

Homology operations for gravity algebras

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Abstract

Let $\mathcal{M}_{0,n+1}$ be the moduli space of genus zero Riemann surfaces with $n+1$ marked points. In this paper we compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ for any $n \in \mathbb{N}$ and any prime p , where $\mathbb{F}_p(\pm 1)$ denotes the sign representation of the symmetric group Σ_n . The interest in these homology groups is twofold: on the one hand classes in these equivariant homology groups parametrize homology operations for gravity algebras. On the other hand the homotopy quotient $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ is a model for the classifying space for $B_n/Z(B_n)$, the quotient of the braid group B_n by its center.

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1 Introduction

The Gravity operad $Grav$ was introduced by Getzler in [12] and [11] as a sub-operad of $H_*(\mathcal{D}_2)$, where \mathcal{D}_2 is the little two disk operad. The space of arity n operations is given by $sH_*(\mathcal{M}_{0,n+1}; \mathbb{Z})$, the (shifted) homology of the moduli space of genus zero marked curves. There are also chain model versions of the Gravity operad, described in the paper [16] by Westerland and in [13] by Getzler-Kapranov. A comparison between these definitions has been written by Dupont and Horel in [9]. Remarkable examples of algebras over this operad are:

- $H_*^{S^1}(X)$, where X is an algebra over the framed little two disk operad $f\mathcal{D}_2$ (see [16]).
- If M is a closed oriented manifold, then the string topology operations on $s^{1-d}H_*^{S^1}(LM)$ assemble to a gravity algebra structure (see [4] and [16]).
- If A is a Frobenius algebra, then $HC^*(A)$ is a gravity algebra. Further examples along these lines can be found in [15].

In the theory of $H_*(\mathcal{D}_2)$ -algebras a key role is played by the Dyer-Lashof operations, which correspond to classes in $H_*^{\Sigma_n}(\mathcal{D}_2(n); \mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{D}_2(n); \mathbb{F}_p(\pm 1))$ (here $\mathbb{F}_p(\pm 1)$ denotes the sign representation of the symmetric group Σ_n). The knowledge of these operations is crucial to describe, for example, the homology of the braid groups B_n and of $\Omega^2 \Sigma^2 X$, as explained in the remarkable work of F. Cohen [6]. Similarly, classes in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ give rise to homology operations for Gravity algebras. More precisely: given a gravity algebra $(A, d_A) \in Ch(\mathbb{F}_p)$ and a class $Q \in H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ (resp. $Q \in H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$) we can construct an operation

$$Q : H_*(A) \rightarrow H_{n*+|Q|+1}(A)$$

which acts on even (resp. odd) degree classes. In this paper we compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ for any $n \in \mathbb{N}$ and any prime number p . The key observation to do this computation is the following: the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $C_n(\mathbb{C})_{S^1}$ (where $C_n(\mathbb{C})$ is the unordered configuration space) are both models for the classifying space of $B_n/Z(B_n)$, the quotient of the braid group by its center. This allows us to do the computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ by looking at the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \longrightarrow BS^1 \tag{1}$$

which is much simpler than

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \longrightarrow B\Sigma_n$$

because in the first case there is not any monodromy. Moreover, in the (homological) Serre spectral sequence associated to (1) the homology of the fiber is well known, and the differential of the second page is given by the BV-operator Δ (Proposition 3.7). So everything is now quite explicit and the main result is the following (Theorem 4.16 and Theorem 4.18):

Theorem 1.1. *Let $n \in \mathbb{N}$ and p a prime number. Then:*

- *If $n = 0, 1 \pmod p$ we have $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_p) \otimes H_*(BS^1; \mathbb{F}_p)$.*
- *Otherwise $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ is isomorphic to $H_*(C_n(\mathbb{C}); \mathbb{F}_p)/\text{Im}(\Delta)$.*

The computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ involves different methods, based on the work of Cohen, Bödigheimer and Peim [2]. To explain the strategy we need some notation: let $\lambda : E \rightarrow B$ be a fiber bundle with fiber F and consider the (ordered) fiberwise configuration space

$$E(\lambda, n) := \{(e_1, \dots, e_n) \in E^n \mid e_i \neq e_j \text{ and } \lambda(e_i) = \lambda(e_j) \text{ if } i \neq j\}$$

Now let X be a connected CW-complex with basepoint $*$. The space of *fiberwise configurations with label in X* is defined as

$$E(\lambda; X) := \bigsqcup_{n=0}^{\infty} E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

where \sim is the equivalence relation determined by

$$(e_1, \dots, e_n) \times (x_1, \dots, x_n) \sim (e_1, \dots, \hat{e}_i, \dots, e_n) \times (x_1, \dots, \hat{x}_i, \dots, x_n)$$

when $x_i = *$. Now the idea is the following: if $\mathbb{C} \hookrightarrow E \xrightarrow{\lambda} \mathbb{C}P^\infty$ is the tautological line bundle, then $E(\lambda; n)/\Sigma_n$ is a model for the classifying space of $B_n/Z(B_n)$. In this situation one can prove that $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$ can be described as a certain subspace of $H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$ (Proposition 5.6). Therefore it suffices to compute $H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$, and this is done looking at the fibration

$$C(\mathbb{C}; S^{2q+1}) \hookrightarrow E(\lambda; S^{2q+1}) \rightarrow \mathbb{C}P^\infty$$

where $C(\mathbb{C}; S^{2q+1})$ is the configuration space of points in the plane with labels in S^{2q+1} . The result is the following (Theorem 5.7):

Theorem 1.2. *For any $q \in \mathbb{N}$ and p a prime number*

$$H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p) \cong H_*(C(\mathbb{C}; S^{2q+1}); \mathbb{F}_p) \otimes H_*(\mathbb{C}P^\infty; \mathbb{F}_p)$$

Plan for the paper: Section 2 is a recollection of well known facts about the gravity operad. In Section 3 we explain in detail the connection between classes in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ (or in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$) and homology operations for gravity algebras. Section 4 and Section 5 contain the computation of these homology groups.

Future directions: the Dyer-Lashof operations satisfy a bunch of relations (e.g. Cartan relations, Adem relations). It would be interesting to understand what kind of relations we get in the context of gravity algebras. This would imply a complete knowledge of the homology of the free gravity algebras. Moreover it would be interesting to see if the computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ can be used to get some information about $H_*^{\Sigma_n}(\overline{\mathcal{M}}_{0,n+1}; \mathbb{F}_p)$ by mean of the morphism $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p) \rightarrow H_*^{\Sigma_n}(\overline{\mathcal{M}}_{0,n+1}; \mathbb{F}_p)$ induced by the inclusion.

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2 The Gravity operad

Let X be a S^1 -space. The action $\theta : S^1 \times X \rightarrow X$ induces an operator $\Delta : H_*(X; \mathbb{Z}) \rightarrow H_{*+1}(X; \mathbb{Z})$ by the composition

$$H_*(X) \longrightarrow H_*(S^1) \otimes H_*(X) \xrightarrow{\times} H_*(S^1 \times X) \xrightarrow{\theta_*} H_*(X)$$

where the first map take a class $x \in H_*(X)$ and send it to $[S^1] \otimes x$. We will call Δ the **BV-operator** (see [10]). In what follows all the homology groups are taken with integer coefficients, unless otherwise stated. To easy the notation we sometimes write $H_*(X)$ instead of $H_*(X; \mathbb{Z})$.

Definition 1 (Getzler, [10]). Let \mathcal{D}_2 be the little two disk operad. S^1 acts on $\mathcal{D}_2(n)$ by rotations, so we get a BV-operator $\Delta : H_*(\mathcal{D}_2(n)) \rightarrow H_{*+1}(\mathcal{D}_2(n))$. This map is compatible with the operadic structure and induces a morphism of operads $\Delta : H_*(\mathcal{D}_2) \rightarrow H_{*+1}(\mathcal{D}_2)$. The kernel of this map is a sub-operad of $H_*(\mathcal{D}_2)$, called the **Gravity operad**. We will denote it by $Grav$.

Remark 2.1. S^1 acts freely on $\mathcal{D}_2(n)$ so we can identify the kernel of $\Delta : H_*(\mathcal{D}_2(n)) \rightarrow H_{*+1}(\mathcal{D}_2(n))$ with $H_*^{S^1}(\mathcal{D}_2(n)) \cong sH_*(\mathcal{M}_{0,n+1})$, where this last isomorphism holds because $\mathcal{M}_{0,n+1}$ and $\mathcal{D}_2(n)/S^1$ are homotopy equivalent. To sum up we have the following identification:

$$Grav(n) = sH_*(\mathcal{M}_{0,n+1})$$

Remark 2.2. The action of Σ_{n+1} on $\mathcal{M}_{0,n+1}$ by relabelling the points induces an action in homology, making $Grav$ a cyclic operad.

Unlike many familiar operads, the Gravity operad is not generated by a finite number of operations. However, it has a nice presentation with infinitely many generators:

Theorem 2.3 (Getzler [12]). *As an operad $Grav$ is generated by (graded) symmetric operations of degree one*

$$\{a_1, \dots, a_n\} \in Grav(n) \quad \text{for } n \geq 2$$

Geometrically, $\{a_1, \dots, a_n\}$ corresponds to the generator of $H_0(\mathcal{M}_{0,n+1}, \mathbb{Z})$. These operations (called brackets) satisfy the so called generalized Jacobi relations: for any $k \geq 2$

and $l \in \mathbb{N}$

$$\sum_{1 \leq i < j \leq k} (-1)^{\epsilon(i,j)} \{\{a_i, a_j\}, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k, b_1, \dots, b_l\} = \{\{a_1, \dots, a_k\}, b_1, \dots, b_l\} \quad (2)$$

where the right hand term is interpreted as zero if $l = 0$ and $\epsilon(i, j) = (|a_1| + \dots + |a_{i-1}|)|a_i| + (|a_1| + \dots + |a_{j-1}|)|a_j| + |a_i||a_j|$.

