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Dispersionless version of the multicomponent KP hierarchy revisited

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Abstract

We revisit dispersionless version of the multicomponent KP hierarchy considered previously by Takasaki and Takebe. In contrast to their study, we do not fix any distinguished component treating all of them on equal footing. We obtain nonlinear equations for dispersionless tau-function (the F-function) and represent them using the trigonometric parametrization. In this trigonometric uniformization the equations considerably simplify and acquire a nice form.

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1 Introduction

The multicomponent KP hierarchy was introduced in the paper [1] in 1981. Since then it was considered in the works [2, 3, 4]. Recently, some generalization of it was suggested in [5] under the name of the universal hierarchy. (The difference of the approach in [5] with the previous works is that the integer variables of the hierarchy are allowed to take arbitrary complex values, with the equations in these variables being still difference.)

In this note we are mainly interested in the dispersionless limit of the hierarchy, tending the small "dispersion parameter" $\hbar$ to zero. The general approach to dispersionless limits of integrable hierarchies was developed in [6] by Takasaki and Takebe. In [3], they analyzed the dispersionless limit of the multicomponent KP hierarchy. In their approach, one of the components was distinguished. We find it appropriate to revisit this problem. In our approach, all components are treated on equal footing. We obtain nonlinear equations for the dispersionless limit of (logarithm of) tau-function (the F-function) which follow from the bilinear equations of the Hirota-Miwa type.

The main result of this paper is a new form of the dispersionless equations for the F-function which emerges after a trigonometric parametrization. In this form, the equations considerably simplify and some of them become equivalent, so the number of independent equations reduces (basically, only one main equation is left).

The paper is organized as follows. In section 2 we present the generating bilinear integral equation for the tau-function and derive different equations of the Hirota-Miwa type as corollaries of it. Section 3 is devoted to the dispersionless limit. We obtain the dispersionless version of all the Hirota-Miwa equations from the previous section. In section 4 we suggest the trigonometric parametrization and rewrite all the equations obtained in the previous section in the nice trigonometric form. Section 5 contains concluding remarks.

2 Bilinear equations for the tau-function

Let us start from the n-component KP hierarchy extended by certain additional integrable flows [1, 2, 3, 4]. In [5] it is called the universal hierarchy. The independent variables are n infinite sets of (in general complex) "times"

$$\mathbf{t} = \{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n\}, \qquad \mathbf{t}_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \dots\}, \qquad \alpha = 1, \dots, n$$

and n additional variables $s_1, \dots, s_n$ such that

$$\sum_{\alpha=1}^{n} s_\alpha = 0. \tag{2.1}$$

By $\mathbf{s}$ we denote the vector $\mathbf{s} = \{s_1, \dots, s_n\}$ and by $\mathbf{e}_\alpha$ the vector whose α's component is 1 and all other components 0. We will also use the following standard notation:

$$\Big(\mathbf{t} \pm [z^{-1}]_\gamma\Big)_{\alpha j} = t_{\alpha,j} \pm \delta_{\alpha\gamma}\frac{z^{-j}}{j}, \tag{2.2}$$

$$\xi(\mathbf{t}_\alpha, z) = \sum_{j\geq 1} t_{\alpha,j} z^j. \tag{2.3}$$

In general we treat $s_1, \ldots, s_n$ as complex variables, as in [5]. If they are restricted to be integers, the hierarchy coincides with the one considered in [1]–[4].

In the bilinear formalism, the dependent variable is the tau-function $\tau(\mathbf{s}, \mathbf{t})$ The universal hierarchy is the infinite set of bilinear equations for the tau-function which are encoded in the basic bilinear relation [1, 4]

$$\begin{aligned} &\sum_{\gamma=1}^{n} \epsilon_{\alpha\gamma}(\mathbf{s})\epsilon_{\beta\gamma}^{-1}(\mathbf{s}') \oint_{C_\infty} dz\, z^{s_\gamma - s'_\gamma + \delta_{\alpha\gamma} + \delta_{\beta\gamma} - 2} e^{\xi(\mathbf{t}_\gamma - \mathbf{t}'_\gamma,\, z)} \\ &\qquad \times \tau\left(\mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\gamma, \mathbf{t} - [z^{-1}]_\gamma\right) \tau\left(\mathbf{s}' + \mathbf{e}_\gamma - \mathbf{e}_\beta, \mathbf{t}' + [z^{-1}]_\gamma\right) = 0 \end{aligned} \tag{2.4}$$

