

Exact descriptonal complexity of determinization of input-driven pushdown automata

Olga Martynova*

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Abstract

The number of states and stack symbols needed to determinize nondeterministic input-driven pushdown automata (NIDPDA) working over a fixed alphabet is determined precisely. It is proved that in the worst case exactly 2^{n^2} states are needed to determinize an n -state NIDPDA, and the proof uses witness automata with a stack alphabet $\Gamma = \{0, 1\}$ working on strings over a 4-symbol input alphabet (Only an asymptotic lower bound was known before in the case of a fixed alphabet). Also, the impact of NIDPDA determinization on the size of stack alphabet is determined precisely for the first time: it is proved that $s(2^{n^2} - 1)$ stack symbols are necessary in the worst case to determinize an n -state NIDPDA working over an input alphabet of size $s + 5$ with s left brackets (The previous lower bound was only asymptotic in the number of states and did not depend on the number of left brackets).

1 Introduction

Input-driven pushdown automata, also known as visibly pushdown automata, are a model of computation equipped with a stack. An automaton has finitely many states. It reads an input string from the left to the right changing its state. The automaton also uses an infinite memory in the form of stack with restricted access. An input alphabet is $\Sigma = \Sigma_0 \cup \Sigma_{+1} \cup \Sigma_{-1}$, with Σ_0 containing neutral symbols, Σ_{+1} left brackets and Σ_{-1} right brackets. When an automaton reads a left bracket, it pushes onto the stack a symbol of its stack alphabet; when it sees a right bracket, it looks at the top symbol of the stack and pops it out; and when the automaton processes a neutral symbol, it makes a transition without looking at the stack.

Deterministic input-driven pushdown automata (DIDPDA) were invented by Mehlhorn [7]. He also proved that every language defined by an n -state DIDPDA can be recognized by an algorithm that uses $O(\frac{(\log n)^2}{\log \log n})$ bits of memory and works in polynomial time. Von Braunmühl and Verbeek [3] considered a nondeterministic version of input-driven pushdown automata (NIDPDA) that at every step may have several possible actions. Such an automaton accepts a string if there is at least one accepting computation

*Department of Mathematics and Computer Science, St. Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg 199034, Russia, olga22mart@gmail.com.

on it. A natural question immediately follows: whether deterministic and nondeterministic input-driven pushdown automata are equal in power. Von Braunmühl and Verbeek [3] made the first determinization construction: for an n -state NIDPDA, working over an alphabet $\Sigma = \Sigma_0 \cup \Sigma_{+1} \cup \Sigma_{-1}$, they constructed a DIDPDA with 2^{n^2} states and with $2^{n^2} |\Sigma_{+1}|$ stack symbols recognizing the same language. Also von Braunmühl and Verbeek [3] improved the result by Mehlhorn [7]: they have shown that any language defined by an n -state DIDPDA is recognized by an algorithm working in logarithmic memory. Later Rytter [13] created a simpler algorithm for this task also using $\log n$ bits of memory.

Alur and Madhusudan [1, 2] reintroduced the model of input-driven pushdown automata under the name of *visibly pushdown automata* and obtained many important results on these automata. They proved that the class of languages recognized by these automata is closed under intersection, union, concatenation and the Kleene star. They also defined input-driven pushdown automata that work on infinite strings and investigated properties of this variant of the model. Alur and Madhusudan [1, 2] established the first lower bound on the number of states needed for NIDPDA determinization: for each n , they constructed an n -state NIDPDA with the fixed input alphabet such that any deterministic automaton recognizing the same language has at least $2^{\Omega(n^2)}$ states; this lower bound is asymptotically tight. Furthermore, Alur and Madhusudan [1, 2] studied decidability and complexity of input-driven pushdown automata and showed that universality and inclusion problems for NIDPDA are NEXP-complete. There is some current research on decision problems for NIDPDA. Han, Ko and Salomaa [4] built an algorithm that for a given NIDPDA decides in polynomial time whether its path size is finite, that is, whether a number of leaves in a tree of its computations on every string is bounded by a common constant. They also proved that deciding whether the path size of a given NIDPDA is less than a given number is EXP-complete.

The research on NIDPDA determinization was continued by Okhotin, Piao and Salomaa [9], who bounded from below not only the number of states in the deterministic automaton but also the number of stack symbols it uses. They showed that using $2^{\Omega(n^2)}$ stack symbols can be necessary to determinize an n -state NIDPDA. Furthermore, Okhotin, Piao and Salomaa [9] and later Okhotin and Salomaa [11] studied the state complexity of different operations on DIDPDA. Jirásková and Okhotin [5] continued improving bounds for operations on DIDPDA, and in addition proved that in the worst case one needs 2^{n^2} states to determinize an n -state NIDPDA. This lower bound is precise, however, witness automata used by Jirásková and Okhotin [5] work over an input alphabet of size exponential in n . So the problem of determining the exact state complexity of NIDPDA determinization in the case of a bounded alphabet has remained open.

The state complexity of determinization was investigated for variants of the classical model of input-driven pushdown automata. Nguyen Van Tang and Ogawa [14] introduced event-clock input-driven pushdown automata and proved that these automata can be determinized. Later Ogawa and Okhotin [8] defined a direct determinization construction for these automata, establishing an upper bound: it is enough to use 2^{n^2} states, $2^{n^2+k} |\Sigma_{+1}|$ stack symbols and k clock constraints to determinize an n -state nondeterministic event-clock input-driven pushdown automaton with k clock constraints. Furthermore, Ogawa and Okhotin [8] proved a lower bound to this transformation which is asymptotically precise both in the number of states and in the size of a stack alphabet. Rose and Okhotin [12] first considered probabilistic input-driven pushdown automata and determined asymptotically precisely the state complexity of determinization for this model. Kutrib, Malcher

and Wendlandt [6] defined a variant of input-driven pushdown automata in which every input string is read twice: first, a deterministic sequential transducer determines the type of each symbol (in this model an alphabet is not initially split into left brackets, right brackets and neutral symbols), and then the string is read by an input-driven pushdown automaton. Kutrib et al. [6] proved that such input-driven pushdown automata with transducers are stronger than without transducers, determinized such automata and established their closure properties.

In this paper I improve the bounds on the complexity of determinization for classical input-driven pushdown automata (NIDPDA) both in the number of states and in the size of the stack alphabet. In Section 3, it is proved that in the worst case one can need 2^{n^2} states to determinize an n -state NIDPDA that uses 2 stack symbols and works over a 4-symbol input alphabet. This is the first precise lower bound on the number of states needed for NIDPDA determinization with bounded alphabet.

In Section 4, the precise lower bound $|\Sigma_{+1}|(2^{n^2} - 1)$ on the number of stack symbols in a deterministic automaton that recognizes a language defined by an n -state NIDPDA is proved in a special case of only one left bracket ($\Sigma_{+1} = \{<\}$). The witness nondeterministic automata in this proof have stack alphabets growing linearly in n , and the input alphabet is bounded.

Finally, in Section 5 I establish the exact lower bound on the complexity of NIDPDA determinization, for any number of left brackets in the input alphabet fewer than 2^{n^2} , both in the number of states and in the number of stack symbols: in the worst case one needs 2^{n^2} states and $|\Sigma_{+1}|(2^{n^2} - 1)$ stack symbols to determinize an n -state NIDPDA. Moreover, these examples of NIDPDA that are hard to determinize work over input alphabets that do not depend on n and use stack alphabets growing linearly in n and logarithmically in the number of left brackets. This lower bound is precise, since the known upper bound of $|\Sigma_{+1}| \cdot 2^{n^2}$ stack symbols is improved to $|\Sigma_{+1}|(2^{n^2} - 1)$ stack symbols in Section 2.

2 Input-driven pushdown automata

Input-driven pushdown automata, also known as visibly pushdown automata, were investigated a lot. These automata were first invented by Mehlhorn [7]; later von Braunmühl and Verbeek [3] introduced the nondeterministic version of input-driven pushdown automata.

This paper uses the definition given by Alur and Madhusudan [1] but with one difference: Alur and Madhusudan [1] allow computations on ill-nested strings, whereas in the definition used in this paper, input strings must be well-nested, as in the first definitions of these automata.

A nondeterministic automaton will be defined first, and a deterministic one is its special case.

Definition 1 (Mehlhorn [7], von Braunmühl and Verbeek [3], Alur and Madhusudan [1]). A *nondeterministic input-driven pushdown automaton (NIDPDA)* is a sextuple $A = (\Sigma, Q, \Gamma, Q_0, (\delta_a)_{a \in \Sigma}, F)$, where

- $\Sigma = \Sigma_0 \cup \Sigma_{+1} \cup \Sigma_{-1}$ is a finite input alphabet split into three disjoint sets: Σ_0 contains neutral symbols, Σ_{+1} consists of left brackets and Σ_{-1} has right brackets;
- Q is a finite set of states of the automaton;

- Γ is a finite stack alphabet;
- $Q_0 \subseteq Q$ is a subset of initial states;
- $(\delta_a)_{a \in \Sigma}$ are functions that for each symbol of the alphabet define possible actions of the automaton at this symbol:
 - for a neutral symbol $a \in \Sigma_0$, the function $\delta_a: Q \rightarrow 2^Q$ for each state gives a set of possible next states of the automaton;
 - for a left bracket $a \in \Sigma_{+1}$, the function $\delta_a: Q \rightarrow 2^{Q \times \Gamma}$ for each state assumed at a symbol a specifies the set of pairs (q, s) in which the automaton can make a transition forward in the state q pushing the symbol s onto the stack;
 - for a right bracket $a \in \Sigma_{-1}$, the function $\delta_a: Q \times \Gamma \rightarrow 2^Q$ for a state and for a symbol popped out of the stack gives a set of all possible next states;
- $F \subseteq Q$ is a subset of accepting states.

Inputs of the automaton A are well-nested strings over an alphabet Σ . A string w is called *well-nested* if it has as many left brackets as right brackets and if every prefix contains at least as many left brackets as right brackets.

Let $w = a_1 \dots a_\ell$ be a well-nested string. A *computation* of the automaton A on the string w is a sequence $(p_0, \alpha_0), \dots, (p_\ell, \alpha_\ell)$ of pairs of a state and of stack contents, with the neighbouring pairs related to each other as follows.

The automaton A begins reading the string in one of the initial states with an empty stack: the initial pair (p_0, α_0) must have the form (q_0, ε) , where $q_0 \in Q_0$ is an initial state. Let the automaton have processed the first i symbols and let a pair (p_i, α_i) describe its state and the stack at the moment. Then, the next pair (p_{i+1}, α_{i+1}) should be reachable from the previous pair by the transition by the next symbol of the string. Consider three cases.