Definition 2. A **Gravity algebra** (in the category of chain complexes) is an algebra over the Gravity operad. To be explicit, it is a chain complex (A, d_A) together with graded symmetric chain maps $\{-, \dots, -\} : A^{\otimes k} \rightarrow A$ of degree one such that for $k \geq 3$, $l \geq 0$ and $a_1, \dots, a_k, b_1, \dots, b_l \in A$ Equation 2 is satisfied.

Gravity algebras and BV -algebras are closely related. The idea is that every time we have a BV -algebra structure on the (co)homology of a space/d.g. algebra, then we get a Gravity algebra structure on the S^1 -equivariant version of our (co)homology theory. The following list of examples should clarify this last sentence:

1. Let X be a $f\mathcal{D}_2$ -algebra. Then $H_*(X)$ is a BV -algebra and $H_*^{S^1}(X)$ is a Gravity algebra (Westerland, [16]).
2. Let M be an oriented d -dimensional manifold. The homology $H_*(LM)$ of the free loop space on M carries a rich algebraic structure: the loop product of Chas-Sullivan [4] endow $s^{-d}H_*(LM)$ with a commutative algebra structure. Moreover the S^1 action on LM is compatible with this product, so $s^{-d}H_*(LM)$ is an BV -algebra (Cohen-Jones [7]). As before, if we switch to S^1 -equivariant homology (and shift the degree appropriately) we get a Gravity algebra structure on $s^{1-d}H_*^{S^1}(LM)$ (Westerland, [16]).
3. If A is a Frobenius algebra, then the Hochschild cohomology $HH^*(A)$ is a BV -algebra. If we switch to the S^1 -equivariant version of Hochschild cohomology (i.e. the cyclic cohomology) we get a Gravity algebra structure on $HC^*(A)$. See the paper by Ward [15] for further details and examples.

The following table summarizes what we said in this section:

| | BV -algebra | $Grav$ -algebra |
|-------------------------------|-----------------|------------------------|
| X $f\mathcal{D}_2$ -algebra | $H_*(X)$ | $H_*^{S^1}(X)$ |
| M closed oriented | $s^{-d}H_*(LM)$ | $s^{1-d}H_*^{S^1}(LM)$ |
| A Frobenius algebra | $HH^*(A)$ | $HC^*(A)$ |

3 Homology operations for gravity algebras

In this section we show that classes in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ give rise to homology operations for Gravity algebras.

3.1 Equivariant operations

Let p be a prime and use \mathbb{F}_p -coefficients for (co)homology from now on. Fix $grav$ any chain model for the Gravity operad.

Remark 3.1. We can suppose that Σ_n acts freely on $grav(n)$ for any $n \in \mathbb{N}$. Indeed it suffices to replace our chain model for the gravity operad with a cofibrant replacement of it. From now on we assume that we are in this situation.

Now let (A, d_A) be a Gravity algebra in the category $Ch(\mathbb{F}_p)$. The structure maps $grav(n) \otimes A^{\otimes n} \rightarrow A$ are Σ_n -equivariant, so they factor through the coinvariants:

$$\begin{array}{ccc} grav(n) \otimes A^{\otimes n} & \xrightarrow{\quad} & A \\ \downarrow & \nearrow \gamma & \\ grav(n) \otimes_{\Sigma_n} A^{\otimes n} & & \end{array}$$

Passing to homology we get

$$\gamma_* : H_*(grav(n) \otimes_{\Sigma_n} A^{\otimes n}) \rightarrow H_*(A)$$

Remark 3.2. Since we are working with coefficients in a field, we can define a quasi-isomorphism $H_*(A) \rightarrow A$ by choosing a basis for $H_*(A)$ and assigning to each element of it a representative cycle in $Z_*(A) \subseteq A$. Therefore we get an isomorphism between $H_*(grav(n) \otimes_{\Sigma_n} H_*(A)^{\otimes n})$ and $H_*(grav(n) \otimes_{\Sigma_n} A^{\otimes n})$.

Remark 3.3. Since Σ_n acts freely on $grav(n)$, the coinvariants $grav(n)_{\Sigma_n}$ compute (up to a degree shift) the Σ_n -equivariant homology of $\mathcal{M}_{0,n+1}$. More explicitly,

$$H_*(grav(n)_{\Sigma_n}) \cong sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$$

Similarly, the quotient of $grav(n)$ by the subspace $\langle x - (-1)^\sigma \sigma \cdot x \mid x \in grav(n), \sigma \in \Sigma_n \rangle$ computes $sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$. So,

$$H_* \left(\frac{grav(n)}{\langle x - (-1)^\sigma \sigma \cdot x \rangle} \right) \cong sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$$

We are now ready to define what is an equivariant homology operation for a gravity algebra:

Definition 3 (Equivariant operations for even classes). Let $Q \in sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ and let $q \in grav(n)$ be an element such that $[q] \in grav(n)_{\Sigma_n}$ is a representative for the class $sQ \in sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$. Since there is a shift of degree, $|q| = |Q| + 1$. If $[a] \in H_*(A)$ is a even degree class, then it is not hard to verify that $q \otimes a^{\otimes n}$ is a cycle in $grav(n) \otimes_{\Sigma_n} A^{\otimes n}$. Then we define

$$Q(a) := \gamma_*(q \otimes a^{\otimes n}) \in H_*(A)$$

It is not hard to see that $Q(a)$ does not depend neither on the choice of q , nor on the choice of a representative cycle for $[a]$. So the definition is well posed. To sum up, any class $Q \in H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ gives rise to a homology operation (defined only for classes of even degrees)

$$Q(-) : H_{2m}(A) \rightarrow H_{2m+|Q|+1}(A)$$

Definition 4 (Equivariant operations for odd classes). Let $Q \in H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ and choose $q \in \text{grav}(n)$ be an element such that $[q] \in \text{grav}(n)/\langle x - (-1)^\sigma \sigma \cdot x \mid x \in \text{grav}(n), \sigma \in \Sigma_n \rangle$ is a representative for the class $sQ \in sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$. Since there is a shift of degree, $|q| = |Q| + 1$. If $[a] \in H_*(A)$ is a odd degree class, then it is not hard to verify that $q \otimes a^{\otimes n}$ is a cycle in $\text{grav}(n) \otimes_{\Sigma_n} A^{\otimes n}$. Then we define

$$Q(a) := \gamma_*(q \otimes a^{\otimes n}) \in H_*(A)$$

It is not hard to see that $Q(a)$ does not depend neither on the the choice of q , nor on the choice of a representative cycle for $[a]$. So the definition is well posed. To sum up, any class $Q \in H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1))$ gives rise to a homology operation (defined only for classes of odd degrees)

$$Q(-) : H_{2m+1}(A) \rightarrow H_{(2m+1)n+|Q|+1}(A)$$

To conclude, if one wants to understand the homology operations for gravity algebras, the first step is to compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p) = H_*(B_n/Z(B_n); \mathbb{F}_p)$ and $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$. This will be the main achievement of Section 4 and Section 5.

3.2 Homotopy models for $(\mathcal{M}_{0,n+1})_{\Sigma_n}$

As explained in the previous section, to understand the equivariant operations on gravity algebras one has to compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$. The first thing one might try to do is to study the Serre spectral sequence associated to the fibration

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \rightarrow B\Sigma_n \quad (3)$$

However the action of Σ_n on the homology of $\mathcal{M}_{0,n+1}$ is not trivial and this complicates the whole computation. To overcome this problem the key observation is the following:

Lemma 3.4. $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ is homotopy equivalent to $C_n(\mathbb{C})_{S^1}$.

Proof. Recall that $\mathcal{M}_{0,n+1}$ is Σ_n -homotopy equivalent to $F_n(\mathbb{C})/S^1$, so the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $(F_n(\mathbb{C})/S^1)_{\Sigma_n}$ are homotopy equivalent. S^1 acts freely on $F_n(\mathbb{C})$ so $(F_n(\mathbb{C})/S^1)_{\Sigma_n}$ is homotopy equivalent to $(F_n(\mathbb{C})_{S^1})_{\Sigma_n}$. The action of Σ_n on $F_n(\mathbb{C})$ and that of S^1 commute so we get a homotopy equivalence between $(F_n(\mathbb{C})_{S^1})_{\Sigma_n}$ and $(F_n(\mathbb{C})_{\Sigma_n})_{S^1}$, which is homotopy equivalent to $C_n(\mathbb{C})_{S^1}$ since Σ_n acts freely on $F_n(\mathbb{C})$. \square

Thanks to this Lemma we have an isomorphism between $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1})$ and $H_*^{S^1}(C_n(\mathbb{C}))$. Section 4 will be dedicated to the computation of $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ for any $n \in \mathbb{N}$ and any prime number p . This is obtained by the Serre spectral sequence associated to

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \rightarrow BS^1 \quad (4)$$

Remark 3.5. The spectral sequence of the fibration 4 is much easier than the one associated to 3. First of all in this case we have trivial monodromy since BS^1 is simply connected. If we fix a field of coefficients \mathbb{F} for (co)homology, the E_2 page of the spectral sequence is given by

$$E_{p,q}^2 = H_p(C_n(\mathbb{C})) \otimes H_q(BS^1)$$

In addition, the differential d^2 of the E^2 page is well known in this case, as we will explain in the rest of this section.