valid for any $\mathbf{t}$, $\mathbf{t}'$, $\mathbf{s}$, $\mathbf{s}'$ such that $\mathbf{s} - \mathbf{s}' \in \mathbb{Z}^n$ (and subject to the constraint (2.1)). In (2.4)

$$\epsilon_{\alpha\gamma}(\mathbf{s}) = \begin{cases} \exp\Big(-i\pi \sum\limits_{\alpha<\mu\leq\gamma} s_\mu\Big), & \alpha < \gamma \\ \quad 1, & \alpha = \gamma \\ -\exp\Big(i\pi \sum\limits_{\gamma<\mu\leq\alpha} s_\mu\Big), & \alpha > \gamma \end{cases} \tag{2.5}$$

(see [5]). The contour C_∞ is a big circle around infinity. It is easy to see that for $\mathbf{s}-\mathbf{s}' \in \mathbb{Z}^n$ the factor $\epsilon_{\alpha\gamma}(\mathbf{s})\epsilon_{\beta\gamma}^{-1}(\mathbf{s}')$ multiplied by $\epsilon_{\alpha\beta}(\mathbf{s})$ is just a sign factor ± 1 depending only on $\mathbf{s} - \mathbf{s}'$.

Different bilinear relations for the tau-function of the Hirota-Miwa type which follow from (2.4) for special choices of $\mathbf{s} - \mathbf{s}'$ and $\mathbf{t} - \mathbf{t}'$ are given in [4]. We present some of them below, with a sketch of the derivation.

First of all we consider (2.4) at $\alpha = \beta$, differentiate it with respect to $t_{\alpha,1}$ and put $\mathbf{s}' = \mathbf{s}$, $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\alpha$, so that

$$e^{\xi(\mathbf{t}_\alpha - \mathbf{t}'_\alpha, z)} = \frac{a\,b}{(a-z)(b-z)}, \quad e^{\xi(\mathbf{t}_\gamma - \mathbf{t}'_\gamma, z)} = 1 \quad \text{for } \gamma \neq \alpha.$$

We get

$$\begin{aligned} &\oint_{C_\infty} dz \frac{abz}{(a-z)(b-z)}\, \tau(\mathbf{s}, \mathbf{t} - [z^{-1}]_\alpha)\tau(\mathbf{s}, \mathbf{t} - [a^{-1}]_\alpha - [b^{-1}]_\alpha + [z^{-1}]_\alpha) \\ &+ \oint_{C_\infty} dz \frac{ab}{(a-z)(b-z)}\, \partial_{t_{\alpha,1}} \tau(\mathbf{s}, \mathbf{t} - [z^{-1}]_\alpha)\tau(\mathbf{s}, \mathbf{t} - [a^{-1}]_\alpha - [b^{-1}]_\alpha + [z^{-1}]_\alpha) = 0. \end{aligned}$$

The residue calculus followed by some simple transformations yields the equation

$$\frac{\tau(\mathbf{s}, \mathbf{t})\tau(\mathbf{s}, \mathbf{t} + [a^{-1}]_\alpha + [b^{-1}]_\alpha)}{\tau(\mathbf{s}, \mathbf{t} + [a^{-1}]_\alpha)\tau(\mathbf{s}, \mathbf{t} + [b^{-1}]_\alpha)} = 1 - \frac{1}{a-b}\, \partial_{t_{\alpha,1}} \log \frac{\tau(\mathbf{s}, \mathbf{t} + [a^{-1}]_\alpha)}{\tau(\mathbf{s}, \mathbf{t} + [b^{-1}]_\alpha)}. \tag{2.6}$$

Now consider (2.4) with $\alpha = \beta$ and $\mathbf{s}' = \mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\mu$ (with $\mu \neq \alpha$) and $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\alpha$. The residue calculus yields:

$$\begin{aligned} &\frac{\tau(\mathbf{s}, \mathbf{t})\tau(\mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\beta, \mathbf{t} + [a^{-1}]_\alpha + [b^{-1}]_\alpha)}{\tau(\mathbf{s}, \mathbf{t} + [a^{-1}]_\alpha)\tau(\mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\beta, \mathbf{t} + [b^{-1}]_\alpha)} \\ &= \frac{1}{a-b}\left(a \frac{\tau(\mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\beta, \mathbf{t} + [b^{-1}]_\alpha)}{\tau(\mathbf{s}, \mathbf{t} + [b^{-1}]_\alpha)} - b \frac{\tau(\mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\beta, \mathbf{t} + [a^{-1}]_\alpha)}{\tau(\mathbf{s}, \mathbf{t} + [a^{-1}]_\alpha)}\right), \end{aligned} \tag{2.7}$$