- If the next symbol is neutral: $a_{i+1} = c$, for $c \in \Sigma_0$, then the stack remains the same: $\alpha_{i+1} = \alpha_i$, and the next state is obtained from the previous state by a transition: $p_{i+1} \in \delta_c(p_i)$.
- If the next symbol in the string is a left bracket: $a_{i+1} = <$, for $< \in \Sigma_{+1}$, then a new stack symbol is pushed onto the stack at the current step and the next pair must have the form $(p_{i+1}, \alpha_{i+1}) = (r, \alpha_i s)$, for a state $r \in Q$ and for a stack symbol $s \in \Gamma$ such that $(r, s) \in \delta_{<}(p_i)$.
- If the next symbol is a right bracket: $a_{i+1} = >$, for $> \in \Sigma_{-1}$, then the symbol at the top of the stack is popped; the stack is not empty at the current moment, since the string w is well-nested. Let $s \in \Gamma$ be a symbol at the top of the stack, that is, let $\alpha_i = \beta s$. Then, the automaton removes the top stack symbol: $\alpha_{i+1} = \beta$, and changes its state to one of the possible next states, while seing the popped symbol: $p_{i+1} \in \delta_{>}(p_i, s)$.

The automaton finishes its computation on a string in a state p_ℓ with an empty stack α_ℓ (the stack is empty since the string w is well-nested). If the final state is accepting: $p_\ell \in F$, then the computation $(p_0, \alpha_0), \dots, (p_\ell, \alpha_\ell)$ of the automaton A_n on the string w

is called *accepting*. The string w is said to be accepted by the automaton if there is at least one accepting computation on this string. And the automaton A defines the language $L(A)$ consisting of all well-nested strings over the alphabet Σ accepted by the automaton.

An input-driven pushdown automaton is called *deterministic* (DIDPDA), if it has a unique initial state, $|Q_0| = 1$, and if in every situation it has exactly one possible action:

- $|\delta_a(q)| = 1$, for all $q \in Q$ and $a \in \Sigma_0 \cup \Sigma_{+1}$;
- $|\delta_a(q, s)| = 1$, for all $q \in Q$, $s \in \Gamma$ and $a \in \Sigma_{-1}$.

Simplified notation is used in the deterministic case: it is said that a deterministic automaton has a unique initial state q_0 and all functions give not a set of possible actions but a deterministically defined action: $\delta_a: Q \rightarrow Q$, for $a \in \Sigma_0$, $\delta_a: Q \rightarrow Q \times \Gamma$, for $a \in \Sigma_{+1}$, and $\delta_a: Q \times \Gamma \rightarrow Q$, for $a \in \Sigma_{-1}$.

A DIDPDA is complete, that is, all functions $(\delta_a)_{a \in \Sigma}$ are fully defined. There are some definitions of DIDPDA in the literature that allow rejecting in the middle of the string. The bounds proved for complete and for partial variants of the definition usually differ by one state.

It is known that NIDPDA can be determinized: von Braunmühl and Verbeek [3] for an arbitrary n -state NIDPDA constructed a deterministic automaton recognizing the same language with 2^{n^2} states and with $2^{n^2}|\Sigma_{+1}|$ stack symbols.

The number of stack symbols in the determinization construction can be slightly improved to $(2^{n^2} - 1)|\Sigma_{+1}|$. This small improvement was not needed before, but since I prove exact lower bounds in Sections 4 and 5, it is useful to make the upper bound match these lower bounds.

Theorem 1. *Let A be an n -state NIDPDA, working over an alphabet $\Sigma = \Sigma_0 \cup \Sigma_{+1} \cup \Sigma_{-1}$. Then, there is a DIDPDA with 2^{n^2} states and with $|\Sigma_{+1}|(2^{n^2} - 1)$ stack symbols that recognizes the language $L(A)$.*

Proof. Consider the modern presentation of the determinization construction by von Braunmühl and Verbeek [3] given in the survey by Okhotin and Salomaa [10, Thm. 1]. Let B be a deterministic automaton obtained from A , as in this survey. The idea of the construction is that the deterministic automaton B calculates the behaviour relation $R \subseteq Q \times Q$ of the nondeterministic automaton on a current well-nested segment of a string, and when the automaton B sees a left bracket, it pushes this relation onto the stack along with the bracket. This is how 2^{n^2} states and $2^{n^2}|\Sigma_{+1}|$ stack symbols occur.

On the other hand, if the automaton B enters the state $\emptyset$ in any situation, then the behavior relation of a current well-nested segment is empty, and this segment cannot be passed by the nondeterministic automaton A . So, the state $\emptyset$ means that the string should be rejected by B . Then, this state is made a rejecting state—the transition function is modified so that all transitions from the state $\emptyset$ lead again to this state $\emptyset$. Is it useful to push onto the stack the empty relation $R = \emptyset$? If the automaton B pushes a symbol of the form $(\emptyset, <)$, then a string contains a well-nested segment that cannot be passed through by A and the string is rejected. Then, the transition that pushes $(\emptyset, <)$ onto the stack can be modified. The automaton can instead assume the rejecting state $\emptyset$ and push onto the stack a symbol $(R, <)$ with any non-empty relation R , and the automaton will reject the string as well. After the transitions of the automaton B are changed this way, the stack symbols of the form $(\emptyset, <)$ are not pushed onto the stack anymore. These symbols can be removed from the stack alphabet, leaving $(2^{n^2} - 1)|\Sigma_{+1}|$ stack symbols. $\square$

3 The exact bound on the number of states

Alur and Madhusudan [1, 2] established the first lower bound on the state complexity of the determinization of NIDPDA. They proved that some nondeterministic input-driven pushdown automata with n states require at least $2^{\Omega(n^2)}$ states in equivalent deterministic automata. Alur and Madhusudan [1, 2] used NIDPDA examples with a fixed input alphabet and with stack alphabet growing linearly in n . Later, Jirásková and Okhotin [5] obtained the exact lower bound of 2^{n^2} states, using nondeterministic automata with input alphabet exponential in n .

In this section, the exact lower bound of 2^{n^2} states on the state complexity of determinization is proved for the first time in the case of a fixed input alphabet. Moreover, the stack alphabet of nondeterministic automata is also fixed and is of size 2. This is the minimal size of the stack alphabet that allows an automaton to get any information from the stack.

Theorem 2. *For each $n \geq 1$, there is an n -state NIDPDA $A_n = (\Sigma, Q, \Gamma, q_0, (\delta_a)_{a \in \Sigma}, F)$, over a 4-symbol input alphabet $\Sigma_{+1} = \{<\}$, $\Sigma_{-1} = \{>\}$, $\Sigma_0 = \{-, \#\}$, with one initial state, and with stack alphabet $\Gamma = \{0, 1\}$, such that any DIDPDA recognizing the language $L(A_n)$ has at least 2^{n^2} states.*

Proof. Let n be fixed. The desired automaton A_n is defined as follows. The set of states is $Q = \{0, \dots, n-1\}$, with the initial state 0. All states are accepting: $Q = F$. The input alphabet Σ and the stack alphabet Γ have already been defined in the theorem statement. And the symbols of the input alphabet act in the following way.

The symbol ‘#’ contains all the nondeterminism of the automaton A_n , the automaton can move from every state to every state by this symbol:

$$\delta_{\#}(i) = \{0, \dots, n-1\}, \quad \text{for } i \in \{0, \dots, n-1\}.$$

The automaton works deterministically on all other symbols of Σ . The symbol ‘-’ decreases the state of A_n by 1 modulo n :

$$\delta_{-}(i) = \{(i-1) \bmod n\}, \quad \text{for } i \in \{0, \dots, n-1\}.$$

By the left bracket ‘<’, the automaton does not change the state and pushes the information, whether the current state is 0 or not, onto the stack, that is, for a state i , it pushes $\text{sgn } i$, which is 0 if the state is 0, and is 1 otherwise:

$$\delta_{<}(i) = \{(i, \text{sgn } i)\}, \quad \text{for } i \in \{0, \dots, n-1\}.$$

By the right bracket ‘>’, the automaton either stays in the same state or rejects. If the current state is not 0, or if the symbol in the stack is not 0, the automaton A_n continues in the same state. And if the current state and the symbol at the top of the stack both equal 0, then the automaton rejects:

$$\delta_{>}(i, s) = \{i\}, \quad \text{for } i \in \{0, \dots, n-1\}, s \in \{0, 1\}, (i, s) \neq (0, 0).$$

Now the automaton A_n has been defined.

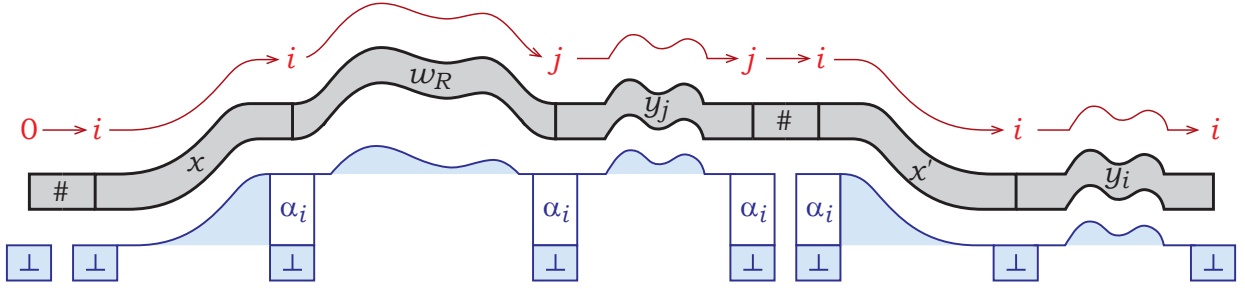


Figure 1: The accepting computation of the automaton A_n on the string $\#xw_Ry_j\#x'y_i$, where $(i, j) \in R$.

If $n = 1$, the theorem is trivial, because a deterministic automaton needs at least $2^1 = 2$ states: an accepting state to accept the empty string and a rejecting state to reject the string $\langle \rangle$. Then, let n be at least 2.

It should be proved that a deterministic automaton must use at least 2^{n^2} states to recognize the language $L(A_n)$.

First, strings with specific behaviour of the automaton A_n are constructed.

