential is simply

$$\mathbf{A}'(X',Y',0,T') = m\frac{-\widehat{x}Y' + \widehat{y}(X'+VT')}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}}, \tag{31}$$

where the magnetic moment is unchanged,

$$\mathbf{m}'(t) = q\left[\frac{\left(\mathbf{r}'_q \times \mathbf{u}'_q\right)}{2c}\right] = \widehat{z}q\frac{\omega R^2}{2c} = \widehat{z}m. \tag{32}$$

since the expression for $\mathbf{m}$ is already first order in $1/c$ and we are dropping terms in $1/c^3$. The vector potential leads to a magnetic field which for field points $\mathbf{r}$,t in S or $\mathbf{r}', t'$ in S' is purely in the z-direction and

$$\mathbf{B}(X',Y',0,T') = -\widehat{z}\frac{m}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}}. \tag{33}$$

F. Lorentz Transformation of the Average Values

We saw above that in the S inertial frame, the ϕ_q-average values at a fixed time t for the point-charge model for a magnetic dipole agreed with the values given for the "ideal" magnetic dipole moment. If we carry out Lorentz transformations for the "ideal" magnetic moment from the S to the S' inertial frame, we find

$$\Phi'_{\mathbf{m}}(\mathbf{r}', t') \approx 0 - \frac{V}{c}A_x(\mathbf{r},t) \tag{34}$$

giving

$$\begin{aligned}\Phi'_{\mathbf{m}}(X',Y',0,T') &\approx 0 - \frac{V}{c}m\frac{-Y}{(X^2+Y^2)^{3/2}} \\ &= -\frac{V}{c}m\frac{-Y'}{\left[(X'+VT')^2+Y'^2\right]^{3/2}} \\ &\approx \frac{V}{c}m\frac{Y'}{[X'^2+Y'^2]^{3/2}}\end{aligned} \tag{35}$$

through first order in V/c. We notice that this result for the scalar potential agrees with the *average* value for the time-varying scalar potential given in Eq. (27). However, the average value does not include the additional time-varying term involving $x'_q(t')$ in Eq. (26). It is this additional time-varying term which, when combined with the relativistic time dependence, gives an average value for the electric field parallel to the velocity of the moving current loop in the S' inertial frame.

Since the ϕ_q-average scalar potential vanishes in S, the ϕ_q-average vector potential (within our approximations) is given by $\mathbf{A}'(\mathbf{r}',t') \approx \mathbf{A}(\mathbf{r},t)$, since the expression is already first order in $u_q/c = \omega R/c$ and we are dropping any terms in $1/c^3$. Since the only corrections to the non-relativistic expressions are already of order $1/c^2$ in Eq. (21), we have

$$\mathbf{A}'_{\mathbf{m}}(X',Y',0,T') \approx m\frac{-\widehat{x}Y' + \widehat{y}(X'+VT')}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}}. \tag{36}$$

Working from the "ideal" magnetic dipole potentials in Eqs. (35) and (36), we find the magnetic field from $B = \nabla \times \mathbf{A}$ giving

$$\mathbf{B}'_{\mathbf{m}}(X',Y',0,T') = -\widehat{z}\frac{m}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}}, \tag{37}$$

and the electric field from $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$ giving

