Symmetries of Quiver schemes

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Abstract

We introduce reflection functors on quiver schemes in the sense of Hausel–Wong–Wyss, generalizing those on quiver varieties. Also we construct some isomorphisms between quiver schemes whose underlying quivers are different.

1 Introduction

Let Q be a finite quiver with no edge-loops and $\mathbf{d} = (d_i)_{i\in I}$ be a collection of positive integers indexed by the vertex set I. We think of each d_i as the "multiplicity" of i and call the pair $(\mathsf{Q}, \mathbf{d})$ a *quiver with multiplicities*.

In [12], the second author associated to $\boldsymbol{\lambda} \in \bigoplus_{i\in I} \mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i})$, $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$ a complex symplectic manifold $\mathcal{N}^{\mathrm{s}}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$, called the quiver variety with multiplicities[1]. In the multiplicity-free case ($d_i = 1$ for all i), it coincides with the quiver variety $\mathfrak{M}^{\mathrm{reg}}_{\zeta}(\mathbf{v}, \mathbf{w})$ in the sense of Nakajima [9] with $\mathbf{w} = 0$, $\zeta = (0, \boldsymbol{\lambda})$. One of the main theorems in [12] says that, in roughly speaking, the quiver with multiplicities $(\mathsf{Q}, \mathbf{d})$ determines a symmetrizable (possibly non-symmetric) generalized Cartan matrix, and the quiver varieties with multiplicities $\mathcal{N}^{\mathrm{s}}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ for various $\boldsymbol{\lambda}, \mathbf{v}$ admit symmetry of the associated Weyl group, which coincides with the Weyl group symmetry of quiver varieties generated by reflection functors [10] in the multiplicity-free case.

On the other hand, Geiss–Leclerc–Schröer [3] associated an algebra Π to each symmetrizable generalized Cartan matrix $\mathbf{C}$ with a symmetrizer. If $\mathbf{C}$ is symmetric (with the trivial symmetrizer), then Π coincides with the usual preprojective algebra of type $\mathbf{C}$. Recall that Nakajima's quiver variety $\mathfrak{M}^{\mathrm{reg}}_0(\mathbf{v}, 0)$ parametrizes isomorphism classes of irreducible representations of a preprojective algebra with dimension vector $\mathbf{v}$. Thus their work also leads to generalization of Nakajima's quiver varieties to the non-symmetric case.

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[1]In [12] the parameter $\boldsymbol{\lambda} = (\lambda_i)$ is supposed to be an element of $\bigoplus_{i\in I} \epsilon_i^{-d_i}\mathbb{C}[\![\epsilon_i]\!]/\mathbb{C}[\![\epsilon_i]\!]$, but it may be regarded as an element of $\bigoplus_{i\in I} \mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i})$ by multiplying each λ_i by $\epsilon_i^{d_i}$.

Based on the work of Geiss–Leclerc–Schröer, Hausel–Wong–Wyss [4] modified the definition of $\mathcal{N}^{\mathrm{s}}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ to introduce an affine scheme, called the *quiver scheme*, which we denote by $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ in this paper. The purpose of this paper is to obtain analogues/generalization of the results obtained in [12] for quiver schemes.

We briefly explain the main results in this paper. We associate to each $(\mathsf{Q},\mathbf{d})$ a symmetrizable generalized Cartan matrix following Geiss–Leclerc–Schröer, and let the associated Weyl group act both on $\bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i})$ and on $\mathbb{Z}^I$. Thus for each $j\in I$, the j-th simple reflection gives rise to linear transformations $r_j\colon \bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i}) \to \bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon^{d_i})$ and $s_j\colon \mathbb{Z}^I\to\mathbb{Z}^I$. The first main result generalizes reflection functors of Lusztig [7], Maffei [8] and Nakajima [10] .

Theorem 1.1 (see Section 3)**.** *Take $j\in I$, $\boldsymbol{\lambda}=(\lambda_i)\in\bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i})$, $\mathbf{v}\in\mathbb{Z}^I_{\geq 0}$ so that λ_j is a unit of $\mathbb{C}[\epsilon_j]/(\epsilon_j^{d_j})$. Then there exists an isomorphism of schemes*

$$\mathcal{F}_j\colon \mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v}) \xrightarrow{\sim} \mathcal{S}_{\mathsf{Q},\mathbf{d}}(r_j(\boldsymbol{\lambda}), s_j(\mathbf{v})).$$

Note that Geiss–Leclerc–Schröer [3] also introduced reflection functors for Π but we cannot use them to show the above since $\boldsymbol{\lambda}\neq 0$ by the assumption.

The second main result is generalization of [12, Theorem 5.8].

Theorem 1.2 (see Section 4)**.** *Suppose that a sequence of pairwise distinct vertices, which we denote by $0,1,\dots,l$ ($l>0$), satisfies the following conditions:*

- *vertices i,j in $\{0,1,\dots,l\}$ are connected by exactly one arrow if $|i-j|=1$, and otherwise no arrow connects them;*
- *no arrow connects any $i\in I\setminus\{0,1,\dots,l\}$ and $j\in\{1,2,\dots,l\}$;*
- *$d_0=1$ and $d_i=d$ ($i=1,2,\dots,l$) for some integer $d>1$.*

Also, suppose that a pair $(\boldsymbol{\lambda},\mathbf{v})\in\bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon_i^{d_i})\times\mathbb{Z}^I_{\geq 0}$ satisfies the following conditions:

- *the sequence $v_0,v_1,\dots,v_l$ is non-increasing;*
- *$\lambda_i(0)+\lambda_{i+1}(0)+\cdots+\lambda_j(0)\neq 0$ for all pairs $i\leq j$ in $\{1,2,\dots,l\}$.*

Then there exist another quiver with multiplicities $(\check{\mathsf{Q}},\check{\mathbf{d}})$ with the same vertex set I and a pair $(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})\in\bigoplus_{i\in I}\mathbb{C}[\epsilon_i]/(\epsilon_i^{\check{d}_i})\times\mathbb{Z}^I_{\geq 0}$ such that $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ and $\mathcal{S}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})$ are isomorphic.

In fact, both $(\check{\mathsf{Q}},\check{\mathbf{d}})$ and $(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})$ are explicitly given and $(\check{\mathsf{Q}},\check{\mathbf{d}})$ does not depend on $(\boldsymbol{\lambda},\mathbf{v})$. Using this theorem we can show that some quiver schemes are (affine) algebraic varieties.

This paper is organized as follows: In Section 2, we recall the definition of preprojective algebra Π in the sense of Geiss–Leclerc–Schröer and quiver schemes $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$. Also, we recall some result of Hausel–Wong–Wyss on coadjoint orbits, which we will use to prove our second main theorem. Sections 3 and 4 are devoted to prove our first and second main theorems, respectively.

Throughout the paper, we write $\otimes$ for $\otimes_{\mathbb{C}}$.

2 Quiver schemes

In this section we recall the definitions of preprojective algebras in the sense of Geiss–Leclerc–Schröer [3] and quiver schemes introduced by Hausel–Wong–Wyss [4].

2.1 Preliminaries

In this subsection we introduce some symplectic vector spaces related to truncated polynomial rings; they are building blocks of quiver schemes.

For a positive integer d, put

$$R_d \coloneqq \mathbb{C}[\![\epsilon]\!]/\epsilon^d\mathbb{C}[\![\epsilon]\!], \quad R^d \coloneqq \epsilon^{-d}\mathbb{C}[\![\epsilon]\!]/\mathbb{C}[\![\epsilon]\!] \subset \mathbb{C}(\!(\epsilon)\!)/\mathbb{C}[\![\epsilon]\!].$$

We also denote the variable ϵ by ϵ_d in order to distinguish it from other variables. The bilinear form

$$\mathbb{C}[\![\epsilon]\!] \times \epsilon^{-d}\mathbb{C}[\![\epsilon]\!] \to \mathbb{C}; \quad (f,g) \mapsto \operatorname*{res}_{\epsilon=0}\left(f(\epsilon)g(\epsilon)d\epsilon\right)$$

induces a non-degenerate pairing $R_d \times R^d \to \mathbb{C}$, by which we may identify the vector space R^d with the $\mathbb{C}$-dual space R_d^* of R_d. On the other hand, the multiplication by ϵ^d induces a $\mathbb{C}$-linear isomorphism $R^d \simeq R_d$. Thus we may also identify R_d with R_d^*; the corresponding pairing $R_d \times R_d \to \mathbb{C}$ is

$$(f,g) \mapsto \langle f,g\rangle_d \coloneqq \operatorname*{res}_{\epsilon=0}\left(f(\epsilon)g(\epsilon)\,\frac{d\epsilon}{\epsilon^d}\right).$$

More generally, for homomorphisms $X\colon \mathbb{W} \to \mathbb{V}$, $Y\colon \mathbb{V} \to \mathbb{W}$ between free R_d-modules $\mathbb{V}, \mathbb{W}$, we define

$$\langle X,Y\rangle_d = \operatorname*{res}_{\epsilon=0}\left(\operatorname{tr}_{R_d}(XY)\,\frac{d\epsilon}{\epsilon^d}\right) = \langle \operatorname{tr}_{R_d}(XY), 1\rangle_d, \tag{2.1}$$

where $\operatorname{tr}_{R_d}\colon \operatorname{End}_{R_d}(\mathbb{V}) \to R_d$ is the trace. It gives an isomorphism $\operatorname{Hom}_{R_d}(\mathbb{V},\mathbb{W}) \simeq \operatorname{Hom}_{R_d}(\mathbb{W},\mathbb{V})^*$.

The $\mathbb{C}$-algebra R_d is d-dimensional with a basis $\{1, \epsilon, \ldots, \epsilon^{d-1}\}$. More generally, if d is a multiple of some positive integer c, the homomorphism

$$R_c \to R_d; \quad \epsilon_c \mapsto \epsilon_d^{d/c}$$

makes R_d into a free R_c-algebra with a basis $\{1, \epsilon_d, \ldots, \epsilon_d^{d/c-1}\}$. In this manner we equip each R_d-module $\mathbb{V}$ with a structure of R_c-module.

Lemma 2.1. *Suppose that $\mathbb{V}$ is a free R_d-module. Then the map*

$$\operatorname{pr}_{c,d}\colon \operatorname{End}_{R_c}(\mathbb{V}) \to \operatorname{End}_{R_d}(\mathbb{V}); \quad Z \mapsto \sum_{k=0}^{d/c-1} \epsilon_d^k Z \epsilon_d^{d/c-1-k}$$

is the transpose of the inclusion $\operatorname{End}_{R_d}(\mathbb{V}) \hookrightarrow \operatorname{End}_{R_c}(\mathbb{V})$*:*

$$\langle \operatorname{pr}_{c,d}(Z), Z'\rangle_d = \langle Z, Z'\rangle_c \quad (Z \in \operatorname{End}_{R_c}(\mathbb{V}),\ Z' \in \operatorname{End}_{R_d}(\mathbb{V})).$$

Proof. Since $\mathrm{pr}_{c,d}(Z)Z' = \mathrm{pr}_{c,d}(ZZ')$ for $Z \in \mathrm{End}_{R_c}(\mathbb{V})$, $Z' \in \mathrm{End}_{R_d}(\mathbb{V})$, it suffices to show

$$\operatorname*{res}_{\epsilon_d=0}\left(\mathrm{tr}_{R_d}\left(\mathrm{pr}_{c,d}(Z)\right)\frac{d\epsilon_d}{\epsilon_d^d}\right) = \operatorname*{res}_{\epsilon_c=0}\left(\mathrm{tr}_{R_c}(Z)\frac{d\epsilon_c}{\epsilon_c^c}\right) \quad (Z \in \mathrm{End}_{R_c}(\mathbb{V})).$$

Take an ordered R_d-basis $(v_1, v_2, \ldots, v_n)$ of $\mathbb{V}$ and let $(Z_{ij}) \in M_n(R_d)$ be the matrix representation of $\mathrm{pr}_{c,d}(Z)$. Also, let $(Z_{(i,k)(j,l)}) \in M_{nd/c}(R_c)$ be the matrix representation of Z with respect to the R_c-basis $v_{i,k} \coloneqq \epsilon_d^k v_i$, $i = 1, 2, \ldots, n$, $k = 0, 1, \ldots, d/c - 1$ of $\mathbb{V}$. Then

$$Z_{ij} = \sum_{k,l=0}^{d/c-1} \epsilon_d^{d/c-1-l+k}\ Z_{(i,k)(j,l)}\big|_{\epsilon_c=\epsilon_d^{d/c}}.$$

From this formula one easily deduces

$$\operatorname*{res}_{\epsilon_d=0}\left(Z_{ij}\frac{d\epsilon_d}{\epsilon_d^d}\right) = \sum_{k=0}^{d/c-1}\operatorname*{res}_{\epsilon_c=0}\left(Z_{(i,k)(j,k)}\frac{d\epsilon_c}{\epsilon_c^c}\right).$$

Hence

$$\begin{aligned}\operatorname*{res}_{\epsilon_d=0}\left(\mathrm{tr}_{R_d}\left(\mathrm{pr}_{c,d}(Z)\right)\frac{d\epsilon_d}{\epsilon_d^d}\right) &= \sum_{i=1}^{n}\operatorname*{res}_{\epsilon_d=0}\left(Z_{ii}\frac{d\epsilon_d}{\epsilon_d^d}\right)\\ &= \sum_{i=1}^{n}\sum_{k=0}^{d/c-1}\operatorname*{res}_{\epsilon_c=0}\left(Z_{(i,k)(i,k)}\frac{d\epsilon_c}{\epsilon_c^c}\right) = \operatorname*{res}_{\epsilon_c=0}\left(\mathrm{tr}_{R_c}(Z)\frac{d\epsilon_c}{\epsilon_c^c}\right).\end{aligned}$$

□

For a finite dimensional $\mathbb{C}$-vector space V, define

$$G_d(V) = \mathrm{Aut}_{R_d}(V \otimes R_d), \quad \mathfrak{g}_d(V) = \mathrm{End}_{R_d}(V \otimes R_d).$$

Since $G_d(V) \subset \mathrm{GL}_{\mathbb{C}}(V \otimes R_d)$ is the centralizer of the multiplication by ϵ, it is a linear algebraic group with Lie algebra $\mathfrak{g}_d(V)$. We have an obvious isomorphism $\mathfrak{g}_d(V) \simeq \mathfrak{gl}_{\mathbb{C}}(V) \otimes R_d$, which enables us to identify each element of $\mathfrak{g}_d(V)$ with a matrix polynomial

$$\xi = \sum_{k=0}^{d-1}\xi_k\epsilon^k, \quad \xi_k \in \mathfrak{gl}_{\mathbb{C}}(V).$$

For instance, the identity $\mathrm{Id}_{V\otimes R_d}$ is identified with Id_V. As a subset of $\mathfrak{g}_d(V)$, the group $G_d(V)$ consists of all $g = \sum_{k=0}^{d-1} g_k\epsilon^k \in \mathfrak{g}_d(V)$ such that $\det g_0 \neq 0$. Also, the pairing (2.1) for $\mathbb{W} = \mathbb{V}$ enables us to identify $\mathfrak{g}_d(V)$ with its $\mathbb{C}$-dual space.

