# A solvable non-unitary fermionic long-range model with extended symmetry 

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#### Abstract

We define and study a long-range version of the xx model, arising as the free-fermion point of the xxz-type Haldane-Shastry (HS) chain. It has a simple realisation via non-unitary fermions, based on the free-fermion Temperley-Lieb algebra, and may also be viewed as an alternating $\mathfrak{g l}(1 \mid 1)$ spin chain. Even and odd length behave very differently; we focus on odd length. The model is integrable, and we explicitly identify two commuting hamiltonians. While non-unitary, their spectrum is real by PT-symmetry. One hamiltonian is chiral and quadratic in fermions, while the other is parityinvariant and quartic. Their one-particle spectra have two linear branches, realising a massless relativistic dispersion on the lattice. The appropriate fermionic modes arise from 'quasi-translation' symmetry, which replaces ordinary translation symmetry. The model exhibits exclusion statistics, like for the isotropic HS chain, with even more 'extended symmetry' and larger degeneracies.


Introduction. Integrability is an invaluable tool for studying strongly-interacting quantum many-body systems and two-dimensional statistical systems at and away from criticality. Integrable spin chains serve as lattice regularisations of integrable QFTs, and are particularly useful for the non-unitary CFTs describing critical disordered or geometric systems such as polymers or percolation. Examples include loop models and the closely related nearest-neighbour Heisenberg XXZ chain [1-3].

Models with long-range interactions are an important chapter of integrability. Prominent examples are Calogero-Sutherland systems [4] and the associated spin chains $[5,6]$, with deep relations to matrix models, exclusion statistics and 2d CFT $[7,8]$. Long-range spin chains also appear in AdS/CFT integrability [9]. While such models do not admit a standard Bethe-ansatz analysis, they can be tackled using algebraic methods. In particular, the trigonometric Calogero-Sutherland system and closely related Haldane-Shastry (HS) chain can be elegantly solved in terms of Jack polynomials [10, 11]. The HS chain furthermore has extended (Yangian) spin symmetry [11, 12], causing high degeneracies. The continuum limit in the antiferromagnetic regime, which is in the same universality class as for the Heisenberg xxx chain, is captured by the $S U(2)_{k=1}$ Wess-Zumino-Witten CFT [12-14]. Until recently there were few, if any, examples of non-unitary spin chains with extended symmetry. Such systems would provide finite discretisations of nonunitary CFTs with current-algebra symmetry, like those arising in disordered critical systems [15].

Main results. We introduce and solve a new integrable long-range model. It can be viewed as a long-range xx model, a long-range model of non-unitary fermions, or an alternating $\mathfrak{g l}(1 \mid 1)$ long-range super-spin chain. It has
i) a family of conserved charges,
ii) extended symmetry,
iii) an extremely degenerate and simple spectrum.

The HS chain already has these properties, but for us (iii) is more extreme still, so we expect (ii) to be so too.

The 'parent model' underlying our model is the xxztype counterpart of the HS chain [11, 16-18], reviewed in [19]. It generalises the isotropic HS chain by breaking the $\mathfrak{s u}(2)$ spin symmetry to $\mathfrak{u}(1)$ without spoiling its key properties. This underpins (i)-(iii). Crucially, the extended spin symmetry of the HS chain persists [11], where the Yangian is replaced by a quantum-affine algebra, and, in particular, $\mathfrak{s u}(2)$ by its 'quantisation' $U_{\mathrm{q}} \mathfrak{s l}(2)$. The deformation parameter q determines the anisotropy parameter of the Heisenberg Xxz chain via $\Delta=\left(q+q^{-1}\right) / 2$, and $\mathrm{q} \rightarrow 1$ yields the isotropic case. For real q most properties of the parent model parallel those of the HS chain. At root of unity, however, new features appear. Here we consider the simple but important case $\mathrm{q}=\mathrm{i}$. For the Heisenberg Xxz chain this gives the Xx model $(\Delta=0)$, equivalent to free fermions via the Jordan-Wigner transformation, with $\left.U_{\mathrm{q}} \mathfrak{s l}(2)\right|_{\mathrm{q}=\mathrm{i}}$ spin symmetry depending on the boundary conditions $[3,20,21]$. We study its longrange counterpart by combining knowledge from the parent model with fermionic techniques.

The model we propose has several striking features. Its properties depend sensitively on the parity of the system size. In this Letter we focus on an odd number of sites. Employing fermionic degrees of freedom that are related to spins through the Temperley-Lieb (TL) algebra, we are able to write down three conserved charges from (i) explicitly. One is a 'quasi-translation' that replaces the lattice translation, since ordinary translational invariance is broken. The second charge is free fermionic, and parity odd (chiral). Since the third charge has quartic interactions and is parity even, it is better suited for the role of hamiltonian. The extended symmetry (ii) requires special 'quasi-periodic' boundary conditions, which break unitarity, cf. [20]. Nevertheless, by PT invariance, the spectrum is real, cf. [22]. The reward for having complicated interactions is that this spectrum is extremely simple as in (iii): sums of quasiparticle energies with linear dispersions, comprising two branches that are associated with even and odd mode numbers. Spin chains with linear dispersions also arise in $\mathrm{AdS}_{3} / \mathrm{CFT}_{2}$ integra-
bility [23]. The interaction in the quartic hamiltonian implements a statistical selection rule, forbidding occupied successive mode numbers, that is inherited from the parent model and matches the description of the HS chain via 'motifs' [10]. This selection rule comes with high degeneracies for the motifs to account for the full Hilbert space. These degeneracies are caused by (ii), arising from the parent model's extended spin symmetry. The latter includes a global symmetry algebra that contains $\mathfrak{g l}(1 \mid 1)$ and is the commutant of the free-fermion TL algebra [2]. Due to the linear dispersions, there are many additional 'accidental' degeneracies between different motifs.
The model. Consider fermions hopping on a onedimensional lattice with an odd number $N$ sites.

The simplest definition of our model uses non-unitary fermionic operators with anticommutation relations [3]

$$
\begin{equation*}
\left\{f_{i}, f_{j}^{+}\right\}=(-1)^{i} \delta_{i j}, \quad\left\{f_{i}, f_{j}\right\}=\left\{f_{i}^{+}, f_{j}^{+}\right\}=0 \tag{1}
\end{equation*}
$$

They are related to canonical Jordan-Wigner fermions as $f_{j}=(-\mathrm{i})^{j} c_{j}, f_{j}^{+}=(-\mathrm{i})^{j} c_{j}^{\dagger}$. The $f$ s will avoid a proliferation of factors of $i$ and make the symmetries more transparent. From the two-site fermionic operators

$$
\begin{equation*}
g_{i} \equiv f_{i}+f_{i+1}, \quad g_{i}^{+}=f_{i}^{+}+f_{i+1}^{+}, \quad 1 \leqslant i<N \tag{2}
\end{equation*}
$$

we construct the quadratic combinations

$$
\begin{equation*}
e_{i} \equiv g_{i}^{+} g_{i}, \quad 1 \leqslant i<N \tag{3}
\end{equation*}
$$

which obey the free-fermion TL algebra relations

$$
\begin{equation*}
e_{i}^{2}=0, \quad e_{i} e_{i \pm 1} e_{i}=e_{i}, \quad\left[e_{i}, e_{j}\right]=0 \text { if }|i-j|>1 \tag{4}
\end{equation*}
$$

Further define the nested TL commutators [19, § C]

$$
\begin{align*}
e_{[i, j]} & \equiv\left[\left[\cdots\left[e_{i}, e_{i+1}\right], \cdots\right], e_{j}\right] \\
& =s_{i j}\left(g_{j}^{+} g_{i}+(-1)^{i-j} g_{i}^{+} g_{j}\right), \quad i \neq j \tag{5}
\end{align*}
$$

where $s_{i j} \equiv(-1)^{(i-j)(i+j-1) / 2}$, and we set $e_{[i, i]} \equiv e_{i}$. Note that (5) is bilinear in the fermions (1). Finally set

$$
\begin{equation*}
t_{k} \equiv \tan \frac{\pi k}{N}, \quad t_{k, l} \equiv \prod_{i=k}^{l-1} t_{i} \quad(k<l), \quad t_{k, k} \equiv 1 \tag{6}
\end{equation*}
$$

Then the chiral hamiltonian reads

$$
\begin{align*}
\mathrm{H}^{\mathrm{L}}= & \frac{\mathrm{i}}{2} \sum_{1 \leqslant i \leqslant j<N} h_{i j}^{\mathrm{L}} e_{[i, j]}, \\
& h_{i j}^{\mathrm{L}} \equiv \sum_{n=j+1}^{N} t_{n-j, n-i}\left(1-(-1)^{i} t_{n-i, n}^{2}\right) . \tag{7}
\end{align*}
$$

This is a quadratic (free-fermion) hamiltonian describing long-range hopping. It is not translationally invariant: the amplitudes $h_{i j}^{\mathrm{L}}$ do not only depend on the distance $i-j$, and sites $N, 1$ are not on the same footing as other neighbours $i, i+1$. Instead, the standard translation operator is replaced by the quasi-translation operator

$$
\begin{equation*}
\mathrm{G}=\left(1+t_{N-1} e_{N-1}\right) \cdots\left(1+t_{1} e_{1}\right), \quad \mathrm{G}^{N}=1 \tag{8}
\end{equation*}
$$

The model is integrable: there exists a hierarchy of conserved charges that commute with each other and (7). We can explicitly write down the next charge, which is a linear combination of anticommutators of the nested commutators (5) with coefficients like in (7),

$$
\begin{align*}
& \mathrm{H}=-\frac{1}{4 N} \sum_{i \leqslant j<k \leqslant l}^{N-1}\left(h_{i j ; k l}^{\mathrm{L}}+h_{i j ; k l}^{\mathrm{R}}\right)\left\{e_{[i, j]}, e_{[k, l]}\right\},  \tag{9}\\
& h_{i j ; k l}^{\mathrm{L}} \equiv(-1)^{k-j} \sum_{n(>l)}^{N} t_{n-l, n-j} t_{n-k, n-i}\left(1-(-1)^{i} t_{n-i, n}^{2}\right), \\
& h_{i j ; k l}^{\mathrm{R}} \equiv(-1)^{l-j+k-i} h_{N-l, N-k ; N-j, N-i}^{\mathrm{L}} .
\end{align*}
$$

Integrability guarantees that these quantities, and higher charges that we do not give here, mutually commute,

$$
\begin{equation*}
\left[\mathrm{G}, \mathrm{H}^{\mathrm{L}}\right]=[\mathrm{G}, \mathrm{H}]=\left[\mathrm{H}^{\mathrm{L}}, \mathrm{H}\right]=0 . \tag{10}
\end{equation*}
$$

The existence of this hierarchy of commuting charges, and their expressions, stem from the parent model [19].

Symmetries. The commuting charges (7)-(9) have various symmetries and transformation properties.

Parity. Parity acts by reversal of the lattice sites $\mathrm{P}\left(f_{i}\right)=f_{N+1-i}$. This preserves the anticommutation relations (1) since $N$ is odd. The TL generators transform as $\mathrm{P}\left(e_{i}\right)=e_{N-i}$. The chiral hamiltonian (7) is not invariant under parity, whence its name. It is a highly nontrivial result that $\mathrm{P}\left(\mathrm{H}^{\mathrm{L}}\right)=-\mathrm{H}^{\mathrm{L}}$ [19]. We have

$$
\begin{equation*}
\mathrm{P}\left(\mathrm{H}^{\mathrm{L}}\right)=-\mathrm{H}^{\mathrm{L}}, \quad \mathrm{P}(\mathrm{H})=\mathrm{H}, \quad \mathrm{P}(\mathrm{G})=\mathrm{G}^{-1} \tag{11}
\end{equation*}
$$

where the last relation uses $t_{i}=-t_{N-i}$.
Time reversal. We define time reversal as complex conjugation of the coefficients with respect to the Fock basis $f_{i_{1}}^{+} \cdots f_{i_{M}}^{+}|0\rangle$. Thus $\mathrm{T}\left(e_{i}\right)=e_{i}$, and

$$
\begin{equation*}
\mathrm{T}\left(\mathrm{H}^{\mathrm{L}}\right)=-\mathrm{H}^{\mathrm{L}}, \quad \mathrm{~T}(\mathrm{H})=\mathrm{H}, \quad \mathrm{~T}(\mathrm{G})=\mathrm{G} \tag{12}
\end{equation*}
$$

Since the hamiltonians (and their eigenstates) are PTinvariant, their spectrum is real [24-27]. The same is true for the 'quasi-momentum' $\mathrm{p}=-\mathrm{i} \log \mathrm{G}$.

Particle-hole transformation. Another simple operation, which we interpret as the charge conjugation, exchanges the creation and annihilation operators

$$
\begin{equation*}
\mathrm{C}\left(f_{i}\right)=f_{i}^{+}, \quad \mathrm{C}\left(f_{i}^{+}\right)=f_{i} \tag{13}
\end{equation*}
$$

Then $\mathrm{C}\left(e_{i}\right)=-e_{i}[19, \S \mathrm{C}]$, which preserves the freefermion TL relations. The transformed conserved charges arise from (7)-(9) by formally replacing $t_{i} \rightarrow-t_{i}$.

Global symmetry. The model can be seen as a longrange spin chain with alternating $\mathfrak{g l}(1 \mid 1)$-representations. Recall that $\mathfrak{g l}(1 \mid 1)$ has two bosonic and two fermionic generators, which we denote by $\mathrm{N}, \mathrm{E}$ and $\mathrm{F}_{1}, \mathrm{~F}_{1}^{+}$respectively. The nontrivial (anti)commutation relations are

$$
\begin{equation*}
\left[\mathrm{N}, \mathrm{~F}_{1}\right]=-\mathrm{F}_{1}, \quad\left[\mathrm{~N}, \mathrm{~F}_{1}^{+}\right]=\mathrm{F}_{1}^{+}, \quad\left\{\mathrm{F}_{1}, \mathrm{~F}_{1}^{+}\right\}=\mathrm{E} \tag{14}
\end{equation*}
$$

and E is central. This is just a fermionic version of the usual spin algebra. Each site $i$ carries a $\mathfrak{g l}(1 \mid 1)$ representation generated by $f_{i}, f_{i}^{+}$, the number operator $(-1)^{i} f_{i}^{+} f_{i}$ and central charge $(-1)^{i}$. From this perspective, (7)-(9) is a long-range $\mathfrak{g l}(1 \mid 1)$ super-spin chain. For odd length the alternating central charge breaks periodic boundaries, which are replaced by (8). Our model has a global $\mathfrak{g l}(1 \mid 1)$-symmetry generated by

$$
\begin{gather*}
\mathrm{F}_{1}=\sum_{i=1}^{N} f_{i}, \quad \mathrm{~F}_{1}^{+}=\sum_{i=1}^{N} f_{i}^{+}  \tag{15}\\
\mathrm{N}=\sum_{i=1}^{N}(-1)^{i} f_{i}^{+} f_{i}, \quad \mathrm{E}=\sum_{i=1}^{N}(-1)^{i}=-1
\end{gather*}
$$

Indeed, these operators anticommute with all $g_{i}, g_{j}^{+}$, and thus commute with the conserved charges (7)-(9).

Since $\mathrm{F}_{1}^{2}=\left(\mathrm{F}_{1}^{+}\right)^{2}=0, \mathfrak{g l}(1 \mid 1)$ produces fewer descendants than $\mathfrak{s u}(2)$ does for isotropic spin chains. This is compensated by additional bosonic generators

$$
\begin{equation*}
\mathrm{F}_{2}=\sum_{i<j}^{N} f_{i} f_{j}, \quad \mathrm{~F}_{2}^{+}=\sum_{i<j}^{N} f_{i}^{+} f_{j}^{+} \tag{16}
\end{equation*}
$$

which commute with the $e_{i}$, whence with (7)-(9). Together, (15)-(16) generate the full global-symmetry algebra, called $\mathcal{A}_{1 \mid 1}$ [2]. It is the $\left.U_{\mathrm{q}} \mathfrak{s l}(2)\right|_{\mathrm{q}=\mathrm{i}}$ symmetry from the parent model in fermionic language, cf. [3, §2.3].

Extended symmetry. The parent model has quantumaffine $\mathfrak{s l}(2)$ symmetry, which underpins the large degeneracies (20). As we will see, these are already visible in the two-particle spectrum. A plausible guess is that it relates to the Yangian of $\mathfrak{g l}(1 \mid 1)$. A detailed study of this extended symmetry will be performed elsewhere.
The spectrum. The spectrum and degeneracies of the parent model are known explicitly [11, 16-18]. Like for the HS chain, the quantum numbers are 'motifs' [10] $\left\{\mu_{m}\right\}$, consisting of integers $1 \leqslant \mu_{m}<N$ increasing as

$$
\begin{equation*}
\mu_{m+1}>\mu_{m}+1, \quad 1 \leqslant m<M \tag{17}
\end{equation*}
$$

Such a motif labels an $M$-fermion state with quasimomentum $p=\frac{2 \pi}{N} \sum_{m} \mu_{m} \bmod 2 \pi$ setting the eigenvalue $\mathrm{e}^{\mathrm{i} p}$ of G. Its energy is additive:

$$
\begin{equation*}
E_{\left\{\mu_{m}\right\}}^{\mathrm{L}}=\sum_{m=1}^{M} \varepsilon_{\mu_{m}}^{\mathrm{L}}, \quad E_{\left\{\mu_{m}\right\}}=\sum_{m=1}^{M} \varepsilon_{\mu_{m}} \tag{18}
\end{equation*}
$$

with dispersions having two linear branches (Fig. 1):

$$
\varepsilon_{n}^{\mathrm{L}}=\left\{\begin{array}{ll}
n, & n \text { even },  \tag{19}\\
n-N, & n \text { odd },
\end{array} \quad \varepsilon_{n}=\left|\varepsilon_{n}^{\mathrm{L}}\right|\right.
$$

This state has (often many) descendants due to the extended symmetry. Its multiplicity is $[12,18] N+1$ for the empty motif (at $M=0$ ), and otherwise

$$
\begin{equation*}
\mu_{1}\left(N-\mu_{M}\right) \prod_{m=1}^{M-1}\left(\mu_{m+1}-\mu_{m}-1\right) \tag{20}
\end{equation*}
$$



Figure 1. The dispersion relations (19) alternate between two linear branches, realising chiral and (up to a shift) 'full' massless relativistic dispersions on the lattice.

Given the extremely simple dispersion, further ('accidental') degeneracies between different motifs occur much more often than even for the HS chain.

The energy levels are equispaced with steps of $2 . E^{\mathrm{L}}$ is bounded by $\pm\left(N^{2}-1\right) / 4$ and reaches these extremes at motifs $\{1,3, \ldots, N-2\}$ and $\{2,4, \ldots, N-1\}$, cf. Fig. 1. H has eigenvalues $\geqslant 0$, with $E=0$ for the empty motif, and maximal energy $E=2 \ell(3 \ell+1)$ or $E=2(\ell+1)(3 \ell+1)$ depending on whether $N=4 \ell+3$ or $N=4 \ell+1$, respectively. This maximum corresponds to the one or two motifs $\{1,3, \ldots, N-4, N-2\}$ switching halfway between the branches of $\varepsilon_{n}$ in Fig. 1. Intriguingly, for H a few levels near the maximum are missing.

