

The General and Finite Satisfiability Problems for PCTL are Undecidable

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ABSTRACT

The general/finite PCTL satisfiability problem asks whether a given PCTL formula has a general/finite model. We show that the finite PCTL satisfiability problem is *undecidable*, and the general PCTL satisfiability problem is even *highly undecidable* (beyond the arithmetical hierarchy). Consequently, there are no sound deductive systems proving all generally/finitely valid PCTL formulae.

CCS CONCEPTS

• **Theory of computation** → **Modal and temporal logics.**

KEYWORDS

Probabilistic Temporal Logics, Satisfiability, PCTL

1 INTRODUCTION

Probabilistic CTL (PCTL) [15] is a temporal logic interpreted over states in discrete Markov chains. PCTL is obtained from the standard CTL (Computational Tree Logic, see, e.g., [12]) by replacing the existential/universal path quantifiers with the probabilistic operator $P(\Phi) \bowtie r$. Here, Φ is a path formula, \bowtie is a comparison such as \geq or $<$, and r is a rational numerical constant. A formula $P(\Phi) \bowtie r$ holds in a state s if the probability of all runs initiated in s satisfying Φ is \bowtie -bounded by r .

Unlike CTL and other non-probabilistic temporal logics, PCTL does not have the small model property guaranteeing the existence of a bounded-size model for every satisfiable formula. In fact, one can easily construct satisfiable PCTL formulae without *any* finite model (see, e.g., [7]). Hence, the PCTL satisfiability problem is studied in two basic variants: (1) *finite satisfiability*, where we ask about the existence of a finite model, and (2) *general satisfiability*, where we ask about the existence of an unrestricted model.

At first glance, the finite satisfiability problem appears simpler. Let φ be a PCTL formula. The existence of a model for φ with a *given* number of states is decidable by encoding the question into first-order theory of the reals (see, e.g., [11]). Hence, the finite satisfiability problem is at least semi-decidable. To prove its decidability, it suffices to establish *some* computable upper bound on the number of states of a model for a finite-satisfiable formula. One is tempted to conjecture the existence of such a bound, because there is no apparent way how a finite-satisfiable PCTL formula φ can “enforce” the existence of $F(|\varphi|)$ distinct states in a model of φ , where F grows faster than every computable function (such as the Ackermann function). Despite numerous research attempts resulting in positive decidability results for various PCTL fragments (see Related work), the decidability of general/finite PCTL satisfiability has remained open for almost 30 years.

Our Contribution. In this paper, we show that the general and the finite PCTL satisfiability problems are *undecidable*.

The undecidability result for finite PCTL satisfiability holds even for a simple PCTL fragment consisting of formulae of the form $\varphi_1 \wedge \mathbf{G}_{=1} \varphi_2$, where φ_1, φ_2 contain only the path connectives \mathbf{X} and \mathbf{F}^2 ($\mathbf{X}\psi$ says that ψ holds in the next state, and $\mathbf{F}^2\psi$ says that ψ holds in a state reachable in at most two steps). An immediate consequence is that the *finite validity* problem for PCTL is not even semi-decidable. Hence, there is no sound & complete deductive system proving all finitely valid PCTL formulae.

For general PCTL satisfiability, we show that the problem is even *highly undecidable* (Σ_1^1 -hard). This result holds even for a PCTL fragment consisting of formulae of the form $\varphi_1 \wedge \mathbf{G}_{=1} \varphi_2 \wedge \mathbf{G}_{=1} \mathbf{F}_{=1} \varphi_3$, where φ_1, φ_2 contain only the path connectives \mathbf{X} and \mathbf{F}^2 , and φ_3 is a Boolean combination of atomic propositions. This implies that the (general) validity problem for PCTL is also highly undecidable.

Paper Organization. The results are obtained by constructing a formula Ψ simulating a computation of a given non-deterministic two-counter Minsky machine. The construction of Ψ is based on combining several novel techniques. To make the construction comprehensible, we explain these techniques gradually and proceed in four main steps.

- (1) In Section 3, we introduce *characteristic vectors* as a way of representing counter values, and two transformations τ, σ representing the *decrement* and *increment* operations on the counter (due to the chosen encoding, testing the counter for zero is trivial).
- (2) In Section 4, we show that there exists a *fixed* PCTL formula enforcing arbitrarily large finite models just by changing numerical constants x and y in its subformulae $\mathbf{X}_{=x} a$ and $\mathbf{X}_{=y} b$. This result is perhaps interesting on its own because it reveals a specific power of probability constraints. Intuitively, the constants x and y encode a characteristic vector representing a counter value n (by choosing appropriate x and y , the n can be arbitrarily large), and the formula implements the function τ decrementing the counter. This enforces the existence of states whose characteristic vectors represent the counter values ranging from n to 0, and such states must be pairwise different.
- (3) In Section 5, we extend the result of (2) by constructing a formula ψ simulating the computation of a non-deterministic one-counter Minsky machine. The crucial new ingredient is the construction implementing the *increment* function σ .
- (4) Finally, in Section 6, we construct a formula Ψ simulating a non-deterministic two-counter Minsky machine \mathcal{M} . Technically, \mathcal{M} is first “translated” into a synchronized product of

two non-deterministic one-counter Minsky machines \mathcal{M}_1 and \mathcal{M}_2 . Then, we use the formulae ψ_1 and ψ_2 constructed for \mathcal{M}_1 and \mathcal{M}_2 by the method of Section 5 and “merge” them into the formula Ψ .

In each step, we re-use the results of the previous steps, possibly after some necessary modifications. This leads to substantial simplifications at the cost of frequent references to previous sections. We compensate for this inconvenience by suggestive notation, writing the re-used formulae consistently in **boldface**. The exact semantics of this notation is explained at the beginning of Section 5 and Section 6.

The presented constructions bring some additional consequences formulated in Section 7.

Related Work. The probabilistic extension of CTL (and also CTL*) has been initially studied in its *qualitative* form, where the range of admissible probability constraints is restricted to $\{=0, >0, =1, <1\}$ [17, 19, 22]. Both general and finite satisfiability for qualitative PCTL are shown decidable in these works. A precise complexity classification of general and finite satisfiability for qualitative PCTL, together with a construction of (a finite description of) a model, are given in [7]. In the same paper, it is also shown that both general and finite satisfiability are undecidable when the class of admissible models is restricted to Markov chains with a k -bounded branching degree, where $k \geq 2$ is an arbitrary constant (this technique is not applicable to general Markov chains). A variant of the bounded satisfiability problem, where transition probabilities are restricted to $\{\frac{1}{2}, 1\}$, is proven **NP**-complete in [3].

The decidability of finite satisfiability for various quantitative PCTL fragments (with general probability constraints) is established in [10, 11, 20]. More concretely, in [10], it is shown that every formula φ of the *bounded fragment* of PCTL, where the validity of φ in a state s depends only on a bounded prefix of a run initiated in s , has a bounded-size tree model. In [20], several PCTL fragments based on F and G operators are studied. For each of these fragments, it is shown that every finite satisfiable formula has a bounded-size model where every non-bottom SCC is a singleton. In [11], a more abstract decidability result based on isolating the progress achieved along a chain of visited SCCs is presented.

The *model-checking* problem for PCTL has been studied both for finite Markov chains (see, e.g., [1, 2, 4, 18]) and for infinite Markov chains generated by probabilistic pushdown automata and their subclasses [9, 13, 14]. PCTL formulae have also been used as *objectives* in Markov decision processes (MDPs) and stochastic games, where the players controlling non-deterministic states strive to satisfy/falsify a given PCTL formula. Positive decidability results exist for finite MDPs and qualitative PCTL formulae [8]. For general PCTL and finite MDPs, the problem becomes undecidable [6].

2 PRELIMINARIES

The sets of non-negative integers, rational numbers, and real numbers are denoted by \mathbb{N} , \mathbb{Q} , and \mathbb{R} , respectively. The intervals of real numbers are written in the standard way, e.g., $[0, 1)$ is the set of all $r \in \mathbb{R}$ such that $0 \leq r < 1$.

We use $\mathbf{v}, \mathbf{u}, \mathbf{\kappa}, \dots$ to denote the elements of $\mathbb{R} \times \mathbb{R}$. The first and the second components of \mathbf{v} are denoted by \mathbf{v}_1 and \mathbf{v}_2 , respectively.

The n -fold composition $f \circ \dots \circ f$ of a function $f : A \rightarrow A$ (where A is some set) is denoted by f^n .

2.1 The Logic PCTL

The logic PCTL [15] is obtained from the standard CTL (Computational Tree Logic [12]) by replacing the existential and universal path quantifiers with the probabilistic operator $P(\Phi) \bowtie r$, where Φ is a path formula, \bowtie is a comparison, and $r \in [0, 1]$ is a rational constant.

Definition 2.1 (PCTL). Let AP be a set of atomic propositions. The syntax of PCTL state and path formulae is defined by the following abstract syntax equations:

$$\begin{aligned} \varphi & ::= a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid P(\Phi) \bowtie r \\ \Phi & ::= \mathbf{X}\varphi \mid \varphi_1 \mathbf{U}\varphi_2 \mid \varphi_1 \mathbf{U}^k \varphi_2 \end{aligned}$$

Here, $a \in AP$, $\bowtie \in \{\geq, >, \leq, <, =\}$, $r \in [0, 1]$ is a rational constant, and $k \in \mathbb{N}$.

