# BILINEAR OSCILLATORY FOURIER MULTIPLIERS 

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#### Abstract

For bilinear Fourier multipliers that contain some oscillatory factors, boundedness of the operators between Lebesgue spaces is given including endpoint cases. Sharpness of the result is also considered.


## 1. Introduction

Throughout this paper, the letter $n$ denotes a positive integer.
For a bounded function $\sigma=\sigma(\xi)$ on $\mathbb{R}^{n}$, the linear Fourier multiplier operator $\sigma(D)$ is defined by

$$
\sigma(D) f(x)=\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} \sigma(\xi) \widehat{f}(\xi) d \xi, \quad x \in \mathbb{R}^{n}
$$

for $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, where $\widehat{f}$ denotes the Fourier transform of $f$. If $X$ is a function space on $\mathbb{R}^{n}$ equipped with the quasi-norm $\|\cdot\|_{X}$ and there exists $c>0$ such that

$$
\|\sigma(D) f\|_{X} \leq c\|f\|_{X} \text { for all } f \in \mathcal{S} \cap X
$$

then we say that $\sigma(D)$ is bounded on $X$.
We recall the result for the multiplier of the form

$$
e^{i|\xi|^{s}} \zeta(\xi)|\xi|^{m}, \quad 0<s<1 \text { or } 1<s<\infty, \quad m \in \mathbb{R}
$$

where $\zeta$ is $C^{\infty}$ function on $\mathbb{R}^{n}$ such that $\zeta(\xi)=0$ for $|\xi| \leq 1$ and $\zeta(\xi)=1$ for $|\xi| \geq 2$ (see also Notation (1.5).

Theorem A ([16, 11]). Let $m \in \mathbb{R}$, and let $0<s<1$ or $1<s<\infty$, and let $1 \leq p \leq \infty$. Then the Fourier multiplier operator $e^{i|D|^{s}} \zeta(D)|D|^{m}$ is bounded on $H^{p}$ when $p<\infty$ and on $B M O$ when $p=\infty$ if and only if $m \leq-n s|1 / p-1 / 2|$.

Here, $H^{p}, 0<p \leq \infty$, denotes the Hardy space and the space $B M O$ denotes the space of bounded mean oscillation. It is known that $H^{p}=L^{p}$ if $1<p \leq \infty$ and $H^{1} \hookrightarrow L^{1}$. For details on these function spaces, see, e.g., [17, Chapters III and IV].

Next, we shall consider the bilinear case. For a bounded function $\sigma=\sigma(\xi, \eta)$ on $\mathbb{R}^{2 n}$, the bilinear Fourier multiplier operator $T_{\sigma}$ is defined by

$$
T_{\sigma}(f, g)(x)=\frac{1}{(2 \pi)^{2 n}} \iint_{\mathbb{R}^{n} \times \mathbb{R}^{n}} e^{i x \cdot(\xi+\eta)} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) d \xi d \eta, \quad x \in \mathbb{R}^{n}
$$

[^0]for $f, g \in \mathcal{S}\left(\mathbb{R}^{n}\right)$. For function spaces on $\mathbb{R}^{n}, X, Y$ and $Z$ equipped with the quasi-norms $\|\cdot\|_{X},\|\cdot\|_{Y}$ and $\|\cdot\|_{Z}$, respectively, we say that $T_{\sigma}$ is bounded from $X \times Y$ to $Z$, or $T_{\sigma}$ is bounded in $X \times Y \rightarrow Z$ if there exists $C>0$ such that
$$
\left\|T_{\sigma}(f, g)\right\|_{Z} \leq C\|f\|_{X}\|g\|_{Y} \text { for all } f \in \mathcal{S} \cap X \text { and all } g \in \mathcal{S} \cap Y
$$

We define the operator norm $\left\|T_{\sigma}\right\|_{X \times Y \rightarrow Z}$ to be the smallest constant $C$ in the above inequality.

In this paper, we especially consider the bilinear Fourier multiplier operator $T_{\sigma}^{s}, 0<s<\infty$, of the following form:

$$
T_{\sigma}^{s}(f, g)(x)=\frac{1}{(2 \pi)^{2 n}} \iint_{\mathbb{R}^{n} \times \mathbb{R}^{n}} e^{i x \cdot(\xi+\eta)} e^{i\left(|\xi|^{s}+|\eta|^{s}\right)} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) d \xi d \eta, \quad x \in \mathbb{R}^{n}
$$

for $f, g \in \mathcal{S}\left(\mathbb{R}^{n}\right)$.
In order to describe the results on the bilinear operators of this type, we define the class $S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ as follows.

Definition 1.1. For $m \in \mathbb{R}$, the class $S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ is defined to be the set of all $C^{\infty}$ functions $\sigma=\sigma(\xi, \eta)$ on $\mathbb{R}^{2 n}$ that satisfy the estimate

$$
\left|\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma(\xi, \eta)\right| \leq C_{\alpha, \beta}(1+|\xi|+|\eta|)^{m-|\alpha|-|\beta|}
$$

for all multi-indices $\alpha, \beta \in\left(\mathbb{N}_{0}\right)^{n}=(\{0,1,2, \ldots\})^{n}$.
For the case $s=1$, Grafakos-Peloso [6] first gave the boundedness results for such kind of operators, which were developed in the series of the papers [13, 14, 15 ] by the authors S. Rodríguez-Lopéz, D. Rule, and W. Staubach. Quite recently, the first, the second and the last authors of the present paper improve these results in [9]. Although the present paper is inspired by [9], since our subject concerns with the case $s \neq 1$, we omit to mention the details on the results of [13, 14, 15, 9].

For the case $s \neq 1$, Bergfeldt-Rodríguez-Lopéz-Rule-Staubach [1] recently considered the bilinear operator $T_{\sigma}^{s}$, and proved the following theorem.
Theorem 1.2 ([1, Theorem 1.4 and Remark 1.5]). Let $0<s<1$ or $1<s<\infty$ and let $1 \leq p, q \leq \infty$ and $1 / r=1 / p+1 / q$. Suppose that $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ with $m=-n s(|1 / p-1 / 2|+$ $|1 / q-1 / 2|)$. Then, $T_{\sigma}^{s}$ is bounded from $H^{p} \times H^{q}$ to $L^{r}$, where $L^{r}$ should be replaced by BMO when $r=\infty$.

Here we give a remark on Theorem 1.2. The verbatim statement of [1, Theorem 1.4 and Remark 1.5] contains the restriction $r>\frac{n}{n+\min \{1, s\}}$. However, if we carefully read the paper, we see that this restriction can be removed. For the reader's convenience, we shall give an independent proof of Theorem 1.2 in Section 2.

Now, the purpose of this paper is to give an improvement of Theorem 1.2, To state our main result, we prepare some notations. We divide the set $\{1 \leq p, q \leq \infty\} \subset \mathbb{R}^{2}$ into the following six subsets:

$$
\begin{aligned}
\mathrm{I} & =\{2 \leq p, q \leq \infty\} \\
\mathrm{II} & =\{1 \leq p, q \leq 2\}, \\
\mathrm{III} & =\{1 \leq p \leq 2 \leq q \leq \infty \text { and } 1 / p+1 / q \leq 1\} \\
\mathrm{IV} & =\{1 \leq p \leq 2 \leq q \leq \infty \text { and } 1 / p+1 / q \geq 1\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V} & =\{1 \leq q \leq 2 \leq p \leq \infty \text { and } 1 / p+1 / q \leq 1\} \\
\mathrm{VI} & =\{1 \leq q \leq 2 \leq p \leq \infty \text { and } 1 / p+1 / q \geq 1\}
\end{aligned}
$$

which satisfy $\mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}=\{1 \leq p, q \leq \infty\}$ and are assigned into the picture below.


Using these sets, we define $m_{s}(p, q)$ by

$$
m_{s}(p, q)= \begin{cases}-n s\left(\left|\frac{1}{p}-\frac{1}{2}\right|+\left|\frac{1}{q}-\frac{1}{2}\right|\right) & \text { for }(p, q) \in \mathrm{I} \cup \mathrm{II}, \\ -n s(1-s)\left|\frac{1}{p}-\frac{1}{2}\right|-n s\left|\frac{1}{q}-\frac{1}{2}\right| & \text { for }(p, q) \in \mathrm{III} \cup \mathrm{VI}, \quad \text { when } 0<s<1 \\ -n s\left|\frac{1}{p}-\frac{1}{2}\right|-n s(1-s)\left|\frac{1}{q}-\frac{1}{2}\right| & \text { for }(p, q) \in \mathrm{IV} \cup \mathrm{~V},\end{cases}
$$

and

$$
m_{s}(p, q)= \begin{cases}-n s\left(\left|\frac{1}{p}-\frac{1}{2}\right|+\left|\frac{1}{q}-\frac{1}{2}\right|\right) & \text { for }(p, q) \in \mathrm{I} \cup \mathrm{II}, \\ -n s\left|\frac{1}{q}-\frac{1}{2}\right| & \text { for }(p, q) \in \mathrm{III} \cup \mathrm{VI}, \quad \text { when } 1<s<\infty \\ -n s\left|\frac{1}{p}-\frac{1}{2}\right| & \text { for }(p, q) \in \mathrm{IV} \cup \mathrm{~V},\end{cases}
$$

The main result of this paper reads as follows.
Theorem 1.3. Let $0<s<1$ or $1<s<\infty$ and let $1 \leq p, q \leq \infty$ and $1 / r=1 / p+1 / q$. Suppose that $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ with $m=m_{s}(p, q)$. Then $T_{\sigma}^{s}$ is bounded from $H^{p} \times H^{q}$ to $L^{r}$, where $L^{r}$ should be replaced by BMO when $r=\infty$.

For $0<s<1$ or $1<s<\infty$, the number $m_{s}(p, q)$ is always bigger than or equal to the number $-n s(|1 / p-1 / 2|+|1 / q-1 / 2|)$ for all $1 \leq p, q \leq \infty$. In particular, if $(p, q) \in \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}$, then $m_{s}(p, q)>-n s(|1 / p-1 / 2|+|1 / q-1 / 2|)$ except for $p=2$ or $q=2$. In this sense, Theorem 1.3 improves Theorem 1.2. Moreover, the number $m_{s}(p, q)$ defined above is optimal for some cases. More precisely, the following theorem holds true.

Theorem 1.4. Let $0<s<1$ or $1<s<\infty$, and let $m \in \mathbb{R},(p, q) \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}$, and $1 / r=1 / p+1 / q$. Suppose that all $T_{\sigma}^{s}$ with $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ are bounded from $H^{p} \times H^{q}$ to $L^{r}$ with $L^{r}$ replaced by BMO when $r=\infty$. Then $m \leq m_{s}(p, q)$.

It should be emphasized that the optimality for the case $(p, q) \in I \cup I I$ is already proved in [1, Section 3.2]. However, we shall also give the proofs of these cases, which will be slightly different from the one in [1, Section 3.2].

The rest of this paper is organized as follows. In Section 2, we give a proof of Theorem 1.2. In Section 3, we consider the asymptotic behavior of the Fourier transform of functions including an oscillator $e^{i|\xi|^{s}}$, which will play important roles in the proofs of Theorems 1.3 and 1.4. In Section 4, we prepare some lemmas which will be used in Section 5. In Section 5. we prove the assertion of Theorem 1.3 in the end point case $(p, q)=(1, \infty)$, which implies Theorem 1.3 with the aid complex interpolation. In Section 6, we prove Theorem 1.4 .

We end this section by preparing some notations.
Notation 1.5. We denote by $\mathbb{N}$ and $\mathbb{N}_{0}$ the sets of positive integers and nonnegative integers, respectively.

The Fourier transform and the inverse Fourier transform on $\mathbb{R}^{n}$ are defined by

$$
\widehat{f}(\xi)=\int_{\mathbb{R}^{n}} e^{-i \xi \cdot x} f(x) d x \quad \text { and } \quad(g)^{\vee}(x)=\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} e^{i \xi \cdot x} g(\xi) d \xi
$$

We take $\varphi, \psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that $\varphi=1$ on $\{|\xi| \leq 1\}$, $\operatorname{supp} \varphi \subset\{|\xi| \leq 2\}$, $\operatorname{supp} \psi \subset\{1 / 2 \leq$ $|\xi| \leq 2\}$, and $\varphi+\sum_{j \in \mathbb{N}} \psi\left(2^{-j}.\right)=1$. In what follows, we will write $\psi_{0}=\varphi, \psi_{j}=\psi\left(2^{-j}.\right)$ for $j \in \mathbb{N}$, and $\varphi_{j}=\varphi\left(2^{-j}.\right)$ for $j \in \mathbb{N}_{0}$. Then, we see that $\varphi_{0}=\psi_{0}=\varphi$ and

$$
\sum_{j=0}^{k} \psi_{j}=\varphi_{k}, \quad k \in \mathbb{N}_{0}
$$

We define the $C^{\infty}$ function $\zeta=1-\varphi$. Then we have $\partial^{\alpha} \zeta \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ for $|\alpha| \geq 1, \zeta=\sum_{j \in \mathbb{N}} \psi_{j}$, and

$$
\zeta=0 \text { on }\{|\xi| \leq 1\}, \quad \zeta=1 \text { on }\{|\xi| \geq 2\} .
$$

For a smooth function $\theta$ on $\mathbb{R}^{n}$ and for $N \in \mathbb{N}_{0}$, we write $\|\theta\|_{C^{N}}=\max _{|\alpha| \leq N} \sup _{\xi}\left|\partial_{\xi}^{\alpha} \theta(\xi)\right|$.
Lastly, we recall the local Hardy space $h^{1}$ (for the definition of the local Hardy space $h^{1}$, see Goldberg [8]). It is known that $H^{1} \hookrightarrow h^{1} \hookrightarrow L^{1}$. As proved in [8], all functions in $h^{1}$ can be decomposed by so-called atoms, which satisfy that

$$
\begin{equation*}
\operatorname{supp} f \subset\left\{y \in \mathbb{R}^{n}| | y-\bar{y} \mid \leq r\right\}, \quad\|f\|_{L^{\infty}} \leq r^{-n} \tag{1.1}
\end{equation*}
$$

and, in addition, if $r<1$,

$$
\begin{equation*}
\int f(y) d y=0 \tag{1.2}
\end{equation*}
$$

It is easily proved that, if $f$ satisfies only (1.1) with $r \geq 1$, then $f$ can be written as a linear combination of the atoms that satisfy (1.1) with $r=1$ (see, e.g., Miyachi-Tomita [12]). In this paper, a function $f$ on $\mathbb{R}^{n}$ is called an $h^{1}$-atom of first kind if $f$ satisfies (1.1)-(1.2) for $r<1$, and is called an $h^{1}$-atom of second kind if $f$ satisfies (1.1) for $r=1$. Atoms of both kinds are simply called $h^{1}$-atoms.

## 2. Proof of Theorem 1.2

In this section, we shall give a proof of Theorem 1.2. The ideas of the proof come from [9, Proof of Theorem 1.3].

