

A NOTE ON GENERALIZED SPHERICAL MAXIMAL MEANS

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ABSTRACT. The goal of this note is to provide an alternative proof of Theorem 1.1 (i) in [4], that is, if $n \geq 2$ and M^α is bounded on $L^p(\mathbb{R}^n)$ for some $\alpha \in \mathbb{C}$ and $p \geq 2$, then we have

$$\operatorname{Re} \alpha \geq \max \left\{ \frac{1-n}{2} + \frac{1}{p}, \frac{1-n}{p} \right\}.$$

1. INTRODUCTION

In [8], Stein introduced the generalized spherical maximal means

$$M^\alpha f(x) := \sup_{t>0} |A_t^\alpha f(x)|,$$

where $A_t^\alpha f(x)$ is defined as

$$A_t^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{|y| \leq 1} (1 - |y|^2)^{\alpha-1} f(x - ty) dy.$$

The generalized spherical means are defined a priori only for $\operatorname{Re} \alpha > 0$. A direct calculation (see [10, p. 171] and [5, Appendix A]) implies

$$\widehat{A_t^\alpha f}(\xi) = \widehat{f}(\xi) \pi^{-\alpha+1} |t\xi|^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi|t\xi|) =: \widehat{f}(\xi) m^\alpha(t\xi), \quad (1.1)$$

where J_β denotes the Bessel function of order β . Recall that for $\beta \in \mathbb{C}$ and $r > 0$, the Bessel function $J_\beta(r)$ is given by

$$J_\beta(r) := \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{\Gamma(j + \beta + 1)} \left(\frac{r}{2}\right)^{2j+\beta}.$$

For more details we refer the readers to [11, Chapter II]. Thus, the definition of $A_t^\alpha f$ can be extended to $\alpha \in \mathbb{C}$ via (1.1). In particular, one can recover the averages over Euclidean balls and the classical spherical means by taking $\alpha = 1$ and $\alpha = 0$, respectively. Stein [8] proved that M^α is bounded on $L^p(\mathbb{R}^n)$ if

$$1 < p \leq 2 \text{ and } \operatorname{Re} \alpha > 1 - n + \frac{n}{p} \quad (1.2)$$

or

$$2 \leq p \leq \infty \text{ and } \operatorname{Re} \alpha > \frac{2-n}{p}. \quad (1.3)$$

The conditions in (1.2) are optimal, see [9, p. 519]. Applying this result, Stein showed that M^0 is bounded on $L^p(\mathbb{R}^n)$ whenever $p > n/(n-1)$ and $n \geq 3$. Later, Bourgain obtained the $L^p(\mathbb{R}^2)$ -boundedness of M^0 for $p > 2$ in [1]. Based on the local smoothing estimate, Mockenhaupt, Seeger and Sogge [6] provided an alternative proof of Bourgain's

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result and improved the range of p in (1.3) when $n = 2$. Miao, Yang and Zheng [5] made a further improvement by using the Bourgain-Demeter ℓ^2 decoupling theorem [2], that is, M^α is bounded on $L^p(\mathbb{R}^n)$ if

$$2 \leq p \leq \frac{2(n+1)}{n-1} \text{ and } \operatorname{Re} \alpha > \frac{1-n}{4} + \frac{3-n}{2p}$$

or

$$\frac{2(n+1)}{n-1} \leq p \leq \infty \text{ and } \operatorname{Re} \alpha > \frac{1-n}{p}.$$

In [7], Nowak, Roncal and Szarek obtained some optimal results for the generalized spherical maximal means on radial functions when $n \geq 2$ and $\alpha > (1-n)/2$. Recently, Liu, Shen, Song and Yan [4] established the following necessary conditions for the $L^p(\mathbb{R}^n)$ -boundedness of M^α and showed these conditions are almost optimal when $n = 2$.