where we have changed $\mu \to \beta$. The next choice in (2.4) with $\alpha = \beta$ is $\mathbf{s}' = \mathbf{s}$ and $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\mu$ $(\mu \neq \alpha)$. The residue calculus yields the equation

$$\begin{aligned} &\tau(\mathbf{s},\mathbf{t})\tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha+[b^{-1}]_\beta) - \tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha)\tau(\mathbf{s},\mathbf{t}+[b^{-1}]_\beta) \\ &= (ab)^{-1}\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)\tau(\mathbf{s}+\mathbf{e}_\beta-\mathbf{e}_\alpha,\mathbf{t}+[b^{-1}]_\beta), \end{aligned} \tag{2.8}$$

where we again have changed $\mu \to \beta$.

Let us now consider the case $\alpha \neq \beta$ in (2.4). First, we apply $\partial_{t_{\alpha,1}}$ and put $\mathbf{s}' = \mathbf{s}$, $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\beta$ after that. The residue calculus yields:

$$a\frac{\tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha+[b^{-1}]_\beta))\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t})}{\tau(\mathbf{s},\mathbf{t}+[b^{-1}]_\beta))\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)} = a + \partial_{t_{\alpha,1}}\log\frac{\tau(\mathbf{s},\mathbf{t}+[b^{-1}]_\beta)}{\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)}. \tag{2.9}$$

The next choice is $\mathbf{s}' = \mathbf{s}$, $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\mu$ with $\mu \neq \alpha, \beta$. In this case equation (2.4) yields:

$$\begin{aligned} &\tau(\mathbf{s},\mathbf{t})\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\mu) - \tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\mu)\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}) \\ &+ \epsilon_{\alpha\beta}\epsilon_{\alpha\mu}\epsilon_{\beta\mu}a^{-1}\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\mu,\mathbf{t})\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\mu) = 0, \end{aligned} \tag{2.10}$$

where $\epsilon_{\alpha\beta} = 1$ at $\alpha \leq \beta$ and $\epsilon_{\alpha\beta} = -1$ at $\alpha > \beta$. Finally, we put $\mathbf{s}' = \mathbf{s} + \mathbf{e}_\alpha - \mathbf{e}_\beta$, $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\alpha$ in (2.4). The residue calculus yields:

$$\begin{aligned} &\frac{a}{b}\,\frac{\tau(\mathbf{s}-\mathbf{e}_\alpha+\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[b^{-1}]_\alpha)}{\tau(\mathbf{s},\mathbf{t})\tau(\mathbf{s},\mathbf{t})+[a^{-1}]_\alpha+[b^{-1}]_\alpha)} \\ &\qquad - \frac{b}{a}\,\frac{\tau(\mathbf{s}-\mathbf{e}_\alpha+\mathbf{e}_\beta,\mathbf{t}+[b^{-1}]_\alpha)\tau(\mathbf{s}+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)}{\tau(\mathbf{s},\mathbf{t})\tau(\mathbf{s},\mathbf{t})+[a^{-1}]_\alpha+[b^{-1}]_\alpha)} \\ &= (a-b)\partial_{t_{\beta,1}}\log\frac{\tau(\mathbf{s},\mathbf{t})+[a^{-1}]_\alpha+[b^{-1}]_\alpha)}{\tau(\mathbf{s},\mathbf{t})}. \end{aligned} \tag{2.11}$$

Another equation can be obtained in the following way. Put $\alpha = \beta = \mu$ in (2.4):

$$\begin{aligned} &\sum_{\gamma=1}^{n}\epsilon_{\mu\gamma}(\mathbf{s})\epsilon^{-1}_{\mu\gamma}(\mathbf{s}')\oint_{C_\infty}dz\, z^{s_\gamma-s'_\gamma+2\delta_{\mu\gamma}-2}e^{\xi(\mathbf{t}_\gamma-\mathbf{t}'_\gamma,\,z)} \\ &\qquad \times\tau\left(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\gamma,\mathbf{t}-[z^{-1}]_\gamma\right)\tau\left(\mathbf{s}'+\mathbf{e}_\gamma-\mathbf{e}_\mu,\mathbf{t}'+[z^{-1}]_\gamma\right) = 0, \end{aligned}$$