For $d \in \mathbb{N}$ and $m \in \mathbb{R}$, the class $\dot{S}_{1,0}^{m}\left(\mathbb{R}^{d}\right)$ consists of all $C^{\infty}$ functions $\sigma$ on $\mathbb{R}^{d} \backslash\{0\}$ such that

$$
\left|\partial_{\xi}^{\alpha} \sigma(\xi)\right| \leq C_{\alpha}|\xi|^{m-|\alpha|}, \quad \xi \in \mathbb{R}^{d} \backslash\{0\}
$$

for all multi-indices $\alpha \in\left(\mathbb{N}_{0}\right)^{d}$. We use the notation $X_{r}$ given by

$$
X_{r}= \begin{cases}L^{r} & \text { if } 0<r<\infty  \tag{2.1}\\ B M O & \text { if } r=\infty\end{cases}
$$

Now, we recall the boundedness result for the bilinear Fourier multiplier operator $T_{\sigma}$ with $\sigma \in \dot{S}_{1,0}^{0}\left(\mathbb{R}^{2 n}\right)$. The following theorem is due to Coifman-Meyer [3, 4], Kenig-Stein [10], Grafakos-Kalton [5], and Grafakos-Torres [7].
Theorem 2.1. Let $0<p, q \leq \infty$ and $1 / r=1 / p+1 / q$. If $\sigma \in \dot{S}_{1,0}^{0}\left(\mathbb{R}^{2 n}\right)$, then the bilinear Fourier multiplier operator $T_{\sigma}$ is bounded from $H^{p} \times H^{q}$ to $X_{r}$.

We will use the following two propositions, whose proofs can be found in [9, Section 6].
Proposition 2.2 ([9, Proposition 2.3]). Let $m_{1}, m_{2} \leq 0, m=m_{1}+m_{2}, a_{0} \in \dot{S}_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$, $a_{1} \in \dot{S}_{1,0}^{-m_{1}}\left(\mathbb{R}^{n}\right)$, $a_{2} \in \dot{S}_{1,0}^{-m_{2}}\left(\mathbb{R}^{n}\right)$, and $\sigma(\xi, \eta)=a_{0}(\xi, \eta) a_{1}(\xi) a_{2}(\eta)$. Then the bilinear Fourier multiplier operator $T_{\sigma}$ is bounded in

$$
\begin{cases}H^{p} \times H^{q} \rightarrow L^{r}, & 0<p, q<\infty, \quad 1 / r=1 / p+1 / q \\ B M O \times H^{q} \rightarrow L^{q}, & 0<q<\infty, \quad \text { if } m_{1}<0 \\ H^{p} \times B M O \rightarrow L^{p}, & 0<p<\infty, \quad \text { if } m_{2}<0 \\ B M O \times B M O \rightarrow B M O & \text { if } m_{1}, m_{2}<0\end{cases}
$$

Proposition 2.3 ([9, Proposition 2.4]). Let $m_{1} \leq 0, a_{0} \in \dot{S}_{1,0}^{m_{1}}\left(\mathbb{R}^{2 n}\right)$, $a_{1} \in \dot{S}_{1,0}^{-m_{1}}\left(\mathbb{R}^{n}\right)$, and let $\sigma(\xi, \eta)=a_{0}(\xi, \eta) a_{1}(\xi)$. Then the bilinear Fourier multiplier operator $T_{\sigma}$ is bounded in

$$
\begin{cases}H^{p} \times L^{\infty} \rightarrow L^{p}, & 0<p<\infty \\ B M O \times L^{\infty} \rightarrow B M O & \text { if } m_{1}<0\end{cases}
$$

Proof of Theorem 1.2. Let $0<s<1$ or $s>1$, and let $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right), m=-n s(\mid 1 / p-$ $1 / 2|+|1 / q-1 / 2|)$. We write $m_{1}=-n s|1 / p-1 / 2|$ and $m_{2}=-n s|1 / q-1 / 2|$. Using the functions $\zeta$ and $\varphi$ given in Notation 1.5, we decompose the bilinear multiplier $\tau$ defined by $\tau(\xi, \eta)=e^{i|\xi| s} e^{i|\eta|^{s}} \sigma(\xi, \eta)$ as

$$
\begin{aligned}
& \tau(\xi, \eta)=\tau_{1}(\xi, \eta)+\tau_{2}(\xi, \eta)+\tau_{3}(\xi, \eta)+\tau_{4}(\xi, \eta), \\
& \tau_{1}(\xi, \eta)=e^{i|\xi|^{s}} \varphi(\xi) e^{i|\eta|^{s}} \varphi(\eta) \sigma(\xi, \eta), \quad \tau_{2}(\xi, \eta)=e^{i|\xi|^{s}} \zeta(\xi) e^{i|\eta|^{s}} \varphi(\eta) \sigma(\xi, \eta), \\
& \tau_{3}(\xi, \eta)=e^{i|\xi|^{s}} \varphi(\xi) e^{i|\eta|^{s}} \zeta(\eta) \sigma(\xi, \eta), \quad \tau_{4}(\xi, \eta)=e^{i|\xi|^{s}} \zeta(\xi) e^{i|\eta|^{s}} \zeta(\eta) \sigma(\xi, \eta) .
\end{aligned}
$$

We show that each $T_{\tau_{i}}, i=1,2,3,4$, is bounded from $H^{p} \times H^{q}$ to $X_{r}, 1 \leq p, q \leq \infty$, $1 / r=1 / p+1 / q$.

We begin with the estimate of $T_{\tau_{1}}$. Since $\left(e^{i|\xi|^{s}} \varphi(\xi)\right)^{\vee} \in L^{1}\left(\mathbb{R}^{n}\right)$ (see (4.1) and (4.8)), the Fourier multiplier operator $e^{i|D|^{s}} \varphi(D)$ is bounded on $H^{p}, 1 \leq p \leq \infty$. Since $m \leq 0$, we have $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right) \subset S_{1,0}^{0}\left(\mathbb{R}^{2 n}\right) \subset \dot{S}_{1,0}^{0}\left(\mathbb{R}^{2 n}\right)$, and hence by Theorem [2.1, $T_{\sigma}$ is bounded from $H^{p} \times H^{q}$ to $X_{r}$. Thus, the desired boundedness of $T_{\tau_{1}}$ is given.

Next, we consider the estimate of $T_{\tau_{2}}$. We write

$$
\tau_{2}(\xi, \eta)=e^{i|\xi|^{s}} \zeta(\xi)|\xi|^{m_{1}} \times e^{i|\eta|^{s}} \varphi(\eta) \times|\xi|^{-m_{1}} \sigma(\xi, \eta)
$$

By Theorem A, the Fourier multiplier operator $e^{i|D|^{s}} \zeta(D)|D|^{m_{1}}$ is bounded on $H^{p}$ if $1 \leq$ $p<\infty$, and on $B M O$ if $p=\infty$. As we showed above, $e^{i|D|^{s}} \varphi(D)$ is bounded on $H^{q}, 1 \leq$
$q \leq \infty$. On the other hand, since $|\xi|^{-m_{1}} \in \dot{S}_{1,0}^{-m_{1}}\left(\mathbb{R}^{n}\right)$ and $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right) \subset \dot{S}_{1,0}^{m_{1}}\left(\mathbb{R}^{2 n}\right)$, from Propositions 2.2 and 2.3, it follows that the bilinear Fourier multiplier operator corresponding to $|\xi|^{-m_{1}} \sigma(\xi, \eta)$ is bounded in $H^{p} \times H^{q} \rightarrow X_{r}$, with $H^{p}$ replaced by $B M O$ if $p=\infty$ (notice that $m_{1}<0$ if $p=\infty$ ). Thus, combining these boundedness, we obtain the $H^{p} \times H^{q} \rightarrow X_{r}$ boundedness of $T_{\tau_{2}}$.

In the same way as above, we see that $T_{\tau_{3}}$ is bounded from $H^{p} \times H^{q}$ to $X_{r}$.
We finally prove that $T_{\tau_{4}}$ is bounded from $H^{p} \times H^{q}$ to $X_{r}$. The multiplier $\tau_{4}$ can be written as

$$
\tau_{4}(\xi, \eta)=e^{i|\xi|^{s}} \zeta(\xi)|\xi|^{m_{1}} \times e^{i|\eta|^{s}} \zeta(\eta)|\eta|^{m_{2}} \times|\xi|^{-m_{1}}|\eta|^{-m_{2}} \sigma(\xi, \eta) .
$$

Since $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right) \subset \dot{S}_{1,0}^{m}\left(\mathbb{R}^{2 n}\right),|\xi|^{-m_{1}} \in \dot{S}_{1,0}^{-m_{1}}$ and $|\eta|^{-m_{2}} \in \dot{S}_{1,0}^{-m_{2}}$, it follows from Proposition 2.2 that the bilinear Fourier multiplier $|\xi|^{-m_{1}}|\eta|^{-m_{2}} \sigma(\xi, \eta)$ gives rise to a bounded operator in $H^{p} \times H^{q} \rightarrow X_{r}$ with $H^{p}$ or $H^{q}$ replaced by $B M O$ if $p=\infty$ or $q=\infty$, respectively. Here, we notice that $m_{1}<0$ if $p=\infty$, and $m_{2}<0$ if $q=\infty$, respectively. Hence, combining this with Theorem A, we obtain the $H^{p} \times H^{q} \rightarrow X_{r}$ boundedness of $T_{\tau_{4}}$. This completes the proof of Theorem 1.2.

## 3. Fourier transform of $e^{i|\xi|^{s}} \psi\left(2^{-j} \xi\right)$

In this section, we investigate the asymptotic behavior of the Fourier transform of the oscillator $e^{i|\xi|^{s}}$ multiplied by Littlewood-Paley's dyadic decompositions. This property is one of the keys to proving our main theorem.

Proposition 3.1. Let $0<s<1$ or $1<s<\infty$. Suppose that $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ satisfies $\operatorname{supp} \psi \subset$ $\{1 / 2 \leq|\xi| \leq 2\}$. Then, for any $N_{1}, N_{2}, N_{3} \geq 0$, there exist $c=c\left(n, s, N_{1}, N_{2}, N_{3}\right)>0$ and $M=M\left(n, s, N_{1}, N_{2}, N_{3}\right) \in \mathbb{N}$ such that

$$
\left|\left(e^{i|\xi|^{s}} \psi\left(2^{-j} \xi\right)\right)^{\vee}(x)\right| \leq c\|\psi\|_{C^{M}} \begin{cases}2^{-j N_{1}}, & \text { if } 2^{j(1-s)}|x|<a  \tag{3.1}\\ 2^{j\left(n-\frac{n s}{2}\right)}, & \text { if } a \leq 2^{j(1-s)}|x| \leq b \\ 2^{-j N_{2}}|x|^{-N_{3}}, & \text { if } 2^{j(1-s)}|x|>b\end{cases}
$$

for $j \in \mathbb{N}_{0}$, where $a=s 4^{-|1-s|}$ and $b=s 4^{|1-s|}$. If in addition $\psi(\xi) \neq 0$ for $2 / 3 \leq|\xi| \leq 3 / 2$, then there exist $c^{\prime}=c^{\prime}(n, s, \psi)>0$ and $j_{0}=j_{0}(n, s, \psi) \in \mathbb{N}$ such that

$$
\begin{align*}
& \frac{1}{c^{\prime}} 2^{j\left(n-\frac{n s}{2}\right)} \leq\left|\left(e^{i|\xi|^{s}} \psi\left(2^{-j} \xi\right)\right)^{\vee}(x)\right| \leq c^{\prime} 2^{j\left(n-\frac{n s}{2}\right)}  \tag{3.2}\\
& \text { if } \quad a^{\prime} \leq 2^{j(1-s)}|x| \leq b^{\prime} \quad \text { and } j>j_{0}
\end{align*}
$$

where $a^{\prime}=s(3 / 2)^{-|1-s|}$ and $b^{\prime}=s(3 / 2)^{|1-s|}$.
To prove this proposition, we first observe that the determinant and the signature ( $=$ (the number of positive eigenvalues) -(the number of negative eigenvalues)) of the matrix

$$
\operatorname{Hess}\left(|\xi|^{s}\right)=\left(\partial_{\xi_{i}} \partial_{\xi_{j}}|\xi|^{s}\right)_{1 \leq i, j \leq n}
$$

are given by

$$
\begin{equation*}
\operatorname{det} \operatorname{Hess}\left(|\xi|^{s}\right)=s^{n}(s-1)|\xi|^{(s-2) n} \tag{3.3}
\end{equation*}
$$

and

$$
\operatorname{sign} \operatorname{Hess}\left(|\xi|^{s}\right)= \begin{cases}n-2 & \text { if } 0<s<1  \tag{3.4}\\ n & \text { if } s>1\end{cases}
$$

In fact this is a simple computation. We have

$$
\partial_{\xi_{i}} \partial_{\xi_{j}}|\xi|^{s}=s|\xi|^{s-2} \delta_{i, j}+s(s-2)|\xi|^{s-2} \frac{\xi_{i}}{|\xi|} \frac{\xi_{j}}{|\xi|} .
$$

Hence if we take an orthogonal matrix $T=\left(t_{i, j}\right)$ that satisfies

$$
\sum_{j=1}^{n} t_{i, j} \frac{\xi_{j}}{|\xi|}= \begin{cases}1 & \text { for } i=1 \\ 0 & \text { for } i=2, \ldots, n\end{cases}
$$

then $T\left(\operatorname{Hess}\left(|\xi|^{s}\right)\right) T^{-1}$ is equal to the diagonal matrix with the diagonal entries

$$
s(s-1)|\xi|^{s-2}, s|\xi|^{s-2}, \ldots, s|\xi|^{s-2}
$$

From this we obtain (3.3) and (3.4).
Proof of Proposition 3.1. By a simple change of variables we can write

$$
H_{j}(x)=\left(e^{i|\xi|^{s}} \psi\left(2^{-j} \xi\right)\right)^{\vee}(x)=\frac{2^{j n}}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} e^{i 2^{j s} \phi_{j}(x, \eta)} \psi(\eta) d \eta
$$

with

$$
\phi_{j}(x, \eta)=2^{j(1-s)} x \cdot \eta+|\eta|^{s} .
$$

The gradient of the phase function $\phi_{j}(x, \eta)$ is given by

$$
\operatorname{grad}_{\eta} \phi_{j}(x, \eta)=2^{j(1-s)} x+s|\eta|^{s-1} \frac{\eta}{|\eta|} .
$$

For each $x \in \mathbb{R}^{n} \backslash\{0\}$, there exists a unique $\eta_{0}=\eta_{0}(x) \in \mathbb{R}^{n} \backslash\{0\}$ such that $\left.\operatorname{grad}_{\eta} \phi_{j}(x, \eta)\right|_{\eta=\eta_{0}}=$ 0 . In fact, $\eta_{0}$ is determined by the equations

$$
2^{j(1-s)}|x|=s\left|\eta_{0}\right|^{s-1}, \quad-\frac{x}{|x|}=\frac{\eta_{0}}{\left|\eta_{0}\right|} .
$$

If $\eta_{0}$ is in a neighborhood of $\operatorname{supp} \psi$ then we can use the stationary phase method to obtain the asymptotic behavior of $H_{j}(x)$. If $\eta_{0}$ is outside a neighborhood of $\operatorname{supp} \psi$ then we can deduce the rapid decay of $H_{j}(x)$ by integration by parts. To be precise, we divide the argument into several cases.