The formulae *true*, *false* and the other Boolean connectives are defined using \neg and \wedge in the standard way. We also use $\mathbf{F}\varphi$ and $\mathbf{F}^k\varphi$ to abbreviate the formulae *true* $\mathbf{U}\varphi$ and *true* $\mathbf{U}^k\varphi$, respectively. Furthermore, we often abbreviate a formula of the form $P(\Phi) \bowtie r$ by omitting P and adjoining the probability constraint directly to the topmost path operator of Φ . For example, we write $\mathbf{X}_{=1}\varphi$ instead of $P(\mathbf{X}\varphi) = 1$. We also write $\mathbf{G}_{=1}\varphi$ instead of $\mathbf{F}_{=0}\neg\varphi$.

PCTL formulae are interpreted over Markov chains where every state s is assigned a subset $v(s) \subseteq AP$ of propositions valid in s .

Definition 2.2 (Markov chain). A Markov chain is a triple $M = (S, P, v)$, where S is a finite or countably infinite set of *states*, $P : S \times S \rightarrow [0, 1]$ is a function such that $\sum_{t \in S} P(s, t) = 1$ for every $s \in S$, and $v : S \rightarrow 2^{AP}$ is a *valuation*. We say that M is finite if S is a finite set.

For $s, t \in S$, we say that t is an *immediate successor* of s if $P(s, t) > 0$. A *path* in M is a finite sequence $w = s_0, \dots, s_n$ of states where $n \geq 0$ and $P(s_i, s_{i+1}) > 0$ for all $i < n$. We say that t is *reachable* from s if there is a path where the first and the last state is s and t , respectively.

A *run* in M is an infinite sequence $\pi = s_0, s_1, \dots$ of states such that every finite prefix of π is a path in M . We also use $\pi(i)$ to denote the state s_i of π .

For every path $w = s_0, \dots, s_n$, let $Run(w)$ be the set of all runs starting with w , and let $\mathbb{P}(Run(w)) = \prod_{i=0}^{n-1} P(s_i, s_{i+1})$. To every state s , we associate the probability space $(Run(s), \mathcal{F}_s, \mathbb{P}_s)$, where \mathcal{F}_s is the σ -field generated by all $Run(w)$ where w starts in s , and \mathbb{P}_s is the unique probability measure obtained by extending \mathbb{P} in the standard way (see, e.g., [5]).

The *validity* of a PCTL state/path formula for a given state/run of M is defined inductively as follows:

| | | |
|--|-----|--|
| $s \models a$ | iff | $a \in v(s)$, |
| $s \models \neg\varphi$ | iff | $s \not\models \varphi$, |
| $s \models \varphi_1 \wedge \varphi_2$ | iff | $s \models \varphi_1$ and $s \models \varphi_2$, |
| $s \models P(\Phi) \bowtie r$ | iff | $\mathbb{P}_s(\{\pi \in \text{Run}(s) \mid \pi \models \Phi\}) \bowtie r$, |
| $\pi \models \mathbf{X}\varphi$ | iff | $\pi(1) \models \varphi$ for some $i \in \mathbb{N}$, |
| $\pi \models \varphi_1 \mathbf{U} \varphi_2$ | iff | there is $j \geq 0$ such that $\pi(j) \models \varphi_2$ and $\pi(i) \models \varphi_1$ for all $0 \leq i < j$, |
| $\pi \models \varphi_1 \mathbf{U}^k \varphi_2$ | iff | there is $0 \leq j \leq k$ such that $\pi(j) \models \varphi_2$ and $\pi(i) \models \varphi_1$ for all $0 \leq i < j$. |

We say that M is a *model* of φ if $s \models \varphi$ for some state s of M . The *general/finite PCTL satisfiability problem* is the question of whether a given PCTL formula has a general/finite model.

2.2 Parameterized PCTL Formulae

A *parameterized PCTL formula* is a PCTL formula where some probability constraints are replaced with *parameters* ranging over rationals in $[0, 1]$. For example, $\xi(x) \equiv \mathbf{F}_{\geq 0.6} a \wedge \mathbf{G}_{=x} \neg a$ is a parameterized PCTL formula with one parameter x . For a parameterized PCTL formula $\varphi(x_1, \dots, x_k)$ and rational constants p_1, \dots, p_k in the interval $[0, 1]$, we use $\varphi[p_1, \dots, p_k]$ to denote the PCTL formula obtained from $\varphi(x_1, \dots, x_k)$ by substituting every x_i with p_i (we say that $\varphi[p_1, \dots, p_k]$ is an *instance* of $\varphi(x_1, \dots, x_k)$). For example, $\xi[0.1]$ is the formula $\mathbf{F}_{\geq 0.6} a \wedge \mathbf{G}_{=0.1} \neg a$.

2.3 Minsky Machines

A *non-deterministic Minsky machine* \mathcal{M} with $k \geq 1$ counters is a finite program

$$1 : \text{Ins}_1; \dots m : \text{Ins}_m;$$

where $m \geq 1$ and every $i : \text{Ins}_i$ is a labeled instruction of one of the following types:

- I. $i : \text{inc } c_j; \text{ goto } L;$
- II. $i : \text{if } c_j=0 \text{ then goto } L \text{ else dec } c_j; \text{ goto } L'$

Here, $j \in \{1, \dots, k\}$ is a counter index and $L, L' \subseteq \{1, \dots, m\}$ are sets of labels with one or two elements. We say that \mathcal{M} is *deterministic* if all L, L' occurring in the instructions of \mathcal{M} are singletons¹.

A *configuration* of \mathcal{M} is a tuple (i, n_1, \dots, n_k) of non-negative integers where $1 \leq i \leq m$ represents the current control position and n_1, \dots, n_k represent the current counter values. A configuration (i', n'_1, \dots, n'_k) is a *successor* of a configuration (i, n_1, \dots, n_k) , written $(i, n_1, \dots, n_k) \mapsto (i', n'_1, \dots, n'_k)$, if the tuple (n'_1, \dots, n'_k) is obtained from (n_1, \dots, n_k) by performing Ins_i , and i' is an element of the corresponding L (or L') in Ins_i . Note that every configuration has either one or two successor(s). A *computation* of \mathcal{M} is an infinite sequence of configurations $\omega \equiv C_0, C_1, \dots$ such that $C_0 = (1, 0, \dots, 0)$ and $C_i \mapsto C_{i+1}$ for all $i \in \mathbb{N}$. We say that ω is *periodic* if there are $i, j \in \mathbb{N}$ such that $i < j$ and the infinite sequences C_i, C_{i+1}, \dots and C_j, C_{j+1}, \dots are the same.

¹Our definition of non-deterministic Minsky machines is equivalent to the standard one where the target sets of labels are singletons, and there is also a Type III instruction of the form $i : \text{goto } u \text{ or } u'$. For purposes of this paper, the adopted definition is more convenient.

Now, we recall the standard undecidability results for Minsky machines. The symbols Σ_1^0 and Σ_1^1 denote the corresponding levels in the arithmetical and the analytical hierarchies, respectively.

(1) The *boundedness problem* for a given deterministic two-counter Minsky machine \mathcal{M} is undecidable and Σ_1^0 -complete [21]. Here, \mathcal{M} is bounded if the unique computation ω contains only finitely many pairwise different configurations (i.e., ω is periodic).

(2) The *recurrent reachability problem* for a given non-deterministic two-counter Minsky machine \mathcal{M} is highly undecidable a Σ_1^1 -complete [16]. Here, the question is whether there exists a *recurrent* computation ω of \mathcal{M} such that the instruction Ins_1 is executed infinitely often along ω .

3 REPRESENTING A COUNTER

In this section, we introduce several “geometrical” concepts underpinning our results. Furthermore, we show how to represent a non-negative counter value by a pair of quantities, and we design functions modeling the decrement/increment operation on the counter. Missing proofs are in the Appendix.

Let us fix a rational constant q such that $\frac{3}{4} < q < 1$ and $\sqrt{4q-3}$ is rational. For example, we can put $q = \frac{13}{16}$. Furthermore, we define

$$I_q = \left(\frac{1 - \sqrt{4q-3}}{2}, \frac{1 + \sqrt{4q-3}}{2} \right)$$

By our choice of q , we immediately obtain that $I_q \subseteq (0, 1)$. Finally, we fix $\boldsymbol{\kappa} \in (0, 1)^2$ with rational components such that $\boldsymbol{\kappa}_1 \in I_q$ and $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \leq 1$.

The rational constants $q, \boldsymbol{\kappa}_1$, and $\boldsymbol{\kappa}_2$ are used as probability constraints in the PCTL formulae constructed in the next sections. The defining properties of q and I_q are explained in Lemma 3.3.

Definition 3.1 (characteristic vector, A_t, B_t , and C_t sets). Let t be a state of a Markov chain with transition function P . Let A_t, B_t , and C_t be the sets of all immediate successors of t satisfying the atomic propositions a, b , and c , respectively.

The *characteristic vector* of t is the vector $\mathbf{v}[t] \in [0, 1]^2$ where $\mathbf{v}[t]_1 = \sum_{u \in A_t} P(t, u)$ and $\mathbf{v}[t]_2 = \sum_{u \in B_t} P(t, u)$.

Observe that $\mathbf{v}[t]_1$ and $\mathbf{v}[t]_2$ is the probability of satisfying the path formula $\mathbf{X}a$ and $\mathbf{X}b$ in t , respectively. Intuitively, we use characteristic vectors to encode non-negative integers, where $\boldsymbol{\kappa}$ represents zero, and the decrement/increment operations correspond to performing the functions τ/σ introduced in our next definition.