Claim 1. *For each relation $R \subseteq Q \times Q$, there is a well-nested string $w_R \in \Sigma^*$, such that for all $i, j \in Q$, the automaton A_n can start reading the string w_R in the state i and finish reading it in the state j if and only if $(i, j) \in R$.*

For each state $i \in Q$, there is such a well-nested string $y_i \in \Sigma^$, that if the automaton A_n enters y_i in some state other than i , then it rejects, and if A_n enters the string y_i in the state i , then it can leave the string in the state i and cannot leave the string y_i in any other state.*

There are two strings $x \in \{-, <\}^$ and $x' \in \{-, >\}^*$, such that x has as many left brackets as there are right brackets in x' . If the automaton A_n begins reading x in the state $i \in Q$, then it leaves this string in the same state i and pushes some string α_i onto the stack. And if A_n enters the string x' in a state $i' \in Q$ having the string α_i on the top of the stack, then, if $i' \neq i$, it rejects, and if $i' = i$, it leaves the string x' in the state i .*

The plan of the proof of Theorem 2 is to build strings with special properties defined in Claim 1 using encoding over a small alphabet, and then to consider computations of the automaton A_n on strings of the form $\#xw_Ry_j\#x'y_i$, for $R \subseteq Q \times Q$ and $i, j \in Q$. It will be proved, that if $(i, j) \in R$, the automaton A_n accepts the string $\#xw_Ry_j\#x'y_i$, as in Figure 1. And if $(i, j) \notin R$, then there will be no accepting computations of A_n on this string. As in the earlier lower bound proofs [2], the final step of the plan is to show that any deterministic automaton should remember the relation R in its state after reading the string w_R to be ready to give the correct answer for every possible pair $(i, j) \in Q \times Q$ in the suffix $y_j\#x'y_i$.

The conditions on strings w_R in Claim 1 can be reformulated as follows.

For a well-nested string $w \in \Sigma^*$, one can define a *behaviour relation* $R(w) \subseteq Q \times Q$: the pair (i, j) is in $R(w)$ if and only if there is a computation of the nondeterministic automaton A_n on the string w , that begins in the state i and ends by leaving the string in the state j . In these terms, for each relation $R \subseteq Q \times Q$, one wants to construct a well-nested string w_R with the behaviour relation $R(w_R) = R$.

For example, for a full behaviour relation, one can take a string $\#$ as $w_{Q \times Q}$, because

$R(\#) = Q \times Q$. To construct strings with all possible relations, it is enough to learn how to eliminate an arbitrary pair of states from the relation of a string. This is done in the following claim.

Claim 2. *There are strings $u_i \in \{-, <\}^*$, for all $i \in Q$, and $v_j \in \{-, >\}^*$, for all $j \in Q$, such that for each well-nested string $w \in \Sigma^*$, and for all states $i, j \in Q$, the equality $R(u_i w v_j) = R(w) \setminus \{(i, j)\}$ holds.*

Furthermore, strings u_i and v_j satisfy the following conditions.

- *Each string u_i , for $i \in Q$, has exactly one left bracket, and there is exactly one right bracket in each string v_j , for $j \in Q$.*
- *If the automaton A_n begins reading the string u_i , for $i \in Q$, or v_j , for $j \in Q$, in some state, then it can leave the string only in this state. On the string v_j the automaton can reject in some cases; it never rejects while reading the string u_i .*

Strings u_i and v_j are defined as follows:

$$\begin{aligned} u_i &= (-)^i < (-)^{n-i}, & \text{for } i \in Q; \\ v_j &= (-)^j > (-)^{n-j}, & \text{for } j \in Q. \end{aligned}$$

There are no symbols ‘#’ in these strings, and therefore the automaton A_n works deterministically on them. When the automaton A_n enters a string u_i in a state i' , first, it passes i symbols ‘-’ decreasing its state i times, and comes to the symbol ‘<’ in a state $(i' - i) \bmod n$. If $i' = i$, this state is 0 and the automaton pushes 0 onto the stack at the left bracket ‘<’, otherwise it pushes 1 onto the stack. Finally, it reads the last $n - i$ symbols ‘-’ and finishes reading the string u_i in the state i' .

Then, consider computations of the automaton A_n on a string v_j . Let A_n enter this string in a state $j' \in Q$ with a symbol $s \in \{0, 1\}$ on the top of the stack. After reading the first j symbols ‘-’ the automaton comes to the symbol ‘>’ in a state $(j' - j) \bmod n$. If $s = 0$ and $j' = j$ at the same time, then A_n rejects, and if the symbol on the top of the stack is not 0 or the state of the automaton at the symbol ‘>’ is not 0, the automaton safely passes through the right bracket. And after reading the remaining $n - j$ symbols ‘-’ the automaton recovers the state j' .

How the behaviour relation on a well-nested string w will change if one wraps it in a pair of strings u_i and v_j , for $i, j \in Q$, that is, if the string w is replaced with $u_i w v_j$?

Can the pair (i, j) lie in the relation $R(u_i w v_j)$? Consider any computation of A_n on the string $u_i w v_j$ that enters the string in the state i and leaves the string in the state j . The automaton begins reading u_i in the state i , and so it pushes 0 at the top of the stack. Since the substring w is well-nested, the automaton will pop this symbol 0 out of the stack while reading the substring v_j , and because the substring v_j does not change the state of the automaton, the automaton A_n enters the substring v_j in the state j . Then, it rejects and $(i, j) \notin R(u_i w v_j)$. The relation $R(u_i w v_j)$ cannot contain any pairs that are not in $R(w)$, because the strings u_i and v_j do not change the state of the automaton, and therefore the automaton should enter the substring w in the state i' and leave this substring in the state j' to enter the whole string $u_i w v_j$ in the state i' and leave it in the state j' . Each pair $(i', j') \in R(w)$, that is not equal to (i, j) , lies in the relation $R(u_i w v_j)$, because there is the following computation on the string $u_i w v_j$. First, the automaton enters the substring u_i in the state i' and leaves it in the state i' , while pushing 1 onto the

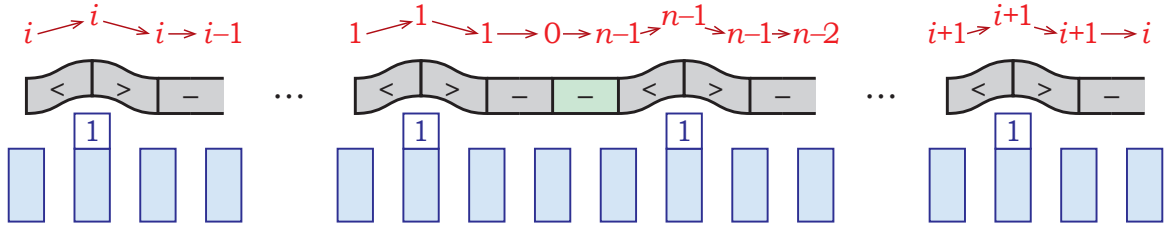


Figure 2: The automaton A_n begins reading the string y_i in the state i and deterministically leaves this string in the same state i .

stack if $i' \neq i$, and 0 otherwise. Then, the automaton reads the substring w and changes its state from i' to j' , it can do this, since $(i', j') \in R(w)$. Finally, it begins reading the string v_j in the state j' . Since $(i', j') \neq (i, j)$, at this moment the automaton cannot at the same time be in the state j and have 0 at the top of the stack, so the automaton does not reject while reading v_j and finishes its computation by leaving the string $u_i w v_j$ in the state j' . Therefore, $R(u_i w v_j) = R(w) \setminus \{(i, j)\}$.

Claim 2 is proved.

Now it is time to construct all special strings in Claim 1. Strings w_R for all $R \subseteq Q \times Q$ can be obtained from a string $\#$ with a full behaviour relation, using Claim 2. Let $R \subseteq Q \times Q$ be an arbitrary relation. Let $(i_1, j_1), \dots, (i_k, j_k)$ be all pairs of states that are not in R . Then, the string w_R can be defined as:

$$w_R = u_{i_k} u_{i_{k-1}} \dots u_{i_1} \# v_{j_1} \dots v_{j_{k-1}} v_{j_k}.$$

By Claim 2, the behaviour relation on this string is obtained from $Q \times Q$ by eliminating pairs $(i_1, j_1), \dots, (i_k, j_k)$, so it equals R . Also, this string w_R is well-nested, since each string u_{i_t} , for $t = 1, \dots, k$, has exactly one left bracket, and each string v_{j_t} , for $t = 1, \dots, k$, has exactly one right bracket.

Next, the well-nested string $y_i \in \Sigma^*$, for $i \in Q$, can be constructed as:

$$y_i = w_R, \text{ where } R = \{(i, i)\}.$$

There is another way to define the string y_i , explicitly and without symbols $\#$:

$$y_i = (<>-)^i - (<>-)^{n-i},$$

In this construction, each pair of brackets $<>$ forbids the current state to be equal to 0 and as a result it is prohibited to enter the string in any state other than i . Figure 2 shows how the automaton A_n reads the string y_i , starting in the state i . If the automaton enters the string y_i in some state $i' \neq i$, then it will be in the state 0 after reading the first i' symbols $'-'$. And since $i' \neq i$, there is a substring $<>$ in the string y_i after the i' -th symbol $'-'$; the automaton reads this substring in the state 0 and rejects.

It remains to construct the strings $x \in \{-, <\}^*$ and $x' \in \{-, >\}^*$. Consider the diagonal relation $R = \{(0, 0), (1, 1), \dots, (n-1, n-1)\}$. The string with this relation is $w_R = u_{i_k} u_{i_{k-1}} \dots u_{i_1} \# v_{j_1} \dots v_{j_{k-1}} v_{j_k}$, where $(i_1, j_1), \dots, (i_k, j_k)$ are all pairs of states that are not in R . Then, the strings x and x' are defined as:

$$\begin{aligned} x &= u_{i_k} u_{i_{k-1}} \dots u_{i_1}, \\ x' &= v_{j_1} \dots v_{j_{k-1}} v_{j_k}. \end{aligned}$$

Then the string x is defined over an alphabet $\{-, <\}$ and contains exactly k left brackets, whereas x' is a string over an alphabet $\{-, >\}$ and has exactly k right brackets. The strings x and x' have no symbols $\#$, so the automaton A_n works on these strings deterministically. For each state $i \in Q$, there is a uniquely defined string $\alpha_i \in \Gamma^*$, which is pushed onto the stack when the automaton reads the string x from the state i . Since the strings u_i , for $i \in Q$, preserve the state of the automaton A_n , the string x also cannot change the state of the automaton. Analogously, the automaton cannot change its state by reading x' , but it may reject on x' , because the strings v_j , for $j \in Q$, have these properties.

It shall be proved that if the automaton reads the string x' with the string α_i at the top of the stack, then it can move through this string in the state i , but the automaton will reject the string if it enters x' in any other state. Consider the behaviour relation of the automaton A_n on the string $x\#x'$. The symbol $\#$ allows the automaton to change each state to each state, whereas the strings x and x' preserve the state of the automaton. Thus, the pair $(i, j) \in Q \times Q$ is in the relation $R(x\#x')$ if and only if the automaton A_n first enters the string x in the state i and pushes the string α_i onto the stack, and then enters the string x' in the state j having the string α_i on the top of the stack and does not reject. On the other hand, by the definition, the relation $R(x\#x')$ is the diagonal relation: $R(x\#x') = \{(0, 0), (1, 1), \dots, (n-1, n-1)\}$. So, for each $i \in Q$, since $(i, i) \in R(x\#x')$, the automaton A_n can enter the string x' in the state i with the string α_i at the top of the stack, and finish reading x' in the state i . Also, the automaton should have no computations on the string x' with α_i at the top of the stack that begin and end in some state $j \neq i$, since $(i, j) \notin R(x\#x')$, for $j \neq i$. Therefore, the constructed strings x and x' are as desired and this finishes the proof of Claim 1.