Now let V, W be two finite dimensional $\mathbb{C}$-vector spaces and d, c be positive integers with $c \mid d$. Put $\mathbb{V} = V \otimes R_d$, $\mathbb{W} = W \otimes R_c$ and consider the vector space

$$\mathbf{M} \coloneqq \mathrm{Hom}_{R_c}(\mathbb{W}, \mathbb{V}) \oplus \mathrm{Hom}_{R_c}(\mathbb{V}, \mathbb{W}),$$

together with the action of the linear algebraic group $G_d(V) \times G_c(W)$ defined by

$$(g,h)\colon (X,Y) \mapsto (gXh^{-1}, hYg^{-1}).$$

Since $\mathbb{V}$ is also free as an R_c-module, the two-form

$$\omega := \langle dX \wedge dY\rangle_c = \operatorname*{res}_{\epsilon_c=0} \left(\epsilon_c^{-c} \operatorname{tr}_{R_c}(dX \wedge dY)\right)$$

is a $G_d(V) \times G_c(W)$-invariant symplectic form on $\mathbf{M}$.

The symplectic form ω has another description. The extension of scalar gives an isomorphism

$$\operatorname{Hom}_{R_c}(\mathbb{W}, \mathbb{V}) \xrightarrow{\sim} \operatorname{Hom}_{R_d}(\mathbb{W} \otimes_{R_c} R_d, \mathbb{V}),$$

which we denote by $X \mapsto X^{R_d}$. Furthermore, the projection

$$\mathbb{W} \otimes_{R_c} R_d = \bigoplus_{k=0}^{d/c-1} \mathbb{W}\epsilon_d^k \to \mathbb{W}\epsilon_d^{d/c-1} \simeq \mathbb{W}$$

induces an isomorphism

$$\operatorname{Hom}_{R_d}(\mathbb{V}, \mathbb{W} \otimes_{R_c} R_d) \xrightarrow{\sim} \operatorname{Hom}_{R_c}(\mathbb{V}, \mathbb{W}),$$

whose inverse is explicitly described as

$$Y \mapsto Y^{R_d} \colon \mathbb{V} \ni v \mapsto \sum_{k=0}^{d/c-1} Y(\epsilon_d^{d/c-1-k} v) \otimes \epsilon_d^k.$$

Observe that for $X \in \operatorname{Hom}_{R_c}(\mathbb{W}, \mathbb{V})$ and $Y \in \operatorname{Hom}_{R_c}(\mathbb{V}, \mathbb{W})$, we have

$$X^{R_d} Y^{R_d} = \sum_{k=0}^{d/c-1} \epsilon_d^k XY \epsilon_d^{d/c-1-k} = \operatorname{pr}_{c,d}(XY).$$

Thus the previous lemma shows

$$\langle X, Y\rangle_c = \langle Y^{R_d}, X^{R_d}\rangle_d \quad ((X,Y) \in \mathbf{M}),$$

and hence

$$\omega = \langle dX^{R_d} \wedge dY^{R_d}\rangle_d.$$

Proposition 2.2. *The map*

$$\mu\colon \mathbf{M} \to \mathfrak{g}_d(V) \simeq \mathfrak{g}_d(V)^*; \quad (X,Y) \mapsto X^{R_d} Y^{R_d}$$

is a moment map generating the $G_d(V)$-action.

Proof. We have

$$\omega = -\langle dY^{R_d} \wedge dX^{R_d} \rangle_d = -d\langle Y^{R_d}, dX^{R_d} \rangle_d.$$

Also, the generating vector fields ξ^*, $\xi \in \mathfrak{g}_d(V)$ are given by $\xi^*_{(X,Y)} = (\xi X, -Y\xi)$. Hence the moment map $\mu\colon \mathbf{M} \to \mathfrak{g}_d(V)$ with $\mu(0,0) = 0$ is

$$\langle \mu(X,Y), \xi \rangle = \langle Y^{R_d}, dX^{R_d}(\xi^*) \rangle_d = \langle Y^{R_d}, \xi X^{R_d} \rangle_d = \langle X^{R_d} Y^{R_d}, \xi \rangle_d \quad (\xi \in \mathfrak{g}_d(V)).$$

□

Remark 2.3. Through the isomorphism

$$\mathbf{M} \simeq \mathrm{Hom}_{R_d}(\mathbb{W} \otimes_{R_c} R_d, \mathbb{V}) \oplus \mathrm{Hom}_{R_d}(\mathbb{V}, \mathbb{W} \otimes_{R_c} R_d),$$

the action of $G_c(W)$ extends to an action of $G_d(W) = \mathrm{Aut}_{R_d}(\mathbb{W} \otimes_{R_c} R_d)$. This action is Hamiltonian with moment map

$$\nu\colon \mathbf{M} \to \mathfrak{g}_d(W); \quad (X,Y) \mapsto -Y^{R_d} X^{R_d}.$$

Since $\mathfrak{g}_d(W) \simeq \mathfrak{g}_c(W) \otimes_{R_c} R_d \simeq \mathfrak{gl}_{\mathbb{C}}(W) \otimes R_d$, we may also identify the dual space $\mathfrak{g}_d(W)^*$ with

$$\mathfrak{gl}_{\mathbb{C}}(W) \otimes R^d \simeq \mathfrak{g}_c(W) \otimes_{R_c} R^d,$$

where we regard R^d as a free R_d-module of rank one using the linear isomorphism $\epsilon^{-d}\colon R_d \xrightarrow{\sim} R^d$. Under this identification, the moment map ν is expressed as

$$\nu(X,Y) = -\sum_{k=0}^{d/c-1} Y \epsilon_d^k X \otimes \epsilon_d^{-k-1}.$$

When $c = 1$, we may rewrite it as

$$\nu(X,Y) = -\sum_{k=0}^{d-1} Y N^k X \otimes \epsilon_d^{-k-1} = -Y(\epsilon_d - N)^{-1} X,$$

where $N \in \mathfrak{gl}_{\mathbb{C}}(\mathbb{V})$ is the multiplication by ϵ_d. Such a moment map appears in [12, 13, 14].

2.2 GLS preprojective algebras and quiver schemes

In this paper a quiver Q is always assumed to be finite, and usually denoted as $\mathsf{Q} = (I, \Omega, s, t)$, where I is the set of vertices, Ω is the set of arrows, and $s, t\colon \Omega \to I$ are the source/target maps. For a quiver $\mathsf{Q} = (I, \Omega, s, t)$, we denote by $\overline{\mathsf{Q}} = (I, \overline{\Omega}, s, t)$ the quiver obtained by reversing the orientation of each arrow of Q; so for each $h \in \Omega$ we have the reversed arrow $\overline{h} \in \overline{\Omega}$, satisfying $s(\overline{h}) = t(h)$, $t(\overline{h}) = s(h)$. Putting together all the arrows of Q and $\overline{\mathsf{Q}}$ we get a quiver $\mathsf{Q} + \overline{\mathsf{Q}} = (I, H, s, t)$ with arrow set $H := \Omega \sqcup \overline{\Omega}$, called the *double* of Q, together with an involution $H \to H$, $h \mapsto \overline{h}$. We define a map $\mathrm{sgn}\colon H \to \{\pm 1\}$ by $\mathrm{sgn}|_\Omega \equiv 1$, $\mathrm{sgn}|_{\overline{\Omega}} \equiv -1$.

Definition 2.4. A *quiver with multiplicities* is a quiver Q with each vertex i equipped with a positive integer d_i, called the *multiplicity* of i.

Take a quiver $\mathsf{Q} = (I, \Omega, s, t)$ with multiplicities $\mathbf{d} = (d_i)_{i \in I}$. For $i, j \in I$, put

$$d_{ij} \coloneqq \gcd(d_i, d_j), \quad f_{ij} \coloneqq \frac{d_j}{d_{ij}},$$

and for $h \in H$, put

$$d_h \coloneqq d_{s(h)t(h)}, \quad f_h \coloneqq f_{s(h)t(h)}.$$

Let $\mathsf{Q}' = (I, H', s, t)$ be the quiver obtained by adding an edge-loop ℓ_i to the double $\mathsf{Q} + \overline{\mathsf{Q}}$ for each $i \in I$. Let us recall the preprojective algebras in the sense of Geiss–Leclerc–Schröer [3]:

Definition 2.5. The *GLS preprojective algebra* Π associated to the quiver with multiplicities $(\mathsf{Q}, \mathbf{d})$ is defined to be the quotient of the path algebra of Q' modulo the following relations:

(P1) $\ell_i^{d_i} = 0$ for any $i \in I$;

(P2) $\ell_{t(h)}^{f_h} h = h \ell_{s(h)}^{f_{\overline{h}}}$ for each arrow h of $\mathsf{Q} + \overline{\mathsf{Q}}$;

(P3) the *mesh relations*

$$\sum_{h \in H; t(h) = i} \sum_{k=0}^{f_h - 1} \operatorname{sgn}(h) \ell_i^k h \overline{h} \ell_i^{f_h - 1 - k} = 0 \quad (i \in I).$$

By definition, a (finite dimensional) representation of Π is given by a datum consisting of

- a finite dimensional $\mathbb{C}$-vector space $\mathbb{V}_i$ for each $i \in I$;
- a linear map $B_h \colon \mathbb{V}_i \to \mathbb{V}_j$ for each arrow $h \colon i \to j$ of $\mathsf{Q} + \overline{\mathsf{Q}}$;
- a linear transformation N_i of $\mathbb{V}_i$ for each $i \in I$,

such that

(P'1) $N_i^{d_i} = 0$ for any $i \in I$;

(P'2) $N_{t(h)}^{f_h} B_h = B_h N_{s(h)}^{f_{\overline{h}}}$ for each arrow h of $\mathsf{Q} + \overline{\mathsf{Q}}$;

(P'3) the mesh relations

$$\sum_{h \in H; t(h) = i} \sum_{k=0}^{f_h - 1} \operatorname{sgn}(h) N_i^k B_h B_{\overline{h}} N_i^{f_h - 1 - k} = 0 \quad (i \in I).$$

Observe that relations (P'1) make each $\mathbb{V}_i$ into a module over R_{d_i}, and then relations (P'2) are equivalent to that each $B_h \colon \mathbb{V}_{s(h)} \to \mathbb{V}_{t(h)}$ is an R_{d_h}-homomorphism. In what follows we consider the case where each $\mathbb{V}_i$ is a free R_{d_i}-module (such a representation is said to be *locally free*), and take an I-graded $\mathbb{C}$-vector space $\mathbf{V} = \bigoplus_{i \in I} V_i$ so that

$$\mathbb{V}_i = V_i \otimes R_{d_i} \quad (i \in I).$$

Then the tuple $\mathbf{B} = (B_h)_{h\in H}$ lives in the vector space

$$\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \coloneqq \bigoplus_{h\in H} \mathrm{Hom}_{R_{d_h}}(\mathbb{V}_{s(h)}, \mathbb{V}_{t(h)}).$$

Put

$$G_{\mathbf{d}}(\mathbf{V}) \coloneqq \prod_{i\in I} G_{d_i}(V_i), \quad \mathfrak{g}_{\mathbf{d}}(\mathbf{V}) \coloneqq \mathrm{Lie}\, G_{\mathbf{d}}(\mathbf{V}) = \bigoplus_{i\in I} \mathfrak{g}_{d_i}(V_i).$$

For simplicity, we use the following notation for variables:

$$\epsilon_i \coloneqq \epsilon_{d_i} \in R_{d_i}, \quad \epsilon_h \coloneqq \epsilon_{d_h} \in R_{d_h}.$$

The observations made in the previous subsection show that $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$ has a symplectic form

$$\omega = \sum_{h\in\Omega} \langle dB_h \wedge dB_{\overline{h}}\rangle_{d_h} = \frac{1}{2}\sum_{h\in H} \mathrm{sgn}(h) \langle dB_h \wedge dB_{\overline{h}}\rangle_{d_h},$$

and the obvious action of $G_{\mathbf{d}}(\mathbf{V})$ on $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$ is Hamiltonian with moment map

$$\mu_{\mathbf{d}} = (\mu_{\mathbf{d},i})_{i\in I} \colon \mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \to \mathfrak{g}_d(\mathbf{V}),$$
$$\mu_{\mathbf{d},i}(\mathbf{B}) \coloneqq \sum_{\substack{h\in H,\\ t(h)=i}} \mathrm{sgn}(h) B_h^{R_{d_i}} B_{\overline{h}}^{R_{d_i}} = \sum_{\substack{h\in H,\\ t(h)=i}} \sum_{k=0}^{f_h-1} \mathrm{sgn}(h)\epsilon_i^k (B_h B_{\overline{h}})\epsilon_i^{f_h-k-1}.$$

Therefore the mesh relations (P'3) are exactly the same as the moment map relation $\mu_{\mathbf{d}}(\mathbf{B}) = 0$.

Since two points on $\mu_{\mathbf{d}}(\mathbf{B})$ are in the same $G_{\mathbf{d}}(\mathbf{V})$-orbit if and only if the corresponding representations of Π are isomorphic, we see that the isomorphism classes of locally free representations of Π with fixed dimension vector are parametrized by the orbit space $\mu_{\mathbf{d}}^{-1}(0)/G_{\mathbf{d}}(\mathbf{V})$. Motivated by this observation, we define the quiver schemes in the sense of Hausel–Wong–Wyss [4] as follows:

Definition 2.6. For $\boldsymbol{\lambda} = (\lambda_i) \in R_{\mathbf{d}} \coloneqq \bigoplus_{i\in I} R_{d_i}$ and a finite dimensional I-graded $\mathbb{C}$-vector space $\mathbf{V}$, we define

$$\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v}) = \mathrm{Spec}\left(\mathbb{C}[\,\mu_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})\,]^{G_{\mathbf{d}}(\mathbf{V})}\right),$$

where $\mathbf{v} \coloneqq (\dim V_i)_{i\in I} \in \mathbb{Z}_{\geq 0}^I$ is the dimension vector of $\mathbf{V}$ and $\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}} \coloneqq (\lambda_i\,\mathrm{Id}_{V_i}) \in \mathfrak{g}_{\mathbf{d}}(\mathbf{V})^{G_{\mathbf{d}}(\mathbf{V})}$. Also, for convenience we put $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v}) = \emptyset$ for $\mathbf{v} \in \mathbb{Z}^I \setminus \mathbb{Z}_{\geq 0}^I$. We call $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ the *quiver scheme* associated to $(\mathsf{Q}, \mathbf{d})$ with dimension vector $\mathbf{v}$ and complex parameter $\boldsymbol{\lambda}$.

When $d_i = 1$ for all $i \in I$, the quiver scheme $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ is Nakajima's quiver variety (with trivial real parameter).

Remark 2.7. If $\mathbf{V}$, $\mathbf{V}'$ are two I-graded $\mathbb{C}$-vector spaces with $\dim \mathbf{V} = \dim \mathbf{V}' = \mathbf{v}$, then we have a *canonical* isomorphism

$$\mathrm{Spec}\left(\mathbb{C}[\,\mu_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})\,]^{G_{\mathbf{d}}(\mathbf{V})}\right) \simeq \mathrm{Spec}\left(\mathbb{C}[\,\mu_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}'})\,]^{G_{\mathbf{d}}(\mathbf{V}')}\right).$$

Thus we are identifying them, which is the reason why we use the notation $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ rather than $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{V})$ for the quiver scheme.