Theorem 1.1. *Let $n \geq 2$ and $p \geq 2$. If M^α is bounded on $L^p(\mathbb{R}^n)$ for some $\alpha \in \mathbb{C}$, then one of the following conditions holds*

- (i) $2 \leq p \leq 2n/(n-1)$ and $\operatorname{Re} \alpha \geq (1-n)/2 + 1/p$;
- (ii) $p \geq 2n/(n-1)$ and $\operatorname{Re} \alpha \geq (1-n)/p$.

The aim of this note is to provide an alternative proof of Theorem 1.1. To show Theorem 1.1, Liu, Shen, Song and Yan [4] tested M^α on some functions whose Fourier transform concentrate on the direction $\vec{e}_1 := (1, 0, \dots, 0)$ to avoid the interference between $e^{it\sqrt{-\Delta}}f$ and $e^{-it\sqrt{-\Delta}}f$. In this note, we will test M^α on

$$\widehat{f}_\lambda(\xi) := e^{-2\pi i|\xi|} \chi(\lambda^{-1}|\xi|) |\xi|^{i\operatorname{Im} \alpha}$$

for $\alpha \in \mathbb{C}$ and large λ . The function $e^{-2\pi i|\xi|}$ allows us to obtain the main term of $A_t^\alpha f_\lambda^\alpha(x)$ for some x and t . To show Theorem 1.1 (i), we employ the main idea in [3]. Observe that

$$\widehat{d\sigma}(\lambda x) = \int_{\mathbb{S}^{n-1}} e^{2\pi i \lambda x \cdot \theta} d\sigma(\theta) \quad (1.4)$$

is almost a constant when $\lambda|x|$ is small. Thus, we obtain the main term of $A_1^\alpha f_\lambda^\alpha(x)$ when $\lambda|x|$ is small, from which Theorem 1.1 (i) follows. For Theorem 1.1 (ii), we choose $2 \leq |x| \leq 3$ and $t_x := |x| + 1$. By the asymptotic expansion of (1.4), we find the main term of $A_{t_x}^\alpha f_\lambda^\alpha(x)$ and further deduce Theorem 1.1 (ii).

Throughout this article, each different appearance of the letter C may represent a different positive constant and is independent of the main parameters. We write $A \lesssim B$ if there is $C > 0$ such that $A \leq CB$, and write $A \approx B$ when $A \lesssim B \lesssim A$. \widehat{f} means the Fourier transform of f . We denote by χ_E the characteristic function of E for any $E \subset \mathbb{R}^n$.

2. THE PROOF OF THEOREM 1.1

In this section, we establish Propositions 2.2 and 2.3, from which Theorem 1.1 follows immediately. We first recall the following asymptotic expansion for the Bessel function

$$J_\beta(r) = r^{-1/2} e^{ir} [b_{0,\beta} + E_{1,\beta}(r)] + r^{-1/2} e^{-ir} [d_{0,\beta} + E_{2,\beta}(r)], \quad r \geq 1, \quad (2.1)$$

where $b_{0,\beta}, d_{0,\beta}$ are suitable coefficients and $E_{1,\beta}(r), E_{2,\beta}(r)$ satisfy

$$\left| \left(\frac{d}{dr} \right)^N E_{1,\beta}(r) \right| + \left| \left(\frac{d}{dr} \right)^N E_{2,\beta}(r) \right| \lesssim r^{-N-1}, \quad r \geq 1$$

for any $N \in \mathbb{N}$. The following fact (see [9, p. 347]) will be useful in the proof,

$$\widehat{d\sigma}(\xi) = 2\pi|\xi|^{(2-n)/2}J_{(n-2)/2}(2\pi|\xi|). \quad (2.2)$$

We first show the following lemma.