and put here $\mathbf{s} - \mathbf{s}' = \mathbf{e}_\alpha + \mathbf{e}_\beta - 2\mathbf{e}_\mu$, $\mathbf{t} - \mathbf{t}' = [a^{-1}]_\alpha + [b^{-1}]_\beta$ with $\alpha \neq \beta$ and $\mu \neq \alpha, \beta$. The residue calculus yields:

$$\begin{aligned} &\epsilon_{\mu\alpha}(\mathbf{s})\epsilon^{-1}_{\mu\alpha}(\mathbf{s}+2\mathbf{e}_\mu-\mathbf{e}_\alpha-\mathbf{e}_\beta)\partial_{t_{\mu,1}}\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\alpha,\mathbf{t}+[b^{-1}]_\beta)\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha) \\ &+\epsilon_{\mu\beta}(\mathbf{s})\epsilon^{-1}_{\mu\beta}(\mathbf{s}+2\mathbf{e}_\mu-\mathbf{e}_\alpha-\mathbf{e}_\beta)\partial_{t_{\mu,1}}\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\beta,\mathbf{t}+[a^{-1}]_\alpha)\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\alpha,\mathbf{t}+[b^{-1}]_\beta) \\ &+\tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha+[b^{-1}]_\beta)\tau(\mathbf{s}+2\mathbf{e}_\mu-\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}) = 0. \end{aligned}$$

The case-by-case inspection shows that

$$\epsilon_{\mu\alpha}(\mathbf{s})\epsilon^{-1}_{\mu\alpha}(\mathbf{s}+2\mathbf{e}_\mu-\mathbf{e}_\alpha-\mathbf{e}_\beta) = -\epsilon_{\mu\beta}(\mathbf{s})\epsilon^{-1}_{\mu\beta}(\mathbf{s}+2\mathbf{e}_\mu-\mathbf{e}_\alpha-\mathbf{e}_\beta) = -\epsilon_{\alpha\beta}\epsilon_{\alpha\mu}\epsilon_{\beta\mu}.$$

Therefore, the equation finally reads:

$$
\begin{aligned}
\epsilon_{\alpha\beta}\frac{\tau(\mathbf{s}+\mathbf{e}_\beta-\mathbf{e}_\mu,\mathbf{t}+[a^{-1}]_\alpha+[b^{-1}]_\beta)\tau(\mathbf{s}+\mathbf{e}_\mu-\mathbf{e}_\alpha,\mathbf{t})}{\tau(\mathbf{s}+\mathbf{e}_\beta-\mathbf{e}_\alpha,\mathbf{t}+[b^{-1}]_\beta)\tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha)}& \\
=\epsilon_{\alpha\mu}\epsilon_{\beta\mu}\partial_{t_{\mu,1}}\log\frac{\tau(\mathbf{s}+\mathbf{e}_\beta-\mathbf{e}_\alpha,\mathbf{t}+[b^{-1}]_\beta)}{\tau(\mathbf{s},\mathbf{t}+[a^{-1}]_\alpha)}&.
\end{aligned}
\tag{2.12}
$$

3 The dispersionless limit

In order to perform the dispersionless limit [6], one should introduce a small parameter $\hbar$ and rescale the times $\mathbf{t}$ and variables $\mathbf{s}$ as $t_{\alpha,k}\to t_{\alpha,k}/\hbar$, $s_\alpha\to s_\alpha/\hbar$. Introduce a function $F(\mathbf{s},\mathbf{t};\hbar)$ related to the tau-function by the formula

$$
\tau(\mathbf{s}/\hbar,\mathbf{t}/\hbar)=\exp\Big(\frac{1}{\hbar^2}\,F(\mathbf{s},\mathbf{t};\hbar)\Big) \tag{3.1}
$$

and consider the limit $F=\lim\limits_{\hbar\to 0}F(\mathbf{s},\mathbf{t};\hbar)$. The function F represents the tau-function in the dispersionless limit $\hbar\to 0$. It satisfies an infinite number of nonlinear differential equations which follow from the bilinear equations for the tau-function. In the dispersionless limit the difference equations in the variables s_α become differential, and the derivative ∂_{s_α} will be denoted simply as ∂_α. Introduce also the differential operators

$$
D_\alpha(z)=\sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_{t_{\alpha,k}}, \tag{3.2}
$$

so that

$$
\tau(\mathbf{s}/\hbar,\mathbf{t}/\hbar+[z^{-1}]_\alpha)=\exp\Big(\frac{1}{\hbar^2}\,e^{\hbar D_\alpha(z)}F(\mathbf{s},\mathbf{t};\hbar)\Big)
$$

and

$$
\tau(\mathbf{s}/\hbar+\mathbf{e}_\alpha-\mathbf{e}_\beta,\mathbf{t}/\hbar)=\exp\Big(\frac{1}{\hbar^2}\,e^{\hbar\partial_\alpha-\hbar\partial_\beta}F(\mathbf{s},\mathbf{t};\hbar)\Big).
$$

