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Status and progress of lattice QCD

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Abstract

We review recent progress from lattice QCD for the determination of the Cabibbo-Kobayashi-Maskawa matrix elements.

1 Introduction

Lattice QCD provides first-principles and non-perturbative estimate of hadronic matrix elements needed to determine Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. Independent realistic simulations have been available for the so-called "gold-plated" processes, where the initial and final states involve at most one hadron stable in QCD without bottom quarks [1].

Simulations of bottom quarks need fine and large lattices, and hence are very time-consuming, because the lattice spacing a and spatial lattice size L must satisfy a condition $a \ll m_b^{-1} \ll M_\pi^{-1} \ll L$ in order to control discretization and finite volume effects (FVEs). Currently available simulations, therefore, employ unphysically small masses or effective-theory-based actions to be matched with QCD for bottom quarks. Another and more essential problem for non-gold-plated processes is that on-shell matrix elements of interest are not straightforward to be extracted from correlation functions on finite-volume Euclidean lattices.

In this article, we introduce selected examples of recent progress with precise realistic simulations for gold-plated processes and that with newly developed methods for non-gold-plated processes. We refer to Refs. [2, 3, 4, 5] for more comprehensive reviews.

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2 Cabibbo angle anomaly

Kaon leptonic and semileptonic decays are gold-plated, and the condition for a and L is relaxed as $a \ll m_s^{-1} < M_\pi^{-1} \ll L$. As shown in the left panel of Fig. 1, there have been independent and realistic simulations in this decade leading to the sub-% accuracy for hadronic matrix elements to determine $|V_{us}|/|V_{ud}|$ and $|V_{us}|$, namely, ratio of decay constants f_K/f_π and $K \to \pi$ form factor at zero momentum transfer $f_+(0)$. Together with a recent dispersive estimate of universal radiative correction $1 + \Delta_R$ to the superallowed nuclear β decays [6, 7], a tension in CKM unitarity in the first row called "Cabibbo angle anomaly" has been reported, while it is alleviated by recent re-analysis of nuclear dependent corrections [8].

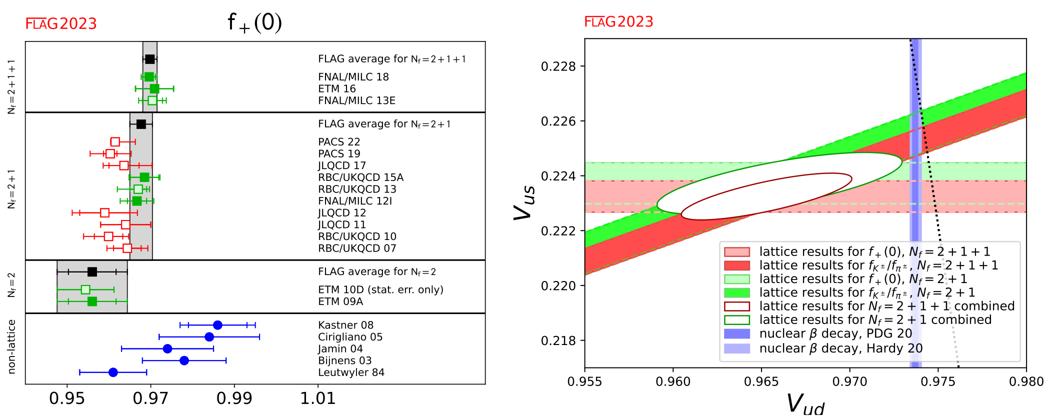


Figure 1: Left: lattice estimates of $K \to \pi$ form factor $f_+(0)$. These shown in green (red) squares (do not) satisfy FLAG's criteria to control systematics. The black square and band for each N_f show the average of the green squares. Blue symbols are phenomenological estimates. Right: $|V_{us}|$ versus $|V_{ud}|$. The slanted and horizontal bands show $|V_{us}|/|V_{ud}|$ from kaon/pion leptonic decays and $|V_{us}|$ from $K \to \pi \ell \nu$ semileptonic decays, respectively. Red and green bands are estimates in $N_f = 2+1+1$ and $2+1$ QCD, respectively. The blue bands show $|V_{ud}|$ from the nuclear β decays. The black dotted line satisfies CKM unitarity. (Both panels from Ref. [1].)