Also, the isomorphism class of $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ does not depend on the orientation of the quiver Q.

Remark 2.8. Note that the subgroup

$$\mathbb{C}^\times \mathrm{Id}_{\mathbf{V}} := \{\, (c\, \mathrm{Id}_{V_i})_{i \in I} \mid c \in \mathbb{C}^\times \,\} \subset G_{\mathbf{d}}(\mathbf{V})$$

trivially acts on $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$. Hence the moment map $\mu_{\mathbf{d}}$ takes values perpendicular to the Lie algebra of $\mathbb{C}^\times \mathrm{Id}_{\mathbf{V}}$, namely, the image of $\mu_{\mathbf{d}}$ is contained in

$$\mathfrak{g}_{\mathbf{d}}(\mathbf{V})_0 := \left\{ (\xi_i)_{i \in I} \in \mathfrak{g}_{\mathbf{d}}(\mathbf{V}) \;\middle|\; \sum_{i \in I} \operatorname*{res}_{\epsilon_i = 0} \left(\mathrm{tr}_{R_{d_i}}(\xi_i) \frac{d\epsilon_i}{\epsilon_i^{d_i}} \right) = 0 \right\}.$$

It follows that the quiver scheme $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ is empty unless

$$\sum_{i \in I} v_i \operatorname*{res}_{\epsilon_i = 0} \left(\lambda_i \frac{d\epsilon_i}{\epsilon_i^{d_i}} \right) = 0. \tag{2.2}$$

Remark 2.9. Let $\mathbf{A} = (a_{ij})_{i,j \in I}$ be the adjacency matrix of the underlying graph of Q, i.e.,

$$a_{ij} = \#\{\, h \in H \mid s(h) = i,\, t(h) = j \,\},$$

and put

$$\mathbf{A}' := \left(\frac{a_{ij}}{d_{ij}} \right)_{i,j \in I}, \quad \mathbf{D} := \mathrm{diag}(d_i)_{i \in I}, \quad \mathbf{C} = (c_{ij})_{i,j \in I} := 2\,\mathrm{Id} - \mathbf{A}'\mathbf{D},$$

where Id denotes the identity matrix. Define a symmetric bilinear form $(\ ,\ )$ on $\mathbb{Z}^I$ by

$$(\mathbf{v}, \mathbf{w}) = {}^t\mathbf{v}\mathbf{D}\mathbf{C}\mathbf{w} \quad (\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I). \tag{2.3}$$

If we formally apply the dimension formula for Hamiltonian reductions, then the dimension of $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ is equal to

$$\begin{aligned}
\dim \mathbf{M}_{\mathsf{Q}}(\mathbf{V}_{\mathbf{d}}) - 2 \dim \bigl(G_{\mathbf{d}}(\mathbf{V}) / \mathbb{C}^\times \mathrm{Id} \bigr) &= \sum_{h \in H} \frac{v_{s(h)} d_{s(h)} v_{t(h)} d_{t(h)}}{d_h} - 2 \sum_{i \in I} v_i^2 d_i + 2 \\
&= \sum_{i,j \in I} \frac{a_{ij}}{d_{ij}} v_i d_i v_j d_j - 2 \sum_{i \in I} v_i^2 d_i + 2 \\
&= 2 - {}^t\mathbf{v}(2\,\mathbf{D} - \mathbf{D}\mathbf{A}'\mathbf{D})\mathbf{v} = 2 - (\mathbf{v}, \mathbf{v}).
\end{aligned}$$

Hence the "expected dimension" of $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ is equal to $2 - (\mathbf{v}, \mathbf{v})$, as in the case of quiver varieties.

Note that if Q has no edge-loops, then $\mathbf{C}$ is a symmetrizable generalized Cartan matrix with symmetrizer $\mathbf{D}$, and $(\ ,\ )$ is the standard symmetric bilinear form on the root lattice (identified with $\mathbb{Z}^I$ using the basis consisting of the simple roots) associated to $\mathbf{D}$. Clearly $\mathbf{C}$, $\mathbf{D}$ do not depend on the orientation of the quiver Q, so they only depend on the underlying "graph with multiplicities". All symmetrizable generalized Cartan matrices may be constructed in this way; see [3] for the inverse construction.

Remark 2.10. Let $\mathbf{V_d}$ be the I-graded vector space $\bigoplus_{i\in I} V_i \otimes R_{d_i}$ and consider the symplectic vector space

$$\mathbf{M}_{\mathsf{Q}}(\mathbf{V_d}) \coloneqq \bigoplus_{h\in H} \operatorname{Hom}_{\mathbb{C}}(V_{s(h)} \otimes R_{d_{s(h)}}, V_{t(h)} \otimes R_{d_{t(h)}})$$

instead of $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$. The group $G_{\mathbf{d}}(\mathbf{V})$ acts on $\mathbf{M}_{\mathsf{Q}}(\mathbf{V_d})$ as a subgroup of

$$\mathrm{GL}(\mathbf{V_d}) \coloneqq \prod_{i\in I} \mathrm{GL}_{\mathbb{C}}(V_i \otimes R_i),$$

and the quiver varieties with multiplicities introduced by the second author [12] are defined as Hamiltonian reductions of $\mathbf{M}_{\mathsf{Q}}(\mathbf{V_d})$ by the action of $G_{\mathbf{d}}(\mathbf{V})$. If $\gcd(d_i, d_j) = 1$ for all $i, j \in I$ with $a_{ij} \geq 1$, then $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) = \mathbf{M}_{\mathsf{Q}}(\mathbf{V_d})$ and hence they are essentially the same as quiver schemes, although they are defined as complex manifolds (not schemes) using geometric invariant theory.

2.3 Some $G_d(V)$-coadjoint orbits

In this subsection we fix a finite dimensional $\mathbb{C}$-vector space V together with a positive integer d, and review a result of Hausel–Wong–Wyss on some $G_d(V)$-coadjoint orbits.

Take any direct sum decomposition $V = \bigoplus_{i=0}^{l} W_i$ and elements $\theta_0, \theta_1, \ldots, \theta_l \in R_d$ ($l > 0$) so that $\theta_i - \theta_j$ is a unit whenever $i \neq j$. Put

$$\Theta \coloneqq \bigoplus_{i=0}^{l} \theta_i \operatorname{Id}_{W_i \otimes R_d} \in \mathfrak{g}_d(V),$$

and let $\mathcal{O}_\Theta \subset \mathfrak{g}_d(V)$ be the $G_d(V)$-coadjoint orbit of Θ.

Hausel–Wong–Wyss proved that $\mathcal{O}_\Theta$ is an example of quiver schemes. Let $(\mathsf{Q}, \mathbf{d})$ be the quiver consisting of l vertices $\{1, 2, \ldots, l\}$ and $(l-1)$ arrows $h_i\colon i \to i+1$, $i = 1, 2, \ldots, l-1$ with multiplicities

$$d_i = d \quad (i = 1, 2, \ldots, l)$$

for some positive integer d. We call it the *d-leg of length l*. Define a graded $\mathbb{C}$-vector space $\mathbf{V} = \bigoplus_{i=1}^{l} V_i$ by

$$V_i \coloneqq \bigoplus_{j\geq i} W_i \quad (i = 1, 2, \ldots, l),$$

and consider the symplectic vector space

$$\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \oplus \operatorname{Hom}_{\mathbb{C}}(V, V_1 \otimes R_d) \oplus \operatorname{Hom}_{\mathbb{C}}(V_1 \otimes R_d, V)$$

acted (diagonally) on by the group $G_{\mathbf{d}}(\mathbf{V})$. An element $\mathbf{B}$ of this space consists of R_d-homomorphisms

$$B_{i+1,i}\colon V_i \otimes R_d \to V_{i+1} \otimes R_d, \quad B_{i,i+1}\colon V_{i+1} \otimes R_d \to V_i \otimes R_d \quad (i = 1, 2, \ldots, l-1),$$

together with $\mathbb{C}$-linear maps

$$a\colon V \to V_1 \otimes R_d, \quad b\colon V_1 \otimes R_d \to V.$$

For such an element we put

$$B_{1,0} \coloneqq a^{R_d} \colon V \otimes R_d \to V_1 \otimes R_d, \quad B_{0,1} \coloneqq b^{R_d} \colon V_1 \otimes R_d \to V \otimes R_d.$$

Observe that the $G_{\mathbf{d}}(\mathbf{V})$-action is Hamiltonian with moment map

$$\tilde{\mu}_{\mathbf{d}} = (\tilde{\mu}_{\mathbf{d},i}), \quad \tilde{\mu}_{\mathbf{d},i}(\mathbf{B}) = B_{i,i-1}B_{i-1,i} - B_{i,i+1}B_{i+1,i} \quad (i = 1, 2, \ldots, l),$$

where $B_{l,l+1}, B_{l+1,l}$ are understood to be zero.

Proposition 2.11 (Hausel–Wong–Wyss [4, Proposition 6.3.4])**.** *Define $\boldsymbol{\lambda} = (\lambda_i) \in R_{\mathbf{d}}$ by*

$$\lambda_i = \theta_i - \theta_{i-1}.$$

Then the $G_{\mathbf{d}}(\mathbf{V})$-action on the level set $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$ is free and has a geometric quotient. Moreover, the map

$$\nu \colon \mathbf{M}_{\mathbf{Q},\mathbf{d}}(\mathbf{V}) \oplus \operatorname{Hom}_{\mathbb{C}}(V, V_1 \otimes R_d) \oplus \operatorname{Hom}_{\mathbb{C}}(V_1 \otimes R_d, V) \to \mathfrak{g}_d(V); \quad \mathbf{B} \mapsto -B_{0,1}B_{1,0} + \theta_0 \operatorname{Id}_V$$

induces a symplectic isomorphism from $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})/G_{\mathbf{d}}(\mathbf{V})$ to the coadjoint orbit $\mathcal{O}_\Theta$.

See e.g. [11] for the definition of geometric quotient. Because our convention and the statement are slightly different to those of [4], we will give a proof below (our proof is different to the proof of [4]).

Lemma 2.12. *Let $\mathbf{B} \in \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$. Then $B_{i,i+1}$ is injective and $B_{i+1,i}$ is surjective for all $i = 0, 1, \ldots, l-1$.*

Proof. First consider the case of $d = 1$. In this case, [2, Lemma 9.1] (with a different sign convention) shows that the affine quotient $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})/G_{\mathbf{d}}(\mathbf{V})$ is isomorphic to the closure of the orbit $\mathcal{O}_\Theta \subset \mathfrak{gl}_{\mathbb{C}}(V)$ via the map ν. Furthermore, it is known (see e.g. [15, 5.1.2, 5.1.4]) that the image of a point $\mathbf{B} \in \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$ lies in $\mathcal{O}_\Theta$ if and only if $B_{i,i+1}$ is injective and $B_{i+1,i}$ is surjective for all $i = 0, 1, \ldots, l-1$. Since $\mathcal{O}_\Theta$ is closed (Θ is semisimple), it follows that $B_{i,i+1}$ is injective and $B_{i+1,i}$ is surjective for all $\mathbf{B} \in \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$ and $i = 0, 1, \ldots, l-1$.

Now consider the general case. Take any $\mathbf{B} \in \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$. Specializing ϵ to zero, we then obtain $\mathbb{C}$-linear maps

$$B_{i+1,i}(0) \colon V_i \to V_{i+1}, \quad B_{i,i+1}(0) \colon V_{i+1} \to V_i \quad (i = 0, 1, \ldots, l-1),$$

where $V_0 \coloneqq V$, such that

$$B_{i,i-1}(0)B_{i-1,i}(0) - B_{i,i+1}(0)B_{i+1,i}(0) = -\lambda_i(0) \operatorname{Id}_{V_i} \quad (i = 1, 2, \ldots, l).$$

Since $\theta_i(0) \neq \theta_j(0)$ whenver $i \neq j$, the above fact in the case of $d = 1$ shows that $B_{i,i+1}(0)$ is injective and $B_{i+1,i}(0)$ is surjective for all $i = 0, 1, \ldots, l-1$. This implies that $B_{i,i+1}$ is injective and $B_{i+1,i}$ is surjective for all $i = 0, 1, \ldots, l-1$. $\square$

Proof of Proposition 2.11. Using the above lemma one can easily check that $G_{\mathbf{d}}(\mathbf{V})$ acts freely on $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})$. Hence $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})$ is non-singular and equidimensional. Also, note that the group $G_d(V)$ acts on the space $\mathbf{M}_{\mathbf{Q},\mathbf{d}}(\mathbf{V})\oplus\mathrm{Hom}_{\mathbb{C}}(V, V_1\otimes R_d)\oplus\mathrm{Hom}_{\mathbb{C}}(V_1\otimes R_d, V)$ in Hamiltonian fashion with moment map ν (see Remark 2.3) and this action preserves $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})$.

Now we take any $\mathbf{B}\in\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}})$ and show that $A\coloneqq\nu(\mathbf{B})$ lies in $\mathcal{O}_\Theta$. For $i=1,2,\dots,l$, put

$$B_{0,i}=B_{0,1}B_{1,2}\cdots B_{i-1,i},\quad B_{i,0}=B_{i,i-1}\cdots B_{2,1}B_{1,0}.$$

Then using the moment map relation iteratively one easily deduces

$$\begin{aligned}&(B_{0,1}B_{0,1}+(\lambda_1+\cdots+\lambda_i)\mathrm{Id}_V)\,B_{0,i}=B_{0,i+1}B_{i+1,i}\quad(i=0,1,\dots,l-1),\\&(B_{0,1}B_{0,1}+(\lambda_1+\cdots+\lambda_l)\mathrm{Id}_V)\,B_{0,l}=0,\end{aligned}$$

that is,

$$(-A+\theta_i\,\mathrm{Id}_V)\,B_{0,i}=B_{0,i+1}B_{i+1,i}\quad(i=0,1,\dots,l-1),\quad(-A+\theta_l\,\mathrm{Id}_V)\,B_{0,l}=0.\tag{2.4}$$

Thus for any $i=0,1,\dots,l-1$, we have

$$\begin{aligned}B_{0,i+1}B_{i+1,0}&=(-A+\theta_i\,\mathrm{Id}_V)\,B_{0,i}B_{i,0}\\&=(-A+\theta_i\,\mathrm{Id}_V)\,(-A+\theta_{i-1}\,\mathrm{Id}_V)\,B_{0,i-1}B_{i-1,0}\\&=\cdots=\prod_{j=1}^{i}(-A+\theta_j\,\mathrm{Id}_V)\,B_{0,1}B_{1,0}=\prod_{j=0}^{i}(-A+\theta_j\,\mathrm{Id}_V)\,,\end{aligned}$$

and

$$\prod_{j=0}^{l}(-A+\theta_j\,\mathrm{Id}_V)=(-A+\theta_l\,\mathrm{Id}_V)\,B_{0,l}B_{l,0}=0.$$

By the above lemma, $B_{0,i}$ is injective and $B_{i,0}$ is surjective for all $i=1,2,\dots,l$. Hence

$$\mathrm{Im}\prod_{j=0}^{i}(-A+\theta_j\,\mathrm{Id}_V)\simeq V_{i+1}\otimes R_d\quad(i=0,1,\dots,l-1).\tag{2.5}$$