Let us obtain dispersionless versions of equations (2.6)–(2.11). Equation (2.6) can be rewritten as

$$
\exp\Big(\frac{1}{\hbar^2}\Big(1+e^{\hbar D_\alpha(a)+\hbar D_\alpha(b)}-e^{\hbar D_\alpha(a)}-e^{\hbar D_\alpha(b)}\Big)F\Big)=1-\frac{\hbar^{-1}}{a-b}\,\partial_{t_{\alpha,1}}\Big(e^{\hbar D_\alpha(a)}-e^{\hbar D_\alpha(b)}\Big)F.
$$

In this form it is ready for the dispersionless limit $\hbar\to 0$ which yields:

$$
e^{D_\alpha(a)D_\alpha(b)F}=1-\frac{D_\alpha(a)\partial_{t_{\alpha,1}}F-D_\alpha(b)\partial_{t_{\alpha,1}}F}{a-b}. \tag{3.3}
$$

The limits of the other equations can be found in a similar way. They are as follows. The limit of equation (2.7):

$$
e^{D_\alpha(a)D_\alpha(b)F}=\frac{ae^{-D_\alpha(a)(\partial_\alpha-\partial_\beta)F}-be^{-D_\alpha(b)(\partial_\alpha-\partial_\beta)F}}{a-b}. \tag{3.4}
$$

The limit of equation (2.8):

$$e^{D_\alpha(a)D_\beta(b)F} = 1 + \frac{1}{ab}\, e^{(D_\alpha(a)-D_\beta(b))(\partial_\alpha-\partial_\beta)F+(\partial_\alpha-\partial_\beta)^2F}. \tag{3.5}$$

The limit of equation (2.9):

$$ae^{D_\alpha(a)D_\beta(b)F-D_\alpha(a)(\partial_\alpha-\partial_\beta)F} = a-(\partial_\alpha-\partial_\beta+D_\alpha(a)-D_\beta(b))\partial_{t_{\alpha,1}}F. \tag{3.6}$$

The limit of equation (2.10):

$$e^{D_\mu(a)(\partial_\alpha-\partial_\beta)F}-1+\epsilon_{\alpha\beta}\epsilon_{\alpha\mu}\epsilon_{\beta\mu}a^{-1}e^{D_\mu(a)(\partial_\mu-\partial_\beta)F+(\partial_\mu-\partial_\alpha)(\partial_\mu-\partial_\beta)F}=0. \tag{3.7}$$

The limit of equation (2.11):

$$\begin{aligned} &e^{-D_\alpha(a)D_\alpha(b)F+(\partial_\alpha-\partial_\beta)^2F}\Big(\frac{a}{b}e^{-(D_\alpha(a)-D_\alpha(b))(\partial_\alpha-\partial_\beta)F}-\frac{b}{a}e^{(D_\alpha(a)-D_\alpha(b))(\partial_\alpha-\partial_\beta)F}\Big)\\ &\qquad=(a-b)(D_\alpha(a)+D_\alpha(b))\partial_{t_{\beta,1}}F. \end{aligned} \tag{3.8}$$

The limit of equation (2.12):

$$\epsilon_{\alpha\beta}e^{(\partial_\alpha-\partial_\mu+D_\alpha(a))(\partial_\beta-\partial_\mu+D_\beta(b))F}=\epsilon_{\alpha\mu}\epsilon_{\beta\mu}\Big(\partial_\beta+D_\beta(b)-\partial_\alpha-D_\alpha(a)\Big)\partial_{t_{\mu,1}}F. \tag{3.9}$$

As is proved in [3], these equations are equivalent to the universal Whithem hierarchy for genus zero.