Proposition 3.6. *Let X be any topological space such that $H_i(X; \mathbb{Z})$ is a finitely generated abelian group for each $i \in \mathbb{N}$. Consider $S^1 \times X$ with the natural action of S^1 by multiplication on the left and fix a field \mathbb{F} of coefficients for (co)homology. Then the homological spectral sequence associated to the fibration $S^1 \times X \rightarrow (S^1 \times X)_{S^1} \rightarrow BS^1$ has the following form:*

1. $E^2 = H_*(S^1) \otimes H_*(X) \otimes H_*(BS^1)$.
2. Let y_{2i} be a generator of $H_{2i}(BS^1; \mathbb{F})$, e_0 a generator of $H_0(S^1)$ and $x \in H_*(X; \mathbb{F})$. Then the differential d^2 of the second page is given by

$$d^2(e_0 \otimes x \otimes y_{2i}) = \begin{cases} 0 & \text{if } i = 0 \\ [S^1] \otimes x \otimes y_{2i-2} & \text{otherwise} \end{cases} \quad d^2([S^1] \otimes x \otimes y_{2i}) = 0$$

3. The spectral sequence degenerates at the third page, which is given by:

$$E_{i,j}^3 = \begin{cases} e_0 \otimes H_j(X) \otimes y_0 & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Proof. Point (1) is clear, (3) follows from (2). So the only thing to prove is the statement of point (2). Since S^1 acts freely on $S^1 \times X$, the homotopy quotient $(S^1 \times X)_{S^1}$ is homotopy equivalent to the strict quotient $(S^1 \times X)/S^1 = X$. The original fiber sequence can be rewritten as $S^1 \times X \rightarrow X \rightarrow BS^1$, where the first map $p : S^1 \times X \rightarrow X$ is the projection on the second factor. We prove the dual statement in order to exploit the multiplicativity of the cohomological spectral sequence. The second page looks as follows:

$$E_2 = \frac{\mathbb{F}[a]}{(a^2)} \otimes H^*(X; \mathbb{F}) \otimes \mathbb{F}[c]$$

where a is a generator of $H^1(S^1)$ and c is a generator of $H^2(BS^1)$. The classes $y \in H^*(X) \subseteq E_2^{0,*}$ are infinite cycles because they belong to the image of $p^* : H^*(X) \rightarrow H^*(S^1 \times X)$. This observation implies that $E_3 = E_\infty$, because the only multiplicative generator which can have non zero differentials is a , which is a class in $E_2^{0,1}$, and therefore $d_n(a) = 0$ for $n \geq 3$. Now we claim that $d_2(a)$ is a generator of $E_2^{2,0} = \mathbb{F}c$: consider the projection $p : S^1 \times X \rightarrow S^1$ on the first factor. This map is S^1 -equivariant, so we get a map of fibrations

$$\begin{array}{ccc} S^1 \times X & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \\ \downarrow & & \downarrow \\ BS^1 & \longrightarrow & BS^1 \end{array}$$

The claim now follows by comparing the spectral sequences of the right and left fibration. \square

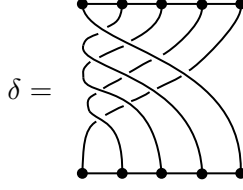


Figure 1: The braid δ .

Proposition 3.7. *Let X be a topological space of finite type equipped with an S^1 action. Fix \mathbb{F} a field of coefficients for (co)homology. Then the differential d^2 of the second page of the homological spectral sequence associated to $X \hookrightarrow X_{S^1} \rightarrow BS^1$ is given by*

$$d^2(x \otimes y_{2i}) = \begin{cases} 0 & \text{if } i = 0 \\ \Delta(x) \otimes y_{2i-2} & \text{otherwise} \end{cases}$$

where y_{2i} is the generator of $H_{2i}(BS^1; \mathbb{F})$.

Proof. Consider the map of fibrations

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{\theta} & X \\ \downarrow & & \downarrow \\ (S^1 \times X)_{S^1} & \longrightarrow & X_{S^1} \\ \downarrow & & \downarrow \\ BS^1 & \longrightarrow & BS^1 \end{array}$$

The statement follows combining the definition of Δ and the formula for d^2 given in Proposition 3.6. \square

We end this Section observing that $(\mathcal{M}_{0,n+1})_{\Sigma_n} \simeq C_n(\mathbb{C})_{S^1}$ is the classifying space for the quotient of the braid group by its center: let B_n be the Braid group on n -strands and let us denote by σ_i the i -th generator of B_n . It is well known that the center of B_n is an infinite cyclic group generated by δ^2 , where $\delta := \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \dots (\sigma_{n-1}\sigma_{n-2} \dots \sigma_1)$; see Figure 1 for a picture.

Remark 3.8. The center of the Pure braid group PB_n is an infinite cyclic group as well, generated by δ^2 .

Proposition 3.9. *The homotopy quotient $C_n(\mathbb{C})_{S^1}$ is the classifying space for the group $B_n/Z(B_n)$. Similarly, $F_n(\mathbb{C})_{S^1}$ is the classifying space for $PB_n/Z(PB_n)$.*

Proof. Consider the long exact sequence for homotopy groups associated to the fibration $S^1 \hookrightarrow ES^1 \times C_n(\mathbb{C}) \rightarrow C_n(\mathbb{C})_{S^1}$. To get the result just observe that the map $i_* : \pi_1(S^1) \rightarrow \pi_1(ES^1 \times C_n(\mathbb{C})) \cong \pi_1(C_n(\mathbb{C}))$ induced by the inclusion of a fiber sends the generator of $\pi_1(S^1)$ to δ^2 . The case of the ordered configurations is completely analogous. \square

We can summarize these observations in the following table:

| Group | Models for the classifying space | |
|----------------|----------------------------------|------------------------------------|
| PB_n | $F_n(\mathbb{C})$ | |
| B_n | $C_n(\mathbb{C})$ | |
| $PB_n/Z(PB_n)$ | $F_n(\mathbb{C})_{S^1}$ | $\mathcal{M}_{0,n+1}$ |
| $B_n/Z(B_n)$ | $C_n(\mathbb{C})_{S^1}$ | $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ |

4 Operations for even degree classes

In this section we compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p)$ for any $n \in \mathbb{N}$ and any prime number p . As explained in Section 3.2 this computation can be done by looking at the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \longrightarrow BS^1$$

instead of

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \longrightarrow B\Sigma_n$$

Before going into the details of the computation, let us review some preliminary results.

4.1 Preliminaires

We start by reviewing the basics of equivariant cohomology. We refer to [8] and [3] for further details.

4.1.1 Equivariant cohomology

let $G = \mathbb{Z}/n$ or S^1 and M be an abelian group which we use as coefficients for (co)homology. We also suppose that X is a finite dimensional G -complex of finite orbit type. Let $c \in H^2(BG; \mathbb{Z})$ be a generator, and consider the multiplicative subset $S := \{1, c, c^2, \dots\}$. Consider the following subspace:

$$FX := \{x \in X \mid \tilde{H}^*(BG_x; M) \neq 0\}$$

where G_x denotes the stabilizer of a point $x \in X$. A crucial result is the so called *Localization Theorem*:

Theorem 4.1 ([8], p.198). *The inclusion $i : FX \rightarrow X$ induces an isomorphism*

$$i^* : S^{-1}H_G^*(X; M) \rightarrow S^{-1}H_G^*(FX; M)$$

where $S^{-1}H_G^*(X; M)$ is the localization of the $H^*(BG; M)$ -module $H_G^*(X; M)$ to the subset S .

A consequence of this Theorem is the following:

Theorem 4.2 ([8], p.199). *Suppose $H^i(X; M) = 0$ for $i > n$. Then the inclusion $i : FX \rightarrow X$ induces an isomorphism*

$$H_G^i(X; M) \rightarrow H_G^i(FX; M)$$

for any $i > n - \dim(G)$ and an epimorphism for $i = n - \dim(G)$.

Now let us restrict to the case $G = \mathbb{Z}/p$, with p a prime number. In the following we will use \mathbb{F}_p coefficients for the \mathbb{Z}/p -equivariant (co)homology of X .

Theorem 4.3 ([8] p. 200). *Suppose $\dim_{\mathbb{F}_p} \bigoplus_{k \in \mathbb{N}} H^k(X)$ is finite and $H^k(X) = 0$ for any $k > n$. Then we have*

$$\dim_{\mathbb{F}_p} \bigoplus_{k \in \mathbb{N}} H^k(X^{\mathbb{Z}/p}) \leq \dim_{\mathbb{F}_p} \bigoplus_{k \in \mathbb{N}} H^k(X) \quad (5)$$

Moreover, the following assertions are equivalent:

1. *Equality holds in (5).*
2. *The map induced by the inclusion $i^* : H_{\mathbb{Z}/p}^*(X) \rightarrow H^*(X)$ is surjective.*
3. *$\dim_{\mathbb{F}_p} H_{\mathbb{Z}/p}^k(X) = \dim_{\mathbb{F}_p} \bigoplus_{i \in \mathbb{N}} H^i(X)$ for $k > n$.*
4. *\mathbb{Z}/p acts trivially on $H^*(X)$ and the Serre spectral sequence of $X \hookrightarrow X_{\mathbb{Z}/p} \rightarrow B\mathbb{Z}/p$ degenerates at the E_2 page.*

Remark 4.4. The previous Theorem holds as well if we replace \mathbb{Z}/p with S^1 and take \mathbb{Q} as field of coefficients.

4.1.2 Labelled configuration spaces

Definition 5 (Bödigheimer, [1]). Let Y be a topological space and $(X, *)$ be a based CW-complex, not necessarily connected. The space of configurations in Y with labels in X is defined as

$$C(Y; X) := \bigsqcup_{n \in \mathbb{N}} F_n(M) \times_{\Sigma_n} X^n / \sim$$

where $(p_1, \dots, p_n; x_1, \dots, x_n) \sim (p_1, \dots, \hat{p}_i, \dots, p_n; x_1, \dots, \hat{x}_i, \dots, x_n)$ if $x_i = *$.

Example. $C(\mathbb{C}; S^0)$ is just the disjoint union $\bigsqcup_{n \in \mathbb{N}} C_n(\mathbb{C})$. to easy the notation we sometimes abbreviate $C(\mathbb{C}; S^0)$ by $C(\mathbb{C})$.

When $Y = \mathbb{R}^n$ the homology of $C(\mathbb{R}^n; X)$ is known, thanks to the work of Cohen [6]. The idea is the following: $C(\mathbb{R}^n; X)$ is homotopy equivalent to the free \mathcal{D}_n -algebra on X . Therefore $H_*(C(\mathbb{R}^n; X); \mathbb{F}_p)$ can be described as a functor of $H_*(X; \mathbb{F}_p)$. For the purpose of this work it is enough to recall the results in the case $n = 2$.