$$\begin{aligned}
\mathbf{E}'_{\mathbf{m}}(X',Y',0,T') &= -\frac{V}{c}m\left\{\frac{-\widehat{y}}{\left[\left((X'+VT')^2\right)^2 + Y'^2\right]^{3/2}} - 3\frac{\widehat{x}(X'+VT')Y' + \widehat{y}(Y')^2}{\left[\left((X'+VT')^2\right)^2 + Y'^2\right]^{5/2}}\right\} \\
&\quad - \frac{1}{c}\frac{\partial}{\partial T'}\left(m\frac{-\widehat{x}Y' + \widehat{y}(X'+VT')}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}}\right) \\
&= \frac{V}{c}m\frac{\widehat{y}}{\left[\left((X'+VT')^2\right)^2 + Y'^2\right]^{3/2}} + 3\frac{\widehat{x}(X'+VT')Y' + \widehat{y}(Y')^2}{\left[\left((X'+VT')^2\right)^2 + Y'^2\right]^{5/2}} \\
&\quad - \frac{Vm}{c}\left\{\frac{\widehat{y}}{\left[(X'+VT')^2 + Y'^2\right]^{3/2}} - 3\frac{\left[-\widehat{x}Y' + \widehat{y}(X'+VT')\right](X'+VT')}{\left[\left((X'+VT')^2\right)^2 + Y'^2\right]^{5/2}}\right\} \\
&= \frac{V}{c}m\left\{\frac{\widehat{y}}{\left[\left((X')^2\right)^2 + Y'^2\right]^{3/2}} + 3\frac{\widehat{x}(X')Y' + \widehat{y}(Y')^2}{\left[\left((X')^2\right)^2 + Y'^2\right]^{5/2}}\right\} \\
&\quad - \frac{Vm}{c}\left\{\frac{\widehat{y}}{\left[(X')^2 + Y'^2\right]^{3/2}} - 3\frac{\left[-\widehat{x}Y' + \widehat{y}(X')\right](X')}{\left[(X')^2 + Y'^2\right]^{5/2}}\right\} \\
&= \widehat{y}\frac{V}{c}m\left\{\frac{3\left(X'^2 + Y'^2\right)}{\left[(X')^2 + Y'^2\right]^{5/2}}\right\} = \widehat{y}\frac{V}{c}m\left\{\frac{3}{\left[(X')^2 + Y'^2\right]^{3/2}}\right\}.
\end{aligned} \tag{38}$$

G. The "Ideal" Magnetic Moment Model Leaves Out Terms in the Electric Potential and Electric Field

We notice that if we use the ϕ_q-average value of the point-charge model or the "ideal" magnetic moment model in the S inertial frame, there is no component of the electric field in the direction of the relative velocity in the S' inertial frame. The tensor transformation $E_x = E'_x$ for the electromagnetic fields[6] [7] shows this same discrepancy depending upon whether or not one takes the average values in the S inertial frame. If one takes the average before making the Lorentz transformation to the S' inertial frame, one loses the time-varying electric dipole which gives a time-varying scalar potential and a time-varying electric field E_x in the x-direction. The time-varying expressions in the new S' inertial frame then lead to new *average* values in this frame at a single time t' because of the space-coordinate dependence of the time transformation given in Eq. (21). Average values at a fixed time lead to different averages in different inertial frames. This inertial-frame dependence of the averages is strikingly illustrated by the absence of any electric dipole moment for our point-charge system in the S inertial frame and the existence of a non-zero average dipole moment for our system in the S' inertial frame. The electromagnetic field tensor at a spacetime point is a mathematical representation of a physical object and the reprentation has tensor transformations between inertial frames. Averages at a single time have no physical existence but depend upon the choice of inertial frame in which the average is evaluated.

IV. COMMENTS ON THE ANALYSIS

A. Straight Line Current and Point-Charge Models

When dealing with relativity and electrodynamics, all textbooks discuss a straight line current. In this case, Lorentz transformations do not betray the importance of using point-charge models for the currents. The order of Lorentz transformations and the limit to continuous currents makes no difference. Of course, when trying to give a physical picture of what is involved in the sudden appearance of a non-zero charge density in a moving inertial frame S' from a neutral wire in the electrically neutral wire, the textbook discussion retreats from the continuous-current limit over to the point-charge picture. The textbook makes the Lorentz transformation in the point-charge picture, and then goes back to the continuous-current limit.

B. Failures of the Continuous-Current Model for a Magnet Moment

The straight-line continuous current which appears in all the textbooks betrays no error in interchanging the continuous-current limit and Lorentz transformation. However, in the case of a current loop, the interchange of the order of continuous-current limits and Lorentz transformation is not successful. Use of the continuous limit before making the Lorentz transformation leaves out an important part of the *time-varying* electric potential and *time-varying* electric field. The use of the average potential or average electric field in the S inertial frame leads to the "ideal" magnetic moment which has no electric field. Then use of Lorentz transformation omits the electric field parallel to the relative velocity between the relativistic inertial frames. Thus, the interchange of averages and Lorentz transformations fails for this reason.

