THE LOWEST DISCRIMINANT IDEAL OF CENTRAL EXTENSIONS OF ABELIAN GROUPS

ZHONGKAI MI

Abstract. In a previous joint paper with Wu and Yakimov, we gave an explicit description of the lowest discriminant ideal in a Cayley-Hamilton Hopf algebra (H, C, tr) of degree d over an algebraically closed field $\Bbbk$, $\mathrm{char}\,\Bbbk \notin [1, d]$ with basic identity fiber, i.e. all irreducible representations over the kernel of the counit of the central Hopf subalgebra C are one-dimensional. Using results developed in that paper, we compute relevant quantities associated with irreducible representations to explicitly describe the zero set of the lowest discriminant ideal in the group algebra of a central extension of the product of two arbitrary finitely generated Abelian groups by any finite Abelian group under some conditions. Over a fixed maximal ideal of C the representations are tensor products of representations each corresponding to a central extension of a subgroup isomorphic to the product of two cyclic groups of the same order. A description of the orbit of the identity, i.e. the kernel of the counit of C, under winding automorphisms is also given.

1. Introduction

Discriminants and discriminant ideals (Definition 2.5) are defined for an algebra with trace (R, C, tr) (Definition 2.1) and usually R is a finite C-module. Discriminants has been a very active area of research in noncommutative algebra in recent years. On one hand, there is work on their computation using techniques such as smash products [9], Poisson geometry [11] [17], cluster algebras [16] and reflexive hulls [6]. On the other hand, it has been used to study automorphism groups of noncommutative algebras [4] [5] [6] and the Zariski cancellation problem [1] [6]. The Zariski cancellation problem is about whether $A \cong B$ as algebras when $A[X] \cong B[X]$. Discriminant ideals are much more general and do not require some ideal to be principal. Currently very little is known about discriminant ideals. When (R, C, tr) satisfies

$(\mathbf{CH_n})$ (R, C, tr) is a finitely generated Cayley-Hamilton algebra (Definition 2.3) of degree n over an algebraically closed field $\Bbbk$ and $\mathrm{char}\,\Bbbk \notin [1, n]$;

there is a description of the zero sets of discriminant ideals by dimensions of irreducible representations [2, Theorem 4.1(b)]:

$$\begin{aligned} \mathcal{V}_k &= \mathcal{V}(D_k(R/C, \mathrm{tr})) = \mathcal{V}(MD_k(R/C, \mathrm{tr})) \\ &= \left\{ \mathbf{m} \in \mathrm{MaxSpec}(C) \middle| \mathrm{Sd}(\mathbf{m}) = \sum_{V \in \mathrm{Irr}(R/\mathbf{m}R)} (\dim(V))^2 < k \right\}. \end{aligned} \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 16G30; Secondary 16T05, 16D60, 16W20.

Key words and phrases. Algebras with trace, discriminant ideals, Cayley–Hamilton Hopf algebras, projective representations of finite groups.

The research of the author has been supported by NSF grant DMS–2200762.

where $\operatorname{Irr}(R/\mathbf{m}R)$ denotes isomorphism classes of irreducible representations of R over $\mathbf{m} \in \operatorname{MaxSpec}(C)$. If R is furthermore prime and, C is the center of R and integrally closed; the zero set of the highest discriminant ideal $D_h(R/C, \operatorname{tr})$ (Definition 2.6) is the complement of the Azumaya locus [2, Main Theorem]. In the case $(H, C, \operatorname{tr})$ is a Cayley-Hamilton Hopf algebra (Definition 2.3) and all irreducible representations over $\mathbf{m}_{\bar{\epsilon}}$, the identity or the kernel of the counit of C, are one-dimensional ($\mathbf{m}_{\bar{\epsilon}}$ has basic fiber); the level l of the lowest discriminant ideal $D_l(H/C, \operatorname{tr})$ (Definition 2.7) is equal to the number of such representations plus one [14, Theorem B(b)]. The zero set of $D_l(H/C, \operatorname{tr})$ contains the orbit of $\mathbf{m}_{\bar{\epsilon}}$ under left and right winding automorphisms of H and $\mathbf{m} \in \operatorname{MaxSpec}(C)$ is in the orbit if and only if its fiber is basic [14, Theorem C(a)]. A special case in [14] is a central extension of $\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ by $\mathbb{Z}/l\mathbb{Z}$. In this case $\mathcal{V}_l = \operatorname{MaxSpec}(C)$, thus $l = h + 1$. We consider a more general setting as follows

Main Theorem. *Let A and B be finitely generated Abelian groups, Δ be a finite Abelian group, $\Lambda = A \times B$ and*

$$1 \longrightarrow \Delta \longrightarrow \Sigma \xrightarrow{f} \Lambda \longrightarrow 1$$

be a central extension of Λ by Δ such that $f^{-1}((A, 0)) \subseteq \mathrm{C}_\Sigma(f^{-1}((A, 0)))$ as well as $f^{-1}((0, B)) \subseteq \mathrm{C}_\Sigma(f^{-1}((0, B)))$, β be a 2-cocycle associated with Σ. There is a central subgroup $\Omega \lhd \Sigma$ containing $\langle \operatorname{Im}(\beta) \rangle$ of finite index, i.e. $m = [\Sigma : \Omega] < \infty$. Choose an algebraically closed field $\Bbbk$ with $\operatorname{char} \Bbbk \notin [1, m]$ and, define $H = \Bbbk\Sigma$ and $C = \Bbbk\Omega$. Let $\mathbf{m} \in \operatorname{MaxSpec}(C)$ and $\operatorname{tr}_{\mathrm{reg}}$ denote the regular trace then