The right panel of Fig. 1 shows recent estimate of $|V_{us}|/|V_{ud}|$ and $|V_{us}|$ by Flavour Lattice Averaging Group (FLAG) [1] including ETM collaboration's result for f_K/f_π [9] after the last CKM workshop. A measure of unitarity violation $\Delta_{\rm CKM} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.0011(7)$ is consistent with zero, when it is estimated from the kaon and pion leptonic and nuclear β decays. On the other hand, $\gtrsim 3\,\sigma$ tension is observed with $|V_{us}|$ from the kaon semileptonic decays: namely $\Delta_{\rm CKM} = -0.0021(4)$ together with nuclear β decay and $-0.0184(64)$ with the leptonic decays. Since the average of $f_+(0)$ for $N_f = 4$ is largely constrained by a single precise study by Fermilab/MILC [10], independent calculations are welcome to firmly establish the lattice prediction of $f_+(0)$. Note also that, for $N_f = 3$, the PACS collaboration reported their preliminary result in Ref. [11].

At the sub-% accuracy of the hadronic inputs, the uncertainty of the conventional calculation of radiative corrections in chiral perturbation theory (ChPT) is no longer negligible. The RM123-Soton collaboration developed a method to calculate the radiative correction by decomposing the photon-inclusive decay rate into its perturbative estimate for the point-like meson and structure dependent correction on the lattice [12]. Their latest result including the strong isospin correction $\delta_{\mathrm{EM+SU(2)}} = -1.26(14)\,\%$ [13] is consitent with the conventional ChPT estimate $-1.12(21)\,\%$ [14]. A recent indpendent calculation using a different lattice formulation also obtained a consistent result $-0.86(^{+41}_{-40})\,\%$ [15]. We also note that a different approach using the so-called infinite volume recunstruction (IVR) method has been proposed, and its applicability to the semileptonic decay is discussed in Ref. [16]. The IVR technique has been also applied to the first lattice calculation of the γW box diagram responsible for the universal correction $1+\Delta_R$ to the nuclear β decay [17]. Their etimate $\Box_{\gamma W}^{VA,\leq 2\,\mathrm{GeV}^2} \times 10^3 = 1.49(8)$ is in good agreement with that from the dispersive analysis 1.62(10) mentioned at the beginning of this section, and is systematically improvable on finer lattices.

3 B meson exclusive semileptonic decays

The conventional decay modes to determine $|V_{cb}|$ and $|V_{ub}|$, namely $B \to D^{(*)}\ell\nu$ and $B \to \pi\ell\nu$, are gold-plated. There has been steady progress in the lattice calculation of relevant form factors to resolve the long standing tension in these CKM elements with estimates from inclusive decays. One of the most interesting progress is that three collaborations have calculated all $B \to D^*$ form factors, h_{A_1}, h_{A_2}, h_{A_3}, h_V [18, 19, 20, 21], while, until recently, only h_{A_1} had been calculated only at zero recoil [22, 23]. However, there is a tension among the new lattice calculations [2, 24] as shown in the left panel of Fig. 2, which shows differential decay rate without some kinematical factors $|\eta_{EW} V_{cb} \mathcal{F}(w)|^2$, where η_{EW} is the electroweak correction, $\mathcal{F}(w)$ is a functional of form factors given as

$$\mathcal{F}^2 \propto \left[2\frac{1-2wr+r^2}{(1-r)^2} \left\{ 1 + \frac{w-1}{w+1} R_1^2 \right\} + \left\{ 1 + \frac{w-1}{1-r}\left(1-R_2\right) \right\}^2 \right] h_{A_1}^2 \qquad (3.1)$$

with a mass ratio $r = M_{D^*}/M_B$, and form factor ratios $R_1 = h_V/h_{A_1}$ and $R_2 = \left(r h_{A_2} + h_{A_3}\right)/h_{A_1}$. The recoil parameter $w = v_B v_{D^*}$ is defined using B and D^* meson four velocities. Fermilab/MILC employed the so-called Fermilab action [25] based on heavy quark effective theory (HQET) to directly simulate physical bottom quark mass. While their data (red band) shows slightly steeper slope than experimental data (green and black bands), these data can be fitted to a model independent parameterization of w dependence [26] with acceptable value of $\chi/\mathrm{dof} \sim 1.5$. HPQCD employed a relativistic approach, namely a relativistic lattice action (highly improved staggered quark action) at unphysically small bottom quark masses to avoid large discretization erorrs, and reported an even steeper slope (cyan band). This also leads to a tension in the londitudinal