The use of the "ideal" magnetic moment model also fails for another reason. The "ideal" magnetic moment may start with a continuous current I and finite radius R, giving a magnetic moment of magnitude $m = I\pi R^2/c$, but it takes the limit to a very small (zero) radius limit. If a point-charge mode of the current loop is used, then one becomes aware that the magnetic moment magnitude is $m = qu_q R/(2c) = q\omega R^2/(2c)$, and the limit of a very small radius means that the current $I = q\omega R/(2\pi R) = q\omega/(2\pi)$ must diverge. If speed of the charges is less than c, then the charge density must diverge.

Use of the "ideal" magnetic moment model also tends to ignore the possibility of Faraday induction because the area πR^2 of the loop is taken as negligible. Thus, the Faraday induction due to the changing magnetic field of a passing charge e tends to be ignored since the area of the "ideal" magnetic moment is so small. At the same time, the magnetic force on the "ideal" magnetic moment due to a passing charge e moving in the same plane as the magnetic dipole is given by $\mathbf{F}_{on\ \mathbf{m}} = -\nabla\left(-\mathbf{m}\cdot\mathbf{B}_e\left(\mathbf{r},t\right)\right) = m\nabla B_z\left(\mathbf{r},t\right)$ and, in general, will have a force component parallel to the relative velocity between the passing charge and the magnetic moment. However, this force on the"ideal" magnetic moment seems unconnected with a force on the passing charge or with ideas of changing electromagnetic energies. The use of "ideal" magnetic moments and "ideal" magnets has obscured the classical electromagnetic interaction between a magnet and a passing charge.

C. Effect First Order in the Relative Velocity Between the Inertial Frames

1. *Relativistic Effects of Order V^2/c^2*

Most relativistic experimental effects involve order $(V/c)^2$ where V is the relative velocity between the inertial frames. Thus, the Michelson-Morley experiment involves length contraction between inertial frames which is second order in the relative velocity between the frames. Similarly, the slowing down of decays for moving unstable particles involves time dilation which is again second order. In contrast, the discrepancy between the results from the point-charge model and the "ideal" magnetic moment model corresponds to a first-order effect in the relative velocity V between the frames.

2. *Relativistic Effects of Order V/c*

In the present point-charge model for a magnetic moment, there is a physical effect which is first order in the relative velocity V between the inertial frames. If we consider an external charge e passing our current loop, the existence of a force on the charge e in the direction of the velocity depends upon describing the system in terms of a *point-charge model for the current loop.* Using the point-charge model, there will be forces upon both the charge e and upon the current loop leading to a relative lag or lead depending upon which side of the loop the charge passes. In the S inertial frame in which the current loop has no average velocity, the forces are associated with magnetic fields of the charge e creating a magnetic force on the current loop and a Faraday induction effect of order Vu_x/c^2 leading to a force back on the charge e. In the frame S' in which the current loop is moving and the charge e is initially at rest, all the forces are electric attractions or repulsions of order Vu_x/c^2. This situation is analyzed in detail in the literature.[3] However, if one goes to the "ideal" magnetic moment model for the magnetic moment, then any classical electromagnetic analysis is obscure at best. Thus the Faraday induction becomes problematic in the S inertial frame because the size of the current loop is neglected; also, in the S' inertial frame, there is no electric field in the direction of the relative velocity of the "ideal" magnetic moment.

3. *The Fizeau Experiment*

The Fizeau experiment[12] involving light traveling in moving water is one of the few natural phenomena involving an effect first order in the relative velocity between the inertial frames. The effect is described in terms of relativistic addition of velocities where at a single spacetime point the transformation of the velocity between S and S' inertial frames is

$$u'_x = \frac{u_x - V}{1 - (Vu_x/c^2)} \approx u_x \left(1 + \frac{Vu_x}{c^2}\right) - V. \tag{39}$$

The *nonrelativistic* addition of velocities gives simply $u'_x = u_x - V$, but the additional relativistic correction in order Vu_x/c^2 is needed to bring high speeds down to values less than c in any inertial frame. In the Fizeau experiment, the water provides the relative motion between the inertial frames of water and lab, and the speed of light c/n relative to the water (where n is the index of refraction of the water) provides the second velocity.