We first consider the case $0<s<1$.
Case I: $0<s<1$ and $2^{j(1-s)}|x|<s 4^{s-1}$. Then $\left|\eta_{0}\right|>4$. In this case, for $\eta \in \operatorname{supp} \psi \subset$ $\{1 / 2 \leq|\eta| \leq 2\}$, we have

$$
\left.\left|\operatorname{grad}_{\eta} \phi_{j}(x, \eta)\right|=\left.\left|2^{j(1-s)} x+s\right| \eta\right|^{s-1} \frac{\eta}{|\eta|}\left|\geq-2^{j(1-s)}\right| x \right\rvert\,+s 2^{s-1}>s\left(2^{s-1}-4^{s-1}\right)
$$

and

$$
\begin{equation*}
\left|\partial_{\eta}^{\alpha} \phi_{j}(x, \eta)\right|=\left.\left|\partial_{\eta}^{\alpha}\right| \eta\right|^{s} \mid \leq c(n, s, \alpha) \quad \text { for } \quad|\alpha| \geq 2 \tag{3.5}
\end{equation*}
$$

Thus integration by parts gives

$$
\left|H_{j}(x)\right| \leq c(n, s, N)\|\psi\|_{C^{N}} 2^{j n}\left(2^{j s}\right)^{-N}
$$

for each $N \in \mathbb{N}$. Since $N$ can be taken arbitrarily large, the desired estimate of $H_{j}(x)$ in this case follows.

Case II: $0<s<1$ and $2^{j(1-s)}|x|>s 4^{-(s-1)}$. In this case, $\left|\eta_{0}\right|<1 / 4$, and, for $\eta \in \operatorname{supp} \psi \subset$ $\{1 / 2 \leq|\eta| \leq 2\}$, we have

$$
\left.\left|\operatorname{grad}_{\eta} \phi_{j}(x, \eta)\right|=\left.\left|2^{j(1-s)} x+s\right| \eta\right|^{s-1} \frac{\eta}{|\eta|}\left|\geq 2^{j(1-s)}\right| x\left|-s 2^{-(s-1)}>2^{j(1-s)}\right| x \right\rvert\,\left(1-2^{s-1}\right)
$$

and we also have (3.5). Thus integration by parts gives

$$
\left|H_{j}(x)\right| \leq c(n, s, N)\|\psi\|_{C^{N}} 2^{j n}\left(2^{j}|x|\right)^{-N}
$$

for each $N \in \mathbb{N}$. Since $N$ can be taken arbitrarily large, the desired estimate of $H_{j}(x)$ in this case follows.

Case III: $0<s<1$ and $s 4^{s-1} \leq 2^{j(1-s)}|x| \leq s 4^{-(s-1)}$. In this case, $1 / 4 \leq\left|\eta_{0}\right| \leq 4$. By (3.3) and (3.4), we have

$$
\begin{equation*}
\operatorname{det} \operatorname{Hess}_{\eta}\left(\phi_{j}(x, \eta)\right)=s^{n}(s-1)|\eta|^{(s-2) n}<0 \tag{3.6}
\end{equation*}
$$

and

$$
\operatorname{sign} \operatorname{Hess}_{\eta}\left(\phi_{j}(x, \eta)\right)=n-2
$$

Also for each multi-index $\alpha$ there exists $c(n, s, \alpha)$ such that

$$
\begin{equation*}
\left|\partial_{\eta}^{\alpha} \phi_{j}(x, \eta)\right|=\left|\partial_{\eta}^{\alpha}\left(2^{j(1-s)} x \cdot \eta+|\eta|^{s}\right)\right| \leq c(n, s, \alpha) \quad \text { for } \quad \frac{1}{10}<|\eta|<10 \tag{3.7}
\end{equation*}
$$

Notice that the constant $c(n, s, \alpha)$ can be taken independent of $j$ and $x$ so long as they are in the range of Case III. Thus by using the stationary phase method (see, for example, [17, Chapter VIII, Section 2.3]), we obtain

$$
\begin{align*}
H_{j}(x)= & (2 \pi)^{-\frac{n}{2}} \exp \left(i|x| \frac{s}{s-1} S^{\frac{-s}{s-1}}(1-s)\right)\left(s^{n}(1-s)\left|\eta_{0}\right|^{(s-2) n}\right)^{-\frac{1}{2}} e^{\frac{\pi i}{4}(n-2)}  \tag{3.8}\\
& \times \psi\left(\eta_{0}\right) 2^{j\left(n-\frac{n s}{2}\right)}+O\left(2^{j\left(n-\frac{n s}{2}-s\right)}\right) .
\end{align*}
$$

Here notice that the oscillating factor $\exp (\cdots)$ comes from

$$
2^{j s} \phi_{j}\left(x, \eta_{0}\right)=|x|^{\frac{s}{s-1}} s^{\frac{-s}{s-1}}(1-s) .
$$

Also notice that, by virtue of (3.6) and (3.7), the $O$-estimate in (3.8) holds uniformly for $(j, x)$ in the range of Case III and for $\psi$ satisfying $\operatorname{supp} \psi \subset\{1 / 2 \leq|\xi| \leq 2\}$ and $\|\psi\|_{C^{M}} \leq 1$ with a sufficiently large $M=M(n)$. From (3.8) the estimate of $H_{j}(x)$ in (3.1) for Case III follows.

The estimate (3.2) also follows from (3.8) since $2 / 3 \leq\left|\eta_{0}\right| \leq 3 / 2$ if $s(3 / 2)^{s-1} \leq 2^{j(1-s)}|x| \leq$ $s(2 / 3)^{s-1}$.

Next we consider the case $s>1$. Since the argument needs only slight modification of the case $0<s<1$, we shall only indicate necessary modifications.

Case I': $s>1$ and $2^{j(1-s)}|x|<s 4^{-(s-1)}$. In this case, $\left|\eta_{0}\right|<1 / 4$ and

$$
\left|\operatorname{grad}_{\eta} \phi_{j}(x, \eta)\right|>s\left(2^{-(s-1)}-4^{-(s-1)}\right) .
$$

for $\eta \in \operatorname{supp} \psi$. Integration by parts yields the desired estimate.
Case $\mathrm{II}^{\prime}: s>1$ and $2^{j(1-s)}|x|>s 4^{s-1}$. In this case, $\left|\eta_{0}\right|>4$ and

$$
\left|\operatorname{grad}_{\eta} \phi_{j}(x, \eta)\right|>2^{j(1-s)}|x|\left(1-2^{-(s-1)}\right)
$$

for $\eta \in \operatorname{supp} \psi$. Integration by parts yields the desired estimate.

Case III': $s>1$ and $s 4^{-(s-1)} \leq 2^{j(1-s)}|x| \leq s 4^{s-1}$. In this case, $1 / 4 \leq\left|\eta_{0}\right| \leq 4$. By (3.3) and (3.4), we have

$$
\operatorname{det} \operatorname{Hess}_{\eta}\left(\phi_{j}(x, \eta)\right)=s^{n}(s-1)|\eta|^{(s-2) n}>0
$$

and

$$
\operatorname{sign} \operatorname{Hess}_{\eta}\left(\phi_{j}(x, \eta)\right)=n
$$

The estimate (3.7) also holds. By the stationary phase method, we obtain

$$
\begin{aligned}
H_{j}(x)= & (2 \pi)^{-\frac{n}{2}} \exp \left(i|x|^{\frac{s}{s-1}} s^{\frac{-s}{s-1}}(1-s)\right)\left(s^{n}(s-1)\left|\eta_{0}\right|^{(s-2) n}\right)^{-\frac{1}{2}} e^{\frac{\pi i}{4} n} \\
& \times \psi\left(\eta_{0}\right) 2^{j\left(n-\frac{n s}{2}\right)}+O\left(2^{j\left(n-\frac{n s}{2}-s\right)}\right),
\end{aligned}
$$

from which the desired estimates follow. This completes the proof of Proposition 3.1.
Corollary 3.2. Suppose that $\theta \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ satisfies $\operatorname{supp} \theta \subset\{|\xi| \leq 2\}$ and the function $\zeta$ is as in Notation 1.5. Then the following hold.
(1) Let $0<s<1$ and $N \geq 0$. Then, there exist $c>0$ and $M \in \mathbb{N}$ such that

$$
\left|\left(e^{i|\xi|^{s}} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x)\right| \leq c\|\theta\|_{C^{M}} \begin{cases}|x|^{-\frac{n}{2}-\frac{n}{2(1-s)}}, & \text { if }|x| \leq 1 \\ |x|^{-N}, & \text { if }|x|>1\end{cases}
$$

for all $j \in \mathbb{N}_{0}$.
(2) Let $1<s<\infty$ and $N \geq 0$. Then, there exist $c>0$ and $M \in \mathbb{N}$ such that

$$
\left|\left(e^{i|\xi|^{s}} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x)\right| \leq c\|\theta\|_{C^{M}} \begin{cases}(1+|x|)^{-\frac{n}{2}+\frac{n}{2(s-1)},}, & \text { if }|x| \leq s 8^{s-1} 2^{j(s-1)}, \\ |x|^{-N}, & \text { if }|x|>s 8^{s-1} 2^{j(s-1)},\end{cases}
$$

for all $j \in \mathbb{N}_{0}$.
Proof. We first put $K_{j}=\left(e^{i|\xi|^{s}} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}$ and decompose $K_{j}$ as

$$
\begin{equation*}
K_{j}(x)=\sum_{k=1}^{j+1} K_{k, j}(x) \quad \text { with } \quad K_{k, j}(x)=\left(e^{i|\xi|^{s}} \psi\left(2^{-k} \xi\right) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x) \tag{3.9}
\end{equation*}
$$

Here, we notice from (3.1) of Proposition 3.1 that

$$
\left|K_{k, j}(x)\right| \lesssim\left\|\psi(\cdot) \theta\left(2^{k-j} \cdot\right)\right\|_{C^{M}} \begin{cases}2^{-k N_{1}} & \text { on } \Omega_{k}^{1}:=\left\{2^{k(1-s)}|x| \leq a\right\}  \tag{3.10}\\ 2^{k\left(n-\frac{n s}{2}\right)} & \text { on } \Omega_{k}^{2}:=\left\{a<2^{k(1-s)}|x| \leq b\right\} \\ 2^{-k N_{2}}|x|^{-N_{3}} & \text { on } \Omega_{k}^{3}:=\left\{2^{k(1-s)}|x|>b\right\}\end{cases}
$$

where $a=s 4^{-|1-s|}$ and $b=s 4^{|1-s|}$. For $1 \leq k \leq j+1,\left\|\psi(\cdot) \theta\left(2^{k-j}\right)\right\|_{C^{M}} \lesssim\|\theta\|_{C^{M}}$. Hence $K_{j}$ is estimated as

$$
\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}}\left(\sum_{k=1}^{j+1} 2^{-k N_{1}} \mathbf{1}_{\Omega_{k}^{1}}(x)+\sum_{k=1}^{j+1} 2^{k\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\Omega_{k}^{2}}(x)+\sum_{k=1}^{j+1} 2^{-k N_{2}}|x|^{-N_{3}} \mathbf{1}_{\Omega_{k}^{3}}(x)\right)
$$

Hence, in the following argument, we shall estimate the above three sums.
(11) Let $0<s<1$ and write $L=\frac{n}{2}+\frac{n}{2(1-s)}$. To prove the estimate mentioned in (1), we first prove that

$$
|x|^{L}\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}}, \quad|x| \leq 1
$$

We assume $|x| \leq 1$. For the sum with $\mathbf{1}_{\Omega_{k}^{1}}(x)$, we have for $N_{1}>0$,

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{-k N_{1}} \mathbf{1}_{\Omega_{k}^{1}}(x)=\sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L} 2^{-k L(1-s)} 2^{-k N_{1}} \mathbf{1}_{\Omega_{k}^{1}}(x) \lesssim 1
$$

For the sum with $\mathbf{1}_{\Omega_{k}^{2}}(x)$, since $-L(1-s)+\left(n-\frac{n s}{2}\right)=0$ and overlaps of $\Omega_{k}^{2}$ are finite,

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{k\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\Omega_{k}^{2}}(x)=\sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L} 2^{-k L(1-s)} 2^{k\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\Omega_{k}^{2}}(x) \approx \sum_{k=1}^{j+1} \mathbf{1}_{\Omega_{k}^{2}}(x) \lesssim 1
$$

For the sum with $\mathbf{1}_{\Omega_{k}^{3}}(x)$, we have by choosing $N_{3}>L$ and $N_{2}>(1-s)\left(N_{3}-L\right)$,

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{-k N_{2}}|x|^{-N_{3}} \mathbf{1}_{\Omega_{k}^{3}}(x)=\sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L-N_{3}} 2^{k\left(N_{3}-L\right)(1-s)} 2^{-k N_{2}} \mathbf{1}_{\Omega_{k}^{3}}(x) \lesssim 1 .
$$

Combining the above inequalities, we obtain the assertion (1) for the case $|x| \leq 1$.
We next prove that

$$
|x|^{N}\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}}, \quad|x|>1,
$$

which can be shown by a similar way. In this case, the sum with respect to $\mathbf{1}_{\Omega_{k}^{1}}$ vanishes. Replacing $L$ by $N$ in the above, and taking $N_{2}, N_{3}>0$ satisfying that $N_{3}>N$ and $N_{2}>$ $(1-s)\left(N_{3}-N\right)$, we have the desired estimate for the sum with respect to $\mathbf{1}_{\Omega_{k}^{3}}$. For the sum with respect to $\mathbf{1}_{\Omega_{k}^{2}}$, since $|x|>1$ gives $2^{k(1-s)}<b$ on $\Omega_{k}^{2}$, the cardinality of $k$ is finite. Furthermore, since $1<|x| \leq 2^{-(1-s)} b$ if $x \in \Omega_{k}^{2} \cap\{|x|>1\}, k \geq 1$, it follows that $|x|^{N} \approx 1$ on $\Omega_{k}^{2} \cap\{|x|>1\}$. Thus, we obtain the assertion (11) for $|x|>1$.
(2) In the case $1<s<\infty$, we first observe that, if $|x| \leq s 2^{-(s-1)}=a 2^{s-1}$ or $|x|>$ $s 8^{s-1} 2^{j(s-1)}=b 2^{(j+1)(s-1)}$, then $|x| \leq a 2^{k(s-1)}$ or $|x|>b 2^{k(s-1)}$ holds for all $1 \leq k \leq j+1$, that is, $x \in \Omega_{k}^{1}$ or $x \in \Omega_{k}^{3}$. By (3.9) and (3.10), this implies that for any $N_{1}>0$

$$
\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}} \sum_{k=1}^{j+1} 2^{-k N_{1}} \lesssim\|\theta\|_{C^{M}}, \quad|x| \leq s 2^{-(s-1)}
$$

and, for any $N_{2}>0$ and $N_{3} \geq 0$,

$$
\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}} \sum_{k=1}^{j+1} 2^{-k N_{2}}|x|^{-N_{3}} \lesssim\|\theta\|_{C^{M}}|x|^{-N_{3}}, \quad|x|>s 8^{s-1} 2^{j(s-1)}
$$

Hence, to obtain the desired result, it suffices to prove that

$$
|x|^{L}\left|K_{j}(x)\right| \lesssim\|\theta\|_{C^{M}} \quad \text { on } \quad \Omega_{j}:=\left\{s 2^{-(s-1)}<|x| \leq s 8^{s-1} 2^{j(s-1)}\right\},
$$

where, we wrote $L=\frac{n}{2}-\frac{n}{2(s-1)}$. Here, we note that $L \leq 0$ for $1<s \leq 2$ and $L \geq 0$ for $2 \leq s<\infty$ and write $L_{+}=\max \{0, L\}$. Assume $x \in \Omega_{j}$. For the sum with $\mathbf{1}_{\Omega_{k}^{1}}(x)$, we have for $N_{1}>L_{+}(s-1)$

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{-k N_{1}} \mathbf{1}_{\Omega_{k}^{1}}(x) \lesssim \sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L_{+}} 2^{-k L_{+}(1-s)} 2^{-k N_{1}} \mathbf{1}_{\Omega_{k}^{1}}(x) \lesssim 1 .
$$

For the sum with $\mathbf{1}_{\Omega_{k}^{2}}(x)$, since $-L(1-s)+\left(n-\frac{n s}{2}\right)=0$ and overlaps of $\Omega_{k}^{2}$ are finite,

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{k\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\Omega_{k}^{2}}(x)=\sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L} 2^{-k L(1-s)} 2^{k\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\Omega_{k}^{2}}(x) \approx \sum_{k=1}^{j+1} \mathbf{1}_{\Omega_{k}^{2}}(x) \lesssim 1
$$

For the sum with $\mathbf{1}_{\Omega_{k}^{3}}(x)$, we have by choosing $N_{2}>0$ and $N_{3}>L_{+}$

$$
|x|^{L} \sum_{k=1}^{j+1} 2^{-k N_{2}}|x|^{-N_{3}} \mathbf{1}_{\Omega_{k}^{3}}(x) \lesssim \sum_{k=1}^{j+1}\left(2^{k(1-s)}|x|\right)^{L_{+}-N_{3}} 2^{k\left(N_{3}-L_{+}\right)(1-s)} 2^{-k N_{2}} \mathbf{1}_{\Omega_{k}^{3}}(x) \lesssim 1 .
$$

Therefore we complete the proof of the assertion (2).