(i) *$(H, C, \operatorname{tr}_{\mathrm{reg}})$ is a Cayley-Hamilton Hopf algebra of rank m with basic identity fiber, $\operatorname{GKdim}(H) = \operatorname{GKdim}(C)$ equals free rank of Ω.*

(ii) *The algebra $R/\mathbf{m}R$ is simple. Let $n = |\Delta|$ and ξ be a primitive n-th root of unity. Then there are positive integers k and l_i, $1 \le i \le k$ such that*

$$l_k \mid l_{k-1} \mid \cdots \mid l_1 \mid n.$$

$$R/\mathbf{m}R \cong R_1 \otimes_\Bbbk \cdots \otimes_\Bbbk R_k \quad \textit{where} \qquad (1.2)$$
$$R_i \cong \frac{\Bbbk < x_i, y_i >}{(x_i^{l_i} - 1, y_i^{l_i} - 1, x_i y_i - \xi^{\frac{n}{l_i}} y_i x_i)}.$$

(iii) *$R/\mathbf{m}R$ has an irreducible representation V with*

$$\dim(V) = \prod_{i=1}^{k} l_i \quad \textit{and} \qquad (1.3)$$
$$\operatorname{Stab}_{G_0}(V) \cong (\mathbb{Z}/l_1\mathbb{Z} \times \cdots \mathbb{Z}/l_k\mathbb{Z})^2.$$

Hence all irreducible representations of H are maximally stable (Definition 2.9) and

(iv)

$$\mathcal{V}(D_k(H/C, \operatorname{tr}_{\mathrm{reg}})) = \begin{cases} \varnothing, & k \le m, \\ \operatorname{MaxSpec}(C), & k > m. \end{cases} \qquad (1.4)$$

(v) *The orbit of* $\mathbf{m}_{\bar{\epsilon}}$ *under* $\mathrm{Aut}_{\Bbbk-\mathrm{Alg}}(H,C)$ *denoted by* $\mathcal{O}$ *satisfies*

$$\mathcal{O} = \mathrm{Aut}_{\Bbbk-\mathrm{Alg}}(H,C)\cdot \mathbf{m}_{\bar{\epsilon}} = \mathrm{W}_{\mathrm{l}}(G(H^\circ))\cdot \mathbf{m}_{\bar{\epsilon}} = \mathrm{W}_{\mathrm{r}}(G(H^\circ))\cdot \mathbf{m}_{\bar{\epsilon}} \tag{1.5}$$
$$\cong \Omega/\langle \mathrm{Im}\beta\rangle. \tag{1.6}$$

As a group $\mathcal{O}$ *is isomorphic to the subgroup of* $G(C^\circ)$ *with basic fibers.*

Acknowledgement The author is thankful to his advisor Prof. Milen Yakimov for posing the problem addressed in this paper as well as his guidance and encouragement.

2. Background

In this section, we review definitions related to discriminant ideals, Cayley-Hamilton Hopf algebras and winding automorphisms of Hopf algebras.

Definition 2.1. Let (R,C) be an algebra R over a field $\Bbbk$ with a central subalgebra C. Following the definition in [2] a nonzero function $\mathrm{tr}: R \mapsto C$ is called a *trace* if it is C-linear and cyclic, i.e.

(i) $\mathrm{tr}(r_1 + cr_2) = \mathrm{tr}(r_1) + c\mathrm{tr}(r_2) \quad \forall c \in C,\ r_1,\ r_2 \in R$;
(ii) $\mathrm{tr}(r_1 r_2) = \mathrm{tr}(r_2 r_1) \quad r_1,\ r_2 \in R$.

If such a function exits, (R, C, tr) is called an *algebra with trace.*

Assuming that $\mathrm{tr}(1)$ has no $\mathbb{Z}$-torsion, then R has invariant basis number (IBN), i.e. $R^n \cong R^m$ implies $n = m$ [19, Proposition 13.29].

Example 2.2. Let R be an algebra over $\Bbbk$ and C a central subalgebra. Suppose R is a free module over C of rank n. Then each $\phi \in \mathrm{End}(_C R)$ can be represented by $\Theta(\phi) \in M_n(C)$ and there is an inclusion $\eta : R \hookrightarrow \mathrm{End}(_C R)$ defined by left multiplication. Let tr be the trace on $M_n(C)$ defined by summing up the diagonal elements. Then the *regular trace* $\mathrm{tr}_{\mathrm{reg}}$ is defined by the composition

$$R \overset{\eta}{\hookrightarrow} \mathrm{End}(_C R) \overset{\Theta}{\hookrightarrow} M_n(C) \xrightarrow{\mathrm{tr}} C. \tag{2.1}$$

Note that in the case of $C = \Bbbk$, $\mathrm{tr}_{\mathrm{reg}}(x)$ is m-times the usual trace for $x \in M_m(\Bbbk)$ because x has m columns [2, 2.2-(4)]. And Θ is proper inclusion for $m > 1$ as $n = m^2$ in (2.1).

In $\Bbbk[x_1, \cdots, x_n]$, the k-th power sum is defined by

$$\psi_k(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^k$$

and for $1 \le j \le n$ define

$$e_j(x_1, \cdots, x_n) = \sum_{\{i_1, \cdots, i_j\} \subseteq \{1, \cdots, n\}} \prod_{l=1}^{j} x_{i_l}.$$

Then by Newton identities there are $p_i \in \mathbb{Z}[(i!)^{-1}][x_1, \cdots, x_n]$ for $1 \le i \le n$ such that

$$p_i(\psi_1, \cdots, \psi_i) = e_i$$

as formal polynomials in $\mathbb{Z}[x_1, \cdots, x_n]$.