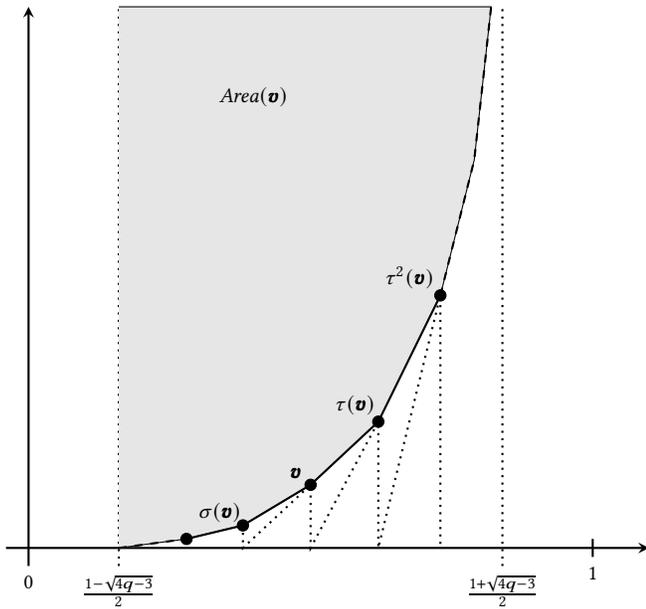
Definition 3.2 (τ, σ , and W). Let $W = I_q \times [0, \infty)$. Furthermore, let $\tau, \sigma : W \rightarrow \mathbb{R}^2$ be functions² defined as follows:

- $\tau(\mathbf{v}) = ((q-1+\mathbf{v}_1)/\mathbf{v}_1, \mathbf{v}_2/\mathbf{v}_1)$
- $\sigma(\mathbf{v}) = ((1-q)/(1-\mathbf{v}_1), (\mathbf{v}_2(1-q))/(1-\mathbf{v}_1))$.

The *slope* of a line or a line segment in \mathbb{R}^2 is defined in the standard way. For all $\mathbf{v}, \mathbf{u} \in \mathbb{R}^2$ where $\mathbf{v}_1 \neq \mathbf{u}_1$, we use $\text{slope}(\mathbf{v}, \mathbf{u})$ to denote the slope of the line containing \mathbf{v}, \mathbf{u} .

In the next lemma, we use the defining properties of q and I_q (this explains their purpose).

² τ and σ are not arbitrary; they must satisfy several properties simultaneously to enable the presented constructions. There is no trivial intuition behind their design.

Figure 1: Points(\mathbf{v}), LSegs(\mathbf{v}), and Area(\mathbf{v}).

LEMMA 3.3. For every $\mathbf{v} \in W$, we have the following:

- $\tau(\mathbf{v}), \sigma(\mathbf{v}) \in W$;
- $\tau(\mathbf{v})_1 > \mathbf{v}_1$ and $\tau(\mathbf{v})_2 \geq \mathbf{v}_2$; if $\mathbf{v}_2 > 0$, then $\tau(\mathbf{v})_2 > \mathbf{v}_2$;
- let $\mathbf{u} = (\mathbf{v}_1, 0)$; then $\text{slope}(\mathbf{u}, \tau(\mathbf{v})) = \text{slope}(\tau(\mathbf{v}), \tau^2(\mathbf{v}))$;
- let $\mathbf{u} = (\mathbf{v}_1, y)$ where $0 \leq y < \mathbf{v}_2$. Then $\text{slope}(\mathbf{u}, \tau(\mathbf{u})) < \text{slope}(\mathbf{v}, \tau(\mathbf{v}))$;
- $\sigma(\tau(\mathbf{v})) = \tau(\sigma(\mathbf{v})) = \mathbf{v}$.

For every $\mathbf{u} \in W$, let $L(\mathbf{u})$ be the line segment between the points \mathbf{u} and $\tau(\mathbf{u})$, including \mathbf{u} and excluding $\tau(\mathbf{u})$, i.e.,

$$L(\mathbf{u}) = \{\mathbf{w} \in \mathbb{R}^2 \mid \mathbf{w} = \lambda \mathbf{u} + (1-\lambda)\tau(\mathbf{u}) \text{ for some } \lambda \in (0, 1]\}.$$

We use $\text{Line}(\mathbf{u})$ to denote the line obtained by prolonging the line segment $L(\mathbf{u})$, and $H(\mathbf{u})$ to denote the closed half-space above $\text{Line}(\mathbf{u})$, i.e.,

$$H(\mathbf{u}) = \{\mathbf{u} + \alpha \mid \alpha \in \mathbb{R}^2, (\tau(\mathbf{u}_2) - \mathbf{u}_2, \mathbf{u}_1 - \tau(\mathbf{u})_1) \cdot \alpha \leq 0\}.$$

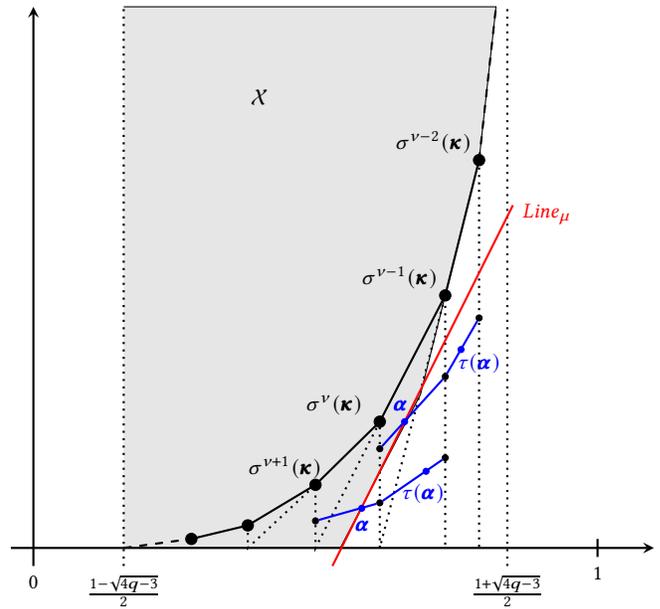
Definition 3.4. For every $\mathbf{v} \in W$ where $\mathbf{v}_2 > 0$, let

$$\begin{aligned} \text{Points}(\mathbf{v}) &= \{\tau^k(\mathbf{v}), \sigma^k(\mathbf{v}) \mid k \in \mathbb{N}\}, \\ \text{LSegs}(\mathbf{v}) &= \bigcup_{\mathbf{u} \in \text{Points}(\mathbf{v})} L(\mathbf{u}), \\ \text{Area}(\mathbf{v}) &= W \cap \bigcap_{\mathbf{u} \in \text{Points}(\mathbf{v})} H(\mathbf{u}). \end{aligned}$$

The structure of $\text{Points}(\mathbf{v})$, $\text{LSegs}(\mathbf{v})$, and $\text{Area}(\mathbf{v})$ is shown in Fig. 1, where the dotted lines illustrate the property of Lemma 3.3 (c).

Now, we present a sequence of technical observations culminating with (crucial) Theorem 3.8.

A convex combination of vectors $\mathbf{u}^1, \mathbf{u}^2, \dots$ is *positive* if all coefficients used in the combination are positive. The next lemma is a trivial corollary to Lemma 3.3 (see Fig. 1).

Figure 2: The construction proving $\text{Out} = \emptyset$.

LEMMA 3.5. For every $\mathbf{v} \in W$ where $\mathbf{v}_2 > 0$ and every $\mathbf{u} \in \text{Points}(\mathbf{v})$, we have the following:

- If \mathbf{u} is a positive convex combination of $\mathbf{u}^1, \mathbf{u}^2, \dots$ where $\mathbf{u}^i \in \text{Area}(\mathbf{v})$ for all $i \in \mathbb{N}$, then $\mathbf{u}^i = \mathbf{u}$ for all $i \in \mathbb{N}$.
- If $\mathbf{w} \in L(\mathbf{u})$ is a positive convex combination of $\mathbf{u}^1, \mathbf{u}^2, \dots$ where $\mathbf{u}^i \in \text{Area}(\mathbf{v})$ for all $i \in \mathbb{N}$, then $\mathbf{u}^i \in L(\mathbf{u}) \cup \{\tau(\mathbf{u})\}$ for all $i \in \mathbb{N}$.

LEMMA 3.6. Let $\mathbf{w} \in W$ and $\mathbf{u} \in L(\mathbf{w})$. Then $\tau(\mathbf{u}) \in L(\tau(\mathbf{w}))$.

LEMMA 3.7. For all $\mathbf{v} \in W \setminus \text{Area}(\kappa)$ where $\mathbf{v}_1 \leq \kappa_1$, there exists $\mathbf{u} \in W \setminus \text{Area}(\kappa)$ such that $\mathbf{u}_1 = \sigma^k(\kappa_1)$ for some $k \geq 0$ and $\mathbf{v} \in L(\mathbf{u})$.

Now we prove the main result of this section.

THEOREM 3.8. Let T be a subset of states of some Markov chain with transition function P such that for every $t \in T$, we have that $\mathbf{v}[t]_1 \leq \kappa_1$ and if $\mathbf{v}[t] \neq \kappa$, then $A_t \subseteq T$ and the following equations are satisfied:

$$q = 1 - \mathbf{v}[t]_1 + \sum_{u \in A_t} P(t, u) \cdot \mathbf{v}[u]_1 \quad (1)$$

$$q = 1 - \mathbf{v}[t]_1 - \mathbf{v}[t]_2 + \sum_{u \in A_t} P(t, u) \cdot (\mathbf{v}[u]_1 + \mathbf{v}[u]_2) \quad (2)$$

Then $\mathbf{v}[t] \in \text{Area}(\kappa)$ for every $t \in T$. Furthermore, for every $t \in T$ such that $\mathbf{v}[t] \neq \kappa$ and every $u \in A_t$, we have that $\mathbf{v}[u] = \tau(\mathbf{v}[t])$.