Definition 6. Let p be a prime number. Fix a basis \mathcal{B} of $H_*(X; \mathbb{F}_p)/[*]$, where $[*] \in H_0(X; \mathbb{F}_p)$ is the class of the basepoint. We define a **basic bracket of weight \mathbf{k}** inductively as follows:

- A basic bracket of weight 1 is just an element $a \in \mathcal{B}$. Its degree is by definition the homological degree of a . Observe that any class of $H_*(X; \mathbb{F}_p)$ can be seen as a class in $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$.
- By induction assume that the basic brackets of weight j have been defined and equipped with a total ordering compatible with weight for $j < k$. Then a basic bracket of weight k is a homology class $[a, b] \in H_*(C(\mathbb{C}; X); \mathbb{F}_p)$, where $[-, -]$ is the Browder bracket and a, b are basic brackets such that:

1. $\text{weight}(a) + \text{weight}(b) = k$.
2. $a < b$ and if $b = [c, d]$ then $c \leq a$.

The degree of $[a, b]$ is by definition $\deg(a) + \deg(b) + 1$.

In the case $p \neq 2$ we also include as basic brackets classes of the form $[a, a]$, where a is a basic bracket of even degree.

Theorem 4.5 (Cohen, [6]). *Let p be any prime, and $Q : H_q(C(\mathbb{C}; X); \mathbb{F}_p) \rightarrow H_{pq+p-1}(C(\mathbb{C}; X); \mathbb{F}_p)$ be the first Dyer-Lashof operation (when p is odd it acts only on classes of odd degree q). Then $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ has the following form:*

$p = 2$: $H_*(C(\mathbb{C}; X); \mathbb{F}_2)$ is the free graded commutative algebra on classes $Q^i(x)$, where Q^i denotes the i -th iteration of Q and x is a basic bracket.

$p \neq 2$: $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ is the free graded commutative algebra on classes $Q^i(x)$ and $\beta Q^i(x)$, where β is the Bockstein operator, Q^i denotes the i -th iteration of Q and x is a basic bracket of odd degree.

Corollary 4.6. *Let $C(\mathbb{C}) := \bigsqcup_{n \in \mathbb{N}} C_n(\mathbb{C})$ be the disjoint union of all unordered configuration spaces of points in the complex plane. Then if p is an odd prime*

$$H_*(C(\mathbb{C}); \mathbb{F}_p) = \mathbb{F}_p[\iota, \beta Q[\iota, \iota], \beta Q^2[\iota, \iota], \dots] \otimes \Lambda[[\iota, \iota], Q[\iota, \iota], Q^2[\iota, \iota], \dots]$$

where ι is the generator of $H_0(C_1(\mathbb{C}))$. When $p = 2$ we have

$$H_*(C(\mathbb{C}); \mathbb{F}_2) = \mathbb{F}_2[\iota, Q\iota, Q^2\iota, \dots]$$

Remark 4.7. In what follows we will adopt the following notation:

$$\begin{aligned} u &:= [\iota, \iota] \\ \beta_i &:= \beta Q^i[\iota, \iota] \\ \alpha_i &:= Q^i[\iota, \iota] \end{aligned}$$

Moreover, the following table will be useful in the next section:

| Homology class | Number of points | Degree |
|----------------|------------------|------------|
| ι | 1 | 0 |
| u | 2 | 1 |
| α_i | $2p^i$ | $2p^i - 1$ |
| β_i | $2p^i$ | $2p^i - 2$ |

4.2 Computation of $H_*^{\mathbb{Z}/p}(C_n(\mathbb{C}); \mathbb{F}_p)$ when $n = 0, 1 \pmod p$

In this section we compute $H_*^{\mathbb{Z}/p}(C_n(\mathbb{C}); \mathbb{F}_p)$ when p is a prime that divides n or $n - 1$. Here we are considering \mathbb{Z}/p as the subgroup of p -th roots of unity inside S^1 , so its generator acts on $C_n(\mathbb{C})$ by the rotation of $2\pi/p$. Some of the statements of this section are stated with the assumption that p is an odd prime, but similar statements holds for $p = 2$ with minor modifications. A different proof for the case $p = 2$ will be included in Section 5 (see Corollary 5.10).

Theorem 4.8. *Let p be a prime, $n \in \mathbb{N}$ such that $p|n$ or $p|n - 1$. Then*

$$H_*^{\mathbb{Z}/p}(C_n(\mathbb{C}); \mathbb{F}_p) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_p) \otimes H_*(B\mathbb{Z}/p; \mathbb{F}_p)$$

Proof. Consider the homological spectral sequence associated to the fibration $C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{\mathbb{Z}/p} \rightarrow B\mathbb{Z}/p$. Since \mathbb{Z}/p acts by rotations on $C_n(\mathbb{C})$, the monodromy action is trivial. Therefore

$$E_{p,q}^2 \cong H_p(C_n(\mathbb{C}); \mathbb{F}_p) \otimes H_q(B\mathbb{Z}/p; \mathbb{F}_p)$$

We will see that

$$\dim_{\mathbb{F}_p} \bigoplus_{k \in \mathbb{N}} H^k(C_n(\mathbb{C})^{\mathbb{Z}/p}) = \dim_{\mathbb{F}_p} \bigoplus_{k \in \mathbb{N}} H^k(C_n(\mathbb{C})) \quad (6)$$

So the result follows applying Theorem 4.3. \square

Now we focus in proving the equality 6. For the moment we restrict to the case $n = pq$, then we will extend the result to the case $n = pq + 1$.

Lemma 4.9. *Let $n = pq$ or $n = pq + 1$. Then the fixed points $C_n(\mathbb{C})^{\mathbb{Z}/p}$ are homeomorphic to $C_q(\mathbb{C}^*)$.*

Proof. Let us prove the statement when $n = pq$, the other case is similar. Let us denote by $\zeta := e^{i2\pi/p}$ the generator of \mathbb{Z}/p . Consider the quotient space

$$H := \{z \in \mathbb{C}^* \mid \arg(z) \in [0, 2\pi/p]\} / \sim$$

where \sim identifies a point $z \in \{z \in \mathbb{C}^* \mid \arg(z) = 0\}$ with ζz . So H is homeomorphic to \mathbb{C}^* . Now observe that any configuration in $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is of the form $\{z_1, \zeta z_1, \dots, \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots, \zeta^{p-1} z_q\}$, where z_1, \dots, z_q are distinct points in $\{z \in \mathbb{C}^* \mid \arg(z) \in [0, 2\pi/p]\}$. The association $\{z_1, \zeta z_1, \dots, \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots, \zeta^{p-1} z_q\} \mapsto \{z_1, \dots, z_q\}$ defines a continuous map

$$f : C_n(\mathbb{C})^{\mathbb{Z}/p} \rightarrow C_q(H)$$

Conversely, if we have a configuration $\{z_1, \dots, z_q\} \in C_q(H)$, we can produce a configuration of $C_n(\mathbb{C})^{\mathbb{Z}/p}$ by taking the \mathbb{Z}/p -orbits of every point. More precisely, the association $\{z_1, \dots, z_q\} \mapsto \{z_1, \zeta z_1, \dots, \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots, \zeta^{p-1} z_q\}$ defines a continuous function $C_q(H) \rightarrow C_n(\mathbb{C})^{\mathbb{Z}/p}$, which is the inverse of f . See Figure 2 for a pictorial description of f . \square

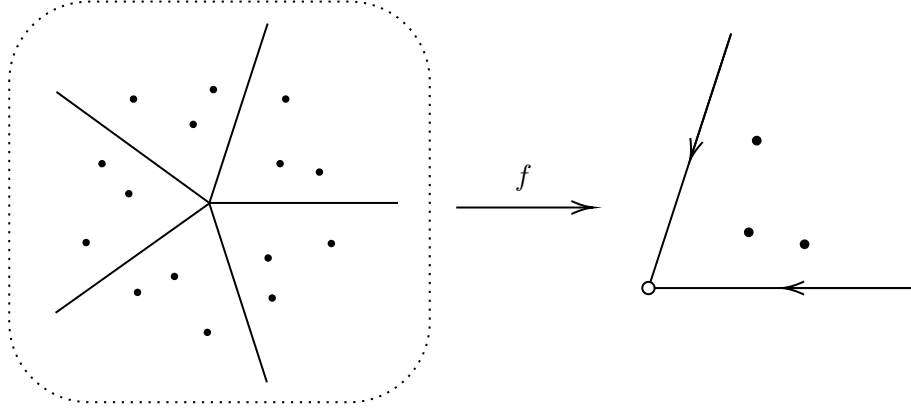


Figure 2: This picture shows how the homeomorphism $f : C_{15}(\mathbb{C})^{\mathbb{Z}/5} \rightarrow C_3(\mathbb{C}^*)$ works.

Remark 4.10. $C_q(\mathbb{C}^*)$ is homotopy equivalent to the configuration space of q black particles and one white particle in the plane. Therefore $H_*(C_q(\mathbb{C}^*); \mathbb{F}_p)$ will be the subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$ spanned by those classes that involve only q black particles and one white particle. Let us denote by a (resp b) the class in $H_0(S^0 \vee S^0)$ which represent a black particle (resp. a white particle). Then $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$ can be computed using Theorem 4.5. The Lemma stated below just identifies explicitly $H_*(C_q(\mathbb{C}^*); \mathbb{F}_p)$ as a subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$.