Definition 2.3. Let (R, C, tr) be an algebra with trace over $\Bbbk$, then the *n-characteristic polynomial* of $r \in R \in C[x]$ is defined by

$$\chi_{n,r}(x) \coloneqq x^n + \sum_1^n (-1)^{n-i} p_i(\mathrm{tr}(r), \cdots, \mathrm{tr}(r^{n-i}))x^i.$$

An algebra with trace (R, C, tr) is a *Cayley-Hamilton algebra of degree d* if it satisfies

(i) $\mathrm{tr}(1) = d$,
(ii) $\chi_{d,r}(r) = 0$, $\forall r \in R$.

It is called a *Cayley-Hamilton Hopf algebra of degree d* if furthermore H is a Hopf algebra and C is a Hopf subalgebra.

A finitely generated Cayley-Hamilton algebra (R, C, tr) is a finite module over $\mathrm{tr}(R)$, so in particular it is a finite module over C [8, Theorem 2.6]. If $\mathrm{char}\,\Bbbk = 0$, it has an injective represention compatible with trace $\phi : R \mapsto M_d(B)$ for some commutative algebra B, i.e. ϕ is an algebra homomorphism s.t. $\mathrm{tr}_{M_d(B)} \circ \phi = \phi \circ \mathrm{tr}_R$, if and only if R is a Cayley-Hamilton algebra of degree d [18, Theorem 0.3]. Here $\mathrm{tr}_{M_d(B)}$ is computed through taking the sum of diagonal elements. If (R, C, tr) satisfies $(\mathbf{CH_d})$, and denote its Jacobson radical by J, then [7, Proposition 4.3]

$$R/(JR) \cong \oplus_{i=1}^k M_{n_i}(\Bbbk)$$

for some positive integers k, n_i and there are positive integer s_i such that

$$\sum_{i=1}^k s_i n_i = d.$$

Example 2.4. Let R be a finitely generate algebra over a field $\Bbbk$ with $\mathrm{char}\,\Bbbk \notin [1, r]$ that is a finite free module over a central subalgebra C of rank r. Then $(R, C, \mathrm{tr}_{\mathrm{reg}})$ is a Cayley-Hamilton algebra of degree r.

Definition 2.5.

(i) The *n-th discriminant ideal* $D_n(R/C, \mathrm{tr})$ is defined as the ideal of C generated by elements of the form
$$\det(\mathrm{tr}(y_i y_j))_{1\le i,j\le n}, \quad (y_1, \cdots, y_n) \in R^n$$
(ii) and the *n-th modified discriminant ideal* $MD_n(R/C, \mathrm{tr})$ is defined as the ideal of C generated by the elements of the form
$$\det(\mathrm{tr}(y_i y_j'))_{1\le i,j\le n}, \quad (y_1, \cdots, y_n), (y_1', \cdots, y_n') \in R^n.$$
(iii) When R is furthermore a free (left) C-module of rank N, the *discriminant* $D(R/C, \mathrm{tr})$ is given by
$$D(R/C, \mathrm{tr}) = \det(\mathrm{tr}(a_i a_j))_{1\le i,j\le N}$$
for a basis $\{a_1, \cdots, a_N\}$ of R as a C-module.

The discriminant is determined up to the square of a unit of C computed from the determinant of the change of basis matrix and

$$D_{n^2}(R/C, \mathrm{tr}) = MD_{n^2}(R/C, \mathrm{tr}) = \langle D(R/C, \mathrm{tr})\rangle.$$

In literature, R is usally a finite module over C, so R is a PI ring. If R is furthermore prime and n is its PI-degree, then $D_k(R/C, \mathrm{tr}) = MD_k(R/C, \mathrm{tr}) = 0$ for $k > n^2$ [2, Corollay 2.4]. Since the determinant of a matrix can be computed using C-linear combination of smaller matrices

$$C = MD_1(R/C, \mathrm{tr}) \supseteq \cdots \supseteq MD_k(R/C, \mathrm{tr}) \supseteq MD_{k+1}(R/C, \mathrm{tr}) \cdots .$$

If (R, C, tr) satisfies $(\mathbf{CH_d})$ for some d, define $\mathcal{V}_k$ as in (1.1); then

$$\varnothing = \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_k \subseteq \mathcal{V}_{k+1} \cdots .$$

Recall (1.1) and R is a finite module over C, so there is a smallest positive integer h such that

$$\varnothing = \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_h \subsetneq \mathcal{V}_{h+1} = \mathrm{MaxSpec}(C).$$

Definition 2.6. Then $D_h(R/C, \mathrm{tr})$ is called the *the highest discriminant ideal*.

Similarly there is a smallest l such that

$$\varnothing = \mathcal{V}_1 = \cdots \subsetneq \mathcal{V}_l \subseteq \cdots .$$

Definition 2.7. Then $D_l(R/C, \mathrm{tr})$ is called the *the lowest discriminant ideal*.

Remark 2.8. In a Hopf algebra H, $\mathrm{Alg}_{\Bbbk}(H, \Bbbk)$ is the same as the group-like elements in its finite dual denoted by $G(H^\circ)$ [15, (1.3.5)].