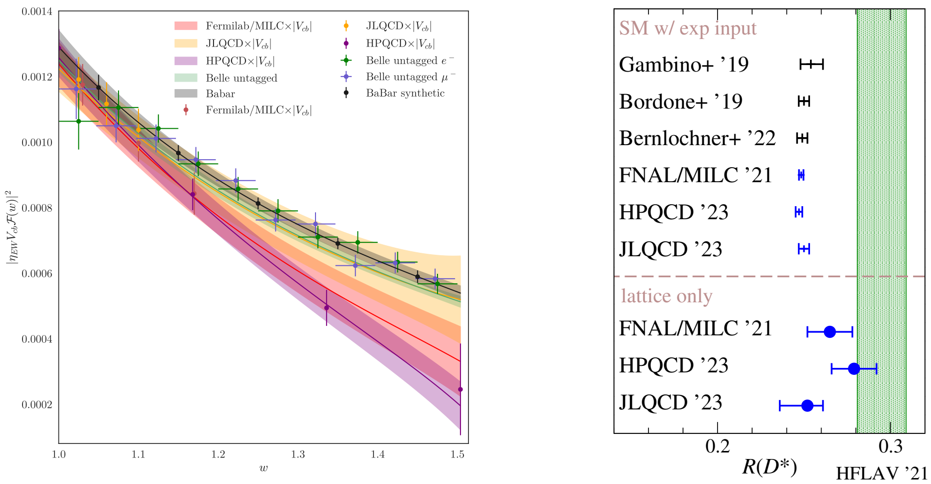


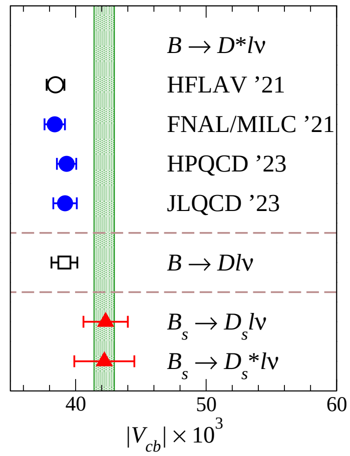
Figure 2: Left: comparison of recent lattice calculations of $B \to D^*\ell\nu$ form factors with experiments (figure from Ref. [24]). The differential decay rate without some kinematical factors, $|\eta_{EW}V_{cb}\mathcal{F}(w)|^2$, is plotted as a function of the recoil parameter w. The red, cyan and orange bands show results from Fermilab/MILC [18], HPQCD [19] and JLQCD [20], respectively, whereas experimental data from Belle (BaBar) are plotted by the green (black) band. Right: recent estimates of $R(D^*)$. Pluses and circles are from analyses with and without experimental data, respectively. The vertical green band shows an average of experimental measurements [32].

polarized fraction for the light lepton channels ($\ell = e, \mu$), which is not easy to explain even if new physics is introduced [27, 28][1]. On the other hand, JLQCD's data (orange band) obtained along a relativistic approach with a chiral fermion action show good consistency in the kinemarical distribution with experiment. The source of this tension is not yet clarified. Extenting their simulations safely to larger recoils is a key to resolve the situation and for more extensive comparison with experiment.

The right panel of Fig. 2 shows recent estimates of $R(D^*)$. The previous phenomenological analyses of experimental data already achieved 1 % accuracy with minimal lattice input $h_{A_1}(w=1)$, and observed $\gtrsim 3\,\sigma$ tension with experiment [29, 30, 31]. A slightly better accuracy of $\gtrsim 0.5\,\%$ is attained using the recent lattice data of other form factors as well [18, 19, 20]. The same figure also shows purely theoretical estimates without experimental input of the differential decay rate. While the uncertainty increases to $\sim 5\,\%$ and the tension with experiment becomes unclear, the accuracy is systematically improvable by more realistic simulations particularly at large recoils.