Now all special strings have been constructed, and it remains to prove that the non-deterministic automaton A_n accepts a string $\#xw_Ry_j\#x'y_i$ if and only if $(i, j) \in R$, and that every deterministic automaton needs many states to do the same.

Claim 3. *Let $i, j \in Q$ and $R \subseteq Q \times Q$, let the well-nested strings $w_R, y_i, y_j \in \Sigma^*$ and the strings $x \in \{-, <\}^*$, $x' \in \{-, >\}^*$ be constructed as in Claim 1. Then, the automaton A_n accepts the string $\#xw_Ry_j\#x'y_i$ if and only if $(i, j) \in R$.*

Let (i, j) be in R . Then there is the following accepting computation of the automaton A_n on the string $\#xw_Ry_j\#x'y_i$, illustrated in Figure 1. At the first symbol $\#$ the automaton guesses the state i . It enters the substring x in the state i and pushes the string α_i onto the stack while reading x . Then, it leaves the substring x in the same state i and enters the substring w_R . Since $(i, j) \in R$, the automaton can leave the substring w_R in the state j after entering it in the state i . So, the automaton changes the state from i to j and finishes reading the substring w_R in the state j . The next substring y_j prohibits all states except j , but the automaton is in the state j , and it safely passes this substring. Then, at the other symbol $\#$ the automaton A_n nondeterministically chooses the state i . Since the substring $w_Ry_j\#$ is well-nested, the string α_i that has been pushed onto the stack while reading the substring x , is at the top of the stack at this moment. The automaton moves through the substring x' popping the string α_i out of the stack, and comes to the next substring in the same state i . The stack is empty after reading x' . And finally the automaton A_n reads the substring y_i in the state i and leaves the whole string $\#xw_Ry_j\#x'y_i$ in the state i . Therefore, the automaton accepts the string, because all states are accepting.

Now let the automaton A_n accept a string $\#xw_Ry_j\#x'y_i$, for some $i, j \in Q$ and $R \in Q \times Q$. It shall be proved that $(i, j) \in R$. Consider an arbitrary accepting computation of the automaton A_n on the string $\#xw_Ry_j\#x'y_i$. Let i' be the state guessed by the automaton at the first symbol '#', let j' be the state assumed by A_n after reading the substring w_R , and let the state i'' be chosen at the symbol '#' before the substring $x'y_i$. The automaton always gets out of the substring x' in the state in which it enters this substring, and also the automaton rejects if it enters the substring y_i in any state other than i . Therefore, $i'' = i$. Not to reject on the substring x' , the automaton should enter it in the same state in which it has entered the substring x , so $i'' = i' = i$. The automaton will reject while reading the substring y_j if it starts reading it in a state other than j . Thus, $j' = j$. However, since the substring x cannot change the state of the automaton, the automaton A_n enters the substring w_R in the state i and leaves it in the state j . This means that $(i, j) \in R$.

It has been shown that the nondeterministic automaton A_n works correctly on strings of the form $\#xw_Ry_j\#x'y_i$, and accepts such a string if and only if $(i, j) \in R$. Now it will be proved that no deterministic automaton recognizes the language $L(A_n)$ using fewer than 2^{n^2} states.

Claim 4. *Let A be a deterministic input-driven pushdown automaton that recognizes the language $L(A_n)$. Then the automaton A has at least 2^{n^2} states.*

By the definition, DIDPDA is complete, so the automaton A cannot reject in the middle of the string. The automaton A accepts a string of the form $\#xw_Ry_j\#x'y_i$ if and only if $(i, j) \in R$, since it recognizes the same language as A_n , and since the automaton A_n has this property by Claim 3. Here all the strings w_R , for $R \subseteq Q \times Q$, y_i , for $i \in Q$, x and x' are constructed by Claim 1. Let q_R , for any relation $R \subseteq Q \times Q$, be the state assumed by the deterministic automaton A after reading the substring $\#xw_R$ from the initial state. It will be shown that all states q_R , for $R \in Q \times Q$, are pairwise distinct, then the automaton A will have at least 2^{n^2} such states.

For the sake of a contradiction, let $R_1, R_2 \subseteq Q \times Q$ be two different relations with $q_{R_1} = q_{R_2}$. Since $R_1 \neq R_2$, there is a pair of states $(i, j) \in Q \times Q$ that lies in exactly one of the relations R_1, R_2 . Then, the automaton A accepts only one of the strings $\#xw_{R_1}y_j\#x'y_i$ and $\#xw_{R_2}y_j\#x'y_i$. While reading the prefix $\#x$ the deterministic automaton A always pushes onto the stack the same string α . And since $q_{R_1} = q_{R_2}$ and since the substrings w_{R_1} and w_{R_2} are well-nested, the automaton A gets out of substrings $\#xw_{R_1}$ and $\#xw_{R_2}$ in the same state $q_{R_1} = q_{R_2}$ with the stack α . Then, in its computations on the strings $\#xw_{R_1}y_j\#x'y_i$ and $\#xw_{R_2}y_j\#x'y_i$ the deterministic automaton enters the suffix $y_j\#x'y_i$ in the state $q_{R_1} = q_{R_2}$ with the stack α . Thus, the automaton either accepts both strings or rejects both, this is a contradiction.

Therefore, all the states q_R , for $R \subseteq Q \times Q$, are pairwise distinct and the deterministic automaton A has at least 2^{n^2} states. Theorem 2 is proven. $\square$

Note that if an n -state nondeterministic input-driven pushdown automaton uses a one-symbol stack alphabet, then it can be determinized like a simple DFA by the well-known subset construction using only 2^n states. Therefore, using two stack symbols in a stack alphabet Γ in the automata in Theorem 2 is optimal.

4 The lower bound on the number of stack symbols: one left bracket

It has been proven in Section 3 that, in the case of a fixed alphabet, 2^{n^2} states can be necessary to determinize an n -state nondeterministic automaton.

However, the size of an input-driven pushdown automaton is described not only by the number of its states but also by the number of stack symbols it uses. The determinization construction by von Braunmühl and Verbeek [3], for an n -state NIDPDA, gives a deterministic automaton with 2^{n^2} states and $2^{n^2}|\Sigma_{+1}|$ stack symbols. This upper bound has been slightly improved to $(2^{n^2} - 1)|\Sigma_{+1}|$ stack symbols and the same 2^{n^2} states in Theorem 1 in Section 2.

How optimal is this determinization construction in the number of stack symbols? The best lower bound of $2^{\frac{n^2}{c}}$ stack symbols, with a constant $c > 1$, was obtained by Okhotin, Piao and Salomaa [9]. They used witness automata working on strings over a bounded input alphabet and with a stack alphabet of size linear in n .

The goal is to prove an exact lower bound on the number of stack symbols needed for NIDPDA determinization, and preferably using a linearly growing stack alphabet and working over a fixed input alphabet. To do this, one needs to work efficiently with small alphabets to avoid division by a constant c , and also needs to learn how to use several left brackets to multiply the bound by $|\Sigma_{+1}|$.

In this section, the exact bound is proved in a simpler case of only one left bracket, and in the next Section 5 it will be shown how to use any number of left brackets less than 2^{n^2} to multiply the number of stack symbols in a deterministic automaton by $|\Sigma_{+1}|$.

Theorem 3. *For each $n \geq 1$, there is an n -state NIDPDA $B_n = (\Sigma, Q, \Gamma, q_0, (\delta_a)_{a \in \Sigma}, F)$, with one initial state, with a stack alphabet of size $|\Gamma| = 2n + 2$, working over a 5-symbol input alphabet: $\Sigma_{+1} = \{<\}$, $\Sigma_{-1} = \{>, \gg\}$, $\Sigma_0 = \{-, \#\}$, such that any DIDPDA recognizing a language $L(B_n)$ has at least 2^{n^2} states and at least $2^{n^2} - 1$ stack symbols.*

Proof. Consider the automaton A_n defined in the proof of Theorem 2. It is an n -state automaton with $Q = \{0, 1, \dots, n-1\}$; the state 0 is initial and all states are accepting; the automaton has stack alphabet $\{0, 1\}$ and works over an input alphabet $\Sigma \setminus \{\gg\}$. The automaton B_n will be obtained from A_n by adding a new right bracket ' $\gg$ ', new stack symbols $\widehat{i}$ and $\vec{i}$, for $i \in Q$, and some new transitions.

The stack alphabet of the automaton B_n is

$$\Gamma = \{0, 1\} \cup \{\widehat{i} \mid i \in Q\} \cup \{\vec{i} \mid i \in Q\}.$$

Transitions by symbols ' $-$ ', ' $\#$ ', ' $>$ ' in the automaton B_n remain the same as in A_n . The automaton A_n does not change its state on a left bracket ' $<$ ', and pushes 0 onto the stack if the state is 0, and 1 if the state is not 0 (in other words, A_n pushes $\text{sgn } i$ in a state i). The automaton B_n can either do the same, or alternatively decide to change its state (maybe, to the same state) and to push onto the stack either the old state i as $\widehat{i}$, or the new state j as $\vec{j}$.

$$\delta_{<}(i) = \{(i, \text{sgn } i)\} \cup \{(j, \widehat{i}) \mid j \in Q\} \cup \{(j, \vec{j}) \mid j \in Q\}, \quad \text{for } i \in \{0, \dots, n-1\}$$

If a new stack symbol is popped out of the stack at an old right bracket ' $>$ ', then the automaton rejects. If 0 or 1 is at the top of the stack when the automaton B_n reads a

double bracket ‘ $\gg$ ’, then it also rejects. If a symbol $\widehat{i}$, for $i \in Q$, is popped out of the stack at a double bracket ‘ $\gg$ ’, then, if the current state is 0, the automaton moves to the state i , and otherwise rejects.

$$\delta_{\gg}(0, \widehat{i}) = \{i\}, \quad \text{for } i \in Q$$

And if a stack symbol is of the form $\overrightarrow{j}$, for $j \in Q$, then at a double bracket ‘ $\gg$ ’ the automaton changes the state 1 to the state j and rejects in all states other than 1.

$$\delta_{\gg}(1, \overrightarrow{j}) = \{j\}, \quad \text{for } j \in Q$$

Now the automaton B_n has been defined.

If B_n begins reading some well-nested string without double brackets ‘ $\gg$ ’ in some state $i \in Q$ and leaves the string in some state $j \in Q$, then in this computation the automaton B_n makes a transition of A_n at each left bracket ‘ $<$ ’. Indeed, if B_n chooses to push onto the stack any new stack symbol while reading the left bracket ‘ $<$ ’, then it will later pop this symbol out of the stack at the matching old right bracket ‘ $>$ ’ and reject. Therefore, the next property holds.