Explicit diagonalisation. Let us (re)derive this spectrum from the fermionic representation. The key to defining a good basis of fermions is to start at one end of the lattice and use the quasi-translation operator:

$$
\begin{equation*}
\Phi_{i} \equiv \mathrm{G}^{1-i} f_{1} \mathrm{G}^{i-1}, \quad \Phi_{i}^{+} \equiv \mathrm{G}^{1-i} f_{1}^{+} \mathrm{G}^{i-1} \tag{21}
\end{equation*}
$$

These dressed fermions obey the periodicity

$$
\begin{equation*}
\Phi_{i+N}=\Phi_{i}, \quad \Phi_{i+N}^{+}=\Phi_{i}^{+} \tag{22}
\end{equation*}
$$

and have non-local anticommutation relations [19]

$$
\begin{equation*}
\left\{\Phi_{i}, \Phi_{j}^{+}\right\}=-\left(1+t_{j-i}\right), \quad\left\{\Phi_{i}, \Phi_{j}\right\}=\left\{\Phi_{i}^{+}, \Phi_{j}^{+}\right\}=0 \tag{23}
\end{equation*}
$$

The nontrivial relation only depends on the distance. For $a_{0} \equiv \mathrm{i}$ and $a_{n} \equiv \mathrm{i}^{n+1 / 2}$ else, the rescaled Fourier modes

$$
\begin{equation*}
\tilde{\Psi}_{n} \equiv \frac{a_{n}}{N} \sum_{j=1}^{N} \mathrm{e}^{-2 \mathrm{i} \pi n j / N} \Phi_{j}, \quad \tilde{\Psi}_{n}^{+} \equiv \frac{a_{n}}{N} \sum_{j=1}^{N} \mathrm{e}^{2 \mathrm{i} \pi n j / N} \Phi_{j}^{+} \tag{24}
\end{equation*}
$$

obey canonical anticommutation relations

$$
\begin{equation*}
\left\{\tilde{\Psi}_{n}, \tilde{\Psi}_{m}^{+}\right\}=\delta_{n m}, \quad\left\{\tilde{\Psi}_{n}, \tilde{\Psi}_{m}\right\}=\left\{\tilde{\Psi}_{n}^{+}, \tilde{\Psi}_{m}^{+}\right\}=0 \tag{25}
\end{equation*}
$$

They are covariant under quasi-translations in the sense

$$
\begin{equation*}
\mathrm{G} \tilde{\Psi}_{n} \mathrm{G}^{-1}=\mathrm{e}^{-2 \mathrm{i} \pi n / N} \tilde{\Psi}_{n}, \quad \mathrm{G} \tilde{\Psi}_{n}^{+} \mathrm{G}^{-1}=\mathrm{e}^{2 \mathrm{i} \pi n / N} \tilde{\Psi}_{n}^{+} \tag{26}
\end{equation*}
$$

The relation to the original fermions is strikingly simple. The zero-modes commute with the hamiltonians: they are just the fermionic $\mathfrak{g l}(1 \mid 1)$ generators from (15) [19],

$$
\begin{equation*}
\frac{1}{a_{0}} \tilde{\Psi}_{0}=\sum_{i=1}^{N} f_{i}=\mathrm{F}_{1}, \quad \frac{1}{a_{0}} \tilde{\Psi}_{0}^{+}=\sum_{i=1}^{N} f_{i}^{+}=\mathrm{F}_{1}^{+} \tag{27}
\end{equation*}
$$

The other modes are explicit linear combinations of the two-site fermions (2), with coefficients given in [19]:

$$
\begin{equation*}
\frac{1}{a_{n}} \tilde{\Psi}_{n}=\sum_{i=1}^{N-1} M_{n i} g_{i}, \quad \frac{1}{a_{n}} \tilde{\Psi}_{n}^{+}=\sum_{i=1}^{N-1} \bar{M}_{n i} g_{i}^{+} \tag{28}
\end{equation*}
$$

In terms of these fermionic modes, $\mathrm{H}^{\mathrm{L}}$ is diagonal:

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}}=\sum_{n=1}^{N-1} \varepsilon_{n}^{\mathrm{L}} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{n} \tag{29}
\end{equation*}
$$

Numerics for low $N$ confirms the equality with (7). If $|\varnothing\rangle$ is the fermionic vacuum, then by (25) the Fock states

$$
\begin{equation*}
\left|n_{1}, \ldots, n_{M}\right\rangle \equiv \tilde{\Psi}_{n_{1}}^{+} \ldots \tilde{\Psi}_{n_{M}}^{+}|\varnothing\rangle \tag{30}
\end{equation*}
$$

form an eigenbasis for $\mathrm{H}^{\mathrm{L}}$ labelled by all $2^{N}$ fermionic mode numbers $\left\{n_{m}\right\}$ with $0 \leqslant n_{1}<\cdots<n_{M}<N$. The quasi-momentum of (30) is $p=\frac{2 \pi}{N} \sum_{m} n_{m} \bmod 2 \pi$, and its chiral energy $E_{\left\{n_{m}\right\}}^{\mathrm{L}}=\sum_{m} \varepsilon_{n_{m}}^{\mathrm{L}}$ matches (18)-(19) when $\left\{n_{m}\right\}$ is a motif. Observe that, cf. Fig. 1,

$$
\begin{equation*}
\varepsilon_{n}^{\mathrm{L}}+\varepsilon_{n+1}^{\mathrm{L}}=\varepsilon_{2 n+1 \bmod N}^{\mathrm{L}} . \tag{31}
\end{equation*}
$$

Next, (9) takes the quartic form [19]:

$$
\begin{equation*}
\mathrm{H}=\sum_{n=1}^{N-1} \varepsilon_{n} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{n}+\sum_{\substack{1 \leqslant m<n<N \\ 1 \leqslant r<s<N}} \tilde{V}_{m n ; r s} \tilde{\Psi}_{m}^{+} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{r} \tilde{\Psi}_{s} \tag{32}
\end{equation*}
$$

The commutation (10) only allows $\tilde{V}_{m n ; r s} \neq 0$ if [19] the quasi-momentum and chiral energy are conserved:

$$
\begin{equation*}
m+n=r+s \bmod N, \quad \varepsilon_{m}^{\mathrm{L}}+\varepsilon_{n}^{\mathrm{L}}=\varepsilon_{r}^{\mathrm{L}}+\varepsilon_{s}^{\mathrm{L}} . \tag{33}
\end{equation*}
$$

Numerics for odd $N \leqslant 9$ suggest the stronger selection rule that $m+n=r+s$ be odd, with nonzero values $\tilde{V}= \pm 4$ determined by $\tilde{V}_{m n ; r s}=\tilde{V}_{r s ; m n}$ and

$$
\begin{equation*}
\tilde{V}_{m n ; m+k, n-k}=(-1)^{k+1} 4 \delta_{m \text { odd }}, \quad 0 \leqslant 2 k<n-m \tag{34}
\end{equation*}
$$

For one-particle states $|n\rangle$ only the quadratic part of (32) contributes, reproducing the non-chiral dispersion (19). The quartic part implements the statistical repulsion rule (17): H is genuinely interacting. Correspondingly, the Fock states (30) are generally not eigenstates of H . We illustrate this for the two-fermion spectrum:

- By global symmetry, the descendant $|0, n\rangle \propto \mathrm{F}_{1}^{+}|n\rangle$, cf. (27), is an H-eigenstate belonging to the motif $\{n\}$.
- Any $|m, n\rangle$ with $0<m<n<N$ and $n-m$ even is protected $(\tilde{V}=0)$ by the selection rules. It is an H -eigenstate with motif $\{m, n\}$.
- Any $\left|1,2 n^{\prime}\right\rangle$ is mixed with $\left|1+k, 2 n^{\prime}-k\right\rangle$ by (34):
- For $|1,2\rangle$ only $k=0$ contributes, so it is again an H-eigenstate, with energy $\varepsilon_{1}+\varepsilon_{2}+\tilde{V}_{12 ; 12}=\varepsilon_{3}$. It is degenerate with $|3\rangle$ for all charges, cf. (31), and belongs to the motif $\{3\}:|1,2\rangle \propto \widehat{\mathrm{F}}_{1}^{+}|3\rangle$ for some extended-symmetry generator $\widehat{\mathrm{F}}_{1}^{+}$.
- All $\left|1,2 n^{\prime}\right\rangle$ with $n^{\prime}>1$ mix with $\left|1+k, 2 n^{\prime}-k\right\rangle$, $k>0$. Diagonalising this $n^{\prime} \times n^{\prime}$ block of H gives eigenstates with 'squeezed' motifs $\left\{1+k, 2 n^{\prime}-k\right\}$, $0 \leqslant k \leqslant n^{\prime}-2$, plus a state that is proportional to $\widehat{\mathrm{F}}_{1}^{+}\left|2 n^{\prime}+1\right\rangle$ or, if $2 n^{\prime}=N-1$, to $\mathrm{F}_{2}^{+}|\varnothing\rangle$.
- Likewise for $\left|N-2 n^{\prime}+1, N-1\right\rangle \propto \mathrm{P}\left|1,2 n^{\prime}\right\rangle$ by parity.

See [19] for examples at low $N$. Besides actually diagonalising the blocks, and matching the result with the parent-model eigenstates at $q=i$, this gives the full twoparticle spectrum. Note the statistical repulsion in action, 'squeezing' adjacent modes to extended-symmetry descendants, cf. [28, §4.1.6]. A fermionic description of the higher spectrum requires a deeper understanding of the extended symmetry.

Outlook. We obtained and studied a long-range fermionic model with extended symmetry from the $\mathrm{q} \rightarrow \mathrm{i}$ limit of the xxz-type HS chain. A full understanding requires identifying the extended-symmetry algebra and explicitly constructing it in the fermionic language. This, and a systematic construction of the eigenvectors, which are known for the parent model, is left for future work.

The case of even $N$ needs separate treatment. Then the parent hamiltonian diverges as $\mathrm{q} \rightarrow \mathrm{i}$, and regularisation sets all energies to zero. However, the wave functions remain non-trivial, and numerics suggests the presence of Jordan blocks up to size $N / 2+1$. While indecomposable representations are expected, the size of the Jordan blocks signal that these are not just zigzag modules appearing for systems with merely global symmetry [29]. We plan to report on this in the near future.

Another important direction is the continuum limit, where we expect the system to exhibit conformal invariance. The explicit identification of the corresponding CFT requires determining the extended symmetry. Based on the isotropic HS chain we expect the CFT limit to have Kac-Moody symmetry, perhaps level-one $\mathfrak{g l}(1 \mid 1)$ [30]. It will be interesting to find the continuum counterparts of the chiral Hamiltonian, which is reminiscent of the Virasoro generator $L_{0}$, and see what happens with the staggering in the dispersion relation. While the relativistic-like dispersion for odd length seems well adapted for the continuum limit, it is at odds with the vanishing spectrum for even length.

For the Heisenberg xxz chain other root-of-unity values of $q$, notably $q^{3}=1$ [31], are special too. It would be exciting to explore their analogues in our parent model. Still more generally, it would be interesting to study $\mathrm{q}=\mathrm{i}$ and other special points of the xxz-type Inozemtsev chain [32], which interpolates between a quasiperiodic Heisenberg xxz chain and our parent model.

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## SUPPLEMENTAL MATERIAL

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## A. The parent model

The model introduced in this Letter is the $q=i$ limit of an integrable long-range version of the Heisenberg Xxz spin chain: the xxz-type Haldane-Shastry (HS) chain with spin $1 / 2$. This 'parent model' was proposed in [11] and explicitly described and studied further in [16-18]. Here we review the basics.

## 1. The Haldane-Shastry chain

To set up our notation we start with the isotropic case. For $N$ spin- $1 / 2$ sites, the HS chain [5] has pairwise spin exchange interactions with an inverse square potential,

$$
\begin{equation*}
\mathrm{H}^{\mathrm{HS}}=\sum_{i<j}^{N} \mathrm{~V}(i-j)\left(1-\mathrm{P}_{i j}\right), \quad \mathrm{V}(k)=\frac{1}{4 \sin (\pi k / N)^{2}} . \tag{A.1}
\end{equation*}
$$

This spin chain has many remarkable properties. For us the most important ones are that it possesses
i) a family of conserved charges: a hierarchy of commuting operators including (A.1) [6, 12, 33-35],
ii) extended (Yangian) $\mathfrak{s u}(2)$ spin symmetry [11, 12],
iii) a very simple and highly degenerate spectrum.

Let us elaborate on the last point. The eigenspaces are labelled by 'motifs' $\left\{\mu_{m}\right\}_{m}$, which are

$$
\begin{equation*}
\text { integers } 1 \leqslant \mu_{1}<\cdots<\mu_{M}<N \quad \text { obeying the statistical repulsion rule } \quad \mu_{m+1}>\mu_{m}+1 \tag{A.2}
\end{equation*}
$$

The momentum is

$$
\begin{equation*}
p=\frac{2 \pi}{N} \sum_{m} \mu_{m} \bmod 2 \pi \tag{A.3}
\end{equation*}
$$

which means that the lattice translation

$$
\begin{equation*}
\mathrm{G}^{\mathrm{HS}}=\mathrm{P}_{N-1, N} \cdots \mathrm{P}_{12} \tag{A.4}
\end{equation*}
$$

which is one of the conserved charges, has eigenvalue $\mathrm{e}^{\mathrm{i} p}$. The energy is (strictly) additive with a quadratic dispersion:

$$
\begin{equation*}
E_{\left\{\mu_{m}\right\}}^{\mathrm{HS}}=\sum_{m=1}^{M} \varepsilon_{\mu_{m}}^{\mathrm{HS}}, \quad \varepsilon_{n}^{\mathrm{HS}}=\frac{1}{2} n(N-n) . \tag{A.5}
\end{equation*}
$$

Therefore all energies are half integers, and the motifs control which energy levels occur. For example, the empty motif corresponds to the ferromagnetic vacuum $|\uparrow \cdots \uparrow\rangle$ with zero momentum and energy, while if $N$ is even the fully packed motif $\{1,3, \ldots, N-3, N-1\}$ corresponds to the antiferromagnetic vacuum with maximal energy. In general, the motif (A.2) has a (Yangian) highest-weight state at $S^{z}=N / 2-M$, with an explicit wave function containing a Jack polynomial [10, 11]. It typically has many (ordinary as well as 'affine') descendants thanks to the extended spin symmetry. The degeneracy equals $[11,12]$

$$
\begin{equation*}
N+1 \quad \text { for the empty motif, } \quad \mu_{1}\left(N-\mu_{M}\right) \prod_{m=1}^{M-1}\left(\mu_{m+1}-\mu_{m}-1\right) \quad \text { else } \tag{A.6}
\end{equation*}
$$

which is (20) from the main text. Additional 'accidental' degeneracies between different motifs occur.

## 2. The xxz-type Haldane-Shastry chain

The HS chain has an xxz-type generalisation where the $\mathfrak{s u}(2)$ spin symmetry is broken to $\mathfrak{u}(1)$, or more precisely: deformed. It still possesses
i) a family of conserved charges [11, 16-18],
ii) extended spin symmetry, now deformed to quantum affine $\mathfrak{s l}(2)$ [11],
iii) a very simple and highly degenerate spectrum $[16,18]$.

The price to pay for keeping (i)-(ii) in the presence of anisotropy is that the long-range spin exchange $1-\mathrm{P}_{i j}$ from (A.1) becomes more complicated. This goes as follows.

We start from the two-site operators

$$
e_{i}(\mathrm{q}) \equiv \mathbb{1}^{\otimes(i-1)} \otimes\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{A.7}\\
0 & \mathrm{q}^{-1} & -1 & 0 \\
0 & -1 & \mathrm{q}^{-1} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \otimes \mathbb{1}^{\otimes(N-i-1)}, \quad 1 \leqslant i<N
$$

where $\mathrm{q} \in \mathbb{C}$ is the anisotropy (deformation) parameter. These local operators obey the commutation relations

$$
\begin{align*}
e_{i}(\mathrm{q})^{2} & =\left(\mathrm{q}+\mathrm{q}^{-1}\right) e_{i}(\mathrm{q}), & & \\
e_{i}(\mathrm{q}) e_{j}(\mathrm{q}) e_{i}(\mathrm{q}) & =e_{i}(\mathrm{q}), & & \text { if }|i-j|=1,  \tag{A.8}\\
{\left[e_{i}(\mathrm{q}), e_{j}(\mathrm{q})\right] } & =0, & & \text { if }|i-j|>1
\end{align*}
$$

That is, (A.7) furnish a representation of the Temperley-Lieb (TL) algebra with 'loop fugacity' $q+q^{-1}$ on the spin chain. This TL representation also appears for the Heisenberg $x x z$ chain with $\Delta=\left(q+q^{-1}\right) / 2$, cf. e.g. [20]. Since $e_{i}(1)=1-\mathrm{P}_{i, i+1}$ the TL generators (A.7) are deformed nearest-neighbour spin antisymmetrisers.

Next we define the trigonometric (xxz) $R$-matrix

$$
\begin{equation*}
\check{\mathrm{R}}_{i, i+1}(u ; \mathrm{q})=\mathrm{P}_{i, i+1} \mathrm{R}_{i, i+1}(u ; \mathrm{q}) \equiv 1-f(u ; \mathrm{q}) e_{i}(\mathrm{q}), \quad f(u ; \mathrm{q}) \equiv \frac{u-1}{\mathrm{q} u-\mathrm{q}^{-1}} \tag{A.9}
\end{equation*}
$$

It obeys the Yang-Baxter equation in the form $\check{\mathrm{R}}_{12}(u / v ; q) \check{\mathrm{R}}_{23}(u ; q) \check{\mathrm{R}}_{12}(v ; q)=\check{\mathrm{R}}_{23}(v ; \mathrm{q}) \check{\mathrm{R}}_{12}(u ; \mathrm{q}) \check{\mathrm{R}}_{23}(u / v ; \mathrm{q})$, as well as the unitarity relation $\check{R}(u ; q) \check{R}(1 / u ; q)=\mathbb{1} \otimes \mathbb{1}$, and 'initial' condition $\check{R}(0 ; q)=\mathbb{1} \otimes \mathbb{1}$. These relations, together with $\check{\mathrm{R}}(u ; 1)=\mathrm{P}$, means that we can think of (A.9) as a fancy version of the spin permutation operator.

The xxz-type long-range spin exchange operator turns out to come in two 'chiral' versions:

$$
\begin{align*}
& \mathrm{S}_{[i, j]}^{\mathrm{L}}(\mathrm{q}) \equiv\left(\prod_{j>k>i} \check{\mathrm{R}}_{k, k+1}\left(\omega^{j-k} ; \mathrm{q}\right)\right) e_{i}(\mathrm{q})\left(\prod_{i<k<j}^{\operatorname{L}} \check{\mathrm{R}}_{k, k+1}\left(\omega^{k-j} ; \mathrm{q}\right)\right)=\underbrace{\ldots}_{\omega^{1}} \underbrace{\uparrow}_{\omega^{i}} \\
& \mathrm{~S}_{[i, j]}^{\mathrm{R}}(\mathrm{q}) \equiv\left(\prod_{i<k<j} \check{\mathrm{R}}_{k-1, k}\left(\omega^{k-i} ; \mathrm{q}\right)\right) e_{j-1}(\mathrm{q})\left(\prod_{j>k>i} \check{\mathrm{R}}_{k-1, k}\left(\omega^{i-k} ; \mathrm{q}\right)\right)=\underbrace{\ldots}_{\omega^{1}} \underbrace{\uparrow}_{\omega^{i}},  \tag{A.10}\\
& i<j .
\end{align*}
$$

Here the arrows on the products indicate the direction in which the subscripts of the factors, which do not commute, increase. The diagrams show how we can think of these interactions: one of the two interacting spins is transported to the site next to the other interacting spin, where it interacts with is neighbour, after which it is transported back. The transport $\mathcal{X}$ is taken care of by the deformed spin permutation (A.9), and the nearest-neighbour exchange $\hat{\sim} \uparrow \hat{\psi}$ by the antisymmetriser $e_{i}(\mathrm{q})$. More precisely, in (A.10) the diagrams are equivalent to the formulas via the rules

At the isotropic point we recover $S_{[i, j]}^{\mathrm{L}}(1)=\mathrm{S}_{[i, j]}^{\mathrm{R}}(1)=1-\mathrm{P}_{i j}$. In general the chiral operators (A.10) differ from each other and involve multi-spin interactions, as the intermediate spins do feel the transport.