Lemma 4.11. *Let p be any prime and use \mathbb{F}_p coefficients for homology. Consider the space $C(\mathbb{C}^*) := \bigsqcup_{n \in \mathbb{N}} C_n(\mathbb{C}^*)$. Then we have:*

$$H_*(C(\mathbb{C}^*)) = b \cdot H_*(C(\mathbb{C})) + [a, b] \cdot H_*(C(\mathbb{C})) + [a, [a, b]] \cdot H_*(C(\mathbb{C})) + \dots$$

Proof. We use Theorem 4.5 to compute $H_*(C(\mathbb{C}; S^0 \vee S^0); \mathbb{F}_p)$ and then we identify $H_*(C(\mathbb{C}^*); \mathbb{F}_p)$ as the subspace spanned by classes involving exactly one white particle. Let us denote by a (resp. b) the class in $H_0(S^0 \vee S^0)$ which represent a black particle (resp. a white particle). The basic brackets involving only one white particle turns out to be b , $[a, b]$, $[a, [a, b]]$, $[a, [a, [a, b]]]$ etc. Now if x is one of these brackets, $Q(x)$ will be a class containing p white particles. So the classes of $H_*(C(\mathbb{C}^*); \mathbb{F}_p)$ are of the form $x \cdot y$, where $y \in H_*(C(\mathbb{C}); \mathbb{F}_p)$ and x is one of the basic bracket listed above, and this proves the statement. \square

Corollary 4.12. *Let q be any natural number and take \mathbb{F}_p -coefficients for homology, for p a fixed prime. Then*

$$H_*(C_q(\mathbb{C}^*)) = b \cdot H_*(C_q(\mathbb{C})) + [a, b] \cdot H_{*-1}(C_{q-1}(\mathbb{C})) + \dots + [a, [a, \dots [a, b]]] \cdot H_0(C_0(\mathbb{C}))$$

Definition 7. Let us denote by $d(q)$ the dimension of $H_*(C_q(\mathbb{C}); \mathbb{F}_p)$ as \mathbb{F}_p -vector space.

Lemma 4.9 and Corollary 4.12 allows us to rewrite equation (6) in the following way:

$$d(pq) = d(q) + d(q-1) + \dots + d(1) + d(0) \quad (7)$$

Before going into the details of the proof of this equation, let us look at an example. The general proof will be a generalization of the methods we are going to use in this specific case.

Example. Let $p = q = 3$. To prove equation 7 one can proceed by induction on q , so by inductive hypothesis it suffices to show that $d(pq) = d(q) + d(p(q-1))$. Let us verify this equality in this specific case. $H_*(C_3(\mathbb{C}); \mathbb{F}_3)$ is generated by ι^3 and ιu , so $d(3) = 2$. The generators of $H_*(C_9(\mathbb{C}); \mathbb{F}_3)$ are listed in the left table, while those of $H_*(C_6(\mathbb{C}); \mathbb{F}_3)$ are in the right one:

| Homology class | Degree | Homology class | Degree |
|--------------------|--------|----------------|--------|
| ι^9 | 0 | ι^6 | 0 |
| $\iota^7 u$ | 1 | $\iota^4 u$ | 1 |
| $\iota^3 \beta_1$ | 4 | β_1 | 4 |
| $\iota u \beta_1$ | 5 | α_1 | 5 |
| $\iota^3 \alpha_1$ | 5 | | |
| $\iota u \alpha_1$ | 6 | | |

Therefore $d(9) = 6 = d(6) + d(3)$ and the equality holds. A more conceptual proof of this equality is the following: observe that four classes of $H_*(C_9(\mathbb{C}); \mathbb{F}_3)$ are obtained just multiplying by ι^3 the generators of $H_*(C_6(\mathbb{C}); \mathbb{F}_3)$. The remaining generators are $\iota u \beta_1$ and $\iota u \alpha_1$ and they can be obtained from those of $H_*(C_3(\mathbb{C}); \mathbb{F}_3)$ by the following change of variables:

$$\begin{aligned} u &\mapsto \alpha_1 \\ \iota^{2l+1} &\mapsto \iota^{p-2} u \beta_1^l \end{aligned}$$

This procedure can be used to prove Equality 7 in general, as we will see in Proposition 4.13.

Proposition 4.13. *For any p prime and $q \in \mathbb{N}$ we have $d(pq) = d(q) + d(q-1) + \dots + d(1) + d(0)$.*

Proof. We proceed by induction on q . If $q = 0$ there is nothing to prove. Let us suppose that the equation holds until q , let us prove it for $q+1$: by induction we have

$$\sum_{i=0}^{q+1} d(i) = d(pq) + d(q+1) \quad (8)$$

Therefore it suffices to show that $d(pq) + d(q+1) = d(p(q+1))$. We will do this by constructing an explicit isomorphism of vector spaces between $H_*(C_{pq}(\mathbb{C})) \oplus H_*(C_{q+1}(\mathbb{C}))$ and $H_*(C_{p(q+1)}(\mathbb{C}))$. Let us suppose p is an odd prime (the case $p = 2$ is similar). We refer to Remark 4.7 for the notation we are going to use. Consider the linear map

$$f : H_*(C_{pq}(\mathbb{C})) \oplus H_*(C_{q+1}(\mathbb{C})) \rightarrow H_*(C_{p(q+1)}(\mathbb{C}))$$

defined on the basis monomials as follows:

- If x is a monomial of $H_*(C_{pq}(\mathbb{C}))$, then $f(x) = \iota^p x$.

- If $x = \iota^k u^\epsilon \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_n}^{a_n}$ is a monomial of $H_*(C_{q+1}(\mathbb{C}))$, with $\epsilon = 0, 1$, then

$$f(x) := \begin{cases} \beta_1^l \alpha_1^\epsilon \alpha_{i_1+1} \cdots \alpha_{i_m+1} \beta_{j_1+1}^{a_1} \cdots \beta_{j_n+1}^{a_n} & \text{if } k = 2l \\ (\beta_1^l u u^{p-2}) \alpha_1^\epsilon \alpha_{i_1+1} \cdots \alpha_{i_m+1} \beta_{j_1+1}^{a_1} \cdots \beta_{j_n+1}^{a_n} & \text{if } k = 2l + 1 \end{cases}$$

In other words, $f(x)$ is the monomial of $H_*(C_{p(q+1)}(\mathbb{C}))$ obtained from x by the following substitution of variables:

- (a) $\alpha_i \mapsto \alpha_{i+1}$
- (b) $\beta_i \mapsto \beta_{i+1}$
- (c) $u \mapsto \alpha_1$
- (d) $\iota^k \mapsto \begin{cases} \beta_1^l & \text{if } k = 2l \\ \beta_1^l u u^{p-2} & \text{if } k = 2l + 1 \end{cases}$

We claim that f is an isomorphism of vector spaces:

- f is well defined: clearly if we multiply by ι^p a monomial of $H_*(C_{pq}(\mathbb{C}))$ we obtain a monomial of $H_*(C_{p(q+1)}(\mathbb{C}))$. So let us pick $x = \iota^k u^\epsilon \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_n}^{a_n} \in H_*(C_{q+1}(\mathbb{C}))$; we prove that $f(x) \in H_*(C_{p(q+1)}(\mathbb{C}))$: by hypothesis we know that

$$q + 1 = k + 2\epsilon + \sum_{r=1}^m 2p^{i_r} + \sum_{s=1}^n a_s 2p^{j_s}$$

If $k = 2l$, then $f(x)$ is a class which involves the following number of points:

$$2lp + 2p\epsilon + \sum_{r=1}^m 2p^{i_r+1} + \sum_{s=1}^n a_s 2p^{j_s+1} = p(q + 1)$$

proving our claim. If $k = 2l + 1$ the computation is analogous.

- f is injective: f restricted to the subspaces $H_*(C_{pq}(\mathbb{C}))$ and $H_*(C_{q+1}(\mathbb{C}))$ is injective by definition. Moreover the intersection of $f(H_*(C_{pq}(\mathbb{C})))$ and $f(H_*(C_{q+1}(\mathbb{C})))$ contains only the zero element: indeed the elements of $f(H_*(C_{pq}(\mathbb{C})))$ are sum of monomials which contain ι^p , while the monomials spanning $f(H_*(C_{q+1}(\mathbb{C})))$ do not contain ι^p .
- f is surjective: to achieve this we need to prove that if x is a basic monomial in $H_*(C_{p(q+1)}(\mathbb{C}))$, then it is of the following forms:

1. $x = \iota^p y$ for some $y \in H_*(C_{pq}(\mathbb{C}))$
2. x contains only the letters from $\{\alpha_i, \beta_i\}_{i \geq 1}$. In other words, x do not contain ι and u .
3. $x = \iota^{p-2} u y$ with y a monomial in $\{\alpha_i, \beta_i\}_{i \geq 1}$.

But this is exactly the content of the following Proposition 4.14, therefore f is surjective.

□

Proposition 4.14. *Let p be an odd prime, $n \in \mathbb{N}$ and $n = k \bmod p$. If $x \in H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ is a basic monomial, then it has one of the following forms:*

1. $x = \iota^p y$ for some $y \in H_*(C_{n-p}(\mathbb{C}))$
2. $x = \iota^k \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_s}^{a_s}$
3. $x = (\iota^{k-2} u) \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_s}^{a_s}$

If $k = 0$ (resp. $k = 1$) replace the exponent $k - 2$ in point (c) with the corresponding class in \mathbb{Z}/p , i.e with $p - 2$ (resp. $p - 1$) to get the correct statement.

Proof. We proceed by induction on n :

- If $n = 1$ the only class in $H_*(C_1(\mathbb{C}); \mathbb{F}_p)$ is ι , so the statement is true. If $n \in \{2, \dots, p\}$ then $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ contains only two classes, ι^n and $\iota^{n-2}u$, and the statement follows.
- Assume the result is true until n , let us prove it for $n + p$: let

$$x = \iota^l u^\epsilon \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_s}^{a_s} \in H_*(C_{n+p}(\mathbb{C}))$$

If $l \geq p$ then we are in the first case. So let us restrict to the case where $l \leq p - 1$. If x contains some β_i then $x = \beta_i \cdot x'$ with $x' \in H_*(C_{n+p-2p^i}(\mathbb{C}))$. Since $i \geq 1$ we have $n + p - 2p^i \leq n$. Moreover $n + p - 2p^i = k \bmod p$ therefore we can use the inductive hypothesis for x' and get the result. We can proceed in the same way when x contains one of the variables $\{\alpha_i\}_{i \geq 1}$. The only case which is not yet considered is when $x = \iota^l u^\epsilon$. In this case $n + p$ must be equal to $l + 2\epsilon$. Therefore $l = n + p - 2\epsilon \leq p - 1$ if and only if $n \leq 2\epsilon - 1 \leq 1$ which is not the case we are considering.