## 4. Lemmas

In this section, we prepare some lemmas for our main theorems. Let $\theta \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ satisfy $\operatorname{supp} \theta \subset\{|\xi| \leq 2\}$ and $\zeta$ be as in Notation 1.5. Then, we define, for $j \in \mathbb{N}$,

$$
\begin{aligned}
S_{j} f(x) & =\left(e^{i|\xi|^{s}} \zeta(\xi) \theta\left(2^{-j} \xi\right) \widehat{f}(\xi)\right)^{\vee}(x), \\
T f(x) & =\left(e^{i|\xi|^{s}} \theta(\xi) \widehat{f}(\xi)\right)^{\vee}(x),
\end{aligned}
$$

which can be represented as follows:

$$
\begin{align*}
& S_{j} f(x)=K_{j} * f(x) \text { with } \\
& T f(x)=L * f(x) \text { with } \quad L(x)=\left(e_{j}^{i|\xi|^{s}} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x),  \tag{4.1}\\
& T \mid \xi^{s} \\
&T(\xi))^{\vee}(x)
\end{align*}
$$

Notice that the kernel $K_{j}$ already appeared in Corollary 3.2.
In the succeeding subsections, we will give several inequalities for the operators $S_{j}$ and $T$. Some of the inequalities are concerned with $h^{1}$-atoms. Recall that, in our definition of $h^{1}$-atom, the radius $r$ of the supporting ball of an $h^{1}$-atom satisfies $r \leq 1$ (see Notation 1.5).
4.1. Inequalities for $s \neq 1$. In this subsection, we show some inequalities which will be used for proving the boundedness in both cases $s<1$ and $s>1$.

Lemma 4.1. Let $0<s<1$ or $1<s<\infty$ and let $1 \leq p \leq \infty$. Then, there exist $c>0$ and $M \in \mathbb{N}$ such that

$$
\left\|S_{j} f\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \leq c\left(2^{j}\right)^{s n\left|\frac{1}{p}-\frac{1}{2}\right|}\|\theta\|_{C^{M}}\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}
$$

for all $j \in \mathbb{N}_{0}$.
Proof. This lemma follows from the trivial $L^{2}$-boundedness, the $L^{1}$-boundedness and $L^{\infty}$ boundedness with the aid of complex interpolation. The $L^{1}$ and $L^{\infty}$-boundedness follow from the kernel estimate below:

$$
\begin{equation*}
\left\|K_{j}\right\|_{L^{1}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{s n}{2}} \tag{4.2}
\end{equation*}
$$

for some constant $M \in \mathbb{N}$. This inequality is derived from the following fact: for $N \geq 0$ with

$$
\begin{cases}0 \leq N<\frac{n}{2(1-s)}, & \text { if } 0<s<1  \tag{4.3}\\ 0 \leq N<\infty, & \text { if } 1<s<\infty\end{cases}
$$

the kernel $K_{j}$ of $S_{j}$ satisfies that

$$
\begin{equation*}
\left\|\left(1+2^{j(1-s)}|x|\right)^{N} K_{j}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}} \tag{4.4}
\end{equation*}
$$

In fact, choosing $N \geq 0$ that satisfies

$$
\begin{cases}\frac{n}{2}<N<\frac{n}{2(1-s)}, & \text { if } 0<s<1 \\ \frac{n}{2}<N<\infty, & \text { if } 1<s<\infty\end{cases}
$$

(notice that $\frac{n}{2}<\frac{n}{2(1-s)}$ if $0<s<1$ ), and using Cauchy-Schwarz inequality and (4.4), we obtain

$$
\begin{aligned}
\left\|K_{j}\right\|_{L^{1}} & \leq\left\|\left(1+2^{j(1-s)}|x|\right)^{-N}\right\|_{L^{2}}\left\|\left(1+2^{j(1-s)}|x|\right)^{N} K_{j}\right\|_{L^{2}} \\
& \lesssim\|\theta\|_{C^{M}}\left(2^{j}\right)^{-\frac{(1-s) n}{2}}\left(2^{j}\right)^{\frac{n}{2}}=\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{s n}{2}}
\end{aligned}
$$

Although both (4.2) and (4.4) can be shown by the use of Corollary 3.2, here we shall give an elementary proof of (4.4), which may be of independent interest. We will also use the inequality (4.4) in the proof of the next lemma.

Hence, we move on proving that (4.4) holds for $N \geq 0$ satisfying (4.3). To this end, it is sufficient to show that

$$
\begin{equation*}
\left\|\left(2^{j(1-s)}|x|\right)^{N} K_{j}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}}, \quad \text { if } N \geq 0 \text { satisfies (4.3). } \tag{4.5}
\end{equation*}
$$

The case $N=0$ obviously follows from Plancherel's theorem, and thus, we shall assume that $N>0$. We recall the decomposition (3.9):

$$
K_{j}(x)=\sum_{k=1}^{j+1} K_{k, j}(x) \quad \text { with } \quad K_{k, j}(x)=\left(e^{i|\xi|^{s}} \psi\left(2^{-k} \xi\right) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x)
$$

A simple calculation gives that for $\alpha \in\left(\mathbb{N}_{0}\right)^{n}$ and $1 \leq k \leq j+1$

$$
\left|\partial_{\xi}^{\alpha}\left(e^{i|\xi|^{s}} \psi\left(2^{-k} \xi\right) \theta\left(2^{-j} \xi\right)\right)\right| \lesssim\|\theta\|_{C^{|\alpha|} \mid}\left(2^{k}\right)^{(s-1)|\alpha|} \mathbf{1}_{\left\{2^{k-1} \leq|\xi| \leq 2^{k+1}\right\}}
$$

and thus, by Plancherel's theorem,

$$
\left\|x^{\alpha} K_{k, j}(x)\right\|_{L^{2}} \lesssim\|\theta\|_{C^{|\alpha|} \mid}\left(2^{k}\right)^{(s-1)|\alpha|}\left(2^{k}\right)^{\frac{n}{2}}
$$

Here, take $0<t<1$ and $\gamma \in \mathbb{N}$ satisfying $N=t \gamma>0$. Then, by Hölder's inequality

$$
\begin{aligned}
& \left\||x|^{N} K_{k, j}\right\|_{L^{2}}=\left\|\left(|x|^{\gamma}\left|K_{k, j}\right|\right)^{t}\left|K_{k, j}\right|^{1-t}\right\|_{L^{2}} \\
& \lesssim \sum_{|\alpha|=\gamma}\left\|x^{\alpha} K_{k, j}\right\|_{L^{2}}^{t}\left\|K_{k, j}\right\|_{L^{2}}^{1-t} \lesssim\|\theta\|_{C^{\gamma}}\left(2^{k}\right)^{(s-1) N}\left(2^{k}\right)^{\frac{n}{2}}
\end{aligned}
$$

Therefore, since the condition (4.3) especially means $\frac{n}{2}+(s-1) N>0$ in the case $0<s<1$,

$$
\begin{aligned}
\left\||x|^{N} K_{j}\right\|_{L^{2}} & \leq \sum_{k=1}^{j+1}\left\||x|^{N} K_{k, j}\right\|_{L^{2}} \lesssim\|\theta\|_{C^{\gamma}} \sum_{k=1}^{j+1}\left(2^{k}\right)^{(s-1) N}\left(2^{k}\right)^{\frac{n}{2}} \\
& \approx\|\theta\|_{C^{\gamma}}\left(2^{j}\right)^{(s-1) N}\left(2^{j}\right)^{\frac{n}{2}} .
\end{aligned}
$$

This implies (4.5), and thus the proof is completed.
Lemma 4.2. Let $0<s<1$ or $1<s<\infty$ and let $0 \leq t \leq 1$. Suppose $f$ is an $h^{1}$-atom supported on a ball of radius $r$ centered at the origin in $\mathbb{R}^{n}$. Then the following hold.
(1) There exists $c>0$ depending only on $n$ such that

$$
\left\|S_{j} f\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \leq c\|\theta\|_{C^{0}}\left(2^{j}\right)^{\frac{n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}
$$

for all $j \in \mathbb{N}_{0}$.
(2) If $A \geq 2 r$ and if $N$ satisfies that

$$
\begin{cases}0 \leq N<\frac{n}{2(1-s)}, & \text { if } 0<s<1 \\ 0 \leq N<\infty, & \text { if } 1<s<\infty\end{cases}
$$

then

$$
\left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)} \leq c\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} A\right)^{-N(1-t)} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}
$$

for all $j \in \mathbb{N}_{0}$, where the constants $c>0$ and $M \in \mathbb{N}$ depend only on $n$, $s$, and $N$.
Proof. (1) We first observe that, by Plancherel's theorem,

$$
\begin{equation*}
\left\|S_{j} f\right\|_{L^{2}}=\left\|K_{j} * f\right\|_{L^{2}} \leq\left\|K_{j}\right\|_{L^{2}}\|f\|_{L^{1}} \lesssim\left(2^{j}\right)^{\frac{n}{2}}\|\theta\|_{C^{0}} \tag{4.6}
\end{equation*}
$$

We next show that

$$
\begin{equation*}
\left\|S_{j} f\right\|_{L^{2}} \lesssim\left(2^{j}\right)^{\frac{n}{2}} \min \left\{2^{j} r,\left(2^{j} r\right)^{-\frac{n}{2}}\right\}\|\theta\|_{C^{0}} \tag{4.7}
\end{equation*}
$$

holds for all $h^{1}$-atoms $f$. If $f$ is an $h^{1}$-atom of second kind (i.e., $r=1$ ), then, by Plancherel's theorem, $\left\|S_{j} f\right\|_{L^{2}} \lesssim\|\theta\|_{C^{0}}$, which is identical with (4.7) for $r=1$. We shall next consider the case that $f$ is an $h^{1}$-atom of first kind (i.e., $r<1$ ). By Plancherel's theorem,

$$
\left\|S_{j} f\right\|_{L^{2}} \leq\|\theta\|_{C^{0}}\|f\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{0}} r^{-\frac{n}{2}}=\|\theta\|_{C^{0}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j} r\right)^{-\frac{n}{2}}
$$

Moreover, since $f$ is an $h^{1}$-atom of first kind and is supported on a ball centered at the origin, Taylor's theorem with the moment condition $\int f=0$ yields that

$$
\begin{aligned}
\left\|S_{j} f\right\|_{L^{2}} & =\left\|\sum_{|\alpha|=1} \int_{|y| \leq r}\left(e^{i|\xi|^{s}} \xi^{\alpha} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x-t y) y^{\alpha} f(y) d y d t\right\|_{L_{x}^{2}} \\
& \leq \sum_{|\alpha|=1}\left\|\left(e^{i|\xi|^{s}} \xi^{\alpha} \zeta(\xi) \theta\left(2^{-j} \xi\right)\right)^{\vee}(x)\right\|_{L_{x}^{2}} \int_{|y| \leq r}|y||f(y)| d y \\
& \lesssim\|\theta\|_{C^{0}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j} r\right),
\end{aligned}
$$

where, in the last inequality, we used Plancherel's theorem. These two estimates imply (4.7).
Finally, interpolating (4.6) and (4.7), we have for $0 \leq t \leq 1$

$$
\begin{aligned}
& \left\|S_{j} f\right\|_{L^{2}}=\left(\left\|S_{j} f\right\|_{L^{2}}\right)^{1-t}\left(\left\|S_{j} f\right\|_{L^{2}}\right)^{t} \\
& \lesssim\left(\left(2^{j}\right)^{\frac{n}{2}}\|\theta\|_{C^{0}}\right)^{1-t}\left(\left(2^{j}\right)^{\frac{n}{2}} \min \left\{2^{j} r,\left(2^{j} r\right)^{-\frac{n}{2}}\right\}\|\theta\|_{C^{0}}\right)^{t} \\
& =\left(2^{j}\right)^{\frac{n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\|\theta\|_{C^{0}},
\end{aligned}
$$

which completes the proof of the assertion (11).
(21) We observe that, if $A \geq 2 r, A \leq|x| \leq 2 A$, and $|y| \leq r \leq \frac{A}{2}$, then $|x-y| \approx|x| \approx A$. Since $f$ is an $h^{1}$-atom supported on a ball centered at the origin, this observation yields that

$$
\left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)}=\left\|\left(2^{j(1-s)}|x|\right)^{-N}\left(2^{j(1-s)}|x|\right)^{N} K_{j} * f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)}
$$

$$
\begin{aligned}
& \lesssim\left(2^{j(1-s)} A\right)^{-N}\left\|\int_{|y| \leq r}\left(2^{j(1-s)}|x-y|\right)^{N}\left|K_{j}(x-y)\right||f(y)| d y\right\|_{L_{x}^{2}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left(2^{j(1-s)} A\right)^{-N}\left\|\left(2^{j(1-s)}|x|\right)^{N} K_{j}(x)\right\|_{L^{2}} .
\end{aligned}
$$

Here, we recall (4.5). Since the assumption in this assertion is identical with (4.3), by utilizing (4.5), we obtain

$$
\left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)} \lesssim\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} A\right)^{-N}
$$

for some constant $M \in \mathbb{N}$. Also, the inequality (4.7) obviously holds if $L^{2}\left(\mathbb{R}^{n}\right)$ is replaced by $L^{2}(A \leq|x| \leq 2 A)$. Therefore, interpolating these two inequalities, we have for $0 \leq t \leq 1$

$$
\begin{aligned}
& \left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)}=\left(\left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)}\right)^{1-t}\left(\left\|S_{j} f(x)\right\|_{L^{2}(A \leq|x| \leq 2 A)}\right)^{t} \\
& \lesssim\left(\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} A\right)^{-N}\right)^{1-t}\left(\|\theta\|_{C^{0}}\left(2^{j}\right)^{\frac{n}{2}} \min \left\{2^{j} r,\left(2^{j} r\right)^{-\frac{n}{2}}\right\}\right)^{t} \\
& \leq\|\theta\|_{C^{M}}\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} A\right)^{-N(1-t)} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\},
\end{aligned}
$$

which completes the proof of the assertion (2).
Lemma 4.3. Let $0<s<\infty$. Then there exist $c>0$ and $M \in \mathbb{N}$ depending only on $n$ and $s$ such that the following hold.
(1) If $1 \leq p \leq q \leq \infty$, then

$$
\|T f\|_{L^{q}\left(\mathbb{R}^{n}\right)} \leq c\|\theta\|_{C^{M}}\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}
$$

(2) If $f$ is an $h^{1}$-atom supported on a ball of radius $r$ centered at the origin in $\mathbb{R}^{n}$ and if $A \geq 2 r$, then

$$
\|T f\|_{L^{\infty}(A \leq|x| \leq 2 A)} \leq c A^{-(n+s)}\|\theta\|_{C^{M}}
$$

Proof. Before beginning with proofs of the assertions, we show that the kernel $L$ defined in (4.1) satisfies the following inequality: there exists $M \in \mathbb{N}$ such that

$$
\begin{equation*}
|L(x)| \lesssim\|\theta\|_{C^{M}}(1+|x|)^{-(n+s)} \tag{4.8}
\end{equation*}
$$

Although the inequality (4.8) is a well-known fact, for the sake of a self-contained proof, we revisit a proof.