Let us rewrite these equations in a more suggestive form. For this purpose, we introduce the notation

$$R_\alpha = e^{\partial_\alpha^2 F}, \qquad R_{\alpha\beta}=R_{\beta\alpha}=e^{\partial_\alpha\partial_\beta F} \tag{3.10}$$

and the functions

$$\begin{aligned} w_\alpha(z) &= ze^{-D_\alpha(z)\partial_\alpha F-\partial_\alpha^2F},\\ w_{\alpha\beta}(z) &= e^{-D_\alpha(z)\partial_\beta F-\partial_\alpha\partial_\beta F}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} p_\alpha(z) &= z-(\partial_\alpha+D_\alpha(z))\partial_{t_{\alpha,1}}F,\\ p_{\alpha\beta}(z) &= -(\partial_\alpha+D_\alpha(z))\partial_{t_{\beta,1}}F \end{aligned} \tag{3.12}$$

(note that $R_{\alpha\alpha}=R_\alpha$ but $w_{\alpha\alpha}(z)\neq w_\alpha(z)$, $p_{\alpha\alpha}(z)\neq p_\alpha(z)$). In this notation, the equations (3.3)–(3.8) read as follows:

$$e^{D_\alpha(a)D_\alpha(b)F}=\frac{p_\alpha(a)-p_\alpha(b)}{a-b}, \tag{3.13}$$

$$e^{D_\alpha(a)D_\alpha(b)F}=\frac{R_\alpha}{R_{\alpha\beta}}\,\frac{w_\alpha(a)w_{\alpha\beta}^{-1}(a)-w_\alpha(b)w_{\alpha\beta}^{-1}(b)}{a-b}, \tag{3.14}$$

$$e^{D_\alpha(a)D_\beta(b)F}=1+\frac{w_{\alpha\beta}(a)w_{\beta\alpha}(b)}{w_\alpha(a)w_\beta(b)}, \tag{3.15}$$

$$\frac{R_\alpha}{R_{\alpha\beta}}\,\frac{w_\alpha(a)}{w_{\alpha\beta}(a)}\,e^{D_\alpha(a)D_\beta(b)F} = p_\alpha(a) - p_{\beta\alpha}(b), \tag{3.16}$$

$$R_{\mu\alpha}^{-1} w_{\mu\alpha}^{-1}(a) - R_{\mu\beta}^{-1} w_{\mu\beta}^{-1}(a) + \epsilon_{\alpha\beta}\epsilon_{\alpha\mu}\epsilon_{\beta\mu} R_{\alpha\beta} R_{\mu\alpha}^{-1} R_{\mu\beta}^{-1} w_\mu^{-1}(a) = 0, \tag{3.17}$$

$$\begin{aligned}\frac{R_\alpha R_\beta}{R_{\alpha\beta}^2}\left(\frac{w_\alpha(a)w_{\alpha\beta}(b)}{w_\alpha(b)w_{\alpha\beta}(a)} - \frac{w_\alpha(b)w_{\alpha\beta}(a)}{w_\alpha(a)w_{\alpha\beta}(b)}\right) e^{-D_\alpha(a)D_\alpha(b)F} \\ = -(a-b)(p_{\alpha\beta}(a) + p_{\alpha\beta}(b) - 2p_{\alpha\beta}(\infty)),\end{aligned} \tag{3.18}$$

$$\epsilon_{\alpha\beta}\frac{R_\mu}{R_{\alpha\beta}}\,\frac{w_{\alpha\mu}(a)w_{\beta\mu}(b)}{w_{\alpha\beta}(a)w_{\beta\alpha}(b)}\,e^{D_\alpha(a)D_\beta(b)F} = \epsilon_{\alpha\mu}\epsilon_{\beta\mu}(p_{\alpha\mu}(a) - p_{\beta\mu}(b)). \tag{3.19}$$

In the next section we show that these seven equations can be reduced to one.

4 The dispersionless equations in trigonometric form

Dividing equations (3.13), (3.14), we get the relation

$$p_\alpha(a) - \frac{R_\alpha w_\alpha(a)}{R_{\alpha\beta} w_{\alpha\beta}(a)} = p_\alpha(b) - \frac{R_\alpha w_\alpha(b)}{R_{\alpha\beta} w_{\alpha\beta}(b)},$$

from which it follows that $p_\alpha(z) - \dfrac{R_\alpha w_\alpha(z)}{R_{\alpha\beta} w_{\alpha\beta}(z)}$ does not depend on z. The limit $z \to \infty$ yields:

$$p_\alpha(z) - \frac{R_\alpha w_\alpha(z)}{R_{\alpha\beta} w_{\alpha\beta}(z)} = p_{\beta\alpha}(\infty). \tag{4.1}$$