Since $\theta_i-\theta_j\in R_d^\times$ $(i\neq j)$ and $\prod_{j=0}^{l}(A-\theta_j\,\mathrm{Id}_V)=0$, the idempotent decomposition

$$\mathrm{Id}_V=\sum_{i=0}^{l}\pi_i,\quad\pi_i\coloneqq\prod_{j\neq i}(\theta_i-\theta_j)^{-1}\prod_{j\neq i}(A-\theta_j\,\mathrm{Id}_V)$$

gives a direct sum decomposition $V\otimes R_d=\bigoplus_{i=0}^{l}\mathrm{Ker}(A-\theta_i\,\mathrm{Id}_V)$, and equalities (2.5) show that $\mathrm{Ker}(A-\theta_i\,\mathrm{Id}_V)\simeq W_i\otimes R_d$. Hence $A\in\mathcal{O}_\Theta$. Furthermore, if we put

$$\mathbb{V}_i\coloneqq\mathrm{Im}\prod_{j=0}^{i-1}(-A+\theta_j\,\mathrm{Id}_V)=\mathrm{Im}\,B_{0,i}\quad(i=1,2,\dots,l),$$

then equalities (2.4) yield the following commutative diagrams for $i = 0, 1, \ldots, m-2$:

$$\begin{tikzcd} V_i \otimes R_d \arrow[r, "B_{i+1,i}"] \arrow[d, "B_{0,i}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{i+1} \otimes R_d \arrow[d, "B_{0,i+1}"] \\ \mathbb{V}_i \arrow[r, "-A+\theta_i \operatorname{Id}_V"'] & \mathbb{V}_{i+1}, \end{tikzcd} \qquad \begin{tikzcd} V_{i+1} \otimes R_d \arrow[r, "B_{i,i+1}"] \arrow[d, "B_{0,i+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_i \otimes R_d \arrow[d, "B_{0,i}"] \\ \mathbb{V}_{i+1} \arrow[r, "\text{inclusion}"'] & \mathbb{V}_i. \end{tikzcd}$$

Here we use the conventions $\mathbb{V}_0 = V \otimes R_d$, $V_0 = V$, $B_{0,0} = \operatorname{Id}_{V\otimes R_d}$ in the case of $i = 0$. Note that the vertical arrows are all isomorphisms. It follows that each fiber of the map $\nu \colon \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}}) \to \mathcal{O}_\Theta$ is a single $G_{\mathbf{d}}(\mathbf{V})$-orbit. Since this map is $G_d(V)$-equivariant and $G_d(V)$ transitively acts on $\mathcal{O}_\Theta$, the group $G_{\mathbf{d}}(\mathbf{V}) \times G_d(V)$ transitively acts on $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$. In particular, $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})$ is irreducible. Thus [11, Theorem 4.2] shows that $\nu \colon \tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}}) \to \mathcal{O}_\Theta$ is a geometric quotient. Also, since ν is a moment map the induced isomorphism $\tilde{\mu}_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})/G_{\mathbf{d}}(\mathbf{V}) \simeq \mathcal{O}_\Theta$ preserves the Poisson structure (and hence the symplectic structure). $\square$

In fact, the geometric quotient in Proposition 2.11 is an example of quiver schemes. Let $(\widetilde{\mathsf{Q}}, \tilde{\mathbf{d}})$ be the quiver with multiplicities obtained from $(\mathsf{Q}, \mathbf{d})$ by adding a new vertex 0 of multiplicity 1 and $\dim V$ arrows from 0 to 1. Define a graded $\mathbb{C}$-vector space $\widetilde{\mathbf{V}} = \bigoplus_{i=0}^m \widetilde{V}_i$ by $\widetilde{V}_0 := \mathbb{C}$, $\widetilde{V}_i := V_i$ $(i > 0)$. Then fixing a linear isomorphism $V \simeq \mathbb{C}^{\dim V}$, we have

$$\begin{aligned} \mathbf{M}_{\widetilde{\mathsf{Q}},\tilde{\mathbf{d}}}(\widetilde{\mathbf{V}}) &= \mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \oplus \bigoplus_{i=1}^{\dim V} \left(\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, V_1 \otimes R_d) \oplus \operatorname{Hom}_{\mathbb{C}}(V_1 \otimes R_d, \mathbb{C})\right) \\ &\simeq \mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \oplus \operatorname{Hom}_{\mathbb{C}}(V, V_1 \otimes R_d) \oplus \operatorname{Hom}_{\mathbb{C}}(V_1 \otimes R_d, V). \end{aligned}$$

Also, the moment map $\mu_{\tilde{\mathbf{d}}}$ is described as

$$\mu_{\tilde{\mathbf{d}},0}(\mathbf{B}) = -\operatorname{tr} B_{0,1}B_{1,0}, \quad \mu_{\tilde{\mathbf{d}},i}(\mathbf{B}) = B_{i,i-1}B_{i-1,i} - B_{i,i+1}B_{i+1,i} = \mu_{\mathbf{d},i}(\mathbf{B}) \quad (i > 0).$$

Let $\boldsymbol{\lambda}$ be as in Proposition 2.11 and define $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_i) \in R_{\tilde{\mathbf{d}}}$ by

$$\tilde{\lambda}_i = \lambda_i \quad (i > 0), \quad \tilde{\lambda}_0 = -\frac{1}{\dim V} \sum_{i>0} v_i \operatorname*{res}_{\epsilon_i=0} \left(\lambda_i \frac{d\epsilon_i}{\epsilon_i^{d_i}} \right),$$

so that (2.2) holds. Then any $\mathbf{B} \in \mathbf{M}_{\widetilde{\mathsf{Q}},\tilde{\mathbf{d}}}(\widetilde{\mathbf{V}})$ satisfying $\mu_{\mathbf{d}}(\mathbf{B}) = -\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}}$ also satisfies $\mu_{\tilde{\mathbf{d}},0}(\mathbf{B}) = -\lambda_0$ as $\mu_{\tilde{\mathbf{d}}}(\mathbf{B})$ lives in $\mathfrak{g}_{\tilde{\mathbf{d}}}(\widetilde{\mathbf{V}})_0$ (see Remark 2.8). Thus

$$\mu_{\tilde{\mathbf{d}}}^{-1}(-\tilde{\boldsymbol{\lambda}} \operatorname{Id}_{\tilde{\mathbf{V}}}) = \mu_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}}).$$

Furthermore, since $\mathbb{C}^\times \operatorname{Id}_{\tilde{\mathbf{V}}} \subset G_{\tilde{\mathbf{d}}}(\widetilde{\mathbf{V}}) = \mathbb{C}^\times \times G_{\mathbf{d}}(\mathbf{V})$ acts trivially, we have

$$\mathbb{C}[\,\mu_{\tilde{\mathbf{d}}}^{-1}(-\tilde{\boldsymbol{\lambda}} \operatorname{Id}_{\tilde{\mathbf{V}}})\,]^{G_{\tilde{\mathbf{d}}}(\tilde{\mathbf{V}})} = \mathbb{C}[\,\mu_{\mathbf{d}}^{-1}(-\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}})\,]^{G_{\mathbf{d}}(\mathbf{V})}.$$

Corollary 2.13. *The orbit $\mathcal{O}_\Theta$ is isomorphic to the quiver scheme $\mathcal{S}_{\widetilde{\mathsf{Q}},\tilde{\mathbf{d}}}(\tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{v}})$.*

Proof. Proposition 2.11 implies that $\mathcal{S}_{\widetilde{\mathsf{Q}},\tilde{\mathbf{d}}}(\tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{v}})$ is isomorphic to the affinization $\operatorname{Spec} \mathbb{C}[\mathcal{O}_\Theta]$ of $\mathcal{O}_\Theta$. On the other hand, $\mathcal{O}_\Theta$ is known to be affine (see [4, Lemma 2.2.4] or Corollary 4.3). Thus $\operatorname{Spec} \mathbb{C}[\mathcal{O}_\Theta] = \mathcal{O}_\Theta$. $\square$

3 Reflection functors for quiver schemes

In this section, we modify the arguments made in [12, Section 4] to generalize the reflection functors of Lusztig [7], Maffei [8] and Nakajima [10] for quiver schemes. Fix a quiver with multiplicities $(\mathsf{Q}, \mathbf{d})$ with Q having no edge-loops.

3.1 Reflection functors

Let $\mathbf{C}$, $\mathbf{D}$ be the symmetrizable generalized Cartan matrix and the symmetrizer defined in Remark 2.9. Fix a realization $(\mathfrak{h}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ of $\mathbf{C}$ in the sense of [6]; so $\mathfrak{h}$ is the Cartan subalgebra, $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ is the set of simple roots, and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$ is the set of simple coroots. Let $Q \coloneqq \sum_{i\in I} \mathbb{Z}\alpha_i$ be the root lattice and identify it with $\mathbb{Z}^I$ using the basis $\{\alpha_i\}_{i\in I}$. Then the dimension vectors of finite dimensional I-graded $\mathbb{C}$-vector spaces live in the subset $Q_+ \coloneqq \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$.

Recall that the Weyl group $W(\mathbf{C})$ of $\mathbf{C}$ is the subgroup of $\mathrm{GL}_{\mathbb{C}}(\mathfrak{h}^*)$ generated by the simple reflections

$$s_i \colon \mathfrak{h}^* \to \mathfrak{h}^*; \quad \boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} - \langle \boldsymbol{\lambda}, \alpha_i^\vee \rangle\, \alpha_i \quad (i \in I).$$

The group $W(\mathbf{C})$ is a Coxeter group with defining relations

$$s_i^2 = \mathrm{Id}_{\mathfrak{h}^*},\ (s_i s_j)^{m_{ij}} = \mathrm{Id}_{\mathfrak{h}^*} \quad (i, j \in I,\ i \neq j),$$

where m_{ij} are determined from $c_{ij}c_{ji}$ by the following table.

$c_{ij}c_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

We define an $W(\mathbf{C})$-action on $R_{\mathbf{d}} \times Q$. The action on the second component Q is just the restriction of the action on $\mathfrak{h}^*$; explicitly,

$$s_i(\mathbf{v}) = \mathbf{v} - \sum_{j\in I} c_{ij} v_j \alpha_i, \quad \mathbf{v} = \sum_{j\in I} v_j \alpha_j \in Q \quad (i \in I).$$

This action is effective; so we may regard $W(\mathbf{C})$ as a subgroup of $\mathrm{GL}_{\mathbb{Z}}(Q)$. On the other hand, the action on the first component $R_{\mathbf{d}}$ is defined by

$$r_i(\boldsymbol{\lambda}) = (r_i(\boldsymbol{\lambda})_j)_{j\in I}, \quad r_i(\boldsymbol{\lambda})_j \coloneqq \begin{cases} -\lambda_i & (j = i), \\ \lambda_j - \sum_{l=0}^{d_{ij}-1} \lambda_{i,(d_i - \frac{d_i}{d_{ij}} l - 1)} c_{ij} \epsilon_j^{d_j - \frac{d_j}{d_{ij}} l - 1} & (j \neq i), \end{cases}$$

where $\boldsymbol{\lambda} = (\lambda_i)_{i\in I} \in R_{\mathbf{d}}$, $\lambda_i = \sum_{k=0}^{d_i-1} \lambda_{i,k} \epsilon_i^k$.

Proposition 3.1. *The above $r_i, i \in I$ satisfy relations* (3.1).

Proof. For each $i \in I$, the transpose $\tilde{s}_i \colon R_{\mathbf{d}} \to R_{\mathbf{d}}$ of r_i is explicitly described as

$$\tilde{s}_i(\mathbf{v}) = \mathbf{v} - \sum_{j \in I} c_{ij} \sum_{m=0}^{d_{ij}-1} v_{j, f_{ij} m} \epsilon_i^{f_{ji} m} \mathbf{e}_i \quad \left(\mathbf{v} = \left(\sum v_{i,k} \epsilon_i^k \right) \in R_{\mathbf{d}} \right),$$

where $\mathbf{e}_i := (\delta_{ij})_{j \in I}$. Put $\tilde{I} = \{ (i,k) \mid i \in I,\ k = 0, 1, \ldots, d_i - 1 \}$ and define a matrix $\widetilde{\mathbf{C}} = \left(\tilde{c}_{(i,k)(j,l)} \right) \in \mathbb{C}^{\tilde{I} \times \tilde{I}}$ so that

$$\tilde{s}_i(\mathbf{v})_{i,k} = v_{i,k} - \sum_{(j,l) \in \tilde{I}} \tilde{c}_{(i,k)(j,l)} v_{j,l};$$

explicitly,

$$\tilde{c}_{(i,k)(j,l)} = \begin{cases} c_{ij} & (k = f_{ji} m,\ l = f_{ij} m \text{ for some } m \in \mathbb{Z}), \\ 0 & (\text{otherwise}). \end{cases}$$

Then the matrix $\widetilde{\mathbf{C}}$ is a symmetrizable generalized Cartan matrix with symmetrizer $\widetilde{\mathbf{D}} = \operatorname{diag}(\tilde{d}_{i,k})$, where $\tilde{d}_{i,k} := d_i$. For $(i,k) \in \tilde{I}$, let $s_{i,k} \colon \mathbb{Z}^{\tilde{I}} \to \mathbb{Z}^{\tilde{I}}$ be the (i,k)-th simple reflection acting on the root lattice $\mathbb{Z}^{\tilde{I}}$ for $\widetilde{\mathbf{C}}$. Then for any $i \in I$, the reflections $s_{i,0}, s_{i,1}, \ldots, s_{i,d_i-1}$ commute pairwise, and $\tilde{s}_i$ coincides with the linear map

$$(s_{i,0} s_{i,1} \cdots s_{i,d_i-1}) \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{C}} \colon \mathbb{C}^{\tilde{I}} \to \mathbb{C}^{\tilde{I}}$$

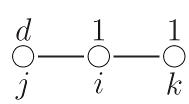
under the obvious identification $\mathbb{C}^{\tilde{I}} = R_{\mathbf{d}}$. Now relations (3.1) follow from the defining relations for the Weyl group $W(\widetilde{\mathbf{C}})$. $\square$

Remark 3.2. Define a linear map $\rho \colon R_{\mathbf{d}} \to \mathbb{C}^I$ by

$$\rho \colon (\lambda_i)_{i \in I} \mapsto \left(\operatorname*{res}_{\epsilon_i = 0} \left(\lambda_i \frac{d\epsilon_i}{\epsilon_i^{d_i}} \right) \right)_{i \in I}.$$

Then one can easily check that $\rho(r_i(\boldsymbol{\lambda})) = {}^t s_i(\rho(\boldsymbol{\lambda}))$ for all $i \in I$. In particular, if $d_i = 1$ for all $i \in I$, then the $W(\mathbf{C})$-action on $R_{\mathbf{d}} = \mathbb{C}^I$ is dual to that on $Q \otimes_{\mathbb{Z}} \mathbb{C}$.

Example 3.3. (i) Suppose that $(\mathsf{Q}, \mathbf{d})$ has the graph with multiplicities given below



Here we assume $d > 1$. Then the corresponding generalized Cartan matrix is

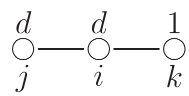
$$2\mathrm{Id} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -d & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

We have

$$r_i(\boldsymbol{\lambda})_i = -\lambda_i, \quad r_i(\boldsymbol{\lambda})_j = \lambda_i - c_{ij} \lambda_i \epsilon_j^{d-1} = \lambda_j + d\lambda_i \epsilon_j^{d-1}, \quad r_i(\boldsymbol{\lambda})_k = \lambda_k - c_{ik} \lambda_i = \lambda_i + \lambda_k.$$

It coincides with the ones in [12, Section 4]. In general, if $\gcd(d_i, d_j) = 1$ for all $j \in I$ joining the vertex i, then the action coincides with the action in [12, Section 4].

