MACDONALD IDENTITIES, WEYL-KAC DENOMINATOR FORMULAS AND AFFINE GRASSMANNIANS

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ABSTRACT. We expand the affine Weyl denominator formulas as signed q-series of ordinary Weyl characters running over the affine Grassmannian. Here the grading in q coincides with the (dual) atomic length of the root system considered as introduced by Chapelier-Laget and Gerber. Next, we give simple expressions of the atomic lengths in terms of self-conjugate core partitions. This permits in particular to rederive, from the general theory of affine root systems, some results of the second author obtained by case-by-case computations on determinants and the use of particular families of strict partitions. These families are proved to be in simple one-to-one correspondences with the previous core partition model and, through this correspondence, the atomic length on cores equates the rank of the strict partitions considered. Finally, we make explicit some interactions between the affine Grassmannian elements and the Nekrasov-Okounkov type formulas.

1. INTRODUCTION

The Weyl-Kac formula is a corner stone in the representation theory of infinite-dimensional Lie algebras (also called Kac-Moody algebras). It permits to compute the character of a highest weight irreducible representation and naturally generalizes the classical Weyl character formula when restricted to the finite-dimensional simple Lie algebras over the complex number field. The Weyl denominator formula (see Section 3 for the notation) can be written

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha} = \sum_{w \in W_a} \varepsilon(w) e^{w(\rho) - \rho}$$

and reflects the fact that the trivial representation has a character equal to 1. When applied to affine root systems and by putting $q = e^{-\delta}$, it yields a rich class of generating q-series. In particular, as proved by Han [Han10] by using the affine root systems of type A, it permits to rederive the Nekrasov-Okounkov formula

$$(1) \qquad \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} (1 - q^k)^{z-1}$$

where $\mathcal{P}$ is the set of partitions, $\mathcal{H}(\lambda)$ the multiset of hook lengths in the partition λ and z is any fixed complex number. When $z = 0$, one recovers the Euler generating series for the set of partitions. By assuming that $z = n$, with n an integer greater or equal to 2, it is also possible to derive interesting generating series running over the set of n-core partitions, that is the subset of $\mathcal{P}$ containing exactly the partitions with no hook length equal to n.

As mentioned in [CRV18, RW18], there exists a u-analogue of the Nekrasov–Okounkov formula, which is a reformulation of a result due to Dehaye–Han [DH11], using a specialization of the Macdonald identity for type $\tilde{A}_{t-1}$, and Iqbal–Nazir–Raza–Saleem [INRS12], using the refined topological vertex. It can be written as follows:

$$(2) \qquad \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1 - zu^h)(1 - z^{-1}u^h)}{(1 - u^h)^2} = \prod_{k,r \geq 1} \frac{(1 - zu^r q^k)^r (1 - z^{-1} u^r q^k)^r}{(1 - u^{r-1} q^k)^r (1 - u^{r+1} q^k)^r}.$$

Note that taking $u = q^z$ and letting $q \to 1$ in (2) yields (1), although it is not immediate for the product side.

In fact, one can deduce numerous generalizations of the Nekrasov-Okounkov formula from the Weyl-Kac denominator formula, not only in type A, but also for the other affine root systems. This was done in particular by Pétréolle for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$ (see [Pé15]) and the second author for u-deformation in any types of unbounded ranks in [Wah23, Section 5.3] to which we refer the reader for a more complete introduction on the history and developments about the generating series on partitions and their links with the Weyl-Kac denominator formula. A central idea of [Wah23] is to get expansions of the Weyl denominator formula associated to classical affine root systems in terms of the irreducible Weyl characters, that is, the characters of the irreducible highest weight representations corresponding to

the underlying finite-dimensional Lie algebras. Another important aspect of the results proved in [Wah23] is the use, as indexation sets for the previous expansion, of families of distinct partitions (i.e. with distinct parts) which can be regarded as natural analogues of cores partitions for each classical affine root system. Nevertheless, the methods used are based on case-by-case computations and tricky combinatorial manipulations on binary codings of core partitions.

In contrast, our goal here is to fully use the powerful machinery of affine root systems, as developed by Kac and exposed in [Car05] or [Kac90], to first expand the affine Weyl-Kac denominator formula in terms of the ordinary Weyl characters by using a summation over the affine Grassmannian elements. Recall here these are the minimal length elements in the cosets of an affine Weyl group by its classical parabolic subgroup. The set of affine Grassmannian elements (called the affine Grassmannian in the literature) comes with a natural statistics called the "atomic length" introduced and studied (in a more general context) by Chapelier-Laget and Gerber in [CLG23]. In affine type $A_{n-1}^{(1)}$, the affine Grassmannian elements are in one-to-one correspondence with the n-cores and the atomic length is just the number of boxes of the associated Young diagram. For the classical affine root systems (twisted or not) and in types $G_2^{(1)}$ and $D_4^{(3)}$ (here we follow the classification of affine root systems in Kac's book [Kac90]), we show that it is possible to parametrize the affine Grassmannian elements by using particular subsets of self-conjugate $2n$-cores. In each case, we are able to write a simple combinatorial formula for their atomic lengths just in terms of their number of boxes and the number of their boxes (or nodes) of residues 0 or n. In particular, in the non twisted cases, we recover the formulas obtained in [STW23]. Observe nevertheless, that our methods, based on foldings of Dynkin diagrams, are different from those of [STW23] and also well-suited to consider the twisted affine cases. Alternatively, one can compute the atomic lengths by counting boxes in the Young diagrams of some self-conjugate $2n$-cores with weights on boxes depending on their residue (in type A, all the weights are equal to 1). This approach has the advantage of homogeneity but it is not well-adapted for considering at the same time a given family $(X_n^{(a)})_{n \geq 2}$ of Dynkin diagrams (i.e. when the type is fixed and one consider all the possible ranks at once). Also in order to get a natural parametrization of the solutions of certain Diophantine equations, as proposed in [BCLG24], it is important to label the affine Grassmannian elements of the previous types by families of partitions for which the atomic length equates the numbers of boxes. This is what we explain in Section 7 where we show how the families of distinguished self-conjugate $2n$-cores obtained in Section 5 are related to the families of partitions introduced in [Wah23] by simple bijections sending the atomic length on the number of boxes. This also give us the dictionary to connect in Section 8, the Nekrasov-Okounkov type formulas as stated in [Wah23] with the affine Grassmannian elements.

The paper is organized as follows. In Section 2, we recall the background on the affine root systems on which is based the decomposition of the affine Weyl denominator formulas in terms of the generalized Weyl characters that we establish in Section 3. Observe that this formula is essentially equivalent to Theorems 20.03 and 20.04 in Carter's book [Car05]. In Section 4, we simplify the previous decomposition in order to get a decomposition in terms of genuine characters, that is labeled by dominant weights. This also makes naturally appear the affine Grassmannian elements. In Section 5, we use foldings of Dynkin diagram techniques to realize each affine root system in an affine root system of type A. This permits in particular to get the desired formulas for the atomic length in terms of weighted boxes in self-conjugate $2n$-cores. Section 6 presents the combinatorics of partitions and the Littlewood decomposition which are the keys to understand the connections between the different combinatorial models that we use. This permits in Section 7 to describe simple bijections between the model of self-conjugate cores and the model of partitions obtained in [Wah23]. The basic idea here is first to identify each $2n$-core with its so-called $2n$-charge β (a vector in $\mathbb{Z}^{2n}$ whose sum of coordinates is equal to zero), next to express the atomic lengths in each types in terms of β and finally add a number a of zero parts to β in order to be able to identify the atomic lengths obtained as a number of boxes in a relevant partition. When the Dynking diagram considered as no subdiagram of classical type D, it suffices to use $(2n + a)$-cores partitions or some of their half-reductions regarded as distinct partitions. Otherwise the situation becomes more complicated but one can yet reduce the problem to simple families of distinct partitions even if they do not come from $(2n+a)$-cores in general. Finally, in Section 8, we show how the dominant weights appearing in the Weyl characters expansion of the affine Weyl denominator formula of Section 4 and obtained from the affine Grassmannian elements can be computed directly from the previous combinatorics of partitions thanks to the notion of $V_{(g,t)}$-coding introduced in [Wah23]. This yields in particular a more uniform presentation of the generalized Nekrasov-Okounkov formulas.

2. Affine root systems and affine Lie algebras

Let $I = \{0, 1, \ldots, n\}$, $I^* = \{1, 2, \ldots, n\}$ and $A = (a_{i,j})_{(i,j) \in I^2}$ be a generalized Cartan matrix of a classical affine root system. These affine root systems are classified in [Kac90] (see the Table 1 page 54). The matrix A has rank n and there exists a unique vector $v = (a_i)_{i \in I} \in \mathbb{Z}^{n+1}$ with $(a_i)_{i \in I}$ relatively primes and a unique vector $v^\vee = (a_i^\vee)_{i \in I} \in \mathbb{Z}^{n+1}$

with $(a_i^\vee)_{i\in I}$ relatively primes such that $v^\vee \cdot A = A \cdot {}^t v = 0$. Note that we have $a_0 = a_0^\vee = 1$ except in the $A_{2n}^{(2)}$ case for which $a_0 = 2$ and $a_0^\vee = 1$. We refer to [Car05, Kac90] for details and proofs of the results presented in this section.

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A that is:
  (1) $\mathfrak{h}$ is a complex vector space of dimension $n + 2$,
  (2) $\Pi = \{\alpha_0, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_0^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ are linearly independent subsets,
  (3) $a_{i,j} = \langle \alpha_j, \alpha_i^\vee \rangle$ for $0 \le i, j \le n$.
Here, $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denotes the pairing $\langle \alpha, h \rangle = \alpha(h)$. Fix an element $d \in \mathfrak{h}$ such that $\langle \alpha_i, d \rangle = \delta_{0,i}$ for all $i \in I$ so that $\Pi^\vee \cup \{d\}$ is a basis of $\mathfrak{h}$. We denote by $\mathfrak{g}$ the affine Lie algebra associated to this datum and refer to [Kac90] for its definition.

Let $\Lambda_0, \ldots, \Lambda_n \subset \mathfrak{h}^*$ be the such that
$$\Lambda_i(\alpha_j^\vee) = \delta_{i,j} \quad \text{and} \quad \Lambda_i(d) = 0 \text{ for all } i, j \in I.$$
We set $\delta = \sum_{j=0}^n a_j \alpha_j$ so that
$$\delta(\alpha_i^\vee) = \sum_{j=0}^n a_j \alpha_j(\alpha_i^\vee) = \sum_{j=0}^n a_j a_{i,j} = [Av]_i = 0 \quad \text{and} \quad \delta(d) = a_0.$$
Then the family $(\Lambda_0, \ldots, \Lambda_n, \delta)$ is the dual basis of $(\alpha_0^\vee, \ldots, \alpha_n^\vee, d) \subset \mathfrak{h}$ and we have
$$\alpha_j = \sum_{i \in I} \langle \alpha_j, \alpha_i^\vee \rangle \Lambda_i + \langle \alpha_j, d \rangle \delta = \sum_{i \in I} a_{i,j} \Lambda_i \quad \text{for all } j \in I^*.$$

For any $i \in I^*$, set
$$\omega_i = \Lambda_i - \frac{a_i^\vee}{a_0^\vee} \Lambda_0.$$

Lemma 2.1. *For any $j \in I^*$, we have*
$$\alpha_j = \sum_{i \in I^*} a_{i,j} \omega_i$$
that is the weights $\omega_i, i \in I^*$ and the roots $\alpha_i, i \in I^*$ can be regarded as the dominants weight and the simple roots of the finite root system with Cartan matrix $(a_{i,j})_{(i,j) \in I^* \times I^*}$.

Proof. For any $j \in I^*$, we have
$$\alpha_j = \sum_{i \in I} a_{i,j} \Lambda_i = \sum_{i \in I^*} a_{i,j} \omega_i + \left(a_{0,j} + \sum_{i \in I^*} \frac{a_{i,j} a_i^\vee}{a_0^\vee} \right) \Lambda_0 =$$
$$\sum_{i \in I^*} a_{i,j} \omega_i + \frac{1}{a_0^\vee} \left(a_{0,j} a_0^\vee + \sum_{i \in I^*} a_{i,j} a_i^\vee \right) \Lambda_0.$$
But now, by definition of $v^\vee$, we have
$$\sum_{i \in I} a_i^\vee a_{i,j} = 0$$
for any $j \in J^*$ hence the expected result.
Let us set
$$\eta^\vee = \sum_{i \in I} \frac{a_i^\vee}{a_0^\vee}.$$
$\square$

There exists an invariant nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$ uniquely defined by
$$\begin{cases} (\alpha_i^\vee, \alpha_j^\vee) = \langle \alpha_i, \alpha_j^\vee \rangle a_i a_i^{\vee -1} = a_j a_j^{\vee -1} a_{j,i} & \text{for } i, j \in I \\ (\alpha_i^\vee, d) = 0 & \text{for } i \in I^* \\ (\alpha_0^\vee, d) = a_0 & \text{for } i \in I^* \\ (d, d) = 0 \end{cases}$$
It can be checked that $(\alpha_i^\vee, \alpha_j^\vee) = (\alpha_j^\vee, \alpha_i^\vee)$ for all $i, j \in I$. Let ν be the associated map from $\mathfrak{h}$ to its dual:
$$\begin{array}{cccc} \nu : & \mathfrak{h} & \to & \mathfrak{h}^* \\ & h & \mapsto & (h, \cdot) \end{array}$$

The form $(\cdot, \cdot)$ on $\mathfrak{h}$ then induces a form on $\mathfrak{h}^*$ via ν. We still denote this form $(\cdot, \cdot)$. Then we have $\nu(d) = \Lambda_0$, $a_{i,j} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ and $\nu(\alpha_i^\vee) = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$ and

$$
\begin{cases}
(\alpha_i, \alpha_j) = a_i^\vee a_i^{-1} a_{i,j} & \text{for } i, j \in I \\
(\alpha_i, \Lambda_0) = 0 & \text{for } i \in I^* \\
(\alpha_0, \Lambda_0) = a_0^{-1} & \\
(\Lambda_0, \Lambda_0) = 0 & \\
(\Lambda_0, \delta) = 1 &
\end{cases}
$$

Warning : The previous scalar product does not yield in general the Euclidean norm on the real part of $\mathfrak{h}^*$ as it appears in many references (see for example [Bou68]). For example in type C_n the long roots have length equal to 2 and the short roots length 1 which is not the most common convention. We indicate in the table below the relation between the previous norm $\|\beta\|^2 = (\beta, \beta)$ and the Euclidean norm $\|\beta\|_2^2$ used in ([Bou68]).

type	$v = (a_0, \ldots, a_n)$	$v^\vee = (a_0^\vee, \ldots, a_n^\vee)$	Q	$Q^\vee$	η	$\eta^\vee$	$\|\beta\|^2$
$A_n^{(1)}$	1^{n+1}	1^{n+1}	A_n	A_n	$n+1$	$n+1$	$\|\beta\|_2^2$
$B_n^{(1)}$	$1^2 2^{n-1}$	$1^2 2^{n-2} 1$	B_n	C_n	$2n$	$2n-1$	$\|\beta\|_2^2$
$C_n^{(1)}$	$1 2^{n-1} 1$	1^{n+1}	C_n	B_n	$2n$	$n+1$	$\frac{1}{2}\|\beta\|_2^2$
$D_n^{(1)}$	$1^2 2^{n-3} 1^2$	$1^2 2^{n-3} 1^2$	D_n	D_n	$2n-2$	$2n-2$	$\|\beta\|_2^2$
$A_{2n-1}^{(2)}$	$1^2 2^{n-2} 1$	$1^2 2^{n-1}$	C_n	B_n	$2n-1$	$2n$	$\|\beta\|_2^2$
$A_{2n}^{(2)}$	$2^n 1$	$1 2^n$	C_n	B_n	$2n+1$	$2n+1$	$\|\beta\|_2^2$
$D_{n+1}^{(2)}$	1^{n+1}	$1 2^{n-1} 1$	B_n	C_n	$n+1$	$2n$	$2\|\beta\|_2^2$
$G_2^{(1)}$	123	121	G_2	G_2^t	6	4	$\frac{1}{3}\|\beta\|_2^2$
$D_4^{(3)}$	121	123	G_2^t	G_2	4	6	$\|\beta\|_2^2$

There are a different objects associated to the datum $(\mathfrak{h}, \Pi, \Pi^\vee)$, some of them lives in $\mathfrak{h}$ other in $\mathfrak{h}^*$. We will as much as possible use the following convention: we will add the suffix "co" to the name of the object to indicate that it naturally lives in $\mathfrak{h}$ (eventhough we sometime think of it as an element of $\mathfrak{h}^*$) and we will add a superscript $^\vee$ to the notation. For instance, $\alpha_i^\vee$ is called a coroot as it is an element of $\mathfrak{h}$. We now introduce various lattices:

- the coweight lattice: $P_a^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee + \mathbb{Z}d \subset \mathfrak{h}$,
- the weight lattice $P_a = \{\gamma \in \mathfrak{h}^* \mid \gamma(P_a^\vee) \subset \mathbb{Z}\} = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i + \mathbb{Z}\delta$,
- the dominant weights in $\mathfrak{h}^*$ are the elements of $P_a^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\delta$,
- the root lattice $Q_a = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$,
- the coroot lattice $Q_a^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$.

For any $i \in I$, we define the simple reflection s_i on $\mathfrak{h}^*$ by

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i \text{ for any } x \in \mathfrak{h}^*.$$

The affine Weyl group W_a is the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections s_i. Since $(\Lambda_0, \ldots, \Lambda_n, \delta)$ is the dual basis of $(\alpha_0^\vee, \ldots, \alpha_n^\vee, d)$, we have for all $i \in I$

$$
\begin{aligned}
s_i(\delta) &= \delta - \langle \alpha_i^\vee, \delta \rangle \alpha_i = \delta \\
s_i(\Lambda_j) &= \Lambda_j \quad \text{if } i \neq j \\
s_i(\alpha_i) &= -\alpha_i
\end{aligned}
$$

The Weyl group W_a is acting on the weight lattice P_a.

Let $\mathring{A}$ be the matrix obtained from A by deleting the row and the column corresponding to 0. Then it is well-known that $\mathring{A}$ is a Cartan matrix of finite type. Let $\mathring{\mathfrak{h}}^*$ and $\mathring{\mathfrak{h}}$ be the vector spaces spanned by the subset $\mathring{\Pi} = \Pi \setminus \{\alpha_0\}$ and $\mathring{\Pi}^\vee = \Pi \setminus \{\alpha_0^\vee\}$. The root and weight lattices associated to $\mathring{A}$ are $Q = \bigoplus_{i \in I^*} \mathbb{Z}\alpha_i$ and $P = \bigoplus_{i \in I^*} \mathbb{Z}\omega_i$ where we have set $\omega_i = \Lambda_i - a_i^\vee \Lambda_0$ for all $i \in I^*$. We denote by W the finite Weyl group associated to $\mathring{A}$: it is generated by the orthogonal reflections s_i with respect to the hyperplane orthogonal to α_i in $\mathring{\mathfrak{h}}^*$. Finally we set $Q^\vee = \bigoplus_{i \in I^*} \mathbb{Z}\alpha_i^\vee \subset \mathring{\mathfrak{h}}^*$.

Note that the reflection $s_i \in W_a$ for $i \in I^*$ stabilizes $\overset{\circ}{\mathfrak{h}}^*$ so that the Weyl group W can be seen as the subgroup of W_a generated by $(s_i)_{i \in I^*}$.

Let

$$\theta = \delta - a_0 \alpha_0.$$

Warning : The root θ does not always coincide with the highest root of the finite root system $\overset{\circ}{\Pi} = \{\alpha_1, \ldots, \alpha_n\}$. This is for example the case in type $A_{2n-1}^{(2)}$. We refer the reader to Proposition 17.18 in [Car05] for more details.

Let s_θ be the orthogonal reflection with respect to θ defined by $s_\theta(\gamma) = \gamma - \langle \theta^\vee, \gamma \rangle \theta$. For all $\gamma \in \mathfrak{h}^*$ we have

$$s_0 s_\theta(\gamma) = \gamma + (\gamma, \delta)\theta - ((\gamma, \theta) + \frac{1}{2}(\theta, \theta)(\gamma, \delta))\delta.$$

Consider $\beta \in \overset{\circ}{\mathfrak{h}}^*$. We define the map t_β on $\mathfrak{h}^*$ by

$$t_\beta(\gamma) = \gamma + (\gamma, \delta)\beta - ((\gamma, \beta) + \frac{1}{2}(\beta, \beta)(\gamma, \delta))\delta$$

for all γ in $\mathfrak{h}^*$. Note that if $(\gamma, \delta) = 0$ (as it is the case when $\gamma \in \overset{\circ}{\mathfrak{h}}^*$) we get a simpler formula

$$t_\beta(\gamma) = \gamma - (\gamma, \beta)\delta.$$

It is important to observe that t_β doesn't act as a translation on $\overset{\circ}{\mathfrak{h}}^*$. Nevertheless it can be shown that

- $t_\beta \circ t_{\beta'} = t_{\beta + \beta'}$ for all $\beta, \beta' \in \mathfrak{h}^*$,
- $w \circ t_\beta \circ w^{-1} = t_{w(\beta)}$ for all $\beta \in \mathfrak{h}^*$ and $w \in W$,
- $s_0 = t_\theta s_\theta = s_\theta t_{-\theta}$.