Next, multiply equations (3.13) and (3.18) and express w_α through p_α in the left hand side with the help of (4.1). This yields the relation

$$R_{\alpha\beta}^2(p_{\alpha\beta}(a) - p_{\alpha\beta}(\infty)) + \frac{R_\alpha R_\beta}{p_\alpha(a) - p_{\beta\alpha}(\infty)} = -R_{\alpha\beta}^2(p_{\alpha\beta}(b) - p_{\alpha\beta}(\infty)) - \frac{R_\alpha R_\beta}{p_\alpha(b) - p_{\beta\alpha}(\infty)},$$

from which it follows that

$$R_{\alpha\beta}^2 p_{\alpha\beta}(z) + \frac{R_\alpha R_\beta}{p_\alpha(z) - p_{\beta\alpha}(\infty)} = R_{\alpha\beta}^2 p_{\alpha\beta}(\infty). \tag{4.2}$$

This equation, connecting $p_{\alpha\beta}$ and p_α, defines a rational curve.

It is natural to uniformize this curve using trigonometric functions. For this purpose, we introduce a function $u_\alpha(z)$ normalized so that $u_\alpha(\infty) = 0$ with expansion around ∞ of the form

$$u_\alpha(z) = \frac{c_{\alpha,1}}{z} + \sum_{k\geq 2}\frac{c_{\alpha,k}}{z^k}, \tag{4.3}$$

with the coefficients depending on the times. In terms of this function the uniformization reads:

$$
\begin{aligned}
p_\alpha(z) &= \gamma_\alpha \frac{\cos u_\alpha(z)}{\sin u_\alpha(z)}, \\
p_{\alpha\beta}(z) &= \gamma_\beta \frac{\cos(u_\alpha(z) + \eta_{\alpha\beta})}{\sin(u_\alpha(z) + \eta_{\alpha\beta})}, \\
R_\alpha &= \gamma_\alpha, \quad R_{\alpha\beta} = \sin \eta_{\alpha\beta}.
\end{aligned} \tag{4.4}
$$

Here $\gamma_\alpha = \gamma_\alpha(\mathbf{t})$, $\eta_{\alpha\beta} = \eta_{\alpha\beta}(\mathbf{t})$ are some functions of the times. Note that it should be $R_{\alpha\beta} = R_{\beta\alpha}$ but, as we shall see later, the assumption that $\eta_{\alpha\beta} = \eta_{\beta\alpha}$ is wrong. Instead,

$$\eta_{\beta\alpha} = \pi - \eta_{\alpha\beta}. \tag{4.5}$$

With this relation, substitution of (4.4) into (4.2) brings the latter to identity.

Plugging (4.4) into (4.1), one finds:

$$
\begin{aligned}
w_\alpha(z) &= \frac{1}{\sin u_\alpha(z)}, \\
w_{\alpha\beta}(z) &= \frac{1}{\sin(u_\alpha(z) + \eta_{\alpha\beta})}.
\end{aligned} \tag{4.6}
$$

Tending $z \to \infty$ in the first of equations (4.4), we get the relation

$$\gamma_\alpha(\mathbf{t}) = c_{\alpha,1}(\mathbf{t}). \tag{4.7}$$

In the trigonometric parametrization equations (3.13), (3.14) and (3.15), (3.16), (3.19) become the same. In order to write them in a compact form, we introduce the differential operator

$$\nabla_\alpha(z) = \partial_\alpha + D_\alpha(z). \tag{4.8}$$

The equations read:

$$e^{\nabla_\alpha(a)\nabla_\alpha(b)F} = \frac{\sin(u_\alpha(a) - u_\alpha(b))}{a^{-1} - b^{-1}}, \tag{4.9}$$

$$e^{\nabla_\alpha(a)\nabla_\beta(b)F} = \sin(u_\alpha(a) - u_\beta(b) + \eta_{\alpha\beta}). \tag{4.10}$$

(In the second equation it is assumed that $\alpha \neq \beta$.) In the limit $b \to \infty$ they become

$$a e^{\nabla_\alpha(a)\partial_\alpha F} = \sin u_\alpha(a), \tag{4.11}$$

$$e^{\nabla_\alpha(a)\partial_\beta F} = \sin(u_\alpha(a) + \eta_{\alpha\beta}). \tag{4.12}$$

Finally, we consider equation (3.17). In the trigonometric parametrization the dependence on a disappears and the equation becomes a constraint for $\eta_{\alpha\beta}$:

$$\epsilon_{\alpha\beta} \sin \eta_{\alpha\beta} = \epsilon_{\alpha\mu}\epsilon_{\beta\mu} \sin(\eta_{\mu\alpha} - \eta_{\mu\beta}), \tag{4.13}$$

which should hold for all distinct α, β, μ. The solution is

$$\eta_{\alpha\beta} = \eta_\alpha - \eta_\beta + \frac{\pi}{2}(\epsilon_{\alpha\beta} + 1) \tag{4.14}$$

with some η_α.