(ii) Suppose that $(\mathsf{Q}, \mathbf{d})$ has the graph with multiplicities given below



Here we assume $d > 1$. Then the Cartan matrix of it is

$$2\mathrm{Id} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -d & -1 \\ -d & 2 & 0 \\ -d & 0 & 2 \end{pmatrix}.$$

We have

$$r_i(\boldsymbol{\lambda})_i = -\lambda_i, \quad r_i(\boldsymbol{\lambda})_j = \lambda_j - c_{ij}\lambda_i = \lambda_j + d\lambda_i, \quad r_i(\boldsymbol{\lambda})_k = \lambda_k - c_{ik}\lambda_{i,d-1} = \lambda_k + \lambda_{i,d-1}.$$

Theorem 3.4. *Suppose that Q has no edge-loops. Take $\boldsymbol{\lambda} \in R_{\mathbf{d}}$, $\mathbf{v} \in Q_+$, $i \in I$ and suppose that $\lambda_i \in R_{d_i}$ is a unit. Then there exists an isomorphism of $\mathbb{C}$-schemes*

$$\mathcal{F}_i \colon \mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v}) \xrightarrow{\sim} \mathcal{S}_{\mathsf{Q},\mathbf{d}}(r_i(\boldsymbol{\lambda}), s_i(\mathbf{v})).$$

The map $\mathcal{F}_i$ generalizes the i-th reflection functor of quiver varieties [10].

We will prove this theorem in the next subsection.

3.2 Proof of Theorem 3.4

For $h \in H$, define $V_h \coloneqq \sum_{l=0}^{f_{\overline{h}}-1} V_{s(h)}\epsilon_{s(h)}^l$ so that $V_h \otimes R_{d_h} = V_{s(h)} \otimes R_{d_{s(h)}}$. Then the extension of scalar gives isomorphisms

$$\begin{aligned} \alpha_h &\colon \mathrm{Hom}_{\mathbb{C}}(V_h, V_{t(h)} \otimes R_{d_{t(h)}}) \xrightarrow{\sim} \mathrm{Hom}_{R_{d_h}}(V_{s(h)} \otimes R_{d_{s(h)}}, V_{t(h)} \otimes R_{d_{t(h)}}), \\ \beta_h &\colon \mathrm{Hom}_{\mathbb{C}}(V_{s(h)} \otimes R_{d_{s(h)}}, V_{\overline{h}}) \xrightarrow{\sim} \mathrm{Hom}_{R_{d_h}}(V_{s(h)} \otimes R_{d_{s(h)}}, V_{t(h)} \otimes R_{d_{t(h)}}). \end{aligned}$$

Fix a vertex $i \in I$ and set $\widetilde{V}_i \coloneqq \bigoplus_{t(h)=i} V_h$, so

$$\dim \widetilde{V}_i = \sum_{t(h)=i} \dim V_h = \sum_{t(h)=i} f_{\overline{h}} v_{s(h)} = \sum_{j \in I} a_{ij} \frac{d_j}{d_{ij}} v_j.$$

Then we can decompose the vector space $\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$ as

$$\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \simeq \mathrm{Hom}_{\mathbb{C}}\left(\widetilde{V}_i, V_i \otimes R_{d_i}\right) \oplus \mathrm{Hom}_{\mathbb{C}}\left(V_i \otimes R_{d_i}, \widetilde{V}_i\right) \oplus \mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}),$$

where $\mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) := \bigoplus_{t(h),s(h)\neq i} \operatorname{Hom}_{\mathbb{C}}\left(V_{s(h)} \otimes R_{d_{s(h)}}, V_{t(h)} \otimes R_{d_{t(h)}}\right)$. Each $\mathbf{B} \in \mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$ corresponds to the triple $(B_{i\leftarrow}, B_{\leftarrow i}, B_{\neq i})$, where

$$\begin{aligned}
B_{i\leftarrow} &:= (\operatorname{sgn}(h)\alpha_h^{-1}(B_h))_{t(h)=i} \in \operatorname{Hom}_{\mathbb{C}}\left(\widetilde{V}_i, V_i \otimes R_{d_i}\right), \\
B_{\leftarrow i} &:= (\beta_{\overline{h}}^{-1}(B_{\overline{h}}))_{t(h)=i} \in \operatorname{Hom}_{\mathbb{C}}\left(V_i \otimes R_{d_i}, \widetilde{V}_i\right), \\
B_{\neq i} &:= (B_h)_{t(h),s(h)\neq i} \in \mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}),
\end{aligned}$$

and the group $G_{d_i}(V_i)$ acts trivially on $\mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V})$.

Applying Proposition 2.11 with $l = 1$ to the symplectic vector space

$$\operatorname{Hom}_{\mathbb{C}}\left(\widetilde{V}_i, V_i \otimes R_{d_i}\right) \oplus \operatorname{Hom}_{\mathbb{C}}\left(V_i \otimes R_{d_i}, \widetilde{V}_i\right),$$

we obtain the following corollary (we also use the description of the moment map given in Remark 2.3).

Corollary 3.5. *Let λ_i be a unit of R_{d_i}.*

(i) *If $\dim \widetilde{V}_i < \dim V_i$, then the set $\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$ is empty.*

(ii) *If $\dim \widetilde{V}_i \geq \dim V_i$, then the $G_{d_i}(V_i)$-action on $\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$ has a geometric quotient, and the map*

$$\Phi_i\colon \mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) \to \mathfrak{g}_{d_i}(\widetilde{V}_i); \quad \mathbf{B} \mapsto -\epsilon_{d_i}^{d_i} B_{\leftarrow i}(\epsilon_{d_i} - N_i)^{-1} B_{i\leftarrow},$$

where $N_i \in \mathfrak{gl}_{\mathbb{C}}(V_i \otimes R_{d_i})$ is the multiplication by ϵ_{d_i}, induces a symplectic isomorphism

$$\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})/G_{d_i}(V_i) \simeq \mathcal{O} \times \mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}),$$

where $\mathcal{O}$ is the $G_{d_i}(\widetilde{V}_i)$-coadjoint orbit consisting of elements having a matrix representation of the form $\operatorname{diag}(\lambda, \ldots, \lambda, 0, \ldots, 0)$ *with λ appearing* $\dim V_i$ *times in the diagonal entries.*

Proof. Suppose that $\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$ is non-empty and take any $\mathbf{B} \in \mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$. Since λ_i is a unit, the moment map condition $B_{i\leftarrow}^{R_{d_i}} B_{\leftarrow i}^{R_{d_i}} = -\lambda_i \operatorname{Id}_{V_i}$ implies that $B_{i\leftarrow}^{R_{d_i}}$ is surjective and $B_{\leftarrow i}^{R_{d_i}}$ is injective. In particular, we have $\dim \widetilde{V}_i \geq \dim V_i$. (ii) follows from Proposition 2.11. □

By Corollary 3.5, the level set $\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$ is non-empty if and only if

$$v_i \leq \dim \widetilde{V}_i = 2v_i - \sum_j c_{ij} v_j,$$

which is equivalent to $s_i(\mathbf{v}) \in \mathbb{Z}^I_{\geq 0}$ as the i-th component of $s_i(\mathbf{v})$ is equal to $\dim \widetilde{V}_i - \dim V_i$. We assume this condition, because otherwise $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ and $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(r_i(\boldsymbol{\lambda}), s_i(\mathbf{v}))$ are both empty. We embed V_i into $\widetilde{V}_i$ as a vector subspace and take any complement V_i', so $\widetilde{V}_i = V_i \oplus V_i'$. By Corollary 3.5, we have an isomorphism

$$\mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})/G_{d_i}(V_i) \xrightarrow{\sim} \mathcal{O} \times \mathbf{M}^{(i)}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}),$$

where $\mathcal{O}$ is the $G_{d_i}(\widetilde{V}_i)$-coadjoint orbit of

$$\Lambda = \begin{pmatrix} \lambda_i \operatorname{Id}_{V_i} & 0 \\ 0 & 0 \operatorname{Id}_{V_i'} \end{pmatrix}.$$

We define an I-graded $\mathbb{C}$-vector space $\mathbf{V}'$ by

$$\mathbf{V}' = \bigoplus_{j \in I} V_j', \quad V_j' = \begin{cases} V_i' & \text{if } j = i, \\ V_j & \text{if } j \neq i. \end{cases}$$

Then $\dim \mathbf{V}' = s_i(\mathbf{v})$. By replacing $\mathbf{V}$ and λ_i with $\mathbf{V}'$ and $-\lambda_i$, respectively in Corollary 3.5, we also have an isomorphism

$$\mu_{\mathbf{d},i}^{-1}(\lambda_i \operatorname{Id}_{V_i'})/G_{d_i}(V_i') \xrightarrow{\sim} \mathcal{O}' \times \mathbf{M}_{\mathbf{Q},\mathbf{d}}^{(i)}(\mathbf{V}),$$

where $\mathcal{O}'$ is the $G_{d_i}(\widetilde{V}_i)$-coadjoint orbit of

$$\Lambda' = \begin{pmatrix} 0 \operatorname{Id}_{V_i} & 0 \\ 0 & -\lambda_i \operatorname{Id}_{V_i'} \end{pmatrix} = \Lambda - \lambda \operatorname{Id}_{\widetilde{V}_i}.$$

Note that $\mathbf{M}_{\mathbf{Q},\mathbf{d}}^{(i)}(\mathbf{V}') = \mathbf{M}_{\mathbf{Q},\mathbf{d}}^{(i)}(\mathbf{V})$. Therefore, the scalar shift $\mathcal{O} \xrightarrow{\sim} \mathcal{O}'$ induces an isomorphism

$$\widetilde{\mathcal{F}}_i \colon \mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})/G_{d_i}(V_i) \xrightarrow{\sim} \mu_{\mathbf{d},i}^{-1}(\lambda_i \operatorname{Id}_{V_i'})/G_{d_i}(V_i').$$

For $\mathbf{B} \in \mu_{\mathbf{d},i}^{-1}(-\lambda_i \operatorname{Id}_{V_i})$, take $\mathbf{B}' \in \mu_{\mathbf{d},i}^{-1}(\lambda_i \operatorname{Id}_{V_i'})$ so that $\widetilde{\mathcal{F}}_i[\mathbf{B}] = [\mathbf{B}']$.

Lemma 3.6. *If* $\mu_{\mathbf{d}}(\mathbf{B}) = -\boldsymbol{\lambda} \operatorname{Id}_{\mathbf{V}}$, *then* $\mu_{\mathbf{d}}(\mathbf{B}') = -r_i(\boldsymbol{\lambda}) \operatorname{Id}_{\mathbf{V}'}$.

Proof. By the definition, $\Phi_i(\mathbf{B}')$ equals to $\Phi_i(\mathbf{B}) - \lambda_i \operatorname{Id}_{\widetilde{V}_i}$. Thus we have

$$\sum_{k=0}^{d_i-1} B'_{\leftarrow i}(N_i')^k B'_{i\leftarrow} \epsilon_i^{-k-1} = \sum_{k=0}^{d_i-1} B_{\leftarrow i} N_i^k B_{i\leftarrow} \epsilon_i^{-k-1} + \epsilon_i^{-d_i} \lambda_i \operatorname{Id}_{\widetilde{V}_i},$$

where $N_i \in \mathfrak{gl}_{\mathbb{C}}(V_i \otimes R_{d_i})$, $N_i' \in \mathfrak{gl}_{\mathbb{C}}(V_i' \otimes R_{d_i})$ are the multiplication by ϵ_i. This implies that for any arrow h with $t(h) = i$, the following equality holds:

$$\sum_{k=0}^{d_i-1} \operatorname{sgn}(h) \beta_{\overline{h}}^{-1}(B'_{\overline{h}})(N_i')^k \alpha_h^{-1}(B'_h) \epsilon_i^{-k-1} = \sum_{k=0}^{d_i-1} \operatorname{sgn}(h) \beta_{\overline{h}}^{-1}(B_{\overline{h}}) N_i^k \alpha_h^{-1}(B_h) \epsilon_i^{-k-1} + \epsilon_i^{-d_i} \lambda_i \operatorname{Id}_{V_h}.$$

For all $l = 0, \ldots, d_h - 1$, comparing the coefficient of $\epsilon_i^{-f_h l - 1}$ in the above equality yields

$$\operatorname{sgn}(h) \beta_{\overline{h}}^{-1}(B'_{\overline{h}})(N_h')^l \alpha_h^{-1}(B'_h) = \operatorname{sgn}(h) \beta_{\overline{h}}^{-1}(B_{\overline{h}}) N_h^l \alpha_h^{-1}(B_h) + \lambda_{i,(d_i - f_h l - 1)} \operatorname{Id}_{V_h},$$

where $N_h = N_i^{f_h}$ and $N'_h = (N'_i)^{f_h}$. On the other hand, for $B \in \mathrm{Hom}_{\mathbb{C}}(V_h, V_{t(h)} \otimes R_{d_h})$ and $\overline{B} \in \mathrm{Hom}_{\mathbb{C}}(V_{s(h)} \otimes R_{s(h)}, V_{\overline{h}})$ we have

$$\beta_{\overline{h}}(\overline{B})\alpha_h(B) = \epsilon_h^{d_h}\overline{B}(\epsilon_h \mathrm{Id} - N_{d_h})^{-1}B = \epsilon_h^{d_h}\sum_{l=0}^{d_h-1}\overline{B}N_h^l B\epsilon_h^{-l-1}.$$

Thus we obtain

$$\mathrm{sgn}(h)B'_{\overline{h}}B'_h = \mathrm{sgn}(h)B_{\overline{h}}B_h + \sum_{l=0}^{d_h-1}\lambda_{i,(d_i-f_h l-1)}\epsilon_h^{d_h-l-1}\mathrm{Id}_{V_h}.$$

Replacing h with $\overline{h}$, we also obtain

$$\mathrm{sgn}(h)B'_h B'_{\overline{h}} = \mathrm{sgn}(h)B_h B_{\overline{h}} - \sum_{l=0}^{d_h-1}\lambda_{i,(d_i-f_{\overline{h}}l-1)}\epsilon_h^{d_h-l-1}\mathrm{Id}_{V_{\overline{h}}}$$

for arrow h with $s(h) = i$. Note that

$$\mathrm{pr}_{d_h,d_{t(h)}}\left(\epsilon_h^{d_h-l-1}\mathrm{Id}_{V_{\overline{h}}}\right) = \sum_{k=0}^{f_h-1}\epsilon_{t(h)}^k\epsilon_h^{d_h-l-1}\epsilon_{t(h)}^{f_h-k-1}\mathrm{Id}_{V_{t(h)}} = f_h\epsilon_{t(h)}^{d_{t(h)}-f_h l-1}\mathrm{Id}_{V_{t(h)}}.$$

Thus

$$\mathrm{pr}_{d_h,d_{t(h)}}\left(\mathrm{sgn}(h)B'_h B'_{\overline{h}}\right) = \mathrm{pr}_{d_h,d_{t(h)}}\left(\mathrm{sgn}(h)B_h B_{\overline{h}}\right) - \sum_{l=0}^{d_h-1}\lambda_{i,(d_i-f_{\overline{h}}l-1)}f_h\epsilon_{t(h)}^{d_{t(h)}-f_h l-1}\mathrm{Id}_{V_{t(h)}}.$$

On the other hand, since $B'_h = B_h$ whenever $t(h), s(h) \neq i$, we have

$$\mathrm{pr}_{d_h,d_{t(h)}}(B'_h B'_{\overline{h}}) = \mathrm{pr}_{d_h,d_{s(h)}}(B_h B_{\overline{h}}).$$

Thus, for all $j \neq i$, we obtain

$$\begin{aligned}\mu_{\mathbf{d},j}(\mathbf{B}') &= \mu_{\mathbf{d},j}(\mathbf{B}) - \sum_{\substack{h\in H\\ s(h)=i,t(h)=j}}\sum_{l=0}^{d_h-1}\lambda_{i,(d_i-f_{\overline{h}}l-1)}f_h\epsilon_j^{d_j-f_h l-1}\mathrm{Id}_{V_{t(h)}}\\ &= \mu_{\mathbf{d},j}(\mathbf{B}) + \sum_{l=0}^{d_{ij}-1}c_{ij}\lambda_{i,(d_i-\frac{d_i}{d_{ij}}l-1)}\epsilon_j^{d_j-\frac{d_j}{d_{ij}}l-1}\mathrm{Id}_{V_j},\end{aligned}$$

whence the result.