The left panel of Fig. 3 shows recent estimate of $|V_{cb}|$. The conventional determinations from $B \to D^*\ell\nu$ (open circle) and $B \to D\ell\nu$ (open square) are

[1]HPQCD's paper [19] has been recently updated, and show better consistency with other lattice and experimental data.



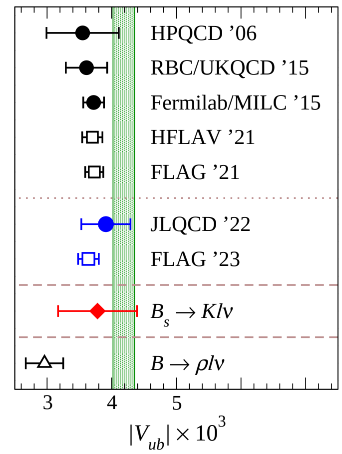


Figure 3: Left: recent estimate of $|V_{cb}|$. Circles, square and triangles are obtained from the $B \to D^*\ell\nu$, $B \to D\ell\nu$ and B_s meson decays, respectively. The open circle shows previous estimate using a minimal lattice input $h_{A_1}(1)$, whereas blue circles are from recent lattice calculations of all form factors. Right: recent estimate of $|V_{ub}|$. Black and blue circles show previous and JLQCD's results, respectively, whereas black and blue squares are world averages using these data. We plot estimates with recent lattice calculation of the $B_s \to K\ell\nu$ form factors (red diamond) and phenomenological estimate of $B \to \rho\ell\nu$ form factors (triangle). In both panel, the vertical bands show results from the inclusive decays.

consistent with each other, and sigificantly smaller than that from the inclusive decay (green band). HPQCD also calculated the form factors for $B_s \to D_s^{(*)}\ell\nu$, although resulting $|V_{cb}|$'s are consistent both with those from the exclusive and inclusive B decays within their 5 %uncertainty [33, 34]. The recent three lattice calculations of all form factors obtained $|V_{cb}|$ in good agreement with the previous determination in spite of the tension of their form factor data. We note that the HPQCD value is obtained from a simultaneous analysis of $B \to D^*\ell\nu$ and $B_s \to D_s^*\ell\nu$ decays, and they obtain a rather large value of $45.5(2.0) \times 10^{-3}$, if they use $\Gamma(B \to D^*\ell\nu)$ alone. The $|V_{cb}|$ tension is, therefore, remains unsolved. It has been suggested that the $|V_{cb}|$ determination may suffer from a bias called D'Agostini effecs [29, 35]. While a model independent parametrization of w dependence is available [26], how to safely apply the Bayesian prior to poorly determined fit parameters is another subtle issue [36, 37].

Recent estimates of $|V_{ub}|$ are shown in the right panel of Fig. 3. JLQCD recently calculated the $B \to \pi\ell\nu$ form factors, where the largest uncertainties come from the statistics and the "chiral extrapolation" from their simulated pion masses $M_\pi \gtrsim 230$ MeV to the physical mass [38, 21]. This is why rather old Fermilab/MILC study with smaller $M_\pi \gtrsim 165$ MeV achieved a better accuracy. World averages of $|V_{ub}|$ has been largely constrained by this study for about 10 years, and hence new independent calculations of the $B \to \pi$ form factors are higly welcome.

It is known that the $B_s \to K\ell\nu$ decay is advantageous in the control of the statistical accuracy [40] and chiral extrapolation to the physical M_π. Figure 3 also shows $|V_{ub}|$ from a recent lattice study of the relevant form factors [41] combined with LHCb data of branching fractions $\mathcal{B} = (B_s \to K\ell\nu)/(B_s \to D_s\ell\nu)$ [42] and $\Gamma(B_s \to D_s\ell\nu)$ [43]. It is, however, consistent both with estimates from the exclusive and inclusive B decays within its 16 % uncertainty. More precise experimental and lattice results are necessary to provide an alternative determination of $|V_{ub}|$ competitive to the conventional one from $B \to \pi\ell\nu$.

We also plot $|V_{ub}|$ obtained fom $B \to \rho\ell\nu$ using a light-cone sum rule estimate of relevant form factors [44]. It is not "gold-plated" with unstable ρ in the final state, and hence the lattice study is not straightforward. Preliminary results based on a newly developed finite volume framework are presented at this workshop [45]. It is interesting to extend this study to the $B \to K^*\ell\ell$ decay, for which a tension between the Standard Model and experiment has been reported [46].