□

This concludes the proof of Theorem 4.8 in the case $p|n$. It remains to prove the statement for $n = pq + 1$. This is equivalent to show that

$$d(pq + 1) = d(q) + d(q - 1) + \cdots + d(0) = d(pq)$$

Where the last equality holds for Proposition 4.13. But this is an easy consequence of Proposition 4.14:

Corollary 4.15. *If p divides n , then $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ and $H_*(C_{n+1}(\mathbb{C}); \mathbb{F}_p)$ have the same dimension as \mathbb{F}_p -vector spaces.*

Proof. We observe that multiplication by ι is an isomorphism between $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ and $H_*(C_{n+1}(\mathbb{C}); \mathbb{F}_p)$. Clearly it is injective. By Proposition 4.14 we have that each monomial in $H_*(C_{n+1}(\mathbb{C}); \mathbb{F}_p)$ contains at least one ι , and this proves the surjectivity.

□

4.3 Computation of $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ when $n = 0, 1 \bmod p$

In this section we compute $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ when $n = 0, 1 \bmod p$.

Theorem 4.16. *Let p be a prime, $n \in \mathbb{N}$ such that $n = 0, 1 \bmod p$. Then*

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_p) \otimes H_*(BS^1; \mathbb{F}_p)$$

Proof. Consider the map of fibrations

$$\begin{array}{ccc} C_n(\mathbb{C}) & \longrightarrow & C_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ C_n(\mathbb{C})_{\mathbb{Z}/p} & \longrightarrow & C_n(\mathbb{C})_{S^1} \\ \downarrow & & \downarrow \\ B\mathbb{Z}/p & \longrightarrow & BS^1 \end{array}$$

Then observe that with coefficients in \mathbb{F}_p this map induces a surjection between the E^2 pages of the homological spectral sequences. The result now follows from Theorem 4.8. \square

Remark 4.17. It would be interesting to understand the ring structure of $H_{S^1}^*(C_n(\mathbb{C}); \mathbb{F}_p)$. However there are some non-trivial extension problems to solve. For example, if we put $n = p = 2$ the homotopy quotient $C_2(\mathbb{C})_{S^1}$ is a model for $B(\mathbb{Z}/2)$, therefore

$$H_{S^1}^*(C_2(\mathbb{C}); \mathbb{F}_2) = \mathbb{F}_2[x]$$

where x is a variable of degree one. Theorem 4.16 tells us that

$$H_{S^1}^*(C_2(\mathbb{C}); \mathbb{F}_2) \cong \frac{\mathbb{F}_2[x]}{(x^2)} \otimes \mathbb{F}_2[c]$$

as $\mathbb{F}_2[c]$ -module, where x (resp. c) is a generator of $H^1(C_2(\mathbb{C}); \mathbb{F}_2)$ (resp. of $H^2(BS^1; \mathbb{F}_2)$). To get the correct ring structure we need to impose the relation $x^2 = c$.

4.4 Computation of $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ when $n \neq 0, 1 \bmod p$

In this section we compute $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$ when $n \neq 0, 1 \bmod p$. Of course this request is not empty only if p is an odd prime. The main result is the following:

Theorem 4.18. *Let p be an odd prime, $n \in \mathbb{N}$ such that $n \neq 0, 1 \bmod p$. Then*

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p) \cong \text{coker}(\Delta)$$

where $\Delta : H_*(C_n(\mathbb{C}); \mathbb{F}_p) \rightarrow H_{*+1}(C_n(\mathbb{C}); \mathbb{F}_p)$ is the BV-operator.

Proof. Consider the homological Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \rightarrow C_n(\mathbb{C})_{S^1} \rightarrow BS^1$$

By Proposition 3.7 the second page is given by

$$E_{i,j}^2 = H_i(C_n(\mathbb{C})) \otimes H_j(BS^1) \quad d^2(x \otimes y_{2j}) = \begin{cases} \Delta(x) \otimes y_{2j-2} & \text{if } j \geq 1 \\ 0 & \text{if } j = 0 \end{cases}$$

where y_{2j} is the generator of $H_{2j}(BS^1)$. By Theorem 4.5 any class of $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ is a product of the variables ι , $[\iota, \iota]$, $Q^i[\iota, \iota]$ and $\beta Q^i[\iota, \iota]$, $i \geq 1$. In particular any class is of the form $\iota^k[\iota, \iota]^l x$, where $k \in \mathbb{N}$, $l = 0, 1$ and x is a monomial which contains only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. We claim that the operator Δ acts as follows:

$$\Delta(\iota^k x) = k(k-1)\iota^{k-2}[\iota, \iota]x \quad \Delta(\iota^k[\iota, \iota]x) = 0 \quad (9)$$

A detailed proof of these formulas will be given in Section 4.5 (Proposition 4.24). Observe that if we have a monomial of the form $\iota^k x$, $x \in H_*(C_m(\mathbb{C}); \mathbb{F}_p)$, then $n = k + m = k \bmod p$ (x contains only letters from $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$, so the number of points m is divisible by p). Therefore $k \neq 0, 1 \bmod p$ and the term $k(k-1)\iota^{k-2}[\iota, \iota]x$ is never zero. Now we can conclude: the formula 9 shows that the third page of the Serre spectral sequence looks as follows: $E_{i,j}^3 = 0$ for any $j \geq 1$, while the first column $E_{0,*}^3$ may contain some non zero elements. To be more precise, $E_{0,*}^3$ is the quotient of $E_{0,*}^2 = H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ by the image of the differential. Since the differential of the second page is given by Δ , we get that

$$E_{0,*}^3 \cong \text{coker}(\Delta)$$

The third page contains only the first column, so the spectral sequence degenerates and we get the statement. \square

Remark 4.19. We can explicitly describe $\text{coker}(\Delta)$: a basis of this vector space is given by (the image of) classes in $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ which do not contain the bracket $[\iota, \iota]$ (Equation 9).

4.5 Auxiliary computations

In this paragraph we recollect some easy algebraic computations which are relevant for Section 4.4. In particular we prove the formulas 9 in Proposition 4.24. In what follows p will be an odd prime. We begin by recalling some basic properties of the bracket and Δ ; to easy the notation we will write $(-1)^x$ instead of $(-1)^{\deg(x)}$, where x is an element in some graded vector space. We refer to [6] and [10] for further details.

1. Graded anticommutativity: $[x, y] = (-1)^{xy+x+y}[y, x]$.
2. Jacobi relation: $[x, [y, z]] = [[x, y], z] - (-1)^{y+x+xy}[y, [x, z]]$.
3. The bracket is a derivation: $[x, yz] = [x, y]z + (-1)^{y+yx}y[x, z]$.

4. $\Delta(xy) = \Delta(x)y + (-1)^x x\Delta(y) + (-1)^x [x, y]$.
5. $\Delta[x, y] = [\Delta x, y] + (-1)^{x+1} [x, \Delta y]$.
6. $[x, Qy] = ad^p(y)(x)$, where $ad(y)(x) := [x, y]$ and for any $n \in \mathbb{N}$ we define $ad^n(y)(x) := ad(y)(ad^{n-1}(y)(x))$. For example, $ad^2(y)(x) = [[x, y], y]$, $ad^3(y)(x) = [[[x, y], y], y]$ and so on.
7. $[x, \beta Qy] = [x, ad^{p-1}(y)(\beta y)]$

Lemma 4.20. *Let p be an odd prime, $k \in \mathbb{N}$. Then $\Delta(\iota^k) = k(k-1)\iota^{k-2}[\iota, \iota]$.*

Proof. It suffices to proceed by induction: $\Delta(\iota) = 0$ since ι is the top class of $H_*(C_1(\mathbb{C}); \mathbb{F}_p)$. $\Delta(\iota^2) = [\iota, \iota]$ by equation (4) at the beginning of this section. The general formula follows easily using equation (4). \square

Lemma 4.21. *Let $x \in H_*(C(\mathbb{C}); \mathbb{F}_p)$ be a monomial containing only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. Then $\Delta(x) = 0$ and $\Delta(\iota x) = 0$.*

Proof. We prove that $\Delta(x) = 0$, the other case is analogous. Since x contains only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$, it is a class of $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ for some n divisible by p . Theorem 4.16 tell us that the homological Serre spectral sequence associated to

$$C_n(\mathbb{C}) \rightarrow C_n(\mathbb{C})_{S^1} \rightarrow BS^1$$

degenerates at the second page. But the differential of the second page is given by Δ (Proposition 3.7) so $\Delta(x) = 0$. \square

Lemma 4.22. *Let $x \in H_*(C(\mathbb{C}); \mathbb{F}_p)$ be a monomial containing only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. Then $[\iota^k, x] = 0$.*

Proof. Since the bracket is a derivation it is enough to prove that $[\iota, x] = 0$. By Lemma 4.21 $\Delta(\iota x) = 0$, therefore

$$0 = \Delta(\iota x) = \Delta(\iota)x \pm \iota\Delta(x) \pm [\iota, x] = \pm[\iota, x]$$

The last equality holds because $\Delta(\iota) = 0$ (ι is the top class of $H_*(C_1(\mathbb{C}); \mathbb{F}_p)$) and $\Delta(x) = 0$ (Lemma 4.21). \square

Lemma 4.23. *Let $x \in H_*(C(\mathbb{C}), \mathbb{F}_p)$ be a monomial which contains only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. Then $[[\iota, \iota], x] = 0$.*

Proof. Since the bracket is a derivation it is enough to show that $[[\iota, \iota], Q^i[\iota, \iota]] = 0$ and that $[[\iota, \iota], \beta Q^i[\iota, \iota]] = 0$. In the first case we proceed by induction: if $i = 1$ we get $[[\iota, \iota], Q[\iota, \iota]] = ad^p([\iota, \iota])([\iota, \iota])$ by property (6), and the latter term is zero by Jacobi. In general we use again property (6) and we get:

$$[[\iota, \iota], Q^i[\iota, \iota]] = ad^p(Q^{i-1}[\iota, \iota])([\iota, \iota]) = 0$$

where the last equality holds by induction. The other formula can be proved as follows: by equation (7) we get

$$[[\iota, \iota], \beta Q^i[\iota, \iota]] = [[\iota, \iota], ad^{p-1}(Q^{i-1}[\iota, \iota])(\beta Q^{i-1}[\iota, \iota])]$$