The final result can be summarized as follows. Let us redefine the functions $u_\alpha(z)$ including into them the constant terms in the expansion as $z \to \infty$. So we introduce the functions $v_\alpha(z) = u_\alpha(z) + \eta_\alpha$. Then all equations of the hierarchy are encoded in the single equation

$$\epsilon_{\beta\alpha} e^{\nabla_\alpha(a)\nabla_\beta(b)F} = \frac{\sin(v_\alpha(a) - v_\beta(b))}{(a^{-1} - b^{-1})^{\delta_{\alpha\beta}}} \tag{4.15}$$

valid for all complex a, b.

The meaning of this equation is that general second order derivatives of the function F in the independent variables, as is usual in the dispersionless equations, are expressed through some special second order derivatives. Indeed, the equation for $\alpha \neq \beta$ can be written as

$$\epsilon_{\beta\alpha} e^{\nabla_\alpha(a)\nabla_\beta(b)F} = \sin\Big(\eta_\alpha - \eta_\beta + \arcsin\left(a^{-1} e^{\nabla_\alpha(a)\partial_\alpha F}\right) - \arcsin\left(b^{-1} e^{\nabla_\beta(b)\partial_\beta F}\right)\Big). \tag{4.16}$$

At the same time, since

$$\eta_\alpha - \eta_\beta = \sum_{\gamma=\alpha}^{\beta-1} (\eta_\gamma - \eta_{\gamma+1}), \qquad \alpha < \beta,$$

$$\eta_\alpha - \eta_\beta = -\sum_{\gamma=\beta}^{\alpha-1} (\eta_\gamma - \eta_{\gamma+1}), \qquad \alpha > \beta,$$

we have:

$$\eta_\alpha - \eta_\beta = -\sum_{\gamma=\alpha}^{\beta-1} \arcsin(e^{\partial_\gamma \partial_{\gamma+1} F}), \qquad \alpha < \beta,$$

$$\eta_\alpha - \eta_\beta = \sum_{\gamma=\beta}^{\alpha-1} \arcsin(e^{\partial_\gamma \partial_{\gamma+1} F}), \qquad \alpha > \beta.$$

Therefore, as equation (4.16) shows, the general second order derivatives of the function F are expressed through the special derivatives $\partial_\alpha^2 F$, $\partial_\alpha \partial_{\alpha+1} F$, $\partial_{t_{\alpha,k}} \partial_\alpha F$.

Finally, by applying $\nabla_\gamma(c)$ to the logarithm of both sides of (4.15), this equation can be written as

$$\begin{aligned} \nabla_\alpha(a) \log \sin(v_\beta(b) - v_\gamma(c)) &= \nabla_\beta(b) \log \sin(v_\gamma(c) - v_\alpha(a)) \\ &= \nabla_\gamma(c) \log \sin(v_\alpha(a) - v_\beta(b)). \end{aligned} \tag{4.17}$$

This symmetry is a manifistation of integrability of the dispersionless multicomponent KP hierarchy.

5 Concluding remarks

We have reconsidered the dispersionless limit of the multicomponent KP hierarchy. Nonlinear differential equations for the dispersionless limit of logarithm of the tau-function have been obtained. We have shown that there is a rational curve built in the structure of the hierarchy. This curve can be uniformized via trigonometric functions. In the trigonometric parametrization, the equations of the hierarchy acquire especially nice form, being encoded in a single equation.

This work was motivated by our study of the multicomponent DKP hierarchy (work in progress). In the fermionic approach, the tau-function of the KP hierarchy is an expectation value of exponent of neutral quadratic form in fermions while for the DKP hierarchy (known also as the Pfaff lattice) the quadratic form can be arbitrary. In this sense the latter is more general than the former. In the dispersionless limit of the DKP hierarchy there is an elliptic curve built in its structure [7, 8], with the elliptic modulus being a dynamical variable. Uniformizing this curve via elliptic functions, one can represent the equations in a nice elliptic form (for the one-component case see [9]). Basically, the sin-function in (4.15) and other equations is replaced by the elliptic sn-function. We have found it instructive to reconsider the dispersionless KP hierarchy within a similar approach, where elliptic functions degenerate to trigonometric ones.

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