4 Inclusive decays

Lattice QCD can provide first principles estimate of B meson matrix elements needed in the conventional calculation of the inclusive decay rate based on the operator product expansion (OPE). However, a direct lattice calculation of the inclusive rate is also attractive subject, since it enables a systematic comparison between the exclusive and inclusive determinations of $|V_{ub}|$ as well as $|V_{cb}|$ in the same simulation to resolve their tensions. The hadronic tensor $W_{\mu\nu}$ describing non-perturbative QCD effects to, for instance, $B \to X_c\ell\nu$ apears in the spectral decomposition of B meson four-point function on the lattice as

$$C_{\mu\nu}(t, \mathbf{q}) = \sum_{\mathbf{x}} \frac{e^{i\mathbf{q}\mathbf{x}}}{2M_B} \langle \bar{B}(\mathbf{0}) \left| J_\mu^\dagger(\mathbf{x}, t) J_\nu(0) \right| B(\mathbf{0}) \rangle = \int_0^\infty d\omega W_{\mu\nu}(\omega, \mathbf{q}) e^{-\omega t}, \quad (4.1)$$

where J_μ represents the weak current, $(t, \mathbf{x})$ and q are coordinates and momentum transfer, respectively, and $\omega = M_B - q^0$ is the haronic final state energy. It is, however, an ill-posed inverse problem to determine $W_{\mu\nu}$ from $C_{\mu\nu}$, because $W_{\mu\nu}$ is largely distorted in a finite volume due to the momentum direcretization: namely, its multi-particle continuum part turns into superposition of the δ function like singularities. Recent breakthrough in direct lattice studies of inclusive processes is based on the idea that, on the lattice, a smeared hadornic tensor[47] and its energy integral [48] can be calculated to a good preision. The latter developed into a direct calculation of the inclusive rate as

$$\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{q^2_{\max}} d\mathbf{q}^2 \sqrt{\mathbf{q}^2} \bar{X}(\mathbf{q}^2), \quad \bar{X}(\mathbf{q}^2) = \int_{\omega_{\min}}^{\omega_{\max}} d\omega K_{\mu\nu}(\omega, \mathbf{q}) W_{\mu\nu}(\omega, \mathbf{q}) \quad (4.2)$$

with integration kernel $K_{\mu\nu}$ specified by the leptonic tensor and kinematical factors for Γ, and by re-writing $\bar{X}$ (but not $W_{\mu\nu}$ itself) using $C_{\mu\nu}$ [49]. Here, $\mathbf{q}^2_{\max}$ ($\omega_{\{\min,\max\}}$) represents kinematically allowed region of $\mathbf{q}^2$ (ω).