The final ingredients that we need are the 'quantum integers'

$$
\begin{equation*}
[N]_{\mathrm{q}} \equiv \frac{\mathrm{q}^{N}-\mathrm{q}^{-N}}{\mathrm{q}-\mathrm{q}^{-1}}=\frac{\sin (N \eta)}{\sin (\eta)} \tag{A.12}
\end{equation*}
$$

and the appropriate modification of the pair potential (A.1),

$$
\begin{equation*}
\mathrm{V}(k ; \mathrm{q}) \equiv \frac{1}{\left(\mathrm{q} \omega^{k}-\mathrm{q}^{-1}\right)\left(\mathrm{q} \omega^{-k}-\mathrm{q}^{-1}\right)}=\frac{1}{4 \sin (\pi k / N+\eta) \sin (\pi k / N-\eta)}, \quad \omega \equiv \mathrm{e}^{2 \pi \mathrm{i} / N} \tag{A.13}
\end{equation*}
$$

where on the right-hand sides we wrote $q=e^{i \eta}$. Then the xxz-type HS chain has chiral hamiltonians

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}}(\mathrm{q})=\frac{[N]_{\mathrm{q}}}{N} \sum_{i<j}^{N} \mathrm{~V}(i-j ; \mathrm{q}) \mathrm{S}_{[i, j]}^{\mathrm{L}}(\mathrm{q}), \quad \mathrm{H}^{\mathrm{R}}(\mathrm{q})=\frac{[N]_{\mathrm{q}}}{N} \sum_{i<j}^{N} \mathrm{~V}(i-j ; \mathrm{q}) \mathrm{S}_{[i, j]}^{\mathrm{R}}(\mathrm{q}), \tag{A.14}
\end{equation*}
$$

where by ' $\sum_{i<j}^{N}$ ' we always mean the sum over all pairs of spins, i.e. over $1 \leqslant i<j \leqslant N$.
Comparing the nearest-neighbour bulk terms $\mathrm{S}_{[i, i+1]}^{\mathrm{L}}(\mathrm{q})=e_{i}(\mathrm{q})=\mathrm{S}_{[i, i+1]}^{\mathrm{R}}(\mathrm{q})$ with the highly non-local boundary terms $\mathrm{S}_{[1, N]}^{\mathrm{L}}(\mathrm{q}) \neq \mathrm{S}_{[1, N]}^{\mathrm{R}}(\mathrm{q})$ shows that (A.14) are not invariant under the usual lattice translation (A.4). Its role is taken over by the deformed shift operator, which is again built from the deformed permutations (A.9) [17]:

$$
\begin{equation*}
\mathrm{G}(\mathrm{q}) \equiv \varlimsup_{N>k \geqslant 1} \check{\mathrm{R}}_{k, k+1}\left(\omega^{-k} ; \mathrm{q}\right)=\check{\mathrm{R}}_{N-1, N}\left(\omega^{1-N} ; \mathrm{q}\right) \cdots \check{\mathrm{R}}_{12}\left(\omega^{-1} ; \mathrm{q}\right)=\underbrace{}_{\omega^{1}}, \mathrm{G}(\mathrm{q})^{N}=1 . \tag{A.15}
\end{equation*}
$$

We will call this operator the 'quasi-translation', and denote its eigenvalues by $\mathrm{e}^{\mathrm{i} p(\mathrm{q})}$ where $p(\mathrm{q})$ is the 'quasimomentum'. At $q=1$, (A.15) reduces to (A.4) and $p(\mathrm{q})$ becomes the usual lattice momentum.

The deformed hamiltonians and quasi-translation are constructed such that the special properties of the HS chain are preserved for $q \neq 1$ : the XxZ-type HS chain is more complicated in position space (hamiltonians) precisely so that it remains extremely simple in momentum space (spectrum). More precisely,
i) The chiral hamiltonians (A.10) and quasi-translation operator (A.15) all commute,

$$
\begin{equation*}
\left[\mathrm{G}(\mathrm{q}), \mathrm{H}^{\mathrm{L}}(\mathrm{q})\right]=\left[\mathrm{G}(\mathrm{q}), \mathrm{H}^{\mathrm{R}}(\mathrm{q})\right]=\left[\mathrm{H}^{\mathrm{L}}(\mathrm{q}), \mathrm{H}^{\mathrm{R}}(\mathrm{q})\right]=0, \tag{A.16}
\end{equation*}
$$

belonging to the conserved charges of the parent model. In particular, the parity-invariant hamiltonian

$$
\begin{equation*}
\mathrm{H}^{\text {full }}(\mathrm{q}) \equiv \frac{\mathrm{H}^{\mathrm{L}}(\mathrm{q})+\mathrm{H}^{\mathrm{R}}(\mathrm{q})}{2} \tag{A.17}
\end{equation*}
$$

is also a conserved charge. At $q=1$, all three deformed hamiltonians reduce to the HS hamiltonian (A.1).
ii) The extended spin symmetry persists, again suitably deformed. We refer to [18] for details.
iii) The spectrum is still given in terms of the motifs (A.2), with degeneracies (A.6). The (highest-weight) eigenstate of each motif is known in closed form [18]. Its quasi-momentum has the same value (A.3) as for the HS chain: the meaning of $p(\mathrm{q})$ depends on q through the definition of $\mathrm{G}(\mathrm{q})$, but its values are constant. The chiral and full energies remain additive as in (A.5), with dispersion relations

$$
\begin{equation*}
\varepsilon_{n}^{\mathrm{L}}(\mathrm{q})=\frac{1}{\mathrm{q}-\mathrm{q}^{-1}}\left(\mathrm{q}^{N-n}[n]_{\mathrm{q}}-\frac{[N]_{\mathrm{q}}}{N} n\right), \quad \varepsilon_{n}^{\text {full }}(\mathrm{q})=\frac{1}{2}[n]_{\mathrm{q}}[N-n]_{\mathrm{q}}, \quad \varepsilon_{n}^{\mathrm{R}}(\mathrm{q})=\varepsilon_{n}^{\mathrm{L}}\left(\mathrm{q}^{-1}\right) \tag{A.18}
\end{equation*}
$$

The 'full' dispersion follows from the chiral ones by (A.17); note that it is real for $q$ real or unimodular $(|q|=1)$. When $\mathrm{q} \rightarrow 1$ all three dispersions reduce to (A.5). For generic q , the 'accidental' degeneracies of the HS chain are lifted, while for $q$ a root of unity there are more accidental degeneracies between different motifs.
For general q we work in the spin representation (A.7) of the TL algebra. However, it is known that this representation is faithful for any $q \in \mathbb{C}([27]$ Section $2 . B)$, which means that many* results carry over to any other representation, or even to the 'abstract' setting directly inside the TL algebra itself. In particular, the commutativity (A.16) of the conserved charges in the xxz representation [18] persists at the level of the TL algebra. This will be important in the following.

## B. The model at $\mathrm{q}=\mathrm{i}$

## 1. Specialising the parent model

Now consider the special point $\mathrm{q}=\mathrm{i}$. The TL generators are regular at $\mathrm{q}=\mathrm{i}$ but

$$
\begin{equation*}
e_{j} \equiv e_{j}(\mathrm{i}), \quad 1 \leqslant j<N \tag{B.1}
\end{equation*}
$$

are nilpotent due to (A.8):

$$
\begin{align*}
e_{i}^{2} & =0, & & \\
e_{i} e_{j} e_{i} & =e_{i}, & & \text { if }|i-j|=1,  \tag{B.2}\\
{\left[e_{i}, e_{j}\right] } & =0, & & \text { if }|i-j|>1 .
\end{align*}
$$

We will call this the free-fermion TL algebra. Since the function (A.9) becomes

$$
\begin{equation*}
f(u ; \mathrm{i})=\mathrm{i} \frac{1-u}{1+u}, \quad f\left(\mathrm{e}^{\mathrm{i} x} ; \mathrm{i}\right)=\tan x \tag{B.3}
\end{equation*}
$$

we shall have ample opportunity to use the short-hand notation

$$
\begin{equation*}
t_{k} \equiv f\left(\omega^{k} ; \mathrm{i}\right)=\tan \frac{\pi k}{N} \tag{B.4}
\end{equation*}
$$

Thus the quasi-translation (A.15) becomes (8) from the main text,

$$
\begin{equation*}
\mathrm{G} \equiv \mathrm{G}(\mathrm{i})=\left(1+t_{N-1} e_{N-1}\right) \cdots\left(1+t_{1} e_{1}\right) \tag{B.5}
\end{equation*}
$$

[^0]Note, however, that $t_{k}$ has a simple pole at $k=N / 2$, which appears in (B.5) when $N$ is even. This is the first hint that the specialisation of the parent model to $q=i$ behaves very differently for even vs odd length. This is more pronounced still for the hamiltonians. On the one hand, the quantum integers (A.12) simplify to

$$
\begin{equation*}
[2 k]_{\mathrm{i}}=0, \quad[2 k+1]_{\mathrm{i}}=(-1)^{k} \tag{B.6}
\end{equation*}
$$

so the prefactor in (A.14) contributes a simple zero when $N$ is even. On the other hand, the potential (A.13) becomes

$$
\begin{equation*}
\mathrm{V}(k ; \mathrm{i})=-\frac{1}{4 \cos (\pi k / N)^{2}}=-\frac{1}{4}\left(1+t_{k}^{2}\right) \tag{B.7}
\end{equation*}
$$

which has a second-order pole at antipodal points, $k=N / 2$. The sign in (B.7) is irrelevant and will be absorbed in a rescaling below. The point is that we see that some of the matrix elements of the hamiltonians also become infinite when $N$ is even. One can renormalise the quasi-translation and hamiltonians in order to remove these poles (by taking residues), but this sets their spectra to zero. Numerical investigations show that there are Jordan blocks of size up to $(M+1) \times(M+1)$ in the $M$-magnon sector, $M \leqslant N / 2$. We will come back to even length, and investigate these intriguing features, in a separate publication.

In everything that follows, we will exclusively consider the case where $N$ is odd. None of the preceding singularities appear, and all matrix elements remain finite. It will require quite some effort to work out what happens with the long-range interactions at $\mathrm{q}=\mathrm{i}$. Let us first see what happens to the energies. The chiral dispersions are purely imaginary and become linear with the mode number $n$ :

$$
\varepsilon_{n}^{\mathrm{L}}(\mathrm{i})=-\varepsilon_{n}^{\mathrm{R}}(\mathrm{i})=\frac{(-1)^{(N+1) / 2}}{2 \mathrm{i} N} \times \begin{cases}n, & n \text { even }  \tag{B.8}\\ n-N, & n \text { odd }\end{cases}
$$

This is the origin of the chiral dispersion (19) from the main text. Next, the full dispersion (A.18) vanishes identically at $\mathrm{q}=\mathrm{i}$ when $N$ is odd, because either $[n]_{\mathrm{q}=\mathrm{i}}$ or $[N-n]_{\mathrm{q}=\mathrm{i}}$ is zero. To extract a nonzero result we divide by $\mathrm{q}+\mathrm{q}^{-1}$ before we specialise,

$$
\lim _{\mathrm{q} \rightarrow \mathrm{i}} \frac{\varepsilon_{n}^{\text {full }}(\mathrm{q})}{\mathrm{q}+\mathrm{q}^{-1}}=\frac{(-1)^{(N+1) / 2}}{4} \times \begin{cases}n, & n \text { even }  \tag{B.9}\\ N-n, & n \text { odd }\end{cases}
$$

This is where the second part of (19) from the main text comes from. More precisely, removing the prefactors from (B.8) and (B.9) we obtain the dispersions (19) from the main text.

It is a highly nontrivial fact that the vanishing of $\varepsilon^{\text {full }}$ at $q=i$ already occurs at the level of the hamiltonian:

$$
\begin{equation*}
H^{\text {full }}(i)=0 \tag{B.10}
\end{equation*}
$$

as will be established in $\S \mathrm{E} 2 \mathrm{a}$. We are thus led to define

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}} \equiv 2 \mathrm{i} N(-1)^{(N+1) / 2} \mathrm{H}^{\mathrm{L}}(\mathrm{i}), \quad \mathrm{H} \equiv 2(-1)^{(N+1) / 2} \lim _{\mathrm{q} \rightarrow \mathrm{i}} \frac{\mathrm{H}^{\mathrm{full}}(\mathrm{q})}{\left(\mathrm{q}+\mathrm{q}^{-1}\right) / 2}, \quad \mathrm{H}^{\mathrm{R}} \equiv 2 \mathrm{i} N(-1)^{(N+1) / 2} \mathrm{H}^{\mathrm{R}}(\mathrm{i}) \tag{B.11}
\end{equation*}
$$

We have $\mathrm{H}^{\mathrm{L}}=-\mathrm{H}^{\mathrm{R}}$ according to (B.10). The commutativity (A.16) of the quasi-translation and chiral hamiltonians survives at $\mathrm{q}=\mathrm{i}$. The vanishing (B.10) guarantees that those operators furthermore commute with H , as one sees by expanding the commutators like (A.16) in $q+q^{-1}$. Therefore we have (10) from the main text, i.e.

$$
\begin{equation*}
\left[\mathrm{G}, \mathrm{H}^{\mathrm{L}}\right]=[\mathrm{G}, \mathrm{H}]=\left[\mathrm{H}^{\mathrm{L}}, \mathrm{H}\right]=0 \tag{B.12}
\end{equation*}
$$

More precisely, the limit in (B.11) is taken in the spin representation (A.7) of the Temperley-Lieb generators, where the dependence on $q$ is meromorphic. It turns out that the result can again be expressed in terms of the TL generators (B.1). It follows* (p.10) that (B.5) and (B.11) correspond to pairwise commutative elements of the TL algebra at the free-fermion point. Next we will provide explicit formulas for the hamiltonians (B.11).

## 2. Explicit formulas for the hamiltonians

Let us expand the chiral spin interactions around $q=i$ as

$$
\begin{equation*}
\mathrm{S}_{[i, j]}^{\mathrm{L}, \mathrm{R}}(\mathrm{q})=\mathrm{S}_{[i, j]}^{\mathrm{L}, \mathrm{R}}+\frac{\mathrm{q}+\mathrm{q}^{-1}}{2} \tilde{\mathrm{~S}}_{[i, j]}^{\mathrm{L}, \mathrm{R}}+\mathcal{O}\left(\left(\mathrm{q}+\mathrm{q}^{-1}\right)^{2}\right), \quad \mathrm{S}_{[i, j]}^{\mathrm{L}, \mathrm{R}} \equiv \mathrm{~S}_{[i, j]}^{\mathrm{L}, \mathrm{R}}(\mathrm{i}),\left.\quad \tilde{\mathrm{S}}_{[i, j]}^{\mathrm{L}, \mathrm{R}} \equiv \partial_{\mathrm{q}}\right|_{\mathrm{q}=\mathrm{i}} \mathrm{~S}_{[i, j]}^{\mathrm{L}, \mathrm{R}}(\mathrm{q}) \tag{B.13}
\end{equation*}
$$

Since (B.7) holds up to quadratic corrections we have

$$
\begin{align*}
\mathrm{H}^{\mathrm{L}, \mathrm{R}} & =\frac{\mathrm{i}}{2} \sum_{i=1}^{N-1} \sum_{k=1}^{N-i}\left(1+t_{k}^{2}\right) \mathrm{S}_{[i, i+k]}^{\mathrm{L}, \mathrm{R}} \\
\mathrm{H} & =\frac{1}{4 N} \sum_{i=1}^{N-1} \sum_{k=1}^{N-i}\left(1+t_{k}^{2}\right)\left(\tilde{\mathrm{S}}_{[i, i+k]}^{\mathrm{L}}+\tilde{\mathrm{S}}_{[i, i+k]}^{\mathrm{R}}\right), \tag{B.14}
\end{align*}
$$

Remarkably, the complicated spin operators can be written explicitly in terms of nested commutators of adjacent Temperley-Lieb generators,

$$
\begin{equation*}
e_{[l, m]} \equiv\left[e_{l},\left[e_{l+1}, \ldots\left[e_{m-1}, e_{m}\right] \ldots\right]\right]=\left[\left[\ldots\left[e_{l}, e_{l+1}\right], \ldots e_{m-1}\right], e_{m}\right], \quad i \leqslant l<m<j \tag{B.15}
\end{equation*}
$$

Note that $e_{[k, k]}=e_{k}$ is just a single Temperley-Lieb generator. The commutators in (B.15) can be nested from left to right or from right to left, as can be proven by induction using the Jacobi identity. Note that it suffices to consider commutators of strings of successive Temperley-Lieb generators, since non-successive generators commute, which implies that the nested commutators vanish if any $e_{k}$ with $l \leqslant k \leqslant m$ is missing from the string.
a. Chiral hamiltonians. In §B3 we will use the recursive structure of the spin interactions to show that the

$$
\begin{align*}
& \mathrm{S}_{[i, i+k]}^{\mathrm{L}}=\sum_{0 \leqslant l \leqslant m<k}(-1)^{l} t_{k-m, k-l} t_{k-l, k}^{2} e_{[i+l, i+m]}, \\
& \mathrm{S}_{[i, i+k]}^{\mathrm{R}}=\sum_{0 \leqslant l \leqslant m<k}(-1)^{k-l-1} t_{l+1, m+1} t_{m+1, k}^{2} e_{[i+l, i+m]}, \tag{B.16}
\end{align*}
$$

where denote products of the tangents (B.4) as

$$
\begin{equation*}
t_{k, l} \equiv \prod_{i=k}^{l-1} t_{i} \quad(k<l), \quad t_{k, k} \equiv 1 \tag{B.17}
\end{equation*}
$$

For our purposes it will be more convenient to rewrite (B.16) in the more symmetric form

$$
\begin{equation*}
\mathrm{S}_{[i, i+k]}^{\mathrm{L}, \mathrm{R}}=\sum_{0 \leqslant l \leqslant m<k} \varsigma_{l, m, k-1}^{\mathrm{L}, \mathrm{R}} e_{[i+l, i+m]}, \quad \varsigma_{l, m, k-1}^{\mathrm{L}} \equiv(-1)^{l} t_{k-l, k} t_{k-m, k}, \quad \varsigma_{l, m, k}^{\mathrm{R}} \equiv \varsigma_{k-l, k-m, k}^{\mathrm{L}} \tag{B.18}
\end{equation*}
$$

Note that (B.17) makes sense for any $k \leqslant l$, and the coefficients in (B.18) for arbitrary $k$ and nonnegative $l, m$.
Plugging (B.18) into the expression (B.14) for the chiral hamiltonians one obtains