Lemma 2.1. *Suppose $\chi \in C_c^\infty(\mathbb{R})$, $\chi \equiv 1$ on $[3/4, 5/4]$ and $\chi \geq 0$. For $\alpha \in \mathbb{C}$ and $\lambda > 0$, define*

$$\widehat{f}_\lambda^\alpha(\xi) := e^{-2\pi i|\xi|}\chi(\lambda^{-1}|\xi|)|\xi|^{i\operatorname{Im}\alpha}.$$

Then for $p \geq 1$,

$$\|f_\lambda^\alpha\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{(n+1)/2-1/p}.$$

Proof. By a change of variable, we deduce

$$f_\lambda^\alpha(x) = \lambda^{n+i\operatorname{Im}\alpha} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{2\pi i r \lambda x \cdot \theta} d\sigma(\theta) e^{-2\pi i r \lambda} \chi(r) r^{n-1+i\operatorname{Im}\alpha} dr.$$

Note that

$$\int_{\mathbb{S}^{n-1}} e^{2\pi i r \lambda x \cdot \theta} d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} e^{2\pi i r \lambda |x| \vec{e}_1 \cdot \theta} d\sigma(\theta) := \vartheta(\lambda|x|r) \quad (2.3)$$

is a smooth function. Thus, a simple integration-by-parts argument shows

$$|f_\lambda^\alpha(x)| \leq C_N \lambda^{-N} \quad (2.4)$$

when $|x| \leq \lambda^{-1}$.

If $|x| \geq \lambda^{-1}$, by (2.1) and (2.2), we have

$$\begin{aligned} |f_\lambda^\alpha(x)| &\lesssim \lambda^{(1+n)/2} |x|^{(1-n)/2} \left(\left| \int_0^\infty e^{2\pi i r \lambda (|x|-1)} a^+(2\pi r \lambda |x|) \chi(r) r^{(n-1)/2+i\operatorname{Im}\alpha} dr \right| \right. \\ &\quad \left. + \left| \int_0^\infty e^{-2\pi i r \lambda (|x|+1)} a^-(2\pi r \lambda |x|) \chi(r) r^{(n-1)/2+i\operatorname{Im}\alpha} dr \right| \right), \end{aligned} \quad (2.5)$$

where a^\pm are standard symbols of order 0. There are now two subcases.

Case (i) If $||x| - 1| \geq \lambda^{-1}$, by (2.5) and a simple integration-by-parts argument, we obtain

$$|f_\lambda^\alpha(x)| \lesssim \lambda^{(n+1)/2} |x|^{(1-n)/2} (\lambda ||x| - 1|)^{-N+(1-n)/2} \lesssim \lambda (\lambda ||x| - 1|)^{-N} \quad (2.6)$$

for any $N \in \mathbb{N}$, where in the second inequality we used

$$|x| ||x| - 1| \geq \frac{1}{4}.$$

Case (ii) If $||x| - 1| \leq \lambda^{-1}$, (2.5) implies that

$$|f_\lambda^\alpha(x)| \lesssim \lambda^{(n+1)/2} |x|^{(1-n)/2}. \quad (2.7)$$

Thus, combining (2.4), (2.6), (2.7) and the fact

$$\frac{r}{\lambda(r-1)} \leq 1$$

for $r \geq 0$ with $|r-1| \geq 1$, we have

$$\|f_\lambda^\alpha\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{(1+n)/2-1/p}.$$

This finishes the proof. \square

Now, we prove the following proposition.

Proposition 2.2. *Let $\alpha \in \mathbb{C}$ and $p \geq 2$. If M^α is bounded on $L^p(\mathbb{R}^n)$, then*

$$\operatorname{Re} \alpha \geq \frac{1-n}{p}.$$

Proof. By (2.1) and (2.3), we have

$$\begin{aligned} A_1^\alpha f_\lambda(x) &= \pi^{-\alpha+1} \lambda^{n/2+1-\operatorname{Re} \alpha} \int_0^\infty \vartheta(\lambda r|x|) e^{-2\pi i \lambda r} \chi(r) J_{n/2+\alpha-1}(2\pi \lambda r) r^{n/2-\operatorname{Re} \alpha} dr \\ &= \lambda^{(n+1)/2-\operatorname{Re} \alpha} \int_0^\infty \vartheta(\lambda r|x|) \chi(r) (b_{1,\alpha} + e^{-4\pi i \lambda r} d_{1,\alpha} + a_1(\lambda r)) r^{(n-1)/2-\operatorname{Re} \alpha} dr \\ &=: \sum_{i=1}^3 I_i(x, \lambda), \end{aligned}$$