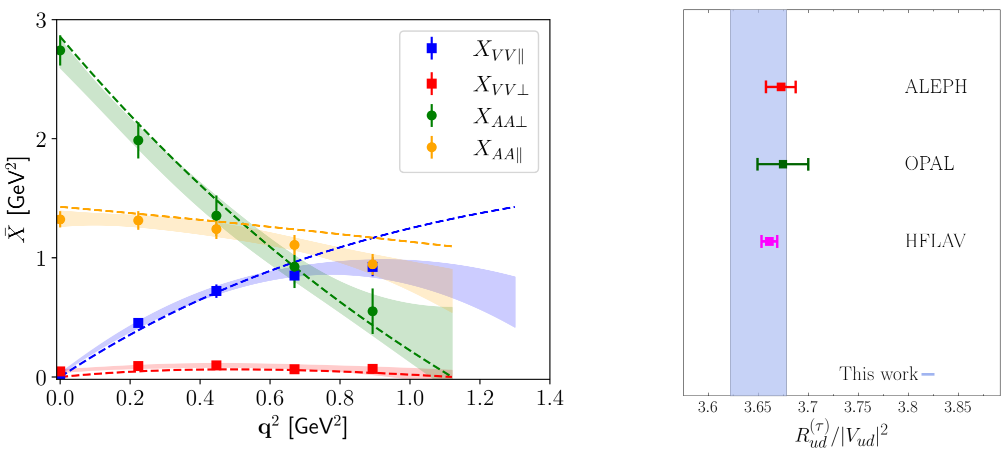


Figure 4: Left: differential decay rate $\bar{X}(\mathbf{q}^2)$ as a function of $\mathbf{q}^2$ (figure from Ref. [50]). Symbols and bands in different colors show inclusive rate for two choices of the weak current ($J_\mu = V_\mu$ or A_μ) and its polarization (parallel or perpendicular to $\mathbf{q}$). These are compared with the ground state contribution from $B_s \to D_s^{(*)}\ell\nu$ shown by dashed lines. Right: comparison of $R_{ud}/|V_{ud}|^2$ (figure from Ref. [56]). The horizontal band represents ETM collaboration's lattice estimate. For experimental estimates (squares), R_{ud}'s from ALEPH [57], OPAL [58] and HFLAV review [32] are combined with $|V_{ud}|$ from the nuclear β decays [8].

An extensive feasibility study of this apporach has been presented in Ref. [50]. The CPU cost is largely reduced by focusing on the $B_s \to X_{cs}\ell\nu$ decay withtout valence light quarks at a single combination of unphysicall heavy pion and light bottom quark masses. The left panel of Fig. 4 shows a comaprison between a differential $\mathbf{q}^2$ distribition of the inclusive rate $\bar{X}(\mathbf{q}^2)$ (symbols and bands) and corresponding ground state contribution from $B_s \to D_s^{(*)}\ell\nu$ decay (dashed lines). Their good consistency demonstrates the validiy of the approach to evaluate the energy integral of the hadronic tensor using lattice four-point functions. The authors also note that the excited state contribution can be suppressed due to the limited phase space at the unphysically small bottom quark mass.

Toward practical application to determine CKM elements, systematics of the approach are under intense investigation in more realistic setups. To evatulate the energy integral (4.2) using $C_{\mu\nu}$ on the lattice, the integration kernel $K_{\mu\nu}$ has to be approximated to a smooth function of ω [47, 49, 51, 52]. Reference [53] studied systematics due to this approximation using an HQET-based bottom quark action at its physical mass. A study of FVEs, which are one of the most non-trivial systematic uncertainties, was reported at this workshop [54, 55]. The authors focus on the $D_s \to X_{ss}\ell\nu$ decay to control discretization errors and chiral extrapolation in valence quark masses. In this safe setup, they develop a model to describe FVEs by taking the contribution for the two-body $K\bar{K}$ final states as an example. Their model shows good consistency with their lattice data at

$\mathbf{q}^2 = 0$. Further studies at other values of $\mathbf{q}^2$ and direct simulations in different volumes are necessary towards good control of FVEs.

Another interesing example of the inclusive anaysis is ETM collaboration's study of the $\tau \to X_{ud}\nu$ decay to determine $|V_{ud}|$ [56]. The corresponding hadornic tensor appears in the spectral representation of the lattice two-point functions, which are computationally much less expensive than the four-point functions for the inclusive $B_{(s)}$ decays. Therefore, both statistical and systematic uncertainties are under better contol: namely, high statistics simulations with all relevant quark masses set to their physical value in two different volumes to study FVEs and at three lattice spacings to take the limit of $a = 0$. Their estimate of the normalized inclusive decay rate $R_{ud} = \Gamma(\tau \to X_{ud}\nu)/\Gamma(\tau \to e\bar{\nu}\nu)$ divided by $|V_{ud}|^2$ is in good agreement with experiment as shown in the right panel of Fig. 4. They achieved 0.4 % determination of $|V_{ud}| = 0.9752(37)_{\rm th}(10)_{\rm ex}$, which is nicely consistent with the precise (0.03 %) estimate 0.97373(31) from the nuclear β decays[8]. The uncertainty is dominated by theoretical ones from isopin corrections, statistics and FVEs. With better control of these errors, the inclusive τ decay would provide deteremination of $|V_{ud}|$ competitive to those from the neutron (0.1–0.2 %) and pion (0.3 %) β decays. An interesting future direction is to extend the ETM study to $\tau \to X_{us}\nu$ to provide an alternative determination of $|V_{us}|$, because a tension with the determinations from excluive kaon decays has been reported [32][2].