4. *The Interaction of a Magnet and a Passing Charge*

A second example of a first-order effect (made famous by the claims of Aharonov and Bohm)[4] involves the interaction of a passing charge e and a magnet. In the case of a magnet and passing charge, the relative velocity V between the charge and the magnet provides the relative velocity between the inertial frames S and S', and the speed u_x of the point charges in the magnet provides the second velocity. This situation, of course, is directly related to the calculations in the present article where the speed of the point charge $+q$ in the S' inertial frame involves exactly the relativistic correction term Vu_x/c^2. In the S' inertial frame, the motion of the charge $+q$ has a larger *relativistic* slowing-down factor when it is moving at high speed (nonrelativistically $\omega R + V$) than when it is moving with the slower speed (nonrelativistically $\omega R - V$). Thus the positive point-charge model for the current loop develops a relativistic electric dipole moving where the relativistic $1/c^2$ correction to high-speed motion is on the positive side of the dipole and the slower-speed relativistic correction is on the negative side of the dipole. This situation involving *relativistic* speed corrections is just the reverse of the V^2/c^2 change in the density of a of a line charge λ of finite length which is Lorentz contracted in the direction of motion, but the total charge is Lorentz invariant. Thus, for the line charge, the faster the line charge moves relative to some inertial frame, the larger the charge density of the finite line charge.

D. Conclusion

Once again, we emphasize that tensor transformations of mathematical representations for electromagnetic quantities hold only at a single spacetime point. Here we have given an example showing that *averages* over an extended point-charge model of a current loop do not give reliable answers under Lorentz transformations between inertial frames. We also emphasize that *continuous* current distributions can disguise the point-charge nature of the Lorentz transformations for the currents at a spacetime point as seen in different inertial frames. Finally, we point out that the use of an "ideal" magnetic dipole or an "ideal" magnet, which appear in the literature, disguises appropriate Lorentz transformations between inertial frames.

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VI. DATA AND CONFLICTS

There is no new data associated with this article.

The author is not aware of any conflicts of interest.

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[1] D. J. Griffiths, *Introduction to Electrodynamics* 4th edn (Pearson, New York 2013), pp. 571-572, Problem 12.64. There is a misprint in Griffiths' text, p. 572; the parentheses do not balance.

[2] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1962), p. 389, Problem 11.9.

[3] T. H. Boyer, "Concerning Classical Forces, Energies, and Potentials for Accelerated Point Charges ," Am. J. Phys. **91**, 74 (2023); "The Classical Aharonov-Bohm Interaction as a Relativity Paradox," Eur. J. Phys. **44**, 035202 (11pp) (2023); "Classical Electromagnetic

Interaction of a Charge with a Solenoid or Toroid," Found. Phys **53,** 71 (29pp) (2023). "Concerning the Direction of the Aharonov-Bohm Deflection," J. Phys. A: Math. Theor. **56**, 455205 (2023).

[4] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in quantum theory," Phys. Rev. **115**, 485-491 (1959).

[5] R. M. Wald, *Advanced Classical Electromagnetism* (Princeton U. Press, Princeton, 2022), Section 1.4 pp.10-12. Indeed, Section 1.4 has the title, "At a Fundamental Level, Classical Charged Matter Must be Viewed as Continuous Rather Than Point-Like." See also, Section 8.3.

[6] See ref. 1, p. 559.

[7] J. D. Jackson, *Classical Electrodynamics 2nd ed* (John Wiley & Sons, New York, 1975), p. 552.

[8] C. G. Darwin, "The dynamical motions of charged particles," Phil. Mag. **39**, 537-551 (1920). See also ref. 7, pp. 593-595.

[9] L. Page and N. I. Adams, *Electrodynamics* (Van Nostrand, New York, 1940), p. 175; "Action and reaction between moving charges," Am. J. Phys. **13**, 141–147 (1945).

[10] See ref. 1, pp. 454, 459-460.

[11] See ref. 7, p. 657.

[12] H. L. Fizeau, "Sur les hypothèse relative à l'éther lumineux," Compt. Rend. **33**, 349-355 (1851). See the Wikipedia article, "Fizeau experiment." See also ref. 7, pp. 506 and 564.

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