$\square$

Proof of Theorem 3.4. Since the morphism

$$\mu^{-1}_{\mu^{-1}(-\lambda_i\,\mathrm{Id}_{V_i})/G_{d_i}(V_i)}(-\boldsymbol{\lambda}\,\mathrm{Id}_{\mathbf{V}}) \to \mu^{-1}_{\mu^{-1}(\lambda_i \mathrm{Id}_{V'_i})/G_{d_i}(V'_i)}(-r_i(\boldsymbol{\lambda})\mathrm{Id}_{\mathbf{V}'})$$

is a $\prod_{j\neq i} G_{d_j}(V_j)$-equivariant, it induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mu^{-1}_{\mu^{-1}(-\lambda_i \operatorname{Id}_{V_i})/G_{d_i}(V_i)}(-\boldsymbol{\lambda}\operatorname{Id}_{\mathbf{V}})]^{\prod_{j\neq i} G_{d_j}(V_j)} \xrightarrow{\sim} \mathbb{C}[\mu^{-1}_{\mu^{-1}(\lambda_i \operatorname{Id}_{V'_i})/G_{d_i}(V'_i)}(-r_i(\boldsymbol{\lambda})\operatorname{Id}_{\mathbf{V}'})]^{\prod_{j\neq i} G_{d_j}(V_j)}.$$

Therefore we obtain an isomorphism of $\mathbb{C}$-schemes

$$\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v}) \xrightarrow{\sim} \mathcal{S}_{\mathsf{Q},\mathbf{d}}(r_i(\boldsymbol{\lambda}), s_i(\mathbf{v})).$$

□

4 Regularization

In this section, we generalize [12, Theorem 5.8] using a result of Hausel–Wong–Wyss.

4.1 Shifting trick

In this subsection we fix a finite dimensional $\mathbb{C}$-vector space V together with a positive integer d, and recall a sort of "shifting trick" found by Boalch [1] relating to the $G_d(V)$-coadjoint orbits considered in Section 2.3. For simplicity, we put $G := \mathrm{GL}_{\mathbb{C}}(V)$ and $\mathfrak{g} := \mathfrak{gl}_{\mathbb{C}}(V)$.

Let $B_d(V)$ be the kernel of the homomorphism

$$G_d(V) \to G; \quad g = \sum_{k=0}^{d-1} g_k \epsilon^k \mapsto g_0,$$

and $\mathfrak{b}_d(V)$ be its Lie algebra. Then we have a direct sum decomposition

$$\mathfrak{g}_d(V) = \mathfrak{g} \oplus \mathfrak{b}_d(V).$$

Taking dual via the pairing on $\mathfrak{g}_d(V)$, we also have a decomposition

$$\mathfrak{g}_d(V) = \epsilon_i^{d-1}\mathfrak{g}_d(V) \oplus \mathfrak{b}_d^*(V), \tag{4.1}$$

where

$$\mathfrak{b}_d^*(V) := \sum_{k=0}^{d-2} \mathfrak{gl}_{\mathbb{C}}(V)\epsilon_i^k \simeq \mathfrak{g}_d(V)/\epsilon^{d-1}\mathfrak{g}_d(V).$$

It may be regarded as the dual space of $\mathfrak{b}_d(V)$, and the coadjoint action of $g \in B_d(V)$ is given by

$$g \cdot \eta = g\eta g^{-1} \mod \epsilon_i^{d-1}\mathfrak{g}_d(V).$$

According to the decomposition (4.1), we can decompose $A = \sum_{k=0}^{d-1} A_k \epsilon^k \in \mathfrak{g}_d(V)$ as

$$A = \epsilon^{d-1}A_{d-1} + A^0, \quad A^0 \in \mathfrak{b}_d^*(V).$$

Now take any direct sum decomposition $V = \bigoplus_{i=0}^{l} W_i$ and elements $\theta_0, \theta_1, \ldots, \theta_l \in R_d$ so that $\theta_i - \theta_j$ is a unit whenever $i \neq j$. Put

$$\Theta \coloneqq \bigoplus_{i=0}^{l} \theta_i \operatorname{Id}_{W_i \otimes R_d} \in \mathfrak{g}_d(V),$$

and consider the $G_d(V)$-coadjoint orbit $\mathcal{O}_\Theta$ of Θ as in Section 2.3. Let $\check{\mathcal{O}}_\Theta \subset \mathfrak{b}_d^*(V)$ be the $B_d(V)$-coadjoint orbit of Θ^0 and put

$$G_\Theta \coloneqq \prod_{i=0}^{l} \mathrm{GL}_{\mathbb{C}}(W_i) \subset G,$$

whose Lie algebra is $\mathfrak{g}_\Theta \coloneqq \bigoplus_{i=0}^{l} \mathfrak{gl}_{\mathbb{C}}(W_i) \subset \mathfrak{g}$. Using the trace pairing we identify the dual space $\mathfrak{g}_\Theta^*$ with $\mathfrak{g}_\Theta$. Since $gbg^{-1} \in B_d(V)$ and $g\Theta^0 g^{-1} = \Theta^0$ for all $g \in G_\Theta$ and $b \in B_d(V)$, we see that the orbit $\check{\mathcal{O}}_\Theta$ is invariant under the conjugation by G_Θ.

Proposition 4.1. *There exists an G_Θ-equivariant symplectic isomorphism*

$$\check{\mathcal{O}}_\Theta \xrightarrow{\sim} \bigoplus_{i<j} \mathrm{Hom}_{\mathbb{C}}(W_i, W_j)^{\oplus(d-2)} \oplus \bigoplus_{i>j} \mathrm{Hom}_{\mathbb{C}}(W_i, W_j)^{\oplus(d-2)}$$

sending Θ^0 to the origin.

Proof. This is a special case of [5, Corollary 3.9]. □

In particular, $\check{\mathcal{O}}_\Theta$ is affine and the G_Θ-action on $\check{\mathcal{O}}_\Theta$ admits a moment map $\mu_{\check{\mathcal{O}}} \colon \check{\mathcal{O}}_\Theta \to \mathfrak{g}_\Theta^* \simeq \mathfrak{g}_\Theta$ with $\mu_{\check{\mathcal{O}}}(\Theta^0) = 0$.

We let G_Θ act on the cotangent bundle T^*G via the left translation and consider the diagonal action on the product $T^*G \times \check{\mathcal{O}}_\Theta$, which has a moment map

$$\mu_{T^*G \times \check{\mathcal{O}}_\Theta} \colon T^*G \times \check{\mathcal{O}}_\Theta \to \mathfrak{g}_\Theta^*; \quad (g, R, B) \mapsto -\operatorname{pr}_{\mathfrak{g}_\Theta}(gRg^{-1}) + \mu_{\check{\mathcal{O}}}(B),$$

where T^*G is identified with $G \times \mathfrak{g}$ via the left translation and $\operatorname{pr}_{\mathfrak{g}_\Theta} \colon \mathfrak{g} \to \mathfrak{g}_\Theta$ is the transpose of the inclusion $\mathfrak{g}_\Theta \hookrightarrow \mathfrak{g}$.

Note that Θ_{d-1} lies in $\mathfrak{g}_\Theta^{G_\Theta}$.

Proposition 4.2. *The G_Θ-action on the level set $\mu_{T^*G \times \check{\mathcal{O}}}^{-1}(-\Theta_{d-1})$ is free and the affine quotient $\mu_{T^*G \times \check{\mathcal{O}}}^{-1}(-\Theta_{d-1})/G_\Theta$ is a geometric quotient. Moreover, the map*

$$\mu_{T^*G \times \check{\mathcal{O}}}^{-1}(-\Theta_{d-1}) \to \mathfrak{g}_d(V); \quad (g, R, B) \mapsto \epsilon^{d-1} R + g^{-1} B g$$

induces a symplectic isomorphism

$$\mu_{T^*G \times \check{\mathcal{O}}}^{-1}(-\Theta_{d-1})/G_\Theta \xrightarrow{\sim} \mathcal{O}_\Theta.$$

Proof. The G_Θ-action on $\mu^{-1}_{T^*G\times\check{\mathcal{O}}}(-\Theta_{d-1})$ is free as G_Θ acts freely on T^*G. Hence all the G_Θ-orbits have equal dimension, and hence are closed. Thus [11, Theorem 4.10] implies that the affine quotient $\mu^{-1}_{T^*G\times\check{\mathcal{O}}}(-\Theta_{d-1})/G_\Theta$ is a geometric quotient. For the rest assertions, see [5, Propositions 2.6, 2.12]. ☐

Corollary 4.3. *The orbit $\mathcal{O}_\Theta$ is affine.*

Corollary 4.4. *Let M be a non-singular affine symplectic variety acted on by G in Hamiltonian fashion with moment map $\mu_M\colon M\to\mathfrak{g}$. Then for each $\zeta\in\mathbb{C}$, the map*

$$\psi\colon \check{\mathcal{O}}_\Theta\times M\to\mathfrak{g}_d(V)\times M;\quad (B,x)\mapsto(B-\epsilon^{d-1}\mu_M(x)-\epsilon^{d-1}\zeta\,\mathrm{Id}_V,x)$$

induces an isomorphism between affine quotients

$$\mu^{-1}_{\check{\mathcal{O}}\times M}(-\Theta_{d-1}-\zeta\,\mathrm{Id}_V)/G_\Theta\xrightarrow{\sim}\mu^{-1}_{\mathcal{O}\times M}(-\zeta\,\mathrm{Id}_V)/G,$$

where $\mu_{\check{\mathcal{O}}\times M}$ and $\mu_{\mathcal{O}\times M}$ are the moment maps

$$\mu_{\check{\mathcal{O}}\times M}(B,x)=\mu_{\check{\mathcal{O}}}(B)+\mathrm{pr}_{\mathfrak{g}_\Theta}(\mu_M(x)),\quad \mu_{\mathcal{O}\times M}(A,x)=A_{d-1}+\mu_M(x)\quad (B\in\check{\mathcal{O}}_\Theta,\ A\in\mathcal{O}_\Theta,\ x\in M).$$

Proof. By the above proposition, the Hamiltonian reduction of $\mathcal{O}\times M$ by the G-action at level $-\zeta\,\mathrm{Id}_V$ is isomorphic to that of $T^*G\times\check{\mathcal{O}}_\Theta\times M$ by the $G_\Theta\times G$-action

$$(u,v)\colon (g,R,B,x)\mapsto(ugv^{-1},vRv^{-1},uBu^{-1},v\cdot x),\quad (u,v)\in G_\Theta\times G$$

at level $(-\Theta_{d-1},-\zeta\,\mathrm{Id}_V)$. If we first perform the Hamiltonian reduction by G, then the result is isomorphic to $\check{\mathcal{O}}_\Theta\times M$ via the map

$$\check{\mathcal{O}}_\Theta\times M\to T^*G\times\check{\mathcal{O}}_\Theta\times M;\quad (B,x)\mapsto(\mathrm{Id}_V,-\mu_M(x)-\zeta\,\mathrm{Id}_V,B,x),$$

with the induced G_Θ-moment map equal to $\mu_{\check{\mathcal{O}}\times M}+\zeta\,\mathrm{Id}_V$. Thus performing further the Hamiltonian reduction by G_Θ, we obtain a desired isomorphism $\mu^{-1}_{\check{\mathcal{O}}\times M}(-\Theta_{d-1}-\zeta\,\mathrm{Id}_V)/G_\Theta\xrightarrow{\sim}\mu^{-1}_{\mathcal{O}\times M}(-\zeta\,\mathrm{Id}_V)/G$, which is explicitly given by $(B,x)\mapsto(-\epsilon^{d-1}(\mu_M(x)+\zeta\,\mathrm{Id}_V)+B,x)$. ☐

4.2 Irregular legs and regularization

Let $\mathsf{Q}=(I,\Omega,s,t)$ be a quiver with multiplicities $\mathbf{d}$. For integers $i<j$ we put $[i,j]:=\{i,i+1,\dots,j\}$.

Definition 4.5. $(\mathsf{Q},\mathbf{d})$ is said to have an *irregular leg* if there exists a sequence of pairwise distinct vertices such that, if we denote it by $0,1,\dots,l$, then $l>0$ and the following hold:

1. vertices i,j in $[0,l]$ are connected by exactly one arrow if $|i-j|=1$, and otherwise no arrow connects them;
2. no arrow connects any $i\in I\setminus[0,l]$ and $j\in[1,l]$;

3. $d_0 = 1$ and $d_i = d$ $(i = 1, 2, \dots, l)$ for some integer $d > 1$.

In what follows we consider such a quiver with multiplicities, and for simplicity, assume that the arrows connecting $0, 1, \dots, l$ are oriented as $0 \to 1 \to \cdots \to l$. We denote by $\mathsf{Q}_{\mathrm{leg}} = ([1, l], \Omega_{\mathrm{leg}}, s, t)$ the subquiver $1 \to 2 \to \cdots \to l$ and call it the *irregular leg* of $(\mathsf{Q}, \mathbf{d})$ with *base* 0.

Definition 4.6. Let $\check{\mathsf{Q}} = (I, \check{\Omega}, s, t)$ be the quiver obtained from Q by the following procedure:

1. first, delete the l arrows $0 \to 1 \to \cdots \to l$; then
2. for each arrow h with $t(h) = 0$ and each $i \in [1, l]$, add an arrow from $s(h)$ to i;
3. for each arrow h with $s(h) = 0$ and each $i \in [1, l]$, add an arrow from i to $t(h)$;
4. finally, for each pair $i < j$ in $[0, l]$, add $(d - 2)$ arrows from i to j.