The case for $|x| \leq 1$ is simple, and so we will consider the case $|x| \geq 1$. Since $e^{i|\xi|^{s}}-1=$ $i|\xi|^{s} \int_{0}^{1} e^{i t \mid \xi^{s}} d t$, the kernel $L$ can be expressed by

$$
L(x)=\int_{\substack{|\xi| \leq 2 \\ 0<t<1}} e^{i x \cdot \xi} e^{i t|\xi|^{s}}\left(i|\xi|^{s}\right) \theta(\xi) d \xi d t+\int_{|\xi| \leq 2} e^{i x \cdot \xi} \theta(\xi) d \xi
$$

Integration by parts yields that the absolute value of the second integral is bounded by $\|\theta\|_{C^{M}}(1+|x|)^{-M}$ for any $M \in \mathbb{N}_{0}$, and thus, in the following, we shall consider the first integral. Using a Littlewood-Paley partition of unity on $\mathbb{R}^{n},\left\{\psi\left(2^{-k}\right)\right\}_{k \in \mathbb{Z}}$, since supp $\theta \subset$ $\{|\xi| \leq 2\}$, we can decompose the first integral into

$$
\sum_{k \leq 1} I_{k}(x) \quad \text { with } \quad I_{k}(x):=\int_{\substack{2^{k-1} \leq|\xi| \leq 2^{k+1} \\ 0<t<1}} e^{i x \cdot \xi} e^{i t|\xi|^{s}}\left(i|\xi|^{s}\right) \psi\left(2^{-k} \xi\right) \theta(\xi) d \xi d t
$$

Here, we have for $\alpha \in\left(\mathbb{N}_{0}\right)^{n}$ and $k \leq 1$

$$
\left|\partial_{\xi}^{\alpha}\left(e^{i t|\xi|^{s}}\left(i|\xi|^{s}\right) \psi\left(2^{-k} \xi\right) \theta(\xi)\right)\right| \lesssim\|\theta\|_{C^{|\alpha|}}\left(2^{k}\right)^{s-|\alpha|} \mathbf{1}_{\left\{2^{k-1} \leq|\xi| \leq 2^{k+1}\right\}}, \quad 0 \leq t \leq 1
$$

which gives that for $M \in \mathbb{N}_{0}$

$$
\left|I_{k}(x)\right| \lesssim\|\theta\|_{C^{M}} \times\left\{\begin{array}{l}
\left(2^{k}\right)^{n+s}=|x|^{-(n+s)}\left(2^{k}|x|\right)^{n+s}, \\
|x|^{-M}\left(2^{k}\right)^{-M+n+s}=|x|^{-(n+s)}\left(2^{k}|x|\right)^{-M+n+s}
\end{array}\right.
$$

Therefore, by choosing $M>n+s$,

$$
\sum_{k \leq 1}\left|I_{k}(x)\right| \lesssim\|\theta\|_{C^{M}}|x|^{-(n+s)} \sum_{k \leq 1} \min \left\{\left(2^{k}|x|\right)^{n+s},\left(2^{k}|x|\right)^{-M+n+s}\right\} \approx\|\theta\|_{C^{M}}|x|^{-(n+s)}
$$

which completes the proof of (4.8). Now, we actually prove the assertions (1) and (21).
(11) Take a function $\tilde{\theta} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ satisfying that $\widetilde{\theta}=1$ on $\{|\xi| \leq 2\}$ and $\operatorname{supp} \widetilde{\theta} \subset\{|\xi| \leq 3\}$. Then, we observe that

$$
T f(x)=T(\widetilde{\theta}(D) f)(x)
$$

and also from (4.8) that $L$, the kernel of $T$, is in $L^{1}\left(\mathbb{R}^{n}\right)$ and $\|L\|_{L^{1}} \lesssim\|\theta\|_{C^{M}}$. Therefore, we see that

$$
\begin{aligned}
\|T f\|_{L^{q}} & \lesssim\|\theta\|_{C^{M}}\|\widetilde{\theta}(D) f\|_{L^{q}} \\
& \lesssim\|\theta\|_{C^{M}}\|\widetilde{\theta}(D) f\|_{L^{p}} \lesssim\|\theta\|_{C^{M}}\|f\|_{L^{p}}
\end{aligned}
$$

where, in the second inequality, we used Nikol'skij's inequality (see, e.g., [18, Section 1.3.2, Remark 1]). This completes the proof of the assertion (11).
(2) We first observe that, if $A \geq 2 r, A \leq|x| \leq 2 A$, and $|y| \leq r \leq \frac{A}{2}$, then $|x-y| \approx|x| \approx A$. By (4.8), we have for $A \leq|x| \leq 2 A$

$$
\begin{aligned}
|T f(x)| & \leq \int_{|y| \leq r}|L(x-y)||f(y)| d y \lesssim\|\theta\|_{C^{M}} \int_{|y| \leq r}|x-y|^{-(n+s)}|f(y)| d y \\
& \lesssim A^{-(n+s)}\|\theta\|_{C^{M}}
\end{aligned}
$$

which implies the assertion (2).
4.2. Inequalities for $s<1$. In this subsection, we show some inequalities which will be used for proving the boundedness for $s<1$.

Lemma 4.4. Let $0<s<1$. Then there exist $c>0$ and $M \in \mathbb{N}$ depending only on $n$ and $s$ such that the following hold.
(1) If $f$ is an $h^{1}$-atom supported on a ball centered at the origin in $\mathbb{R}^{n}$, then

$$
\left\|S_{j} f(x)\right\|_{L^{1}(|x| \geq 2)} \leq c\|\theta\|_{C^{M}}
$$

for all $j \in \mathbb{N}_{0}$.
(2) If $0<A \leq 10$, then

$$
\left\|S_{j} f(x)\right\|_{L^{2}(|x| \leq A)} \leq c A^{\frac{n(1-s)}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}
$$

for all $j \in \mathbb{N}_{0}$.

Proof. (1) If $|x| \geq 2$ and $|y| \leq r \leq 1$, then $|x-y| \geq \frac{|x|}{2} \geq 1$. Hence, by Corollary 3.2 (1), we obtain for $N>n$

$$
\begin{aligned}
\left\|S_{j} f(x)\right\|_{L^{1}(|x| \geq 2)} & \lesssim\|\theta\|_{C^{M}} \int_{|x| \geq 2} \int_{|y| \leq r}|x-y|^{-N}|f(y)| d y d x \\
& \lesssim\|\theta\|_{C^{M}}\left(\int_{|x| \geq 2}|x|^{-N} d x\right) \int_{|y| \leq r}|f(y)| d y \lesssim\|\theta\|_{C^{M}}
\end{aligned}
$$

which completes the proof of the assertion (11). (We don't need the moment condition $\int f=0$ here.)
(2) We decompose $f$ by

$$
f=f \mathbf{1}_{\left\{|y| \leq C A^{1-s}\right\}}+f \mathbf{1}_{\left\{|y|>C A^{1-s}\right\}}=: f_{A}^{1}+f_{A}^{2},
$$

where $C=2 \cdot 10^{s}$. For the estimate with respect to $f_{A}^{1}$, we see from Plancherel's theorem that

$$
\left\|S_{j} f_{A}^{1}\right\|_{L^{2}(|x| \leq A)} \leq\left\|S_{j} f_{A}^{1}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \leq\|\theta\|_{C^{0}}\left\|f_{A}^{1}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{0}} A^{\frac{n(1-s)}{2}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}
$$

We next consider the estimate with respect to $f_{A}^{2}$. In the situation here, since $A \leq 10^{s} A^{1-s}$ for $0<A \leq 10$ and $0<s<1$, we realize that, if $|x| \leq A$ and $|y| \geq C A^{1-s}$, then

$$
|x-y| \geq\left(1-\frac{10^{s}}{C}\right)|y|=\frac{|y|}{2}
$$

Hence, by Corollary 3.2 (1),

$$
\begin{aligned}
& \left\|S_{j} f_{A}^{2}\right\|_{L^{2}(|x| \leq A)} \lesssim A^{\frac{n}{2}}\left\|_{|y|>C A^{1-s}}\left|K_{j}(x-y)\right||f(y)| d y\right\|_{L_{x}^{\infty}(|x| \leq A)} \\
& \lesssim A^{\frac{n}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \int_{|y|>C A^{1-s}}|y|^{-\frac{n}{2}-\frac{n}{2(1-s)}} d y \approx A^{\frac{n(1-s)}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

where, in the last inequality, we used $-\frac{n}{2}-\frac{n}{2(1-s)}<-n$. This completes the proof.
4.3. Inequalities for $s>1$. In this subsection, we show some inequalities which will be used for proving the boundedness in the case $s>1$.
Lemma 4.5. Let $1<s<\infty$. If $j \in \mathbb{N}_{0}$ and $A \geq 2^{j(s-1)}$, then

$$
\left\|S_{j} f(x)\right\|_{L^{2}(|x| \leq A)} \leq c A^{\frac{n}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}
$$

where the constants $c>0$ and $M \in \mathbb{N}$ depend only on $n$ and $s$.
Proof. We decompose $f$ as follows:

$$
f=f \mathbf{1}_{\{|y| \leq C A\}}+f \mathbf{1}_{\{|y|>C A\}}=: f_{A}^{1}+f_{A}^{2},
$$

where $C=2 s 8^{s-1}$. For the estimate involved in $f_{A}^{1}$, we have by Plancherel's theorem

$$
\left\|S_{j} f_{A}^{1}\right\|_{L^{2}(|x| \leq A)} \leq\left\|S_{j} f_{A}^{1}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \leq\|\theta\|_{C^{0}}\left\|f_{A}^{1}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\|\theta\|_{C^{0}} A^{\frac{n}{2}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}
$$

Next, we consider the estimate involved in $f_{A}^{2}$. Observe that, for $|x| \leq A$ and $|y| \geq C A$,

$$
|x-y| \geq\left(1-\frac{1}{C}\right)|y| \geq \frac{|y|}{2} \geq \frac{C A}{2} \geq s 8^{s-1} 2^{j(s-1)}
$$

holds, which is possible from the choice of the constant $C \geq 2$. Hence, by utilizing Corollary 3.2 (2), we have for large $N>n$

$$
\begin{aligned}
& \left\|S_{j} f_{A}^{2}\right\|_{L^{2}(|x| \leq A)} \lesssim A^{\frac{n}{2}}\left\|\int_{|y|>C A}\left|K_{j}(x-y)\right||f(y)| d y\right\|_{L_{x}^{\infty}(|x| \leq A)} \\
& \lesssim A^{\frac{n}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \int_{|y|>C A}|y|^{-N} d y \lesssim A^{\frac{n}{2}}\|\theta\|_{C^{M}}\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} .
\end{aligned}
$$

Combining the above estimates, we complete the proof of this lemma.

## 5. Boundedness in $H^{1} \times L^{\infty} \rightarrow L^{1}$

In this section, we shall give a proof of Theorem 1.3. To this end, we will prove the following theorem.

Theorem 5.1. Let $0<s<1$ or $1<s<\infty$. Suppose that $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ with

$$
m= \begin{cases}-\frac{s n}{2}-\frac{s(1-s) n}{2}, & \text { if } 0<s<1 \\ -\frac{s n}{2}, & \text { if } 1<s<\infty\end{cases}
$$

Then $T_{\sigma}^{s}$ is bounded from $h^{1} \times L^{\infty}$ to $L^{1}$.
We notice that Theorem 1.3 can be derived from Theorems 1.2 and 5.1 by virtue of complex interpolation. Thus, it suffices to show Theorem 5.1.