Claim 5. *For each well-nested string without double brackets ‘ $\gg$ ’ and for all states $i, j \in Q$, the automaton B_n can change its state from i to j by reading this string if and only if so can A_n .*

Using Claim 5, one can easily prove that determinization of B_n requires at least as many states as determinization of A_n .

Claim 6. *Every DIDPDA recognizing the language $L(B_n)$ has at least 2^{n^2} states.*

First, by Claim 5, automata A_n and B_n accept the same sets of strings without double brackets ‘ $\gg$ ’, that is, $L(B_n) \cap (\Sigma \setminus \{\gg\})^* = L(A_n)$. It is also used here that A_n and B_n have the same initial and accepting states. For the sake of a contradiction, assume that there is a deterministic automaton A that recognizes the language $L(B_n)$ using less than 2^{n^2} states. Then, one can delete the double bracket ‘ $\gg$ ’ from the alphabet of A getting a new deterministic automaton A' , recognizing the language $L(A_n)$ with fewer than 2^{n^2} states. And it is impossible by Theorem 2.

If $n = 1$, then Claim 6 is enough to prove Theorem 3, since the stack alphabet of any DIDPDA has at least $2^{1^2} - 1 = 1$ symbol. From this moment, let n be at least 2; the following proof uses the existence of states 0 and 1.

Let y_i , for $i \in Q$, and w_R , for $R \subseteq Q \times Q$, be well-nested strings constructed in Claim 1 in the proof of Theorem 2. These strings contain no double brackets ‘ $\gg$ ’, therefore the automaton B_n works on them as A_n . And the next claim is a direct corollary of Claim 1.

Claim 7. *All strings y_i , for $i \in Q$, and w_R , for $R \subseteq Q \times Q$, are well-nested, and the automaton B_n works on them as follows.*

1. *If B_n enters a string y_i , for $i \in Q$, in any state other than i , then it rejects, and if it enters this string in the state i , then it can leave it in the state i and cannot leave it in any other state.*
2. *The automaton B_n can enter a string w_R in a state i and leave it in a state j if and only if $(i, j) \in R$.*

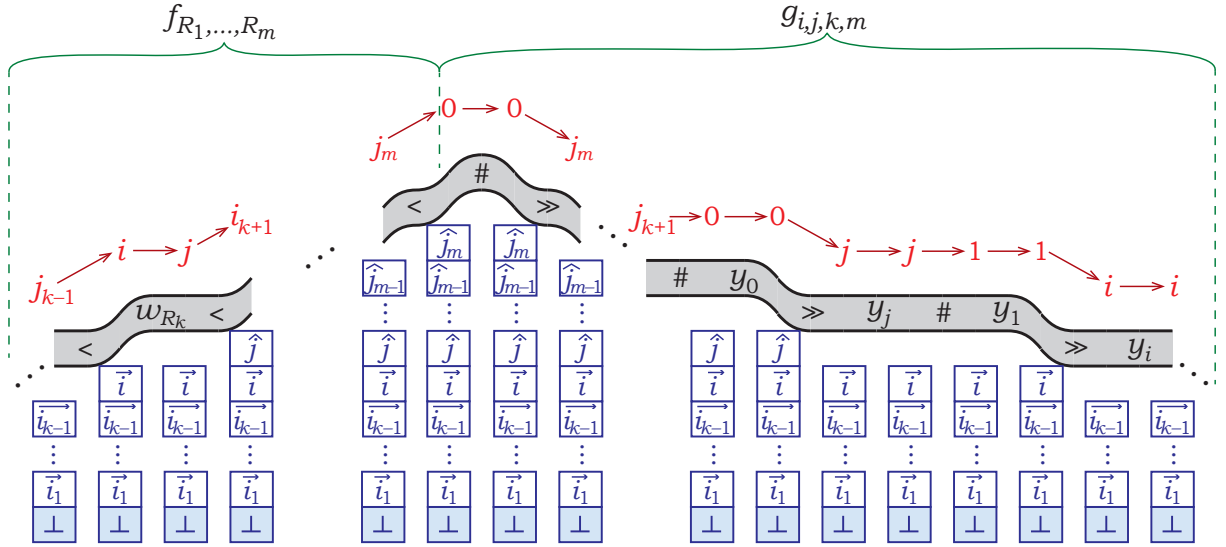


Figure 3: An accepting computation of the automaton B_n on a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$, for $(i, j) \in R_k$. First, it is shown in the figure how the automaton moves through the substring $\langle w_{R_k} \rangle$ of $f_{R_1, \dots, R_m} = \langle w_{R_1} \rangle \langle w_{R_2} \rangle \dots \langle w_{R_m} \rangle$, pushing onto the stack $\vec{i} \hat{j}$. Then B_n enters the second part $g_{i, j, k, m}$ of the string, that begins in the figure with a symbol ‘#’ at the top, and it is illustrated how B_n gets through a substring $\# y_0 \gg y_j \# y_1 \gg y_i$ of $g_{i, j, k, m}$ popping symbols $\hat{j}$ and $\vec{i}$ out of the stack.

It shall be proved that every deterministic automaton recognizing the language of the automaton B_n has at least $2^{n^2} - 1$ stack symbols. To this end, special strings comprised of any substrings w_R , with $R \subseteq Q \times Q$, and y_i , with $i \in Q$, and also of symbols ‘<’, ‘>’, ‘#’, are constructed. For a sequence of m non-empty relations $R_1, R_2, \dots, R_m \subseteq Q \times Q$, a string $f_{R_1, \dots, R_m}$ is defined:

$$f_{R_1, \dots, R_m} = \langle w_{R_1} \rangle \langle w_{R_2} \rangle \dots \langle w_{R_m} \rangle \langle .$$

Also, for all $i, j \in Q$, for each $m \geq 1$ and for each $k = 1, \dots, m$, a string $g_{i, j, k, m}$ is defined:

$$g_{i, j, k, m} = (\# \gg)^{m-k} \# y_0 \gg y_j \# y_1 \gg y_i (\# \gg)^{k-1}$$

The goal is to prove that the nondeterministic automaton B_n accepts a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ if and only if $(i, j) \in R_k$, and that any deterministic automaton needs at least $2^{n^2} - 1$ stack symbols to do the same.

Claim 8. *Let $m \geq 1$ and $k = 1, \dots, m$ be integers, let $R_1, R_2, \dots, R_m \subseteq Q \times Q$ be non-empty relations, let $i, j \in Q$ be two states. Then, the automaton B_n accepts a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ if and only if $(i, j) \in R_k$.*

First, let (i, j) lie in R_k . Then there is the following accepting computation of the automaton B_n on a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ (see Figure 3). Since relations $R_1, \dots, R_m$ are non-empty, one can choose pairs $(i_1, j_1) \in R_1, (i_2, j_2) \in R_2, \dots, (i_m, j_m) \in R_m$, taking $i_k = i$ and $j_k = j$. For convenience, let also $j_0 = 0$ and $i_{m+1} = 0$. The string $f_{R_1, \dots, R_m} = \langle w_{R_1} \rangle \langle w_{R_2} \rangle \dots \langle w_{R_m} \rangle \langle$ has $m + 1$ unmatched left brackets and the automaton B_n starts its computation at the first of them in the state 0. The computation will be

constructed so that the automaton will get to each t -th unmatched left bracket in the state j_{t-1} , for all $t = 1, \dots, m+1$. On t -th unmatched bracket the automaton B_n moves from the state $\widehat{j_{t-1}}$ to the state i_t , pushing onto the stack either the symbol $\widehat{i_t}$, for $t \leq k$, or the symbol $\widehat{j_{t-1}}$, for $t > k$. Next, if $t \neq m+1$, the automaton enters the substring w_{R_t} in the state i_t . This is a well-nested string, so reading it does not change the stack. Since $(i_t, j_t) \in R_t$, there is a computation on w_{R_t} that begins in the state i_t and ends in the state j_t . This computation is used as a part of an accepting computation on the full string. As a result, the automaton gets from the state j_{t-1} at the t -th unmatched left bracket to the state j_t at the next unmatched left bracket. Thus, the correct computation of the automaton B_n on the substring $f_{R_1, \dots, R_m}$ has been constructed, and it ends with the automaton leaving the substring in the state 0 with the following stack contents:

$$\overrightarrow{i_1} \overrightarrow{i_2} \dots \overrightarrow{i_k} \widehat{j_k} \widehat{j_{k+1}} \dots \widehat{j_m}.$$

Next, the automaton B_n reads a string $g_{i,j,k,m} = (\# \gg)^{m-k} \# y_0 \gg y_j \# y_1 \gg y_i (\# \gg)^{k-1}$, and the computation should be continued to make the automaton accept the full string. The automaton can pass by a substring $(\# \gg)^{m-k}$ assuming the state 0 after reading each symbol $\#$. Since the last $m-k$ symbols pushed onto the stack are $\widehat{j_{k+1}}, \dots, \widehat{j_m}$, which are symbols of the form $\widehat{q}$, for $q \in Q$, the automaton can pop these symbols out of the stack in the state 0 at double right brackets $\gg$ without rejecting. Then, B_n needs to get through a substring $\# y_0 \gg y_j \# y_1 \gg y_i$. This can be done as follows: at the first symbol $\#$ the automaton assumes the state 0, in this state it moves through y_0 without rejecting and enters the first double bracket $\gg$ of this substring. The automaton pops the symbol $\widehat{j_k}$ out of the stack in the state 0 at the double bracket $\gg$ and moves to the state $j_k = j$. This allows the automaton B_n to pass through the substring y_j . At the second symbol $\#$ the automaton chooses the state 1 and passes y_1 . Then it pops the symbol $\widehat{i_k}$ out of the stack in the state 1 at $\gg$ and changes its state to $i_k = i$. In this state i it safely moves through y_i . It remains to read the last substring $(\# \gg)^{k-1}$, and this can be done by choosing the state 1 at every symbol $\#$. The remaining string in the stack is $\overrightarrow{i_1} \overrightarrow{i_2} \dots \overrightarrow{i_{k-1}}$, and the automaton can pop symbols of this form at double brackets $\gg$ in the state 1. Thus, the automaton B_n completes its computation on the string $f_{R_1, \dots, R_m} g_{i,j,k,m}$ and accepts, since all states are accepting.