PROOF. We start by proving the following claim: for every $t \in T$ such that $\mathbf{v}[t] \neq \kappa$, the vector $\tau(\mathbf{v}[t])$ is a positive convex combination of the vectors in $\{\mathbf{v}[u] \mid u \in A_t\}$.

By rewriting (1), we obtain

$$\frac{q - 1 + \mathbf{v}[t]_1}{\mathbf{v}[t]_1} = \sum_{u \in U_t} \frac{P(t, u)}{\mathbf{v}[t]_1} \mathbf{v}[u]_1 \quad (3)$$

Note that the left-hand side of (3) is equal to $\tau(\mathbf{v}[t])_1$. Furthermore, by simplifying the right-hand side of (2) using (1), we obtain

$$q = q - \mathbf{v}[t]_2 + \sum_{u \in U_t} P(t, u) \cdot \mathbf{v}[u]_2 \quad (4)$$

Thus,

$$\frac{\mathbf{v}[t]_2}{\mathbf{v}[t]_1} = \sum_{u \in U_t} \frac{P(t, u)}{\mathbf{v}[t]_1} \cdot \mathbf{v}[u]_2 \quad (5)$$

Note that the left-hand side of (5) is equal to $\tau(\mathbf{v}[t])_2$. By combining (3) and (5), we have that

$$\tau(\mathbf{v}[t]) = \sum_{u \in U_t} \frac{P(t, u)}{\mathbf{v}[t]_1} \cdot \mathbf{v}[u] \quad (6)$$

which proves the claim.

Now we prove the first part of the theorem, i.e., $\mathbf{v}[t] \in \text{Area}(\boldsymbol{\kappa})$ for every $t \in T$. Suppose the converse, i.e., the set Out consisting of all $t \in T$ such that $\mathbf{v}[t] \notin \text{Area}(\boldsymbol{\kappa})$ is non-empty. We show that this assumption leads to a contradiction. The arguments are illustrated in Fig. 2.

Let $\mathbf{v}[\text{Out}] = \{\mathbf{v}[t] \mid t \in \text{Out}\}$, and let ν be the *minimal* $k \in \mathbb{N}$ such that $\mathbf{v}[\text{Out}] \setminus H(\sigma^k(\boldsymbol{\kappa})) \neq \emptyset$. Furthermore, for every $0 \leq y \leq \sigma^\nu(\boldsymbol{\kappa})_2$, let Line_y be the line with the same slope as $\text{Line}(\sigma^\nu(\boldsymbol{\kappa}))$ containing the point $(\sigma^\nu(\boldsymbol{\kappa})_1, y)$. We use $H(\text{Line}_y)$ to denote the closed half-space above Line_y . Let μ be the supremum of all $y \leq \sigma^\nu(\boldsymbol{\kappa})_2$ such that $\mathbf{v}[\text{Out}] \subseteq H(\text{Line}_y)$. Clearly, $0 \leq \mu < \sigma^\nu(\boldsymbol{\kappa})_2$. Let

$$\mathcal{X} = W \cap H(\text{Line}_\mu) \cap \bigcap_{k=0}^{\nu-1} H(\sigma^k(\boldsymbol{\kappa}))$$

Note \mathcal{X} is a convex set and $\text{Area}(\boldsymbol{\kappa}) \cup \mathbf{v}[\text{Out}] \subseteq \mathcal{X}$. We show that there exists $t \in \text{Out}$ such that $\tau(\mathbf{v}[t]) \notin \mathcal{X}$. By the claim proven above, $\tau(\mathbf{v}[t])$ is a positive convex combination of the vectors in $\{\mathbf{v}[u] \mid u \in A_t\}$. However, this is impossible because all these vectors are in \mathcal{X} , and \mathcal{X} is a convex set; we have a contradiction.

The existence of $t \in \text{Out}$ such that $\tau(\mathbf{v}[t]) \notin \mathcal{X}$ is proven as follows. By the definition of μ , there is an infinite sequence t_1, t_2, \dots such that

- $t_i \in \text{Out}$ and $\mathbf{v}[t_i]_1 \leq \boldsymbol{\kappa}_1$ for all $i \in \mathbb{N}$,
- the distance of $\mathbf{v}[t_i]$ from Line_μ approaches zero as $i \rightarrow \infty$.

This sequence must contain an infinite subsequence converging to some point $\boldsymbol{\alpha} \in \text{Line}_\mu \cap \mathcal{X}$.

Since τ is continuous, it suffices to show that $\tau(\boldsymbol{\alpha}) \notin \mathcal{X}$ for every $\boldsymbol{\alpha} \in \text{Line}_\mu \cap \mathcal{X}$. So, let us fix some $\boldsymbol{\alpha} \in \text{Line}_\mu \cap \mathcal{X}$. By Lemma 3.7, there exists $\mathbf{u} \in W \setminus \text{Area}(\boldsymbol{\kappa})$ such that $\mathbf{u}_1 = \sigma^k(\boldsymbol{\kappa})_1$ for some $k \geq 0$ and $\boldsymbol{\alpha} \in L(\mathbf{u})$. By the definition of Line_μ , we have that k is either $\nu+1$ or ν , depending on whether $\boldsymbol{\alpha}_1 < \sigma^\nu(\boldsymbol{\kappa})_1$ or not, respectively. In the first case, the slope of both $L(\mathbf{u})$ and $L(\tau(\mathbf{u}))$ is strictly smaller than the slope of Line_μ (see Lemma 3.3). This implies $\tau(\mathbf{u}) \notin \mathcal{X}$ and $L(\tau(\mathbf{u})) \cap \mathcal{X} = \emptyset$, hence $\tau(\boldsymbol{\alpha}) \notin \mathcal{X}$ because $\tau(\boldsymbol{\alpha}) \in L(\tau(\mathbf{u}))$ by Lemma 3.6. In the second case, the slope of $L(\mathbf{u})$ is strictly smaller than the slope of Line_μ , which implies $\tau(\mathbf{u}) \notin \mathcal{X}$. Furthermore, the slope of $L(\tau(\mathbf{u}))$ is strictly smaller than the slope of $L(\sigma^{\nu-1}(\boldsymbol{\kappa}))$, and hence $L(\tau(\mathbf{u})) \cap \mathcal{X} = \emptyset$. Since $\tau(\boldsymbol{\alpha}) \in L(\tau(\mathbf{u}))$ by Lemma 3.6, we obtain $\tau(\boldsymbol{\alpha}) \notin \mathcal{X}$.

The second part of the theorem follows easily. Let $t \in T$ such that $\mathbf{v}[t] \neq \boldsymbol{\kappa}$. By the above claim, $\tau(\mathbf{v}[t])$ is a positive convex

combination of the vectors in $\{\mathbf{v}[u] \mid u \in A_t\}$ and all of these vectors belong to $\text{Area}(\boldsymbol{\kappa})$ by the first part of the theorem. Hence, $\mathbf{v}[u] = \tau(\mathbf{v}[t])$ for all $u \in A_t$ by applying Lemma 3.5 (a) \square

4 A PARAMETERIZED PCTL FORMULA WITH ARBITRARILY LARGE MODELS

In this section, we prove the following result:

THEOREM 4.1. *There exists a parameterized PCTL formula $\psi(x, y)$ such that for every $n \in \mathbb{N}$, there exists an instance $\psi[c, d]$ satisfying the following:*

- every model of $\psi[c, d]$ has at least n states;
- $\psi[c, d]$ has a finite-state model with $O(n)$ states.

4.1 Constructing $\psi(x, y)$

Let us fix q and $\boldsymbol{\kappa}$ in the same way as in Section 3. The set of atomic propositions³ occurring in $\psi(x, y)$ is $A = \{a, b, c, h, r_0, r_1, r_2, r_3, r_4\}$. For every $B \subseteq A$, we use $\langle B \rangle$ to denote the formula saying that exactly the propositions of B are satisfied, i.e., $\langle B \rangle \equiv \bigwedge_{p \in B} p \wedge \bigwedge_{p \in A \setminus B} \neg p$. Slightly abusing our notation, we write, e.g., $\langle a, r_i \rangle$ instead of $\langle \{a, r_i\} \rangle$. Furthermore, for every $r_i \in A$, we use $S(r_i)$ to denote the ‘‘successor’’ proposition $r_j \in A$ such that $j = i+1 \pmod 5$. For example, $S(S(r_3)) = r_0$. Recall that S^k denotes the k -fold composition $S \circ \dots \circ S$.

We put

$$\psi(x, y) \equiv \text{Init}(x, y) \wedge \mathbf{G}_{=1} \text{Invariant}$$

where

$$\text{Init}(x, y) \equiv \langle a, r_0 \rangle \wedge \mathbf{X}_{=x} a \wedge \mathbf{X}_{=y} b$$

is a parameterized *initial condition* that has to be valid in a state s satisfying an instance of $\psi(x, y)$, and

$$\text{Invariant} \equiv \text{Fin} \vee \text{Trans} \vee \text{Free}$$

is a formula (with no parameters) that must be valid in every state reachable from s . The formula

$$\text{Free} \equiv h \wedge \bigvee_{B \subseteq A} (\langle B \rangle \wedge \mathbf{X}_{=1}(B))$$

ensures that every reachable state t satisfying the predicate h has only immediate successors satisfying the same subset of A as t . This enforces that *Free* is valid in all states reachable from t (intuitively, these states are ‘‘free’’ in the sense that they do not require further attention).