These relations tell us that W_a is the semi-direct product of the finite Weyl group W with the lattice that is generated by the W-orbit of $\frac{1}{a_0}\theta$. Let us denote by M^* this lattice. So that we have

$$W_a \simeq W \ltimes M^*.$$

The table below gives the lattice M^* expressed as a sublattice of the underlying finite root system Q thanks to the simple roots (observe that M^* is a sublattice of Q by definition).

type	Q	M^*
$A_n^{(1)}$	A_n	$\bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$
$B_n^{(1)}$	B_n	$\bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \oplus 2\mathbb{Z}\alpha_n$
$C_n^{(1)}$	C_n	$\bigoplus_{i=1}^{n-1} 2\mathbb{Z}\alpha_i \oplus \mathbb{Z}\alpha_n$
$D_n^{(1)}$	D_n	$\bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$
$A_{2n-1}^{(2)}$	C_n	$\bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$
$A_{2n}^{(2)}$	C_n	$\bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \oplus \frac{1}{2}\mathbb{Z}\alpha_n$
$D_{n+1}^{(2)}$	B_n	$\bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$
$G_2^{(1)}$	G_2	$\mathbb{Z}\alpha_1 \oplus 3\mathbb{Z}\alpha_2$
$D_4^{(3)}$	G_2^t	$\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$

3. MACDONALD FORMULA TYPES

We now examine the Macdonald formulas from the point of view of the Weyl-Kac denominator formula. The computations of this section are essentially the same as in Section 20 of Carter's book [Car05]. For any affine root system, the Weyl denominator formula can be written

$$(3) \qquad \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha} = \sum_{w \in W_a} \varepsilon(w) e^{w(\rho) - \rho}$$

where ρ is any element of P_a such that $\langle\rho,\alpha_i^\vee\rangle = 1$ for any $i \in I$. Here the numbers m_α are the multiplicity of the positive roots of the affine root system considered. Contrary to the classical Weyl denominator formula (for a non affine finite root system), ρ is not unique here and cannot be defined as the half sum of the positive roots. Nevertheless, with the previous notation, we can take

$$\rho = \sum_{i\in I} \Lambda_i = \eta^\vee \Lambda_0 + \mathring{\rho}$$

where

$$\mathring{\rho} = \frac{1}{2} \sum_{\alpha\in\mathring{R}_+} \alpha = \sum_{i\in I^*} \omega_i$$

is the half sum of the positive roots associated to the underlying finite root system. Recall the following notation: for any classical weight $\gamma \in \mathring{P}$

$$a_\gamma = \sum_{u\in W} \varepsilon(w)e^{w(\gamma)}.$$

In particular

$$a_{\mathring{\rho}} = \sum_{u\in W} \varepsilon(w)e^{w(\mathring{\rho})} = e^{\mathring{\rho}} \prod_{\alpha\in\mathring{R}_+} (1 - e^{-\alpha})$$

from the Weyl denominator formula for the non affine root system. From the (usual, that is non affine) Weyl Kac character formula, we can define the virtual characters

$$s_\gamma = \frac{a_{\gamma+\mathring{\rho}}}{a_{\mathring{\rho}}}, \gamma \in \mathring{P}.$$

Then $s_\gamma = 0$ or there exists a unique dominant weight $\lambda \in P_+$ and an element $u \in W$ such that

(4) $$s_\gamma = \varepsilon(u)s_\lambda$$

with $\lambda = u(\gamma + \mathring{\rho}) - \mathring{\rho} = u \circ \gamma$ (the so-called "dot"-action). Recall also that each element w of the affine Weyl group W_a admits a unique decomposition on the form

$$w = t_\gamma u \text{ with } \gamma \in M^* \text{ and } u \in W.$$

We will now compute the right-hand side of (3) by using the previous decomposition of the elements in W_a. In fact, we will compute the convenient renormalization below:

$$\Delta = \prod_{\alpha\in R_+\backslash\mathring{R}_+} (1 - e^{-\alpha})^{m_\alpha} = \frac{1}{e^{-\mathring{\rho}}a_{\mathring{\rho}}} \sum_{w\in W_a} \varepsilon(w)e^{w(\rho)-\rho}.$$

First, fix $w = t_\gamma u$ in W_a. We have

$$w(\rho) - \rho = t_\gamma u(\eta^\vee\Lambda_0 + \mathring{\rho}) - \eta^\vee\Lambda_0 - \mathring{\rho} = t_\gamma\left(\eta^\vee\Lambda_0 + u(\mathring{\rho})\right) - \eta^\vee\Lambda_0 - \mathring{\rho} = \eta^\vee\left(t_\gamma(\Lambda_0) - \Lambda_0\right) + t_\gamma u(\mathring{\rho}) - \mathring{\rho}$$

where the first equality uses the fact that $u(\Lambda_0) = \Lambda_0$ for any element u in W the finite Weyl group. We can now apply the formula recalled in the previous section giving the action of a translation t_γ on the element of P_a. We get

$$t_\gamma(\Lambda_0) - \Lambda_0 = \Lambda_0 + \gamma - \frac{1}{2}\|\gamma\|^2\delta - \Lambda_0 = \gamma - \frac{1}{2}\|\gamma\|^2\delta$$

and also

$$t_\gamma u(\mathring{\rho}) - \mathring{\rho} = u(\mathring{\rho}) - (\gamma, u(\mathring{\rho}))\delta - \mathring{\rho} = u(\mathring{\rho}) - \mathring{\rho} - (u^{-1}(\gamma), \mathring{\rho})\delta.$$

This yields

$$w(\rho) - \rho = \eta^\vee\gamma + u(\mathring{\rho}) - \mathring{\rho} - \left(\frac{\eta^\vee}{2}\|\gamma\|^2 + (u^{-1}(\gamma), \mathring{\rho})\right)\delta$$

and by setting $q = e^{-\delta}$

$$\Delta = \frac{1}{e^{-\mathring{\rho}}a_{\mathring{\rho}}} \sum_{\gamma\in M^*} \sum_{u\in W} \varepsilon(t_\gamma)\varepsilon(u)q^{\frac{\eta^\vee}{2}\|\gamma\|^2+(u^{-1}(\gamma),\mathring{\rho})}e^{\eta^\vee\gamma+u(\mathring{\rho})-\mathring{\rho}}.$$

Now, since the classical root lattice M^* is W-invariant, we can set $\beta = u^{-1}(\gamma) \in M^*$ in the previous expression and obtain

$$\Delta = \frac{1}{e^{-\mathring{\rho}}a_{\mathring{\rho}}} \sum_{\beta\in M^*} \sum_{u\in W} \varepsilon(t_\beta)\varepsilon(u)q^{\frac{\eta^\vee}{2}\|\beta\|^2+(\beta,\mathring{\rho})}e^{u(\eta^\vee\beta)+u(\mathring{\rho})-\mathring{\rho}}$$

because $\varepsilon(t_\beta) = \varepsilon(t_\gamma)$ and $\|\beta\|^2 = \|\gamma\|^2$. This can be rewritten

$$(5) \quad \Delta = \frac{1}{e^{-\mathring{\rho}}a_{\mathring{\rho}}} \sum_{\beta \in M^*} \varepsilon(t_\beta) q^{\frac{\eta^\vee}{2}\|\beta\|^2 + (\beta,\mathring{\rho})} \sum_{u \in W} \varepsilon(u) e^{u(\eta^\vee\beta+\mathring{\rho})-\mathring{\rho}} =$$

$$\sum_{\beta \in M^*} \varepsilon(t_\beta) q^{\frac{\eta^\vee}{2}\|\beta\|^2 + (\beta,\mathring{\rho})} \frac{a_{\eta^\vee\beta}}{a_{\mathring{\rho}}} = \sum_{\beta \in M^*} \varepsilon(t_\beta) q^{\frac{\eta^\vee}{2}\|\beta\|^2 + (\beta,\mathring{\rho})} s_{\eta^\vee\beta}.$$

In this last formula, we use an indexation by the affine translations instead of the affine Grassmannians elements (which are the minimal length representatives of the cosets in W_a/W). Observe that each coset in W_a/W also contains a unique translation t_β but it is not of minimal length in general. In fact, we will go further in the following section by applying the straightening rules for the virtual characters (4) and also associating to each translation β, the unique affine Grassmannian element $c(\beta)$ such that

$$c(\beta) \text{ is of minimal length in the coset } t_\beta W.$$

Observe that $t_\beta(\Lambda_0) = c(\beta)(\Lambda_0)$. In the following, it will be convenient to rewrite our formula (5) by changing β into $-\beta$ (which is clearly possible since M^* is a lattice). Since $\|\beta\|^2 = \|-\beta\|^2$ and $\varepsilon(t_{-\beta}) = \varepsilon((t_\beta)^{-1}) = \varepsilon(t_\beta)$, this gives the expression

$$(6) \qquad \Delta = \sum_{\beta \in M^*} \varepsilon(t_\beta) q^{\frac{\eta^\vee}{2}\|\beta\|^2 - (\beta,\mathring{\rho})} s_{-\eta^\vee\beta}.$$

4. CONNECTION WITH THE AFFINE GRASSMANNIAN

As already mentioned the affine Grassmannian is the set of mimimal length elements in the left cosets of W_a/W. We shall denote by W_a^0 these affine Grassmannian elements. Let us recall that any $w = t_\beta u$ in W_a with $\beta \in M^*$ and $u \in W$ can be also written

$$w = t_\beta u = u(u^{-1} t_\beta u) = u t_{u^{-1}(\beta)}.$$

Therefore, we can use decompositions of w of both forms $t_\beta u$ or $u t_\gamma$. The second one is particularly well-adapted for computing the length of w since we have the formula

$$\ell(u t_\gamma) = \sum_{\alpha \in \mathring{R}_+} |(\gamma,\alpha) + \chi(u(\alpha)| \text{ with } \chi(\alpha) = \left\{ \begin{array}{l} 0 \text{ on } \mathring{R}_+ \\ 1 \text{ on } - \mathring{R}_+ \end{array} \right. .$$

From this, it is not difficult to check that a translation t_γ belongs to W_a^0 if and only if $\gamma \in M^* \cap (-P_+)$, that is is an antidominant weight for the finite root system. Indeed, we then have $(\gamma,\alpha) \leq 0$ whereas $\chi(w(\alpha) \geq 0$. We can in fact completely characterize the elements in W_a^0 by the following lemma (which generalizes the previous observation on the translations in W_a^0).

Lemma 4.1. *The element w belongs to W_a^0 if and only if its admits a decomposition $w = u t_\nu$ such that $\nu \in M^* \cap (-P_+)$ and $u \in W$ is of minimal length in a left coset of W/W_ν where W_ν is the stabilizer of ν under the action of the finite Weyl group.*

Now let us consider a translation t_β with $\beta \in M^*$ as in (6). In general, we do not have t_β in W_a^0 but this can be corrected. To do this, observe that $W \cdot \beta$, the orbit of β under the action of the finite Weyl group W, intersects $-P_+$ in a unique antidominant weight ν with $-\nu \in P_+$. In general, one rather uses the intersection with P_+, this works similarly with $-P_+$ just by composing with w_0, the maximal length element in W. Write as usual W^ν for the set of minimal length elements in the cosets of W/W_ν. Then, there exists a unique $u \in W^\nu$ such that $\beta = u(\nu)$. We have then

$$t_\beta = (u t_\nu) u^{-1} = c u^{-1}$$

with $c = u t_\nu \in W_a^0$ and $u^{-1} \in W$. Since $\beta = u(\nu)$ and $t_\beta = (u t_\nu) u^{-1}$, we get

$$\|\beta\|^2 = \|\nu\|^2 \text{ and } \varepsilon(t_\beta) = \varepsilon(t_\nu).$$

We can also compute the analogue of the number of boxes in a core from

$$\Lambda_0 - c(\Lambda_0) = \Lambda_0 - u\left(\Lambda_0 + \nu - \frac{1}{2}\|\nu\|^2 \delta\right) = \Lambda_0 - \Lambda_0 - u(\nu) + \frac{1}{2}\|\nu\|^2 \delta = \frac{1}{2}\|\beta\|^2 \delta - \beta$$

by setting

$$L^\vee(c) = (\Lambda_0 - c(\Lambda_0), \rho) = \eta^\vee(\Lambda_0 - ut_\nu(\Lambda_0), \Lambda_0) + (\Lambda_0 - ut_\nu(\Lambda_0), \mathring{\rho}) =$$

$$\eta^\vee\left(\frac{1}{2}\|\beta\|^2\delta - \beta, \Lambda_0\right) + \left(\frac{1}{2}\|\beta\|^2\delta - \beta, \mathring{\rho}\right)$$

and by using the equalities $(\Lambda_0, \delta) = 1$, $(\delta, \mathring{\rho}) = 0$ and $(\Lambda_0, \beta) = 0$, we obtain

$$L^\vee(c) = \frac{\eta^\vee}{2}\|\beta\|^2 - (\beta, \mathring{\rho}) \text{ with } c = ut_\nu = t_\beta u.$$

Remark 4.2. Observe that the definition of $L^\vee(c)$ is very close to the so-called "Atomic Length" introduced in by Chapelier-Laget and Gerber and generalizing the number of boxes in a core partition (see [CLG23] Corollary 8.2) which satisfies

$$L(c) = \frac{\eta}{2}\|\beta\|^2 - (\beta, \mathring{\rho}^\vee).$$

In particular $L(c)$ counts the number of simple roots appearing in the decomposition of the weight $\Lambda_0 - c(\Lambda_0)$ on the basis of simple roots. Therefore we have $L^\vee(c)) = L(c)$ in the simply laced cases. More generally, one can observe that the lattice M^* and the norm $\|\cdot\|^2$ are the same in type $B_n^{(1)}$ and $A_{2n-1}^{(2)} = (B_n^{(1)})^t$. For types $C_n^{(1)}$ and $D_{n+1}^{(2)} = (C_n^{(1)})^t$, we only have $M^*_{C_n^{(1)}} = 2M^*_{D_{n+1}^{(2)}}$ but $\|\cdot\|^2_{C_n^{(1)}} = \frac{1}{4}\|\cdot\|^2_{D_{n+1}^{(2)}}$. This permits to conclude that the statistics L and $L^\vee$ take exactly the same values up to transposition of the Cartan matrices.

To rewrite Δ in terms of the elements in W_a^0, observe that the previous construction gives a bijection

(7)
$$\begin{cases} M^* \to W_a^0 \\ \beta \longmapsto c = ut_\nu = t_\beta u \end{cases}$$

such that $t_\beta = ut_\nu u^{-1} = cu^{-1}$. We so have $\varepsilon(t_\beta) = \varepsilon(c)\varepsilon(u^{-1})$. Now for each term in the sum (6), we get by the previous computations and remarks the equality

$$\varepsilon(t_\beta)q^{\frac{\eta^\vee}{2}\|\beta\|^2 - (\beta, \mathring{\rho})}s_{-\eta^\vee\beta} = \varepsilon(c)\varepsilon(u^{-1})q^{|c|}s_{-\eta^\vee u(\nu)} = \varepsilon(c)q^{|c|}s_{u^{-1}(-\eta^\vee u(\nu) + \mathring{\rho}) - \mathring{\rho}} = \varepsilon(c)q^{|c|}s_{-\eta^\vee\nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}}$$

where $-\eta^\vee\nu \in P_+$. This gives the theorem below.

Theorem 4.3. *With the previous notation, the renormalized Kac-Weyl denominator formula can be written*

$$\Delta = \sum_{c \in W_a^0} \varepsilon(c)q^{L^\vee(c)}s_{-\eta^\vee\nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}}$$

where we set $c = ut_\nu$ with $\nu \in M^* \cap (-P_+)$ and $u \in W^\nu$ for any element of the affine Grassmannian W_a^0. Moreover, each weight $-\eta^\vee\nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}$ belongs to P_+, the set of dominants weights for $\mathring{A}$.

It remains to prove the last claim of the theorem. First observe that we always have $M^* \subset P$ since $M^* \subset Q \subset P$. Thus $-\eta^\vee\nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}$ belongs to P and it suffices to prove that it is dominant. This is done in the lemma below.

Lemma 4.4. *For any dominant weight $\lambda \in M^*$ and any $u \in W^\lambda$ (i.e. of minimal length in the cosets of W/W_λ), the weight*

$$\eta^\vee\lambda + (u^{-1}(\mathring{\rho}) - \mathring{\rho})$$

is dominant.

Proof. We need to prove that

$$(\eta^\vee\lambda + (u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) = \eta^\vee(\lambda, \alpha_i) + (u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) \geq 0$$

for any $i \in I^*$. Observe that we have

$$(u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) = (u^{-1}(\mathring{\rho})), \alpha_i) - (\mathring{\rho}, \alpha_i) = (\mathring{\rho}, u(\alpha_i)) - \frac{1}{2}\|\alpha_i\|^2.$$

If $u(\alpha_i) \in R_+$, then $(\mathring{\rho}, u(\alpha_i)) \geq \frac{1}{2}\|\alpha_i\|^2$ and therefore $(u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) \geq 0$. We are done because λ is dominant and thus $\eta^\vee(\lambda, \alpha_i) \geq 0$. Now assume $u(\alpha_i) \in -R_+$ is a negative root. Then, we know that there exists a reduced expression of u ending by s_i, that is of the form $u = u's_i$ with $\ell(u) = \ell(u') + 1$. Since $u \in W^\lambda$, this implies that $(\lambda, \alpha_i) \geq p$ where $p = 1$ in all affine types except in types $C_n^{(1)}$ and $G_2^{(1)}$ where $p = 2$ and $p = 3$, respectively (because

$\lambda \in M^* \cap P_+$). Indeed, we would have otherwise $u(\lambda) = u's_i(\lambda) = u'$ and u would not be of minimal of length. We thus have

$$(\eta^\vee \lambda + (u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) \geq p\eta^\vee - (\mathring{\rho}, -u(\alpha_i)) + \frac{1}{2}\|\alpha_i\|^2 \geq 0$$

where $\alpha \in R_+$. One can then check that for any $\alpha \in R_+$, we have $(\mathring{\rho}, -u(\alpha_i)) \leq p\eta^\vee$ and thus the desired inequality

$$(\eta^\vee \lambda + (u^{-1}(\mathring{\rho}) - \mathring{\rho}), \alpha_i) \geq 0.$$

$\square$

Remark 4.5. Recall that the Euler product is $\prod_{k \geq 1} (1 - q^k)$. The Nekrasov–Okounkov formula (1) gives an expansion of powers of the Euler product for any complex z. The milestone in Han's proof to derive (1) is a specialization of the Macdonald identity. For every semisimple Lie algebra $\mathfrak{g}$ of rank n, Macdonald [Mac72] proves that by setting $q = e^{-\delta}$ and $e^{\alpha_i} \mapsto \pm 1$ for $1 \leq i \leq n$, the left-hand side of Theorem 4.3 is equal to

$$\delta(q) = \prod_{k \geq 1} \left(1 - q^k\right)^{\dim \mathfrak{g}}.$$

Moreover a variation of the Euler product, called the Dedekind η-function is defined by

$$\eta(q) = q^{1/24} \prod_{k \geq 1} \left(1 - q^k\right)$$

so that $q^{\dim \mathfrak{g}/24}\delta(q) = \eta(q)^{\dim \mathfrak{g}}$. The term $q^{\dim \mathfrak{g}/24}$ arises from the "strange formula" ([FdV69][p.243]): $\Phi_R(\rho, \rho) = \dim \mathfrak{g}/24$, where Φ_R is the scalar product on R induced by the Killing form on $\mathfrak{g}$. Therefore one so gets an explicit connection between powers of Dedekind η-function and weights $-\eta^\vee \nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}$ associated with elements of the affine Grassmannian, which we do not detail here.

5. ATOMIC LENGTH AND CORES

The goal of this section is to give a simple description of the affine Grassmannian elements of the previous classical affine root systems (twisted or not) with underlying finite root system of rank n in terms of families of $2n$-core partitions. We will also consider the affine root system of type $G_2^{(1)}$ and $D_4^{(3)}$ where our description will use 6-core partitions. For each of the previous affine types, we provide an embedding of its associated root and weight lattices in a weight lattice of affine type A. Incidentally, this yields an embedding of the corresponding affine Weyl group in a group of affine permutations (i.e. a Weyl group of affine type A). Similar results for the previous non twisted affine types also appeared in [STW23] where they are obtained by different techniques. We give below the labelling of the affine Dynkin diagrams that we shall use in the following.