Now let (i, j) be not in R_k . It shall be proved, that the string $f_{R_1, \dots, R_m} g_{i,j,k,m}$ is not accepted by B_n . Consider an arbitrary computation of the automaton B_n on the string $f_{R_1, \dots, R_m} g_{i,j,k,m}$. There are exactly $m+1$ unmatched left brackets in a substring $f_{R_1, \dots, R_m}$, and if the automaton did not reject while reading this substring, then it will leave the substring with some string $\alpha = s_1 \dots s_{m+1}$ of length $m+1$ in the stack. Symbols s_k and s_{k+1} were pushed onto the stack at the first and at the last left brackets in the substring $\langle w_{R_k} \rangle$. If $s_k = \widehat{i}$ and $s_{k+1} = \widehat{j}$, then the automaton B_n leaves the first symbol $\langle$ and enters the substring w_{R_k} in the state i and leaves this substring and enters the last left bracket in the state j . Since $(i, j) \notin R_k$, it is impossible to enter w_{R_k} in the state i and leave it in the state j . Therefore, either $s_k \neq \widehat{i}$, or $s_{k+1} \neq \widehat{j}$. It will be shown that in both cases the automaton rejects while reading the substring $g_{i,j,k,m} = (\# \gg)^{m-k} \# y_0 \gg y_j \# y_1 \gg y_i (\# \gg)^{k-1}$. The symbol s_{k+1} is popped out of the stack while reading the substring $\# y_0 \gg y_j$. Strings y_0 and y_j wrapping the double right bracket $\gg$ require that the automaton enters $\gg$ in the state 0 and leaves it in the state j . Such a change of a state at $\gg$ is possible only with the symbol $\widehat{j}$ at the top

of the stack. Therefore, if $s_{k+1} \neq \widehat{j}$, the automaton rejects. The second case is when $s_k \neq \overrightarrow{i}$. The symbol s_k is popped out of the stack while reading the substring $\#y_1 \gg y_i$, here the double right bracket ' $\gg$ ' is wrapped in substrings y_1 and y_i , and not to reject the automaton B_n should change the state 1 to the state i at the double bracket ' $\gg$ '. But this is impossible since the symbol at the top of the stack is not $\overrightarrow{i}$. Therefore, if $(i, j) \notin R_k$, the automaton B_n rejects the string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ in any computation.

Claim 8 has been proven. It remains to show that any deterministic automaton needs a lot of stack symbols to work as B_n .

Claim 9. *Let DIDPDA A work over an alphabet Σ and let it accept a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ if and only if $(i, j) \in R_k$, for all $m \geq 1$, for all non-empty relations $R_1, R_2, \dots, R_m \subseteq Q \times Q$, for all states $i, j \in Q$ and for all $k \in \{1, \dots, m\}$. Then, there are at least $2^{n^2} - 1$ stack symbols in the stack alphabet of the automaton A .*

Let N be the number of states in the automaton A and let M be the size of its stack alphabet. Then the automaton A finishes reading each string of the form $f_{R_1, \dots, R_m}$ in one of N states with a string of length $m + 1$ in the stack, since a string $f_{R_1, \dots, R_m}$ has exactly $m + 1$ unmatched left brackets. For a fixed m , there are at most $N \cdot M^{m+1}$ possible outcomes of reading a string $f_{R_1, \dots, R_m}$, for m arbitrary non-empty relations. On the other hand, for a fixed m , a number of strings of the form $f_{R_1, \dots, R_m}$ is $(2^{n^2} - 1)^m$, since each of m non-empty relations $R_1, \dots, R_m \subseteq Q \times Q$ can be chosen in $2^{n^2} - 1$ ways.

For the sake of a contradiction, assume that the deterministic automaton A has a small number of stack symbols: $M < 2^{n^2} - 1$. Since the number of states N of the automaton A is fixed and $M < 2^{n^2} - 1$, one can choose m large enough, so that the number of possible outcomes of reading a substring $f_{R_1, \dots, R_m}$ will be fewer than the number of such substrings: $N \cdot M^{m+1} < (2^{n^2} - 1)^m$. Therefore, there are two distinct sequences of non-empty relations $R_1, \dots, R_m$ and $S_1, \dots, S_m$, such that the automaton A finishes reading the strings $f_{R_1, \dots, R_m}$ and $f_{S_1, \dots, S_m}$ in the same state q and with the same stack contents α .

Since the sequences of relations $R_1, \dots, R_m$ and $S_1, \dots, S_m$ are different, there is a pair of states $(i, j) \in Q \times Q$ and there is a number $k \in \{1, \dots, m\}$, such that the pair (i, j) lies in exactly one of the relations R_k and S_k . Without loss of generality, one can assume that $(i, j) \in R_k$ and $(i, j) \notin S_k$. Then the deterministic automaton A accepts the string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ and rejects the string $f_{S_1, \dots, S_m} g_{i, j, k, m}$. However, on both strings the automaton enters the second part $g_{i, j, k, m}$ in the state q with the string α in the stack, and therefore it finishes reading both strings $f_{R_1, \dots, R_m} g_{i, j, k, m}$ and $f_{S_1, \dots, S_m} g_{i, j, k, m}$ in the same state, and either accepts both strings, or rejects both. This is a contradiction, and Claim 9 is proved.

By Claim 8, the nondeterministic automaton B_n accepts a string $f_{R_1, \dots, R_m} g_{i, j, k, m}$ if and only if $(i, j) \in R_k$. Then, every deterministic automaton recognizing the language of B_n does the same and satisfies the conditions of Claim 9. Therefore it has at least $2^{n^2} - 1$ stack symbols, and Theorem 3 has been proven. $\square$

5 The lower bound on the number of stack symbols: several left brackets

The number of stack symbols sufficient for determinizing an n -state NIDPDA is $(2^{n^2} - 1)|\Sigma_{+1}|$, established in Section 2. In the case of only one left bracket, the exact lower

bound on the size of stack alphabet needed for determinization is $2^{n^2} - 1$, proved in Section 4. How to multiply the number of stack symbols needed for determinization by the number of left brackets, if $|\Sigma_{+1}| > 1$? The exact lower bound in the case of several left brackets is proved in the next theorem.

Theorem 4. *For any integers n and s , such that $n \geq 1$ and $1 \leq s \leq 2^{n^2}$, there is an n -state NIDPDA $B_{n,s} = (\Sigma, Q, \Gamma, q_0, (\delta_a)_{a \in \Sigma}, F)$, with one initial state, with a stack alphabet of the size $|\Gamma| = 2 + 2n + \lfloor \log_2(2s - 1) \rfloor$ and working over an $(s + 5)$ -symbol input alphabet: $\Sigma_{+1} = \{<_0, <_1, \dots, <_{s-1}\}$, $\Sigma_{-1} = \{>, \gg, \ggg\}$, $\Sigma_0 = \{-, \#\}$, such that every DIDPDA recognizing the language $L(B_{n,s})$ has at least 2^{n^2} states and at least $s(2^{n^2} - 1)$ stack symbols.*

Proof. Let $n \geq 2$ (a degenerate case of $n = 1$ will be handled at the end of the proof). Also let $s \geq 2$ (if $s = 1$, then one can take the automaton B_n from Theorem 3 as $B_{n,s}$). Then $\lfloor \log_2(2s - 1) \rfloor = \lfloor \log_2(s - 1) \rfloor + 1$.

The automaton $B_{n,s}$ is constructed as a more complicated version of the automaton B_n in Theorem 3. It has a set of states $Q = \{0, \dots, n - 1\}$, with the initial state 0 and with all states accepting, as the automaton B_n . The input alphabet of $B_{n,s}$ is defined in the statement of Theorem 4. The stack alphabet is

$$\Gamma = \{0, 1\} \cup \{\hat{i} \mid i \in Q\} \cup \{\vec{i} \mid i \in Q\} \cup \{\textcircled{x} \mid x = 0, \dots, \lfloor \log_2(s - 1) \rfloor\}.$$

Stack symbols from the set $\{0, 1\} \cup \{\hat{i} \mid i \in Q\} \cup \{\vec{i} \mid i \in Q\}$ are already used by the automaton B_n in Theorem 3; the automaton $B_{n,s}$ has also some new stack symbols: $\textcircled{x}$, for $x \in \{0, \dots, \lfloor \log_2(s - 1) \rfloor\}$. And $|\Gamma| = 2 + 2n + \lfloor \log_2(s - 1) \rfloor + 1 = 2 + 2n + \lfloor \log_2(2s - 1) \rfloor$, as required.

Next, the transition function of the automaton $B_{n,s}$ is defined.

The possible transitions of $B_{n,s}$ at the neutral symbols ‘#’ and ‘-’ and at the old right brackets ‘>’ and ‘>>’ coincide with the transitions of B_n at these symbols.

For each left bracket ‘<_ℓ’, for $\ell \in \{0, \dots, s - 1\}$, the automaton $B_{n,s}$ has all transitions that are defined for B_n at the left bracket ‘<’. Also there are new transitions, that arbitrarily change the state of the automaton and push onto the stack the position of any unary bit in the binary representation of the number ℓ of the left bracket. Denote the coefficient at 2^x in the binary representation of the number ℓ as $\ell[x]$, and denote the transition function of B_n as $\delta^{(B_n)}$. Then the transition function of the automaton $B_{n,s}$ at left brackets is defined as:

$$\delta_{<_\ell}(q) = \delta_{<}^{(B_n)}(q) \cup \{(r, \textcircled{x}) \mid r \in Q, \ell[x] = 1\}, \quad \text{for } \ell \in \{0, \dots, s - 1\}, q \in Q.$$

It remains to define the transition function of $B_{n,s}$ at a new triple right bracket ‘>>>’. No transitions are defined at this bracket for old stack symbols from sets $\{0, 1\}$, $\{\hat{i} \mid i \in Q\}$ and $\{\vec{i} \mid i \in Q\}$. If a new stack symbol of the form $\textcircled{x} \in \{0, \dots, \lfloor \log_2(s - 1) \rfloor\}$ is at the top of the stack, then the automaton $B_{n,s}$ at a triple bracket ‘>>>’ can only change its state from the state $(x \bmod n)$ to the state $\lfloor \frac{x}{n} \rfloor$. Such a definition of transitions at a triple bracket ‘>>>’ allows the automaton to extract all the information about a symbol $\textcircled{x}$ from the stack. The automaton cannot simply put x in its state while reading a triple bracket ‘>>>’, since, for $s = 2^{n^2}$, the number x can reach the value of $\lfloor \log_2(2^{n^2} - 1) \rfloor = n^2 - 1$, and the automaton has only n states. Thus, a number x at the top of the stack is read in

two parts: the state $(x \bmod n)$ should be assumed before reading the bracket ' $\ggg$ ', and the state $\lfloor \frac{x}{n} \rfloor$ emerges after reading this bracket. Since $0 \leq x \leq n^2 - 1$, it holds that $x = \lfloor \frac{x}{n} \rfloor \cdot n + (x \bmod n)$.

$$\delta_{\ggg}(x \bmod n, (\textcircled{x})) = \left\{ \left\lfloor \frac{x}{n} \right\rfloor \right\}$$

The automaton $B_{n,s}$ has been completely defined.