Now, we begin with the proof of Theorem 5.1. We decompose the multiplier $\sigma$ following the idea of Coifman-Meyer [3, 4]. We write

$$
\begin{aligned}
& \sigma(\xi, \eta)=\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sigma(\xi, \eta) \psi_{j}(\xi) \psi_{k}(\eta) \\
& =\sigma(\xi, \eta) \varphi(\xi) \varphi(\eta)+\sum_{j=1}^{\infty} \sum_{k=0}^{j} \sigma(\xi, \eta) \psi_{j}(\xi) \psi_{k}(\eta)+\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sigma(\xi, \eta) \psi_{j}(\xi) \psi_{k}(\eta) \\
& =\sigma(\xi, \eta) \varphi(\xi) \varphi(\eta)+\sum_{j \in \mathbb{N}} \sigma(\xi, \eta) \psi_{j}(\xi) \varphi_{j}(\eta)+\sum_{k \in \mathbb{N}} \sigma(\xi, \eta) \varphi_{k-1}(\xi) \psi_{k}(\eta) \\
& =\sigma_{0}(\xi, \eta)+\sigma_{\mathrm{I}}(\xi, \eta)+\sigma_{\mathrm{II}}(\xi, \eta) .
\end{aligned}
$$

We first consider the multiplier $\sigma_{\mathrm{I}}$. Taking functions $\widetilde{\psi}, \widetilde{\varphi} \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ such that

$$
\begin{array}{ll}
\widetilde{\psi}=1 \text { on }\left\{2^{-1} \leq|\xi| \leq 2\right\}, & \operatorname{supp} \tilde{\psi} \subset\left\{3^{-1} \leq|\xi| \leq 3\right\}, \\
\widetilde{\varphi}=1 \text { on }\{|\xi| \leq 2\}, & \operatorname{supp} \widetilde{\varphi} \subset\{|\xi| \leq 3\},
\end{array}
$$

we can write $\sigma_{\mathrm{I}}$ as

$$
\sigma_{\mathrm{I}}(\xi, \eta)=\sum_{j \in \mathbb{N}} \sigma(\xi, \eta) \widetilde{\psi}\left(2^{-j} \xi\right) \widetilde{\varphi}\left(2^{-j} \eta\right) \psi_{j}(\xi) \varphi_{j}(\eta)
$$

since $\widetilde{\psi}\left(2^{-j} \xi\right) \widetilde{\varphi}\left(2^{-j} \eta\right)$ equals 1 on the support of $\psi_{j}(\xi) \varphi_{j}(\eta)$. Since $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$ and

$$
\operatorname{supp} \sigma\left(2^{j} \xi, 2^{j} \eta\right) \widetilde{\psi}(\xi) \widetilde{\varphi}(\eta) \subset\left\{3^{-1} \leq|\xi| \leq 3\right\} \times\{|\eta| \leq 3\}
$$

the following estimate holds:

$$
\left|\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta}\left(\sigma\left(2^{j} \xi, 2^{j} \eta\right) \widetilde{\psi}(\xi) \widetilde{\varphi}(\eta)\right)\right| \leq C_{\alpha, \beta} 2^{j m}
$$

with $C_{\alpha, \beta}$ independent of $j \in \mathbb{N}_{0}$. Hence, by the Fourier series expansion with respect to the variables $\xi$ and $\eta$, we can write

$$
\sigma\left(2^{j} \xi, 2^{j} \eta\right) \widetilde{\psi}(\xi) \widetilde{\varphi}(\eta)=\sum_{a, b \in \mathbb{Z}^{n}} c_{\mathrm{I}, j}^{(a, b)} e^{i a \cdot \xi} e^{i b \cdot \eta}, \quad|\xi|<\pi, \quad|\eta|<\pi,
$$

with the coefficient satisfying that for $L>0$

$$
\begin{equation*}
\left|c_{\mathrm{I}, j}^{(a, b)}\right| \lesssim 2^{j m}(1+|a|)^{-L}(1+|b|)^{-L} . \tag{5.1}
\end{equation*}
$$

Changing variables $\xi \rightarrow 2^{-j} \xi$ and $\eta \rightarrow 2^{-j} \eta$ and multiplying $\psi_{j}(\xi) \varphi_{j}(\eta)$, we obtain

$$
\sigma(\xi, \eta) \psi_{j}(\xi) \varphi_{j}(\eta)=\sum_{a, b \in \mathbb{Z}^{n}} c_{\mathrm{I}, j}^{(a, b)} e^{i a \cdot 2^{-j} \xi} e^{i b \cdot 2^{-j} \eta} \psi_{j}(\xi) \varphi_{j}(\eta)
$$

Hence, by the definitions of $\psi_{j}$ and $\varphi_{j}$ in Notation 1.5, the multiplier $\sigma_{\mathrm{I}}$ is written as

$$
\begin{aligned}
\sigma_{\mathrm{I}}(\xi, \eta) & =\sum_{a, b \in \mathbb{Z}^{n}} \sum_{j \in \mathbb{N}} c_{\mathrm{I}, j}^{(a, b)} e^{i a \cdot 2^{-j} \xi} e^{i b \cdot 2^{-j} \eta} \psi\left(2^{-j} \xi\right) \varphi\left(2^{-j} \eta\right) \\
& =\sum_{a, b \in \mathbb{Z}^{n}} \sum_{j \in \mathbb{N}} c_{\mathrm{I}, j}^{(a, b)} \psi^{(a)}\left(2^{-j} \xi\right) \varphi^{(b)}\left(2^{-j} \eta\right),
\end{aligned}
$$

where we wrote as

$$
\psi^{(\nu)}(\xi)=e^{i \nu \cdot \xi} \psi(\xi), \quad \varphi^{(\nu)}(\eta)=e^{i \nu \cdot \eta} \varphi(\eta), \quad \nu \in \mathbb{Z}^{n} .
$$

By similar arguments, the multipliers $\sigma_{0}$ and $\sigma_{\text {II }}$ can be written as

$$
\begin{aligned}
\sigma_{0}(\xi, \eta) & =\sum_{a, b \in \mathbb{Z}^{n}} c_{0}^{(a, b)} \varphi^{(a)}(\xi) \varphi^{(b)}(\eta), \\
\sigma_{\mathrm{II}}(\xi, \eta) & =\sum_{a, b \in \mathbb{Z}^{n}} \sum_{j \in \mathbb{N}} c_{\mathrm{I}, j}^{(a, b)} \varphi^{(a)}\left(2^{-(j-1)} \xi\right) \psi^{(b)}\left(2^{-j} \eta\right),
\end{aligned}
$$

where the coefficient $c_{0}^{(a, b)}$ satisfies the same condition as in (5.1) with $j=0$, and the coefficient $c_{\mathrm{II}, j}^{(a, b)}$ satisfies the same condition as in (5.1).

Hereafter we shall consider slightly general multipliers $\widetilde{\sigma}_{0}$ and $\widetilde{\sigma}$ defined by

$$
\begin{align*}
& \widetilde{\sigma}_{0}(\xi, \eta)=c_{0} \theta_{1}(\xi) \theta_{2}(\eta)  \tag{5.2}\\
& \widetilde{\sigma}(\xi, \eta)=\sum_{j \in \mathbb{N}} c_{j} \theta_{1}\left(2^{-j} \xi\right) \theta_{2}\left(2^{-j} \eta\right) \tag{5.3}
\end{align*}
$$

where $\left(c_{j}\right)_{j \in \mathbb{N}_{0}}$ is a sequence of complex numbers satisfying

$$
\begin{equation*}
\left|c_{j}\right| \leq 2^{j m} A, \quad j \in \mathbb{N}_{0} \tag{5.4}
\end{equation*}
$$

with some $A \in(0, \infty)$, and $\theta_{1}, \theta_{2} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ satisfy that

$$
\begin{equation*}
\operatorname{supp} \theta_{1}, \operatorname{supp} \theta_{2} \subset\{|\xi| \leq 2\} \tag{5.5}
\end{equation*}
$$

For such $\widetilde{\sigma}_{0}$ and $\widetilde{\sigma}$, we shall prove that there exist $c>0$ and $M \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|T_{\widetilde{\sigma}_{0}}^{s}\right\|_{h^{1} \times L^{\infty} \rightarrow L^{1}},\left\|T_{\widetilde{\sigma}}^{s}\right\|_{h^{1} \times L^{\infty} \rightarrow L^{1}} \leq c A\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.6}
\end{equation*}
$$

If this is proved, by applying (5.6) for $\widetilde{\sigma}$ to $c_{j}=c_{\mathrm{I}, j}^{(a, b)}, \theta_{1}=\psi^{(a)}$, and $\theta_{2}=\varphi^{(b)}$, we have

$$
\left\|T_{\sigma_{\mathrm{I}}}^{s}\right\|_{h^{1} \times L^{\infty} \rightarrow L^{1}} \lesssim \sum_{a, b \in \mathbb{Z}^{n}}(1+|a|)^{-L}(1+|b|)^{-L}\left\|e^{i a \cdot \xi} \psi(\xi)\right\|_{C_{\xi}^{M}}\left\|e^{i b \cdot \eta} \varphi(\eta)\right\|_{C_{\eta}^{M}}
$$

$$
\lesssim \sum_{a, b \in \mathbb{Z}^{n}}(1+|a|)^{-L+M}(1+|b|)^{-L+M}
$$

and, thus, taking $L$ sufficiently large, we see that $T_{\sigma_{\mathrm{I}}}^{s}$ is bounded from $h^{1} \times L^{\infty}$ to $L^{1}$. In the same way, we see that $T_{\sigma_{0}}^{s}$ and $T_{\sigma_{\mathrm{I}}}^{s}$ are bounded from $h^{1} \times L^{\infty}$ to $L^{1}$. The above three estimates complete the proof of the $h^{1} \times L^{\infty} \rightarrow L^{1}$ boundedness of $T_{\sigma}^{s}$.

Thus the proof is reduced to showing (5.6) for $\widetilde{\sigma}_{0}$ and $\widetilde{\sigma}$ given by (5.2)-(5.5). However, the estimate for $\widetilde{\sigma}_{0}$ is simple. In fact, from Lemma 4.3 (1),

$$
\begin{aligned}
& \left\|T_{\widetilde{\sigma}_{0}}^{s}(f, g)\right\|_{L^{1}}=\left\|c_{0} e^{i|D|^{s}} \theta_{1}(D) f(x) e^{i|D|^{s}} \theta_{2}(D) g(x)\right\|_{L^{1}} \\
& \leq A\left\|e^{i|D|^{s}} \theta_{1}(D) f\right\|_{L^{1}}\left\|e^{i|D|^{s}} \theta_{2}(D) g\right\|_{L^{\infty}} \lesssim A\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}}
\end{aligned}
$$

Hence, in what follows, we concentrate on proving (5.6) for $\widetilde{\sigma}$ given by (5.3)-(5.5).
We shall make further reductions. Using $\varphi$ and $\zeta$ defined in Notation 1.5, we decompose the function 1 on $\mathbb{R}^{n} \times \mathbb{R}^{n}$ into

$$
1=\varphi(\xi) \varphi(\eta)+\varphi(\xi) \zeta(\eta)+\zeta(\xi) \varphi(\eta)+\zeta(\xi) \zeta(\eta)
$$

Then, $T_{\tilde{\sigma}}^{s}$ can be expressed by the following four parts:

$$
\begin{align*}
T_{\widetilde{\sigma}}^{s}(f, g)(x)=\sum_{j \in \mathbb{N}} c_{j}\{ & T_{j}^{1} f(x) T_{j}^{2} g(x)+T_{j}^{1} f(x) S_{j}^{2} g(x)  \tag{5.7}\\
& \left.+S_{j}^{1} f(x) T_{j}^{2} g(x)+S_{j}^{1} f(x) S_{j}^{2} g(x)\right\}
\end{align*}
$$

where, for $\ell=1,2$, we wrote

$$
\begin{aligned}
S_{j}^{\ell} f(x) & =\left(e^{i|\xi|^{s}} \zeta(\xi) \theta_{\ell}\left(2^{-j} \xi\right) \widehat{f}(\xi)\right)^{\vee}(x) \\
T_{j}^{\ell} f(x) & =\left(e^{i|\xi|^{s}} \varphi(\xi) \theta_{\ell}\left(2^{-j} \xi\right) \widehat{f}(\xi)\right)^{\vee}(x)
\end{aligned}
$$

Considering the $L^{1}$-norm of (5.7) and using the assumption (5.4), we see that

$$
\left\|T_{\tilde{\sigma}}^{s}(f, g)\right\|_{L^{1}} \leq A \sum_{U, V \in\{S, T\}} \sum_{j \in \mathbb{N}} 2^{j m}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}}
$$

and thus, in the following argument, it is sufficient to prove that

$$
\sum_{j \in \mathbb{N}} 2^{j m}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}} \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}\|f\|_{h^{1}}\|g\|_{L^{\infty}}, \quad U, V \in\{S, T\}
$$

To prove this, by virtue of the atomic decomposition of $h^{1}$, stated in Notation 1.5, and by translation invariance, it suffices to obtain the uniform estimates for $h^{1}$-atoms $f$ supported on balls centered at the origin. Furthermore, we may assume that $\|g\|_{L^{\infty}}=1$. Therefore, in order to obtain the desired boundedness result in Theorem 5.1, we shall prove that

$$
\begin{equation*}
\sum_{j \in \mathbb{N}} 2^{j m}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}} \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}, \quad U, V \in\{S, T\} \tag{5.8}
\end{equation*}
$$

holds for such $f$ and $g$ and for some $M \in \mathbb{N}$. In the rest of this section, the letter $r$ always denotes the radius of a ball including the support of $f ; f$ is assumed to be an $h^{1}$-atom supported on a ball in $\mathbb{R}^{n}$ of radius $r$ centered at the origin.
5.1. Case $0<s<1$. We recall that the critical order for $0<s<1$ is

$$
m=-\frac{s n}{2}-\frac{s(1-s) n}{2}
$$

and notice that $m<-\frac{s n}{2}$. We also remark that this $m$ can be written as

$$
m=-\frac{n}{2}+\frac{(1-s)^{2} n}{2}
$$

5.1.1. Estimate for $T_{j}^{1} f T_{j}^{2} g$. By Lemma 4.3 (1), it holds that

$$
\begin{aligned}
\left\|T_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}} & \leq\left\|T_{j}^{1} f\right\|_{L^{1}}\left\|T_{j}^{2} g\right\|_{L^{\infty}} \\
& \lesssim\left\|\varphi(\cdot) \theta_{1}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\left\|\varphi(\cdot) \theta_{2}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}} \\
& \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
\end{aligned}
$$

Since $m<0$, we obtain (5.8) with $(U, V)=(T, T)$ from this estimate.
5.1.2. Estimate for $T_{j}^{1} f S_{j}^{2} g$. It follows from Lemmas 4.3 (11) and 4.1 that

$$
\begin{aligned}
\left\|T_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}} & \leq\left\|T_{j}^{1} f\right\|_{L^{1}}\left\|S_{j}^{2} g\right\|_{L^{\infty}} \\
& \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\varphi(\cdot) \theta_{1}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}} \\
& \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}},
\end{aligned}
$$

which gives (5.8) with $(U, V)=(T, S)$ because $m<-\frac{s n}{2}$.
5.1.3. Estimate for $S_{j}^{1} f T_{j}^{2} g$. We use Lemmas 4.1 and 4.3 (1) to obtain

$$
\begin{aligned}
\left\|S_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}} & \leq\left\|S_{j}^{1} f\right\|_{L^{1}}\left\|T_{j}^{2} g\right\|_{L^{\infty}} \\
& \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\varphi(\cdot) \theta_{2}\left(2^{-j}\right)\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}} \\
& \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{aligned}
$$

Since $m<-\frac{s n}{2}$, this yields that (5.8) holds with $(U, V)=(S, T)$.
5.1.4. Estimate for $S_{j}^{1} f S_{j}^{2} g$. We divide the $L^{1}$ norm in (5.8) into the following three parts;

$$
\begin{equation*}
\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(\mathbb{R}^{n}\right)}=\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(|x| \leq 2 r)}+\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(2 r<|x| \leq 4)}+\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(|x|>4)} . \tag{5.9}
\end{equation*}
$$

We first consider the norm $L^{1}(|x| \leq 2 r)$. By the Cauchy-Schwarz inequality, we obtain

$$
\left\|S_{j}^{1} f\right\|_{L^{1}(|x| \leq 2 r)} \lesssim r^{n / 2}\left\|S_{j}^{1} f\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim r^{n / 2}\left\|\theta_{1}\right\|_{C^{0}}\|f\|_{L^{2}} \lesssim\left\|\theta_{1}\right\|_{C^{0}} .
$$