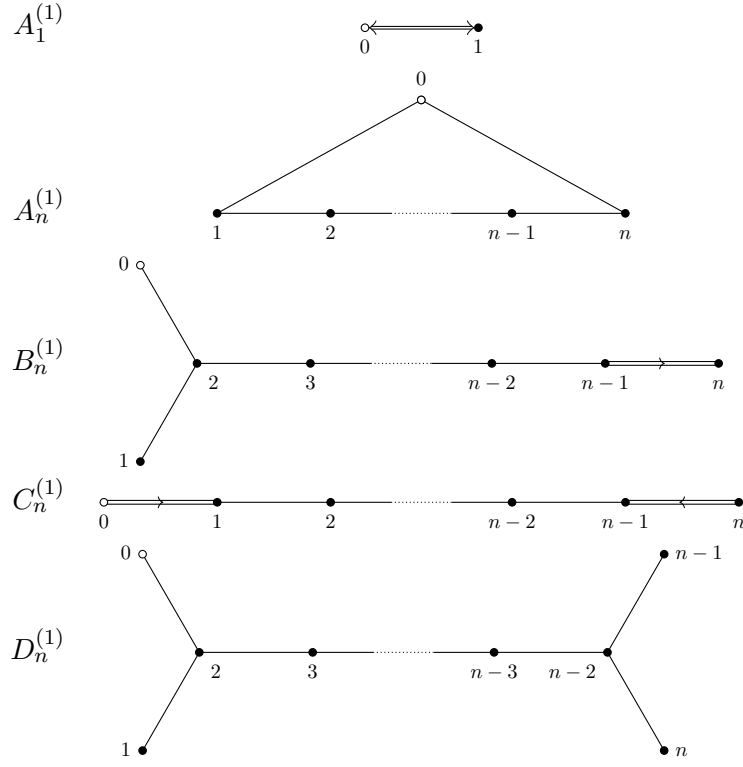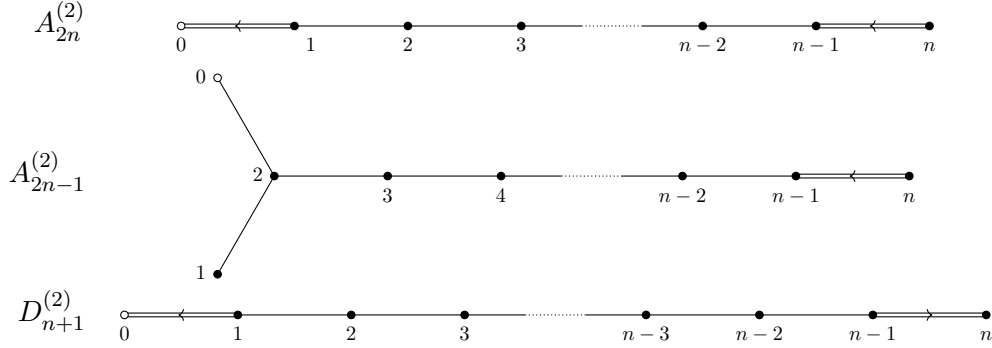FIGURE 1. The Dynkin diagrams of the extended simple root systems

FIGURE 2. The Dynkin diagrams of the twisted simple root systems

5.1. **Affine type A and cores partitions.** Consider an affine root system of type $A_{n-1}^{(1)}$.

A *partition* λ of a positive integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The λ_i's are the *parts* of λ, the number ℓ of parts being the *length* of λ, denoted by $\ell(\lambda)$. For convenience, set $\lambda_i = 0$ for all $i > \ell(\lambda)$.

Each partition can be represented by its Ferrers diagram, which consists in a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. The *Durfee square* of λ is the maximal square fitting in the Ferrers diagram. Its diagonal, denoted by Δ, will be called the main diagonal of λ. Its size will be denoted $d = d_\lambda := \max(s|\lambda_s \geq s)$. The partition $\lambda^{\mathrm{tr}} = (\lambda_1^{\mathrm{tr}}, \lambda_2^{\mathrm{tr}}, \ldots, \lambda_{\lambda_1}^{\mathrm{tr}})$ is the *conjugate* of λ, where λ_j^{tr} denotes the number of boxes in the column j.

For each box v in the Ferrers diagram of a partition λ (for short we will say for each box v in λ), one defines the *arm-length* (respectively *leg-length*) as the number of boxes in the same row (respectively in the same column) as v strictly to the right of (respectively strictly below) the box v. The *hook length* of v, denoted by $h_v(\lambda)$ or h_v, is the number of boxes u such that either $u = v$, or u lies strictly below (respectively to the right) of v in the same column (respectively row).

The *hook lengths multiset* of λ, denoted by $\mathcal{H}(\lambda)$, is the multiset of all hook lengths of λ. For any positive integer n, the multiset of all hook lengths that are congruent to 0 (mod n) is denoted by $\mathcal{H}_n(\lambda)$. Note that $\mathcal{H}(\lambda) = \mathcal{H}_1(\lambda)$. A partition ω is a n-core if $n \notin \mathcal{H}(\omega)$ or equivalent, thanks to the Littlewood decomposition introduced in Section

6, if $\mathcal{H}_n(\omega) = \emptyset$. For example, the only 2-cores are the "staircase" partitions $(k, k-1, \ldots, 1)$, where k is any positive integer.

Given a partition λ, an addable (resp. removable) node is a node b such that $\lambda \sqcup \{b\}$ (resp. $\lambda \setminus \{b\}$) is yet the diagram of a partition. The content of a node b appearing in λ at the intersection of its j-th column and its i-th row is defined as $c(b) = j - i$. The residue of b is the value of $c(b)$ modulo n, that is $r(b) = c(b) \bmod n$. The nodes with the same residue i are called the i-nodes of λ. It is then easy to check that for any $i = 0, \ldots, n-1$, a n-core cannot contain a mix of addable (A) and removable (R) i-nodes: they are either i-nodes A, or i-nodes R.

Let us denote by $\mathcal{C}_n$ the set of n-cores and write $\widetilde{\mathfrak{S}}_n$ for the affine Weyl group of type $A_{n-1}^{(1)}$. The group $\widetilde{\mathfrak{S}}_n$ is a Coxeter group with simple generating reflections $s_0, \ldots, s_{n-1}$. There is classical action of $\widetilde{\mathfrak{S}}_n$ on the set $\mathcal{C}_n$: for any $i = 0, \ldots, n-1$ and any $c \in \mathcal{C}_n$, the core $s_i \cdot c$ is obtained by removing all the addable i-nodes of c when c contains only removable i-nodes and adding all the possible addable i-nodes in c when c does not contain any addable i-node. Observe that $\mathcal{C}_n$ is stable by the transposition (or conjugation) tr operation on partitions (exchanging the rows and the columns in the Young diagrams). It is in fact easy to see that for any core $c \in \mathcal{C}_n$ such that $c = s_{i_k} \cdots s_{i_1} \cdot \emptyset$, we have $c^{\mathrm{tr}} = s_{n-i_k} \cdots s_{n-i_1} \cdot \emptyset$.

Example 5.1. Assume $n = 3$ and consider the 3-core $c = (4, 2, 1, 1)$. We give below the action of the simple reflexions s_0 and s_1 on c (the number indicated are the residues of the nodes).

A partition μ is *self-conjugate* if $\mu = \mu^{\mathrm{tr}}$ or equivalently its Ferrers diagram is symmetric along the main diagonal. Let $\mathcal{SC}$ be the set of self-conjugate partitions and $\mathcal{C}_n^s$ denote the set of self-conjugate n-cores.

Proposition 5.2. *We have $\mathcal{C}_n = \widetilde{\mathfrak{S}}_n \cdot \emptyset$, that is the previous action is transitive on $\mathcal{C}_n$. Moreover, the stabilizer of the empty node $\emptyset$ is the symmetric group $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ and the set $\mathcal{C}_n$ gives a parametrization of the affine Grassmannian elements of type $A_{n-1}^{(1)}$.*

Proof. It follows easily from the fact that any n-core partition has at least on removable i-node for an integer $i \in I$. Also it is clear that the stabilizer of the fundamental weight Λ_0 is the symmetric group $\mathfrak{S}_n$. $\square$

For simplicity, we will slightly abuse the notation and identify each affine Grassmannian elements with its corresponding n-core. In this case, for any $c \in W_a^0 = \mathcal{C}_n$, we get

$$(8) \qquad \Lambda_0 - c(\Lambda_0) = \sum_{i=0}^{n-1} a_i(c)\alpha_i$$

where $a_i(c)$ is the number of i-nodes in the core c. Therefore, the atomic length of c satisfies

$$L(c) = \sum_{i=0}^{n-1} a_i(c) = |c|$$

where $|c|$ is just the number of nodes in the core c. In the following paragraphs, we will see that it is possible to get a similar combinatorial interpretation for any of the previous affine root systems. For any integer $N \geq 2$, we will denote by $\alpha_0, \ldots, \alpha_{N-1}$ and $\Lambda_0, \ldots, \Lambda_N$ (without superscript) the simple roots and the fundamental weights of the affine root system of type $A_{N-1}^{(1)}$. We will then express the simple roots $\alpha_i^{X_n^{(a)}}$ of the affine root system $X_n^{(a)}$ in terms of the α_j's, and similarly the fundamental weights $\Lambda_i^{X_n^{(a)}}$ in terms of the Λ_j's. We will also decompose the simple reflections generating each affine Weyl group $W_{X_n^{(a)}}$ in terms of $s_0, \ldots, s_{N-1}$, the simple reflections generating $\widetilde{\mathfrak{S}}_N$. The same convention will be used for the atomic lengths: we will express $L_{X_n^{(a)}}$ in terms of L the atomic length of affine type A.

subsectionType $C_n^{(1)}$

We can realize the affine root system of type $C_n^{(1)}$ as the subsystem of the root system of type $A_{2n-1}^{(1)}$ such that

$$(9) \qquad \begin{cases} \alpha_i^{C_n^{(1)}} = \alpha_i + \alpha_{2n-i}, i = 1, \ldots, n-1 \\ \alpha_n^{C_n^{(1)}} = 2\alpha_n \\ \alpha_0^{C_n^{(1)}} = 2\alpha_0 \end{cases} \qquad \begin{cases} \Lambda_i^{C_n^{(1)}} = \Lambda_i + \Lambda_{2n-i}, i = 1, \ldots, n-1 \\ \Lambda_n^{C_n^{(1)}} = 2\Lambda_n \\ \Lambda_0^{C_n^{(1)}} = 2\Lambda_0 \end{cases}$$

Indeed, one can then check that the previous relations are compatible with the Dynkin diagram of type $C_n^{(1)}$. The affine Weyl group $W_{C_n^{(1)}}$ can then be seen as the subgroup of $\widetilde{\mathfrak{S}}_{2n}$ such that

$$W_{C_n^{(1)}} = \langle s_i^{C_n^{(1)}}, i = 0, \ldots, n \rangle \text{ with } \begin{cases} s_i^{C_n^{(1)}} = s_i s_{2n-i}, i = 1, \ldots, n-1 \\ s_n^{C_n^{(1)}} = s_n \\ s_0^{C_n^{(1)}} = s_0 \end{cases}$$

The following Lemma is easy to prove. Set

$$\mathcal{C}_{2n}^s = \{c \in \mathcal{C}_{2n}^s \mid c^{\mathrm{tr}} = c\}$$

Lemma 5.3. *We have* $W_{C_n^{(1)}} \cdot \emptyset = \mathcal{C}_{2n}^s$, *that is the affine Grassmannian elements of type* $C_n^{(1)}$ *are parametrized by the self conjugate $2n$-cores.*

Proof. One first check that $\mathcal{C}_{2n}^s$ is stable under the action of $W_{C_n^{(1)}}$. Next, it suffices to observe that for any nonempty $c \in \mathcal{C}_{2n}^s$, there exists at least an integer $i = 0, \ldots, n$ such that all the i and the $2n-i$ nodes of c are removable because c is self conjugate. This implies that $s_i^{C_n^{(1)}} \cdot c = c'$ where $c' \neq c$ belongs to $\mathcal{C}_{2n}^s$ and has a number of nodes strictly less than c. $\qquad\square$

We will identify the elements of the affine Grassmannian $W_{C_n^{(1)}}^0$ with the self conjugate $2n$-cores in $\mathcal{C}_{2n}^s$. Recall also that the atomic length $L(c)$ in affine type A then counts the number of boxes in the Young diagram of c. Then for each c in $\mathcal{C}_{2n}^s$, we can write

$$\Lambda_0^{C_n^{(1)}} - c(\Lambda_0^{C_n^{(1)}}) = \sum_{i=0}^n a_i^{C_n^{(1)}}(c)\alpha_i^{C_n^{(1)}}$$

which gives by using (9)

$$2\Lambda_0 - c(2\Lambda_0) \;=\; 2a_0^{C_n^{(1)}}(c)\alpha_0 + 2a_n^{C_n^{(1)}}(c)\alpha_n + \sum_{i=1}^{n-1} a_i^{C_n^{(1)}}(c)(\alpha_i + \alpha_{2n-i}) \text{ i.e.}$$

$$\Lambda_0 - c(\Lambda_0) \;=\; a_0^{C_n^{(1)}}(c)\alpha_0 + a_n^{C_n^{(1)}}(c)\alpha_n + \sum_{i=1}^{n-1}\frac{1}{2}a_i^{C_n^{(1)}}(c)(\alpha_i + \alpha_{2n-i}).$$

By comparing with (8), we get

$$\begin{cases} a_i^{C_n^{(1)}}(c) = 2a_i(c) = 2a_{2n-i}(c) \text{ for } i = 1, \ldots, n-1 \\ a_n^{C_n^{(1)}}(c) = a_n(c) \text{ and } a_0^{C_n^{(1)}}(c) = a_0(c) \end{cases}$$

The following proposition is obtained by interpreting $L_{C_n^{(1)}}(c)$ as the number of simple roots of type $C_n^{(1)}$ in the previous decomposition of $\Lambda_0 - c(\Lambda_0)$.

Proposition 5.4. *For any* $c \in W_{C_n^{(1)}}^0 = \mathcal{C}_{2n}^s$, *we have*

$$(10) \qquad\qquad\qquad\qquad L_{C_n^{(1)}}(c) = L(c)$$

that is $L_{C_n^{(1)}}(c)$ *is equal to the number of boxes in the self conjugate $2n$-core c.*

5.2. Type $D_{n+1}^{(2)}$. This time, we can realize the affine root system of type $D_{n+1}^{(2)}$ as the subsystem of the root system of type $A_{2n-1}^{(1)}$ such that

$$(11) \qquad \begin{cases} \alpha_i^{D_{n+1}^{(2)}} = \alpha_i + \alpha_{2n-i}, i = 1, \ldots, n-1 \\ \alpha_n^{D_{n+1}^{(2)}} = \alpha_n \\ \alpha_0^{D_{n+1}^{(2)}} = \alpha_0 \end{cases} \qquad \begin{cases} \Lambda_i^{D_{n+1}^{(2)}} = \Lambda_i + \Lambda_{2n-i}, i = 1, \ldots, n-1 \\ \Lambda_n^{D_{n+1}^{(2)}} = \Lambda_n \\ \Lambda_0^{D_{n+1}^{(2)}} = \Lambda_0 \end{cases}$$

The affine Weyl group $W_{D_{n+1}^{(2)}}$ is equal to $W_{C_n^{(1)}}$ and we yet have $W_{D_{n+1}^{(2)}} \cdot \emptyset = \mathcal{C}_{2n}^s$. For each c in $\mathcal{C}_{2n}^s$, we can write

$$\Lambda_0^{D_{n+1}^{(2)}} - c(\Lambda_0^{D_{n+1}^{(2)}}) = \sum_{i=0}^n a_i^{D_{n+1}^{(2)}}(c)\alpha_i^{D_{n+1}^{(2)}}$$

which gives by using (11)

$$\Lambda_0 - c(\Lambda_0) = a_0^{D_{n+1}^{(2)}}(c)\alpha_0 + a_n^{D_{n+1}^{(2)}}(c)\alpha_n + \sum_{i=1}^{n-1} a_i^{D_{n+1}^{(2)}}(c)(\alpha_i + \alpha_{2n-i}) \ .$$

By comparing with (8), we get

$$(12) \qquad \begin{cases} a_i^{D_{n+1}^{(2)}}(c) = a_i(c) = a_{2n-i}(c) \text{ for } i = 1, \ldots, n-1 \\ a_n^{D_{n+1}^{(2)}}(c) = a_n(c) \text{ and } a_0^{D_{n+1}^{(2)}}(c) = a_0(c) \end{cases}$$

Proposition 5.5. *For any* $c \in W_{D_{n+1}^{(2)}}^0 = \mathcal{C}_{2n}^s$, *we have*

$$L_{D_{n+1}^{(2)}}(c) = \frac{1}{2}(L(c) + a_0(c) + a_n(c)).$$

5.3. Type $A_{2n}^{(2)}$. We can realize the affine root system of type $A_{2n}^{(2)}$ as the subsystem of the root system of type $A_{2n-1}^{(1)}$ such that

$$(13) \qquad \begin{cases} \alpha_i^{A_{2n}^{(2)}} = \alpha_i + \alpha_{2n-i}, i = 1, \ldots, n-1 \\ \alpha_n^{A_{2n}^{(2)}} = 2\alpha_n \\ \alpha_0^{A_{2n}^{(2)}} = \alpha_0 \end{cases} \qquad \begin{cases} \Lambda_i^{A_{2n}^{(2)}} = \Lambda_i + \Lambda_{2n-i}, i = 1, \ldots, n-1 \\ \Lambda_n^{A_{2n}^{(2)}} = 2\Lambda_n \\ \Lambda_0^{A_{2n}^{(2)}} = \Lambda_0 \end{cases}$$

The affine Weyl group $W_{A_{2n}^{(2)}}$ is equal to $W_{C_n^{(1)}}$ and we thus have $W_{A_{2n}^{(2)}} \cdot \emptyset = \mathcal{C}_{2n}^s$. For each c in $\mathcal{C}_{2n}^s$, we can write

$$\Lambda_0^{A_{2n}^{(2)}} - c(\Lambda_0^{A_{2n}^{(2)}}) = \sum_{i=0}^{n} a_i^{A_{2n}^{(2)}}(c)\alpha_i^{A_{2n}^{(2)}}$$

which gives by using (11)

$$\Lambda_0 - c(\Lambda_0) = a_0^{A_{2n}^{(2)}}(c)\alpha_0 + 2a_n^{A_{2n}^{(2)}}(c)\alpha_n + \sum_{i=1}^{n-1} a_i^{A_{2n}^{(2)}}(c)(\alpha_i + \alpha_{2n-i}).$$

By comparing with (8), we get

$$\begin{cases} a_i^{A_{2n}^{(2)}}(c) = a_i(c) = a_{2n-i}(c) \text{ for } i = 1, \ldots, n-1 \\ a_n^{A_{2n}^{(2)}}(c) = \frac{1}{2}a_n(c) \text{ and } a_0^{A_{2n}^{(2)}}(c) = a_0(c) \end{cases}$$

Proposition 5.6. *For any* $c \in W_{A_{2n}^{(2)}}^0 = \mathcal{C}_{2n}^s$, *we have*

$$L_{A_{2n}^{(2)}}(c) = \frac{1}{2}(L(c) + a_0(c)).$$

5.4. Type $A_{2n}^{\prime(2)}$. We denote by $A_{2n}^{\prime(2)}$ the affine root system obtained by transposing the Cartan matrix of type $A_{2n}^{\prime(2)}$. The types $A_{2n}^{(2)}$ and $A_{2n}^{\prime(2)}$ coincides up to relabelling of the nodes of their Dynkin diagram. Nevertheless, this relabelling does not fix the affine Grassmannian elements. Equivalently, we could also consider the orbit of the fundamental weight $\Lambda_n^{A_{2n}^{(2)}}$. We realize the affine root system of type $A_{2n}^{\prime(2)}$ as the subsystem of the root system of type $A_{2n-1}^{(1)}$ such that

$$\begin{cases} \alpha_i^{A_{2n}^{\prime(2)}} = \alpha_i + \alpha_{2n-i}, i = 1, \ldots, n-1 \\ \alpha_n^{A_{2n}^{\prime(2)}} = \alpha_n \\ \alpha_0^{A_{2n}^{\prime(2)}} = 2\alpha_0 \end{cases} \qquad \begin{cases} \Lambda_i^{A_{2n}^{\prime(2)}} = \Lambda_i + \Lambda_{2n-i}, i = 1, \ldots, n-1 \\ \Lambda_n^{A_{2n}^{\prime(2)}} = \Lambda_n \\ \Lambda_0^{A_{2n}^{\prime(2)}} = 2\Lambda_0 \end{cases}$$