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}, \mathrm{R}}=\frac{\mathrm{i}}{2} \sum_{1 \leqslant i \leqslant j<N} h_{i j}^{\mathrm{L}, \mathrm{R}} e_{[i, j]}, \tag{B.19}
\end{equation*}
$$

with the nested commutators have coefficients

$$
\begin{equation*}
h_{i j}^{\mathrm{L}, \mathrm{R}} \equiv \sum_{l=1}^{i} \sum_{m=j}^{N-1}\left(1+t_{m-l+1}^{2}\right) \varsigma_{i-l, j-l, m-l}^{\mathrm{L}, \mathrm{R}}, \quad h_{j i}^{\mathrm{L}, \mathrm{R}} \equiv(-1)^{j-i} h_{i j}^{\mathrm{L}, \mathrm{R}}, \quad i \leqslant j . \tag{B.20}
\end{equation*}
$$

Now, using the L-R-symmetry $\varsigma_{l, m, k}^{\mathrm{R}}=\varsigma_{k-l, k-m, k}^{\mathrm{L}}$ and changing indices, it can be checked that the coefficients of the chiral hamiltonians (B.19) are related by

$$
\begin{equation*}
h_{i j}^{\mathrm{R}}=(-1)^{j-i} h_{N-j, N-i}^{\mathrm{L}} . \tag{B.21}
\end{equation*}
$$

Furthermore, $h_{i j}^{\mathrm{L}}$ can be simplified using $t_{m-l+1}^{2} \varsigma_{i-l, j-l, m-l}^{\mathrm{L}}=-\varsigma_{i-(l-1), j-(l-1), m-(l-1)}^{\mathrm{L}}$, which follows from the definitions. As such, coefficients in $h_{i j}^{\mathrm{L}}$ telescope over the $l$ variable to yield

$$
\begin{equation*}
h_{i j}^{\mathrm{L}}=\sum_{m=j}^{N-1}\left(\varsigma_{0, j-i, m-i}^{\mathrm{L}}-\varsigma_{i, j, m}^{\mathrm{L}}\right) . \tag{B.22}
\end{equation*}
$$

This can be more explicitly factorised as

$$
\begin{equation*}
h_{i j}^{\mathrm{L}}=\sum_{n=j+1}^{N} t_{n-j, n-i}\left(1-(-1)^{i} t_{n-i, n}^{2}\right), \tag{B.23}
\end{equation*}
$$

which is useful for numerical computations. This proves (7) from the main text once we show (B.16). In §E 2 a we will furthermore establish, using a rather technical analytic proof, that for any $i, j$

$$
\begin{equation*}
h_{i j}^{\mathrm{R}}=-h_{i j}^{\mathrm{L}}, \tag{B.24}
\end{equation*}
$$

which means that (B.10) holds coefficient by coefficient in terms of the nested TL commutators. This is the origin of the first equality in (11).
b. Full hamiltonian. To find an explicit expression for H we need the next term in the expansion (B.13) of the spin operators. As we will outline in § B 3, they can be written as anticommutators of the nested commutators (B.15):

$$
\begin{align*}
& \tilde{\mathrm{S}}_{[i, i+k]}^{\mathrm{L}}=\sum_{0 \leqslant j \leqslant l<m \leqslant n<k}(-1)^{n-1} t_{k-n, k-l} t_{k-m, k-j} t_{k-j, k}^{2}\left\{e_{[i+n, i+m]}, e_{[i+l, i+j]}\right\}, \\
& \tilde{\mathrm{S}}_{[i, i+k]}^{\mathrm{R}}=\sum_{0 \leqslant j \leqslant l<m \leqslant n<k}(-1)^{k-j} t_{j+1, m+1} t_{l+1, n+1} t_{n+1, k}^{2}\left\{e_{[i+j, i+l]}, e_{[i+m, i+n]}\right\} . \tag{B.25}
\end{align*}
$$

After combining all the factors and telescoping the sums like before we obtain

$$
\begin{equation*}
\mathrm{H}=-\frac{1}{4 N} \sum_{1 \leqslant i \leqslant j<k \leqslant l<N}\left(h_{i j ; k l}^{\mathrm{L}}+h_{i j ; k l}^{\mathrm{R}}\right)\left\{e_{[i, j]}, e_{[k, l]}\right\}, \tag{B.26}
\end{equation*}
$$

with

$$
\begin{align*}
h_{i j ; k l}^{\mathrm{L}} & =(-1)^{k-j} \sum_{n=l+1}^{N} t_{n-l, n-j} t_{n-k, n-i}\left(1-(-1)^{i} t_{n-i, n}^{2}\right), \\
h_{i j ; k l}^{\mathrm{R}} & =(-1)^{l-i} \sum_{n=0}^{i-1} t_{i-n, k-n} t_{j-n, l-n}\left(1-(-1)^{N-l} t_{l-n, N-n}^{2}\right)  \tag{B.27}\\
& =(-1)^{l-j+k-i} h_{N-l, N-k ; N-j, N-i}^{\mathrm{L}} .
\end{align*}
$$

This is (9) from the main text.
c. Example. For instance, at $N=3$, we have

$$
\begin{align*}
\mathrm{H}^{\mathrm{L}} & =\frac{\mathrm{i}}{2}\left(t_{1}\left(1+t_{2}^{2}\right)\left[e_{1}, e_{2}\right]+\left(2+t_{1}^{2}+t_{2}^{2}\right) e_{1}+\left(1-t_{1}^{2} t_{2}^{2}\right) e_{2}\right)=2 \mathrm{i}\left(\sqrt{3}\left[e_{1}, e_{2}\right]+2\left(e_{1}-e_{2}\right)\right) \\
\mathrm{H} & =\frac{1}{6}\left(1+t_{2}^{2}\right) t_{1}^{2}\left\{e_{1}, e_{2}\right\}=2\left\{e_{1}, e_{2}\right\} \tag{B.28}
\end{align*}
$$

## 3. Sketch of the derivation of the spin interaction

In order to prove (B.16) it is useful to introduce the slight generalisation of the spin interaction,

$$
\begin{align*}
& \mathrm{S}_{[i, j] ; n}^{\mathrm{L}}(\mathrm{q}) \equiv\left(\prod_{j>k>i} \check{\mathrm{R}}_{k, k+1}\left(\omega^{n-k} ; \mathrm{q}\right)\right) e_{i}(\mathrm{q})\left(\prod_{i<k<j} \check{\mathrm{R}}_{k, k+1}\left(\omega^{k-n} ; \mathrm{q}\right)\right)  \tag{B.29}\\
& \mathrm{S}_{[i, j] ; n}^{\mathrm{R}}(\mathrm{q}) \equiv\left(\prod_{j>k>i} \check{\mathrm{R}}_{k-1, k}\left(\omega^{k-n} ; \mathrm{q}\right)\right) e_{j-1}(\mathrm{q})\left(\prod_{i<k<j} \check{\mathrm{R}}_{k-1, k}\left(\omega^{n-k} ; \mathrm{q}\right)\right)
\end{align*}
$$

so that

$$
\begin{equation*}
\mathrm{S}_{[i, j]}^{\mathrm{L}}(\mathrm{q})=\mathrm{S}_{[i, j] ; j}^{\mathrm{L}}(\mathrm{q}), \quad \mathrm{S}_{[i, j]}^{\mathrm{R}}(\mathrm{q})=\mathrm{S}_{[i, j] ; i}^{\mathrm{R}}(\mathrm{q}) \tag{B.30}
\end{equation*}
$$

Now set $q=i$ and write

$$
\begin{equation*}
\mathrm{S}_{[i, j] ; n}^{\mathrm{L}} \equiv \mathrm{~S}_{[i, j] ; n}^{\mathrm{L}}(\mathrm{i}), \quad \mathrm{S}_{[i, j] ; n}^{\mathrm{R}} \equiv \mathrm{~S}_{[i, j] ; n}^{\mathrm{R}}(\mathrm{i}) \tag{B.31}
\end{equation*}
$$

These operators obey the recursion relations

$$
\begin{align*}
\mathrm{S}_{[i, j+1] ; n}^{\mathrm{L}} & =\left(1-t_{n-j} e_{j}\right) \mathrm{S}_{[i, j] ; n}^{\mathrm{L}}\left(1+t_{n-j} e_{j}\right)  \tag{B.32}\\
\mathrm{S}_{[i, j] ; n}^{\mathrm{R}} & =\left(1-t_{i-n} e_{i}\right) \mathrm{S}_{[i+1, j] ; n}^{\mathrm{R}}\left(1+t_{i-n} e_{i}\right) .
\end{align*}
$$

Let us show, using this structure, that $\mathrm{S}_{[i, j] ; n}^{\mathrm{L}}$ and $\mathrm{S}_{[i, j] ; n}^{\mathrm{R}}$ are linear combinations of nested commutators of TL generators. We will use induction on the distance $j-i$. The base case is

$$
\begin{equation*}
\mathrm{S}_{[i, i+1] ; n}^{\mathrm{L}}=\mathrm{S}_{[i, i+1] ; n}^{\mathrm{R}}=e_{i} \tag{B.33}
\end{equation*}
$$

The recursion relations (B.30) can be rewritten as

$$
\begin{align*}
& \mathrm{S}_{[i, j+1] ; n}^{\mathrm{L}}=\mathrm{S}_{[i, j] ; n}^{\mathrm{L}}+t_{n-j}\left[\mathrm{~S}_{[i, j] ; n}^{\mathrm{L}}, e_{j}\right]-t_{n-j}^{2} e_{j} \mathrm{~S}_{[i, j] ; n}^{\mathrm{L}} e_{j},  \tag{B.34}\\
& \mathrm{~S}_{[i-1, j] ; n}^{\mathrm{R}}=\mathrm{S}_{[i, j] ; n}^{\mathrm{R}}-t_{i-n}\left[e_{i-1}, \mathrm{~S}_{[i, j] ; n}^{\mathrm{R}}\right]-t_{i-n}^{2} e_{i-1} \mathrm{~S}_{[i, j] ; n}^{\mathrm{R}} e_{i-1}
\end{align*}
$$

The first terms on the right-hand sides of (B.34) reproduce the structure of (B.30). The second terms increase the length of the nested commutators by one to the right or left, respectively. The third terms give a result proportional to $e_{j}$ or $e_{i-1}$, respectively, and thus move a TL generator to the right or left, respectively. At each level of nesting, we get a factor $\pm t$, and every time the index of a solitary TL generator is shifted by one to the right or left we get a factor of $-t^{2}$. In this way we obtain linear combinations of nested TL commutators, as we claimed.

For example,

$$
\begin{align*}
& \mathrm{S}_{[i, i+2]}^{\mathrm{L}}=e_{[i, i]}-t_{1}^{2} e_{[i+1, i+1]}+t_{1} e_{[i, i+1]} \\
& \mathrm{S}_{[i, i+3]}^{\mathrm{L}}=e_{[i, i]}-t_{2}^{2} e_{[i+1, i+1]}+t_{2}^{2} t_{1}^{2} e_{[i+2, i+2]}+t_{2} e_{[i, i+1]}-t_{2}^{2} t_{1} e_{[i+1, i+2]}+t_{2} t_{1} e_{[i, i+2]} \tag{B.35}
\end{align*}
$$

and

$$
\begin{align*}
& \mathrm{S}_{[j-2, j]}^{\mathrm{R}}=e_{[j-1, j-1]}-t_{1}^{2} e_{[j-2, j-2]}-t_{1} e_{[j-2, j-1]}  \tag{B.36}\\
& \mathrm{S}_{[j-3, j]}^{\mathrm{R}}=e_{[j-1, j-1]}-t_{2}^{2} e_{[j-2, j-2]}+t_{2}^{2} t_{1}^{2} e_{[j-3, j-3]}-t_{2} e_{[j-2, j-1]}+t_{2}^{2} t_{1} e_{[j-3, j-2]}+t_{2} t_{1} e_{[j-3, j-1]}
\end{align*}
$$

This structure generalises for any separation between the spins to the result (B.16) for the chiral hamiltonians.
In order to compute $H$ one has to expand the spin interactions $S_{[i, j]}^{\mathrm{L}, \mathrm{R}}(\mathrm{q})$ to linear order in $\mathrm{q}+\mathrm{q}^{-1}$ as in (B.13). It can be proved that, thanks to (B.24), we may keep the TL generators as they are, and expand

$$
\begin{align*}
f(u ; \mathrm{q}) & =f(u)-\frac{\mathrm{q}+\mathrm{q}^{-1}}{2} f(u)^{2}+\ldots, \\
f\left(u^{-1} ; \mathrm{q}\right) & =f\left(u^{-1}\right)-\frac{\mathrm{q}+\mathrm{q}^{-1}}{2} f\left(u^{-1}\right)^{2}=-f(u)-\frac{\mathrm{q}+\mathrm{q}^{-1}}{2} f(u)^{2}+\ldots \tag{B.37}
\end{align*}
$$

Following the recursion relations (B.30), the typical structure of the terms linear in $q+q^{-1}$ in $S_{[i, j]}^{\mathrm{L}}(\mathrm{q})$ is

$$
\begin{equation*}
f(u)^{2} e_{i+1} N_{i}\left(1+f(u) e_{i+1}\right)+\left(1-f(u) e_{i+1}\right) N_{i} f(u)^{2} e_{i+1}=f(u)^{2}\left\{e_{i+1}, N_{i}\right\} \tag{B.38}
\end{equation*}
$$

where $N_{i}$ contains nested commutators, and $u$ will be specified to a particular value. The next term is of the form

$$
\begin{align*}
f^{2}(u)\left(1-f(u / \omega) e_{i+2}\right)\left\{e_{i+1}, N_{i}\right\}\left(1-f(u / \omega) e_{i+2}\right)= & f(u)^{2}\left\{e_{i+1}, N_{i}\right\}-f(u)^{2} f(u / \omega)^{2}\left\{e_{i+2}, N_{i}\right\}  \tag{B.39}\\
& -f(u)^{2} f(u / \omega)\left[e_{i+2},\left\{e_{i+1}, N_{i}\right\}\right]
\end{align*}
$$

In conclusion, the typical structure of each term consists of nested commutators and one anticommutator. As is clear from the preceding formula, the index of the TL generator in the anticommutator can be transported away from those in the commutators. Using the properties of the TL operators at the free-fermion point we conclude that

$$
\begin{equation*}
\left[e_{k}, \ldots,\left\{e_{l},\left[e_{m}, \ldots,\left[e_{n-1}, e_{n}\right] \ldots\right\} \ldots\right]=\left\{e_{[k, l]}, e_{[m, n]}\right\}=(-1)^{k-l+m-n}\left\{e_{[n, m]}, e_{[l, k]}\right\}, \quad k \leqslant l<m \leqslant n\right. \tag{B.40}
\end{equation*}
$$

Then, considering again carefully the recursion relations (B.30), we are able to identify the coefficients in front of these structures to obtain (B.25).

## C. Fermionic representation

So far we have mostly considered the spin representation (A.7) of the TL generators. In this setting we do not only know the spectrum in closed form, but also the corresponding exact (highest-weight) eigenstates. In addition, to make sense of the limit in (B.11), we strictly speaking worked abstractly, i.e. inside the TL algebra.* (p. 10) Now we will be interested in a fermionic realisation of our model.

Let $\sigma_{j}^{ \pm} \equiv\left(\sigma_{j}^{x} \pm \mathrm{i} \sigma_{j}^{y}\right) / 2$ and $\sigma_{j}^{z}$ denote the Pauli matrices acting at site $j$ of the spin chain. The translation to the fermionic setting goes via the Jordan-Wigner transformation

$$
\begin{equation*}
c_{j} \equiv\left(\prod_{i=1}^{j-1}\left(-\sigma_{i}^{z}\right)\right) \sigma_{j}^{-}, \quad c_{j}^{\dagger} \equiv\left(\prod_{i=1}^{j-1}\left(-\sigma_{i}^{z}\right)\right) \sigma_{j}^{+}, \quad\left\{c_{i}, c_{j}^{\dagger}\right\}=\delta_{i j}, \quad\left\{c_{i}, c_{j}\right\}=\left\{c_{i}^{\dagger}, c_{j}^{\dagger}\right\}=0 \tag{C.1}
\end{equation*}
$$

providing canonical fermionic creation and annihilation operators that act on the usual fermionic Fock space. For our purposes it will be convenient to follow [3] and rescale by defining $f_{j}=(-\mathrm{i})^{j} c_{j}$ and $f_{j}^{+}=(-\mathrm{i})^{j} c_{j}^{\dagger}$. Here and below we use superscript '+' (rather than ${ }^{\dagger} \dagger$ ') since we do not mean the hermitian conjugate or adjoint for some scalar product. We thus start from non-unitary fermionic creation and annihilation operators with anticommutation relations

$$
\begin{equation*}
\left\{f_{i}, f_{j}^{+}\right\}=(-1)^{i} \delta_{i j}, \quad\left\{f_{i}, f_{j}\right\}=\left\{f_{i}^{+}, f_{j}^{+}\right\}=0 \tag{C.2}
\end{equation*}
$$

Then the two-site fermions

$$
\begin{equation*}
g_{j} \equiv f_{j}+f_{j+1}, \quad g_{j}^{+} \equiv f_{j}^{+}+f_{j+1}^{+} \tag{C.3}
\end{equation*}
$$

have nontrivial anticommutation relations

$$
\begin{equation*}
\left\{g_{j}, g_{j}^{+}\right\}=0, \quad\left\{g_{j}, g_{j+1}^{+}\right\}=(-1)^{j+1}, \quad\left\{g_{j}, g_{j-1}^{+}\right\}=(-1)^{j} \tag{C.4}
\end{equation*}
$$

It is straightforward to check that

$$
\begin{equation*}
e_{j} \equiv g_{j}^{+} g_{j}=-g_{j} g_{j}^{+}, \quad 1 \leqslant j<N \tag{C.5}
\end{equation*}
$$

provides a representation of the free-fermion TL algebra (B.2) on the fermionic Fock space. It turns out that this representation is isomorphic to the xxz representation (A.7) [3]. Therefore, everything that we already know about the spectrum and eigenstates of our model transports from the spin-chain setting into the fermionic set-up. We note that while a particle-hole transformation of the non-unitary fermions $f_{j} \leftrightarrow f_{j}^{+}$changes the signs of the TL operators (C.5), the result still obeys the free-fermion TL relations (B.2).

We will need some explicit (anti)commutation relations for the operators we have thus defined. Using $[A B, C]=$ $A\{B, C\}-\{A, C\} B$ one obtains the following nontrivial commutation relations between TL generators and two-site fermions:

$$
\begin{align*}
& {\left[e_{j}, g_{j}^{+}\right]=\left[e_{j}, g_{j}\right]=0,} \\
& {\left[e_{j}, g_{j+1}\right]=(-1)^{j} g_{j}, \quad\left[e_{j}, g_{j-1}\right]=(-1)^{j-1} g_{j},}  \tag{C.6}\\
& {\left[e_{j}, g_{j+1}^{+}\right]=(-1)^{j+1} g_{j}^{+}, \quad\left[e_{j}, g_{j-1}^{+}\right]=(-1)^{j} g_{j}^{+} .}
\end{align*}
$$

Note that these relations hold as long as the subscripts of $e$ and $g$ lie between 1 and $N-1$. The commutation relations between the TL generators and the one-site fermions will be given in (E.7).