Hence, by this inequality and Lemma 4.1, we have

$$
\begin{equation*}
\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(|x| \leq 2 r)} \leq\left\|S_{j}^{1} f\right\|_{L^{1}(|x| \leq 2 r)}\left\|S_{j}^{2} g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\theta_{1}\right\|_{C^{0}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.10}
\end{equation*}
$$

for some $M \in \mathbb{N}$. For the norm $L^{1}(|x|>4)$, we also have by Lemmas 4.4 (1) and 4.1

$$
\begin{equation*}
\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(|x|>4)} \leq\left\|S_{j}^{1} f\right\|_{L^{1}(|x|>4)}\left\|S_{j}^{2} g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \lesssim\left(2^{j}\right)^{\frac{s n}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.11}
\end{equation*}
$$

Thus, we shall consider the estimate of the second term in the right hand side of (5.9). We decompose it as follows;

$$
\begin{align*}
& \left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(2 r<|x| \leq 4)} \leq \sum_{k \in \mathbb{N}, 2^{k} r \leq 4}\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{\left.L^{1} 2^{k} r \leq|x| \leq 2^{k+1} r\right)} \\
& =\left(\sum_{k \in \mathbb{N}, 2^{k} r<2^{-j(1-s)}}+\sum_{k \in \mathbb{N}, 2^{-j(1-s) \leq 2^{k} r \leq 4}}\right)\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} . \tag{5.12}
\end{align*}
$$

Here, we remark that the first sum in the second line vanishes if $2^{-j(1-s)} \leq 2 r$.
We show that the following estimate holds; for $0 \leq t \leq 1$ and $0 \leq N<\frac{n}{2(1-s)}$,

$$
\begin{align*}
& \left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} \\
& \lesssim 2^{-j m}\left(2^{j(1-s)} 2^{k} r\right)^{-N(1-t)+\frac{(1-s) n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.13}
\end{align*}
$$

for $k \in \mathbb{N}$ satisfying that $2^{k} r \leq 4$. In fact, it follows from Lemmas 4.2 (2) and 4.4 (2) that

$$
\begin{aligned}
& \left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} \leq\left\|S_{j}^{1} f\right\|_{L^{2}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right.}\left\|S_{j}^{2} g\right\|_{L^{2}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} \\
& \lesssim\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} 2^{k} r\right)^{-N(1-t)} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left(2^{k} r\right)^{\frac{(1-s) n}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}\|g\|_{L^{\infty}} \\
& =\left(2^{j}\right)^{\frac{n}{2}-\frac{(1-s)^{2} n}{2}}\left(2^{j(1-s)} 2^{k} r\right)^{-N(1-t)+\frac{(1-s) n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \\
& =2^{-j m}\left(2^{j(1-s)} 2^{k} r\right)^{-N(1-t)+\frac{(1-s) n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}},
\end{aligned}
$$

where we remark that Lemmas 4.2 (2) and 4.4 (22) are applicable to the factor involved in $f$ and $g$, respectively, since $k \in \mathbb{N}$ implies that $2^{k} r \geq 2 r$ and also $2^{k} r \leq 4$ implies that $|x| \leq 2^{k+1} r \leq 8$.

The former sum in the second line of (5.12) is estimated as follows. Since $0<s<1$, it follows from (5.13) with $0 \leq t \leq 1$ and $N=0$ that

$$
\begin{align*}
& \sum_{k \in \mathbb{N}, 2^{k} r<2^{-j(1-s)}}\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} \\
& \lesssim 2^{-j m} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \sum_{k: 2^{j(1-s)} 2^{k} r<1}\left(2^{j(1-s)} 2^{k} r\right)^{\frac{(1-s) n}{2}}  \tag{5.14}\\
& \lesssim 2^{-j m} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
\end{align*}
$$

On the other hand, the latter sum in the second line of (5.12) is estimated as follows. Since $0<s<1$, we have $\frac{(1-s) n}{2}<\frac{n}{2(1-s)}$, and consequently we can choose $0<t<1$ and $N>0$ satisfying $\frac{(1-s) n}{2(1-t)}<N<\frac{n}{2(1-s)}$. Therefore, by (5.13) with such $t$ and $N$, it follows that

$$
\begin{align*}
& \quad \sum_{k \in \mathbb{N}, 2^{-j(1-s) \leq 2^{k} r \leq 4}}\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} r \leq|x| \leq 2^{k+1} r\right)} \\
& \lesssim 2^{-j m} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \sum_{k: 2^{j(1-s)} 2^{k} r \geq 1}\left(2^{j(1-s)} 2^{k} r\right)^{-N(1-t)+\frac{(1-s) n}{2}}  \tag{5.15}\\
& \lesssim 2^{-j m} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{align*}
$$

Thus, taking the same $0<t<1$ for (5.14) and (5.15) and then combining them with (5.12), we obtain

$$
\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}(2 r \leq|x| \leq 4)} \lesssim 2^{-j m} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
$$

Hence, combining this with (5.10) and (5.11), we conclude that (5.8) holds with $(U, V)=$ $(S, S)$. This completes the proof for the case $0<s<1$.
5.2. Case $s>1$. Before beginning with the proof, let us recall the critical order for $s>1$ :

$$
m=-\frac{n s}{2} .
$$

For this $m$, in what follows, we will prove that (5.8) holds. In this subsection, we take a $j_{0} \in \mathbb{N}$ such that $2^{j_{0}(s-1)} \geq 2$. For such $j_{0} \in \mathbb{N}$, we have by Lemma 4.3 (1) or 4.1

$$
\begin{aligned}
\sum_{1 \leq j \leq j_{0}} 2^{j m}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}} & \lesssim \sum_{1 \leq j \leq j_{0}}\left(2^{j}\right)^{m+n s}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}} \\
& \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
\end{aligned}
$$

for $U, V \in\{S, T\}$. Therefore, in order to achieve (5.8), it is sufficient to show that

$$
\begin{equation*}
\sum_{j>j_{0}} 2^{j m}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}} \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}, \quad U, V \in\{S, T\} \tag{5.16}
\end{equation*}
$$

To this end, except for the case $(U, V)=(T, T)$, we split the norm of $L^{1}\left(\mathbb{R}^{n}\right)$ as follows:

$$
\begin{align*}
& \left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}\left(\mathbb{R}^{n}\right)}=\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}\left(|x| \leq 2^{j(s-1)+1)}\right.}+\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}\left(|x| \geq 2^{j(s-1)+1}\right)} \\
& \quad \leq\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}\left(|x| \leq 2^{j(s-1)+1}\right)}+\sum_{k \in \mathbb{N}, 2^{k} \geq 2^{j(s-1)}}\left\|U_{j}^{1} f V_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \tag{5.17}
\end{align*}
$$

where, the sum over $2^{k} \geq 2^{j(s-1)}$ should be read as the sum over $k \geq k_{0}$ with a positive integer $k_{0}=k_{0}(j)$ satisfying that $2^{k_{0}-1}<2^{j(s-1)} \leq 2^{k_{0}}$. Here, we are able to choose such $k_{0} \in \mathbb{N}$, since $2^{j(s-1)} \geq 2$ for $j>j_{0}$. Now, we shall prove (5.16).
5.2.1. Estimate for $T_{j}^{1} f T_{j}^{2} g$. By Lemma 4.3 (11),

$$
\begin{aligned}
\left\|T_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}} & \leq\left\|T_{j}^{1} f\right\|_{L^{1}}\left\|T_{j}^{2} g\right\|_{L^{\infty}} \\
& \lesssim\left\|\varphi(\cdot) \theta_{1}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\left\|\varphi(\cdot) \theta_{2}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\|f\|_{L^{1}}\|g\|_{L^{\infty}} \\
& \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
\end{aligned}
$$

and thus, since $m=-\frac{n s}{2}<0$, (5.16) holds for $(U, V)=(T, T)$.
5.2.2. Estimate for $T_{j}^{1} f S_{j}^{2} g$. We use the decomposition (5.17). For the first term in (5.17), using Lemma 4.3 (1) with $(p, q)=(1,2)$ and Lemma 4.5 with $A=2^{j(s-1)+1}$ to the factors for $f$ and $g$ respectively, we have

$$
\begin{align*}
& \left\|T_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(|x| \leq 2^{j(s-1)+1}\right)} \leq\left\|T_{j}^{1} f\right\|_{L^{2}\left(\mathbb{R}^{n}\right)}\left\|S_{j}^{2} g\right\|_{L^{2}\left(|x| \leq 2^{j(s-1)+1}\right)} \\
& \lesssim\left\|\varphi(\cdot) \theta_{1}\left(2^{-j} \cdot\right)\right\|_{C^{M}}\|f\|_{L^{1}}\left(2^{j(s-1)}\right)^{\frac{n}{2}}\left\|\theta_{2}\right\|_{C^{M}}\|g\|_{L^{\infty}}  \tag{5.18}\\
& \lesssim\left(2^{j}\right)^{\frac{n(s-1)}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{align*}
$$

Next, for the summand in the sum of (5.17), we have by Hölder's inequality, Lemma 4.3 (2)) with $A=2^{k}$, and Lemma 4.5 with $A=2^{k+1}$

$$
\begin{aligned}
& \left\|T_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \lesssim\left(2^{k}\right)^{\frac{n}{2}}\left\|T_{j}^{1} f\right\|_{L^{\infty}\left(2^{k} \leq|x| \leq 2^{k+1}\right)}\left\|S_{j}^{2} g\right\|_{L^{2}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \\
& \lesssim\left(2^{k}\right)^{\frac{n}{2}} \cdot\left(2^{k}\right)^{-(n+s)}\left\|\varphi(\cdot) \theta_{1}\left(2^{-j}\right)\right\|_{C^{M}} \cdot\left(2^{k}\right)^{\frac{n}{2}}\left\|\theta_{2}\right\|_{C^{M}}\|g\|_{L^{\infty}} \\
& \lesssim 2^{-k s}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}},
\end{aligned}
$$

where, it should be remarked that Lemma 4.5 is applicable to the factor involved in $g$, since $2^{k+1} \geq 2^{j(s-1)}$ in the sum of (5.17). This yields from $s>1$ that

$$
\begin{equation*}
\sum_{k \in \mathbb{N}, 2^{k} \geq^{j(s-1)}}\left\|T_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \lesssim\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \sum_{k \in \mathbb{N}} 2^{-k s} \approx\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.19}
\end{equation*}
$$

Therefore, combining (5.18) and (5.19) with (5.17) we obtain

$$
\left\|T_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}} \lesssim\left(2^{j}\right)^{\frac{n(s-1)}{2}}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
$$

which implies that (5.16) holds for $(U, V)=(T, S)$ since $m<-\frac{n(s-1)}{2}$.
5.2.3. Estimate for $S_{j}^{1} f T_{j}^{2} g$. For the first term in (5.17), Lemmas 4.2 (1) and 4.3 (1) yield that, for $0 \leq t \leq 1$,

$$
\begin{align*}
\left\|S_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}\left(|x| \leq 2^{j(s-1)+1}\right)} & \lesssim\left(2^{j}\right)^{\frac{n(s-1)}{2}}\left\|S_{j}^{1} f\right\|_{L^{2}\left(\mathbb{R}^{n}\right)}\left\|T_{j}^{2} g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}  \tag{5.20}\\
& \lesssim\left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{0}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{align*}
$$

We shall next consider the sum of (5.17). By Hölder's inequality and Lemmas 4.2 (22) and 4.3 (11), the summand in (5.17) is estimated by

$$
\begin{aligned}
& \left\|S_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \lesssim\left(2^{k}\right)^{\frac{n}{2}}\left\|S_{j}^{1} f\right\|_{L^{2}\left(2^{k} \leq|x| \leq 2^{k+1}\right)}\left\|T_{j}^{2} g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left(2^{k}\right)^{\frac{n}{2}} \cdot\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} 2^{k}\right)^{-N(1-t)} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \\
& =\left(2^{j}\right)^{\frac{n s}{2}}\left(2^{j(1-s)} 2^{k}\right)^{-N(1-t)+\frac{n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}},
\end{aligned}
$$

where we notice that $2^{k} \geq 2 r$ holds in the sum of (5.17), since $k$ is restricted to $\mathbb{N}$ and $r \leq 1$ : this allows us to apply Lemma 4.2 (2) with $A=2^{k}$ to the factor with respect to $f$. Then, choosing $0<t<1$ and $N>0$ such that $-N(1-t)+\frac{n}{2}<0$, we have

$$
\begin{align*}
& \sum_{k \in \mathbb{N}, 2^{k} \geq 2^{j(s-1)}}\left\|S_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \\
\lesssim & \left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \sum_{k: 2^{j(1-s)} 2^{k} \geq 1}\left(2^{j(1-s)} 2^{k}\right)^{-N(1-t)+\frac{n}{2}}  \tag{5.21}\\
\approx & \left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{align*}
$$

Thus, by choosing the same $0<t<1$ for (5.20) and (5.21), and by combining them with (5.17), we obtain for such $0<t<1$

$$
\left\|S_{j}^{1} f T_{j}^{2} g\right\|_{L^{1}} \lesssim\left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
$$

which implies that (5.16) holds for $(U, V)=(S, T)$.
5.2.4. Estimate for $S_{j}^{1} f S_{j}^{2} g$. For the first term in (5.17), by Lemmas 4.2 (1) and 4.5, we have for $0 \leq t \leq 1$

$$
\begin{align*}
& \left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(|x| \leq 2^{j(s-1)+1}\right)} \lesssim\left\|S_{j}^{1} f\right\|_{L^{2}\left(\mathbb{R}^{n}\right)}\left\|S_{j}^{2} g\right\|_{L^{2}\left(|x| \leq 2^{j(s-1)+1}\right)} \\
& \lesssim\left(2^{j}\right)^{\frac{n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}} \cdot\left(2^{j}\right)^{\frac{n(s-1)}{2}}\left\|\theta_{2}\right\|_{C^{M}}  \tag{5.22}\\
& =\left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} .
\end{align*}
$$

For the summand of (5.17), we have by Lemmas 4.2 (21) and 4.5

$$
\begin{aligned}
& \left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \leq\left\|S_{j}^{1} f\right\|_{L^{2}\left(2^{k} \leq|x| \leq 2^{k+1}\right)}\left\|S_{j}^{2} g\right\|_{L^{2}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \\
& \lesssim\left(2^{j}\right)^{\frac{n}{2}}\left(2^{j(1-s)} 2^{k}\right)^{-N(1-t)} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}} \cdot\left(2^{k}\right)^{\frac{n}{2}}\left\|\theta_{2}\right\|_{C^{M}} \\
& =\left(2^{j}\right)^{\frac{n s}{2}}\left(2^{j(1-s)} 2^{k}\right)^{-N(1-t)+\frac{n}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}},
\end{aligned}
$$

which yields that, since there exist $N>0$ and $0<t<1$ such that $-N(1-t)+\frac{n}{2}<0$,

$$
\begin{equation*}
\sum_{k \in \mathbb{N}, 2^{k} \geq 2^{j(s-1)}}\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}\left(2^{k} \leq|x| \leq 2^{k+1}\right)} \lesssim\left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}} \tag{5.23}
\end{equation*}
$$

Therefore, choosing the same $0<t<1$ for (5.22) and (5.23) and then combining them with (5.17), we obtain for such $0<t<1$ that

$$
\left\|S_{j}^{1} f S_{j}^{2} g\right\|_{L^{1}} \lesssim\left(2^{j}\right)^{\frac{n s}{2}} \min \left\{\left(2^{j} r\right)^{t},\left(2^{j} r\right)^{-\frac{n t}{2}}\right\}\left\|\theta_{1}\right\|_{C^{M}}\left\|\theta_{2}\right\|_{C^{M}}
$$

This gives (5.16) for $(U, V)=(S, S)$, and we complete the proof for $s>1$.