If the automaton $B_{n,s}$ pops out of the stack a new stack symbol of the form $(\textcircled{x})$ at one of the old right brackets ' $>$ ' and ' $\gg$ ', it immediately rejects. So, new stack symbols can be popped only at a triple right bracket ' $\ggg$ '. Thus, accepting computations of $B_{n,s}$ on well-nested strings without triple brackets ' $\ggg$ ' are the same as accepting computations of B_n , if each left bracket in the input alphabet of $B_{n,s}$ is considered as the only left bracket ' $<$ ' of the automaton B_n . Then one can bound from below the number of states needed for determinization of the automaton $B_{n,s}$.

Claim 10. *Each DIDPDA recognizing the language $L(B_{n,s})$ has at least 2^{n^2} states.*

Indeed, if one eliminates left brackets $<_1, \dots, <_s$ from the alphabet of the DIDPDA, leaving only the symbol ' $<_0$ ' as ' $<$ ', and if one also deletes the triple right bracket ' $\ggg$ ', then the resulting deterministic automaton will recognize exactly the language of the automaton B_n and will have the same number of states as the original deterministic automaton. And the language $L(B_n)$ cannot be recognized by a DIDPDA with fewer than 2^{n^2} states.

Furthermore, since the automaton $B_{n,s}$ works on well-nested strings without triple brackets ' $\ggg$ ', as the automaton B_n , if all left brackets of $B_{n,s}$ are considered as the bracket ' $<$ ' of B_n , one can define strings w_R , for $R \subseteq Q \times Q$, and y_i , for $i \in Q$, for the automaton $B_{n,s}$. These strings are constructed like in Claim 1 in the proof of Theorem 2; the left bracket ' $<_0$ ' is used as the symbol ' $<$ '. These strings are well-nested and do not contain double and triple right brackets ' $\gg$ ' and ' $\ggg$ ', therefore the automaton $B_{n,s}$ works on them as B_n , and the automaton B_n works on them as the automaton A_n in Theorem 2. Then, computations of $B_{n,s}$ on these strings are described by the next claim that directly follows from Claims 1 and 7.

Claim 11. *The automaton $B_{n,s}$ works on well-nested strings y_i , for $i \in Q$, and w_R , for $R \subseteq Q \times Q$, as follows.*

1. *If the automaton enters a string y_i , for $i \in Q$, in the state i , then it can leave this string in the state i and cannot leave it in any other state. If the automaton $B_{n,s}$ begins reading y_i in any state other than i , it rejects.*
2. *For every two states $i, j \in Q$, the automaton $B_{n,s}$ can begin reading a string w_R in the state i and leave this string in the state j , if and only if $(i, j) \in R$.*

Strings $f_{R_1, \dots, R_m}$ and $g_{i,j,k,m}$ were defined for the automaton B_n in Theorem 3. Each string $f_{R_1, \dots, R_m}$ has exactly $m + 1$ unmatched left brackets and m intermediate substrings with non-empty behaviour relations $R_1, \dots, R_m$. Now, more elaborate strings will be defined, in which every unmatched left bracket can be any numbered bracket: $<_0, <_1, \dots, <_{s-1}$. Such a string is determined not only by a sequence of m non-empty relations, but also by a sequence of $m + 1$ left bracket numbers. For every integer $m \geq 1$, for a

sequence of m non-empty relations $R_1, R_2, \dots, R_m \subseteq Q \times Q$ and for a sequence of indices $\ell_1, \dots, \ell_{m+1} \in \{0, \dots, s-1\}$, the following string is defined:

$$f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} = <_{\ell_1} w_{R_1} <_{\ell_2} w_{R_2} \dots <_{\ell_m} w_{R_m} <_{\ell_{m+1}}.$$

Strings $g_{i,j,k,m}$, for $i, j \in Q$, for $m \geq 1$ and for $k = 1, \dots, m$, are exactly the same as for the automaton B_n :

$$g_{i,j,k,m} = (\# \gg)^{m-k} \# y_0 \gg y_j \# y_1 \gg y_i (\# \gg)^{k-1}$$

If a well-nested string has no triple right brackets ' $\ggg$ ', then, while moving through this string, the automaton $B_{n,s}$ either rejects, or works as B_n , considering every left bracket as a symbol ' $<$ ' of the automaton B_n . Then, since all strings of the form $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} g_{i,j,k,m}$ are well-nested and have no triple brackets ' $\ggg$ ', Claim 8 for the automaton B_n and for strings of the form $f_{R_1, \dots, R_m} g_{i,j,k,m}$ implies the same claim for the automaton $B_{n,s}$ and for strings of the form $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} g_{i,j,k,m}$.

Claim 12. *Let $m \geq 1$ be an integer, let $R_1, R_2, \dots, R_m \subseteq Q \times Q$ be any non-empty relations, let $\ell_1, \dots, \ell_{m+1} \in \{0, \dots, s-1\}$ be indices of left brackets, let $i, j \in Q$ be any two states and let $k = 1, \dots, m$ be an integer. Then the automaton $B_{n,s}$ accepts the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} g_{i,j,k,m}$ if and only if $(i, j) \in R_k$.*

For every pair of strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ and $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}}$ with different sequences of non-empty relations, Claim 12 allows one to choose such a continuation $g_{i,j,k,m}$, that the automaton $B_{n,s}$ will give different answers on the strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} g_{i,j,k,m}$ and $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}} g_{i,j,k,m}$, accepting one and rejecting the other. But to distinguish in this way all pairs of strings of the form $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$, one also needs to construct such suffixes that will separate any two strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ and $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}}$ with different indices of left brackets in some position: $\ell_k \neq \ell'_k$.

The next goal is to define strings $h_{k,x,m}$, for $m \geq 1$, for a left bracket number $k \in \{1, \dots, m+1\}$ and for a bit number $x \in \{0, \dots, \lfloor \log_2(s-1) \rfloor\}$, such that the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$ is accepted by the automaton $B_{n,s}$ if and only if $\ell_k[x] = 1$, that is, if the coefficient at 2^x in the binary representation of the number ℓ_k equals 1. Such strings $h_{k,x,m}$ will distinguish any two strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ and $f_{R_1, \dots, R_m, \ell'_1, \dots, \ell'_{m+1}}$, in which, for some number k , the indices of unmatched left brackets differ: $\ell_k \neq \ell'_k$, that is, if there is a bit number x , such that $\ell_k[x] \neq \ell'_k[x]$.

A string $h_{k,x,m}$, for $m \geq 1$, for $k \in \{1, \dots, m+1\}$ and for $x \in \{0, \dots, \lfloor \log_2(s-1) \rfloor\}$, is defined as:

$$h_{k,x,m} = (\# \gg)^{m-k+1} \# y_{x \bmod n} \ggg y_{\lfloor \frac{x}{n} \rfloor} (\# \gg)^{k-1}.$$

The next claim is that strings $h_{k,x,m}$ have the desired properties.

Claim 13. *Let $m \geq 1$ be an integer, let $R_1, R_2, \dots, R_m \subseteq Q \times Q$ be any non-empty relations, let $\ell_1, \dots, \ell_{m+1} \in \{0, \dots, s-1\}$ be indices of left brackets, let $k \in \{1, \dots, m+1\}$ be an integer and let $x \in \{0, \dots, \lfloor \log_2(s-1) \rfloor\}$ be a bit number. Then the automaton $B_{n,s}$ accepts the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$ if and only if $\ell_k[x] = 1$, that is, if the coefficient at 2^x in the binary representation of the number ℓ_k is 1.*

First, let $\ell_k[x] = 1$. Then there is the following accepting computation of the automaton $B_{n,s}$ on the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$, illustrated in Figure 4. Since relations $R_1, \dots, R_m$ are non-empty, one can choose a pair of states of $B_{n,s}$ for each relation:

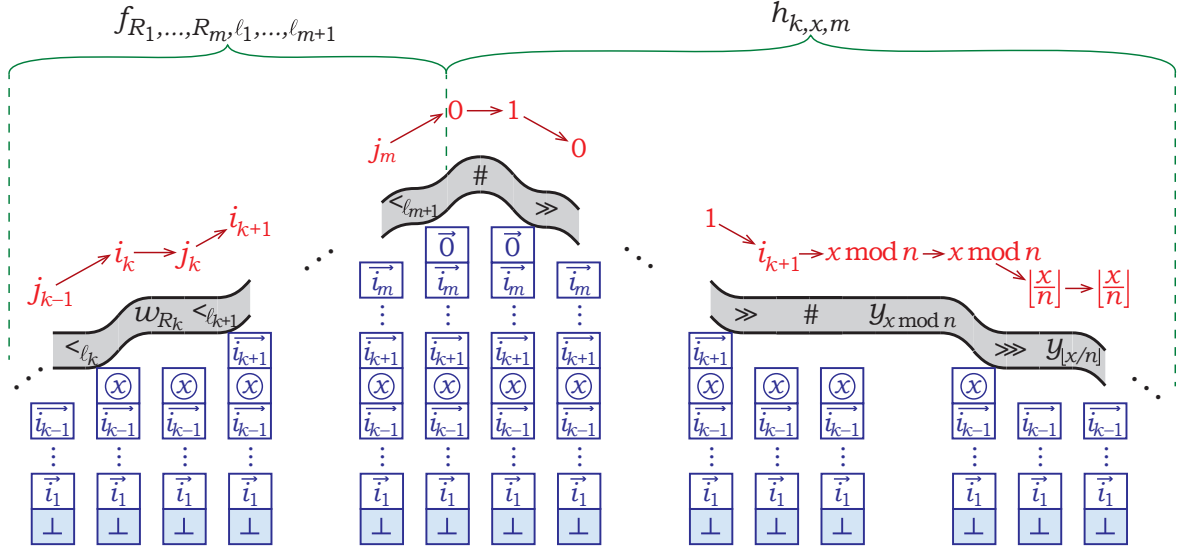


Figure 4: An accepting computation of the automaton $B_{n,s}$ on a string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$, in a case of $\ell_k[x] = 1$.

$(i_1, j_1) \in R_1, \dots, (i_m, j_m) \in R_m$. The automaton reads the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} = <_{\ell_1} w_{R_1} <_{\ell_2} w_{R_2} \dots <_{\ell_m} w_{R_m} <_{\ell_{m+1}}$ as follows: it begins in the state $j_0 = 0$ and comes to every t -th unmatched left bracket ' $<_{\ell_t}$ ', for $t = 1, \dots, m$, in the state j_{t-1} , and then, at each bracket ' $<_{\ell_t}$ ' it changes its state from j_{t-1} to i_t , reads w_{R_t} changing the state i_t to j_t (this is possible, since $(i_t, j_t) \in R_t$), and comes to the next unmatched left bracket ' $<_{\ell_{t+1}}$ ' in the state j_t . Finally, the automaton gets to the last unmatched left bracket of the substring $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ in the state j_m and leaves the substring in the state $i_{m+1} = 0$. During this computation, the automaton pushes symbols $\vec{i}_t$ onto the stack at every unmatched left bracket ' $<_{\ell_t}$ ', for $t \neq k$, and pushes onto the stack the symbol $\textcircled{x}$ at the k -th unmatched left bracket (this can be done, since $\ell_k[x] = 1$). As a result, the automaton leaves the first part of the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ in the state 0 with the stack contents $\alpha = \vec{i}_1 \dots \vec{i}_{k-1} \textcircled{x} i_{k+1} \dots i_{m+1}$.