A special feature of the point $\mathrm{q}=\mathrm{i}$ is that nested commutators of TL generators are quadratic in fermions: using $[A B, C D]=A\{B, C\} D-\{A, C\} B D+C A\{B, D\}-C\{A, D\} B$ successively, the previous expressions yield

$$
\begin{align*}
{\left[e_{j}, e_{j+1}\right] } & =(-1)^{j}\left(g_{j+1}^{+} g_{j}-g_{j}^{+} g_{j+1}\right), \\
{\left[\left[e_{j}, e_{j+1}\right], e_{j+2}\right] } & =-\left(g_{j+2}^{+} g_{j}+g_{j}^{+} g_{j+2}\right),  \tag{C.7}\\
{\left[\left[\left[e_{j}, e_{j+1}\right], e_{j+2}\right], e_{j+3}\right] } & =(-1)^{j+1}\left(g_{j+3}^{+} g_{j}-g_{j}^{+} g_{j+3}\right) .
\end{align*}
$$

Continuing in this we we induce that in general

$$
\begin{equation*}
e_{[j, j+m]} \equiv\left[\ldots\left[e_{j}, e_{j+1}\right] \ldots, e_{j+m}\right]=s_{j, j+m}\left(g_{j+m}^{+} g_{j}+(-1)^{m} g_{j}^{+} g_{j+m}\right), \quad m>0 \tag{C.8}
\end{equation*}
$$

where we define the signs

$$
\begin{equation*}
s_{i j} \equiv(-1)^{i+\cdots+j-1}=(-1)^{(i-j)(i+j-1) / 2} \tag{C.9}
\end{equation*}
$$

Armed with these relations we turn to the relation between the fermionic operators and our conserved charges.

## 1. Fermions and quasi-translations

First we consider the quasi-translation operator $G$ defined in (B.5).
a. Quasi-translated fermions. In order to find the fermionic degrees of freedom in which the model becomes as simple as possible, we seek a basis with good transformation properties under quasi-translations. It is shown by formula (E.12) that conjugating by G preserves the space of one-particle fermionic operator. Thus we are led to define the alternative fermionic basis (21)

$$
\begin{equation*}
\Phi_{i} \equiv \mathrm{G}^{1-i} f_{1} \mathrm{G}^{i-1}, \quad \Phi_{i}^{+} \equiv \mathrm{G}^{1-i} f_{1}^{+} \mathrm{G}^{i-1} \tag{C.10}
\end{equation*}
$$

That is, $\Phi_{1}=f_{1}$ and $\Phi_{i+1}=\mathrm{G}^{-1} \Phi_{i} \mathrm{G}$, and likewise for $\Phi_{j}^{+}$. Since $\mathrm{G}^{N}=1$ these quasi-translated fermions are (formally) periodic by definition,

$$
\begin{equation*}
\Phi_{i+N}=\Phi_{i}, \quad \Phi_{i+N}^{+}=\Phi_{i}^{+} \tag{C.11}
\end{equation*}
$$

As expected for some kind of translated fermions, this new basis is triangular in the $f_{i}$ in the sense that $\Phi_{i}$ is a linear combination of $f_{1}, \ldots, f_{i}$, and $\Phi_{i}^{+}$of $f_{1}^{+}, \ldots, f_{i-1}^{+}$.

In $\S \mathrm{E} 1$ we obtain an explicit formula for the action of $\mathrm{G}^{-1}$ by conjugation on the two-site fermions, see (E.16). That formula can be expressed via $(N-1) \times(N-1)$ matrices $\gamma=\left(\gamma_{i j}\right)_{1 \leqslant i, j \leqslant N-1}$ and $\bar{\gamma}=\left(\bar{\gamma}_{i j}\right)_{1 \leqslant i, j \leqslant N-1}$ defined by the relations

$$
\begin{equation*}
\mathrm{G}^{-1} g_{i}^{+} \mathrm{G}=\sum_{j=1}^{N-1} \bar{\gamma}_{i j} g_{j}^{+}, \quad \mathrm{G}^{-1} g_{i} \mathrm{G}=\sum_{j=1}^{N-1} \gamma_{i j} g_{j} \tag{C.12}
\end{equation*}
$$

Explicitly, the matrix elements read

$$
(-1)^{i-j} \gamma_{i j}=\bar{\gamma}_{i j}= \begin{cases}0, & i<j-1  \tag{C.13}\\ (-1)^{i} t_{i+1}, & i=j-1 \\ s_{i j}\left(1+t_{i} t_{i+1}\right) t_{j, i} & i \geqslant j\end{cases}
$$

In addition, since $\mathrm{G}^{N}=1$, we have $\gamma^{N}=\bar{\gamma}^{N}=1$. It is now possible to express the quasi-translated fermions in those terms, since it is shown in (E.13) that

$$
\begin{equation*}
\mathrm{G}^{-1} f_{1} \mathrm{G}=f_{1}-t_{1} g_{1}, \quad \mathrm{G}^{-1} f_{1}^{+} \mathrm{G}=f_{1}^{+}+t_{1} g_{1}^{+} \tag{C.14}
\end{equation*}
$$

which immediately leads to the relation

$$
\begin{equation*}
\Phi_{i+1}=\Phi_{i}-t_{1} \sum_{k=1}^{N-1}\left(\gamma^{i-1}\right)_{1 k} g_{k}, \quad \Phi_{i+1}^{+}=\Phi_{i}^{+}+t_{1} \sum_{k=1}^{N-1}\left(\bar{\gamma}^{i-1}\right)_{1 k} g_{k}^{+} \tag{C.15}
\end{equation*}
$$

By induction, we then obtain

$$
\begin{equation*}
\Phi_{i+1}=f_{1}-t_{1} \sum_{k=1}^{N-1}\left(\gamma^{0}+\cdots+\gamma^{i-1}\right)_{1 k} g_{k}, \quad \Phi_{i+1}^{+}=f_{1}^{+}+t_{1} \sum_{k=1}^{N-1}\left(\bar{\gamma}^{0}+\cdots+\bar{\gamma}^{i-1}\right)_{1 k} g_{k}^{+} \tag{C.16}
\end{equation*}
$$

It is remarkable that the transformed fermionic operators can be expressed in terms of the $g_{k}^{+}$and $f_{1}^{+}$in such a way.
Using definition (C.10) it is easy to see that anticommutation relations between the quasi-translated fermions are determined by their distances only, i.e.

$$
\begin{equation*}
\left\{\Phi_{i}^{+}, \Phi_{j}\right\}=\alpha(i-j) \tag{C.17}
\end{equation*}
$$

where $\alpha(k) \equiv\left\{\Phi_{k+1}^{+}, f_{1}\right\}$ satisfies

$$
\begin{equation*}
\alpha(k)=\alpha(N+k) \tag{C.18}
\end{equation*}
$$

A more explicit expression for this quantity follows from (C.16):

$$
\begin{equation*}
\alpha(k)=-1-t_{1}\left(\gamma^{0}+\cdots+\gamma^{k-1}\right)_{11}, \quad \alpha(-k)=-1+t_{1}\left(\bar{\gamma}^{0}+\cdots+\bar{\gamma}^{k-1}\right)_{11}, \quad k \geqslant 0 \tag{C.19}
\end{equation*}
$$

Numerical evidence suggests that the result is exceedingly simple:

$$
\begin{equation*}
\alpha(k)=-\left(1+t_{k}\right) . \tag{C.20}
\end{equation*}
$$

This gives the anticommutation-relations (23) in the main text. However, note that we do not yet have a complete proof of this fact, see (C.27).
b. Fourier-transformed fermions. To build an eigenbasis of the quasi-translation in the one-particle sector, we can use the periodicity (C.11) to introduce the Fourier transforms

$$
\begin{equation*}
\tilde{\Phi}_{n}=\frac{1}{N} \sum_{j=1}^{N} \omega^{-n j} \Phi_{j}, \quad \tilde{\Phi}_{n}^{+}=\frac{1}{N} \sum_{j=1}^{N} \omega^{n j} \Phi_{j}^{+} \tag{C.21}
\end{equation*}
$$

where we recall the notation $\omega \equiv \mathrm{e}^{2 \pi \mathrm{i} / N}$ from (A.13). Since $\mathrm{G} \Phi_{j} \mathrm{G}^{-1}=\Phi_{j-1}$ by construction, under quasi-translation we indeed have

$$
\begin{equation*}
\mathrm{G} \tilde{\Phi}_{n} \mathrm{G}^{-1}=\omega^{-n} \tilde{\Phi}_{n}, \quad \mathrm{G} \tilde{\Phi}_{n}^{+} \mathrm{G}^{-1}=\omega^{n} \tilde{\Phi}_{n}^{+} \tag{C.22}
\end{equation*}
$$

It is shown in (E.32) that

$$
\begin{equation*}
\tilde{\Phi}_{0}=\sum_{i=1}^{N} f_{i}, \quad \tilde{\Phi}_{0}^{+}=\sum_{i=1}^{N} f_{i}^{+} \tag{C.23}
\end{equation*}
$$

yielding (27) from the main text up to the rescaling (C.28) discussed below. It follows that (C.23) commute with every $e_{j}$. These zero modes thus do not only commute with G , but also with the hamiltonians.

As proved in (E.33), the remaining fermionic modes $\Phi_{n}$ and $\tilde{\Phi}_{n}(n>0)$, which have positive quasi-momentum, can be expressed as linear combinations of the $N-1$ two-site fermions:

$$
\begin{equation*}
\tilde{\Phi}_{n}=\sum_{j=1}^{N-1} M_{n j} g_{j}, \quad \tilde{\Phi}_{n}^{+}=\sum_{j=1}^{N-1} \bar{M}_{n j} g_{j}^{+} \tag{C.24}
\end{equation*}
$$

These coefficients can readily be expressed in terms of $\gamma$ by Fourier-transforming both sides of (C.15)

$$
\begin{equation*}
M_{n j} \equiv \frac{\omega^{-2 n} t_{1}}{N\left(1-\omega^{-n}\right)} \sum_{k=0}^{N-1}\left[\left(\omega^{-n} \gamma\right)^{k}\right]_{1 j}, \quad \bar{M}_{n j} \equiv \frac{\omega^{2 n} t_{1}}{N\left(1-\omega^{n}\right)} \sum_{k=0}^{N-1}\left[\left(\omega^{n} \bar{\gamma}\right)^{k}\right]_{1 j} \tag{C.25}
\end{equation*}
$$

A numerically more tractable description of these matrix elements will be given in (E.34) and (E.36) below. If we take the numerical conjecture (C.20) as granted, we can obtain the anticommutation relations between the Fourier modes by Fourier transforming the periodic sequence $\alpha$ as follows:

$$
\begin{align*}
\left\{\tilde{\Phi}_{n}^{+}, \tilde{\Phi}_{m}\right\} & =\frac{1}{N^{2}} \sum_{i, j=1}^{N} \omega^{n i-m j}\left\{\Phi_{i}^{+}, \Phi_{j}\right\} \\
& =\frac{1}{N^{2}} \sum_{i, j=1}^{N} \omega^{n i-m j} \alpha(i-j)  \tag{C.26}\\
& =\frac{1}{N^{2}} \sum_{i=1}^{N} \omega^{(n-m) i} \sum_{p=i-N}^{i-1} \omega^{m p} \alpha(p)=\delta_{n m} \tilde{\alpha}(m)
\end{align*}
$$

with

$$
\tilde{\alpha}(n) \equiv \frac{1}{N} \sum_{p=1}^{N} \omega^{n p} \alpha(p)= \begin{cases}-1 & n=0  \tag{C.27}\\ (-1)^{n} \mathrm{i} & 1 \leqslant n<N\end{cases}
$$

Note that (C.27) is equivalent as a statement to the numerical conjecture (C.20) it is based on. While this result was only numerically observed at the moment, it will be proven up to a sign in (E.44) and (E.45). We do not have a way to derive this sign as a function of $n$ yet.

We see that, even though we have a long-range interacting model, we can still Fourier transform the suitably translated fermions as usual. Note the orthogonality $\left\{\tilde{\Phi}_{n}^{+}, \tilde{\Phi}_{m}\right\} \propto \delta_{n m}$. The nearly canonical anticommutation relations are the reason for replacing the modes (C.21) by the rescaled fermionic modes

$$
\begin{equation*}
\tilde{\Psi}_{n} \equiv a_{n} \tilde{\Phi}_{n}, \quad \tilde{\Psi}_{n}^{+} \equiv a_{n} \tilde{\Phi}_{n}^{+}, \quad a_{0} \equiv \mathrm{i}, \quad a_{n} \equiv \mathrm{i}^{n+1 / 2} \quad(n \neq 0) \tag{C.28}
\end{equation*}
$$

as in (24) to absorb the signs and factor of i in (C.27) and obtain the canonical anticommutation relations (25) from the main text.

## 2. The hamiltonians in terms of fermionic modes

It is now relatively straightforward to write the hamiltonians in terms of the eigenmodes of the quasi-translations. First, we can use (B.22) and (C.5) to express $\mathrm{H}^{\mathrm{L}}$ in terms of the two site fermions as

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}}=\frac{\mathrm{i}}{2} \sum_{1 \leqslant i \leqslant j<N} s_{i j} h_{i j}^{\mathrm{L}}\left(g_{j}^{+} g_{i}+(-1)^{i-j} g_{i}^{+} g_{j}\right)=\frac{\mathrm{i}}{2} \sum_{i, j=1}^{N-1} s_{i j} h_{i j}^{\mathrm{L}} g_{j}^{+} g_{i}, \tag{C.29}
\end{equation*}
$$

where we used the symmetry property (B.20) of $h_{i j}^{\mathrm{L}}$ in the second equality. Passing to the new fermionic modes yields the quadratic expression

$$
\begin{equation*}
\mathrm{H}^{\mathrm{L}}=\sum_{n=1}^{N-1} \varepsilon_{n}^{\mathrm{L}} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{n}, \quad \varepsilon_{n}^{\mathrm{L}}=\frac{(-1)^{n+1}}{2} \sum_{i, j=1}^{N-1} s_{i j} h_{i j}^{\mathrm{L}} M_{i n}^{-1} \bar{M}_{j n}^{-1} \tag{C.30}
\end{equation*}
$$

The final equality, between the coefficients coming from (C.24) and the chiral dispersion from (B.8), is not trivial, but follows since it reproduces the one-magnon spectrum known from the parent model. While we do not have a direct proof of this equality, we have checked it numerically for low $N$. We thus obtain the quadratic (free-fermionic) expression (29) from the main text.

The next step is rewrite $H$. This can be done by rearranging the creation and annihilation operators in the anticommutators of nested commutators in (B.26). In the process one gets quadratic terms (when $k=j+1$ ) as well as quartic terms. Let us split the hamiltonian accordingly,

$$
\begin{equation*}
\mathrm{H}=\mathrm{H}_{2}+\mathrm{H}_{4} \tag{C.31}
\end{equation*}
$$

Let us combine the coefficients $h_{i j ; k l}^{\mathrm{L}, \mathrm{R}}$ from (B.27) into

$$
\begin{equation*}
h_{i j ; k l} \equiv h_{i j ; k l}^{\mathrm{L}}+h_{i j ; k l}^{\mathrm{R}} . \tag{C.32}
\end{equation*}
$$

Then the quadratic terms read

$$
\begin{gather*}
\mathrm{H}_{2}=-\frac{1}{4 N} \sum_{i \leqslant j}^{N-1} s_{i j} h_{i j}\left(g_{j}^{+} g_{i}+(-1)^{j-i+1} g_{i}^{+} g_{j}\right)=-\frac{1}{4 N} \sum_{i, j=1}^{N-1} s_{i j} h_{i j} g_{j}^{+} g_{i}  \tag{C.33}\\
h_{i j} \equiv \sum_{k=i}^{j-1} h_{i k ; k+1, j}=(-1)^{i-j+1} h_{j i}
\end{gather*}
$$

and

$$
\begin{equation*}
\mathrm{H}_{2}=\sum_{n=1}^{N-1} \varepsilon_{n} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{n}, \quad \varepsilon_{n}=\frac{(-1)^{n} \mathrm{i}}{4 N} \sum_{i, j=1}^{N-1} s_{i j} h_{i j} M_{i n}^{-1} \bar{M}_{j n}^{-1} \tag{C.34}
\end{equation*}
$$

This gives the first part of (32) in the main text. As before, we do not have an independent proof of the final equality, but it follows from the parent model and has been checked numerically. Regarding the quartic hamiltonian, after reordering the different contributions we get

$$
\begin{equation*}
\mathrm{H}_{4}=\frac{1}{2 N} \sum_{i, j<k, l}^{N-1} s_{i j} s_{k l} h_{i j ; k l} g_{j}^{+} g_{l}^{+} g_{i} g_{k}, \quad h_{i j ; k l}=(-1)^{l-k} h_{i j ; l k}=(-1)^{i-j} h_{j i ; k l}=(-1)^{i-j+l-k} h_{j i ; l k} \tag{C.35}
\end{equation*}
$$

where the sum is over all $1 \leqslant i, j, k, l<N$ such that $\max (i, j)<\min (k, l)$. After the transformation (28) this becomes

$$
\begin{equation*}
\mathrm{H}_{4}=\sum_{m<n, r<s}^{N-1} \tilde{V}_{m n ; r s} \tilde{\Psi}_{m}^{+} \tilde{\Psi}_{n}^{+} \tilde{\Psi}_{r} \tilde{\Psi}_{s}, \quad \tilde{V}_{m n ; r s} \equiv \frac{1}{a_{m} a_{n} a_{r} a_{s}} \frac{2}{N} \sum_{i, j<k, l}^{N-1} s_{i j} s_{k l} h_{i j ; k l} \bar{M}_{j[m \mid}^{-1} \bar{M}_{l \mid n]}^{-1} M_{i[r \mid}^{-1} M_{k \mid s]}^{-1} \tag{C.36}
\end{equation*}
$$

where the square brackets signify antisymmetrisation in the indices $n, m$ and $r, s$ (but not $l$ or $k$ ). Again, we have not been able to compute (C.36) explicitly, and this time we do not know how to obtain a simplification from the parent
model. However, numerical evaluation suggests that the coefficients simplify drastically: nonzero values occur when $m+n$ is odd and equals $r+s$, and are given by

$$
\begin{equation*}
\tilde{V}_{m n ; m+k, n-k}=(-1)^{k+1} 4 \delta_{m, \text { odd }}, \quad 0 \leqslant k<\frac{n-m}{2} \tag{C.37}
\end{equation*}
$$

together with values for $k<0$ following from the symmetry $\tilde{V}_{m n ; r s}=\tilde{V}_{r s ; m n}$.
Note that these selection rules are stronger than the condition (33) for the conservation of quasi-momentum and chiral energy, as required by the commutativity (B.12). For example, when $N=9$ those charges take the same value for $(m, n)=(1,2)$ and $(r, s)=(5,7)$, but $m+n$ equals $r+s$ only $\bmod N$, and indeed we observe $\tilde{V}_{12 ; 57}=0$. There are more and more such examples for larger $N$. The potential (C.37) is corroborated by the two-particle spectrum, which is our next topic.