One of the brackets contained in $ad^{p-1}(Q^{i-1}[\iota, \iota])(\beta Q^{i-1}[\iota, \iota])$ is $[\beta Q^{i-1}[\iota, \iota], Q^{i-1}[\iota, \iota]]$ which we claim is zero:

$$\begin{aligned} 0 &= \Delta(\beta Q^{i-1}[\iota, \iota]Q^{i-1}[\iota, \iota]) \\ &= \Delta(\beta Q^{i-1}[\iota, \iota])Q^{i-1}[\iota, \iota] \pm \beta Q^{i-1}[\iota, \iota]\Delta(Q^{i-1}[\iota, \iota]) \pm [\beta Q^{i-1}[\iota, \iota], Q^{i-1}[\iota, \iota]] \\ &= \pm[\beta Q^{i-1}[\iota, \iota], Q^{i-1}[\iota, \iota]] \end{aligned}$$

where in the first and third equality we used Lemma 4.21. \square

Proposition 4.24. *Let $n \in \mathbb{N}$ and suppose $n \not\equiv 0, 1 \pmod{p}$. Let $\iota^k x$ and $\iota^k[\iota, \iota]x$ be classes in $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$, where x be a monomial which contains only the variables $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. Then*

$$\begin{cases} \Delta(\iota^k x) = k(k-1)\iota^{k-2}[\iota, \iota]x \\ \Delta(\iota^k[\iota, \iota]x) = 0 \end{cases}$$

Proof. By Lemma 4.20 we have $\Delta(\iota^k) = k(k-1)\iota^{k-2}[\iota, \iota]$. Using property (4) of Δ together with Lemma 4.21 and Lemma 4.22 we get

$$\Delta(\iota^k x) = \Delta(\iota^k)x \pm \iota^k \Delta(x) \pm [\iota^k, x] = k(k-1)\iota^{k-2}[\iota, \iota]x$$

and this proves the first part of the statement. Similarly,

$$\Delta(\iota^k[\iota, \iota]x) = \Delta(\iota^k[\iota, \iota])x \pm \iota^k[\iota, \iota]\Delta(x) \pm [\iota^k[\iota, \iota], x]$$

The last term is zero since the bracket is a derivation and $[\iota^k, x] = 0 = [[\iota, \iota], x]$ (Lemma 4.22 and Lemma 4.23). The middle term is zero by Lemma 4.21. The first term is zero as well for the following reason:

$$\Delta(\iota^k[\iota, \iota]) = k(k-1)\iota^{k-2}[\iota, \iota][\iota, \iota] \pm \iota^k \Delta[\iota, \iota] \pm [\iota^k, [\iota, \iota]]$$

The first term is zero since $[\iota, \iota]$ is a variable of odd degree, so it squares to zero. The second term is zero since $[\iota, \iota]$ is the top class of $H_*(C_2(\mathbb{C}); \mathbb{F}_p)$. To see that the last term is zero just use the fact that the bracket is a derivation and that $[\iota, [\iota, \iota]] = 0$ (if $p \neq 3$ the Jacobi relation imply $[\iota, [\iota, \iota]] = 0$, while for $p = 3$ this iterated bracket is zero by definition). \square

5 Operations for odd degree classes

As we saw in Section 3.1 the homology operations for odd degree elements in a gravity algebra are governed by $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_p(\pm 1)) \cong H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$, where p is any fixed prime. In this Section we compute this homology (Theorem 5.7). Observe that if $p = 2$ the sign representation is actually the trivial representation, therefore we get an alternative computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}; \mathbb{F}_2)$. The techniques involved comes from the theory of fiberwise configurations spaces, which we will quickly review in Section 5.1 (further details can be found in [5] and [2]).

5.1 Fiberwise configuration spaces

Let $\lambda : E \rightarrow B$ be a fiber bundle with fiber Y . We consider the following space of **(ordered) fiberwise configurations** of points

$$E(\lambda, n) := \{(e_1, \dots, e_n) \in E^n \mid e_i \neq e_j \text{ and } \lambda(e_i) = \lambda(e_j) \text{ if } i \neq j\}$$

The symmetric group Σ_n acts on $E(\lambda, n)$ by permuting the coordinates, so we can also define the space of **unordered fiberwise configurations** as the quotient $E(\lambda, n)/\Sigma_n$. In particular there are fiber bundles

$$F_n(Y) \hookrightarrow E(\lambda, n) \rightarrow B \quad C_n(Y) \hookrightarrow E(\lambda, n)/\Sigma_n \rightarrow B$$

Now let X be a connected CW-complex with basepoint $*$. We consider the following space of **fiberwise configurations with label in X**

$$E(\lambda; X) := \bigsqcup_{n=0}^{\infty} E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

where \sim is the equivalence relation determined by

$$(e_1, \dots, e_n) \times (x_1, \dots, x_n) \sim (e_1, \dots, \hat{e}_i, \dots, e_n) \times (x_1, \dots, \hat{x}_i, \dots, x_n)$$

when $x_i = *$.

Remark 5.1. If the fiber bundle is given by a constant map $\lambda : E \rightarrow \{*\}$ the space $E(\lambda; X)$ is usually denoted by $C(E; X)$. These are just configuration of points on E with label in X .

The spaces $E(\lambda; X)$ are equipped with a natural filtration by the number of points

$$\{*\} \subseteq E_1(\lambda; X) \subseteq E_2(\lambda; X) \subseteq \dots \subseteq E(\lambda; X)$$

where $E_k(\lambda; X)$ is the subspace

$$E_k(\lambda; X) := \bigsqcup_{n=0}^k E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

In the literature it is standard to denote the quotient $E_k(\lambda; X)/E_{k-1}(\lambda; X)$ by $D_k(\lambda; X)$.

Theorem 5.2 ([2]). *Let X be a connected CW-complex and $F \hookrightarrow E \xrightarrow{\lambda} B$ be a fiber bundle. Then $E(\lambda; X)$ is stably equivalent to $\bigvee_{k \in \mathbb{N}} D_k(\lambda; X)$. In particular we have a homology isomorphism*

$$\tilde{H}_*(E(\lambda; X); \mathbb{Z}) \cong \bigoplus_{k=1}^{\infty} \tilde{H}_*(D_k(\lambda; X); \mathbb{Z})$$

Proposition 5.3 ([5], p. 8). *Let X be a connected CW complex with basepoint $*$ $\in X$ and \mathbb{F} be any field. Then we have a quasi-isomorphism*

$$C_*(E(\lambda, k)) \otimes_{\Sigma_k} \tilde{H}_*(X)^{\otimes k} \rightarrow C_*(D_k(\lambda; X))$$

where $C_*(-)$ are the singular chains with \mathbb{F} -coefficients and Σ_k acts on the graded \mathbb{F} -vector space $\tilde{H}_*(X)^{\otimes k}$ by permutation of variables with the usual sign convention.

5.2 A model for $B(B_n/Z(B_n))$ with fiberwise configuration spaces

Let $\mathbb{C} \hookrightarrow E \xrightarrow{\lambda} \mathbb{C}P^\infty$ be the tautological line bundle. Explicitly, the total space is

$$E := \{(v, l) \in \mathbb{C}^\infty \times \mathbb{C}P^\infty \mid v \in l\}$$

and $\lambda : E \rightarrow \mathbb{C}P^\infty$ is the projection on the second coordinate. The most important observation of this section is the following proposition:

Proposition 5.4. *The unordered fiberwise configurations $E(\lambda, n)/\Sigma_n$ is a model for the classifying space $B(B_n/Z(B_n))$. Similarly, $E(\lambda, n)$ is a model for the classifying space $B(PB_n/Z(PB_n))$.*

Proof. We prove the statement for $E(\lambda, n)/\Sigma_n$, the other case is analogous. Consider the fibration $C_n(\mathbb{C}) \hookrightarrow E(\lambda, n)/\Sigma_n \rightarrow \mathbb{C}P^\infty$. The long exact sequence for homotopy groups shows that $\pi_i(E(\lambda, n)/\Sigma_n) = 0$ for all $i \geq 3$. Moreover, we get the following exact sequence:

$$0 \rightarrow \pi_2(E(\lambda, n)/\Sigma_n) \rightarrow \mathbb{Z} \xrightarrow{\partial} B_n \rightarrow \pi_1(E(\lambda, n)/\Sigma_n) \rightarrow 0$$

Now we claim that the connecting homomorphism ∂ includes \mathbb{Z} as the center of B_n . To prove this, let us consider the map

$$\begin{aligned} f : S^\infty &\rightarrow E(\lambda, n)/\Sigma_n \\ v &\mapsto \{(v, l_v), (\zeta \cdot v, l_v), \dots, (\zeta^{n-1} \cdot v, l_v)\} \end{aligned}$$

where $\zeta := e^{2\pi i/n}$ acts on $S^\infty \subseteq \mathbb{C}^\infty$ by multiplication and l_v denotes the line spanned by v . This is a map of fibrations

$$\begin{array}{ccc} C_n(\mathbb{C}) & \longleftarrow & S^1 \\ \downarrow & & \downarrow \\ E(\lambda, n)/\Sigma_n & \xleftarrow{f} & S^\infty \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty & \longleftarrow & \mathbb{C}P^\infty \end{array}$$

so we get the following commutative diagram, whose rows are exact:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \pi_2(E(\lambda, n)/\Sigma_n) & \rightarrow & \mathbb{Z} & \xrightarrow{\partial} & B_n & \rightarrow & \pi_1(E(\lambda, n)/\Sigma_n) & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \text{ id } & & \uparrow \text{ f* } & & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Finally observe that f_* includes \mathbb{Z} as the center of B_n , so the same holds for ∂ . \square

Remark 5.5. The Braid group B_n is equipped by a natural morphism to Σ_n , whose kernel is the pure braid group PB_n . Since this morphism sends the generator of the

center δ^2 to the identity permutation, there is a factorization

$$\begin{array}{ccccccccc} 1 & \longrightarrow & PB_n & \hookrightarrow & B_n & \longrightarrow & \Sigma_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & PB_n/Z(PB_n) & \hookrightarrow & B_n/Z(B_n) & \longrightarrow & \Sigma_n & \longrightarrow & 1 \end{array}$$

Therefore we can regard any Σ_n -module as a $B_n/Z(B_n)$ module.