The affine Weyl group $W_{A_{2n}^{(2)}}$ is equal to $W_{C_n^{(1)}}$ and we have $W_{A_{2n}^{(2)}} \cdot \emptyset = \mathcal{C}_{2n}^s$. For each c in $\mathcal{C}_{2n}^s$, we can write

$$\Lambda_0^{A_{2n}^{\prime(2)}} - c(\Lambda_0^{A_{2n}^{\prime(2)}}) = \sum_{i=0}^{n} a_i^{A_{2n}^{\prime(2)}}(c)\alpha_i^{A_{2n}^{\prime(2)}}$$

which gives

$$2\Lambda_0 - 2c(\Lambda_0) = 2a_0^{A_{2n}'^{(2)}}(c)\alpha_0 + a_n^{A_{2n}'^{(2)}}(c)\alpha_n + \sum_{i=1}^{n-1} a_i^{A_{2n}'^{(2)}}(c)(\alpha_i + \alpha_{2n-i}) \text{ i.e.}$$

$$\Lambda_0 - c(\Lambda_0) = a_0^{A_{2n}'^{(2)}}(c)\alpha_0 + \frac{1}{2}a_n^{A_{2n}'^{(2)}}(c)\alpha_n + \sum_{i=1}^{n-1} \frac{1}{2}a_i^{A_{2n}'^{(2)}}(c)(\alpha_i + \alpha_{2n-i})$$

By comparing with (8), we get

$$\begin{cases} a_i^{A_{2n}'^{(2)}}(c) = 2a_i(c) = 2a_{2n-i}(c) \text{ for } i = 1, \ldots, n-1 \\ a_n^{A_{2n}'^{(2)}}(c) = 2a_n(c) \text{ and } a_0^{A_{2n}'^{(2)}}(c) = a_0(c) \end{cases}$$

Proposition 5.7. *For any* $c \in W^0_{A_{2n}'^{(2)}} = \mathcal{C}_{2n}^s$, *we have*

$$L_{A_{2n}'^{(2)}}(c) = L(c) + a_n(c).$$

5.5. Type $B_n^{(1)}$. We will proceed in two steps by first embedding the affine root system of type $B_n^{(1)}$ into the root system of type $D_{n+1}^{(2)}$ and next by using § 5.2. We first write

$$(14) \quad \begin{cases} \alpha_i^{B_n^{(1)}} = \alpha_i^{D_{n+1}^{(2)}}, i = 1, \ldots, n \\ \alpha_0^{B_n^{(1)}} = 2\alpha_0^{D_{n+1}^{(2)}} + \alpha_1^{D_{n+1}^{(2)}} \end{cases} \quad \text{and} \quad \begin{cases} \Lambda_i^{B_n^{(1)}} = \Lambda_i^{D_{n+1}^{(2)}}, i = 2, \ldots, n \\ \Lambda_0^{B_n^{(1)}} = \Lambda_0^{D_{n+1}^{(2)}} \\ \Lambda_1^{B_n^{(1)}} = \Lambda_1^{D_{n+1}^{(2)}} - \Lambda_0^{D_{n+1}^{(2)}} \end{cases}$$

which indeed gives a root system of type $B_n^{(1)}$. The affine Weyl group $W_{B_n^{(1)}}$ can then be realized as the subgroup of $W_{D_{n+1}^{(2)}} = W_{C_n^{(1)}}$ such that

$$W_{B_n^{(1)}} = \langle s_0^{B_n^{(1)}} = s_0^{C_n^{(1)}} s_1^{C_n^{(1)}} s_0^{C_n^{(1)}}, s_1^{C_n^{(1)}}, \ldots, s_n^{C_n^{(1)}} \rangle$$

or equivalently, $W_{B_n^{(1)}}$ is the subgroup of $\widetilde{\mathfrak{S}}_n$

$$W_{B_n^{(1)}} = \langle s_0 s_1 s_{2n-1} s_0, s_1 s_{2n-1}, \ldots, s_{n-1} s_{n+1}, s_n \rangle.$$

Write $\mathcal{C}_{2n}^{s,p}$ for the subset of $\mathcal{C}_{2n}^s$ of self-conjugate $2n$-cores with an even number of nodes on its main diagonal (i.e. an even number of nodes with content equal to 0).

Lemma 5.8. *We have* $W_{B_n^{(1)}} \cdot \emptyset = \mathcal{C}_{2n}^{s,p}$, *that is the affine Grassmannian elements of type* $B_n^{(1)}$ *are parametrized by the self conjugate* $2n$-*cores with an even diagonal.*

proof: The proof is similar to that of Lemma 5.3. One first check that $\mathcal{C}_{2n}^{s,p}$ is stable under the action of $W_{B_n^{(1)}}$. Next, one observes that for any nonempty $c \in \mathcal{C}_{2n}^{s,p}$, there exists at least an integer $i = 0, \ldots, n$ such that all the i nodes of c are removable. Since c belongs to $\mathcal{C}_{2n}^{s,p}$, this implies that $s_i^{B_n^{(1)}} \cdot c = c'$ where $c' \neq c$ belongs to $\mathcal{C}_{2n}^s$ and has a number of nodes stricly less than c. ∎

For each c in $\mathcal{C}_{2n}^{s,p}$, we can write

$$\Lambda_0^{B_n^{(1)}} - c(\Lambda_0^{B_n^{(1)}}) = \sum_{i=0}^{n} a_i^{B_n^{(1)}}(c)\alpha_i^{B_n^{(1)}}$$

which gives by using (14)

$$\Lambda_0^{D_{n+1}^{(2)}} - c(\Lambda_0^{D_{n+1}^{(2)}}) = a_0^{B_n^{(1)}}(c)(\alpha_1^{D_{n+1}^{(2)}} + 2\alpha_0^{D_{n+1}^{(2)}}) + \sum_{i=1}^{n} a_i^{B_n^{(1)}}(c)\alpha_i^{D_{n+1}^{(2)}}.$$

We get by using (12)

$$\begin{cases} a_i^{B_n^{(1)}}(c) = a_i^{D_{n+1}^{(2)}}(c) = a_i(c) = a_{2n-i}(c) \text{ for } i = 2, \ldots, n \\ a_0^{B_n^{(1)}}(c) = \frac{1}{2}a_0^{D_{n+1}^{(2)}}(c) = \frac{1}{2}a_0(c) \text{ and } a_1^{B_n^{(1)}}(c) = a_1^{D_{n+1}^{(2)}}(c) - \frac{1}{2}a_0^{D_{n+1}^{(2)}}(c) = a_1(c) - \frac{1}{2}a_0(c) \end{cases}$$

The following proposition can then be deduced from Proposition 5.5.

Proposition 5.9. *For any $c \in W^0_{B_n^{(1)}} = \mathcal{C}_{2n}^{s,p}$, we have*

$$L_{B_n^{(1)}}(c) = L_{D_{n+1}^{(2)}}(c) - a_0^{D_{n+1}^{(2)}}(c) = \frac{1}{2}(L(c) - a_0(c) + a_n(c)).$$

5.6. Type $A_{2n-1}^{(2)}$. Here again, we will proceed in two steps by first embedding the affine root system of type $A_{2n-1}^{(2)}$ into the root system of type $A_{2n}^{(2)}$ and next by using § 5.3. We first write

(15)
$$\begin{cases} \alpha_i^{A_{2n-1}^{(2)}} = \alpha_i^{A_{2n}^{(2)}}, i = 1, \dots, n \\ \alpha_0^{A_{2n-1}^{(2)}} = 2\alpha_0^{A_{2n}^{(2)}} + \alpha_1^{A_{2n}^{(2)}} \end{cases} \quad \text{and} \quad \begin{cases} \Lambda_i^{A_{2n-1}^{(2)}} = \Lambda_i^{A_{2n}^{(2)}}, i = 2, \dots, n \\ \Lambda_0^{A_{2n-1}^{(2)}} = \Lambda_0^{A_{2n}^{(2)}} \\ \Lambda_1^{A_{2n-1}^{(2)}} = \Lambda_1^{A_{2n}^{(2)}} - \Lambda_0^{A_{2n}^{(2)}} \end{cases}$$

which indeed gives a root system of type $A_{2n-1}^{(2)}$. The affine Weyl group $W_{A_{2n-1}^{(2)}}$ can then be realized as the subgroup of $W_{A_{2n}^{(2)}} = W_{C_n^{(1)}}$ such that

$$W_{A_{2n-1}^{(2)}} = \langle s_0^{A_{2n-1}^{(2)}} = s_0^{C_n^{(1)}} s_1^{C_n^{(1)}} s_0^{C_n^{(1)}}, s_1^{C_n^{(1)}}, \dots, s_n^{C_n^{(1)}} \rangle = W_{B_n^{(1)}}.$$

We thus get:

Lemma 5.10. *We have $W_{A_{2n-1}^{(2)}} \cdot \emptyset = \mathcal{C}_{2n}^{s,p}$.*

For each c in $\mathcal{C}_{2n}^{s,p}$, we can write

$$\Lambda_0^{A_{2n-1}^{(2)}} - c(\Lambda_0^{A_{2n-1}^{(2)}}) = \sum_{i=0}^n a_i^{A_{2n-1}^{(2)}}(c) \alpha_i^{A_{2n-1}^{(2)}}$$

which gives by using (15)

$$\Lambda_0^{A_{2n}^{(2)}} - c(\Lambda_0^{A_{2n}^{(2)}}) = a_0^{A_{2n-1}^{(2)}}(c)(\alpha_1^{A_{2n}^{(2)}} + 2\alpha_0^{A_{2n}^{(2)}}) + \sum_{i=1}^n a_i^{A_{2n-1}^{(2)}}(c)\alpha_i^{A_{2n}^{(2)}}.$$

We get by using (12)

$$\begin{cases} a_i^{A_{2n-1}^{(2)}}(c) = a_i^{A_{2n}^{(2)}}(c) = a_i(c) = a_{2n-i}(c) \text{ for } i = 2, \dots, n-1 \\ a_n^{A_{2n-1}^{(2)}}(c) = a_n^{A_{2n}^{(2)}}(c) = \frac{1}{2}a_n(c) \\ a_0^{A_{2n-1}^{(2)}}(c) = \frac{1}{2}a_0^{A_{2n}^{(2)}}(c) = \frac{1}{2}a_0(c) \text{ and } a_1^{A_{2n-1}^{(2)}}(c) = a_1^{A_{2n}^{(2)}}(c) - \frac{1}{2}a_0^{A_{2n}^{(2)}}(c) = a_1(c) - \frac{1}{2}a_0(c) \end{cases}$$

The following proposition can then be deduced from Proposition 5.5.

Proposition 5.11. *For any $c \in W^0_{A_{2n-1}^{(2)}} = \mathcal{C}_{2n}^{s,p}$, we have*

$$L_{A_{2n-1}^{(2)}}(c) = L_{A_{2n}^{(2)}}(c) - a_0^{A_{2n}^{(2)}}(c) = \frac{1}{2}(L(c) - a_0(c)).$$

5.7. Type $D_n^{(1)}$. Here again we use an embedding in the affine root system of type $D_{n+1}^{(2)}$ and write

(16)
$$\begin{cases} \alpha_i^{D_n^{(1)}} = \alpha_i^{D_{n+1}^{(2)}}, i = 1, \dots, n-1 \\ \alpha_0^{D_n^{(1)}} = 2\alpha_0^{D_{n+1}^{(2)}} + \alpha_1^{D_{n+1}^{(2)}} \\ \alpha_n^{D_n^{(1)}} = 2\alpha_n^{D_{n+1}^{(2)}} + \alpha_{n-1}^{D_{n+1}^{(2)}} \end{cases} \quad \text{and} \quad \begin{cases} \Lambda_i^{D_n^{(1)}} = \Lambda_i^{D_{n+1}^{(2)}}, i = 2, \dots, n-1 \\ \Lambda_0^{D_n^{(1)}} = \Lambda_0^{D_{n+1}^{(2)}}, \Lambda_n^{D_n^{(1)}} = \Lambda_n^{D_{n+1}^{(2)}} \\ \Lambda_1^{D_n^{(1)}} = \Lambda_1^{D_{n+1}^{(2)}} - \Lambda_0^{D_{n+1}^{(2)}}, \Lambda_{n-1}^{D_n^{(1)}} = \Lambda_{n-1}^{D_{n+1}^{(2)}} - \Lambda_n^{D_{n+1}^{(2)}} \end{cases}$$

which gives a root system of type $D_n^{(1)}$. The affine Weyl group $W_{D_n^{(1)}}$ can then be realized as the subgroup of $W_{D_{n+1}^{(1)}} = W_{C_n^{(1)}}$ such that

$$W_{D_n^{(1)}} = \langle s_0^{D_n^{(1)}} = s_0^{C_n^{(1)}} s_1^{C_n^{(1)}} s_0^{C_n^{(1)}}, s_1^{C_n^{(1)}}, \dots, s_{n-1}^{C_n^{(1)}}, s_n^{D_n^{(1)}} = s_n^{C_n^{(1)}} s_{n-1}^{C_n^{(1)}} s_n^{C_n^{(1)}} \rangle$$

or equivalently, $W_{D_n^{(1)}}$ is the subgroup of $\widetilde{\mathfrak{S}}_n$

$$W_{D_n^{(1)}} = \langle s_0 s_1 s_{2n-1} s_0, s_1 s_{2n-1}, \dots, s_{n-1} s_{n+1}, s_n s_{n-1} s_{n+1} s_n \rangle.$$

According to Table (2), the affine Grassmannian $W^0_{D_n^{(1)}} = W_{D_n^{(1)}}/W_{D_n}$ is in one-to-one correspondence with the sublattice lattice of $\mathbb{Z}^n$ of vectors $\beta = (\beta_1, \dots, \beta_n)$ such that $\beta_1 + \dots + \beta_n$ is even. On the other hand we must have $W^0_{D_n^{(1)}} \subset W^0_{B_n^{(1)}}$ for $W_{D_n^{(1)}} \subset W_{B_n^{(1)}}$. But by using Table (2) again, the affine Grassmannian $W^0_{B_n^{(1)}} = W_{B_n^{(1)}}/W_{B_n}$ is again in one-to-one correspondence with the vectors in $\mathbb{Z}^n$ whose sum of coordinates is even. Therefore, we have $W^0_{D_n^{(1)}} = W^0_{B_n^{(1)}}$ and $W^0_{D_n^{(1)}}$ is yet parametrized by the elements of $\mathcal{C}^{s,p}_{2n}$. Observe nevertheless that the length functions in $W^0_{D_n^{(1)}}$ and $W^0_{B_n^{(1)}}$ do not coincide.

For each c in $\mathcal{C}^{s,p}_{2n}$, we can write

$$\Lambda_0^{D_n^{(1)}} - c(\Lambda_0^{D_n^{(1)}}) = \sum_{i=0}^n a_i^{D_n^{(1)}}(c)\alpha_i^{D_n^{(1)}}$$

which gives by using (16)

$$\Lambda_0^{D_{n+1}^{(2)}} - c(\Lambda_0^{D_{n+1}^{(2)}}) = a_0^{D_n^{(1)}}(c)(\alpha_1^{D_{n+1}^{(2)}} + 2\alpha_0^{D_{n+1}^{(2)}}) + \sum_{i=1}^{n-1} a_i^{D_n^{(1)}}(c)\alpha_i^{D_{n+1}^{(2)}} + a_n^{D_n^{(1)}}(c)(\alpha_{n-1}^{D_{n+1}^{(2)}} + 2\alpha_n^{D_{n+1}^{(2)}}).$$

We get by using (12)

$$\begin{cases} a_i^{D_n^{(1)}}(c) = a_i^{D_{n+1}^{(2)}}(c) = a_i(c) = a_{2n-i}(c) \text{ for } i = 2, \dots, n-1 \\ a_0^{D_n^{(1)}}(c) = \frac{1}{2}a_0^{D_{n+1}^{(2)}}(c) = \frac{1}{2}a_0(c) \text{ and } a_1^{D_n^{(1)}}(c) = a_1^{D_{n+1}^{(2)}}(c) - \frac{1}{2}a_0^{D_{n+1}^{(2)}}(c) = a_1(c) - \frac{1}{2}a_0(c) \\ a_n^{D_n^{(1)}}(c) = \frac{1}{2}a_n^{D_{n+1}^{(2)}}(c) = \frac{1}{2}a_n(c) \text{ and } a_{n-1}^{D_n^{(1)}}(c) = a_{n-1}^{D_{n+1}^{(2)}}(c) - \frac{1}{2}a_n^{D_{n+1}^{(2)}}(c) = a_{n-1}(c) - \frac{1}{2}a_n(c) \end{cases}$$

The following proposition can then be deduced from Proposition 5.11.

Proposition 5.12. *For any $c \in W^0_{D_n^{(1)}} = \mathcal{C}^{s,p}_{2n}$, we have*

$$L_{D_n^{(1)}}(c) = L_{D_{n+1}^{(2)}}(c) - a_0^{D_{n+1}^{(2)}}(c) - a_n^{D_{n+1}^{(2)}}(c) = \frac{1}{2}(L(c) - a_0(c) - a_n(c)).$$

5.8. **Weighted boxes in cores.** The previous formulas for the atomic lengths can be expressed shortly by defining a weight function π on the relevant cores depending only on the residues of the nodes. We then have

$$L_{X_n^{(a)}}(c) = \sum_{\square \in c} \pi(\square)$$

where the map π takes the same value π_i on the set of nodes having residue i. This is equivalent to associate a weight π_i on each node of the Dynkin diagram of the root system considered. When $\pi_i = 1$ for any i, the previous sum is just equal to the number of nodes in c.

type	$A_{n-1}^{(1)}$	$C_n^{(1)}$	$B_n^{(1)}$	$D_n^{(1)}$
core set	$\mathcal{C}_n$	$\mathcal{C}^s_{2n}$	$\mathcal{C}^{s,p}_{2n}$	$\mathcal{C}^{s,p}_{2n}$
π	$\pi_i = 1$ any i	$\pi_i = 1$ any i	$\pi_0 = 0, \pi_n = 1$ $\pi_i = \frac{1}{2}$, o.t.w	$\pi_0 = \pi_n = 0$ $\pi_i = \frac{1}{2}$, o.t.w

type	$D_n^{(2)}$	$A_{2n}^{(2)}$	$A'^{(2)}_{2n}$	$A_{2n-1}^{(2)}$
core set	$\mathcal{C}^s_{2n}$	$\mathcal{C}^s_{2n}$	$\mathcal{C}^s_{2n}$	$\mathcal{C}^{s,p}_{2n}$
π	$\pi_0 = \pi_n = 1$ $\pi_i = \frac{1}{2}$, o.t.w	$\pi_0 = 1$ $\pi_i = \frac{1}{2}$, o.t.w	$\pi_0 = 2$ $\pi_i = 1$, o.t.w	$\pi_0 = 0$ $\pi_i = \frac{1}{2}$, o.t.w

5.9. **Type $D_4^{(3)}$.** We realize the affine root system of type $D_4^{(3)}$ from the root system of type $A_5^{(2)}$ by setting

$$(17) \qquad \begin{cases} \alpha_0^{D_4^{(3)}} = \alpha_0^{A_5^{(2)}} \\ \alpha_1^{D_4^{(3)}} = \alpha_2^{A_5^{(2)}} \\ \alpha_2^{D_4^{(3)}} = \alpha_1^{A_5^{(2)}} + \alpha_3^{A_5^{(2)}} \end{cases} \quad \text{and} \quad \begin{cases} \Lambda_0^{D_4^{(3)}} = \Lambda_0^{A_5^{(2)}} \\ \Lambda_1^{D_4^{(3)}} = \Lambda_2^{A_5^{(2)}} \\ \Lambda_2^{D_4^{(3)}} = \Lambda_1^{A_5^{(2)}} + \Lambda_3^{A_5^{(2)}} \end{cases}$$

The affine Weyl group $W_{D_4^{(3)}}$ can then be realized as the subgroup of $W_{A_5^{(2)}}$ such that

$$W_{D_4^{(3)}} = \langle s_0^{D_4^{(3)}} = s_0^{A_5^{(2)}}, s_1^{D_4^{(3)}} = s_2^{A_5^{(2)}}, s_2^{D_4^{(3)}} = s_1^{A_5^{(2)}} s_3^{A_5^{(2)}} \rangle = \langle s_0 s_1 s_5 s_0, s_2 s_4, s_1 s_3 s_5 \rangle.$$

Let $\mathcal{C}_6^g$ be the set of 6-cores associated to the elements in $\mathbb{Z}^6$ of the form

(18)
$$(\beta_1, \beta_2, \beta_1 - \beta_2, \beta_2 - \beta_1, -\beta_2, -\beta_1).$$

Observe in particular that $\mathcal{C}_6^g \subset \mathcal{C}_6^{s,p}$.

Lemma 5.13. *We have* $W_{D_4^{(3)}} \cdot \emptyset = \mathcal{C}_6^g$.