## D. Examples of the two-fermion spectrum for small systems

To understand the hamiltonian (C.31) better let us describe the few-particle spectrum in the fermionic language, using the Fock basis (30),

$$
\begin{equation*}
\left|n_{1}, \ldots, n_{M}\right\rangle=\tilde{\Psi}_{n_{1}}^{+} \ldots \tilde{\Psi}_{n_{M}}^{+}|\varnothing\rangle \tag{D.1}
\end{equation*}
$$

These are eigenvectors for the quasi-translation $G$ and $H^{L}$, but not necessarily for $H$ because of its quartic part. Let us show for $M \leq 2$ how it fits with the description of the spectrum via motifs coming from the parent model.

The Fock vacuum $|\varnothing\rangle$ is an eigenvector for all conserved charges, with vanishing quasi-momentum and energies. It is labelled by the empty motif, which has degeneracy $N+1$. All its descendants come from the global symmetry only.

The one-particle spectrum consists of the global descendant $|0\rangle=\mathrm{F}_{1}^{+}|\varnothing\rangle$ along with $N-1$ Fock states $|n\rangle$ that correspond to the motifs $\{n\}$. They have quasi-momentum $p=2 \pi n / N$ and energies given by the dispersions (19). For the $\binom{N}{2}$-dimensional two-particle sector we see the quartic part $\mathrm{H}_{4}$ in action, and some Fock states cease to be eigenstates for $H$. In the following we work out the two-fermion spectrum explicitly for small systems, and link it to the description coming from the parent model in terms of the motifs (A.6), to illustrate the description for general $N$ given at the end of the main text.
$N=3$. The 3 two-particle Fock states lie beyond the 'equator' $M=N / 2$. They are descendants for the global symmetry: $|0,1\rangle \propto \mathrm{F}_{1}^{+}|1\rangle$, its parity-conjugate $|0,2\rangle \propto \mathrm{F}_{1}^{+}|2\rangle$, and $|1,2\rangle \propto \mathrm{F}_{2}^{+}|\varnothing\rangle$. In this case the Fock basis happens to be an eigenbasis for the conserved charges, including H , thus yielding a fermionic description of the full Hilbert space. This is illustrated in Fig. 2.

|  |  | \{1\} $\{2\}$ |
| :---: | :---: | :---: |
| 0 | $\varnothing$ |  |
| 1. | 0 | 12 |
| 2 | 12 | 0102 |
| 3 | 012 |  |
| N |  |  |


| motif | $\}$ | $\{1\}$ | $\{2\}$ |
| :--- | :--- | ---: | :--- |
| $p$ | 0 | $\frac{2 \pi}{3}$ | $\frac{4 \pi}{3}$ |
| $E^{\mathrm{L}}$ | 0 | -2 | 2 |
| $E$ | 0 | 2 | 2 |
| degen. | 4 | 2 | 2 |

Figure 2. Left. The structure of the Hilbert space for $N=3$. Each label represents an eigenstate, which in this case are just Fock states. The vertical axis records the fermion number. The eigenspaces are labelled by motifs, with the parity-conjugate pair linked by a ' ${ }^{\prime}$ '. The lines ' $\mid$ ' and ' $\backslash$ ' indicate the action of the global-symmetry generators $\mathrm{F}_{2}^{ \pm}$and $\mathrm{F}_{1}^{ \pm}$, respectively. Right. The corresponding spectrum: quasi-momenta (note that $\frac{4 \pi}{3}=-\frac{2 \pi}{3} \bmod 2 \pi$ ), energies, and degeneracies.
$N=5$. We seek 10 two-particle eigenstates. Their fermionic description is as follows.

- The 4 Fock states $|0, n\rangle \propto \mathrm{F}_{1}^{+}|n\rangle$ with $1 \leqslant n \leqslant 4$ are global descendants from the one-particle sector, belonging to the motifs $\{n\}$.
- The 2 parity-conjugate Fock states $|1,3\rangle$ and $|2,4\rangle$ are protected by the selection rules. Indeed, 'squeezing' produces coinciding mode numbers, which is not allowed for fermions, and $\tilde{V}_{13 ; 13}=\tilde{V}_{24 ; 24}=0$. Both are eigenvectors for all conserved charges, corresponding to the motifs $\{1,3\}$ and $\{2,4\}$.
- The remaining two-particle eigenstates feel the quartic interaction:
- The Fock state $|1,2\rangle$ is an eigenstate for $H_{4}$ and therefore H. From the parent model we know that it is an extended-symmetry descendants of the motif $\{3\}:|1,2\rangle \propto \widehat{\mathrm{F}}_{1}^{+}|3\rangle$ for some generator $\widehat{\mathrm{F}}_{1}^{+}$of the extendedsymmetry algebra. This is consistent with the chiral dispersion thanks to (31), while for H it relies on $\tilde{V}_{12 ; 12}=-4$ contributing to $\varepsilon_{1}+\varepsilon_{2}+\tilde{V}_{12 ; 12}=4+2-4=2=\varepsilon_{3}$.
- The Fock states $|1,4\rangle$ and $|2,3\rangle$ are mixed by $\mathrm{H}_{4}$. The parent model tells us that diagonalising this $2 \times 2$ block of H produces two eigenvectors that we denote by $|1,4\rangle^{\prime}$, with motif $\{1,4\}$, and $|2,3\rangle^{\prime} \propto \mathrm{F}_{2}^{+}|\varnothing\rangle$ a global descendant of the empty motif.
- The Fock state $|3,4\rangle$ is the parity-conjugate of $|1,2\rangle$, and is an extended-symmetry descendant of the motif $\{2\}$; note that $2=3+4 \bmod 5$
Note that the one-particle motifs $\{2\}$ and $\{3\}$ both have two descendants in the two-particle sector, one for the global symmetry and one for the extended symmetry, while the empty motif, $\{1\}$ and $\{4\}$ each only have a global descendant in this sector. This matches the description known from the parent model. In this case the spectrum with $\leqslant 2$ particles determines the whole Hilbert space by particle-hole symmetry, so we again have complete fermionic description, shown in Fig. 3.


Figure 3. Left. The structure of the Hilbert space for $N=5$. The states with $\leqslant 2$ particles are labelled by their fermionic mode numbers, with a prime for the two eigenstates resulting from diagonalising a $2 \times 2$ block of H , also indicated. Global symmetry and parity are shown as in Fig. 2. Dotted lines like ' ' represent the action of extended-symmetry generators $\widehat{\mathrm{F}}_{1}^{ \pm}$, joining together representations of the global symmetry algebra. Right. The corresponding spectrum.
$N=7$. The two-particle sector has dimension 21. Its states have the following fermionic description.

- The 6 global descendants $|0, n\rangle \propto \mathrm{F}_{1}^{+}|n\rangle, 1 \leqslant n \leqslant 6$, come from the motifs $\{n\}$.
- The 6 Fock states $|1,3\rangle,|1,5\rangle,|2,4\rangle,|2,6\rangle,|3,5\rangle,|4,6\rangle$ are protected by the selection rules, corresponding to the motifs $\{1,3\}$ etc.
- The other two-particle eigenstates correspond to blocks:
- The Fock state $|1,2\rangle \propto \widehat{\mathrm{F}}_{1}|3\rangle$ is an extended-symmetry descendant for the motif $\{3\}$.
- The 2 Fock states $|1,4\rangle$ and $|2,3\rangle$ are mixed. Its diagonalisation produces eigenstates $|1,4\rangle^{\prime}$, corresponding to the motif $\{1,4\}$, along with $|2,3\rangle^{\prime} \propto \widehat{\mathrm{F}}_{1}|5\rangle$ belonging to $\{5\}$.
- The 3 Fock states $|1,6\rangle,|2,5\rangle,|3,4\rangle$ correspond to a $3 \times 3$ block of H. Diagonalisation yields eigenstates with motifs $\{1,6\}$ and $\{2,5\}$, plus a global descendant $|3,4\rangle^{\prime} \propto \mathrm{F}_{2}^{+}|\varnothing\rangle$ belonging to the empty motif.
- The 2 Fock states $|3,6\rangle,|4,5\rangle$ give, by diagonalisation, an eigenstate with motif $\{3,6\}$, and $|4,5\rangle^{\prime} \propto \widehat{\mathrm{F}}_{1}|2\rangle$.
- The Fock state $|5,6\rangle \propto \widehat{\mathrm{F}}_{1}|4\rangle$ is again an extended-symmetry descendant for the motif $\{4\}$.

Observe that the blocks of the same size are related by parity, and the Fock- and eigenstates in the middle block are parity-selfconjugate. The one-fermion Fock states with motifs $\{2\},\{3\},\{4\},\{5\}$ have a global as well as an extendedsymmetry descendant. From this length onwards, the sectors with $\leqslant 2$ particles do no longer determine the full Hilbert space. In addition, we start to get 'accidental degeneracies' between different motifs: from Table I we see that $\{1\}$ and $\{3,5\}$ have the same values for the conserved charges, and by parity the same is true for $\{6\}$ and $\{2,4\}$.

Continuing in this way one arrives at the general description of the two-particle sector as in the main text.

## E. Technical proofs

This section contains various technical proofs of claims from the preceding sections.

| motif | $\}$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{4\}$ | $\{5\}$ | $\{6\}$ | $\{1,3\}$ | $\{1,4\}$ | $\{1,5\}$ | $\{1,6\}$ | $\{2,4\}$ | $\{2,5\}$ | $\{2,6\}$ | $\{3,5\}$ | $\{3,6\}$ | $\{4,6\}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $p$ | 0 | $\frac{2 \pi}{7}$ | $\frac{4 \pi}{7}$ | $\frac{6 \pi}{7}$ | $\frac{8 \pi}{7}$ | $\frac{10 \pi}{7}$ | $\frac{12 \pi}{7}$ | $\frac{8 \pi}{7}$ | $\frac{10 \pi}{7}$ | $\frac{12 \pi}{7}$ | 0 | $\frac{12 \pi}{7}$ | 0 | $\frac{2 \pi}{7}$ | $\frac{2 \pi}{7}$ | $\frac{4 \pi}{7}$ | $\frac{6 \pi}{7}$ |
| $E^{\text {L }}$ | 0 | -6 | 2 | -4 | 4 | -2 | 6 | -10 | -2 | -8 | 0 | 6 | 0 | 8 | -6 | 2 | 10 |
| $E$ | 0 | 6 | 2 | 4 | 4 | 2 | 6 | 10 | 10 | 8 | 12 | 6 | 4 | 8 | 6 | 10 | 10 |
| degen. | 8 | 6 | 10 | 12 | 12 | 10 | 6 | 4 | 6 | 6 | 4 | 6 | 8 | 6 | 6 | 6 | 4 |

Table I. The spectrum for $N=7$ in the sectors of the Fock space with $\leqslant 2$ particles.

## 1. Action of G on the fermions

The aim of this part is to give further details about the results mentioned in § C 1 that give intel about the action of the conjugation operator $x \mapsto \mathrm{G}^{-1} x \mathrm{G}$ on the one-particle fermionic operators, i.e. the $N$-dimensional space spanned by $f_{1}^{+}, \ldots, f_{N}^{+}$in the case of creation operators and the one spanned by $f_{1}, \ldots, f_{N}$ in the case of annihilation operators.

In order to fully appreciate the symmetry in formulas between creation and annihilation operators we will use the notation $f_{k}^{-} \equiv f_{k}$ for annihilation operators (and similarly for $g_{k}^{-}$et cetera) in the following.
a. Ingredients. Since $N$ is odd, let us first note that if we complete the family $\left(g_{1}^{ \pm}, \ldots, g_{N-1}^{ \pm}\right)$defined in (2) with

$$
\begin{equation*}
g_{N}^{ \pm} \equiv f_{N}^{ \pm}+f_{1}^{ \pm} \tag{E.1}
\end{equation*}
$$

we get an alternative basis $\left(g_{1}^{ \pm}, \ldots, g_{N-1}^{ \pm}, g_{N}^{ \pm}\right)$for the span of $\left(f_{1}^{ \pm}, \ldots, f_{N}^{ \pm}\right)$as, for example, we have

$$
\begin{equation*}
f_{1}^{ \pm}=-\frac{1}{2} \sum_{k=1}^{N}(-1)^{k} g_{k}^{ \pm} \tag{E.2}
\end{equation*}
$$

In these bases, we can express the P transformation as

$$
\begin{equation*}
\mathrm{P}\left(f_{k}^{ \pm}\right) \equiv f_{N+1-k}^{ \pm}, \quad \mathrm{P}\left(g_{k}^{ \pm}\right) \equiv g_{N-k}^{ \pm} \tag{E.3}
\end{equation*}
$$

where we considered that $P\left(g_{N}^{ \pm}\right)=g_{0}^{ \pm} \equiv g_{N}^{ \pm}$, the $\mathbb{C}$-antilinear T transformation as

$$
\begin{equation*}
\mathrm{T}\left(f_{k}^{ \pm}\right) \equiv f_{k}^{ \pm}, \quad \mathrm{T}\left(g_{k}^{ \pm}\right) \equiv g_{k}^{ \pm} \tag{E.4}
\end{equation*}
$$

and the C transformation as

$$
\begin{equation*}
\mathrm{C}\left(f_{k}^{ \pm}\right) \equiv f_{k}^{\mp}, \quad \mathrm{C}\left(g_{k}^{ \pm}\right) \equiv g_{k}^{\mp} \tag{E.5}
\end{equation*}
$$

Now consider the quasi-translation. Since $e_{i}^{2}=0$ we have $\left(1+t_{k} e_{k}\right)\left(1-t_{k} e_{k}\right)=\left(1+t_{k} e_{k}\right)\left(1-t_{k} e_{k}\right)=1$, so

$$
\begin{equation*}
\mathrm{G}^{-1}=\left(1-t_{1} e_{1}\right) \cdots\left(1-t_{N-1} e_{N-1}\right) \tag{E.6}
\end{equation*}
$$

To work out $\mathrm{G}^{-1} f_{k}^{ \pm} \mathrm{G}$, it is then natural to compute the result of successively applying conjugation by $1-t_{j} e_{j}$ in increasing order. These computations can be mechanically performed using the following commutation relations, that arise from the formula $[A B, C]=A\{B, C\}-\{A, C\} B$ :

$$
\begin{equation*}
\left[e_{j}, f_{j}^{ \pm}\right]= \pm(-1)^{j} g_{j}^{ \pm}, \quad\left[e_{j}, f_{j+1}^{ \pm}\right]=\mp(-1)^{j} g_{j}^{ \pm}, \quad 1 \leqslant j \leqslant N-1 \tag{E.7}
\end{equation*}
$$

Note that we retrieve in particular the analogous formulas for the $\left[e_{j}, g_{k}^{ \pm}\right]$that were given in (C.6).
When doing so, the following partially conjugated quantities naturally arise:

$$
\begin{equation*}
y_{k}^{ \pm} \equiv\left(1-t_{1} e_{1}\right) \cdots\left(1-t_{k-1} e_{k-1}\right) g_{k}^{ \pm}\left(1+t_{k-1} e_{k-1}\right) \cdots\left(1+t_{1} e_{1}\right), \quad 1 \leqslant k \leqslant N \tag{E.8}
\end{equation*}
$$

A few facts about this family will be of use in what comes next. First and foremost, it is more elegantly characterised by the recurrence relation

$$
\begin{equation*}
y_{k+1}^{ \pm}=g_{k+1}^{ \pm} \pm(-1)^{k} t_{k} y_{k}^{ \pm}, \quad y_{1}^{ \pm}=g_{1}^{ \pm} \tag{E.9}
\end{equation*}
$$

Moreover, as $t_{N}=0$, it is easily seen that this recursion relation is compatible with periodic boundary conditions. Namely, extending indices to more generally have $y_{N+k}^{ \pm}=y_{k}^{ \pm}$results in an $N$-periodic sequence that satisfies (E.9) for any $k \in \mathbb{Z}$. In addition, an explicit formula for the $y_{k}^{ \pm}$follows by induction:

$$
\begin{equation*}
y_{k}^{ \pm}=\sum_{l=1}^{k}( \pm 1)^{l-k} s_{l, k} t_{l, k} g_{l}^{ \pm}, \quad 1 \leqslant k \leqslant N \tag{E.10}
\end{equation*}
$$

where $t_{l, k}$ was defined in (B.17) and $s_{l, k}$ in (C.9). Beware, though, that $y_{0}^{ \pm} \equiv y_{N}^{ \pm} \neq 0$. It is now clear that the $y_{k}^{ \pm}$ form a basis of the space that is triangular in the $g_{k}^{ \pm}$. In particular, we have

$$
\begin{equation*}
y_{k}^{ \pm}=f_{k+1}^{ \pm}+\text {lower }, \quad 1 \leqslant k \leqslant N-1 \tag{E.11}
\end{equation*}
$$

where 'lower' stands for terms that are supported by $f_{1}^{+}, \ldots, f_{k}^{+}$.
b. Action of quasi-translation. Using the ingredients introduced above, a direct calculation leads to the main formula

$$
\begin{equation*}
\mathrm{G}^{-1} f_{k}^{ \pm} \mathrm{G}=f_{k}^{ \pm} \pm(-1)^{k-1}\left(t_{k-1} y_{k-1}^{ \pm}+t_{k} y_{k}^{ \pm}\right) \tag{E.12}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\Phi_{2}^{ \pm} \equiv \mathrm{G}^{-1} f_{1}^{ \pm} \mathrm{G}=f_{1}^{+} \pm t_{1} g_{1}^{+} \tag{E.13}
\end{equation*}
$$

On the two-site operators $g_{k}$, it follows directly from (E.12) that

$$
\begin{equation*}
\mathrm{G}^{-1} g_{k}^{ \pm} \mathrm{G}=g_{k}^{ \pm} \pm(-1)^{k}\left(t_{k+1} y_{k+1}^{ \pm}-t_{k-1} y_{k-1}^{ \pm}\right) \tag{E.14}
\end{equation*}
$$

Equivalently, using the recursion relation (E.9) a first time on the first two terms gives

$$
\begin{equation*}
\mathrm{G}^{-1} g_{k}^{ \pm} \mathrm{G}=y_{k}^{ \pm} \pm(-1)^{k} t_{k+1} y_{k+1}^{ \pm} \tag{E.15}
\end{equation*}
$$

and using it a second time on the last term leads to

$$
\begin{equation*}
\mathrm{G}^{-1} g_{k}^{ \pm} \mathrm{G}=\left(1+t_{k} t_{k+1}\right) y_{k}^{+} \pm(-1)^{k} t_{k+1} g_{k+1}^{+} \tag{E.16}
\end{equation*}
$$

Due to the explicit expression (E.10) for the $y_{k}^{ \pm}$, (E.16) exactly gives the matrix formula with $\gamma, \bar{\gamma}$ given in (C.12).
When $1 \leqslant k \leqslant N-1$, it follows from the structure (E.11) of the $y_{k}$ that formula (E.12) can be decomposed as

$$
\begin{equation*}
\mathrm{G}^{-1} f_{k}^{ \pm} \mathrm{G}=\mp(-1)^{k} t_{k} f_{k+1}^{ \pm}+\text {lower } \tag{E.17}
\end{equation*}
$$

By induction, the quasi-translated fermions $\Phi_{i}^{ \pm}$defined in (C.10) are therefore triangular in the $f_{k}^{ \pm}$:

$$
\begin{equation*}
\Phi_{i}=s_{1 i} t_{1 i} f_{i}+\text { lower }, \quad \Phi_{i}^{+}=(-1)^{i-1} s_{1 i} t_{1, i} f_{i}^{+}+\text {lower }, \quad 1 \leqslant i \leqslant N \tag{E.18}
\end{equation*}
$$

where $t_{l, k}$ was defined in (B.17), $s_{l k}$ in (C.9). In particular, we get

$$
\begin{equation*}
\Phi_{N}^{ \pm}=s_{1, N} t_{1, N} f_{N}^{+}+\text {lower } \tag{E.19}
\end{equation*}
$$