Also, define $\check{\mathbf{d}} = (\check{d}_i)$ by

$$\check{d}_i = \begin{cases} 1 & (i \in [1, l]), \\ d_i & (i \in I \setminus [1, l]). \end{cases}$$

We call $(\check{\mathsf{Q}}, \check{\mathbf{d}})$ the *regularization* of $(\mathsf{Q}, \mathbf{d})$ along the irregular leg $\mathsf{Q}_{\mathrm{leg}}$.

Remark 4.7. When $l = 1$, the regularization is the same as the *normalization* introduced by the second author in [12].

We define a map $R_{\mathbf{d}} \times \mathbb{Z}^I \to R_{\check{\mathbf{d}}} \times \mathbb{Z}^I$ as follows. For $\mathbf{v} = (v_i) \in \mathbb{Z}^I$, define $\check{\mathbf{v}} = (\check{v}_i) \in \mathbb{Z}^I$ by

$$\check{v}_i = \begin{cases} v_i - v_{i+1} & (i \in [0, l-1]), \\ v_i & (\text{otherwise}). \end{cases}$$

Also, for $\boldsymbol{\lambda} = (\lambda_i) \in R_{\mathbf{d}}$, define $\check{\boldsymbol{\lambda}} = (\check{\lambda}_i) \in R_{\check{\mathbf{d}}}$ by

$$\check{\lambda}_i = \begin{cases} \lambda_0 & (i = 0), \\ \lambda_0 + \sum_{j=1}^{i} \lambda_{j,d-1} & (i \in [1, l]), \\ \lambda_i & (\text{otherwise}). \end{cases}$$

The following theorem generalizes [12, Theorem 5.8].

Theorem 4.8. *Let $(\mathsf{Q}, \mathbf{d})$ be a quiver with multiplicities having an irregular leg $\mathsf{Q}_{\mathrm{leg}}$ as above, and let $(\check{\mathsf{Q}}, \check{\mathbf{d}})$ be the regularization of $(\mathsf{Q}, \mathbf{d})$ along $\mathsf{Q}_{\mathrm{leg}}$. Take a pair $(\boldsymbol{\lambda}, \mathbf{v}) \in R_{\mathbf{d}} \times \mathbb{Z}^I_{\geq 0}$ satisfying the following conditions:*

1. *$\check{v}_i \geq 0$ for all $i \in [0, l-1]$;*
2. *$\lambda_i + \lambda_{i+1} + \cdots + \lambda_j \in R_d^{\times}$ for all pairs $i \leq j$ in $[1, l]$.*

Then $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ and $\mathcal{S}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\boldsymbol{\lambda}}, \check{\mathbf{v}})$ are isomorphic.

Proof. By the first condition on $(\boldsymbol{\lambda}, \mathbf{v})$, the sequence $v_0, v_1, \dots, v_l$ is non-increasing. We take I-graded $\mathbb{C}$-vector spaces $\mathbf{V}, \check{\mathbf{V}}$ so that $\dim \mathbf{V} = \mathbf{v}$, $V_0 \supset V_1 \supset \cdots \supset V_l$, and

$$V_i = \begin{cases} \check{V}_i \oplus V_{i+1} & (i \in [0, l-1]), \\ \check{V}_i & (\text{otherwise}). \end{cases}$$

Then $\dim \check{\mathbf{V}} = \check{\mathbf{v}}$ and $V_0 = \bigoplus_{i=0}^{l} \check{V}_i$.

In what follows, for a subset $L \subset I$, the suffix L means the restriction of the index set to L; for instance,

$$\mathbf{V}_L = \bigoplus_{i \in L} V_i, \quad \mathbf{d}_L = (d_i)_{i \in L}, \quad \boldsymbol{\lambda}_L = (\lambda_i)_{i \in L}, \quad \mu_{\mathbf{d},L} = (\mu_{\mathbf{d},i})_{i \in L}.$$

Let Q_J be the maximal subquiver of Q with vertex set $J \coloneqq I \setminus [1, l]$. Then

$$\mathbf{M}_{\mathsf{Q},\mathbf{d}}(\mathbf{V}) = \mathbf{M}_{\mathsf{Q}_J,\mathbf{d}_J}(\mathbf{V}_J) \oplus \mathbf{M}_{\mathsf{Q}_{\mathrm{leg}},\mathbf{d}_{[1,l]}}(\mathbf{V}_{[1,l]}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_0, V_1 \otimes R_d) \oplus \mathrm{Hom}_{\mathbb{C}}(V_1 \otimes R_d, V_0).$$

Also, let Q_K be the maximal subquiver of Q with vertex set $K \coloneqq J \setminus \{0\}$. Then Q_K is also a subquiver of $\check{\mathsf{Q}}$ and $\check{d}_i = d_i$, $\check{V}_i = V_i$ for all $i \in K$. Hence

$$\mathbf{M}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\mathbf{V}}) = \mathbf{M}_{\mathsf{Q}_K,\mathbf{d}_K}(\mathbf{V}_K) \oplus \bigoplus_{i=0}^{l} \left(\bigoplus_{\substack{t(h) \in K \\ s(h) = i}} \mathrm{Hom}_{\mathbb{C}}(\check{V}_i, \check{V}_{t(h)} \otimes R_{\check{d}_{t(h)}}) \oplus \bigoplus_{\substack{s(h) \in K \\ t(h) = i}} \mathrm{Hom}_{\mathbb{C}}(\check{V}_{s(h)} \otimes R_{\check{d}_{s(h)}}, \check{V}_i) \right)$$
$$\oplus \bigoplus_{i,j \in [0,l];\, i \neq j} \mathrm{Hom}_{\mathbb{C}}(\check{V}_i, \check{V}_j)^{\oplus(d-2)}.$$

By the definition of $\check{\mathsf{Q}}$ and the equality $V_0 = \bigoplus_{i=0}^{l} \check{V}_i$, we obtain a canonical isomorphism

$$\mathbf{M}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\mathbf{V}}) \simeq \mathbf{M}_{\mathsf{Q}_J,\mathbf{d}_J}(\mathbf{V}_J) \oplus \bigoplus_{i,j \in [0,l];\, i \neq j} \mathrm{Hom}_{\mathbb{C}}(\check{V}_i, \check{V}_j)^{\oplus(d-2)}.$$

Now define $\Theta \in \mathfrak{g}_d(V_0)$ by

$$\Theta = \bigoplus_{i=0}^{l} \theta_i \, \mathrm{Id}_{\check{V}_i}, \quad \theta_i = \begin{cases} 0 & (i = 0), \\ \lambda_1 + \cdots + \lambda_i & (i > 0). \end{cases}$$

Then $\theta_i - \theta_{i-1} = \lambda_i$ for $i \in [1, l]$ and

$$G_\Theta = \prod_{i=0}^{l} \mathrm{GL}_{\mathbb{C}}(\check{V}_i) = G_{\check{\mathbf{d}}_{[0,l]}}(\check{\mathbf{V}}_{[0,l]}).$$

The second condition on $(\boldsymbol{\lambda}, \mathbf{v})$ implies that $\theta_i - \theta_j \in R_d^\times$ whenever $i \neq j$. Therefore Proposition 4.1 implies that there exists an isomorphism

$$\mathbf{M}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\mathbf{V}}) \simeq \mathbf{M}_{\mathsf{Q}_J,\mathbf{d}_J}(\mathbf{V}_J) \times \check{\mathcal{O}}_\Theta.$$

On the other hand, Proposition 2.11 implies that the $G_{\mathbf{d}_{[1,l]}}(\mathbf{V}_{[1,l]})$-action on $\mu^{-1}_{\mathbf{d},[1,l]}(-\boldsymbol{\lambda}_{[1,l]}\operatorname{Id}_{\mathbf{V}_{[1,l]}})$ has a geometric quotient isomorphic to the affine variety $\mathbf{M}_{\mathsf{Q}_J,\mathbf{d}_J}(\mathbf{V}_J)\times\mathcal{O}_\Theta$. Therefore Corollary 4.4 shows that there exists an isomorphism between affine varieties

$$\mu^{-1}_{\mathbf{d},[0,l]}(-\boldsymbol{\lambda}_{[0,l]}\operatorname{Id}_{\mathbf{V}_{[0,l]}})/G_{\mathbf{d}_{[0,l]}}(\mathbf{V}_{[0,l]}) \simeq \mu^{-1}_{\check{\mathbf{d}},[0,l]}(-\check{\boldsymbol{\lambda}}_{[0,l]}\operatorname{Id}_{\mathbf{V}_{[0,l]}})/G_{\check{\mathbf{d}}_{[0,l]}}(\check{\mathbf{V}}_{[0,l]}).$$

Taking the affine quotients (as schemes) of the level sets of the $G_{\mathbf{d}_K}(\mathbf{V}_K)$-moment maps on both sides, we obtain a desired isomorphism $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})\simeq\mathcal{S}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})$. ☐

The following corollary is useful.

Corollary 4.9. *Let $(\mathsf{Q},\mathbf{d})$ be a quiver with multiplicities having an irregular leg $\mathsf{Q}_{\mathrm{leg}}$ as above with $l=1$, and let $(\check{\mathsf{Q}},\check{\mathbf{d}})$ be the regularization of $(\mathsf{Q},\mathbf{d})$ along $\mathsf{Q}_{\mathrm{leg}}$. Take a pair $(\boldsymbol{\lambda},\mathbf{v})\in R_{\mathbf{d}}\times\mathbb{Z}^I_{\geq 0}$ so that $\lambda_1\in R_d^\times$. Then $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ and $\mathcal{S}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})$ are isomorphic.*

Proof. If $\check{v}_0<0$, then Corollary 3.5 implies that both $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ and $\mathcal{S}_{\check{\mathsf{Q}},\check{\mathbf{d}}}(\check{\boldsymbol{\lambda}},\check{\mathbf{v}})$ are empty. If $\check{v}_0\geq 0$, then they are isomorphic by the above theorem. ☐

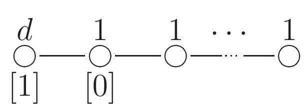
Using the above corollary we can show that some quiver schemes are algebraic varieties.

Corollary 4.10. *Let $(\mathsf{Q},\mathbf{d})$ be a quiver with multiplicities and put $I_{\mathrm{irr}}:=\{\, i\in I \mid d_i>1\,\}$. Suppose that each $i\in I_{\mathrm{irr}}$ is an irregular leg of length one, and any distinct pair $i\neq j$ in I_{irr} has distinct bases. Take a pair $(\boldsymbol{\lambda},\mathbf{v})\in R_{\mathbf{d}}\times\mathbb{Z}^I_{\geq 0}$ so that $\lambda_i\in R^\times_{d_i}$ for any $i\in J$. Then $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ is a variety.*

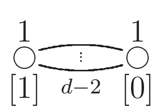
Proof. Applying Corollary 4.9 to each $i\in I_{\mathrm{irr}}$, we obtain an isomorphism from $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ to Nakajima's quiver variety. ☐

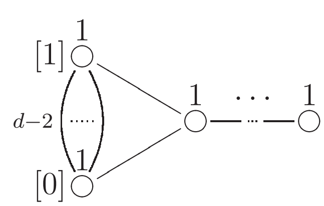
Here are some examples.

Example 4.11. (i) Consider the quiver with multiplicities $(\mathsf{Q},\mathbf{d})$ given in [12, Example 5.6 (i),(ii)], which has the following underlying graph with multiplicities.



Here the number of vertices is $n\geq 2$ and $[0],[1]$ are labels of vertices ($d,1,1,\dots,1$ are the multiplicities). Then the regularization $(\check{\mathsf{Q}},\check{\mathbf{d}})$ has the following underlying graph with multiplicities.

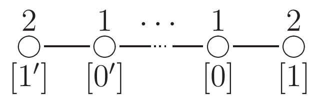


$(n=2)$,


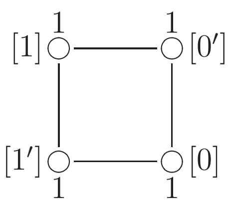
$(n\geq 3)$

Since it is multiplicity-free, Corollary 4.9 implies that $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda},\mathbf{v})$ is a variety if $\lambda_i(0)\neq 0$.

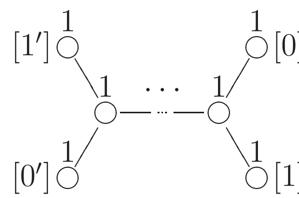
(ii) Consider the quiver with multiplicities $(\mathsf{Q}, \mathbf{d})$ given in [12, Example 5.6 (iii)], which has the following underlying graph with multiplicities.



Here the number of vertices is $n \geq 4$. Then one can perform the regularization twice, and the resulting quiver with multiplicities $(\check{\mathsf{Q}}, \check{\mathbf{d}})$ has the following graph with multiplicities.



$(n = 4)$,



$(n \geq 5)$

Since it is multiplicity-free, Corollary 4.9 implies that $\mathcal{S}_{\mathsf{Q},\mathbf{d}}(\boldsymbol{\lambda}, \mathbf{v})$ is a variety if $\lambda_{[1]}(0), \lambda_{[1']}(0) \neq 0$.

4.3 Weyl groups and regularization

Let $(\mathsf{Q}, \mathbf{d})$ be a quiver with multiplicities having an irregular leg $\mathsf{Q}_{\mathrm{leg}} = ([1, l], \Omega_{\mathrm{leg}}, s, t)$, and let $(\check{\mathsf{Q}}, \check{\mathbf{d}})$ be the one obtained by the regularization of $(\mathsf{Q}, \mathbf{d})$ along $\mathsf{Q}_{\mathrm{leg}}$. We denote by $\check{\mathbf{C}} = 2\mathrm{Id} - \check{\mathbf{A}}'\check{\mathbf{D}}$ the generalized Cartan matrix associated to $(\check{\mathsf{Q}}, \check{\mathbf{d}})$, and by $\check{\mathfrak{h}}, \check{Q}, \check{\alpha}_k, \check{s}_k$ the Cartan subalgebra, the root lattice, the simple roots and simple reflections, of the Kac–Moody algebra with Cartan matrix $\check{\mathbf{C}}$. In this subsection we give some relationship between the two Weyl groups $W(\mathbf{C})$ and $W(\check{\mathbf{C}})$ as in [12, Section 5.3].

Define a homomorphism $\varphi \colon Q \to \check{Q}$ by $\mathbf{v} \mapsto \check{\mathbf{v}} = \mathbf{v} - \sum_{i \in [0, l-1]} v_{i+1} \check{\alpha}_i$.