Proof. One checks easily that $\mathcal{C}_6^g$ is stable under the action of $W_{D_4^{(3)}}$. Moreover when c is not empty in $\mathcal{C}_6^g$, at least one of the actions of $s_0^{D_4^{(3)}}$ or $s_1^{D_4^{(3)}}$ on c makes decrease its number of boxes. $\square$

Remark 5.14. Note that $s_3^{A_6^{(1)}}$ acts on $\mathcal{C}_6^g$ such that $s_3^{A_6^{(1)}} \mathcal{C}_6^g$ correspond to the set of self-conjugate 6-cores such that the 2-quotient is a 3-core partition.

For each c in $\mathcal{C}_6^g$, we can write
$$\Lambda_0^{D_4^{(3)}} - c(\Lambda_0^{D_4^{(3)}}) = \sum_{i=0}^{2} a_i^{D_4^{(3)}}(c)\alpha_i^{D_4^{(3)}}$$

which gives by using (17)
$$\Lambda_0^{A_5^{(2)}} - c(\Lambda_0^{A_5^{(2)}}) = a_0^{D_4^{(3)}}(c)\alpha_0^{A_5^{(2)}} + a_1^{D_4^{(3)}}(c)\alpha_2^{A_5^{(2)}} + a_2^{D_4^{(3)}}(c)(\alpha_1^{A_5^{(2)}} + \alpha_3^{A_5^{(2)}})$$

We get
$$\begin{cases} a_0^{D_4^{(3)}}(c) = a_0^{A_5^{(2)}}(c) = \frac{1}{2}a_0(c) \\ a_1^{D_4^{(3)}}(c) = a_2^{A_5^{(2)}}(c) = a_2(c) \\ a_2^{D_4^{(3)}}(c) = a_1^{A_5^{(2)}}(c) = a_3^{A_5^{(2)}}(c) = a_1(c) - \frac{1}{2}a_0(c) = \frac{1}{2}a_3(c) \end{cases}$$

The following proposition can then be deduced from Proposition 5.11.

Proposition 5.15. *For any* $c \in W_{D_4^{(3)}}^0 = \mathcal{C}_6^g$, *we have*
$$L_{D_4^{(3)}}(c) = L_{A_5^{(2)}}(c) - a_3^{A_5^{(2)}}(c) = \frac{1}{2}(L(c) - a_0(c) - a_3(c)).$$

5.10. Type $G_2^{(1)}$.

We realize the affine root system of type $G_2^{(1)}$ from the root system of type $A_5^{(2)}$ by setting

(19)
$$\begin{cases} \alpha_0^{G_2^{(1)}} = 3\alpha_0^{A_5^{(2)}} \\ \alpha_1^{G_2^{(1)}} = 3\alpha_2^{A_5^{(2)}} \\ \alpha_2^{G_2^{(1)}} = \alpha_1^{A_5^{(2)}} + \alpha_3^{A_5^{(2)}} \end{cases} \quad \text{and} \quad \begin{cases} \Lambda_0^{G_2^{(1)}} = 3\Lambda_0^{A_5^{(2)}} \\ \Lambda_1^{G_2^{(1)}} = 3\Lambda_2^{A_5^{(2)}} \\ \Lambda_2^{G_2^{(1)}} = \Lambda_1^{A_5^{(2)}} + \Lambda_3^{A_5^{(2)}} \end{cases}$$

The affine Weyl group $W_{G_2^{(1)}}$ is the same as $W_{D_4^{(3)}}$
$$W_{G_2^{(1)}} = \langle s_0^{G_2^{(1)}} = s_0^{A_5^{(2)}}, s_1^{G_2^{(1)}} = s_2^{A_5^{(2)}}, s_2^{G_2^{(1)}} = s_1^{A_5^{(2)}} s_3^{A_5^{(2)}} \rangle = \langle s_0 s_1 s_5 s_0, s_2 s_4, s_1 s_3 s_5 \rangle.$$

and we yet have $W_{G_2^{(1)}} \cdot \emptyset = \mathcal{C}_6^g$

For each c in $\mathcal{C}_6^g$, we can write
$$\Lambda_0^{G_2^{(1)}} - c(\Lambda_0^{G_2^{(1)}}) = \sum_{i=0}^{2} a_i^{G_2^{(1)}}(c)\alpha_i^{G_2^{(1)}}$$

which gives by using (19)
$$3\Lambda_0^{A_5^{(2)}} - 3c(\Lambda_0^{A_5^{(2)}}) = 3a_0^{G_2^{(1)}}(c)\alpha_0^{A_5^{(2)}} + 3a_1^{G_2^{(1)}}(c)\alpha_2^{A_5^{(2)}} + a_2^{G_2^{(1)}}(c)(\alpha_1^{A_5^{(2)}} + \alpha_3^{A_5^{(2)}}) =$$
$$3a_0^{A_5^{(2)}}(c)\alpha_0^{A_5^{(2)}} + 3a_1^{A_5^{(2)}}(c)\alpha_1^{A_5^{(2)}} + 3a_2^{A_5^{(2)}}(c)\alpha_2^{A_5^{(2)}} + 3a_3^{A_5^{(2)}}(c)\alpha_3^{A_5^{(2)}}$$

We get
$$\begin{cases} a_0^{G_2^{(1)}}(c) = a_0^{A_5^{(2)}}(c) = \frac{1}{2}a_0(c) \\ a_1^{G_2^{(1)}}(c) = a_2^{A_5^{(2)}}(c) = a_2(c) \\ a_2^{G_2^{(1)}}(c) = 3a_1^{A_5^{(2)}}(c) = 3a_3^{A_5^{(2)}}(c) = 3a_1(c) - \frac{3}{2}a_0(c) = \frac{3}{2}a_3(c) \end{cases}$$

The following proposition can then be deduced from Proposition 5.11.

Proposition 5.16. *For any $c \in W^0_{G_2^{(1)}} = \mathcal{C}^g_6$, we have*

$$L_{G_2^{(1)}}(c) = L_{A_5^{(2)}}(c) + a_3^{A_5^{(2)}}(c) = \frac{1}{2}(L(c) - a_0(c) + a_3(c)).$$

6. Combinatorics of integer partitions

The goal of this section is first to recall a couple of results about the Littlewood decomposition, a bijection mapping an integer partition to its n-core and a n-tuple of partitions. Then we study the restriction of this application to the set of distinct partitions. The goal in what follows is to map bijectively the elements c of the subsets of cores arising in the previous section to elements c' of subsets of partitions defined implicitly in terms of their abaci such that the atomic length of c is equal to (half) the weight of c', i.e. $L(c) = |c'|$.

6.1. **The Littlewood decomposition and bi-infinite binary words.** Recall that there is a natural correspondence between $\mathcal{P}$ and the set of bi-infinite words indexed by $\mathbb{Z}$ over the alphabet $\{0, 1\}$.

Definition 6.1. Define

$$\psi : \left| \begin{array}{ccc} \mathcal{P} & \to & \{0,1\}^{\mathbb{Z}} \\ \lambda & \mapsto & (c_k)_{k \in \mathbb{Z}}, \end{array} \right.$$

such that

$$c_k = \begin{cases} 0 & \text{if } k \in \{\lambda_i - i, i \in \mathbb{N}\}, \\ 1 & \text{if } k \in \{j - \lambda_j^{\text{tr}} - 1, j \in \mathbb{N}\}. \end{cases}$$

Moreover by definition of $\psi(\lambda)$, one has:

(20) $$\#\{k \leq -1, c_k = 1\} = \#\{k \geq 0, c_k = 0\} = d_\lambda.$$

The application ψ is a bijection from the set of integer partitions $\mathcal{P}$ and the subset of bi-infinite binary words

$$\{(c_k) \in \{0,1\}^{\mathbb{Z}} \mid \#\{k \leq -1, c_k = 1\} = \#\{k \geq 0, c_k = 0\} = d_\lambda\}.$$

Let $\partial\lambda$ be the border of the Ferrers diagram of λ. Each step on $\partial\lambda$ is either horizontal or vertical. The above correspondence amounts to encode the walk along the border from the South-West to the North-East as depicted in Figure 3: take "0" for a vertical step and "1" for a horizontal step. The resulting word is indexed by $\mathbb{Z}$. In order to keep this correspondence bijective, one needs to set the index 0. The choice within this framework is to set the letter of index 0 to be the first step after the corner of the Durfee square, the largest square that can fit within the Ferrers diagram of λ. The sequence $\psi(\lambda)$ has many names across literature with slight variations on the alphabet or the binary labels such as Maya diagrams, edge sequences (note that the edge sequence defined in [LLMS10] corresponds to the word obtained when labelling horizontal, respectively vertical, steps with letter "0", respectively with letter "1", dirac sea, abacus.

FIGURE 3. $\partial\lambda$ and its binary correspondence for $\lambda = (5, 5, 3, 2)$ with a hook.

As illustrated in Figure 3, the boxes of λ are in bijective correspondence with letters of $\psi(\lambda)$.

Lemma 6.2. [Wah23, Lemma 2.1] *The map ψ (see Definition 6.1) associates bijectively a box s of hook length h_s of the Ferrers diagram of λ to a pair of indices $(i_s, j_s) \in \mathbb{Z}^2$ of the word $\psi(\lambda)$ such that*

(1) $i_s < j_s$,
(2) $c_{i_s} = 1$, $c_{j_s} = 0$,
(3) $j_s - i_s = h_s$,
(4) s is a box above the main diagonal in the Ferrers diagram of λ if and only if the number of letters "1" with negative index greater than i_s is lower than the number of letters "0" with nonnegative index lower than j_s.

Hook lengths formulas are useful enumerative tools bridging combinatorics with other fields such as representation theory, probability, gauge theory or algebraic geometry. A much more recent identity is the Nekrasov–Okounkov formula. It was discovered independently by Nekrasov and Okounkov in their work on random partitions and Seiberg–Witten theory [NO06], by Westbury [Wes06] in his work on universal characters for $\mathfrak{sl}_n$, and later by Han [Han10] based on one of the identities for affine type $\tilde{A}_t$ in [Mac72, Appendix 1] and a polynomiality argument. This formula is commonly stated as follows:

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} \left(1 - q^k\right)^{z-1},$$

where z is a fixed complex number.

Han's proof is based on the crucial observation that if we take $z = n^2$ in (1), the products $\prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{n^2}{h^2}\right)$ cancel whenever λ does not belong to $\mathcal{C}_n$. Han's proof is to show that in this case, the equality (1) corresponds to a specialization of the Weyl–Kac denominator formula in type $\tilde{A}_{n-1}^{(1)}$. Since the equality holds for an infinite number of n, Han's proof is based on a polynomiality argument.
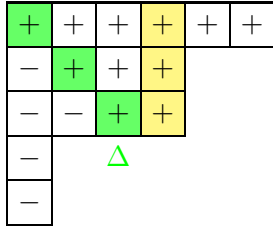
(a) Shifted Young diagram of $\bar{\lambda} = (5, 2, 1) \in \mathcal{D}$
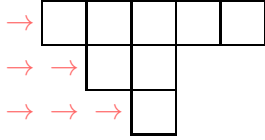
(b) $\lambda = (5, 3, 3, 1, 1) \in \mathcal{SC}$
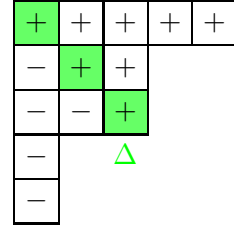
(c) $\lambda = (6, 4, 4, 1, 1) \in \mathcal{DD}$

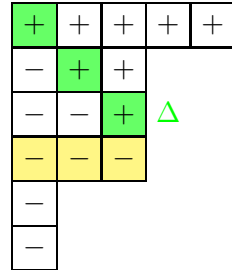(d) $\lambda = (5, 3, 3, 3, 1, 1) \in \mathcal{DD}^{\mathrm{tr}}$

FIGURE 4. Distinct partition, a self-conjugate partition, a doubled distinct partition and its conjugate filled with ε.

The set of distinct partitions, denoted by $\mathcal{D}$, is the set of partitions such that no consecutive parts are equal (see [HX19, Mac95, Sta99]). A distinct partition $\bar{\lambda}$ is identified with its *shifted Young diagram*, which means the i-th row of the usual Young diagram is shifted by i boxes to the right. The *doubled distinct partition* of $\bar{\lambda} \in \mathcal{D}$, denoted by $\bar{\lambda}\bar{\lambda}$, is defined to be the usual partition whose Young diagram is obtained by adding $\bar{\lambda}_i$ boxes to the i-th column of the shifted Young diagram of $\bar{\lambda}$ for $1 \leq i \leq \ell(\bar{\lambda})$. We denote the set of doubled distinct partitions by $\mathcal{DD}$.

Warning: in Figure 4a, the shifted Young diagram of $\bar{\lambda}$ should have every row shifted by 1 to the right in order to obtain $\bar{\lambda}\bar{\lambda}$ from Figure 4c.

Following [HX19], the leftmost box of the i-th row of the shifted Young diagram of $\bar{\lambda}$ has coordinate $(i, i+1)$. The *hook length* of a box of coordinate (i, j) in the shifted diagram is the number of boxes strictly to the right, strictly below, the box itself plus $\bar{\lambda}_j$. Let us denote by $\mathcal{H}(\bar{\lambda})$ the multiset of hook lengths for the strict partition $\bar{\lambda}$. One has then the following relation on mulitsets:

$$(21) \qquad \mathcal{H}(\bar{\lambda}\bar{\lambda}) = \mathcal{H}(\bar{\lambda}) \uplus \mathcal{H}(\bar{\lambda}) \uplus \{2\bar{\lambda}_1, \ldots 2\bar{\lambda}_\ell\} \setminus \{\bar{\lambda}_1, \ldots \bar{\lambda}_\ell\}.$$

where we use the symbol $\uplus$ for the union of multisets. Let us give an alternative definition of the set $\mathcal{DD}$, is that of all partitions λ of Durfee square size d such that $\lambda_i = \lambda_i^{\mathrm{tr}} + 1$ for all $i \in \{1, \ldots, d\}$. We also define the set of conjugate of doubled distinct partitions $\mathcal{DD}^{\mathrm{tr}} := \{\lambda^{\mathrm{tr}} \mid \lambda \in \mathcal{DD}\}$. These constructions maps $\bar{\lambda} = (1) \in \mathcal{D}$ to $\bar{\lambda}\bar{\lambda} = (2) \in \mathcal{DD}$ and to $(1,1) \in \mathcal{DD}^{\mathrm{tr}}$.

Recall that a partition μ is *self-conjugate* if its Ferrers diagram is symmetric along the main diagonal, but it can also be seen equivalently if $\bar{\lambda} \in \mathcal{D}$, μ is defined to be the usual partition whose Young diagram is obtained by adding $\bar{\lambda}_i - 1$ boxes to the i-th column of the shifted Young diagram of $\bar{\lambda}$ for $1 \leq i \leq \ell(\bar{\lambda})$. One can go from a self-conjugate partition to a doubled distinct partition, respectively conjugate doubled distinct partition, by adding a vertical strip, respectively a horizontal strip, of length of the size of the Durfee square (shaded in yellow in Figure 4c, respectively in Figure 4d).

For instance, in Figure 4, take $\bar{\lambda} = (5, 2, 1) \in \mathcal{D}$, the corresponding element in the set of self-conjugate partitions $\mathcal{SC}$ $\lambda = (5, 3, 3, 1, 1)$ in Figure 4a has its main diagonal Δ shaded in green while in Figure 4c $\bar{\lambda}\bar{\lambda} = (6, 4, 4, 1, 1) \in \mathcal{DD}$ has its main diagonal shaded in green as for the strip shaded in yellow, it corresponds to the boxes added to a self-conjugate partition to obtain a doubled distinct partition. The conjugate of a doubled distinct partition is also illustrated in Figure 4d. Note that if we take μ in one of the sets $\mathcal{SC}, \mathcal{DD}, \mathcal{DD}^{\mathrm{tr}}$ and $\bar{\lambda} \in \mathcal{D}$ the corresponding distinct partition, one has the following relation:

$$(22) \qquad\qquad d_\mu = \ell(\bar{\lambda}).$$

Note that the hook lengths on the main diagonal Δ are $2\bar{\lambda}_1, \ldots, 2\bar{\lambda}_\ell$. Moreover the multiset of hook lengths of the $(l+1)$-th column (respectively row) of the Young diagram of the doubled distinct partition (respectively conjugate doubled distinct partition) coloured in yellow in Fig 4c (respectively Fig 4d) is $\{\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell\}$.

(a) $\lambda = (5, 3, 3, 1, 1) \in \mathcal{SC}$ (b) $\lambda = (6, 4, 4, 1, 1) \in \mathcal{DD}$ (c) $\lambda = (5, 3, 3, 3, 1, 1) \in \mathcal{DD}^{\mathrm{tr}}$
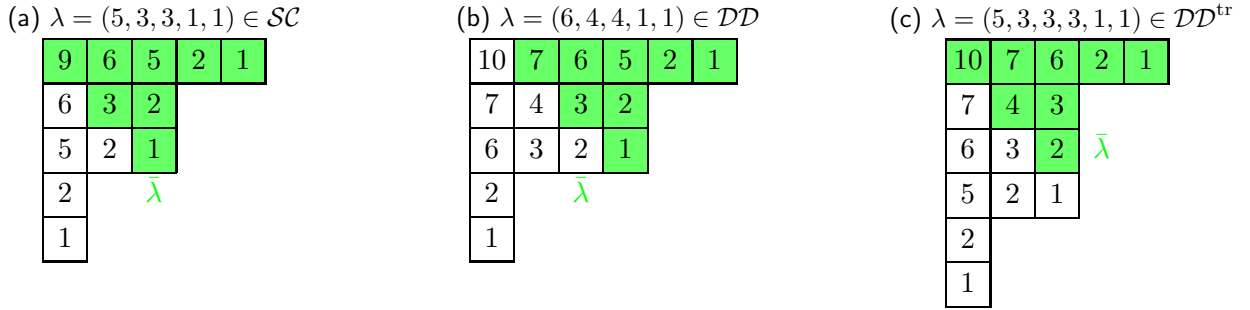

FIGURE 5. A self-conjugate partition, a doubled distinct partition and its conjugate filled with its hook lengths.

Let us introduce here a signed statistic ε_s, for a box s of $\bar{\lambda}\bar{\lambda}$, is defined as $-$ if s is strictly below the main diagonal of the Ferrers diagram of λ and as $+$ otherwise, as depicted in Figure 4. This signed statistic already appears algebraically within the work of King [Kin90] and combinatorially within the work of Pétréolle [Pé15]. Note that the boxes filled with $+$ signs correspond to the shifted diagram of $\bar{\lambda} \in \mathcal{D}$. Note that the multiset $\mathcal{H}(\bar{\lambda})$ of hook lengths of $\bar{\lambda}$ is equal to the multiset of hook lengths of $\bar{\lambda}\bar{\lambda}$ strictly above the main diagonal Δ, which is $\{h_s, s \in \bar{\lambda}\bar{\lambda} \setminus \Delta, \varepsilon = +\} = \mathcal{H}(\bar{\lambda})$.

Remark 6.3. Let λ be a partition and $\psi(\lambda) = (c_k)_{k\in\mathbb{Z}}$ be its corresponding word, as introduced in Definition 6.1. Let λ^{tr} be the conjugate of λ and $\psi(\lambda^{\mathrm{tr}}) = (c_i^{\mathrm{tr}})_{i\in\mathbb{Z}}$. We have

$$\forall k \in \mathbb{Z}, c_k^{\mathrm{tr}} = 1 - c_{-k-1}.$$

Given the properties of symmetries of self-conjugate and doubled distinct partitions, they can alternatively be characterized by:

$$(23) \qquad \lambda \in \mathcal{DD} \iff \psi(\lambda) = (c_k)_{k\in\mathbb{Z}} \mid c_0 = 1 \text{ and } \forall k \in \mathbb{N}^*, c_{-k} = 1 - c_k,$$

and

$$(24) \qquad \lambda \in \mathcal{SC} \iff \psi(\lambda) = (c_k)_{k\in\mathbb{Z}} \mid \forall k \in \mathbb{N}, c_{-k-1} = 1 - c_k.$$

From Remark 6.3 and (23), the set $\mathcal{DD}^{\mathrm{tr}}$ also admits a similar characterization:

$$(25) \qquad \lambda \in \mathcal{DD}^{\mathrm{tr}} \iff \psi(\lambda) = (c_k)_{k\in\mathbb{Z}} \mid c_{-1} = 0 \text{ and } \forall k \in \mathbb{N}, c_{-k-2} = 1 - c_k.$$

From the above relations and the definition of ψ (see Definition 6.1), one has the following lemma.