5 Conclusions

In this article, we review status and progress of lattice QCD for the determination of CKM elements. For gold-plated processes with light, strange and charm quarks, independent realistic similations are becoming available leading to, for instance, sub-% determination of relevant hadronic inputs for the kaon (semi)leptonic decays. More independent studies to crosscheck such precision calculations are highly welcome. Strong isospin and radiative corrections are also under active investiations.

While simulations of bottom quarks are still limited to unphysically small masses or effective-theory-based actions, many interesting quantites are being studied including the $B \to D^*\ell\nu$ and B_s decay form factors. Systematics due to the unphysical setup are expected to be removed in the next decade. We also note that close interplay between theorists and experimentalists is important to resolve the long standing problem including $|V_{cb}|$ tension and to maximaize the sensitivy of on-going experiments to new physics.

It is encouraging that wider applications are becoming avialable with newly developed finite volume frameworks for inclusive decays as well as decays involving unstable particles, such as $B \to \rho\ell\nu$. A practical application to determine $|V_{ud}|$

[2]After the workshop, the ETM collaboration performed the determination of $|V_{us}|$ [59], which is consistent with the conventional determination [60], and hence confirms the tension with the kaon decays.

(and $|V_{us}|$) from the inclusive τ decay has been already avilable.

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References

[1] Y. Aoki *et al.* (Flavour Lattice Averaging Group), Eur. Phys. J. C **82** (2022) 869 [2111.09849].

[2] A. Vaquero, PoS **LATTICE2022** (2023) 250 [2212.10217].

[3] T. Kaneko, PoS **LATTICE2022** (2023) 238 [2304.01618].

[4] S. Meinel, PoS **LATTICE2023** (2024) 126 [2401.08006].

[5] J.T. Tsang and M. Della Morte, Eur. Phys. J. Spec. Top. (2023) [2310.02705].

[6] C.Y. Seng *et al.*, Phys. Rev. Lett. **121** (2018) 241804 [1807.10197].

[7] C.Y. Seng, M. Gorchtein and M.J. Ramsey-Musolf, Phys. Rev. D **100** (2019) 013001 [1812.03352].

[8] J.C. Hardy and I.S. Towner, Phys. Rev. C **102** (2020) 045501.

[9] C. Alexandrou *et al.*, Phys. Rev. D **104** (2021) 074520 [2104.06747].

[10] A. Bazavov *et al.* (Fermilab/MILC collaborations), Phys. Rev. D **99** (2019) 114509 [1809.02827].

[11] T. Yamazaki *et al.* (PACS collaboration), PoS **Lattice 2023** (2024) 276 [2311.16755].

[12] N. Carrasco *et al.*, Phys. Rev. D **91** (2015) 074506 [1502.00257].

[13] M. Di Carlo *et al.*, Phys. Rev. D **100** (2019) 034514 [1904.08731].

[14] V. Cirigliano and H. Neufeld, Phys. Lett. B **700** (2011) 7 [1102.0563].

[15] P. Boyle *et al.*, JHEP **02** (2023) 242 [2211.12865].

[16] N.H. Christ *et al.*, Phys. Rev. D **108** (2023) 014501 [2304.08026].

[17] P.-X. Ma *et al.*, arXiv:2308.16755 [hep-lat].

[18] A. Bazavov *et al.* (Fermilab/MILC collaborations), Eur. Phys. J. C **82** (2022) 1141 [2105.14019].

[19] J. Harrison and C.T.H. Davies (HPQCD collaboration), arXiv:2304.03137 [hep-lat].

[20] Y. Aoki *et al.* (JLQCD collaboration), Phys. Rev. D **109** (2024) 074503 [2306.05657].

[21] For JLQCD's study, see also B. Colquhoun, in these proceedings.

[22] J.A. Bailey *et al.* (Fermilab/MILC collaborations), Phys. Rev. D **89** (2014) 114504 [1403.0635].

[23] J. Harrison, C.T.H. Davies and M. Wingate (HPQCD collaboration), Phys. Rev. D **97** (2018) 054502 [1711.11013].

[24] A. Vaquero, in these proceedings.

[25] A.X. El-Khadra, A.S. Kronfeld and P.B. Mackenzie, Phys. Rev. D **55** (1997) 3933 [hep-lat/9604004].