Suppose (H, C, tr) is a Cayley-Hamilton Hopf algebra with basic identity fiber, denote the subset of $G(H^\circ)$ over $\mathbf{m}_{\bar{\epsilon}}$ as G_0. An action of G_0 on $\mathrm{Irr}(H/\mathbf{m}H)$ can be defined by tensor product as [14, Section 3.1]

$$(\chi, V) \mapsto \chi \otimes_{\Bbbk} V \quad \text{for } \chi \in G_0, V \in \mathrm{Irr}(H/\mathbf{m}H).$$

Definition 2.9. Then the *stabilizer* of $V \in \mathrm{Irr}(H/\mathbf{m}H)$ in G_0 is

$$\mathrm{Stab}_{G_0}(V) \coloneqq \{\chi \in G_0 : \chi \otimes V \cong V\}$$

It can be shown that $|\mathrm{Stab}_{G_0}(V)| \leq \dim(V)^2$ [14, Proposition 3.5(b)]. The irreducible module V is *maximally stable* if the equality holds.

If $\mathbf{m} \in \mathrm{MaxSpec}(C)$, then $\mathbf{m} \in \mathcal{V}_l \iff \exists V \in \mathrm{Irr}(H/\mathbf{m}H)$, V maximally stable [14, Theorem 4.2(c)].

Definition 2.10. Let $\phi \in G(H^\circ)$, the left winding $(\mathrm{W_l})$ and right winding $(\mathrm{W_r})$ automorphisms in $\mathrm{Aut}_{\Bbbk-\mathrm{Alg}}(H)$ are defined by

$$\mathrm{W_l}(\phi)(h) = \sum \phi(h_1)h_2, \quad \mathrm{W_r}(\phi)(h) = \sum \phi(h_2)h_2, \quad \forall h \in H.$$

One of the features of winding automorphisms is that both of the automorphism groups $\mathrm{W_l}(G(H^\circ))$ and $\mathrm{W_r}(G(H^\circ))$ act transitively on kernels of elements in $G(H^\circ)$ or equivalently the set of one-dimensional representations of H. This follows from the fact the $G(H^\circ)$ is a group under convolution and

$$\psi \circ \mathrm{W_l}(\phi) = \phi * \psi, \quad \psi \circ \mathrm{W_r}(\phi) = \psi * \phi, \quad \forall \phi, \psi \in G(H^\circ).$$

Winding automorphisms are very important in the discussions about homological integrals in [12] and dualising complexes in [3].

3. The Second Cohomology Group

The first part of this section is a short introduction to classifying central extensions of groups using the second cohomology group. We refer the read to [13] and its reference for more detailed background on the topic. In the second part we show that for a specific class of central extensions in our setting, the associated second cohomology group can be represented by some matrices.

Definition 3.1. Let G be a group, and M be an Abelian group in additive notation. A functions $f : G \times G \to M$ is called a *2-cocycle* if

$$\beta(h,k) - \beta(gh,k) + \beta(g,hk) - \beta(g,h) = 0 \quad \forall g,h,k \in G$$

and a 2-*coboundary* if there a function $f : G \to M$ such that

$$\beta(g,h) = f(h) - f(gh) + f(g) \quad \forall g,h \in G.$$

Denote the set of 2-cocyles and 2-coboundaries as $Z^2(G,M)$ and $B^2(G,M)$. Then these have structures of Abelian groups from M. And the *2nd-cohomology group* is defined as the quotient group

$$H^2(G,M) = \frac{Z^2(G,M)}{B^2(G,M)}.$$

Two 2-cocyles are called *cohomologous* if they are in the same cohomology class, i.e. the same in $H^2(G,M)$. Every 2-cocycle satisfies [13, Lemma 1.2.1]

$$\beta(1,g) = \beta(1,1) = \beta(g,1) \quad \text{and} \tag{3.1}$$

$$\beta(g,g^{-1}) = \beta(g^{-1},g) \quad \forall g \in G. \tag{3.2}$$

Let G be a group, M be an Abelian group and $\beta \in Z^2(G,M)$, denote by G_β the central extension of G by M with

$$G_\beta := \{(m,g) : m \in M,\ g \in G\}, \tag{3.3}$$

$$(m_1,g_1)(m_2,g_2) := (m_1 + m_2 + \beta(g_1,g_2), g_1g_2), \quad m_i \in M, g_i \in G. \tag{3.4}$$

Remark 3.2. Any central extension of G by M is equivalent to G_β for some $\beta \in Z^2(G,M)$ and two such extensions are equivalent if and only if the cocyles are cohomologoues [13, Theorem 3.2.3].

Notation 3.3. We will use the notation that $\Lambda = A \times B$ where A and B are finitely generated abelian groups, Δ is a finite abelian group and $\Sigma = \Delta \rtimes_\beta \Lambda = \Lambda_\beta$ defined in (3.3) and (3.4) for some $\beta \in Z^2(\Lambda,\Delta)$. In $\Delta \rtimes_\beta \Lambda$, use the shorthand $(A,0) = \{(0,(a,0)) : a \in A\}$ and $(0,B) = \{(0,(0,b)) : b \in B\}$.

Notation 3.4. Denote the centralizers of a set S in a group G by $\mathrm{C}_G(S)$. Define a set

$$\mathcal{B} = \{\beta \in Z^2(\Lambda,\Delta) : (A,0) \subseteq \mathrm{C}_\Sigma((A,0)) \text{ and } (0,B) \subseteq \mathrm{C}_\Sigma((0,B))\}.$$

Then $\mathcal{B}$ is a group. Define another group

$$\begin{aligned} N^2(\Lambda,\Delta) = \{\beta \in Z^2(\Lambda,\Delta) : \beta((a_1,0),(a_2,0)) = \beta((0,b_1),(0,b_2)) \\ = \beta((a_1,0),(0,b_1)) = 0 \quad \forall a_1,\ a_2 \in A \text{ and } b_1,b_2 \in B\}. \end{aligned}$$

Lemma 3.5. *Using the notation above, there is a split short exact sequence of Abelian groups*

$$0 \longrightarrow B^2(\Lambda, \Delta) \cap \mathcal{B} \xrightarrow{i} \mathcal{B} \xrightarrow{\Phi} N^2(\Lambda, \Delta) \longrightarrow 0.$$

where i and the splitting homomorphism are inclusions. Thus if $(A, 0)$ and $(0, B)$ in Σ are Abelian, then $\Sigma \cong \Delta \rtimes_\gamma \Lambda$ for a unique $\gamma \in N^2(\Lambda, \Delta)$.