Lemma 6.4. *Set $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell) \in \mathcal{D}$ and let $\psi(\bar{\lambda}\bar{\lambda}) = (c_k)_{k \in \mathbb{Z}}$ and $\psi(\bar{\lambda}\bar{\lambda}^{\mathrm{tr}}) = (c_k^{\mathrm{tr}})_{k \in \mathbb{Z}}$ and μ the self-conjugate partition corresponding to $\bar{\lambda}$. Set $\psi(\mu) = (c_k^{\mathrm{s}})_{k \in \mathbb{Z}}$. Then*

$$\{k \in \mathbb{N} \mid c_k = 0\} = \{\bar{\lambda}_i, i \in \{1, \ldots, \ell\}\},$$
$$\{k \in \mathbb{N} \mid c_k^{\mathrm{tr}} = 0\} = \{\bar{\lambda}_i - 1, i \in \{1, \ldots, \ell\}\},$$
$$\{k \in \mathbb{N} \mid c_k^{\mathrm{s}} = 0\} = \{\bar{\lambda}_i - 1, i \in \{1, \ldots, \ell\}\}.$$

Take a partition λ and a strictly positive integer n. Obtaining what is called the n-quotient of λ is straightforward from $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}}$: we just look at subwords with indices congruent to the same values modulo n. In this case, the equality (20) is not necessarily verified. To be able to apply ψ^{-1}, the index has to be shifted by $\#\{i \in \mathbb{N} \mid c_{ni+k} = 1\} - \#\{i \in \mathbb{N}^* \mid c_{-ni+k} = 1\}$.

The sequence 10 within these subwords are replaced iteratively by 01 until the subwords are all the infinite sequence of "0"'s before the infinite sequence of "1"'s (in fact it consists in removing all rim hooks in λ of length congruent to $0 \pmod{n}$). Let ω be the partition corresponding to the word which has the subwords $\pmod{n}$ obtained after the removal of the 10 sequences. Note that ω is a n-core.

Now we recall the following classical map, often called the Littlewood decomposition (see for instance [GKS90, HJ11]).

Definition 6.5. *Let $n \geq 2$ be an integer and consider:*

$$\Phi_n : \left| \begin{array}{ccc} \mathcal{P} & \to & \mathcal{C}_n \times \mathcal{P}^n \\ \lambda & \mapsto & (\omega, \nu^{(0)}, \ldots, \nu^{(n-1)}), \end{array} \right.$$

where if we set $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}}$, then for all $k \in \{0, \ldots, n-1\}$, one has $\nu^{(k)} := \psi^{-1}\left((c_{ni+k+m_i})_{i \in \mathbb{Z}}\right)$, where $m_i = \#\{i \in \mathbb{N} \mid c_{ni+k} = 1\} - \#\{i \in \mathbb{N}^* \mid c_{-ni+k} = 1\}$. The tuple $\underline{\nu} = \left(\nu^{(0)}, \ldots, \nu^{(n-1)}\right)$ is called the n-quotient of λ and is denoted by $\mathrm{quot}_n(\lambda)$, while ω is the n-core of λ denoted by $\mathrm{core}_n(\lambda)$.

Proposition 6.6. *Let $n \geq 2$ be an integer. The application Phi_n is a bijection between $\mathcal{P}$ and $\mathcal{C}_n \times \mathcal{P}^n$.*

For example, if we take $\lambda = (4, 4, 3, 2)$ and $n = 3$, then $\psi(\lambda) = \ldots 001101|010011 \ldots$

(26)
$$\begin{array}{cc} \psi\left(\nu^{(0)}\right) = \ldots 001|001 \ldots & \psi(w_0) = \ldots 000|011 \ldots, \\ \psi\left(\nu^{(1)}\right) = \ldots 000|111 \ldots \longmapsto & \psi(w_1) = \ldots 000|111 \ldots, \\ \psi\left(\nu^{(2)}\right) = \ldots 011|011 \ldots & \psi(w_2) = \ldots 001|111 \ldots. \end{array}$$

Thus

$$\psi(\omega) = \ldots 000001|011111 \ldots$$

and

$$\mathrm{quot}_3(\lambda) = \left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}\right) = ((1,1), \emptyset, (2)), \quad \mathrm{core}_3(\lambda) = \omega = (1).$$

Now we discuss the Littlewood decomposition for $\mathcal{DD}^{\mathrm{tr}}$. Let n be a positive integer, take $\lambda \in \mathcal{DD}^{\mathrm{tr}}$, and set $\psi(\lambda) = (c_i)_{i \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, as introduced in Definition 6.1, and $(\omega, \underline{\nu}) = (\mathrm{core}_n(\lambda), \mathrm{quot}_n(\lambda))$. Using (23), one has the equivalence (see for instance [Wah23]):

$$\lambda \in \mathcal{DD}^{\mathrm{tr}} \iff \forall i_0 \in \{0, \ldots, n-1\}, \forall j \in \mathbb{N}, c_{i_0+jn} = 1 - c_{-i_0-jn-2}$$
$$\iff \forall i_0 \in \{0, \ldots, n-1\}, \forall j \in \mathbb{N}, c_{i_0+jn} = 1 - c_{n-(i_0+2)-n(j-1)}$$
$$\iff \forall i_0 \in \{0, \ldots, n-1\}, \nu^{(i_0)} = \left(\nu^{(n-i_0-2)}\right)^{\mathrm{tr}} \quad \text{and} \quad \omega \in \mathcal{DD}^{\mathrm{tr}}_{(n)}.$$

Therefore λ is uniquely defined if its n-core is known as well as the $\lfloor n/2 \rfloor$ first elements of its n-quotient, which are partitions without any constraint. It implies that if n is odd, there is a one-to-one correspondence between a $\mathcal{DD}^{\mathrm{tr}}$ and a triplet made of one $\mathcal{DD}^{\mathrm{tr}}$ n-core, an element of $\mathcal{DD}^{\mathrm{tr}}$ and $(n-3)/2$ generic partitions. If n is even, the Littlewood decomposition is a one to one correspondence between a self-conjugate partition and a quadruplet made of one $\mathcal{DD}^{\mathrm{tr}}$ n-core, $(n-2)/2$ generic partitions, a partition of $\mathcal{DD}^{\mathrm{tr}}$ and a self-conjugate partition $\mu = \nu^{((n-1)/2)}$. Hence the restriction of the Littlewood decomposition when applied to elements of $\mathcal{DD}^{\mathrm{tr}}$ are as follows.

Lemma 6.7. [Wah23, Lemma 2.8] *Let n be a positive integer. The Littlewood decomposition Φ_n (see Definition 6.5) maps a conjugate doubled distinct partition λ to $\left(\omega, \nu^{(0)}, \ldots, \nu^{(n-1)}\right) = (\omega, \underline{\nu})$ such that:*

$(DD'1)$ *the first component ω is a $\mathcal{DD}^{\mathrm{tr}}$ n-core and $\nu^{(0)}, \ldots, \nu^{(n-1)}$ are partitions,*

$(DD'2)$ $\forall j \in \{0, \ldots, \lfloor n/2 \rfloor - 1\}, \nu^{(j)} = \left(\nu^{(n-2-j)}\right)^{\mathrm{tr}}, \nu^{(n-1)} \in \mathcal{DD}^{\mathrm{tr}},$

   and if n is even, $\nu^{(n/2-1)} = \left(\nu^{(n/2-1)}\right)^{\mathrm{tr}} \in \mathcal{SC}$,

$(DD'3)$ $|\lambda| = \begin{cases} |\omega| + 2n \displaystyle\sum_{i=0}^{(n-3)/2} |\nu^{(i)}| + n|\nu^{(n-1)}| & \text{if } n \text{ is odd,} \\ |\omega| + 2n \displaystyle\sum_{i=0}^{n/2-2} |\nu^{(i)}| + n|\nu^{(n-1)}| + n|\nu^{(n/2-1)}| & \text{if } n \text{ is even,} \end{cases}$

$(DD'4)$ $\mathcal{H}_n(\lambda) = n\mathcal{H}(\underline{\nu}).$

where a set S,

$$tS := \{ns, s \in S\}$$

and

$$\mathcal{H}(\underline{\nu}) := \bigcup_{i=0}^{n/2-1} \left(\mathcal{H}\left((\nu^{(i)})\right) \uplus \mathcal{H}\left(\nu^{(n-2-i)}\right)\right).$$

6.2. **Cores and multicharges.** Let us first start by defining the multicharge associated to a core.

Definition 6.8. Let $n \geq 2$ be an integer and consider:

$$\phi_n : \begin{vmatrix} \mathcal{C}_n & \to & \mathbb{Z}^n \\ \omega & \mapsto & (m_0, \ldots, m_{n-1}), \end{vmatrix}$$

with $m_i := \min\{k \in \mathbb{Z} \mid c_{kn+i} = 1\} = \min\{k - \lambda_k^{\mathrm{tr}} - 1, k \in \mathbb{N} \mid k - \lambda_k^{\mathrm{tr}} - 1 \equiv i \pmod{n}\}.$

We then get the following theorem.

Theorem 6.9. [Joh18, Theorem 2.10][GKS90, Bijection 2] *Let ω be a n-core and $\psi(\omega) = (c_k)_{k \in \mathbb{Z}}$ be its corresponding word (see Definition 6.1). The map ϕ_n is a bijection from $\mathcal{C}_n$ to $\mathbb{Z}_0^n := \{(m_i)_{0 \leq i \leq n-1} \in \mathbb{Z}^n \mid \sum_{i=0}^{n-1} m_i = 0\}$. Moreover, we have:*

$$(27) \qquad |\omega| = \frac{n}{2} \sum_{i=0}^{n-1} m_i^2 + \sum_{i=0}^{n-1} im_i.$$

Given an n-core ω, recall that a_i counts the number of nodes (or boxes) in the Young diagram of c with residues i. One can also define the vector $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$.

$$(28) \qquad \beta_i := a_{(i-1) \bmod n} - a_{i \bmod n}$$

for any $i = 1, \ldots, n-1$. It can be proven (see for instance [Ros23]) that the two vectors $(\beta_1, \ldots, \beta_n)$ and $(a_0, \ldots, a_{n-1})$ are related according to the following formulas

$$a_0 = \frac{1}{2} \|\beta\|_2^2 = \frac{1}{2}(\beta_1^2 + \cdots + \beta_n^2)$$

and for any $i = 1, \ldots, n-1$

$$a_i = a_0 - \beta_1 - \cdots - \beta_i.$$

In fact both vectors $(\beta_1, \ldots, \beta_n)$ and $(m_0, \ldots, m_{n-1})$ are related by the relation

$$m_i = \beta_{n-i} \text{ for any } i = 0, \ldots, n-1.$$

Remark 6.10. Observe also that when ω is regarded as an element of the affine Grassmannian of type $A_n^{(1)}$, the vector β is the one appearing in (7).

In the case of a self-conjugate n-core, we get the following corollary of Theorem 6.9 (see [GKS90, Section 7 and 8] and [Wah23, Section 4.1] for details).

Corollary 6.11. *Let* $\omega \in \mathcal{C}_n$ *be a n-core. Then* $\omega \in \mathcal{C}_n^s$ *(i.e. is a self-conjugate n-core) if and only if* $\phi_n(\omega) = (m_0, \ldots, m_{n-1}) \in \mathbb{Z}^n$ *(see Definition 6.8) satisfies* $m_{n-1-i} = -m_i$ *for any* $i = 0, \ldots, \lfloor n/2 \rfloor$. *In particular, if n is odd, we have* $m_{\lfloor n/2 \rfloor} = -m_{\lfloor n/2 \rfloor} = 0$. *Moreover, the map from $\mathcal{C}_n^s$ to $\mathbb{Z}^{\lfloor n/2 \rfloor}$ that sends ω to* $(m_0, \ldots, m_{\lfloor n/2 \rfloor - 1})$ *is a bijection and we have the equality*

$$|\omega| = n \sum_{i=0}^{\lfloor n/2 \rfloor - 1} m_i^2 + \sum_{i=0}^{\lfloor n/2 \rfloor - 1} (2i - n + 1)m_i = n \sum_{i=1}^{\lfloor n/2 \rfloor} \beta_i^2 - \sum_{i=1}^{\lfloor n/2 \rfloor} (2i - 1)\beta_i.$$

Observe that in the right-hand side of the above equality seen as a polynomial in $(m_1, \ldots, m_{\lfloor (n-1)/2 \rfloor})$, the coefficients of the m_i's have a parity opposite to the leading coefficient. Moreover, none of the coefficients of the monomial is divisible by the leading coefficient. A particular case important for the paper is that of the core c in $\mathcal{C}_{2n}^s$. We then get $\phi_{2n}(c) = (m_0, m_1, \ldots, m_{2n-1})$ with $m_{2n-1-i} = -m_i$ for any $i = 0, \ldots, n-1$ and by Proposition 5.4

$$(29) \qquad |c| = 2n \sum_{i=0}^{n-1} m_i^2 + \sum_{i=0}^{n-1} (2i - 2n + 1)m_i = L_{A_{2n-1}^{(1)}}(c) = L_{C_n^{(1)}}(c)$$

We get a similar corollary for the set of "doubled distinct n cores": $\mathcal{C}_n^{dd} := \mathcal{C}_n \cap \mathcal{DD}$.

Corollary 6.12. *Let* $\omega \in \mathcal{C}_n$ *be a n-core. Then ω belongs to $\mathcal{C}_n^{dd}$ if and only if* $\phi_n(\omega) = (m_0, \ldots, m_{n-1}) \in \mathbb{Z}^n$ *satisfies* $m_0 = 0$ *and* $m_{n-i} = -m_i$ *for all* $i \in \{1, \ldots, \lfloor (n-1)/2 \rfloor\}$. *Moreover, the map from $\mathcal{C}_n^{dd}$ to $\mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$ that sends ω to* $(m_1, \ldots, m_{\lfloor (n-1)/2 \rfloor})$ *is a bijection and we have the equality*

$$(30) \qquad |\omega| = n \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} m_i^2 + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} (2i - n)m_i.$$

Note that in the right-hand side of the above equality seen as a polynomial in $(m_1, \ldots, m_{\lfloor (n-1)/2 \rfloor})$, the coefficients of the m_i's share the same parity as the leading coefficient.

Since the elements of $\mathcal{C}_n^{dd} := \mathcal{C}_n \cap \mathcal{DD}$ are not self-conjugate in general, it is relevant to consider $\mathcal{C}_n^{dd,\mathrm{tr}} = \left(\mathcal{C}_n^{dd}\right)^{\mathrm{tr}}$, the conjugate set of doubled distinct n cores. Let $\phi_n(\omega^{\mathrm{tr}}) = (-m_{n-1}, \ldots, -m_0)$, with $\omega \in \mathcal{C}_n^{dd}$. Then by setting $m_i' = -m_{n-i-1}, i = 0, \ldots n-1$, one gets $\phi_n(\omega^{\mathrm{tr}}) = (m_0', \ldots, m_{n-1}')$ with $m_{n-1}' = 0$ and $m_i' = m_{i+1}$ for all $i \in \{0, \ldots, \lfloor (n-1)/2 \rfloor - 1\}$. Therefore the map from $\mathcal{C}_n^{dd,\mathrm{tr}}$ to $\mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$ that sends ω^{tr} to $(m_0', \ldots, m_{\lfloor (n-1)/2 \rfloor - 1}') = (m_1, \ldots, m_{\lfloor (n-1)/2 \rfloor}')$ is a bijection and we have this time equality

$$|\omega| = n \sum_{i=0}^{\lfloor (n-1)/2 \rfloor - 1} (m_i')^2 + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor - 1} (2i - n + 2)m_i'.$$

6.3. **Distinct core partitions and abaci.** Recall that we already introduced the sets of core-partitions $\mathcal{C}_n^s$, $\mathcal{C}_n^{dd} := \mathcal{C}_n \cap \mathcal{DD}$ and $\mathcal{C}_n^{dd,\mathrm{tr}} = \left(\mathcal{C}_n^{dd}\right)^{\mathrm{tr}}$. In order to get simple expressions of the atomic lengths as a number of boxes in all types, we will need to consider new sets of partitions obtained by doubling distinct partitions. They will not be $2n$-cores in general. To do that let us introduce the following set of partitions. Write

- $\mathcal{C}_n^d$ for the set of distinct partitions $\bar{\lambda}$ such that $n \notin \mathcal{H}(\bar{\lambda}) = \{h_s, s \in \lambda, \bar{\lambda}\bar{\lambda} \mid \varepsilon = +\}$
- $\mathcal{C}_n^{d,r}$ be the subset of $\mathcal{C}_n^d$ such that $n/2$ is not a part of $\bar{\lambda}$,
- $\mathcal{C}_n^{d,\mathrm{tr}}$ be the set of distinct partitions such that $n \notin \mathcal{H}(\bar{\lambda}^{\mathrm{tr}}) = \{h_s, s \in \bar{\lambda}\bar{\lambda}, \varepsilon = -\} = \{h_s, s \in (\bar{\lambda}\bar{\lambda})^{\mathrm{tr}} \backslash \Delta, \varepsilon = +\} \backslash \bar{\lambda}$
- $\mathcal{C}_n^{d,\mathrm{tr},r}$ be the subset of $\mathcal{C}_n^{d,\mathrm{tr}}$ such that $n/2 \notin \bar{\lambda}$. Note that the last condition is equivalent to the fact that the main diagonal Δ of $(\bar{\lambda}\bar{\lambda})^{\mathrm{tr}}$ does not contain a box whose hook length is equal to n.

By abuse of notation, we will refer to the sets $\mathcal{C}_n^{d,\mathrm{tr}}$ and $\mathcal{C}_n^{d,\mathrm{tr},r}$ as sets of distinct core partitions. The following lemma obtained from ((21)), shows that how the sets $\mathcal{C}_n^{d,r}$ and $\mathcal{C}_n^{dd}$ can be simply related.

Lemma 6.13. *Let n be an integer greater or equal to 2. Take $\bar{\lambda} \in \mathcal{D}$. Then $\bar{\lambda}$ belongs to $\mathcal{C}_n^{d,r}$ if and only if $\bar{\lambda}\bar{\lambda}$ belongs to $\mathcal{C}_n^{dd}$, the set of doubled distinct n-core partitions.*

Warning: when $\bar{\lambda}$ belongs to $\mathcal{C}_n^{d,\mathrm{tr}}$, its doubled version $\bar{\lambda}\bar{\lambda}$ does not belong to $\mathcal{C}_n^{dd,\mathrm{tr}}$ in general.

As a consequence of the previous lemma, one derives the following proposition

Proposition 6.14. *Let n be an integer greater or equal to 2. Take $\bar{\lambda} \in \mathcal{C}_{2n+a}^{d,r}$ and $a \in \{1, 2\}$. Set $\phi_{2n+a}(\bar{\lambda}\bar{\lambda}) = (m_0, \ldots, m_{2n+a-1})$ (see Definition 6.8). Then $\bar{\lambda}$ is in bijective correspondence with $\phi_{2n+a}(\bar{\lambda}\bar{\lambda})$ and*

$$\{\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell\} = \cup_{i=1}^{2n+a-1}\{\{(2n + a)k + i, 0 \leq k \leq m_i - 1 \mid m_i > 0\}.$$

Proof. Let a be either 1 or 2. Now let describe the connection between the word corresponding to $\bar{\lambda}\bar{\lambda} \in \mathcal{C}^{dd}_{2n+a}$ and $\bar{\lambda}$. Set $\phi_{2n+a}(\bar{\lambda}\bar{\lambda}) = (m_0, m_1, \ldots, m_n, m_{n+1}, m_{n+2}, \ldots, m_{2n+a-1})$ (see Definition 6.8) and $\psi(\bar{\lambda}\bar{\lambda}) = (c_k)_{k\in\mathbb{Z}}$ (see Definition 6.1), then by definition of ϕ_{2n+a}, $(2n+a)m_i + i = \max\{(k+1)g + i \mid c_{kg+i} = 0\}$ for any $0 \leq i \leq 2n + a - 1$. Take $1 \leq i \leq n$. If $\max(m_{2n+a-i}, m_i) > 0$, then Lemma 6.4 guarantees that there exists $k \in \{1, \ldots, \ell\}$ such that $\bar{\lambda}_k = \max((2n+a)(m_{2n+a-i} - 1) + 2n + a - i, (2n+a)(m_i - 1) + i)$. Moreover if $m_i = 0$, by Lemma 6.4, this is then equivalent to

$$\{\bar{\lambda}_k, k \in \{1, \ldots, \ell\}\{\mid \bar{\lambda}_k \equiv \pm i \pmod{2n+a}\} = \emptyset.$$

Therefore we have the set equality:

$$\{\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell\} = \cup_{i=1}^{2n+a-1}\{\{(2n+a)k + i, 0 \leq k \leq m_i - 1 \mid m_i > 0\}.$$

$\square$

As already mentioned, when $\bar{\lambda}$ belongs to $\mathcal{C}^{d,\mathrm{tr}}_n$, its doubled version $\bar{\lambda}\bar{\lambda}$ does not belong to $\mathcal{C}^{dd,\mathrm{tr}}_n$. Nevertheless, the core and the quotient of $\bar{\lambda}\bar{\lambda}$ take a very particular form as explained in the folloowing proposition which can be deduced from Lemma 6.7.