The formula *Fin* is defined as follows:

$$\text{Fin} \equiv \bigvee_{i \in \{0, \dots, 4\}} \langle a, r_i \rangle \wedge \text{FSuc}_i \wedge \text{Zero}$$

where

$$\text{FSuc}_i \equiv \mathbf{X}_{=1}(\langle h, a, S(r_i) \rangle \vee \langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle),$$

$$\text{Zero} \equiv \mathbf{X}_{=\boldsymbol{\kappa}_1} a \wedge \mathbf{X}_{=\boldsymbol{\kappa}_2} b.$$

Finally, we put

$$\text{Trans} \equiv \bigvee_{i \in \{0, \dots, 4\}} \langle a, r_i \rangle \wedge \text{Suc}_i \wedge \text{Interval} \wedge \text{Eq}_i$$

³For purposes of this section, it suffices to use only three r_i propositions instead of five (this is apparent when inspecting Fig. 3). However, using five r_i 's allows us to reuse some of the constructed formulae in the next sections with only trivial modifications.

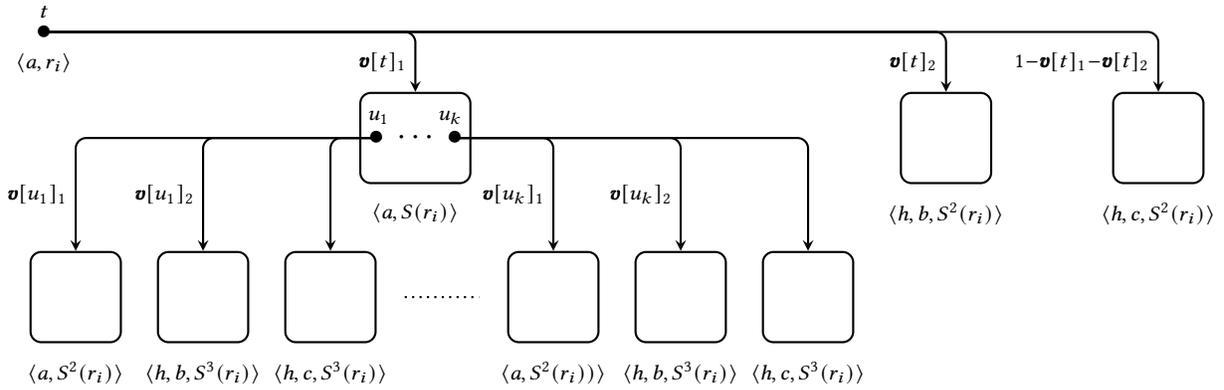


Figure 3: The structure of transient states in a model of $\psi[c, d]$

where

$$Suc_i \equiv \mathbf{X}_{=1}(\langle a, S(r_i) \rangle \vee \langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle)$$

$$Interval \equiv \mathbf{X}_{>(1-\sqrt{4q-3})/2} a \wedge \mathbf{X}_{\leq \kappa_2} a \wedge \mathbf{X}_{>0} b$$

$$Eq_i \equiv \mathbf{F}_{=q}^2 S^2(r_i) \wedge \mathbf{F}_{=q}^2((S^2(r_i) \wedge \neg b) \vee (S^3(r_i) \wedge b))$$

This completes the definition of $\psi(x, y)$.

4.2 A Proof of Theorem 4.1

For a given $n \in \mathbb{N}$, let $c = \sigma^n(\kappa)_1$, $d = \sigma^n(\kappa)_2$, and consider the instance $\psi[c, d]$ of $\psi(x, y)$ (note that c, d are rational). We show that $\psi[c, d]$ satisfies the two claims of Theorem 4.2.

To prove the first claim, let us fix a Markov chain $M = (S, P, \nu)$ such that $s \models \psi[c, d]$ for some $s \in S$. If S is infinite, we are done immediately. We show that if S is finite, then S contains at least n states. Without restrictions, we assume that all states of S are reachable from s .

Note that for every $t \in S$, we have that *exactly one*⁴ of the following possibilities holds: $t \models Free$, $t \models Fin$, or $t \models Trans$. Thus, S is partitioned into three disjoint subsets of *free*, *final*, and *transient* states, respectively.

Properties of free states. Observe that a state t is free iff $t \models h$. The formula *Free* ensures that all immediate successors of a free state t (and consequently also all states reachable from t) satisfy the same set of atomic propositions as t .

Properties of final states. The formula *Fin* ensures that all immediate successors of every final state are free and the characteristic vector of a final state is equal to κ .

Properties of transient states. Let t be a transient state, i.e., $t \models \langle a, r_i \rangle \wedge Suc_i \wedge Interval \wedge Eq_i$ for exactly one $i \in \{0, \dots, 4\}$. The formula *Suc_i* says that t has precisely three types of immediate successors:

- I. transient or final states satisfying $\langle a, S(r_i) \rangle$,
- II. free states satisfying $\langle h, b, S^2(r_i) \rangle$,
- III. free states satisfying $\langle h, c, S^2(r_i) \rangle$.

⁴In particular, note that assuming $t \models Fin \wedge Trans$ implies $t \models \mathbf{X}_{>0} \langle a, S(r_i) \rangle$ and $t \models \mathbf{X}_{>0} \langle h, a, S(r_i) \rangle$, hence $t \not\models FSuc_i$ and $t \not\models Suc_i$ (contradiction).

Note that the first/second component of $\mathbf{v}[t]$ is precisely the total probability of entering an immediate successor of Type I/II from t , respectively (see Fig. 3, where u_1, \dots, u_k are Type I successors of t , and $\mathbf{v}[t]_1 = \sum_{j=1}^k P(t, u_j)$).

Let T be the set of all final and transient states. We show that T satisfies the conditions of Theorem 3.8.

For every final state $t \in T$ we have that $t \models Zero$, and hence $\mathbf{v}[t] = \kappa$. Now let $t \in T$ be a transient state. Since every state of A_t is transient or final, we have that $A_t \subseteq T$. Furthermore,

$$t \models \langle a, r_i \rangle \wedge Suc_i \wedge Interval \wedge Eq_i$$

for precisely one $i \in \{0, \dots, 4\}$. Since $t \models Interval$, we obtain $\mathbf{v}[t]_1 \leq \kappa_1$. Now consider the formula

$$Eq_i \equiv \mathbf{F}_{=q}^2 S^2(r_i) \wedge \mathbf{F}_{=q}^2((S^2(r_i) \wedge \neg b) \vee (S^3(r_i) \wedge b))$$

The first conjunct says that the probability of all runs initiated in t satisfying the path formula $\mathbf{F}^2 S^2(r_i)$ is equal to q . By inspecting the structure of transient states enforced by *Suc_i* (see Fig. 3), we obtain

$$q = 1 - \mathbf{v}[t]_1 + \sum_{u \in A_t} P(t, u) \cdot \mathbf{v}[u]_1.$$

The second conjunct of *Eq_i* says that the probability of satisfying $\mathbf{F}^2((S^2(r_i) \wedge \neg b) \vee (S^3(r_i) \wedge b))$ in t is equal to q . Thus, we obtain

$$q = 1 - \mathbf{v}[t]_1 - \mathbf{v}[t]_2 + \sum_{u \in A_t} P(t, u) \cdot (\mathbf{v}[u]_1 + \mathbf{v}[u]_2).$$

Since $s \models \psi[c, d]$, we have that s is a transient state satisfying $\mathbf{v}[s] = \sigma^n(\kappa)$. By applying Theorem 3.8, we obtain that if t is a transient state satisfying $\mathbf{v}[t] = \sigma^k(\kappa)$ where $1 \leq k \leq n$, then every $u \in A_t$ satisfies $\mathbf{v}[u] = \tau(\sigma^k(\kappa)) = \sigma^{k-1}(\kappa)$. Consequently, for every $i \in \{0, \dots, n\}$, there must be a state of S with characteristic vector $\sigma^i(\kappa)$. These states must be pairwise different because $\sigma^i(\kappa) \neq \sigma^j(\kappa)$ for $i \neq j$ (see Lemma 3.3).

To prove the second claim of Theorem 4.1, realize that $\psi[c, d]$ has a model with states $t_0, \dots, t_n, b_0, \dots, b_{n-1}, c_0, \dots, c_{n-1}$ where

- for all $1 \leq i \leq n$, $t_i \models \langle a, r_j \rangle$ where $j = (n-i) \bmod 5$;
- $t_0 \models \langle h, a, r(j) \rangle$ where $j = n \bmod 5$;
- for all $1 \leq i \leq n$, we have that $b_{i-1} \models \langle h, b, S^2(r_j) \rangle$ and $c_{i-1} \models \langle h, c, S^2(r_j) \rangle$ where $j = (n-i) \bmod 5$;
- for all $0 \leq i \leq n-1$, we have that $P(b_i, b_i) = P(c_i, c_i) = 1$;

- for all $1 \leq i \leq n$, we have that $P(t_i, t_{i-1}) = \sigma^i(\boldsymbol{\kappa})_1$, $P(t_i, b_{i-1}) = \sigma^i(\boldsymbol{\kappa})_2$, and $P(t_i, c_{i-1}) = 1 - \sigma^i(\boldsymbol{\kappa})_1 - \sigma^i(\boldsymbol{\kappa})_2$;
- $P(t_0, t_0) = 1$.

It is easy to check that $t_n \models \psi[c, d]$.

5 SIMULATING MINSKY MACHINES WITH ONE COUNTER

For the rest of this section, we fix q , l_q , and $\boldsymbol{\kappa}$ in the same way as in Section 3, but we additionally require that $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 < q - \frac{1}{2}$. Furthermore, we fix a rational constant γ such that $(1-q)\boldsymbol{\kappa}_2 < \gamma < \frac{3}{4}q - \frac{5}{4}q + \frac{1}{2}q^2$. For example, we can put $q = \frac{13}{16}$, $\boldsymbol{\kappa} = (\frac{17}{64}, \frac{1}{32})$, and $\gamma = 0.06$. These additional assumptions are used in the proof of Theorem 5.1.