It turns out that we have the identity $s_{1, N} t_{1, N}=N$, that is trigonometric in nature, and can be derived from the upcoming result (E.56) by noticing that

$$
\begin{equation*}
t_{1, N}=(-1)^{(N-1) / 2} \lim _{z \rightarrow 1} \frac{f\left(z^{N}\right)}{f(z)} \tag{E.20}
\end{equation*}
$$

As such, we are able to write the Fourier-transformed fermions as

$$
\begin{equation*}
\tilde{\Phi}_{n}^{ \pm}=f_{N}^{ \pm}+\text {lower }, \quad 1 \leqslant n \leqslant N \tag{E.21}
\end{equation*}
$$

In other words, we know explicitly one of their coefficients, and we will see in the next part how to obtain all the others by induction.
c. Recursive equation for the Fourier-transformed fermions. Recall the definition of $g_{N}, g_{N}^{+}$from (E.1). We will establish a necessary condition for a one-particle fermionic operator $\psi \equiv \sum_{i=1}^{N} b_{i} g_{i}^{+}$to be eigen for the conjugation $\mathrm{G} \psi \mathrm{G}^{-1}$ with eigenvalue $\rho$, i.e.

$$
\begin{equation*}
\mathrm{G} \psi \mathrm{G}^{-1}=\rho \psi \tag{E.22}
\end{equation*}
$$

To be able to use more directly what we established in the previous part, it is convenient to rewrite this equation as $\mathrm{G}^{-1} \psi \mathrm{G}=\rho^{-1} \psi$. Using formula (E.15) on the left-hand side and the recursion relation (E.9) on the right-hand side, we then get the following equation in the basis of the $y_{k}^{+}$:

$$
\begin{equation*}
\sum_{k=1}^{N} b_{k}\left(y_{k}^{+}+(-1)^{k} t_{k+1} y_{k+1}^{+}\right)=\rho^{-1} \sum_{k=1}^{N} b_{k}\left(y_{k}^{+}+(-1)^{k} t_{k-1} y_{k-1}^{+}\right) \tag{E.23}
\end{equation*}
$$

Recall that the $y_{k}^{+}$were extended by periodicity. It is natural to do the same for the $b_{k}$ so that we can identify coefficients in the last equation for all $k$ to obtain

$$
\begin{equation*}
(1-\rho) b_{k}=(-1)^{k} t_{k}\left(b_{k+1}-\rho b_{k-1}\right), \quad b_{k+N} \equiv b_{k}, \quad k \in \mathbb{Z} \tag{E.24}
\end{equation*}
$$

Now, two cases arise:

1. When $\rho=1$, then $b_{i}$ is a constant sequence.
2. Otherwise, since $t_{N}=0$

$$
\begin{equation*}
b_{N}=0 \tag{E.25}
\end{equation*}
$$

Note that in both cases, there is only one solution to the (E.24) up to a multiplicative constant.
On another note, it turns out that this equation features a certain form of PT-symmetry. Namely if $\psi$ satisfies (E.24) for eigenvalue $\rho$, then so do the P -transformed $\mathrm{P}(\psi)=\sum_{k=1}^{N} b_{N-k} g_{k}^{+}$and the T - $\operatorname{transformed} \mathrm{T}(\psi)=\sum_{k=1}^{N} b_{k}^{*} g_{k}^{+}$, where we recall that ${ }^{* *}$, denotes complex conjugation, but for eigenvalue $\rho^{-1}$.

However, the case charge conjugation C is slightly different. First, the same calculations as before yield that if an annihilation operator $\chi=\sum \beta_{j} g_{j}$ satisfies

$$
\begin{equation*}
\mathrm{G} \chi \mathrm{G}^{-1}=\rho \chi \tag{E.26}
\end{equation*}
$$

then the following equation holds:

$$
\begin{equation*}
(\rho-1) \beta_{k}=(-1)^{k} t_{k}\left(\beta_{k+1}-\rho \beta_{k-1}\right) . \tag{E.27}
\end{equation*}
$$

When $\rho \neq 1$, it now appears that if $\psi$ satisfies (E.24) the quantity of interest will not be $\mathrm{C}(\psi)=\sum_{k=1}^{N} b_{k} g_{k}$ but rather $\chi \equiv \sum_{k=1}^{N}(-1)^{k} \beta_{k} g_{k}$. Indeed, $\beta_{0}=\beta_{N}=0$ is then well-defined despite the discrepant $\operatorname{sign}(-1)^{0} \neq(-1)^{N}$, and the $\beta_{k}$ do satisfy (E.27) with eigenvalue $\rho$.

Let us now apply these results to the Fourier modes $\tilde{\Phi}_{n}^{+}$(resp. $\tilde{\Phi}_{n}$ ). In the basis of the $g_{j}^{+}$(resp. $g_{j}$ ) a priori we have

$$
\begin{equation*}
\tilde{\Phi}_{n}=\sum_{j=1}^{N} M_{n j} g_{j}, \quad \tilde{\Phi}_{n}^{+}=\sum_{j=1}^{N} \bar{M}_{n j} g_{j}, \quad 0 \leqslant n<N \tag{E.28}
\end{equation*}
$$

We will now show that the row with $n=0$ is very simple, and the column with $j=N$ has only zero entries which is the reason why for $n>0$ we can restrict ourselves to the nontrivial $(N-1) \times(N-1)$ submatrix as in the main text and $\S \mathrm{C}$. By construction, the Fourier modes are eigenstates for conjugation by G with eigenvalue $\omega^{n}$ (resp. $\omega^{-n}$ ). Hence the previously established (E.24) and (E.27) hold true:

$$
\begin{align*}
\left(1-\omega^{n}\right) \bar{M}_{n j} & =(-1)^{j} t_{j}\left(\omega^{n} \bar{M}_{n, j-1}-M_{n, j+1}\right)  \tag{E.29}\\
\left(\omega^{-n}-1\right) M_{n j} & =(-1)^{j} t_{j}\left(\omega^{-n} M_{n, j-1}-M_{n, j+1}\right) \tag{E.30}
\end{align*}
$$

Note that these equations are also valid for $n=0$ and for $j=N$, and are $N$-periodic in both indices. On the other hand, we have established in (E.21) that

$$
\begin{equation*}
\bar{M}_{n, N-1}+\bar{M}_{n, N}=M_{n, N-1}+M_{n, N}=1 \tag{E.31}
\end{equation*}
$$

By the previous discussion, we readily get

$$
\begin{equation*}
\bar{M}_{0 j}=M_{0 j}=\frac{1}{2}, \quad 1 \leqslant j \leqslant N \tag{E.32}
\end{equation*}
$$

which yields the expression for $\tilde{\Phi}_{0}$ and $\tilde{\Phi}_{0}^{+}$in terms of the $f$ s as announced in (C.23). For the case $n \neq 0$, we have at the moment established that these Fourier-transformed fermions $\tilde{\Phi}_{n}\left(\right.$ resp. $\left.\tilde{\Phi}_{n}^{+}\right)$are characterised by relations (E.30) (resp. (E.29)) and by

$$
\begin{equation*}
\bar{M}_{n N}=M_{n N}=0, \quad \bar{M}_{n, N-1}=M_{n, N-1}=1, \quad 1 \leqslant n \leqslant N-1 \tag{E.33}
\end{equation*}
$$

The vanishing of the coefficients with $j=N$ restricts the sum in (E.28) to the $(N-1) \times(N-1)$ submatrix yielding (28). In summary we thus have obtained the simple recursive description

$$
\begin{equation*}
\left(1-\omega^{n}\right) \bar{M}_{n j}=(-1)^{j} t_{j}\left(\omega^{n} \bar{M}_{n, j-1}-\bar{M}_{n, j+1}\right), \quad \bar{M}_{n N}=0, \quad \bar{M}_{n, N-1}=1 \tag{E.34}
\end{equation*}
$$

and similar for $M_{n j}$.
We are now in a position to establish a few of their symmetry properties:

1. We have seen that $\mathrm{T}\left(\tilde{\Phi}_{n}^{+}\right)$satisfies the same recursion relation (E.24) as $\tilde{\Phi}_{N-n}^{+}$. As such, both quantities must be identical up to a multiplicative constant, which is given by (E.33) to be 1. The story is identical for annihilation operators, hence

$$
\begin{equation*}
\mathrm{T}\left(\tilde{\Phi}_{n}\right)=\tilde{\Phi}_{N-n}, \quad \mathrm{~T}\left(\tilde{\Phi}_{n}^{+}\right)=\tilde{\Phi}_{N-n}^{+} \tag{E.35}
\end{equation*}
$$

2. Furthermore, it follows from what we have seen that $\tilde{\Phi}_{n}$ satisfies the same relation as $\sum_{j=1}^{N}(-1)^{j} \bar{M}_{n j}^{*} g_{j}$, and the same argument leads to

$$
\begin{equation*}
M_{n j}=(-1)^{j} \bar{M}_{n j}^{*} \tag{E.36}
\end{equation*}
$$

3. Lastly, $\mathrm{P} \mathrm{T}\left(\tilde{\Phi}_{n}^{+}\right)$must be proportional to $\tilde{\Phi}_{n}^{+}$and since the $\mathbb{C}$-antilinear transformation P T satisfies $(\mathrm{P} \mathrm{T})^{2}=1$, we can conclude that $\mathrm{P} \mathrm{T}\left(\tilde{\Phi}_{n}^{+}\right)$coincides with $\tilde{\Phi}_{n}^{+}$up to a phase. Since $\mathrm{P} T=\mathrm{TP}$, we get from (E.35) that $\mathrm{P}\left(\tilde{\Phi}_{n}^{+}\right)$itself coincides with $\tilde{\Phi}_{N-n}^{+}$up to a phase. Note that, once again, the same conclusion holds in the case of annihilation operators. This phase is then easily determined since we know that $\bar{M}_{n, N-1}=M_{n, N-1}=1$, and we obtain

$$
\begin{equation*}
\mathrm{P}\left(\tilde{\Phi}_{n}\right)=M_{n 1} \tilde{\Phi}_{N-n}, \quad \mathrm{P}\left(\tilde{\Phi}_{n}^{+}\right)=\bar{M}_{n 1} \tilde{\Phi}_{N-n}^{+} \tag{E.37}
\end{equation*}
$$

with both coefficients $\bar{M}_{n 1}$ and $M_{n 1}$ unimodular. Note that any phase is possible because of the antilinearity.
d. Anticommutation relations of Fourier-transformed fermions. Although we have not yet been able to find a complete proof of anticommutation relations (C.27), they are formally established up to a sign as follows, independently of any numerical conjecture.

Firstly, the anticommutation relations (C.2) that hold for the $f_{k}^{ \pm}$guarantee that for any $n, m$,

$$
\begin{gather*}
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{m}\right\}=\left\{\tilde{\Phi}_{n}^{+}, \tilde{\Phi}_{m}^{+}\right\}=0 \\
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{m}^{+}\right\} \in \mathbb{C} \tag{E.38}
\end{gather*}
$$

where by $\mathbb{C}$ we mean the space of multiples of the identity. Moreover, as the $\tilde{\Phi}_{n}$ have distinct eigenvalues, we easily get

$$
\begin{equation*}
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{m}^{+}\right\}=0, \quad n \neq m \tag{E.39}
\end{equation*}
$$

It is then a matter of determining the scalars $\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}$for all $n$. In the case when $n=0$, we have the explicit formula (C.23) at hand, and we immediately obtain

$$
\begin{equation*}
\left\{\tilde{\Phi}_{0}, \tilde{\Phi}_{0}^{+}\right\}=-1 \tag{E.40}
\end{equation*}
$$

In the other case, it follows from (E.39) and the inverse Fourier transform

$$
\begin{equation*}
f_{1}^{ \pm}=\sum_{n=1}^{N} \omega^{\mp n} \tilde{\Phi}_{n} \tag{E.41}
\end{equation*}
$$

that $\left\{f_{1}, \tilde{\Phi}_{\underline{n}}^{+}\right\}=\omega^{n}\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{\underline{n}}^{+}\right\}$. Due to the anticommutation relations (C.2) of the $f_{k}^{ \pm}$, it is clear that $\left\{f_{1}, \tilde{\Phi}_{n}^{+}\right\}=$ $-\left(\bar{M}_{n N}+\bar{M}_{n 1}\right)$ and since $\bar{M}_{n N}=0$, it follows that

$$
\begin{equation*}
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}=-\omega^{-n} \bar{M}_{n 1} \tag{E.42}
\end{equation*}
$$

To say a bit more about these scalars, let us turn to what we know about the symmetries of Fourier-translated fermions. First, our discussion on the P-transformation has established that $\bar{M}_{n 1}$ and $M_{n 1}$ are unimodular. Moreover, note that we can use (E.41) dually as $\left\{f_{1}^{+}, \tilde{\Phi}_{n}\right\}=\omega^{-n}\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}$so that we alternatively get

$$
\begin{equation*}
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}=-\omega^{n} M_{n 1} \tag{E.43}
\end{equation*}
$$

Comparing this expression to (E.42) using relation (E.36), it follows that the anticommutator is equal to minus its complex conjugate, so is a purely imaginary complex number. Since following (E.37) we saw it is moreover unimodular, we have

$$
\begin{equation*}
\left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}= \pm \mathrm{i} \tag{E.44}
\end{equation*}
$$

We have only been able to establish the actual sign, as given in (C.27), using numerics. Furthermore, the T-symmetry (E.35) incarnates as

$$
\begin{align*}
& \left\{\tilde{\Phi}_{n}, \tilde{\Phi}_{n}^{+}\right\}=-\left\{\tilde{\Phi}_{N-n}, \tilde{\Phi}_{N-n}^{+}\right\} \text {. }  \tag{E.45}\\
& \text { 2. Proof that } H^{\text {full }} \text { vanishes at } \mathrm{q}=\mathrm{i}
\end{align*}
$$

a. Main discussion. The purpose of this part is to show that for any $1 \leqslant i \leqslant j \leqslant N-1$, the coefficients $h_{i j}^{\mathrm{L}, \mathrm{R}}$ defined in (B.20) satisfy (B.24), i.e.

$$
\begin{equation*}
h_{i j}^{\mathrm{L}}+h_{i j}^{\mathrm{R}}=0 \tag{E.46}
\end{equation*}
$$

This will imply that $H^{L}=-H^{R}$ and thus $H^{\text {full }}=0$ as in (B.10).
Introduce the notation

$$
\begin{equation*}
\eta_{i j} \equiv \sum_{k=1}^{N} \varsigma_{i, j, k}^{\mathrm{L}} \tag{E.47}
\end{equation*}
$$

where the summands are the coefficients of the nested TL commutators of $\mathrm{S}_{[i, i+k]}^{\mathrm{L}, \mathrm{R}}$, see (B.18). Note that the definition of $\eta_{i j}$ makes sense for any nonnegative $i, j$. Then it will be shown in $\S \mathrm{E} 2 \mathrm{~b}(\mathrm{p} .26)$ that the sums in the expression (B.22) for $h_{i j}^{\mathrm{L}}$ can be 'closed' in terms of $\eta$ as follows:

$$
\begin{equation*}
h_{i j}^{\mathrm{L}}+h_{i j}^{\mathrm{R}}=\eta_{0, j-i}-\eta_{i j}-(-1)^{j-i} \eta_{N-j, N-i} . \tag{E.48}
\end{equation*}
$$

To prove that this quantity vanishes, we will use an analytic argument based on an extension of our notations to the following meromorphic functions for $k \leqslant l$ and arbitrary nonnegative $i, j$ :

$$
\begin{align*}
t_{k, l}(z) & \equiv \prod_{m=k}^{l-1} f\left(z \omega^{m}\right) \\
\varsigma_{i j}(z) & \equiv(-1)^{l} t_{-i, 0}(z) t_{-j, 0}(z)  \tag{E.49}\\
\eta_{i j}(z) & \equiv \sum_{k=1}^{N} \varsigma_{i j}\left(z \omega^{k}\right)
\end{align*}
$$

It is clear from definition (B.4) that $t_{k, l}\left(\omega^{r}\right)=t_{k+r, l+r}$ for any integer $r$, hence $\varsigma_{i j}\left(\omega^{r}\right)=\varsigma_{i, j, r}^{\mathrm{L}}$ by definition (B.18) and $\eta_{i j}(1)=\eta_{i j}$.

The following two major results proved in §E2c and §E2d respectively will now allow us to compute $\eta_{i j}$ as the limit of the meromorphic function $\eta_{i j}(z)$ at $z \rightarrow 1$ :

- As rational fractions, we have the following analogue of L-R symmetry:

$$
\begin{equation*}
\eta_{N-i, N-j}(z)=(-1)^{j-i} f\left(z^{N}\right)^{2} \eta_{i j}\left(-z^{-1}\right) \tag{E.50}
\end{equation*}
$$

- There exists constants $C_{i j}$ in terms of which the following equality holds true for $z \in \mathbb{C}$ generic:

$$
\begin{equation*}
\eta_{i j}(z)=\frac{N}{\mathrm{i}^{j-i}}\left(1-\frac{1}{1+z^{N}}\left(1-(-1)^{j-i}\right)\right)+\frac{z^{N-1}}{\left(1+z^{N}\right)^{2}} C_{i j} \tag{E.51}
\end{equation*}
$$

Moreover, $C_{i j}=0$ when $i=j$.
Before proving these results, let us use them to derive an expression for $h_{i j}^{\mathrm{L}}+h_{i j}^{\mathrm{R}}$ that will let us establish (B.24). To begin with, it directly follows from (E.51) that

$$
\begin{equation*}
\eta_{i j}(1)-\eta_{0, j-i}(1)=\frac{C_{i j}}{4} . \tag{E.52}
\end{equation*}
$$

Besides, straightforward asymptotic expansions of (E.51) around 1 lead to

$$
\begin{equation*}
\eta_{i j}\left(-z^{-1}\right)=\frac{C_{i j}}{N^{2}(z-1)^{2}}+\mathcal{O}\left(\frac{1}{z-1}\right) \tag{E.53}
\end{equation*}
$$

Now notice that $\frac{f(z)}{z-1} \rightarrow_{z \rightarrow 1}-\mathrm{i} N / 2$ so by left/right symmetry (E.50), we get

$$
\begin{equation*}
\eta_{N-j, N-i}(1)=(-1)^{j-i+1} \frac{C_{i j}}{4} \tag{E.54}
\end{equation*}
$$

This leads us to (B.24) as expected, since by (E.48) we can write $h_{i j}^{\mathrm{L}}+h_{i, j}^{\mathrm{R}}=\eta_{0, j-i}-\eta_{i j}-(-1)^{j-i} \eta_{N-j, N-i}=0$.
b. Proof of the closure of the summation loop (E.48). We have seen in (B.22) that $h_{i j}^{\mathrm{L}}$ can be written as $A_{i j}-B_{i j}$ with $A_{i j}=\sum_{m=j}^{N-1} \varsigma_{0, j-i, m-i}^{\mathrm{L}}$ and $B_{i j}=\sum_{m=j}^{N-1} \varsigma_{i, j, m}^{\mathrm{L}}$. Moreover, by L-R symmetry (B.21), we see that

$$
\begin{equation*}
h_{i j}^{\mathrm{L}}+h_{i j}^{\mathrm{R}}=\left(A_{i j}+(-1)^{j-i} A_{N-j, N-i}\right)-\left(B_{i j}+(-1)^{j-i} B_{N-j, N-i}\right) . \tag{E.55}
\end{equation*}
$$

As soon as $0 \leqslant m \leqslant j-1$, the value 0 is within the bounds of the products in the definition (B.18) of $\varsigma_{i j m}^{\mathrm{L}}$. Since $t_{0}=0$, it follows that $\varsigma_{i, j, m}^{\mathrm{L}}=0$ for these values of $m$, and the sum in $B_{i j}$ can be formally completed to obtain $B_{i j}=\eta_{i j}$ and $B_{N-j, N-i}=\eta_{N-j, N-i}$.