Proof. $(A, 0)$ and $(0, B)$ being Abelian is equivalent to

$$\beta((a_1, 0), (a_2, 0)) = \beta((a_2, 0), (a_1, 0)) \quad \forall a_1,\, a_2 \in A \quad \text{and} \tag{3.5}$$

$$\beta((0, b_1), (0, b_2)) = \beta((0, b_2), (0, b_1)) \quad \forall b_1, b_2 \in B. \tag{3.6}$$

Define $f : \Sigma \to \Delta$ by

$$f((a, b)) = \beta((a, 0), (0, b)) - \beta((0, 0), (0, 0))$$

and Φ by

$$\Phi(\beta)(e_1, e_2) = \beta(e_1, e_2) - \beta((0, 0), (0, 0)) + f(e_1) + f(e_2) - f(e_1 + e_2).$$

□

In this work the field $\Bbbk$ is assumed to be algebraically closed and we use $\Bbbk G$ to denote the group algebra of G over $\Bbbk$.

Notation 3.6. In $A \times B$, for brevity denote $a = (a, 0)$ and $b = (0, b)$ for any $a \in A$ and $b \in B$. In a general group G, let $|g|_G$ denote the order of the element g. If no finite positive integer power of g equals the identity, define $|g|_G = \infty$ and use the convention that any positive integer l divides ∞ and $\gcd(l, \infty) = l$.

Lemma 3.7 (compatibility)**.** *Let A, B be finitely generated Abelian groups, $\Lambda = A \times B$ and Δ be a finite Abelian group; $\beta \in N^2(\Lambda, \Delta)$.*

(i) *Then $\beta(a_i + a_j, b_k) = \beta(a_i, b_k) + \beta(a_j, b_k)$ and $\beta(a_i, b_k + b_l) = \beta(a_i, b_k) + \beta(a_i, b_l)$ for any $a_i, a_j \in A$ and $b_k, b_l \in B$.*

(ii) *If $a \in A$, $|a|_A = l$ and $b \in B$, $|b|_B = k$ then $|\beta(a, b)|_\Delta$ divides* $\gcd(l, k)$.

(iii) *There is a group isomorphism $\Phi : N^2(\Lambda, \Delta) \to \mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, \Delta)$.*

Proof. (i) Since Λ is Abelian and Δ is central, in the group algebra of the extension $\Sigma \cong \Delta \rtimes_\beta \Lambda$

$$\begin{aligned} a_i a_j b_k b_l &= a_i \beta(a_j, b_k) b_k a_j b_l = a_i (\beta(a_j, b_k) + \beta(a_j, b_l)) b_k b_l a_j \\ &= (\beta(a_i, b_k) + \beta(a_i, b_l) + \beta(a_j, b_k) + \beta(a_j, b_l)) b_k b_l a_i a_j \\ &= \beta(a_i + a_j, b_k + b_l)(b_k b_l)(a_i a_j) \\ &= \beta(a_i + a_j, b_k + b_l) b_k b_l a_i a_j. \end{aligned}$$

Hence

$$\beta(a_i + a_j, b_k + b_l) = \beta(a_i, b_k) + \beta(a_i, b_l) + \beta(a_j, b_k) + \beta(a_j, b_l).$$

(ii) Recall $\gcd(l, k) = xl + yk$ for some integers x and y. From (i) in $\Bbbk\Delta$

$$1 \cdot b = a^{|a|_A} \cdot b = |a|_A\, \beta(a, b) b \cdot a^{|a|_A} = |a|_A\, \beta(a, b) b$$

So

$$|a|\, \beta(a, b) = 0 \quad \text{in } \Delta.$$

This shows $|\beta(a, b)|_\Delta$ divides $|a|_A$.

(iii) Use the standard identification of Abelian groups with $\mathbb{Z}$-(bi)modules. Set

$$\Phi(\beta)(a,b) = \beta((b,0),(0,a)) \quad \forall \beta \in N^2(\Lambda,\Delta), a \in A, b \in B,$$
$$\Theta(\beta)((b,0),(0,a)) = f(a,b) \quad \forall f \in \mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, \Delta), a \in A, b \in B.$$

Well-definedness and injectivity of Φ follow from (i)(ii) and universal property of tensor products. It is easy to verify that Θ is well-defined and $\Phi \circ \Theta$ is identity, so Φ is surjective as well. $\square$

Corollary 3.8. *Let $A \cong B \cong A_1 \times \cdots A_l$ in the elementary divisor form of finite Abelian groups, i.e. $|A_j| = p_j^{\alpha_j}$ for distinct primes p_j. Set $\Lambda_i = A_i \times A_i$. If $\Sigma = \Delta \rtimes_\beta \Lambda$ for some $\beta \in N^2(\Lambda,\Delta)$ then*

$$\Sigma \cong (\Delta_1 \rtimes_{\beta_1} \Lambda_1) \times \cdots \times (\Delta_l \rtimes_{\beta_l} \Lambda_l)$$
$$\Bbbk\Sigma \cong \Bbbk(\Delta_1 \rtimes_{\beta_1} \Lambda_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \Bbbk(\Delta_l \rtimes_{\beta_l} \Lambda_l)$$

for suitable Abelian groups Δ_i with $|\Delta_i| = p_i^{n_i}$ for some integers n_i and $\beta_i \in N^2(\Lambda_i,\Delta_i)$ defined by restrictions.