Proposition 6.15. *Let n be an integer greater or equal to 2. Take $\bar{\lambda} \in \mathcal{D}$ and set $\Phi_n((\bar{\lambda}\bar{\lambda})^{\mathrm{tr}}) = (\omega, \nu^{(0)}, \ldots, \nu^{(n-1)})$ (see Definition 6.5). Then $\bar{\lambda}$ belongs to $\mathcal{C}^{d,\mathrm{tr},r}_n$ if and only if:*

- *$\nu^{(j)} = \emptyset$ for any $0 \leq j \leq n - 2$,*
- *let $m = \max\left(\{1\} \cup \{(n + \bar{\lambda}_i)/n \mid \bar{\lambda}_i \equiv 0 \pmod{n}\}\right)$. Then $\nu^{(n-1)}$ is the rectangle $(m-1) \times m$ partition.*

Moreover, in the odd case, setting $\phi_{2n-1}(\omega) = (m_0, \ldots, m_{n-2}, -m_{n-2}, \ldots, -m_0, 0)$ one gets

$$(31) \qquad |\bar{\lambda}| = \frac{2n-1}{2}\left(m^2 + \sum_{i=0}^{n-2} m_i^2\right) + \left(-\frac{2n-1}{2}m + \sum_{i=0}^{n-2}(i - n + \frac{1}{2})m_i\right).$$

In the even case, by setting $\phi_{2n}(\omega) = (m_0, \ldots, m_{n-2}, 0, -m_{n-2}, \ldots, -m_0, 0)$ one gets

$$(32) \qquad |\bar{\lambda}| = n\left(m^2 + \sum_{i=0}^{n-2} m_i^2\right) + \left(-nm + \sum_{i=0}^{n-2}(i - n + 1)n_i\right).$$

Proof. Consider $\bar{\lambda} \in \mathcal{C}^{d,\mathrm{tr},r}_n$, the Littlewood decomposition $\Phi_n(\bar{\lambda}\bar{\lambda})^{\mathrm{tr}}) = (\omega, \nu^{(0)}, \ldots, \nu^{(n-1)})$ (see Definition 6.5) and $\psi(\bar{\lambda}\bar{\lambda})^{\mathrm{tr}} = (c_k)_{k\in\mathbb{Z}}$ its corresponding word (see Definition 6.1). Since $n/2 \notin \bar{\lambda}$, the main diagonal Δ of $\bar{\lambda}\bar{\lambda})^{\mathrm{tr}}$ has no box of hook length n. If there exists $i \in \{0, \ldots, n-2\}$ such that $\nu^{(i)} \neq \emptyset$, there exists at least one box in $\nu^{(k)}$ whose hook length is equal to 1. By Lemma 6.2 and by property $(DD'4)$ from Lemma 6.7, there exists $m \in \mathbb{Z}$ such that $c_{mn+i} = 1$ and $c_{(m+1)n+i} = 0$. Note that the hook length of the box $(mn+i, (m+1)n+i)$ is equal to n. If $i \not\equiv -i - 2$ (mod n), that is if $i \neq n/2$, by (25), one gets that $c_{-(m+1)n-i-2} = 1$ and $c_{-mn-i-2} = 0$. Moreover one of the two boxes corresponding to the pairs of indices $(mn+i, (m+1)n+i)$ and $(-(m+1)n-i-2, -mn-i-2)$ is strictly above the main diagonal Δ, which contradicts the fact that $\bar{\lambda} \in \mathcal{C}^{d,\mathrm{tr},r}_n$. If $i = n/2$, which implies that n is even in this case, using the same arguments, $\bar{\lambda} \in \mathcal{C}^{d,\mathrm{tr},r}_n$ implies that there is no hook of length n apart maybe on the diagonal Δ. Since that $n/2 \notin \bar{\lambda}$, we have that $\nu^{(n/2)} = \emptyset$. Therefore $\bar{\lambda} \in \mathcal{C}^{d,\mathrm{tr},r}_n$ implies that $\nu^{(i)} = \emptyset$ for any $0 \leq j \leq n-2$.

Now we prove the second part of the statement. Let d be the size of the Durfee square of $\nu^{(n-1)}$. Using the same arguments as above, if $\nu^{(n-1)}$ has a hook length equal to 1 in any row but the $d+1$-th row, the properties of symmetry along the main diagonal and the property $(DD'4)$ from Lemma 6.7 imply that $\nu^{(n-1)}$ contains a unique box whose hook length is equal to 1. Therefore $\nu^{(n-1)}$ is a rectangle of size $m \times (m-1)$ with $m = \max\left(\{1\} \cup \{(n + \bar{\lambda}_i)/n \mid \bar{\lambda}_i \equiv 0 \pmod{n}\}\right)$ or equivalently $m = \max(k + 1 \mid c_{kn+n-1} = 0)$. Note that the word to $\nu^{(n-1)}$ is:

$$\psi(\nu^{(n-1)}) = \ldots 0\underbrace{1 \ldots 1}_{m-1}0 \mid \underbrace{0 \ldots 0}_{m}1 \ldots$$

Equalities (31) and (32) are derived using (30) and Lemma 6.7 $(DD'3)$. $\square$

Using the same arguments as for Proposition 6.15, we obtain similarly for $\mathcal{C}^{d,\mathrm{tr}}_n$.

Proposition 6.16. *Let n be an integer greater or equal to 2. Take $\bar{\lambda} \in \mathcal{D}$ and set the Littlewood decomposition $\Phi_n((\bar{\lambda}\bar{\lambda})^{\mathrm{tr}}) = (\omega, \nu^{(0)}, \ldots, \nu^{(n-1)})$. Then $\bar{\lambda}$ belongs to $\mathcal{C}^{d,\mathrm{tr}}_n$ if and only if:*

- *$\nu^{(j)} = \emptyset$ for any $0 \leq j \leq n - 2 \setminus \{\frac{n}{2}\}$,*

- let $m = \max\left(\{1\} \cup \{(n + \bar{\lambda}_i)/n \mid \bar{\lambda}_i \equiv 0 \pmod{n})\}\right)$. Then $\nu^{(n-1)}$ is the rectangle partition $(m - 1) \times m$. Moreover, when n is even, then $\nu^{(n/2-1)}$ is the $m' \times m'$ with

$$m' = \max\left(\{0\} \cup \{\bar{\lambda}_i/n \mid \bar{\lambda}_i \equiv n/2 \pmod{n}\}\right)$$

or equivalently $m' = \max(k + 1 \mid c_{kn+n/2-1} = 0)$.

In addition, by setting $\phi_{2n-2}(\omega) = (m_0, \ldots, m_{n-3}, 0, -m_{n-3}, \ldots, -m_0, 0)$, we get

$$(33) \qquad |\bar{\lambda}| = (2n - 2)\left(\sum_{i=0}^{n-3} m_i^2 + m^2 + m'^2\right) + \left(-(2n - 2)m + \sum_{i=0}^{n-2} 2(i - n + 1)m_i\right).$$

The fore coming subsections use the same arguments. We first use the result from Section 5 giving the correspondence between the affine Grassmannian elements and some subsets of self-conjugate $(2n)$-cores. Next, by considering the bijection described in Theorem 6.9, we exhibit the correspondence between the affine Grassmannian elements and some subsets of integer partitions so that the atomic length of any affine Grassmannian element coincides with the weight (i.e. the number of boxes) of the corresponding partition.

6.4. Type $C_n^{(1)}$.

This case is easy since for any $c \in W^0_{C_n^{(1)}}$ identified to $\mathcal{C}^s_{2n}$, we have $L_{C_n^{(1)}}(c) = L_{A_{2n-1}^{(1)}}(c) = |c|$ i.e. the atomic length is given by the number of boxes of c regarded as a self-conjugate $2n$-core. Therefore as already seen in (29)

$$(34) \qquad L_{C_n^{(1)}}(c) = 2n\sum_{i=0}^{n-1} m_i^2 + \sum_{i=0}^{n-1}(2i - 2n + 1)m_i = 2n\sum_{i=0}^{n-1}\beta_i^2 - \sum_{i=0}^{n-1}(2i - 1)\beta_i.$$

Observe also that we have $a_0 = \sum_{i=0}^{n-1}\beta_i^2$ and $a_n = a_0 - \beta_1 - \cdots - \beta_n$ from which it becomes easy to compute the atomic length in any type from the results of Section 5.

6.5. Type $D_{n+1}^{(2)}$.

Recall from Proposition 5.5, that if $c \in W^0_{D_{n+1}^{(2)}} = \mathcal{C}^s_{2n}$, we get:

$$L_{D_{n+1}^{(2)}}(c) = \frac{1}{2}\left((2n + 2)\sum_{i=1}^{n}\beta_i^2 - \sum_{i=1}^{n} 2i\beta_i\right) = (n + 1)\sum_{i=1}^{n}\beta_i^2 - \sum_{i=1}^{n} i\beta_i$$

$$= (n + 1)\sum_{i=0}^{n-1} m_i^2 + \sum_{i=0}^{n-1}(i - n)m_i.$$

Set $c \in W^0_{D_{n+1}^{(2)}} = \mathcal{C}^s_{2n}$ and $\phi_{2n}(c) = (m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -m_0)$ (see Definition 6.8). Set $c' = \phi_{2n+2}^{-1}(0, m_0, \ldots, m_{n-1}, 0, \ldots)$ One easily sees by using that $L_{D_{n+1}^{(2)}}(c) = \frac{1}{2}L_{C_n^{(1)}}(c')$ which suggest that c could be replaced by c' in order to get an atomic length counted by the number of boxes in a partition. In fact, we have the following stronger statement:

Proposition 6.17. *The core c' belongs to $\in \mathcal{C}^{dd}_{2n+2}$ and there exists $\bar{\lambda} \in \mathcal{C}^{d,r}_{2n+2}$ such that $c' = \bar{\lambda}\bar{\lambda}$. Moreover, the map which associates to any $c \in \mathcal{C}^s_{2n}$ the so obtained distinct partition $\bar{\lambda} \in \mathcal{C}^{d,r}_{2n+2}$ is a bijection such that $L_{D_{n+1}^{(2)}}(c) = |\bar{\lambda}|$.*

Proof. The first sentence of the proposition is a consequence of Corollary 6.12 (even case) and Lemma 6.13. Now the map which sends $c \in \mathcal{C}^s_{2n}$ to $\bar{\lambda} \in \mathcal{C}^{d,r}_{2n+2}$ is bijective by composition. The equality $L(c) = |\bar{\lambda}|$ derives from Proposition 6.14, (30) and (29) setting $\beta_i = m_{2n+2-i}$. $\square$

Example 6.18. Take $n = 2$ and $c = (3, 2, 1) \in W^0_{D_3^{(2)}} = \mathcal{C}^s_4$ and $\phi_4(c) = (1, -1, 1, -1)$. Then we obtain $(c_6')^{-1}(0, 1, -1, 0, 1, -1)$ $(5, 3, 1, 1)$ and $\bar{\lambda} = (4, 1)$. We have $L_{D_{n+1}^{(2)}}(c) = |c'| = 5$.

6.6. Type $A_{2n}^{(2)}$.

Recall from Proposition 5.6, that if $c \in W^0_{A_{2n}^{(2)}} = \mathcal{C}^s_{2n}$, then by using (28), one derives:

$$L_{A_{2n}^{(2)}}(c) = \frac{1}{2}\left((2n + 1)\sum_{i=1}^{n}\beta_i^2 - \sum_{i=1}^{n}(2i - 1)\beta_i\right).$$

One can observe that in this case the coefficient of the monomial β_n is $-(2n+1)/2$ which happens to be the opposite of the coefficient of β_n^2. Set $\phi_{2n}(c) = (m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -m_0)$ and $c' = \phi_{2n+1}^{-1}(0, m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -m_0)$

(see Definition 6.8). Using (30), c' is a doubled distinct $(2n+1)$-core. Therefore, by Lemma 6.13, there exists a unique $\bar{\lambda} \in C^{d,r}_{2n+1}$ such that $c' = \bar{\lambda}\lambda$. Moreover we have that:

$$L_{A^{(2)}_{2n}}(c) = |\bar{\lambda}|.$$

Example 6.19. Take $n = 2$ and $c = (3, 2, 1) \in W^0_{A'^{(2)}_4} = C^s_4$ and $\phi_4(c) = (1, -1, 1, -1)$. Then $c'^{-1}_5(0, 1, -1, 1, -1) = (4, 3, 1)$ and $\bar{\lambda} = (3, 1)$. We have $L(c') = 8$.

6.7. Type $A'^{(2)}_{2n}$. Recall from Proposition 5.7, that if $c \in W^0_{A^{(2)}_{2n}} = C^s_{2n}$, then by using (28), one derives:

$$L_{A'^{(2)}_{2n}}(c) = (2n+1) \sum_{i=1}^{n} \beta_i^2 - \sum_{i=1}^{n} 2i\beta_i.$$

Set $\phi_{2n}(c) = (m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -m_0)$ and $c' = \phi^{-1}_{2n+1}(m_0, \ldots, m_{n-1}, 0, -m_{n-1}, \ldots, -m_0)$. The map sending $c \in C^s_n$ to $c' \in C^s_{2n+1}$ is a bijection. Moreover, by Corollary 6.11, we have:

$$L_{A'^{(2)}_{2n}}(c) = |c'|.$$

Example 6.20. Take $n = 2$ and $c = (3, 2, 1) \in W^0_{A'^{(2)}_4} = C^s_4$ and $\phi_4(c) = (1, -1, 1, -1)$. Then $c' = \phi^{-1}_5(1, -1, 0, 1, -1) = (4, 2, 1, 1)$. We have $L(c) = |c'| = 8$.

6.8. Type $B^{(1)}_n$. For all the remaining types in this section, the affine Grassmannian elements are in bijection with the subset $C^{s,p}_{2n}$ of self-conjugate $(2n)$-cores with an even diagonal. So take $c \in C^{s,p}_{2n}$ and set $\phi_{2n}(c) = (m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -)$. A core has an even diagonal if and only if the number of horizontal steps after the corner of the Durfee square is even. This is equivalent to the fact that the number of letters "0" of positive index in the corresponding word $\psi(c)$ is even. Therefore $c \in C^{s,p}_{2n}$ is equivalent to

$$\sum_{i=0}^{n-1} |m_i| = \sum_{i=0}^{n-1} |\beta_i| \equiv 0 \pmod 2.$$

One derives from Proposition 5.9 that for any $c \in W^0_{B^{(1)}_n} = C^{s,p}_{2n}$

$$L_{B^{(1)}_n}(c) = \frac{1}{2}\left(2n \sum_{i=1}^{n} \beta_i^2 - \sum_{i=1}^{n} 2i\beta_i\right) = n \sum_{i=1}^{n} \beta_i^2 - \sum_{i=1}^{n} i\beta_i.$$

Consider $\bar{\lambda} \in C^{d,\mathrm{tr},r}_{2n}$ and set $\Phi_{2n}((\bar{\lambda}\lambda)^{\mathrm{tr}}) = (\omega, \nu^{(0)}, \ldots, \nu^{(n-1)})$. By Proposition 6.15, we obtain $\phi_{2n}(\omega) = (m_0, \ldots, m_{n-2}, 0, -)$ and $\nu^{(n-1)}$ is a rectangle $(m-1) \times m$ with $m \in \mathbb{N}^*$. This permits to define a bijection f_{2n} associating to each $\bar{\lambda}$ its corresponding vector $(m, m_0, \ldots, m_{n-2}) \in \mathbb{N}^* \times \mathbb{Z}^{n-1}$. We can now consider the map

$$g : \begin{array}{|ccc} & \mathbb{N}^* \times \mathbb{Z}^{n-1} & \to & \mathbb{Z}^{2n} \\ & (m, m_0, \ldots, m_{n-2}) & \mapsto & \begin{cases} (m, m_0, \ldots, m_{n-2}, -m_{n-2}, \ldots, -m_0, -m) & \text{if } m \equiv \sum_{i=0}^{n-2} m_i \pmod 2, \\ (-m+1, m_0, \ldots, m_{n-2}, -m_{n-2}, \ldots, -m_0, m-1) & \text{otherwise.} \end{cases} \end{array}$$

which is a bijection from $\mathbb{N}^* \times \mathbb{Z}^{n-1}$ to

$$\{(m_0, \ldots, m_{n-1}, -m_{n-1}, \ldots, -m_0) \in \mathbb{Z}^{2n} \mid \sum_{i=0}^{n-1} |m_i| \equiv 0 \pmod 2\}.$$

So, the map $\phi^{-1}_{2n} \circ g \circ f_{2n}$ is the desired bijection from $C^{d,\mathrm{tr},r}_{2n}$ to $C^{s,p}_{2n}$. Moreover by Proposition 6.15, by setting $\bar{\lambda} = (\phi^{-1}_{2n} \circ g \circ f_{2n})^{-1}(c)$, we get:

$$L_{B^{(1)}_n}(c) = |\bar{\lambda}|.$$

Example 6.21. Take $n = 3$ and $c = (10, 6, 6, 4, 3, 3, 1, 1, 1, 1) \in W^0_{B^{(1)}_3} = C^{s,p}_6$ and $\phi_6(c) = (1, -1, -2, 2, 1, -1)$. Then $g^{-1} \circ \phi_6(c) = (1, -1, -2)$. Then $c' = (10, 6, 6, 3, 3, 3, 3, 1, 1, 1, 1)$ and $\bar{\lambda} = (10, 5, 4)$. We have $L_{B^{(1)}_3}(c) = |\bar{\lambda}| = 19$.

6.9. Type $A_{2n-1}^{(2)}$. Recall from Proposition 5.11 that if $c \in W^0_{A_{2n-1}^{(2)}} = \mathcal{C}_{2n}^{s,p}$, we have:

$$L_{A_{2n-1}^{(2)}}(c) = \frac{1}{2}\left((2n-1)\sum_{i=1}^{n}\beta_i^2 - \sum_{i=1}^{n}(2i-1)\beta_i \right).$$

As in the previous subsection, by Proposition 6.15, $\bar{\lambda} \in \mathcal{C}_{2n-1}^{d,\mathrm{tr},r}$ is mapped bijectively to the vector $(m, m_0, \ldots, m_{n-2}) \in \mathbb{N}^* \times \mathbb{Z}^{n-1}$. Let us denote this bijection by f_{2n-1}.

Therefore c is mapped bijectively to $\bar{\lambda} \in \mathcal{C}_{2n-1}^{d,\mathrm{tr},r}$ with $\bar{\lambda} = (\phi_{2n}^{-1} \circ g \circ f_{2n-1})^{-1}(c)$. Moreover

$$L_{A_{2n-1}^{(2)}}(c) = |\bar{\lambda}|.$$

Example 6.22. Take $n = 3$ and $c = (10, 6, 6, 4, 3, 3, 1, 1, 1, 1) \in W^0_{A_5^{(2)}} = \mathcal{C}_6^{s,p}$ and $\phi_6(c) = (1, -1, -2, 2, 1, -1)$. Then $g^{-1} \circ \phi_6(c) = (1, -1, -2)$. Then $c' = (8, 5, 5, 3, 3, 3, 1, 1, 1)$ and $\bar{\lambda} = (8, 4, 3)$. We have $L_{A_5^{(2)}}(c) = |\bar{\lambda}| = 15$.

6.10. Type $D_n^{(1)}$. Recall from Proposition 5.12 that if $c \in W^0_{D_n^{(1)}} = \mathcal{C}_{2n}^{s,p}$, we have then by using (28)

$$L_{D_n^{(1)}}(c) = (n-1)\sum_{i=1}^{n}\beta_i^2 - \sum_{i=1}^{n}(i-1)\beta_i.$$

By Proposition 6.16, $\bar{\lambda} \in \mathcal{C}_{2n}^{d,\mathrm{tr}}$ is mapped bijectively to the vector $(m, m_0, \ldots, m_{n-3}, m') \in \mathbb{N}^* \times \mathbb{Z}^{n-1} \times \mathbb{N}$. Let us denote this bijection by f_{2n-2}. Therefore c is mapped bijectively to $\bar{\lambda} \in \mathcal{C}_{2n-2}^{d,\mathrm{tr}}$ with $\bar{\lambda} = (\phi_{2n}^{-1} \circ g \circ f_{2n-2})^{-1}(c)$. Moreover

$$L_{D_n^{(1)}}(c) = |\bar{\lambda}|.$$

6.11. Type $D_4^{(3)}$. From Proposition 5.15, for any $c \in W^0_{D_4^{(3)}}$, note that $L_{D_4^{(3)}}(c)$ has the same expression as the atomic length of an element of type $D_3^{(1)}$, which does not exist. Nevertheless the subset $\mathcal{C}_{2n-2}^{d,\mathrm{tr}}$ is well-defined for $n = 3$: it is the subset of distinct partitions such that $4 \notin \mathcal{H}(\bar{\lambda}^{\mathrm{tr}}) = \{h_s, s \in \bar{\lambda}\bar{\lambda}, \varepsilon = -\}$. Moreover the equation (18) implies that $\beta_3 = \beta_1 - \beta_2$. By Proposition 6.16, $c \in W^0_{D_4^{(3)}} = \mathcal{C}_6^g$ is mapped bijectively to $\bar{\lambda} \in \mathcal{C}_4^{d,\mathrm{tr}}$ with $\bar{\lambda} = (\phi_6^{-1} \circ g \circ f_4)^{-1}(c)$ such that $m = \pm m' \pm m_0 - \varphi$, where m, m' and m_0 are as defined in Proposition 6.16 and $\varphi \in \{0, 1\}$. Moreover

$$L_{D_4^{(3)}}(c) = |\bar{\lambda}|.$$

6.12. Type $G_2^{(1)}$. Similarly to the previous subsection, from Proposition 5.16), note that this time for any $c \in W^0_{G_2^{(1)}}$, $L_{G_2^{(1)}}(c)$ has the same expression as the atomic length of an element type $B_3^{(1)}$ with an additional restriction.