Moreover, recall that by definition (B.18), $\varsigma_{0, j-i, m-i}^{\mathrm{L}}=t_{m-j+1, m-i+1}$, which implies that $A_{i j}$ can be reindexed as $A_{i j}=\sum_{m=1}^{N-j} t_{m, m+j-i}$. At the same time, we get $(-1)^{j-i} A_{N-j, N-i}=(-1)^{j-i} \sum_{m=1}^{i} t_{m, m+j-i}$ and, using the property $t_{k}=-t_{N-k}$, successive changes of indices lead to $(-1)^{j-i} A_{N-j, N-i}=\sum_{m=N-j+1}^{N-(j-i)} t_{m, m+j-i}$. Adding up the expressions, we have established that $A_{i j}+(-1)^{j-i} A_{N-j, N-i}=\sum_{m=1}^{N-(j-i)} t_{m, m+j-i}$. It is moreover easily seen that for the remaining values $N-(j-i)+1 \leqslant m \leqslant N$ of the summation index, the summand $t_{m, m+j-i}$ vanishes since the product (B.17) defining it contains $t_{N}=0$. We then obtain $A_{i j}+(-1)^{j-i} A_{N-j, N-i}=\sum_{m=1}^{N} t_{m, m+j-i}$, which is precisely $\eta_{0, j-i}$. Finally, we get the equation (E.48).
c. Proof of L-R symmetry (E.50). Recall that we take $N$ to be odd. By definitions (B.3) and (E.49), $t_{1, N+1}(z)=$ $\prod_{k=1}^{N}\left(-\mathrm{i} \frac{z \omega^{k}-1}{z \omega^{k}+1}\right)$, which can be reindexed by $N$-periodicity as $\prod_{k=1}^{N}\left(-\mathrm{i} \frac{z-\omega^{k}}{z+\omega^{k}}\right)$. Recognising $\prod_{k=1}^{N}\left(z-\omega^{k}\right)$ as the well-known factorisation of the polynomial $z^{N}-1$ and taking note that $(-z)^{N}=-z^{N}$ since $N$ is odd, we get $t_{1, N+1}(z)=(-\mathrm{i})^{N} \prod_{k=1}^{N} \frac{z-\omega^{k}}{z+\omega^{k}}=(-\mathrm{i})^{N} \frac{z^{N}-1}{z^{N}+1}$, which leads us by definition (B.3) to

$$
\begin{equation*}
t_{1, N+1}(z)=(-1)^{(N-1) / 2} f\left(z^{N}\right) \tag{E.56}
\end{equation*}
$$

Let us now fix $j \geqslant 0$. A straightforward calculation shows that $f\left(-z^{-1}\right)=f(z)^{-1}$ and a suitable reindexing of the product (E.49) leads to $t_{-j, 0}\left(-z^{-1}\right)=t_{1, j+1}^{-1}(z)$. It follows that $t_{j+1, N+1}(z)=t_{1, N+1}(z) t_{-j, 0}\left(-z^{-1}\right)$, and the previous property (E.56) leads to

$$
\begin{equation*}
t_{j, N}(z \omega)=(-1)^{(N-1) / 2} f\left(z^{N}\right) t_{-j, 0}\left(-z^{-1}\right) \tag{E.57}
\end{equation*}
$$

after shifting indices in (E.49).
We are now in a position to establish (E.50). Fix $i$ and $j$. The last result (E.57) precisely gives $t_{i, N}(z \omega) t_{j, N}(z \omega)=$ $f\left(z^{N}\right)^{2} t_{-i, 0}\left(-z^{-1}\right) t_{-j, 0}\left(-z^{-1}\right)$. By definition (E.49), this can be rewritten as

$$
\begin{equation*}
(-1)^{N-j} \varsigma_{N-j, N-i}(z \omega)=f\left(z^{N}\right)^{2}(-1)^{i} \varsigma_{i j}\left(-z^{-1}\right) . \tag{E.58}
\end{equation*}
$$

Replacing $z$ by $z \omega^{k}$ and summing over all $k$, we obtain

$$
\begin{equation*}
\sum_{k=1}^{N} \varsigma_{N-j, N-i}\left(z \omega^{k+1}\right)=-(-1)^{j-i} \sum_{k=1}^{N} f\left(\left(z \omega^{k}\right)^{N}\right)^{2} \varsigma_{i j}\left(-z^{-1} \omega^{-k}\right) \tag{E.59}
\end{equation*}
$$

Finally, $f\left(\left(z \omega^{k}\right)^{N}\right)=f\left(z^{N}\right)$ can be factored out and sums can be reindexed to obtain

$$
\eta_{N-i, N-j}(z)=(-1)^{j-i} f\left(z^{N}\right)^{2} \eta_{i j}\left(-z^{-1}\right)
$$

which is (E.50), as we wished to show.
d. Proof of $\eta_{i j}(z)$ from expression (E.51). In what follows, we will fix indices $1 \leqslant i \leqslant j \leqslant N-1$ and establish that when $\alpha$ is taken in what we will call $A$, we have

$$
\eta_{i j}(\alpha)=\frac{N}{\mathrm{i}^{j-i}}\left(1-\frac{1}{1+\alpha^{N}}\left(1-(-1)^{j-i}\right)\right)+\frac{\alpha^{N-1}}{\left(1+\alpha^{N}\right)^{2}} C_{i j}
$$

where $C_{i j}$ will be properly defined below in such a way that $C_{i j}=0$ when $i=j$. This equality between rational fractions was precisely (E.51), and since it will hold for an infinite number of $\alpha$, namely on $A$, this will be enough to conclude that it is also true generically on $\mathbb{C}$ as a whole. The proof consists of two steps.

Step 1: $\eta_{i j}(\alpha)$ in terms of residues. First, let us list out the singularities of the function $\varsigma_{i j}(z)$. Recall that by (B.3) $f(z)$ has exactly one pole, namely -1 , which is simple, and has exactly one zero at $z=1$. By (E.49), for any $k \leqslant l$, all the poles of $t_{k, l}(z)$ are simple and exactly given by $z=-\omega^{-p}$ for $k \leqslant p \leqslant l-1$. It follows from (E.49) that the zeroes (resp. poles) of $\varsigma_{i j}(z)$ lie in the set $\mathcal{Z}$ consisting of $\omega^{k}$ for $1 \leqslant k \leqslant N$ (resp. in the set $\mathcal{P} \equiv-\mathcal{Z}$ consisting of the $\left.-\omega^{k}\right)$. Notice that $\mathcal{P}$ and $\mathcal{Z}$ are disjoint since $N$ is odd. More precisely, the poles of $\varsigma_{i j}(z)$ are exactly as follows:

- double poles $\mathcal{P}_{2}=\left\{-\omega^{k} \mid 1 \leqslant k \leqslant i\right\}$,
- simple poles $\mathcal{P}_{1}=\left\{-\omega^{k} \mid i+1 \leqslant k \leqslant j\right\}$.

For any $\alpha$, let us now define $\varphi_{i j ; \alpha}(z)$ as the meromorphic function

$$
\begin{equation*}
\varphi_{i j ; \alpha}(z) \equiv \varsigma_{i j}(\alpha z) \sum_{\omega^{k} \in \mathcal{Z}} \frac{1}{z-\omega^{k}} \tag{E.60}
\end{equation*}
$$

If $\alpha$ is taken in a certain neighbourhood $A$ of 1 inside the unit circle (excluding 1 itself), it is clear that $\alpha^{-1} \mathcal{P}, \alpha^{-1} \mathcal{Z}$ and $\mathcal{Z}$ are pairwise disjoint sets.

In what follows, we will always suppose that $\alpha \in A$. Under this assumption, $\varsigma_{i j}(\alpha z)$ is regular at $\omega^{i} \in \mathcal{Z}$, so that Cauchy's integral formula and definition (E.47) lead to

$$
\begin{equation*}
\eta_{i j}(\alpha)=\sum_{z \in \mathcal{Z}} \operatorname{Res}_{z}\left(\varphi_{i j ; \alpha}\right) \tag{E.61}
\end{equation*}
$$

To compute this quantity, note that the poles of $\varphi_{i j ; \alpha}$ can be exactly identified as follows:

- simple poles: $\alpha^{-1} \mathcal{P}_{1} \cap \mathcal{Z}$,
- double poles: $\alpha^{-1} \mathcal{P}_{2}$.

Now, recall that the residue at infinity $\operatorname{Res}_{\infty}(f)$ of any meromorphic function $f(z)$ is defined as $\operatorname{Res}_{0}\left(-\frac{1}{z^{2}} f\left(z^{-1}\right)\right)$. A version of the residue theorem then yields that the sum of all residues of $\varphi_{i j ; \alpha}$ vanishes, i.e.

$$
\begin{equation*}
\operatorname{Res}_{\infty}\left(\varphi_{i j ; \alpha}\right)+\sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \operatorname{Res}_{\alpha^{-1} p}\left(\varphi_{i j ; \alpha}\right)+\sum_{z \in \mathcal{Z}} \operatorname{Res}_{z}\left(\varphi_{i j ; \alpha}\right)=0 . \tag{E.62}
\end{equation*}
$$

We finally get the formula

$$
\begin{equation*}
\eta_{i j}(\alpha)=-\operatorname{Res}_{\infty}\left(\varphi_{i j ; \alpha}\right)-\sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \operatorname{Res}_{p}\left(\varphi_{i j ; \alpha}\right) \tag{E.63}
\end{equation*}
$$

In the following, we will establish that this identity can be rewritten exactly as (E.51), which was

$$
\eta_{i j}(\alpha)=\frac{N}{\mathrm{i}^{j-i}}\left(1-\frac{1}{1+\alpha^{N}}\left(1-(-1)^{j-i}\right)\right)+\frac{\alpha^{N-1}}{\left(1+\alpha^{N}\right)^{2}} C_{i j}
$$

where $C_{i j}$ will be properly defined below. More precisely, we will prove that this equality between rational fractions holds for an infinite number of points, namely on $A$, which is enough to conclude that it is also true generically on $\mathbb{C}$ as a whole.

Step 2: Computing residues. For the commodity of the reader, let us recall here that for any meromorphic function $f$ and point $a \in \mathbb{C}$, if $\lim _{z \rightarrow a}(z-a) f(z)$ is finite and nonzero, then it is equal to $\operatorname{Res}_{a}(f)$. This result is readily adapted to $a=\infty$ following the definition $\operatorname{Res}_{\infty}(f) \equiv \operatorname{Res}_{0}\left(-\frac{1}{z^{2}} f\left(z^{-1}\right)\right)$ : if $\lim _{z \rightarrow \infty}(-z f(z))$ is finite and nonzero, then it is equal to $\operatorname{Res}_{\infty}(f)$.

In addition, it will be worth noticing for what follows that $\sum_{j=1}^{N} \frac{1}{z-\omega^{j}}=\frac{N z^{N-1}}{z^{N}-1}$, which can be checked by comparing the logarithmic derivatives of both sides of the identity $\prod_{j=1}^{N}\left(z-\omega^{j}\right)=z^{N}-1$. With this result, it becomes easy to obtain $\operatorname{Res}_{\infty}\left(\varphi_{i j ; \alpha}\right)$ since on one hand, $z \frac{N z^{N-1}}{z^{N}-1} \rightarrow N$ as $z \rightarrow \infty$ and on the other, (B.3) and (E.49) respectively lead to $f(z) \rightarrow-\mathrm{i}$ and $\varsigma_{i j}(z) \rightarrow(-1)^{i}(-\mathrm{i})^{i+j}=(-\mathrm{i})^{j-i}$. Explicitly,

$$
\begin{equation*}
\operatorname{Res}_{\infty}\left(\varphi_{i j ; \alpha}\right)=-\frac{N}{\mathrm{i}^{j-i}} \tag{E.64}
\end{equation*}
$$

As for the other terms of the sum, let us show that

$$
\begin{equation*}
\sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \operatorname{Res}_{p}\left(\varphi_{i j ; \alpha}\right)=\frac{1}{\mathrm{i}^{j-i}} \frac{N}{1+\alpha^{N}}\left(1-(-1)^{j-i}\right)-\frac{\alpha^{N-1}}{\left(1+\alpha^{N}\right)^{2}} C_{i j} \tag{E.65}
\end{equation*}
$$

Provided the constants $C_{i j}$ are properly defined and respect $C_{i i}=0$, this will conclude the proof of (E.51).
Surprisingly enough, the key to get to the result is to write $\varphi_{i j ; \alpha}(z)=\varsigma_{i j}(\alpha z) \sum_{\omega^{k} \in \mathcal{Z}} \frac{1}{z-\omega^{k}}$ as the product

$$
\begin{equation*}
\varphi_{i j ; \alpha}(z)=\alpha \sigma_{i j}(\alpha z) \kappa(z) \tag{E.66}
\end{equation*}
$$

where $\sigma_{i j}(z) \equiv \varsigma_{i j}(z) / z$ and $\kappa(z) \equiv z \sum_{j=1}^{N} \frac{1}{z-\omega^{j}}=\frac{N}{1-z^{-N}}$. In this setting, since $\alpha \in A, \sigma_{i j}(\alpha z)$ exhibits a simple (resp. double) pole at any $\alpha^{-1} p$ for $p \in \mathcal{P}_{1}$ (resp. $\mathcal{P}_{2}$ ) while $\kappa$ stays regular at these values. By usual formulas in the case of lower-order poles, we can calculate residues of $\varphi_{i j ; \alpha}$ as the sum of two terms

$$
\begin{equation*}
\sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \operatorname{Res}_{\alpha^{-1} p}\left(\varphi_{i j ; \alpha}\right)=R_{1}+R_{2} \tag{E.67}
\end{equation*}
$$

where

$$
\begin{align*}
& R_{1} \equiv \sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \kappa\left(\alpha^{-1} p\right) \alpha \operatorname{Res}_{\alpha^{-1} p}\left(\sigma_{i j}(\alpha z)\right)  \tag{E.68}\\
& \left.R_{2} \equiv \sum_{p \in \mathcal{P}_{2}} \kappa^{\prime}\left(\alpha^{-1} p\right) \alpha \frac{d}{d z}\left(z-\alpha^{-1} p\right)^{2} \sigma_{i j}(\alpha z)\right|_{z=\alpha^{-1} p} \tag{E.69}
\end{align*}
$$

Both terms can be substantially simplified and will lead to (E.65).
Let us start with $R_{1}$. It is straightforward to check that $\kappa$ takes the same value at each $\alpha^{-1} p=-\alpha^{-1} \omega^{k} \in \mathcal{P}_{1} \cup \mathcal{P}_{2}$, namely $\kappa(p)=\frac{N}{1+\alpha^{N}}$, which leads to the factorised expression

$$
\begin{equation*}
R_{1}=\frac{N}{1+\alpha^{N}} \sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \alpha \operatorname{Res}_{\alpha^{-1} p}\left(\sigma_{i j}(\alpha z)\right) \tag{E.70}
\end{equation*}
$$

Moreover, we have $\alpha \operatorname{Res}_{\alpha^{-1} p}\left(\sigma_{i j}(\alpha z)\right)=\operatorname{Res}_{p}\left(\sigma_{i j}(z)\right)$ by the usual change of variable formula. Dressing the list of its poles, an application of the residue theorem on the function $\sigma_{i j}$ readily yields

$$
\begin{equation*}
\sum_{p \in \mathcal{P}_{1} \cup \mathcal{P}_{2}} \operatorname{Res}_{p}\left(\sigma_{i j}\right)=-\operatorname{Res}_{\infty}\left(\sigma_{i j}\right)-\operatorname{Res}_{0}\left(\sigma_{i j}\right) \tag{E.71}
\end{equation*}
$$

The right-hand side of the last expression is then easily computed:

- $\operatorname{Res}_{\infty}\left(\sigma_{i j}\right)=\operatorname{Res}_{0}\left(\frac{-1}{z^{2}} \sigma_{i j}\left(z^{-1}\right)\right)$ by definition and $\operatorname{Res}_{\infty}\left(\sigma_{i j}\right)=-1 / \mathrm{i}^{j-i}$ since $\varsigma_{i j} \rightarrow(-\mathrm{i})^{j-i}$ as $z \rightarrow \infty$,
- $\operatorname{Res}_{0}\left(\sigma_{i j}\right)=\varsigma_{i j}(0)$ by Cauchy's integral formula, and we get $\varsigma_{i j}(0)=(-1)^{i} \mathrm{i}^{j+i}=\mathrm{i}^{j-i}$ by (E.49) using $f(0)=\mathrm{i}$. Finally

$$
\begin{equation*}
R_{1}=\frac{1}{\mathrm{i}^{j-i}} \frac{N}{1+\alpha^{N}}\left(1-(-1)^{j-i}\right) . \tag{E.72}
\end{equation*}
$$

It only remains to compute $R_{2}$. First, working out $\kappa^{\prime}(z)=-N^{2} \frac{z^{1-N}}{\left(1-z^{-N}\right)^{2}}$ explicitly at any $\alpha^{-1} p$ for $p \in \mathcal{P}_{2}$ leads to $\kappa^{\prime}\left(\alpha^{-1} p\right)=N^{2} \frac{\alpha^{N-1}}{\left(1+\alpha^{N}\right)^{2}} p$ since $p^{N}=-1$. Based on these calculations, (E.69) can be rewritten as

$$
\begin{equation*}
R_{2}=-\frac{\alpha^{N-1}}{\left(1+\alpha^{N}\right)^{2}} C_{i j},\left.\quad C_{i j} \equiv \sum_{p \in \mathcal{P}_{2}} N^{2} p \alpha \frac{d}{d z}\left(z-\alpha^{-1} p\right)^{2} \sigma_{i j}(\alpha z)\right|_{z=\alpha^{-1} p} \tag{E.73}
\end{equation*}
$$

Moreover, a change of variables in the derivative yields

$$
\begin{equation*}
\left.\alpha \frac{d}{d z}\left(z-\alpha^{-1} p\right)^{2} \sigma_{i j}(\alpha z)\right|_{z=\alpha^{-1} p}=\left.\frac{d}{d z}(z-p)^{2} \sigma_{i j}(z)\right|_{z=p}, \tag{E.74}
\end{equation*}
$$

revealing that $C_{i j}$ is independent of $\alpha$. Finally, $C_{i j}$ clearly vanishes when $i=j$ since $\mathcal{P}_{2}$ is then empty, which concludes the proof of (E.65) and of (E.51).


[^0]:    * To be precise, any equality established in the representation carries over to the TL algebra as long as it is algebraic, i.e. involving addition and multiplication only. This is relevant for us because
    of the limit (B.11), which does not make sense in the TL algebra properly speaking, but will still give results that are valid in the TL algebra.