Let \mathcal{M} be a non-deterministic one-counter Minsky machine with m instructions, and let $Labels = \{\ell_i \mid 1 \leq i \leq m\}$ be a set of fresh atomic propositions. Let $M = (S, P, \nu)$ be a Markov chain. We say that a state $t \in S$ represents a configuration (i, n) of \mathcal{M} iff $t \models \ell_i$, $t \not\models \ell_j$ for all $j \neq i$, and $\nu[t] = \sigma^n(\boldsymbol{\kappa})$. Furthermore, we say that a state $s \in S$ simulates \mathcal{M} if s represents $(1, 0)$ and every state t reachable from s satisfies the following condition: If t represents a configuration C of \mathcal{M} , then at least one immediate successor of t represents a successor configuration of C . Furthermore, for every immediate successor t' of t that does not represent a successor of C we have that $t' \models \mathbf{G}_{=1} \bigwedge_{i=1}^m \neg \ell_i$.

Let $s \in S$ be a state simulating \mathcal{M} , and let s_0, s_1, \dots be a run such that $s_0 = s$ and every s_i represents a configuration C_i of \mathcal{M} . Then C_0, C_1, \dots is a computation of \mathcal{M} covered by s . Note that s covers at least one but not necessarily all computations of \mathcal{M} . In this section, we prove the following theorem:

THEOREM 5.1. *Let \mathcal{M} be a non-deterministic one-counter Minsky machine. Then there is an effectively constructible PCTL formula ψ satisfying the following conditions:*

- For every Markov chain M and every state s of M , we have that if $s \models \psi$, then s simulates \mathcal{M} .
- For every computation ω of \mathcal{M} , there exists a Markov chain M and a state s of M such that $s \models \psi$ and s covers ω . Furthermore, if ω is periodic, then M has finitely many states.

5.1 Constructing ψ

For the rest of this section, we fix a non-deterministic one-counter Minsky machine $\mathcal{M} \equiv 1 : Ins_1; \dots m : Ins_m$. We show how to construct the formula ψ of Theorem 5.1.

Recall that the counter values $0, 1, 2, \dots$ are represented by characteristic vectors $\boldsymbol{\kappa}$, $\sigma(\boldsymbol{\kappa})$, $\sigma^2(\boldsymbol{\kappa})$, \dots . Hence, decrementing and incrementing the counter corresponds to performing the functions τ and σ , respectively. Testing the counter for zero is no issue because both components of $\boldsymbol{\kappa}$ are rational and can be directly used as probability constraints. The function τ is implemented in the same way as in the parameterized formula $\psi(x, y)$ constructed in Section 4. Consequently, some subformulae of $\psi(x, y)$ are re-used in ψ , possibly after small adjustments. The main challenge is to implement the function σ and orchestrate everything so that Theorem 3.8 is still applicable.

The set of propositions used in ψ is $\mathcal{A} = A \cup Labels \cup \{d, e\}$ where A is the set of propositions occurring in $\psi(x, y)$ (see Section 4), and d, e are fresh propositions used for implementing the function σ . For every $B \subseteq \mathcal{A}$, we use $\langle B \rangle$ to denote the formula $\bigwedge_{p \in B} p \wedge \bigwedge_{p \in \mathcal{A} \setminus B} \neg p$. The intuitive meaning of $\langle B \rangle$ is the same as of $\langle B \rangle$ defined in Section 4.1. The only difference is that the set A is replaced with the richer set \mathcal{A} .

The structure of ψ closely resembles the structure of $\psi(x, y)$, and the overall intuition behind the subformulae stays essentially the same. We use the same identifiers for denoting adjusted versions of subformulae defined in Section 4.1. If a subformula does not require any adjustment *except for replacing every occurrence of $\langle \cdot \rangle$ with $\langle \cdot \rangle$* , then we re-use this formula and write its identifier in **boldface**. For example, \mathbf{FSuc}_i denotes the formula

$$\mathbf{X}_{=1}(\langle h, a, S(r_i) \rangle \vee \langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle).$$

We put

$$\psi \equiv \mathbf{Init} \wedge \mathbf{G}_{=1} \mathbf{Invariant}$$

where

$$\mathbf{Init} \equiv \langle a, r_0, \ell_1 \rangle \wedge \mathbf{Zero}$$

$$\mathbf{Invariant} \equiv \mathbf{Fin} \vee \mathbf{Transient} \vee \mathbf{Free}$$

Note that if $s \models \psi$, then every state t reachable from s can again be classified as either *final*, *transient*, or *free*, depending on whether t satisfies **Fin**, **Transient**, or **Free**, respectively.

Furthermore, we put

$$\mathbf{Transient} \equiv \mathbf{Trans} \vee \mathbf{CTrans} \vee \mathbf{LTrans}.$$

As we shall see, every transient state t satisfies precisely one of the formulae $\langle a, r_i \rangle$, $c \wedge r_i \wedge \neg h$, or $\langle x, r_i, \ell \rangle$, where $i \in \{0, \dots, 4\}$, $\ell \in Labels$, and $x \in \{a, b\}$. The formulae **Trans**, **CTrans**, and **LTrans** define the properties of t in the three respective cases. We define

$$\mathbf{CTrans} \equiv \bigvee_{i \in \{0, \dots, 4\}} c \wedge r_i \wedge \neg h \wedge \mathbf{CSuc}_i \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i$$

where

$$\mathbf{CSuc}_i \equiv \mathbf{X}_{=1}(\langle a, S(r_i) \rangle \vee \langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i), d \rangle)$$

Furthermore, we define

$$\mathbf{LTrans} \equiv \bigvee_{i \in \{0, \dots, 4\}} \bigvee_{\ell \in Labels} \bigvee_{x \in \{a, b\}} (\langle x, r_i, \ell \rangle \wedge \mathbf{Step}_{i, \ell})$$

where the formula $\mathbf{Step}_{i, \ell}$ is constructed as follows. Let Ins_j be the instruction associated with ℓ , i.e., $\ell = \ell_j$. We distinguish two cases.

- If $Ins_j \equiv \text{if } c=0 \text{ then goto } L \text{ else dec } c; \text{ goto } L'$, then

$$\mathbf{Step}_{i, \ell} \equiv (\mathbf{Zero} \Rightarrow (\mathbf{ZSuc}_{i, \ell} \wedge \mathbf{X}_{=1}(b \Rightarrow \mathbf{Zero}))) \wedge (\neg \mathbf{Zero} \Rightarrow (\mathbf{PSuc}_{i, \ell} \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i))$$

where

$$\begin{aligned} \mathbf{ZSuc}_{i, \ell} &\equiv \bigvee_{\ell' \in L} \mathbf{X}_{=1}(\langle h, a, S(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i), e \rangle \\ &\quad \vee \langle b, S^2(r_i), \ell' \rangle) \wedge \mathbf{X}_{=1-q} \langle h, c, S^2(r_i), e \rangle \\ \mathbf{PSuc}_{i, \ell} &\equiv \bigvee_{\ell' \in L'} \mathbf{X}_{=1}(\langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i), e \rangle \\ &\quad \vee \langle a, S(r_i), \ell' \rangle) \wedge \mathbf{X}_{=1-q} \langle h, c, S^2(r_i), e \rangle \end{aligned}$$

(B) If $Ins_j \equiv inc\ c; \text{ goto } L$, then

$$\begin{aligned} Step_{i,\ell} &\equiv (\mathbf{Zero} \Rightarrow IZSuc_{i,\ell}) \\ &\wedge (\neg \mathbf{Zero} \Rightarrow (IPSuc_{i,\ell} \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i)) \\ &\wedge \mathbf{F}_{=1-q}^2(a \wedge S^3(r_i)) \\ &\wedge \mathbf{F}_{=y}^2((b \wedge S^4(r_i)) \vee d) \\ &\wedge \mathbf{F}_{=y}^2((c \wedge S^4(r_i) \wedge e) \vee d) \end{aligned}$$

where γ is the constant fixed at the beginning of Section 5 and

$$\begin{aligned} IZSuc_{i,\ell} &\equiv \bigvee_{\ell' \in L} \mathbf{X}_{=1} (\langle \langle h, a, S(r_i) \rangle \vee \langle c, S^2(r_i) \rangle \vee \langle c, S^2(r_i), e \rangle \\ &\quad \vee \langle b, S^2(r_i), \ell' \rangle \rangle \wedge \mathbf{X}_{=1-q} \langle c, S^2(r_i), e \rangle) \\ IPSuc_{i,\ell} &\equiv \bigvee_{\ell' \in L} \mathbf{X}_{=1} (\langle \langle a, S(r_i) \rangle \vee \langle c, S^2(r_i) \rangle \vee \langle c, S^2(r_i), e \rangle \\ &\quad \vee \langle b, S^2(r_i), \ell' \rangle \rangle \wedge \mathbf{X}_{=1-q} \langle c, S^2(r_i), e \rangle) \end{aligned}$$

This completes the construction of ψ .