Remark 3.9. Let $\{e_1, \cdots, e_m\}$, $\{f_1, \cdots, f_n\}$ be the generators of A and B, respectively. Then $\beta \in N^2(\Lambda,\Delta)$ is uniquely determined by $T \in M_{m,n}(\Delta)$ with entries $t_{ij} = \beta(e_i, f_j)$ for $1 \le i \le m$, $1 \le j \le n$ such that $|t_{ij}|_\Delta$ divides $\gcd(|e_i|_A, |f_j|_B)$.

Let $n = |\Delta|$, C be a central subalgebra of $\Bbbk\Sigma$ containing $\Bbbk\Delta$, $\mathbf{m} \in \mathrm{MaxSpec}(C)$ and ξ be a primitive n-th root of unity in $\Bbbk$. Then the images of $t_{ij} \in \Delta$ above under the natural projection $C \to C/\mathbf{m}$ are $\phi(t_{ij}) = \xi^{s_{ij}}$ for some integers $0 \le s_{ij} < n$.

Remark 3.10. By the fundamental theorem of finitely generated Abelian groups

$$A \cong \mathbb{Z}/\tilde{l}_1\mathbb{Z} \times \cdots \mathbb{Z}/\tilde{l}_m\mathbb{Z} \times \mathbb{Z}^{r_1},$$
$$B \cong \mathbb{Z}/\hat{l}_1\mathbb{Z} \times \cdots \mathbb{Z}/\hat{l}_n\mathbb{Z} \times \mathbb{Z}^{r_2}$$

for some integers $\tilde{l}_i$, $\hat{l}_i$, r_1 and r_2. Let $\{e_1, \cdots, e_{m+r_1}\}$, $\{f_1, \cdots, f_{n+r_2}\}$ be the associated standard generators. Define

$$x_i = \gcd_{1 \le j \le n+r_2} |t_{ij}|_\Delta,$$
$$y_j = \gcd_{1 \le i \le m+r_1} |t_{ij}|_\Delta.$$

Then Δ and $\{e_1^{x_1}, \cdots, e_{m+r_1}^{x_{m+r_1}}\}$, $\{f_1^{y_1}, \cdots, f_{n+r_2}^{y_{n+r_2}}\}$ generate a central subgroup Ω of $\Sigma = \Delta \rtimes_\beta \Lambda$, denote $H = \Bbbk\Sigma$ and $C = \Bbbk\Omega$. Set

$$l = \prod_{i=1}^{m+r_1} x_i \prod_{j=1}^{n+r_2} y_j.$$

Then H is a free C-module of rank l and $\langle \mathrm{Im}(\beta) \rangle \subseteq \Delta \subseteq C$. Hence if $\mathrm{char}\,\Bbbk \notin [1,l]$, $(H, C, \mathrm{tr}_{\mathrm{reg}})$ is a Cayley-Hamilton Hopf algebra of degree l.

4. Discriminant Ideals and Orbit of Identity

This last section culminates in the proof of the Main Theorem, which describes the zero set of the lowest discriminant ideal and the orbit of $\mathbf{m}_{\bar{\epsilon}}$ under winding automorphisms or any $\Bbbk$-algebra automorphism of H that fixes C. A simple example is also given to showcase the complexity arising from different choices of $\mathbf{m} \in \mathrm{MaxSpec}(C)$ when A and B are not cyclic.

Lemma 4.1. *Let*

$$H = \frac{\Bbbk < x, y >}{(x^l - 1, y^m - 1, xy - \xi yx)}, \quad C = \Bbbk$$

where ξ is a primitive root of unity of order n and $n|\gcd(l,m)$, then

(i) *R has an irreducible representation V of dimension n;*
(ii) *The stabilizer of V is given by*

$$\begin{aligned}\mathrm{Stab}_{G_0}(V) =& \{\chi \in G(H^\circ) : \ \chi(x) = \xi^i, \, \chi(y) = \xi^j, \, i, j \in \mathbb{Z}\} \\ \cong& \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.\end{aligned}$$

Proof. (i)Let V be a vector space with basis $\{v_0, \cdots, v_{n-1}\}$. Define

$$x \cdot v_i = \xi^i v_i, \quad y \cdot v_i = v_j, \quad j = i + 1 \bmod n.$$

Then V is an n-dimensional representation of R.
(ii) [14, Section 5.1]. □

Lemma 4.2. *Let A and B be finitely generated Abelian groups, Δ be a finite cyclic group of order d, $\Lambda = A \times B$ and $\Sigma = \Delta \rtimes_\beta \Lambda$ for some $\beta \in \mathcal{B}$. Then in some minimal sets of generators of A and B the matrix T defined in Remark 3.9 is in the form*

$$\begin{bmatrix} t_{11} & & & & \\ & t_{22} & & 0 & \\ & & \ddots & & \\ & & & t_{kk} & \\ & 0 & & 0 & \\ & & & & \ddots \end{bmatrix}$$

for some $t_{11}|\cdots|t_{kk}|d$. Minimal means there is no proper subset of the generators that generate the group or the generators are $\mathbb{Z}$-linearly indepedent.