Now it will be shown, how the automaton $B_{n,s}$ with such a stack gets through the substring $h_{k,x,m} = (\# \gg)^{m-k+1} \# y_{x \bmod n} \gg y_{\lfloor x/n \rfloor} (\# \gg)^{k-1}$. The first segment $(\# \gg)^{m-k+1}$ and the last segment $(\# \gg)^{k-1}$ are easy to pass: the automaton chooses the state 1 at symbols ' $\#$ ' and does not reject while popping the symbols $\vec{i}_t$, for $t \neq k$. The symbol $\textcircled{x}$ is popped out of the stack while reading the substring $\# y_{x \bmod n} \gg y_{\lfloor x/n \rfloor}$. The automaton can go through this substring as follows: it chooses the state $(x \bmod n)$ at the first symbol ' $\#$ ', passes the substring $y_{x \bmod n}$, pops the symbol $\textcircled{x}$ out of the stack at the triple right bracket ' $\gg$ ' and changes the state from $(x \bmod n)$ to $\lfloor x/n \rfloor$. The state $\lfloor x/n \rfloor$ allows the automaton to pass the substring $y_{\lfloor x/n \rfloor}$. Therefore, the automaton $B_{n,s}$ can move through the second part $h_{k,x,m}$, entering it in the state 0 and with stack contents α , and accept, since all states are accepting. Thus, the automaton accepts the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$.

Now let the automaton $B_{n,s}$ accept the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$, it shall be proved that $\ell_k[x] = 1$. Consider an arbitrary accepting computation of the automaton $B_{n,s}$ on this string. First, the automaton reads the substring $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} = <_{\ell_1} w_{R_1} <_{\ell_2} w_{R_2} \dots <_{\ell_m} w_{R_m} <_{\ell_{m+1}}$, pushing onto the stack some string α of length $m+1$. It pushes the k -th symbol of α onto the stack at the left bracket ' $<_{\ell_k}$ ', denote this

stack symbol as c . Next, the automaton reads the second part of the string, which is $h_{k,x,m} = (\# \gg)^{m-k+1} \# y_{x \bmod n} \gg y_{\lfloor \frac{x}{n} \rfloor} (\# \gg)^{k-1}$, and pops the k -th symbol of the string α out of the stack while reading the substring $\# y_{x \bmod n} \gg y_{\lfloor \frac{x}{n} \rfloor}$. Since the automaton rejects at the triple right bracket ' $\gg$ ' if it pops a stack symbol not of the form $\textcircled{z}$, therefore $c = \textcircled{z}$, for some $z \in \{0, \dots, \lfloor \log_2(s-1) \rfloor\}$. Since the triple bracket ' $\gg$ ' is wrapped in substrings $y_{x \bmod n}$ and $y_{\lfloor \frac{x}{n} \rfloor}$, and since each of them allows only one state, the automaton comes to the triple bracket ' $\gg$ ' in the state $(x \bmod n)$ and leaves this bracket in the state $\lfloor \frac{x}{n} \rfloor$. Such a change of a state at the triple bracket ' $\gg$ ' is possible only if the symbol at the top of the stack is $\textcircled{x}$, that is, if $c = \textcircled{x}$. And the symbol c was pushed onto the stack while reading the left bracket ' $<_{\ell_k}$ ', therefore, since $c = \textcircled{x}$, it holds that $\ell_k[x] = 1$. Claim 13 has been proven.

It remains to prove that any deterministic automaton needs a lot of stack symbols to simulate the automaton $B_{n,s}$.

Claim 14. *Let DIDPDA A work over the alphabet Σ and recognize the language $L(B_{n,s})$. Then the automaton A has at least $s(2^{n^2} - 1)$ stack symbols.*

Let N be the number of states in the automaton A , and let M be the number of stack symbols it uses. For the sake of a contradiction, suppose that $M < s(2^{n^2} - 1)$.

Then, for a fixed m , the automaton A can get at most $N \cdot M^{m+1}$ combinations of a state and of stack contents after reading strings of the form $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ with $m+1$ unmatched left brackets. There are exactly $(2^{n^2} - 1)^m \cdot s^{m+1}$ strings of the form $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$, where $R_1, \dots, R_m \subseteq Q \times Q$ are non-empty relations, and $\ell_1, \dots, \ell_{m+1} \in \{0, \dots, s-1\}$ are left bracket numbers. Since $M < s(2^{n^2} - 1)$, for values of m large enough, the number $N \cdot M^{m+1}$ is less than $(2^{n^2} - 1)^m \cdot s^{m+1}$. Let m be a sufficiently large integer. Then, there are two different strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ and $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}}$, such that the deterministic automaton A begins reading both strings in its initial state and leaves both strings in the same state q , with the same stack contents α .

Let us first consider the case when the two strings differ in some relation: $R_k \neq R'_k$, for some k . Then there are two states $i, j \in Q$, such that the pair (i, j) lies in exactly one of the relations R_k and R'_k ; without loss of generality, assume that $(i, j) \in R_k$ and $(i, j) \notin R'_k$. Then, by Claim 12, the nondeterministic automaton $B_{n,s}$ accepts the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} g_{i,j,k,m}$ and rejects the string $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}} g_{i,j,k,m}$. Then, the deterministic automaton A does the same, but it cannot give different answers on these strings, since on both strings it enters the suffix $g_{i,j,k,m}$ in the state q with a stack α . Therefore, the case of $R_k \neq R'_k$ is impossible.

Then, the two strings $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}}$ and $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}}$ differ in some left bracket numbers: $\ell_k \neq \ell'_k$. Then one can choose a bit x , such that the coefficients at 2^x in numbers ℓ_k and ℓ'_k differ: $\ell_k[x] \neq \ell'_k[x]$. Without loss of generality, let $\ell_k[x] = 1$ and $\ell'_k[x] = 0$. Then, by Claim 13, the automaton $B_{n,s}$ accepts the string $f_{R_1, \dots, R_m, \ell_1, \dots, \ell_{m+1}} h_{k,x,m}$ and rejects the string $f_{R'_1, \dots, R'_m, \ell'_1, \dots, \ell'_{m+1}} h_{k,x,m}$. That means that the deterministic automaton A also gives different answers on these strings, but this is impossible, since on both strings it enters the suffix $h_{k,x,m}$ in the same state q with the same stack α . This is a contradiction, and therefore the automaton A has at least $s(2^{n^2} - 1)$ stack symbols.

This finishes the proof of Claim 14 and of Theorem 4.

It has not been explained yet, why Theorem 4 holds in the degenerate case of only one state: $n = 1$. If $s = 1$, one can use Theorem 3. Since $s \leq 2^{n^2}$, the remaining case is $n = 1$ and $s = 2$. The desired NIDPDA $B_{1,2}$ can be constructed as follows. An input alphabet

is: $\Sigma_0 = \emptyset$, $\Sigma_{+1} = \{<, \ll\}$, $\Sigma_{-1} = \{>, \gg\}$. At the single left bracket ' $<$ ' the automaton pushes the symbol 0 onto the stack, at the double left bracket ' $\ll$ ' it pushes 1. At the single right bracket ' $>$ ' the automaton $B_{1,2}$ moves forward if there is 0 at the top of the stack, and rejects, if there is 1; and works vice versa at the double right bracket ' $\gg$ '. The automaton $B_{1,2}$ thus defined accepts a string if and only if each single left bracket ' $<$ ' corresponds to a single right bracket ' $>$ ', and every double left bracket ' $\ll$ ' corresponds to a double bracket ' $\gg$ '.

Let A be a deterministic automaton recognizing the language $L(B_{1,2})$. Then, it has at least 2 states: an accepting state to accept the empty string, and a rejecting one to reject the string $<\gg$. It shall be proved that the automaton A has at least $2(2^{1^2} - 1) = 2$ stack symbols. Assume the opposite, that there is only one stack symbol in the stack alphabet of A . There are two different strings u and u' of the same length over an alphabet $\{<, \ll\}$, such that the automaton A finishes reading them in the same state q . Consider the string v , that matches the string u with right brackets of the same types, so that $B_{1,2}$ accepts the string uv . On the other hand, since $u \neq u'$, there is a left bracket in u' that differs from the left bracket at the same position in u , and therefore this bracket in u' does not correspond to the matching right bracket in v , and the automaton $B_{1,2}$ rejects the string $u'v$. Then, the automaton A also accepts the string uv and rejects the string $u'v$, but it leaves the substrings u and u' in the same state q with the same stack contents, this is a contradiction. $\square$

Note that the size of the stack alphabet in Theorem 4 of $B_{n,s}$ is linear in the number of states and is logarithmic in the number of left brackets. The theorem is stated for a number of left brackets not exceeding 2^{n^2} . As a matter of fact, this lower bound cannot possibly be extended for an unbounded number of brackets, because an n -state nondeterministic automaton has only $2^{\Gamma(n^2)}$ possible transition functions by left brackets, and the simulating automaton could remember this function instead of a bracket. Thus, the restriction on the number of left brackets in the theorem is close to the best possible.

6 Conclusion

There are some open questions on the state complexity of input-driven pushdown automata. The lower bound on the number of states needed for determinization is precise, the input alphabet is fixed and the stack alphabet is two-symbol, so it is hard to improve anything here. However, in the lower bound on the number of stack symbols in a deterministic automaton obtained by NIDPDA determinization, witness automata use a stack alphabet linear in n . Is it possible to obtain the same lower bound using a stack alphabet smaller than linear in n ?

Jirásková and Okhotin [5] investigated the state complexity of operations on DIDPDA. Lower and upper bounds on the number of states for concatenation and for the Kleene star are asymptotically tight: mn^n and $m(n^n + 2^n)$ for concatenation of automata with m and n states, and n^n and $n^n + 2^n + 1$ for the Kleene star. One can try to find the exact state complexity.

Ogawa and Okhotin [8] established asymptotically precise bounds for determinization of event-clock input-driven pushdown automata, but one can try to make these bounds even more precise.

Rose and Okhotin [12] introduced probabilistic input-driven pushdown automata, determinized them and proved the asymptotically precise lower bound on the number of states needed for determinization. It remains an open problem to prove a lower bound on the number of stack symbols in a deterministic automaton needed for determinization of probabilistic input-driven pushdown automata.

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