Intuitively, if $t \models \langle x, r_i, \ell_j \rangle$ and $\mathbf{v}[t]$ encodes the current counter value, then $Step_{i,\ell_j}$ enforces the simulation of Ins_j . More concretely,

- if $Ins_j \equiv if\ c=0\ \text{ then goto } L\ \text{ else dec } c; \text{ goto } L'$, then
 - if $\mathbf{v}[t] = \kappa$ encodes zero, then each $t' \in B_t$ satisfies some $\ell' \in L$ and $\mathbf{v}[t'] = \kappa$. The simulation continues in the states of B_t .
 - if $\mathbf{v}[t]$ encodes a positive value, then each $t' \in A_t$ satisfies some $\ell' \in L'$ and $\mathbf{v}[t'] = \tau(\mathbf{v}[t])$. The counter is decremented, and the simulation continues in the states of A_t .
- If $Ins_j \equiv inc\ c; \text{ goto } L$, then each $t' \in B_t$ satisfies some $\ell' \in L$. Furthermore, $\mathbf{v}[t'] = \sigma(\mathbf{v}[t])$ for all $t' \in B_t \cup C_t$ (this part is tricky). The counter is incremented and the simulation continues in the states of B_t .

Furthermore, all transient states t where $\mathbf{v}[t] \neq \kappa$ satisfy the formula $\mathbf{Interval} \wedge \mathbf{Eq}_i$ (for an appropriate i) so that the conditions of Theorem 3.8 are fulfilled for the set of all transient and final states.

5.2 A Proof of Theorem 5.1

Let $\mathcal{M} \equiv 1 : Ins_1; \dots m : Ins_m$ be a non-deterministic one-counter Minsky machine, and let ψ be the formula constructed for \mathcal{M} in Section 5.1. The two claims of Theorem 5.1 are proven separately in the following subsections.

5.2.1 A Proof of Theorem 5.1 (A). Let $M = (S, P, v)$ be Markov chain and $s \in S$ such that $s \models \psi$. Let T be the set of all transient states and all final states of M . One can easily verify that the conditions of Theorem 3.8 are satisfied for T by inspecting the structure of transient states similarly as in Section 4.2 (in particular, the formula \mathbf{Eq}_i still implies Equalities (1) and (2)). Thus, by applying Theorem 3.8, we obtain

- I. For every $t \in T$, we have that $\mathbf{v}[t] \in \mathbf{Area}(\kappa)$.
- II. For every $t \in T$ where $\mathbf{v}[t] \neq \kappa$ and every $t' \in A_t$, we have that $\mathbf{v}[t'] = \tau(\mathbf{v}[t])$.

We show that s simulates \mathcal{M} . Since $s \models \mathbf{Init}$, we have that s represents the configuration (1, 0). Let t be a state reachable from s such that t represents a configuration (j, n) of \mathcal{M} , i.e., $t \models \ell_j$ and $\mathbf{v}[t] = \sigma^n(\kappa)$. Then $t \models LTrans$ and hence also $t \models Step_{i,\ell_j}$. We distinguish two cases.

(A) $Ins_j \equiv if\ c=0\ \text{ then goto } L\ \text{ else dec } c; \text{ goto } L'$. Then,

$$\begin{aligned} Step_{i,\ell} &\equiv (\mathbf{Zero} \Rightarrow (ZSuc_{i,\ell} \wedge \mathbf{X}_{=1}(b \Rightarrow \mathbf{Zero}))) \\ &\wedge (\neg \mathbf{Zero} \Rightarrow (PSuc_{i,\ell} \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i)) \end{aligned}$$

and there are two subcases.

- $n = 0$. Since $\mathbf{v}[t] = \sigma^n(\kappa)$, we obtain $t \models \mathbf{Zero}$. Hence, $t \models ZSuc_{i,\ell} \wedge \mathbf{X}_{=1}(b \Rightarrow \mathbf{Zero})$. This implies that every $t' \in B_t$ satisfies $t' \models \langle \langle b, S^2(r_i), \ell' \rangle \rangle \wedge \mathbf{Zero}$ for some $\ell' \in L$. Hence, t' represents a successor configuration of $(j, 0)$. Since $\mathbf{v}[t]_2 = \kappa_2 > 0$, we have $B_t \neq \emptyset$, and hence at least one such t' exists. Furthermore, all states of $A_t \cup C_t$ are free and hence they satisfy the formula $\mathbf{G}_{=1} \wedge_{i=1}^m \neg \ell_i$.
- $n > 0$. Since $\mathbf{v}[t] = \sigma^n(\kappa)$, we obtain $t \models \neg \mathbf{Zero}$ and hence $t \models PSuc_{i,\ell} \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i$. The formula $PSuc_{i,\ell}$ ensures that every $t' \in A_t$ satisfies $t' \models \langle \langle a, S(r_i), \ell' \rangle \rangle$ for some $\ell' \in L'$. Furthermore, we have that $\mathbf{v}[t'] = \tau(\sigma^n(\kappa)) = \sigma^{n-1}(\kappa)$ by Observation II. above. Hence, t' represents a successor configuration of (j, n) . Since $\mathbf{v}[t]_1 = \sigma^n(\kappa)_1 > 0$, at least one such t' must exist. Note that all states of $B_t \cup C_t$ satisfy the formula $\mathbf{G}_{=1} \wedge_{i=1}^m \neg \ell_i$.

(B) $Ins_j \equiv inc\ c; \text{ goto } L$. Then,

$$\begin{aligned} Step_{i,\ell} &\equiv (\mathbf{Zero} \Rightarrow IZSuc_{i,\ell}) \\ &\wedge (\neg \mathbf{Zero} \Rightarrow (IPSuc_{i,\ell} \wedge \mathbf{Interval} \wedge \mathbf{Eq}_i)) \\ &\wedge \mathbf{F}_{=1-q}^2(a \wedge S^3(r_i)) \\ &\wedge \mathbf{F}_{=y}^2((b \wedge S^4(r_i)) \vee d) \\ &\wedge \mathbf{F}_{=y}^2((c \wedge S^4(r_i) \wedge e) \vee d) \end{aligned}$$

The structure of immediate successors of t enforced by this formula is shown in Fig. 4. The formulae $IZSuc_{i,\ell}$ and $IPSuc_{i,\ell}$ ensure that for every $t' \in B_t$, we have that $t' \models \langle \langle b, S^2(r_i), \ell' \rangle \rangle$ where $\ell' \in L$. Observe that $B_t \neq \emptyset$ and all states of $A_t \cup C_t$ satisfy the formula $\mathbf{G}_{=1} \wedge_{i=1}^m \neg \ell_i$. Hence, it remains to show that $\mathbf{v}[t'] = \sigma(\mathbf{v}[t])$ for all $t' \in B_t$. We prove a stronger result saying that $\mathbf{v}[t'] = \sigma(\mathbf{v}[t])$ for all $t' \in B_t \cup C_t$.

Due to Observation I above and Lemma 3.5.A, it suffices to prove that $\sigma(\mathbf{v}[t])$ is a positive convex combination of the vectors in $\{\mathbf{v}[t'] \mid t' \in B_t \cup C_t\}$. For a path formula Φ , we use $R[\Phi]$ to denote the set of all $w \in Run(t)$ such that $w \models \Phi$.

Since $t \models \mathbf{F}_{=1-q}^2(a \wedge S^3(r_i))$, we have $\mathbb{P}_t(R[\mathbf{F}^2(a \wedge S^3(r_i))]) = 1-q$. By inspecting the structure of immediate successors of t (see Fig. 4), we obtain

$$1-q = \mathbb{P}_t(R[\mathbf{F}^2(a \wedge S^3(r_i))]) = \sum_{t' \in B_t \cup C_t} P(t, t') \cdot \mathbf{v}[t']_1. \quad (7)$$

Observe that $\sum_{t' \in B_t \cup C_t} P(t, t') = 1 - \mathbf{v}[t]_1$. Hence, (7) implies

$$\frac{1-q}{1 - \mathbf{v}[t]_1} = \sum_{t' \in B_t \cup C_t} \frac{P(t, t')}{1 - \mathbf{v}[t]_1} \cdot \mathbf{v}[t']_1. \quad (8)$$

Note that the left-hand side of (8) is equal to $\sigma(\mathbf{v}[t])_1$.

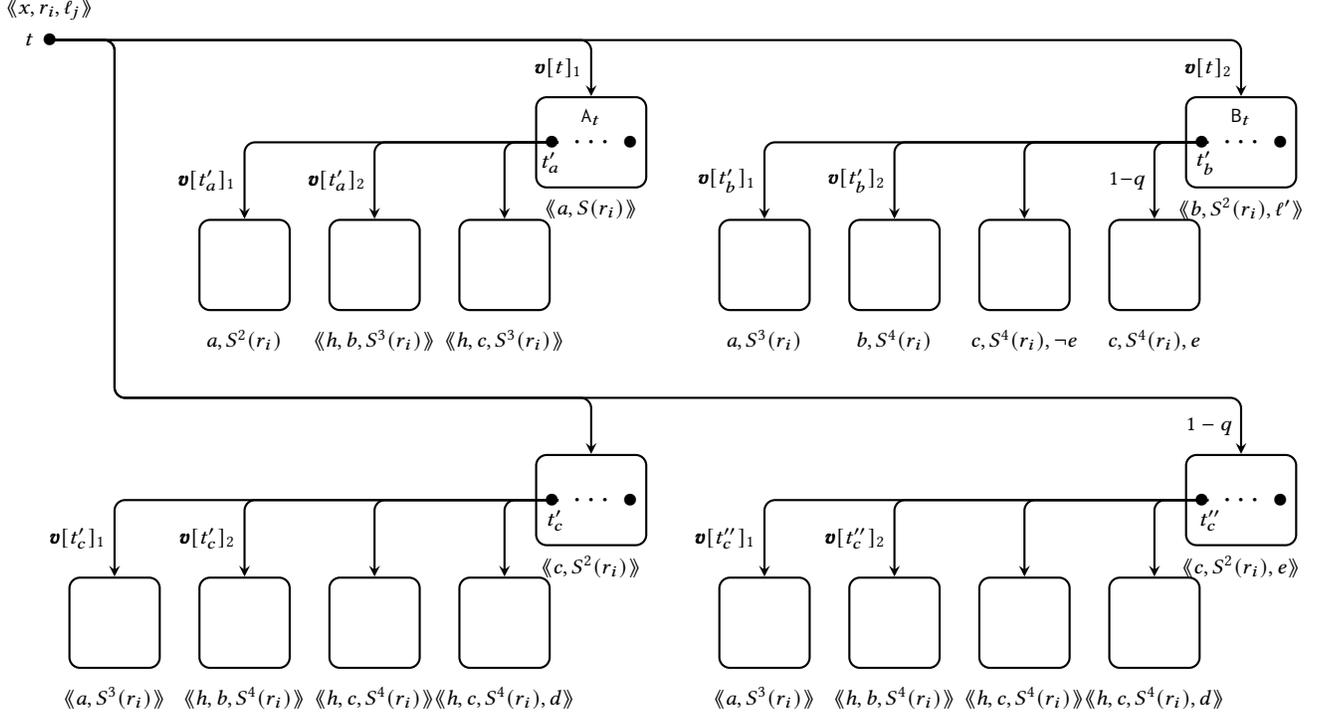


Figure 4: The structure of transient states satisfying $\langle x, r_i, \ell_j \rangle$ where $x \in \{a, b\}$ and $Ins_j \equiv inc\ c; goto\ u$