Now let V be a graded vector space over some field \mathbb{F} , and assume that V is concentrated in degree greater than 1. In this case we can always find a bouquet of spheres S_V such that

$$V \cong \tilde{H}_*(S_V; \mathbb{F})$$

We assume that the symmetric group Σ_n acts on $V^{\otimes n}$ with the usual sign conventions. By the previous remark we can see $V^{\otimes n}$ as a $B_n/Z(B_n)$ -module.

Proposition 5.6. *Let \mathbb{F} be any field, $q \in \mathbb{N}$. Then we have an isomorphism*

$$H_*(B_n/Z(B_n); V^{\otimes n}) \cong \tilde{H}_*(D_n(\lambda; S_V); \mathbb{F})$$

In particular, if we choose V to be a copy of \mathbb{F} concentrated in degree $2q+1$ (resp. $2q$) we get

$$H_*(B_n/Z(B_n); \mathbb{F}(\pm 1)) \cong H_{*+(2q+1)n}(D_n(\lambda; S^{2q+1}); \mathbb{F}) \quad (10)$$

$$H_*(B_n/Z(B_n); \mathbb{F}) \cong H_{*+2qn}(D_n(\lambda; S^{2q}); \mathbb{F}) \quad (11)$$

Proof. To ease the notation, let us call $G_n := B_n/Z(B_n)$ and $H_n := PB_n/Z(PB_n)$. By definition $H_*(G_n; V^{\otimes n})$ is computed by $C_*^{cell}(EG_n) \otimes_{G_n} V^{\otimes n}$, where if G is a discrete group $C_*^{cell}(EG_n)$ denotes the standard resolution of \mathbb{Z} over $\mathbb{Z}[G]$. Since H_n is a subgroup of G_n , we can take EG_n as a model for EH_n . Therefore:

$$C_*^{cell}(EG_n) \otimes_{G_n} V^{\otimes n} \cong \frac{C_*^{cell}(EG_n) \otimes_{H_n} V^{\otimes n}}{\Sigma_n} \cong C_*^{cell}(EG_n)_{H_n} \otimes_{\Sigma_n} V^{\otimes n}$$

where the last isomorphism holds because H_n acts trivially on $V^{\otimes n}$. Note that $C_*^{cell}(EG_n)_{H_n}$ computes the homology of H_n . Proposition 5.4 tells us that $E(\lambda, n)$ is a model for the classifying space of $PB_n/Z(PB_n)$ and Proposition 5.3 gives us the quasi-isomorphism

$$C_*(E(\lambda, n)) \otimes_{\Sigma_n} \tilde{H}_*(S_V)^{\otimes n} \rightarrow C_*(D_n(\lambda; S_V))$$

which allows us to conclude the proof. \square

5.3 Computations

By Proposition 5.6 and Theorem 5.2 the computation of $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$ is reduced to the computation of $H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$.

Theorem 5.7. *Let p be an odd prime. Then*

$$H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p) \cong H_*(C(\mathbb{C}; S^{2q+1}); \mathbb{F}_p) \otimes H_*(BS^1; \mathbb{F}_p)$$

Proof. We get the statement by proving that the Serre spectral sequence with \mathbb{F}_p -coefficients associated to the fibration $C(\mathbb{C}; S^{2q+1}) \hookrightarrow E(\lambda; S^{2q+1}) \rightarrow BS^1$ degenerates at page E_2 . In particular, it suffices to show that the map induced by the inclusion $i_* : H_*(C(\mathbb{C}; S^{2q+1}); \mathbb{F}_p) \rightarrow H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$ is injective. Consider the following map

$$\begin{aligned} \psi : E(\lambda; S^{2q+1}) &\rightarrow C(\mathbb{C}^\infty; S^{2q+1}) \\ [(v_1, l), \dots, (v_n, l)] \times (p_1, \dots, p_n) &\mapsto [(v_1, \dots, v_n) \times (p_1, \dots, p_n)] \end{aligned}$$

where (v_i, l) are points in the total space of the tautological line bundle $\mathbb{C} \hookrightarrow E \rightarrow \mathbb{C}P^\infty$, and $p_i \in S^{2q+1}$ are the corresponding labels. Now observe that the inclusion $j : C(\mathbb{C}; S^{2q+1}) \rightarrow C(\mathbb{C}^\infty; S^{2q+1})$ factors through $E(\lambda; S^{2q+1})$:

$$\begin{array}{ccc} C(\mathbb{C}; S^{2q+1}) & \xrightarrow{j} & C(\mathbb{C}^\infty; S^{2q+1}) \\ \downarrow i & \nearrow \psi & \\ E(\lambda; S^{2q+1}) & & \end{array}$$

Since j induces an injective map in mod p homology (see [6] or [14]), the above commutative diagram shows that i_* is injective as well. \square

Remark 5.8. If we replace S^{2q+1} with S^{2q} the proof written above does not work, indeed the map $i_* : H_*(C(\mathbb{C}; S^{2q}); \mathbb{F}_p) \rightarrow H_*(C(\mathbb{C}^\infty; S^{2q}); \mathbb{F}_p)$ is not injective anymore, except when $p = 2$. For details about the homology of labelled configuration spaces we refer to the work of F. Cohen [6].

Theorem 5.9. *Let $q \geq 1$. Then*

$$H_*(E(\lambda; S^{2q}); \mathbb{F}_2) \cong H_*(C(\mathbb{C}; S^{2q}); \mathbb{F}_2) \otimes H_*(BS^1; \mathbb{F}_2)$$

Proof. Follow the proof of Theorem 5.7, just replace $2q + 1$ with $2q$ and p with 2. \square

We end this section with a proof of Theorem 4.16 based on labelled configuration spaces:

Corollary 5.10. *For any $n \in \mathbb{N}$ we have*

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_2) \otimes H_*(BS^1; \mathbb{F}_2)$$

Proof. By Proposition 5.6 we have

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2) = H_{*+2qn}(D_n(\lambda; S^{2q}); \mathbb{F}_2)$$

By Theorem 5.2 we get

$$\tilde{H}_*(E(\lambda; S^{2q}); \mathbb{F}_2) = \bigoplus_{n=1}^{\infty} H_*(D_n(\lambda; S^{2q}); \mathbb{F}_2)$$

so $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2)$ can be seen as a subspace of $H_*(E(\lambda; S^{2q}); \mathbb{F}_2)$. Finally, the Theorem 5.9 tell us that

$$H_*(E(\lambda; S^{2q}); \mathbb{F}_2) = \mathbb{F}_2[\iota, Q\iota, Q^2\iota, \dots] \otimes H_*(BS^1; \mathbb{F}_2)$$

where $\iota \in H_{2q}(S^{2q}; \mathbb{F}_2)$ is the fundamental class, and this is enough to get the statement. \square

References

- [1] C.-F. Bödigheimer. *Stable splittings of mapping spaces*. English. Algebraic topology, Proc. Workshop, Seattle/Wash. 1985, Lect. Notes Math. 1286, 174-187 (1987). 1987.
- [2] C.-F. Bödigheimer, F. R. Cohen, and M. D. Peim. “Mapping class groups and function spaces”. In: *Homotopy methods in algebraic topology (Boulder, CO, 1999)*. Vol. 271. Contemp. Math. Amer. Math. Soc., Providence, RI, 2001, pp. 17–39.
- [3] Glen E. Bredon. *Introduction to compact transformation groups*. Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972, pp. xiii+459.
- [4] Moira Chas and Dennis Sullivan. “String topology”. In: *arXiv preprint math/9911159* (1999).
- [5] Fred R Cohen and Miguel A Maldonado. “Mapping class groups and function spaces: a survey”. In: *arXiv preprint arXiv:1410.2200* (2014).
- [6] Frederick R Cohen et al. “The homology of C_{n+1} -spaces, $n \geq 0$ ”. In: *The homology of iterated loop spaces* (1976), pp. 207–351.
- [7] Ralph L. Cohen and John D. S. Jones. “A homotopy theoretic realization of string topology”. In: *Math. Ann.* 324.4 (2002), pp. 773–798.
- [8] Tammo tom Dieck. *Transformation groups*. English. Vol. 8. De Gruyter Stud. Math. De Gruyter, Berlin, 1987.
- [9] Clément Dupont and Geoffroy Horel. “On two chain models for the gravity operad”. In: *Proc. Amer. Math. Soc.* 146.5 (2018), pp. 1895–1910.
- [10] E. Getzler. “Batalin-Vilkovisky algebras and two-dimensional topological field theories”. In: *Comm. Math. Phys.* 159.2 (1994), pp. 265–285.
- [11] E. Getzler. “Operads and moduli spaces of genus 0 Riemann surfaces”. In: *The moduli space of curves (Texel Island, 1994)*. Vol. 129. Progr. Math. Birkhäuser Boston, Boston, MA, 1995, pp. 199–230.
- [12] E. Getzler. “Two-dimensional topological gravity and equivariant cohomology”. In: *Comm. Math. Phys.* 163.3 (1994), pp. 473–489.
- [13] E. Getzler and M. M. Kapranov. “Cyclic operads and cyclic homology”. In: *Geometry, topology, & physics*. Conf. Proc. Lecture Notes Geom. Topology, IV. Int. Press, Cambridge, MA, 1995, pp. 167–201.
- [14] Paolo Salvatore. “Configuration spaces on the sphere and higher loop spaces”. In: *Math. Z.* 248.3 (2004), pp. 527–540.

- [15] Benjamin C. Ward. “Maurer-Cartan elements and cyclic operads”. In: *J. Noncommut. Geom.* 10.4 (2016), pp. 1403–1464.
- [16] Craig Westerland. “Equivariant operads, string topology, and Tate cohomology”. In: *Math. Ann.* 340.1 (2008), pp. 97–142.