Proof. These can be derived from the Smith normal forms in the theory of finitely generated modules over PIDs [10, Theorem 3.8]. □

Corollary 4.3. *Under the same assumptions as in the lemma and let C be a central subalgebra of $\Bbbk\Sigma$ containing $\Bbbk\Delta$, $\mathbf{m} \in \mathrm{MaxSpec}(C)$. In some minimal or $\mathbb{Z}$-linearly independent sets of generators of A and B the matrix S defined in Remark 3.9 for*

$R/\mathbf{m}R$ is in the form

$$\begin{bmatrix} s_{11} & & & & & \\ & s_{22} & & & 0 & \\ & & \ddots & & & \\ & & & s_{kk} & & \\ & 0 & & & 0 & \\ & & & & & \ddots \end{bmatrix}$$

for some $s_{11}|\cdots|s_{kk}|n = |\Delta|$.

Proof. This can either be proved directly similar to Lemma 4.2 or one can use the fact that the quotient $R/\mathbf{m}R$ for $\mathbf{m} \in \operatorname{MaxSpec}(C)$ factors through the group algebra of $(\mathbb{Z}/n\mathbb{Z}) \rtimes_\gamma \Lambda$ for $\gamma \in Z^2(\Lambda, \mathbb{Z}/n\mathbb{Z})$ induced from the quotient. □

Proof of Main Theorem. By Lemma 3.5, we may assume $\Sigma \cong \Delta \rtimes_\beta \Lambda$ for some $\beta \in N^2(\Lambda, \Delta)$. The existence of Ω is shown in Remark 3.10.

(i) The GK-dimension is routine and $H = \Bbbk\Sigma$ is a free module over $C = \Bbbk\Omega$ of rank $m = [\Sigma : \Omega]$.

(ii) Corollary 4.3 implies (1.2). To shown $R/\mathbf{m}R$ is simple, define V_i as an irreducible representations of R_i with $\dim V_i = l_i$ and denote the left annihilators of V_i in R_i by $\operatorname{lann}_{R_i}(V_i)$, then

$$R/\mathbf{m}R \cong R_1/\operatorname{lann}_{R_1}(V_1) \otimes_\Bbbk \cdots \otimes_\Bbbk R_k/\operatorname{lann}_{R_k}(V_k) \cong R/\operatorname{lann}_R(V_1 \otimes_\Bbbk \cdots \otimes_\Bbbk V_k)$$

is simple as a tensor product of simple $\Bbbk$-algebras whose centers are $\Bbbk$ [19, Theorem 1.7.27]. The rest is similar to the discussion in [14, Section 5.1] using Lemma 4.1.

(iii) This is an immediate consequence of (ii).

(iv) It is easy to see from the invariant factor form of finitely generated modules over PID that $G_0 \cong \Sigma/\Omega$ and $|G_0| = m$.

(v) (1.5) is the same as [14, Theorem 4.3(a)]. By Corollary 4.3, $\mathbf{m} \in \mathcal{O} \iff$ all s_{ij} defined in Remark 3.9 are zero $\iff \operatorname{Im}(\Phi(\beta)) = \langle \operatorname{Im}(\beta) \rangle \subset \mathbf{m}C$ for Φ defined in the proof of Lemma 3.7(iii). □

Remark 4.4.

(i) For studying representations, one can take $\Delta = \langle \operatorname{Im}(\beta) \rangle$ w.l.o.g. and then $\mathcal{O} \cong \Omega/\Delta$.

(ii) In general $G(H^\circ)$ does not act transitively on irreducible representations of higher dimensions by

$$\chi \cdot V = \chi \otimes V, \quad \chi \in G(H^\circ),\ V \in \operatorname{Irr}(H),$$

e.g. when any of l_i in (1.3) is greater than 2. This can be seen as follows: Denote the corresponding irreducible representation as V and let $c = \operatorname{ann}_C(V)$, then $c \in \operatorname{MaxSpec}(C)$ and any irreducible representation over c^{-1} has the same dimension. However any irrreducible module over c^2 has dimension greater than one since by Lemma 4.1 c^2 does not have basic fiber [14, Theorem C(a)].

Example 4.5. Let $A = B = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, $\Delta = (\mathbb{Z}/2\mathbb{Z})^2$ and $\Lambda = A \times B$. Denote the standard generators as $a_1, a_2 \in A$, $b_1, b_2 \in B$, $c_1, c_2 \in \Delta$. Let $\beta \in N^2(\Lambda, \Delta)$ such that the associated matrix defined in Remark 3.9 is

$$\begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}.$$

Case	$\begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}$	$\{e_1, e_2\}$	$\{f_1, f_2\}$	$\begin{bmatrix} s_{11} & 0 \\ 0 & \ddots \end{bmatrix}$
I	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\{a_1, a_2\}$	$\{b_1, b_2\}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
II	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\{a_1, a_2\}$	$\{b_1, b_2\}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
III	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\{a_1, a_2\}$	$\{b_2, b_1\}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
IV	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\{a_1, a_2\}$	$\{b_1, b_2 b_1\}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

TABLE 1. Bases for matrices in invariant form

Choose $\Omega = \langle c_1, c_2, a_2, b_2^2 \rangle$ and $\Bbbk = \mathbb{C}$ then $\operatorname{MaxSpec}(C) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{G}_{\mathrm{m}}^2)$ where $\mathbb{G}_{\mathrm{m}} = (\Bbbk, *)$, more concretely define $b \coloneqq b_2^2$ then

$$\begin{aligned} \operatorname{MaxSpec}(C) = \{ \mathbf{m}_{u,v,w,x} = \langle c_1 - u, c_2 - v, a_2 - w, b - x \rangle : \\ u, v \in \{\pm 1\}; w,\, x \in \Bbbk \setminus \{0\} \}. \end{aligned}$$