where

$$|a_1(\lambda r)| \lesssim (\lambda r)^{-1}, \quad \lambda r \geq 1.$$

Suppose c_0 is small enough and $|x| \leq c_0 \lambda^{-1}$. Obviously,

$$|I_3(x, \lambda)| \lesssim \lambda^{(n-1)/2-\operatorname{Re} \alpha}. \quad (2.8)$$

It remains to estimate $I_1(x, \lambda)$ and $I_2(x, \lambda)$. Without loss of generality, we may assume $b_{1,\alpha} = d_{1,\alpha} = 1$. Integrating by parts, we obtain

$$|I_2(x, \lambda)| \lesssim \lambda^{-N} \quad (2.9)$$

for any $N \in \mathbb{N}$.

For $I_1(x, \lambda)$, by the mean value theorem, we deduce

$$\left| \int_0^\infty (\vartheta(\lambda r|x|) - 1) \chi(r) r^{(n-1)/2-\operatorname{Re} \alpha} dr \right| \leq c_0 C,$$

from which it follows that

$$|I_1(x, \lambda)| \geq C \lambda^{(n+1)/2-\operatorname{Re} \alpha} \quad (2.10)$$

By (2.8)-(2.10), we conclude

$$|A_1^\alpha f_\lambda(x)| \geq C \lambda^{(n+1)/2-\operatorname{Re} \alpha},$$

which further implies

$$\|A_1^\alpha f_\lambda\|_{L^p(\mathbb{R}^n)} \geq C \lambda^{(n+1)/2-\operatorname{Re} \alpha - n/p}.$$

Combining this fact with the assumption and Lemma 2.1, we see

$$\lambda^{(n+1)/2-\operatorname{Re} \alpha - n/p} \lesssim \lambda^{(n+1)/2-1/p}.$$

Thus, the proof is complete. \square

Finally, we consider $x \approx 2$ and obtain the following proposition.

Proposition 2.3. *Let $\alpha \in \mathbb{C}$ and $p \geq 2$. If M^α is bounded on $L^p(\mathbb{R}^n)$, then*

$$\operatorname{Re} \alpha \geq \frac{1-n}{2} + \frac{1}{p}.$$

Proof. As in Proposition 2.2, we have

$$A_t^\alpha f_\lambda^\alpha(x) = \pi^{-\alpha+1} \lambda^{n/2+1-\operatorname{Re}\alpha} t^{-n/2-\alpha+1} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{2\pi i x \cdot \lambda r \theta} d\sigma(\theta) J_{n/2+\alpha-1}(2\pi t \lambda r) e^{-2\pi i \lambda r} \chi(r) r^{d/2-\operatorname{Re}\alpha} dr.$$

It follows from (2.1) that

$$J_{(n-2)/2}(r) = r^{-1/2} e^{ir} c_1 + r^{-1/2} e^{-ir} e_1 + a_2(r), \quad r \geq 1,$$

where

$$|a_2(r)| \lesssim r^{-3/2}, \quad r \geq 1.$$

Based on this fact and (2.2), for $|2\pi \lambda x r| \geq 1$, we deduce

$$\int_{\mathbb{S}^{n-1}} e^{2\pi i x \cdot \lambda r \theta} d\sigma(\theta) = (2\pi)^{1/2} |\lambda x r|^{(1-n)/2} (e^{i|2\pi \lambda x r|} c_1 + e^{-i|2\pi \lambda x r|} e_1) + a_3(|2\pi \lambda x r|),$$