## 6. Necessary conditions on $m$

In this section, we shall give a proof of Theorem 1.4. In this section, we use the notation $X_{r}$ given in (2.1).

Proof of Theorem 1.4. Let $0<s<1$ or $1<s<\infty$, and let $m \in \mathbb{R},(p, q) \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}$, and $1 / r=1 / p+1 / q$. If all bilinear operators $T_{\sigma}^{s}, \sigma \in S_{1,0}^{m}\left(\mathbb{R}^{2 n}\right)$, are bounded from $H^{p} \times H^{q}$ to $X_{r}$, then, by virtue of the closed graph theorem, there exist $c>0$ and $N \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|T_{\sigma}^{s}\right\|_{H^{p} \times H^{q} \rightarrow X_{r}} \leq c \max _{|\alpha|,|\beta| \leq N}\left\|(1+|\xi|+|\eta|)^{-m+|\alpha|+|\beta|} \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma(\xi, \eta)\right\|_{L^{\infty}\left(\mathbb{R}^{2 n}\right)} \tag{6.1}
\end{equation*}
$$

holds for all $\sigma \in S_{1,0}^{m}\left(\mathbb{R}^{n}\right)$ (see Bényi-Bernicot-Maldonado-Naibo-Torres [2, Lemma 2.6] for the argument using the closed graph theorem).

Now, we take two functions $\theta$ and $\phi$ such that

$$
\begin{array}{ll}
\theta \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right), & \operatorname{supp} \theta \subset\left\{3^{-1} \leq|\xi| \leq 3\right\}, \quad \theta(\xi)=1 \quad \text { on } \quad\left\{2^{-1} \leq|\xi| \leq 2\right\}, \\
\phi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right), & \operatorname{supp} \phi \subset\{|\xi| \leq 3\}, \quad \phi(\xi)=1 \text { on }\{|\xi| \leq 2\} .
\end{array}
$$

For $j \in \mathbb{N}$, we set

$$
\sigma_{j}(\xi, \eta)=2^{j m} \theta\left(2^{-j} \xi\right) \phi\left(2^{-j} \eta\right)
$$

Then we have

$$
\left|\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma_{j}(\xi, \eta)\right| \leq C_{\alpha, \beta}(1+|\xi|+|\eta|)^{m-|\alpha|-|\beta|}, \quad \alpha, \beta \in \mathbb{N}_{0}^{n}
$$

uniformly in $j \in \mathbb{N}$. Hence, by (6.1), we see that there exists $C>0$ such that

$$
\begin{equation*}
\left\|T_{\sigma_{j}}^{s}\right\|_{H^{p} \times H^{q} \rightarrow X_{r}} \leq C, \quad j \in \mathbb{N} \tag{6.2}
\end{equation*}
$$

We shall prove that (6.2) holds only if $m \leq m_{s}(p, q)$.
Take a function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ such that $\operatorname{supp} \psi \subset\left\{2^{-1} \leq|\xi| \leq 2\right\}$ and $\psi(\xi) \neq 0$ on $\{2 / 3 \leq|\xi| \leq 3 / 2\}$, and set

$$
\begin{aligned}
& f_{j}^{+}(x)=\left(e^{i|\xi|^{s}} \psi\left(2^{-j} \xi\right)\right)^{\vee}(x), \\
& f_{j}^{-}(x)=\left(e^{-i|\xi| s} \psi\left(2^{-j} \xi\right)\right)^{\vee}(x), \\
& f_{j}(x)=\left(\psi\left(2^{-j} \xi\right)\right)^{\vee}(x)=2^{j n}(\psi)^{\vee}\left(2^{j} x\right) .
\end{aligned}
$$

Then we have the following estimates;

$$
\begin{align*}
& \left\|f_{j}^{ \pm}\right\|_{H^{p}} \approx\left\|f_{j}^{ \pm}\right\|_{L^{p}} \approx 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j(1-s) \frac{n}{p}} \text { for } 1 \leq p \leq \infty  \tag{6.3}\\
& \left\|f_{j}\right\|_{H^{p}} \approx\left\|f_{j}\right\|_{L^{p}} \approx 2^{j\left(n-\frac{n}{p}\right)} \text { for } 1 \leq p \leq \infty,  \tag{6.4}\\
& \left\|\left(f_{j}\right)^{2}\right\|_{B M O} \approx 2^{2 j n} \tag{6.5}
\end{align*}
$$

In fact, since the Fourier transform of $f_{j}^{ \pm}$are supported in the annulus $\left\{2^{j-1} \leq|\xi| \leq 2^{j+1}\right\}$, we have the first inequality in (6.3), and the second $\approx$ in (6.3) follows from Proposition 3.1, On the other hand, by a straightforward calculation, we see that (6.4) and (6.5) hold.

Proof of the case $(p, q) \in \mathrm{I}$. In this case we use the function $f_{j}^{-}$. Since $\psi\left(2^{-j} \xi\right) \theta\left(2^{-j} \xi\right)=$ $\psi\left(2^{-j} \xi\right)$ and $\psi\left(2^{-j} \eta\right) \phi\left(2^{-j} \eta\right)=\psi\left(2^{-j} \eta\right)$, we have

$$
T_{\sigma_{j}}^{s}\left(f_{j}^{-}, f_{j}^{-}\right)(x)=2^{j m}\left(f_{j}(x)\right)^{2},
$$

and hence, we obtain

$$
\left\|T_{\sigma_{j}}^{s}\left(f_{j}^{-}, f_{j}^{-}\right)\right\|_{X_{r}}=2^{j m}\left\|\left(f_{j}\right)^{2}\right\|_{X_{r}} \approx 2^{j\left(m+2 n-\frac{n}{r}\right)}, \quad j \in \mathbb{N},
$$

where, in the last inequality, we used (6.4) combined with the identity $\left\|\left(f_{j}\right)^{2}\right\|_{L^{r}}=\left\|f_{j}\right\|_{L^{2 r}}^{2}$ if $r<\infty$, and used (6.5) if $r=\infty$. Combining this with (6.2) and (6.3), we obtain

$$
2^{j\left(m+2 n-\frac{n}{r}\right)} \lesssim 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j\left(1-s \frac{n}{p}\right.} 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j(1-s) \frac{n}{q}}, \quad j \in \mathbb{N} .
$$

This is possible only when $m \leq-\operatorname{sn}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{2}-\frac{1}{q}\right)$. This completes the proof of the case $(p, q) \in \mathrm{I}$.

Proof of the case $(p, q) \in$ II. By the same reasons as above, we have

$$
T_{\sigma_{j}}^{s}\left(f_{j}, f_{j}\right)(x)=2^{j m}\left(f_{j}^{+}(x)\right)^{2}
$$

and thus (6.3) implies that, since $2 r \geq 1$ for $(p, q) \in I$,

$$
\left\|T_{\sigma_{j}}^{s}\left(f_{j}, f_{j}\right)\right\|_{L^{r}}=2^{j m}\left\|\left(f_{j}^{+}\right)^{2}\right\|_{L^{r}}=2^{j m}\left\|f_{j}^{+}\right\|_{L^{2 r}}^{2} \approx 2^{j m}\left(2^{j\left(n-\frac{s n}{2}\right)}\right)^{2} 2^{-j(1-s) \frac{n}{r}} .
$$

Hence, if (6.2) holds, then this estimate and (6.4) imply that

$$
2^{j m}\left(2^{j\left(n-\frac{s n}{2}\right)}\right)^{2} 2^{-j(1-s) \frac{n}{r}} \lesssim 2^{j\left(n-\frac{n}{p}\right)} 2^{j\left(n-\frac{n}{q}\right)}, \quad j \in \mathbb{N},
$$

which holds only if $m \leq-\operatorname{sn}\left(\frac{1}{p}-\frac{1}{2}+\frac{1}{q}-\frac{1}{2}\right)$. This completes the proof of the case $(p, q) \in \mathrm{II}$.

Proof of the case $(p, q) \in \mathrm{IV}$. In addition to the functions $f_{j}^{ \pm}$and $f_{j}$, we use the following functions;

$$
\begin{aligned}
& g_{j}(x)=\left(e^{-i|\eta|^{s}} \psi\left(2^{-j(1-s)} \eta\right)\right)^{\vee}(x), \\
& h_{j}(x)=\left(e^{-i 2|\eta|^{s}} \psi\left(2^{-j} \eta\right)\right)^{\vee}(x)
\end{aligned}
$$

where the function $\psi$ is the same given in the definition of $f_{j}^{ \pm}$and $f_{j}$. Since the support of $\widehat{g}_{j}$ is included in the annulus $\left\{2^{j(1-s)-1} \leq|\eta| \leq 2^{j(1-s)+1}\right\}$, if $0<s<1$, then we have by Proposition 3.1

$$
\begin{equation*}
\left\|g_{j}\right\|_{H^{q}} \approx\left\|g_{j}\right\|_{L^{q}} \approx 2^{j(1-s)\left(n-\frac{s n}{2}\right)} 2^{-j(1-s)^{2} \frac{n}{q}} \text { for } 1 \leq q \leq \infty \tag{6.6}
\end{equation*}
$$

(we notice that this holds when $s<1$, since Proposition 3.1 treats the function whose Fourier support locates far from the origin; the support of $\widehat{g}_{j}$ locates near the origin when $s>1$ ). On the other hand, in the same way as for the functions $f_{j}^{ \pm}$, we also have

$$
\begin{equation*}
\left\|h_{j}\right\|_{H^{q}} \approx\left\|h_{j}\right\|_{L^{q}} \approx 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j(1-s) \frac{n}{q}} \text { for } 1 \leq q \leq \infty . \tag{6.7}
\end{equation*}
$$

Now, we first consider the case $0<s<1$. Observe that $\theta\left(2^{-j} \xi\right) \psi\left(2^{-j} \xi\right)=\psi\left(2^{-j} \xi\right)$ and $\psi\left(2^{-j(1-s)} \eta\right) \phi\left(2^{-j} \eta\right)=\psi\left(2^{-j(1-s)} \eta\right)$. Hence we have

$$
T_{\sigma_{j}}^{s}\left(f_{j}, g_{j}\right)(x)=2^{j m} 2^{j(1-s) n} f_{j}^{+}(x)(\psi)^{\vee}\left(2^{j(1-s)} x\right)
$$

Then, from (3.2), it holds that

$$
\left|T_{\sigma_{j}}^{s}\left(f_{j}, g_{j}\right)(x)\right| \mathbf{1}_{\left\{a^{\prime}<2^{\left.j(1-s)|x| \leq b^{\prime}\right\}}\right.} \approx 2^{j m} 2^{j(1-s) n} 2^{j\left(n-\frac{n s}{2}\right)}\left|(\psi)^{\vee}\left(2^{j(1-s)} x\right)\right| \mathbf{1}_{\left\{a^{\prime}<2^{j(1-s)}|x| \leq b^{\prime}\right\}}
$$

for all $j>j_{0}$, where $a^{\prime}, b^{\prime}$ and $j_{0}$ are the same given in Proposition 3.1(see (3.2)). Thus, we obtain

$$
\begin{aligned}
\left\|T_{\sigma_{j}}^{s}\left(f_{j}, g_{j}\right)\right\|_{L^{r}} & \gtrsim 2^{j m} 2^{j\left(n-\frac{n s}{2}\right)} 2^{j(1-s) n}\left\|(\psi)^{\vee}\left(2^{j(1-s)} x\right) \mathbf{1}_{\left\{a^{\prime}<2^{j(1-s)}|x| \leq b^{\prime}\right\}}\right\|_{L_{x}^{r}} \\
& =c 2^{j m} 2^{j\left(n-\frac{n s}{2}\right)} 2^{j(1-s) n} 2^{-j(1-s) \frac{n}{r}}, \quad j>j_{0},
\end{aligned}
$$

with $c=\left\|(\psi)^{\vee} \mathbf{1}_{\left\{a^{\prime}<|x| \leq b^{\prime}\right\}}\right\|_{L^{r}}>0$. Hence it follows from (6.2), (6.4) and (6.6) that

$$
2^{j m} 2^{j\left(n-\frac{n s}{2}\right)} 2^{j(1-s) n} 2^{-j(1-s) \frac{n}{r}} \lesssim 2^{j\left(n-\frac{n}{p}\right)} 2^{j(1-s)\left(n-\frac{s n}{2}\right)} 2^{-j(1-s)^{2} \frac{n}{q}}, \quad j>j_{0},
$$

which is possible only when $m \leq-s n\left(\frac{1}{p}-\frac{1}{2}\right)-s(1-s) n\left(\frac{1}{2}-\frac{1}{q}\right)$.
Finally, we shall consider the case $s>1$. Since

$$
T_{\sigma_{j}}^{s}\left(f_{j}, h_{j}\right)(x)=2^{j m} f_{j}^{+}(x) f_{j}^{-}(x)
$$

it follows from (3.2) that

$$
\left|T_{\sigma_{j}}^{s}\left(f_{j}, h_{j}\right)(x)\right| \mathbf{1}_{\left\{a^{\prime}<2^{j(1-s)}|x| \leq b^{\prime}\right\}} \approx 2^{j m} 2^{j\left(n-\frac{n s}{2}\right)} 2^{j\left(n-\frac{n s}{2}\right)} \mathbf{1}_{\left\{a^{\prime}<2^{j(1-s)}|x| \leq b^{\prime}\right\}}
$$

for all $j>j_{0}$. Hence, we obtain

$$
\left\|T_{\sigma_{j}}^{s}\left(f_{j}, h_{j}\right)\right\|_{L^{r}} \gtrsim 2^{j m} 2^{j\left(n-\frac{n s}{2}\right)} 2^{j\left(n-\frac{n s}{2}\right)} 2^{-j(1-s) \frac{n}{r}}, \quad j>j_{0}
$$

Combining this with (6.2), (6.4) and (6.7), we have

$$
2^{j m} 2^{j\left(n-\frac{s n}{2}\right)} 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j(1-s) \frac{n}{r}} \lesssim 2^{j\left(n-\frac{n}{p}\right)} 2^{j\left(n-\frac{s n}{2}\right)} 2^{-j(1-s) \frac{n}{q}}, \quad j>j_{0},
$$

which is possible only when $m \leq-\operatorname{sn}\left(\frac{1}{p}-\frac{1}{2}\right)$. This completes the proof of the case $(p, q) \in \mathrm{IV}$.

Proof of the case $(p, q) \in \mathrm{VI}$. Since the situation is symmetrical, we obtain the desired conclusion in the same way as for the case $(p, q) \in$ IV. Thus Theorem 1.4 is proved.

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