[26] C.G. Boyd, B. Grinstein and R.F. Lebed, Phys. Rev. D **56** (1997) 6895 [hep-ph/9705252].

[27] M. Fedele *et al.*, Phys. Rev. D **108** (2023) 055037 [2305.15457].

[28] M. Fedele, in these proceedins [2403.20053].

[29] P. Gambino and M. Jung and S. Schacht, Phys. Lett. B **795** (2019) 386 [1905.08209]. o

[30] M. Bordone, M. Jung and D. van Dyk, Eur. Phys. J. C **80** (2020) 74 [1908.09398].

[31] F. Bernlochner *et al.*, Phys. Rev. D **106** (2022) 096015 [2206.11281].

[32] Y. Amhis *et al.* (HFLAV collaboration), Phys. Rev. D **107** (2023) 052008 [2206.07501].

[33] E. McLean *et al.* (HPQCD collaboration), Phys. Rev. D **101** (2020) 074513 [1906.00701].

[34] J. Harrison and C.T.H. Davies (HPQCD collaboration), Phys. Rev. D **105** (2020) 094506 [2105.11433].

[35] G. D'Agostini, Nucl. Instrum. Meth. A **346** (1994) 306.

[36] J.M. Flynn, A. Jüttner, J.T. Tsang, JHEP **12** (2023) 175 [2303.11285].

[37] J.M. Flynn, A. Jüttner, J.T. Tsang, PoS **LATTICE2023** (2024) 239 [2312.14631].

[38] B. Colquhoun *et al.* (JLQCD collaboration), Phys. Rev. D **106** (2022) 054502 [2203.04938].

[39] A. Bailey *et al.* (Fermilab/MILC collaborations), Phys. Rev. D **92** (2015) 014024 [1503.07839].

[40] G.P. Lepage, "The Analysis of Algorithms for Lattice Field Theory" in Boulder ASI (1989) 97.

[41] J.M. Flynn *et al.* (RBC/UKQCD collaboratons), Phys. Rev. D **107** (2023) 114512 [2303.11280].

[42] R. Aaij *et al.* (LHCb collaboraton), Phys. Rev. Lett. **126** (2021) 081804 [2012.05143].

[43] R. Aaij *et al.* (LHCb collaboraton), Phys. Rev. D **101** (2020) 072004 [2001.03225].

[44] F.U. Bernlochner, M.T. Prim and D.J. Robinson Phys. Rev. D **104** (2021) 034032 [2104.05739].

[45] L. Leskovec *et al.*, in these proceedings.

[46] R. Aaij *et al.* (LHCb collaboration), Phys. Rev. Lett. **111** (2013) 191801 [1308.1707].

[47] M.T. Hansen, H.B. Meyer and D. Robaina, Phys. Rev. D **96** (2017) 094513 [1704.08993].

[48] S. Hashimoto, PTEP **2017** (2017) 053B03 [1703.01881].

[49] P. Gambino and S. Hashimoto, Phys. Rev. Lett. **125** (2020) 032001 [2005.13730].

[50] P. Gambino *et al.*, JHEP **07** (2022) 083 [2203.11762].

[51] M. Hansen, A. Lupo and N. Tantalo, Phys. Rev. D **99** (2019) 094508 [1903.06476].

[52] G. Bailas, S. Hashimoto and T. Ishikawa, PTEP **2020** (2020) 043B07 [2001.11779].

[53] A. Barone *et al.*, JHEP **07** (2023) 145 [2305.14092].

[54] R. Kellermann, in these proceedings.

[55] For details, see also R. Kellermann *et al.*, PoS **LATTICE2023** (2024) 272 [2312.16442].

[56] A. Evangelista *et al.* (ETM collaboration), Phys Rev D **108** 074513 [2308.03125].

[57] S. Schael *et al.* (ALEPH collaboration), Phys. Rept. **421** (2005) 191 [hep-ex/0506072].

[58] K. Ackerstaff *et al.* (OPAL collaboration), Eur. Phys. J. C **7** (1999) 571 [hep-ex/9808019].

[59] C. Alexandrou *et al.* (ETM collaboration), arXiv:2403.05404 [hep-lat].

[60] E. Gamiz *et al.*, Nucl. Phys. Proc. Suppl. **169** (2007) 85 [hep-ph/0612154].