By Proposition 6.15, $c \in W^0_{G_2^{(1)}}$ is mapped bijectively to $\bar{\lambda} = (\phi_6^{-1} \circ g \circ f_6)^{-1}(c)$ with the additional condition that m is equal to $m_0 + m_1$ if $m \equiv m_0 + m_1 \pmod 2$ or to $-m_0 - m_1 - 1$ otherwise, where m, m_0, m_1 are defined as in Proposition 6.15. Moreover

$$L_{G_2^{(1)}}(c) = |\bar{\lambda}|.$$

7. More on n-cores and their n-abacus

The goal of this section is first to use the connection between the affine Grassmannian elements and the combinatorial models described in Section 6 to compute $\varepsilon(c)$ as its appears in Theorem 4.3. We next reinterpret the dominant weights appearing in the Weyl characters of the decomposition obtained in Theorem 4.3. These results can be deduced from those obtained in [Wah23] and we omit the proofs for short.

Take n a positive integer. Let us introduce

$$H_n(\lambda) := \{s \in \lambda \mid h_s < n\},$$
$$H_n(\bar{\lambda}) := \{h_s \in \mathcal{H}(\bar{\lambda}) \mid h_s < n\},$$
$$H_n(\bar{\lambda}^{\mathrm{tr}}) := \{h_s \in \mathcal{H}(\bar{\lambda}^{\mathrm{tr}}) \mid h_s < n\},$$
$$H_n(\bar{\lambda} \cup \Delta) := \{h_s \in \mathcal{H}(\bar{\lambda}) \uplus \{2\bar{\lambda}_1, \ldots, 2\bar{\lambda}_\ell\} \mid h_s < n\},$$
$$H_n(\bar{\lambda}^{\mathrm{tr}} \cup \Delta) := \{h_s \in \mathcal{H}(\bar{\lambda}^{\mathrm{tr}}) \uplus \{2\bar{\lambda}_1, \ldots, 2\bar{\lambda}_\ell\} \mid h_s < n\}.$$

We can in fact precise the results of the previous section and use the sets above to compute the signature of the affine Grassmannian elements as detailed in Table 1.

T	$\mathcal{C}_T$	$\varepsilon(c)$
$A_{n-1}^{(1)}$	$\mathcal{C}_n$	$(-1)^{\#H_n(\lambda)}$
$B_n^{(1)}$	$\mathcal{C}_{2n}^{d,\mathrm{tr},r}$	$(-1)^{\#H_{2n}(\bar{\lambda}^{\mathrm{tr}}\cup\Delta)+\ell(\bar{\lambda})}$
$A_{2n-1}^{(2)}$	$\mathcal{C}_{2n-1}^{d,\mathrm{tr},r}$	$(-1)^{\#H_{2n-1}(\bar{\lambda})^{\mathrm{tr}}}$
$C_n^{(1)}$	$\mathcal{C}_{2n}^{s}$	$(-1)^{\#H_{2n}(\bar{\lambda})}$
$D_{n+1}^{(2)}$	$\mathcal{C}_{2n+2}^{d,r}$	$(-1)^{\#H_{2n+2}(\bar{\lambda}\cup\Delta)+\ell(\bar{\lambda})}$
$A_{2n}^{(2)}$	$\mathcal{C}_{2n+1}^{s}$	$(-1)^{\#H_{2n+1}(\bar{\lambda})}$
$A_{2n}'^{(2)}$	$\mathcal{C}_{2n+1}^{d,r}$	$(-1)^{\#H_{2n+1}(\bar{\lambda}\cup\Delta)+\ell(\bar{\lambda})}$
$D_n^{(1)}$	$\mathcal{C}_{2n-2}^{d,\mathrm{tr}}$	$(-1)^{\#H_{2n-2}(\bar{\lambda}^{\mathrm{tr}}\cup\Delta)}$

TABLE 1. Table of affine types with their corresponding partition family

We recall here the following definition that bridges the abacus model appearing in [Wah23] and the affine Grassmannian elements.

Definition 7.1. Let n and g be two positive integers such that $n \leq g$. Set $\lambda \in \mathcal{P}$ and $\psi(\lambda) = (c_k)_{k \in \mathbb{Z}}$ its corresponding binary word. For $i \in \{0, \ldots, g-1\}$, define $m_i := \max\{(k+1)g + i \mid c_{kg+i} = 0\}$. Let $\sigma : \{1, \ldots, g\} \to \{0, \ldots, g-1\}$ be the unique bijection such that $\beta_{\sigma(1)} > \cdots > \beta_{\sigma(g)}$. The vector $\mathbf{v} := (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)})$ is called the $V_{g,n}$-coding corresponding to λ.

By definition of the elements of $\mathcal{C}_T$ in Table 1 and Proposition 6.14, one has the following lemma.

Lemma 7.2. *Take T in Table 1 such that $T \neq A_{n-1}^{(1)}$ and $\bar{\lambda} \in \mathcal{C}_T = \mathcal{C}_g^a$. Let μ be the corresponding partition in either $\mathcal{DD}$ (if a is d, r), $\mathcal{DD}^{\mathrm{tr}}$ (if a is d, tr, r or a is d, tr) or $\mathcal{SC}$ (in the other cases). Let $\mathbf{v} := (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)})$ be the $V_{g,n}$-coding corresponding to μ. Then for any $1 \leq i \leq n$ such that $v_i > g$, we have:*

- *if μ is a doubled distinct partition, $v_i = \max\{\bar{\lambda}_k + g, k \geq 1, \bar{\lambda}_k \equiv \sigma(i) \pmod{g}\}$*
- *otherwise, $v_i = \max\{\bar{\lambda}_k - 1, k \geq 1, \bar{\lambda}_k - 1 + g \equiv \sigma(i) \pmod{g}\}$.*

Recall from Theorem 4.3 that if $c \in W_a^0$, we can set $c = ut_\nu$ with $\nu \in M^* \cap (-P_+)$ and $u \in W^\nu$ for any element of the affine Grassmannian W_a^0. Moreover, each weight $-\eta^\vee \nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}$ belongs to P_+, the set of dominants weights for $\mathring{A}$. The purpose of the following theorem is to bridge the gaps between our reformulation of the Weyl-Kac formula as a sum over elements of the affine Grassmannian and the formulation of the same formula within the language of multivariate series and θ function (see [RS06, Proposition 6.1]), where the sum is over $\mathbb{Z}^n$. Given Remark 4.2, we set for any affine algebra $\mathfrak{g}$ of rank n from Table 2:

$$\tilde{\eta}^\vee = \begin{cases} \eta^\vee & \text{if } \mathfrak{g} \neq C_n^{(1)}, \\ 2\eta^\vee & \text{if } \mathfrak{g} = C_n^{(1)}. \end{cases}$$

Note that $\tilde{\eta}^\vee$ corresponds to the parameter g in [Mac72, Wah23].

Theorem 7.3. *With the previous notation the one-to-one correspondences described in Section 6 associate to any affine Grassmannian element c its corresponding partition λ of $\mathcal{C}_T$ in Table 1 so that by writing $\mathbf{v} = (v_i)_{1 \leq i \leq n}$ for the $V_{\tilde{\eta}^\vee, n}$ coding of λ, we get:*

$$-\tilde{\eta}^\vee \nu + u^{-1}(\mathring{\rho}) - \mathring{\rho} = (v_i + i - \tilde{\eta}^\vee)_{1 \leq i \leq n}.$$

Recall that the proof of the Nekrasov–Okounkov formula as performed by Han [Han08] heavily relies on a specialization of the Weyl-Kac in type $\tilde{A}_{n-1}^{(1)}$ and polynomiality argument. This approach can be extended to derive (u-analogues of) Nekrasov–Okounkov formulas for any classical affine type T with an unbounded rank. In order to do so from Theorem 4.3, one needs to be able to convert the characters on the right-hand side of Theorem 4.3 (or more precisely their associated u-dimension formulas) as a product over hook lengths of elements of $\mathcal{C}_T$. Actually one can reformulate the results from [Wah23, Section 4.2] for a set of Laurent variables $(X_k)_{k \in \mathbb{Z}}$ as the bridge between hook length products on the one hand and products over weights of the form $-\eta^\vee \nu + u^{-1}(\mathring{\rho}) - \mathring{\rho}$ associated with an affine Grassmannian element as in Theorem 4.3.

Theorem 7.4. *Let $(X_k)_{k\in\mathbb{Z}}$ be formal variables. For any affine type T of rank n in Table 1 and any element of the affine Grassmannian $c \in W_a^0$ such that $c = ut_\nu$ with $\nu \in Q \cap (-P_+)$ and $u \in W^\nu$. Set $\mathbf{r} = (r_i)_{1\leq i\leq n}$ the vector such that $r_i = (-\tilde{\eta}^\vee\nu + u^{-1}(\mathring{\rho}) - \mathring{\rho})_i$ for all i. Let $\lambda \in \mathcal{C}_T$ and $\mathcal{H}(\lambda)_T$ its corresponding elements in Table 1 and Table 2, respectively. Set $\alpha_T(i) = \#\{h_s \in \mathcal{H}(\lambda)_T, h_s = \tilde{\eta}^\vee - i\}$ and where $\alpha'_T(i) = \alpha_T(i) + \#\{h \in \bar{\lambda}, 2h = \tilde{\eta}^\vee - i\} + \#\{h \in \bar{\lambda}, h = \tilde{\eta}^\vee - i\}$. Then we have:*

- *if $T = A_{n-1}^{(1)}$*

$$\tag{35} \prod_{h\in\mathcal{H}(\lambda)_T} \frac{X_{h-\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_h^2} = \prod_{i=1}^{\tilde{\eta}^\vee-1}\left(\frac{X_{-i}}{X_i}\right)^{\alpha_T(i)}\Delta_T(\mathbf{r}),$$

- *if $T \in C_n^{(1)} \cup D_{n+1}^{(2)} \cup A_{2n}^{(2)} \cup A_{2n}'^{(2)}$:*

$$\tag{36} \prod_{h\in\bar{\lambda}} \frac{X_{2h-\tilde{\eta}^\vee}X_{h-\tilde{\eta}^\vee}}{X_{2h}X_h} \prod_{h\in\mathcal{H}(\lambda)_T} \frac{X_{h-\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_h^2} = \prod_{i=1}^{\tilde{\eta}^\vee-1}\left(\frac{X_{-i}}{X_i}\right)^{\alpha'_T(i)}\Delta_T(\mathbf{r}),$$

- *if $T \in B_n^{(1)} \cup A_{2n-1}^{(2)}$:*

$$\tag{37} \prod_{h\in\bar{\lambda}} \frac{X_{2h-\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_{2h}X_h} \prod_{h\in\mathcal{H}(\lambda)_T} \frac{X_{h-\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_h^2} = \prod_{i=1}^{\tilde{\eta}^\vee-1}\left(\frac{X_{-i}}{X_i}\right)^{\alpha'_T(i)}\Delta_T(\mathbf{r}),$$

- *if $T \in D_n^{(1)}$:*

$$\tag{38} \prod_{h\in\bar{\lambda}} \frac{X_{2h+\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_{2h}X_h} \prod_{h\in\mathcal{H}(\lambda)_T} \frac{X_{h-\tilde{\eta}^\vee}X_{h+\tilde{\eta}^\vee}}{X_h^2} = \prod_{i=1}^{\tilde{\eta}^\vee-1}\left(\frac{X_{-i}}{X_i}\right)^{\alpha'_T(i)}\Delta_T(\mathbf{r}),$$

T	$\mathcal{H}(\lambda)_T$	$\Delta_T(\mathbf{r})$
$A_{n-1}^{(1)}$	$\mathcal{H}(\lambda)$	$\displaystyle\prod_{1\leq i<j\leq n}\frac{X_{r_i-r_j}}{X_{j-i}}$
$B_n^{(1)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\displaystyle\prod_{1\leq i<j\leq n}\frac{X_{r_i-r_j}X_{r_i+r_j}}{X_{j-i}X_{\tilde{\eta}^\vee+2-i-j}}$
$A_{2n-1}^{(2)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\prod_{i=1}^n \frac{X_{r_i}}{X_i}\Delta_{B_n^{(1)}}(\mathbf{r})$
$C_n^{(1)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\prod_{i=1}^n \frac{X_{r_i}}{X_i}\displaystyle\prod_{1\leq i<j\leq n}\frac{X_{r_i-r_j}X_{r_i+r_j}}{X_{j-i}X_{\tilde{\eta}^\vee-i-j}}$
$D_{n+1}^{(2)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\displaystyle\prod_{1\leq i<j\leq n}\frac{X_{r_i-r_j}X_{r_i+r_j}}{X_{j-i}X_{\tilde{\eta}^\vee+1-i-j}}$
$A_{2n}^{(2)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\prod_{i=1}^n \frac{X_{r_i}}{X_i}\Delta_{D_{n+1}^{(2)}}(\mathbf{r})$
$A_{2n}'^{(2)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\displaystyle\prod_{1\leq i<j\leq n}\frac{X_{r_i-r_j}X_{r_i+r_j}}{X_{j-i}X_{\tilde{\eta}^\vee-i-j}}$
$D_n^{(1)}$	$\mathcal{H}(\bar{\lambda}^{\mathrm{tr}})$	$\prod_{i=1}^n \frac{X_{2r_i}X_i}{X_{2i}X_{r_i}}\Delta_{B_n^{(1)}}(\mathbf{r})$

TABLE 2. Table of affine types with their corresponding hook length product

Remark 7.5. The case $T = A_{n-1}^{(1)}$ in the above theorem is a reformulation of Theorem 1 in [DH11].

Remark 7.6. In the left-hand side products of the Theorem 7.4, one can note that if we take $(X_k)_{k\in\mathbb{Z}} = k$, the product of hook length is equal to 0 for any partition $\lambda \in \mathcal{P} \setminus \mathcal{C}_T$. For instance, in type $A_{n-1}^{(1)}$, the product over all the hook lengths $(1 - n^2/h^2)$ of a partition λ is equal to 0 if and only if λ is not a n-core.

The integers $\alpha_T(i)$ and $\alpha'_T(i)$ can be expressed as a refinement of the cardinal of affine inversion sets.

The remaining steps in order to derive Nekrasov–Okounkov type identities is to consider Theorem 4.3 as an equality of multivariate power series setting $x_i = e^{-\varepsilon_i}$ for $1 \leq i \leq n$. Then we specialize the set of variables (x_i) so that we can use the Weyl u-dimension formula for , then we prove that the Nekrasov–Okounkov type identities hold for an infinite

number of ranks and conclude using a polynomiality argument. The Theorem 7.4 is the corner stone to bridge the gap between the Weyl formula for characters on the one hand and the hook length products on the other.

Let u be a formal variable. The specializations of the indeterminates x_i's as well as the corresponding hook length product and the specialization of the Laurent variables used in Theorem 7.4 are summarized in the following Table, where $\varepsilon(c)$ is the signature of the corresponding Grassmannian element (see Table 1)

T	$(x_i)_{1\le i\le n}$	$(X_k)_{k\in\mathbb{Z}}$	hook length for the group character $s_{-\tilde{\eta}^\vee \nu + u^{-1}(\mathring{\rho})-\mathring{\rho}}$
$A_{n-1}^{(1)}$	$(u^{i-1})_{1\le i\le n}$	$X_k = 1 - u^k$	$\varepsilon(c)\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{h+\tilde{\eta}^\vee})(1-u^{h-\tilde{\eta}^\vee})}{(1-u^h)^2}$
$B_n^{(1)}$	$(u^{2i-1})_{1\le i\le n}$	$X_k = 1 - u^{2k}$	$\varepsilon(c)u^{-\tilde{\eta}^\vee\ell(\bar{\lambda})}\prod\limits_{h\in\bar{\lambda}}\dfrac{(1+u^{(2h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})(1-u^{2h+\tilde{\eta}^\vee})}{(1-u^{4h})(1-u^{2h})}\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{2(h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})}{(1-u^{2h})^2}$
$A_{2n-1}^{(2)}$	$(u^i)_{1\le i\le n}$	$X_k = 1 - u^k$	$\varepsilon(c)(-1)^{\ell(\bar{\lambda})}u^{-\tilde{\eta}^\vee/2\ell(\bar{\lambda})}\prod\limits_{h\in\bar{\lambda}}\dfrac{(1-u^{h-\tilde{\eta}^\vee/2})(1-u^{h+\tilde{\eta}^\vee})(1+u^{h+\tilde{\eta}^\vee/2})}{(1-u^{2h})(1-u^h)(1+u^{h-\tilde{\eta}^\vee/2})}\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{h-\tilde{\eta}^\vee})(1-u^{h+\tilde{\eta}^\vee})}{(1-u^h)^2}$
$C_n^{(1)}$	$(u^i)_{1\le i\le n}$	$X_k = 1 - u^k$	$\varepsilon(c)u^{\tilde{\eta}^\vee/2\ell(\bar{\lambda})}\prod_{h\in\bar{\lambda}}\dfrac{(1-u^{h-\tilde{\eta}^\vee/2})(1-u^{h-\tilde{\eta}^\vee})(1+u^{h+\tilde{\eta}^\vee/2})}{(1-u^{2h})(1-u^h)}\prod_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{h-\tilde{\eta}^\vee})(1-u^{h+\tilde{\eta}^\vee})}{(1-u^h)^2}$
$D_{n+1}^{(2)}$	$(u^{2i-1})_{1\le i\le n}$	$X_k = 1 - u^{2k}$	$\varepsilon(c)(-1)^{\bar{\lambda}}\prod\limits_{h\in\bar{\lambda}}\dfrac{(1+u^{(2h-\tilde{\eta}^\vee)})(1-u^{2h+\tilde{\eta}^\vee})}{(1-u^{4h})}\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{2(h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})}{(1-u^{2h})^2}$
$A_{2n}^{(2)}$	$(u^{2i-1})_{1\le i\le n}$	$X_k = 1 - u^{2k}$	$\varepsilon(c)u^{\tilde{\eta}^\vee\ell(\bar{\lambda})}\prod\limits_{h\in\bar{\lambda}}\dfrac{(1+u^{(2h-\tilde{\eta}^\vee)})(1-u^{2(h-\tilde{\eta}^\vee)})(1-u^{2h+\tilde{\eta}^\vee})}{(1-u^{4h})(1-u^{2h})}\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{2(h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})}{(1-u^{2h})^2}$
$D_n^{(1)}$	$(u^{2i-1})_{1\le i\le n}$	$X_k = 1 - u^{2k}$	$\varepsilon(c)(-1)^{\ell(\bar{\lambda})}u^{-2\tilde{\eta}^\vee\ell(\bar{\lambda})}\prod\limits_{h\in\bar{\lambda}}\dfrac{(1+u^{(2h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})(1-u^{2h+\tilde{\eta}^\vee})}{(1-u^{4h})(1-u^{2h})}\prod\limits_{h\in\mathcal{H}(\lambda)_T}\dfrac{(1-u^{2(h-\tilde{\eta}^\vee)})(1-u^{2(h+\tilde{\eta}^\vee)})}{(1-u^{2h})^2}$

TABLE 3. Table of affine types with their corresponding specializations and hook length product

To get the u-deformation of the Nekrasov-Okounkov formula as stated in (2) (and its generalizations in any affine classical type), we start with Theorem 4.3 and specialize each group character appearing in the left-hand side as prescribed in the second column of Table 3. Then, one gets the hook length formulas of the fourth column by using Theorem 7.4. Observe the signatures $\epsilon(c)$ of the affine Grassmannian elements simplify. In affine type A this yields (2). For the other types, we refer to [Wah23] § 5.3 for the precise formulas.

Warning: Theorem 4.3 uses the **dual** atomic length $L^\vee$. This subtlety imposes to switch each affine root system and its dual (obtained by transposing its generalized Cartan matrix) between Table 1 and Tables 2 and 3.

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