where

$$|a_3(|2\pi \lambda x r|)| \lesssim |\lambda x r|^{-(n+1)/2}.$$

Without loss of generality, we assume $c_1 = e_1 = 1$. Similarly,

$$J_{n/2+\alpha-1}(2\pi t \lambda r) = (2\pi t \lambda r)^{-1/2} (e^{i2\pi t \lambda r} c_{2,\alpha} + e^{-i2\pi t \lambda r} e_{2,\alpha}) + a_{4,\alpha}(2\pi t \lambda r), \quad 2\pi t \lambda r \geq 1,$$

where

$$|a_{4,\alpha}(2\pi t \lambda r)| \lesssim (t \lambda r)^{-3/2}.$$

For the sake of simplicity, we may assume $c_{2,\alpha} = e_{2,\alpha} = 1$. Thus, we get

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} e^{2\pi i x \cdot \lambda r \theta} d\sigma(\theta) J_{n/2+\alpha-1}(2\pi t \lambda r) \\ &= (t \lambda r)^{-1/2} |\lambda x r|^{(1-n)/2} (e^{i2\pi \lambda r(|x|+t)} + e^{i2\pi \lambda r(|x|-t)} + e^{i2\pi \lambda r(-|x|+t)} + e^{i2\pi \lambda r(-|x|-t)}) \\ & \quad + a_{5,\alpha}(t, \lambda, |x|, r), \end{aligned}$$

where

$$|a_{5,\alpha}(t, \lambda, |x|, r)| \lesssim \lambda^{-(n+2)/2}.$$

Hence,

$$\begin{aligned} A_t^\alpha f_\lambda^\alpha(x) &= \pi^{-\alpha+1} \lambda^{n/2+1-\operatorname{Re}\alpha} t^{-n/2-\alpha+1} \int_0^\infty [(t \lambda r)^{-1/2} |\lambda x r|^{(1-n)/2} (e^{i2\pi \lambda r(|x|+t)} + e^{i2\pi \lambda r(|x|-t)} \\ & \quad + e^{i2\pi \lambda r(-|x|+t)} + e^{i2\pi \lambda r(-|x|-t)}) + a_{5,\alpha}(t, \lambda, |x|, r)] e^{-i2\pi \lambda r} \chi(r) r^{n/2-\operatorname{Re}\alpha} dr \\ &=: \sum_{i=1}^5 I_i(x, t, \lambda). \end{aligned}$$

For $2 \leq |x| \leq 3$, we choose $t_x := |x| + 1$. For $I_5(x, t_x, \lambda)$, we have

$$|I_5(x, t_x, \lambda)| \lesssim \lambda^{n/2+1-\operatorname{Re}\alpha} \int_0^\infty \lambda^{-(n+2)/2} \chi(r) r^{n/2-\operatorname{Re}\alpha} dr \lesssim \lambda^{-\operatorname{Re}\alpha}.$$

Note that the phase functions of $I_1(x, t_x, \lambda)$, $I_2(x, t_x, \lambda)$ and $I_4(x, t_x, \lambda)$ do not have critical points, which implies

$$|I_i(x, t_x, \lambda)| \lesssim \lambda^{-N}, \quad i = 1, 2, 4.$$

For the main term $I_3(x, t_x, \lambda)$, we obtain

$$|I_3(x, t_x, \lambda)| = \pi^{-\operatorname{Re} \alpha + 1} \lambda^{1 - \operatorname{Re} \alpha} t_x^{(1-n)/2 - \operatorname{Re} \alpha} |x|^{(1-n)/2} \int_0^\infty r^{-\operatorname{Re} \alpha} \chi(r) dr \geq C_1 \lambda^{1 - \operatorname{Re} \alpha}$$

for some $C_1 > 0$. Thus, we deduce

$$|A_{t_x}^\alpha f_\lambda^\alpha(x)| \geq C_2 \lambda^{1 - \operatorname{Re} \alpha}$$

when λ is large enough. By the assumption and Lemma 2.1, we conclude

$$\frac{1}{p} \leq \frac{n-1}{2} + \operatorname{Re} \alpha.$$

This completes the proof. □

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