Lemma 4.12. *If we regard φ as an element of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^I, \mathbb{Z}^I) = \mathbb{Z}^{I \times I}$, then ${}^t\varphi \check{\mathbf{D}} \check{\mathbf{C}} \varphi = \mathbf{D}\mathbf{C}$.*

Proof. To prove it, we express the matrices in block form with respect to the decomposition of the index set $I = [0, l] \sqcup K$. First, φ is expressed as

$$\varphi = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

By the properties of $\mathsf{Q}_{\mathrm{leg}}$ and the definition of regularization, the matrices $\mathbf{D}$, $\check{\mathbf{D}}$, $\mathbf{A}'$ and $\check{\mathbf{A}}'$ are respectively expressed as

$$\mathbf{D} = \begin{pmatrix} \mathrm{diag}(1, d, \ldots, d) & 0 \\ 0 & \widetilde{\mathbf{D}} \end{pmatrix}, \quad \check{\mathbf{D}} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \widetilde{\mathbf{D}} \end{pmatrix},$$

$$\mathbf{A}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & {}^t\mathbf{a} \\ 1 & 0 & \frac{1}{d} & \ddots & \vdots & 0 \\ 0 & \frac{1}{d} & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{d} & 0 \\ 0 & \cdots & 0 & \frac{1}{d} & 0 & 0 \\ \mathbf{a} & 0 & \cdots & 0 & 0 & \widetilde{\mathbf{A}'} \end{pmatrix}, \quad \check{\mathbf{A}}' = \begin{pmatrix} 0 & d-2 & \cdots & d-2 & {}^t\mathbf{a} \\ d-2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & d-2 & \vdots \\ d-2 & \cdots & d-2 & 0 & {}^t\mathbf{a} \\ \mathbf{a} & \cdots & \cdots & \mathbf{a} & \widetilde{\mathbf{A}'} \end{pmatrix},$$

where $\widetilde{\mathbf{D}}$ (resp. $\widetilde{\mathbf{A}'}$) is the sub-matrix of $\mathbf{D}$ (resp. $\mathbf{A}'$) obtained by restricting the index set to K, and $\mathbf{a} = (a_{k0})_{k\in K}$. Now we check the equality. We have

$$\mathbf{DC} = 2\mathbf{D} - \mathbf{DA}'\mathbf{D} = \begin{pmatrix} 2 & -d & 0 & \cdots & 0 & -{}^t\mathbf{a}\widetilde{\mathbf{D}} \\ -d & 2d & \ddots & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & -d & 0 \\ 0 & \cdots & 0 & -d & 2d & 0 \\ -\widetilde{\mathbf{D}}\mathbf{a} & 0 & \cdots & 0 & 0 & \widetilde{\mathbf{D}}\widetilde{\mathbf{C}} \end{pmatrix},$$

where $\widetilde{\mathbf{C}} = 2\mathrm{Id} - \widetilde{\mathbf{A}'}\widetilde{\mathbf{D}}$. On the other hand,

$$\check{\mathbf{D}}\check{\mathbf{C}} = 2\check{\mathbf{D}} - \check{\mathbf{D}}\check{\mathbf{A}}'\check{\mathbf{D}} = \begin{pmatrix} 2 & 2-d & \cdots & 2-d & -{}^t\mathbf{a}\widetilde{\mathbf{D}} \\ 2-d & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 2-d & -{}^t\mathbf{a}\widetilde{\mathbf{D}} \\ 2-d & \cdots & 2-d & 2 & -{}^t\mathbf{a}\widetilde{\mathbf{D}} \\ -\widetilde{\mathbf{D}}\mathbf{a} & \cdots & -\widetilde{\mathbf{D}}\mathbf{a} & -\widetilde{\mathbf{D}}\mathbf{a} & \widetilde{\mathbf{D}}\widetilde{\mathbf{C}} \end{pmatrix}.$$

By direct calculation, we obtain ${}^t\varphi\check{\mathbf{D}}\check{\mathbf{C}}\varphi = \mathbf{DC}$. □

The above lemma implies that φ preserves the symmetric bilinear form (2.3).

Let S_{l+1} be the symmetric group of $[0,l]$. It effectively acts on $Q = \mathbb{Z}^I$ via permutations of coordiates.

Lemma 4.13. *$\sigma\check{s}_k\sigma^{-1} = \check{s}_{\sigma(k)}$ for any $\sigma \in S_{l+1}$ and $k \in I$.*

Proof. Observe that the matrices $\check{\mathbf{C}}$, $\check{\mathbf{D}}$ are invariant under permutations of indices in $[0,l]$. Hence the action of S_{l+1} on $\check{Q}$ preserves the symmetric bilinear form. Recall that the simple reflections satisfy

$$\check{s}_i(\beta) = \beta - \frac{2(\beta,\check{\alpha}_i)}{(\check{\alpha}_i,\check{\alpha}_i)}\check{\alpha}_i \quad (i \in I,\ \beta \in \check{Q}).$$

For $k \in I$ and $\beta \in \check{Q}$, we thus have

$$\check{s}_{\sigma(k)}(\beta) = \beta - \frac{2(\beta,\check{\alpha}_{\sigma(k)})}{(\check{\alpha}_{\sigma(k)},\check{\alpha}_{\sigma(k)})}\check{\alpha}_{\sigma(k)}$$

$$
\begin{aligned}
&= \beta - \frac{2(\sigma^{-1}(\beta), \check{\alpha}_k)}{(\check{\alpha}_k, \check{\alpha}_k)} \check{\alpha}_{\sigma(k)} \\
&= \sigma \left(\sigma^{-1}(\beta) - \frac{2(\sigma^{-1}(\beta), \check{\alpha}_k)}{(\check{\alpha}_k, \check{\alpha}_k)} \check{\alpha}_k \right) = \left(\sigma \check{s}_k \sigma^{-1} \right)(\beta).
\end{aligned}
$$

□

If we regard $W(\check{\mathbf{C}})$ and S_{l+1} as subgroups of $\mathrm{GL}_{\mathbb{Z}}(\check{Q})$, then the above lemma implies that $W(\check{\mathbf{C}})S_{l+1}$ is a semi-direct product $W(\check{\mathbf{C}}) \rtimes S_{l+1}$.

Proposition 4.14. *Under the isomorphism φ, the Weyl group $W(\mathbf{C})$ is isomorphic to the semidirect product $W(\check{\mathbf{C}}) \rtimes S_{l+1}$.*

Proof. We calculate the subgroup $\varphi W(\mathbf{C})\varphi^{-1} \subset \mathrm{GL}_{\mathbb{Z}}(\check{Q})$. Since φ preserves the symmetric bilinear form, the automorphism $\varphi s_i \varphi^{-1}$ of $\check{Q}$ satisfies

$$
\varphi s_i \varphi^{-1}(\beta) = \beta - \frac{2\,(\beta, \varphi(\alpha_i))}{(\varphi(\alpha_i), \varphi(\alpha_i))} \varphi(\alpha_i) \quad (i \in I,\ \beta \in \check{Q}).
$$

By the defintition of φ, we have

$$
\varphi(\alpha_i) = \begin{cases} \check{\alpha}_i - \check{\alpha}_{i-1} & (i \in [1, l]), \\ \check{\alpha}_i & (\text{otherwise}). \end{cases}
$$

It follows that $\varphi s_i \varphi^{-1} = s_i$ if $i \notin [1, l]$. For $i \in [1, l]$, a direct calculation shows

$$
(\varphi(\alpha_i), \varphi(\alpha_i)) = (\alpha_i, \alpha_i) = 2d, \quad (\check{\alpha}_k, \check{\alpha}_i - \check{\alpha}_{i-1}) = \begin{cases} d & (k = i), \\ -d & (k = i-1), \\ 0 & (\text{otherwise}), \end{cases}
$$

which imply that $\varphi s_i \varphi^{-1}(\check{\alpha}_k) = \check{\alpha}_{\sigma_i(k)}$ for all $k \in I$, where $\sigma_i \in S_{l+1}$ is the transposition of $i-1$ and i. Hence $\varphi s_i \varphi^{-1} = \sigma_i$. As a conclusion, $\varphi W(\mathbf{C})\varphi^{-1}$ is equal to the subgroup generated by σ_i, $i \in [1, l]$ and s_k, $k \notin [1, l]$, which coincides with $W(\check{\mathbf{C}})S_{l+1} \simeq W(\check{\mathbf{C}}) \rtimes S_{l+1}$ by Lemma 4.13. □

Let S_{l+1} act on $R_{\check{\mathbf{d}}}$ by permutations of components. Then it is straightforward to show (using the S_{l+1}-invariance of $\check{\mathbf{D}}, \check{\mathbf{C}}$) that $\sigma \check{r}_k \sigma^{-1} = \check{r}_{\sigma(k)}$ for any $\sigma \in S_{l+1}$ and $k \in I$, where $\check{r}_k \colon R_{\check{\mathbf{d}}} \to R_{\check{\mathbf{d}}}$ is the linear map corresponding to the simple reflection $\check{s}_k$ for the action of $W(\check{\mathbf{C}})$. Thus we obtain an action of the semi-direct product $W(\check{\mathbf{C}}) \rtimes S_{l+1}$ on $R_{\check{\mathbf{d}}}$.

Proposition 4.15. *Let $W(\mathbf{C})$ act on $R_{\check{\mathbf{d}}} \times \check{Q}$ through the isomorphism $W(\mathbf{C}) \simeq W(\check{\mathbf{C}}) \rtimes S_{l+1}$. Then the map $R_{\mathbf{d}} \times Q \to R_{\check{\mathbf{d}}} \times \check{Q}$, $(\boldsymbol{\lambda}, \mathbf{v}) \mapsto (\check{\boldsymbol{\lambda}}, \check{\mathbf{v}})$ is $W(\mathbf{C})$-equivariant.*

Proof. Let $\psi \colon R_{\mathbf{d}} \to R_{\check{\mathbf{d}}}$ be the map $\boldsymbol{\lambda} \mapsto \check{\boldsymbol{\lambda}}$. Then the transpose ${}^t\psi \colon R_{\check{\mathbf{d}}} \to R_{\mathbf{d}}$ is

$$
\check{\mathbf{v}} = (\check{v}_i) \mapsto \mathbf{v} = (v_i), \quad v_i = \begin{cases} \check{v}_i & (i \notin [1, l]), \\ \sum_{k=i}^{l} \check{v}_k & (i \in [1, l]). \end{cases}
$$

For the assertion it is sufficient to show that ${}^t\psi$ is equivariant with respect to the dual actions. For $i \in I$, let $\tilde{s}_i \colon R_{\mathbf{d}} \to R_{\mathbf{d}}$, $\tilde{s}'_i \colon R_{\check{\mathbf{d}}} \to R_{\check{\mathbf{d}}}$ be the actions of the i-th simple reflection. In the proof of the above lemma we checked that

$$\varphi s_i \varphi^{-1} = \begin{cases} s_i & (i \notin [1,l]), \\ \sigma_i & (i \in [1,l]). \end{cases}$$

Thus it is sufficient to show that

$$\tilde{s}_i({}^t\psi(\check{\mathbf{v}})) = \begin{cases} {}^t\psi(\sigma_i(\check{\mathbf{v}})) & (i \in [1,l]), \\ {}^t\psi(\tilde{s}'_i(\check{\mathbf{v}})) & (i \notin [1,l]) \end{cases}$$

for any $\check{\mathbf{v}} \in R_{\check{\mathbf{d}}}$.

Fix $\check{\mathbf{v}} \in R_{\check{\mathbf{d}}}$ and put $\mathbf{v} = {}^t\psi(\check{\mathbf{v}})$. First, suppose $i \in [1,l]$. In the proof of Proposition 3.1, we already calculated $\tilde{s}_i(\mathbf{v})$ as follows:

$$\tilde{s}_i(\mathbf{v}) = \mathbf{v} - \sum_{j \in I} c_{ij} \sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m} \epsilon_{d_i}^{f_{ji}m} \mathbf{e}_i, \quad \mathbf{v} = \left(\sum v_{i,k} \epsilon_{d_i}^{k} \right).$$

Since $i \in [1,l]$, we have

$$c_{ij} = \begin{cases} 2 & (j = i), \\ -1 & (j \in [0,l],\ |i-j| = 1), \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$\sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m} \epsilon_{d_i}^{f_{ji}m} = v_j$$

whenever $c_{ij} \neq 0$. Thus

$$\sum_{j \in I} c_{ij} \sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m} \epsilon_{d_i}^{f_{ji}m} = 2v_i - \sum_{j \in [0,l];\, |i-j|=1} v_j = \check{v}_i - \check{v}_{i-1},$$

and hence $\tilde{s}_i(\mathbf{v}) = \mathbf{v} - (\check{v}_i - \check{v}_{i-1})\mathbf{e}_i$. On the other hand, a direct calculateion shows that the i-th component of ${}^t\psi(\sigma_i(\check{\mathbf{v}}))$ is equal to

$$\sum_{k=i}^{l} \check{v}_{\sigma_i^{-1}(k)} = \check{v}_{i-1} + \sum_{k=i+1}^{l} \check{v}_k = v_i - (\check{v}_i - \check{v}_{i-1}),$$

while the other components are the same as those of $\mathbf{v}$. Hence ${}^t\psi(\sigma_i(\check{\mathbf{v}})) = \mathbf{v} - (\check{v}_i - \check{v}_{i-1})\mathbf{e}_i = \tilde{s}_i(\mathbf{v})$.

Next, suppose $i \notin [1,l]$. For $j \in I$, let $\check{c}_{ij}$ be the (i,j)-entry of $\check{\mathbf{C}}$ and

$$\check{d}_{ij} = \gcd(\check{d}_i, \check{d}_j), \quad \check{f}_{ij} = \check{d}_j / \check{d}_{ij}.$$

Then we have

$$
\begin{aligned}
{}^t\psi(\tilde{s}_i'(\check{\mathbf{v}})) &= {}^t\psi\left(\check{\mathbf{v}} - \sum_{j\in I}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m}\mathbf{e}_i\right) \\
&= \mathbf{v} - \sum_{j\in I}\check{c}_{ij}{}^t\psi\left(\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m}\mathbf{e}_i\right).
\end{aligned}
$$

Since $i \notin [1,l]$, the description of ${}^t\psi$ shows

$$
{}^t\psi\left(\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m}\mathbf{e}_i\right) = \sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m}\mathbf{e}_i.
$$

On the other hand,

$$
\tilde{s}_i(\mathbf{v}) = \mathbf{v} - \sum_{j\in I} c_{ij}\sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m}\epsilon_{d_i}^{f_{ji}m}\mathbf{e}_i.
$$

Therefore it is sufficient to show

$$
\sum_{j\in I}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m} = \sum_{j\in I} c_{ij}\sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m}\epsilon_{d_i}^{f_{ji}m}.
$$

If $i \neq 0$, then

$$
\sum_{j\in[0,l]}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m} = \check{c}_{i0}\sum_{j\in[0,l]}\check{v}_j = c_{i0}v_0,
$$

and hence

$$
\begin{aligned}
\sum_{j\in I}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m} &= \sum_{j\notin[0,l]}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m} + c_{i0}v_0 \\
&= \sum_{j\notin[1,l]} c_{ij}\sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m}\epsilon_{d_i}^{f_{ji}m} \\
&= \sum_{j\in I} c_{ij}\sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m}\epsilon_{d_i}^{f_{ji}m}.
\end{aligned}
$$

If $i = 0$, then

$$
\sum_{j\in[0,l]}\check{c}_{ij}\sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m}\epsilon_{\check{d}_i}^{\check{f}_{ji}m} = 2\check{v}_0 + (2-d)\sum_{j\in[1,l]}\check{v}_j
$$

$$= 2v_0 - dv_1 = \sum_{j \in [0,l]} c_{ij} \sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m} \epsilon_{d_i}^{f_{ji}m},$$

while

$$\sum_{j \in I} \check{c}_{ij} \sum_{m=0}^{\check{d}_{ij}-1} v_{j,\check{f}_{ij}m} \epsilon_{\check{d}_i}^{\check{f}_{ji}m} = \sum_{j \in I} c_{ij} \sum_{m=0}^{d_{ij}-1} v_{j,f_{ij}m} \epsilon_{d_i}^{f_{ji}m}.$$

Thus we obtain the desired equality. □

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