Characters over $\mathbf{m}_{\bar{\epsilon}}$ are

$$\begin{aligned} G_0 &= \{ \chi \in G(H^\circ) : \chi(c_1) = \chi(c_2) = \chi(a_2) = 1;\ \chi(a_1), \chi(b_1), \chi(b_2) \in \pm 1 \} \\ &\cong \Sigma/\Omega \cong (\mathbb{Z}/2\mathbb{Z})^3. \end{aligned}$$

And (1.4) becomes

$$\mathcal{V}(D_k(H/C, \mathrm{tr}_{\mathrm{reg}})) = \begin{cases} \varnothing, & k \leq 8, \\ \operatorname{MaxSpec}(C) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{G}_{\mathrm{m}}^2, & k > 8. \end{cases}$$

H is a free C-module of rank 8 with basis

$$\{ a_1^i b_1^j b_2^k : i, j, k \in \{0, 1\} \}.$$

Pick $\mathbf{m}_{u,v,w,x} \in \operatorname{MaxSpec}(C)$, then

$$H/(\mathbf{m}_{u,v,w,x} H) \cong \frac{\Bbbk\langle X, Y, Z^{\pm 1} \rangle}{(XY - uYX, XZ - vZX, Z^2 - x)}.$$

There are four cases depending on the choice of u and v. The bases for matrices S in invariant factor form are given in (Table 1). Thus

$$\begin{aligned} \mathcal{O} &= \operatorname{Aut}_{\Bbbk-\mathrm{Alg}}(H, C) \cdot \mathbf{m}_{\bar{\epsilon}} = \mathrm{W}_{\mathrm{l}}(G(H^\circ)) \cdot \mathbf{m}_{\bar{\epsilon}} = \mathrm{W}_{\mathrm{r}}(G(H^\circ)) \cdot \mathbf{m}_{\bar{\epsilon}} \\ &= G(C^\circ) = \{ \mathbf{m}_{1,1,w,x} : w,\, x \in \mathbb{G}_{\mathrm{m}} \} \cong \mathbb{G}_{\mathrm{m}}^2. \end{aligned}$$

Choose $\mathbf{m} = \mathbf{m}_{-1,-1,w,x}$ in case (iv), the irreducible modules are two dimensional. In some basis $\{v_1, v_2\}$ the matrices

$$c_1 = c_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, a_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, a_2 = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}, b_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$b_2 = \begin{bmatrix} 0 & \sqrt{x} \\ \sqrt{x} & 0 \end{bmatrix};$$

where $\sqrt{x}$ is some square root of x, is an irreducible module V of H over $\mathbf{m}$. The stabilzers are given by

$$\begin{aligned} \mathrm{Stab}_{G_0}(V) = \{\chi \in G(H^\circ) : & \chi(c_1) = \chi(c_2) = \chi(a_2) = 1; \\ & \chi(a_1), \chi(b_1) = -\chi(b_2) \in \{\pm 1\}\}. \end{aligned}$$

References

[1] J. Bell and J. J. Zhang, *Zariski cancellation problem for noncommutative algebras*, Selecta Math. (N.S.) **23** (2017), 1709–1737.

[2] K. A. Brown and M. T. Yakimov, *Azumaya loci and discriminant ideals of PI algebras*, Adv. Math **340** (2018), 1219–1255.

[3] K. A. Brown and J. J. Zhang, *Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras*, J. Algebra **320** (2008), no. 5, 1814–1850.

[4] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant controls automorphism groups of noncommutative algebras*, Adv. Math. **269** (2015), 551–584.

[5] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant criterion and automorphism groups of quantized algebras*, Adv. Math. **286** (2016), 754–801.

[6] K. Chan, J. Gaddis, R. Won, and J. J. Zhang, *Reflexive hull discriminants and applications*, Selecta Math. (NS.) **28** (2022), no. 2, 40.

[7] C. De Concini and C. Procesi, *Quantum groups*, Lec. Notes in Math., vol. 1565, pp. 31–140, Springer, Boston, MA, 1993.

[8] C. De Concini, C. Procesi, N. Reshetikhin, and M. Rosso, *Hopf algebras with trace and representations*, Invent. Math. **16** (2005), 1–44.

[9] J. Gaddis, E. Kirkman, and W.F. Moore, *On the discriminants of twisted tensor products*, J. Algebra **477** (2017), 29–55.

[10] N. Jacobson, *Basic algebra*, 2nd ed., vol. I, Dover Publications, Mineola, NY, 2009.

[11] J. Levitt and M. Yakimov, *Quantized Weyl algebras at roots of unity*, Israel J. Math. **225** (2018), 681–719.

[12] D.-M. Lu, Q.-S. Wu, and J. J. Zhang, *Homological integral of Hopf algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4945–4975.

[13] E.M. Mendonca, *Projective representations of groups*, `https://alistairsavage.ca/pubs/Mendonca-Projective_Representation_of_Groups.pdf`, accessed: July-11-2023.

[14] Z. Mi, Q.-S. Wu, and M. Yakimov, *The lowest discriminant ideal of a Cayley-Hamilton Hopf algebra*, preprint arXiv:2307.15477[math.RT], 2023.

[15] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conf. Ser. Math., no. 82, Amer. Math. Soc., Providence, RI, 1993.

[16] B. Nguyen, K. Trampel, and M. Yakimov, *Root of unity quantum cluster algebras and discriminants*, preprint arXiv:2012.02314[math.QT].

[17] B. Nguyen, K. Trampel, and M. Yakimov, *Noncommutative discriminants via Poisson primes*, Adv. Math. **322** (2017), 269–307.

[18] C. Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Algebra **107** (1987), 63–74.

[19] L.H. Rowen, *Ring theory*, Pure and applies mathematics, vol. I, Academic Press, 1988.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

Email address: `zmi1@lsu.edu`