Furthermore, $t \models \mathbf{F}_{=y}^2((b \wedge S^4(r_i)) \vee d)$. By inspecting the structure of immediate successors of t (see Fig. 4), we obtain⁵

$$\begin{aligned}
 \gamma &= \mathbb{P}_t(R[\mathbf{F}^2((b \wedge S^4(r_i)) \vee d)]) \\
 &= \mathbb{P}_t(R[\mathbf{F}^2(b \wedge S^4(r_i))] \uplus R[\mathbf{F}^2(d)]) \\
 &= \mathbb{P}_t(R[\mathbf{F}^2(b \wedge S^4(r_i))]) + \mathbb{P}_t(R[\mathbf{F}^2(d)]) \\
 &= \sum_{t' \in B_t \cup C_t} P(t, t') \cdot \mathbf{v}[t']_2 + \mathbb{P}_t(R[\mathbf{F}^2(d)]) \quad (9)
 \end{aligned}$$

Similarly, $t \models \mathbf{F}_{=y}^2((c \wedge S^4(r_i) \wedge e) \vee d)$ implies

$$\begin{aligned}
 \gamma &= \mathbb{P}_t(R[\mathbf{F}^2((c \wedge S^4(r_i) \wedge e) \vee d)]) \\
 &= \mathbb{P}_t(R[\mathbf{F}^2(c \wedge S^4(r_i) \wedge e)] \uplus R[\mathbf{F}^2(d)]) \\
 &= \mathbb{P}_t(R[\mathbf{F}^2(c \wedge S^4(r_i) \wedge e)]) + \mathbb{P}_t(R[\mathbf{F}^2(d)]) \\
 &= \sum_{t' \in B_t} P(t, t') \cdot (1-q) + \mathbb{P}_t(R[\mathbf{F}^2(d)]) \\
 &= \mathbf{v}[t]_2 \cdot (1-q) + \mathbb{P}_t(R[\mathbf{F}^2(d)]) \quad (10)
 \end{aligned}$$

Since the right-hand sides of (9) and (10) are equal, we have that

$$\mathbf{v}[t]_2 \cdot (1-q) = \sum_{t' \in B_t \cup C_t} P(t, t') \cdot \mathbf{v}[t']_2$$

Hence,

$$\frac{\mathbf{v}[t]_2 \cdot (1-q)}{1 - \mathbf{v}[t]_1} = \sum_{t' \in B_t \cup C_t} \frac{P(t, t')}{1 - \mathbf{v}[t]_1} \cdot \mathbf{v}[t']_2 \quad (11)$$

⁵Here, by writing $C = A \uplus B$ we mean that $C = A \cup B$ and $A \cap B = \emptyset$.

Observe that the left-hand side of (11) is equal to $\sigma(\mathbf{v}[t])_2$. By combining (8) and (11), we finally obtain

$$\sigma(\mathbf{v}[t]) = \sum_{t' \in B_t \cup C_t} \frac{P(t, t')}{1 - \mathbf{v}[t]_1} \cdot \mathbf{v}[t']_2.$$

5.2.2 *A Proof of Theorem 5.1 (B).* For the rest of this proof, we fix a computation $\omega = C_0, C_1, \dots$ of \mathcal{M} . Furthermore, if ω is periodic, we fix α, β such that $\alpha < \beta$ and the infinite sequences $C_{\alpha-1}, C_\alpha, C_{\alpha+1}, \dots$ and $C_{\beta-1}, C_\beta, C_{\beta+1}, \dots$ are the same. If ω is not periodic, we put $\beta = \infty$ and leave α undefined.

Observe that if ω is periodic, then our choice of α and β ensures that the infinite sequences $C_\alpha, C_{\alpha+1}, \dots$ and $C_\beta, C_{\beta+1}, \dots$ are also the same, and the computational steps $C_{\alpha-1} \mapsto C_\alpha$ and $C_{\beta-1} \mapsto C_\beta$ are generated by the same instruction of \mathcal{M} . As we shall see, this ensures that C_α and C_β are represented by the same state in the constructed model of ψ .

We show that there exist a Markov chain $M = (S, P, \nu)$ and a state $s \in S$ such that $s \models \psi$ and s covers ω . Furthermore, if $\beta < \infty$, then S is finite.

Every state of S is a triple of the form $[i, \mathcal{L}, n]$ where i is an index ranging over $\{i \in \mathbb{N} \mid 0 \leq i < \beta\} \cup \{\star\}$, $\mathcal{L} \subseteq \mathcal{A}$ is the set of atomic propositions satisfied in the state, and $n \in \mathbb{N} \cup \{\star\}$ is a counter value. The \star symbol is used when the index or the counter value (or both) are not relevant.

The Markov chain M is the least Markov chain M' such that $s \equiv [0, \{a, r_0, \ell_1\}, 0]$ is a state of M' , and if t is a state of M' then M' contains all immediate successors of t defined by the following rules:

Rule I. If $t = [k, \{x, r_i, \ell_j\}, n]$ where $x \in \{a, b\}$, $C_k = (j, n)$, and $k < \beta$, then the immediate successors of t are determined as follows. First, let k' be either α or $k+1$ depending on whether $k = \beta-1$ or $k < \beta-1$, respectively. Furthermore, let $C_{k'} = (j', n')$, and let t' be either $[k', \{a, S(r_i), \ell_{j'}\}, n']$ or $[k', \{b, S^2(r_i), \ell_{j'}\}, n']$, depending on whether $n' = n-1$ or $n' \geq n$, respectively.

Now we distinguish four possibilities (the cases when Ins_j is a Type II and Type I instruction are covered by (A)–(B) and (C)–(D), respectively. In each case, we distinguish between zero and positive counter values represented by the n).

(A) $n' = n = 0$. Then,

$$\begin{aligned} P(t, [\star, \{h, a, S(r_i)\}, \star]) &= \kappa_1, \\ P(t, t') &= \kappa_2, \\ P(t, [\star, \{h, c, S^2(r_i), e\}, \star]) &= 1-q, \\ P(t, [\star, \{h, c, S^2(r_i)\}, \star]) &= q - \kappa_1 - \kappa_2. \end{aligned}$$

(B) $n' = n - 1$. We put

$$\begin{aligned} P(t, t') &= \sigma^n(\kappa)_1, \\ P(t, [\star, \{h, b, S^2(r_i)\}, \star]) &= \sigma^n(\kappa)_2, \\ P(t, [\star, \{h, c, S^2(r_i), e\}, \star]) &= 1-q, \\ P(t, [\star, \{h, c, S^2(r_i)\}, \star]) &= q - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2. \end{aligned}$$

(C) $n' = 1$ and $n = 0$. Then,

$$\begin{aligned} P(t, [\star, \{h, a, S(r_i)\}, \star]) &= \kappa_1, \\ P(t, t') &= \kappa_2, \\ P(t, [\star, \{c, S^2(r_i), e\}, 1]) &= 1-q, \\ P(t, [\star, \{c, S^2(r_i)\}, 1]) &= q - \kappa_1 - \kappa_2 \end{aligned}$$

(D) $n' = n + 1$ and $n > 0$. Then,

$$\begin{aligned} P(t, [\star, \{a, S(r_i)\}, n-1]) &= \sigma^n(\kappa)_1, \\ P(t, t') &= \sigma^n(\kappa)_2, \\ P(t, [\star, \{c, S^2(r_i), e\}, n+1]) &= 1-q, \\ P(t, [\star, \{c, S^2(r_i)\}, n+1]) &= q - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2 \end{aligned}$$

Rule II. If $t = [\star, \{a, r_i\}, n]$ where $n \geq 0$, we have the following:

$$\begin{aligned} \text{if } n = 0, \text{ then } P(t, [\star, \{h, a, S(r_i)\}, \star]) &= \sigma^n(\kappa)_1 = \kappa_1, \\ \text{if } n > 0, \text{ then } P(t, [\star, \{a, S(r_i)\}, n-1]) &= \sigma^n(\kappa)_1, \\ P(t, [\star, \{h, b, S^2(r_i)\}, \star]) &= \sigma^n(\kappa)_2, \\ P(t, [\star, \{h, c, S^2(r_i)\}, \star]) &= 1 - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2 \end{aligned}$$

Rule III. If $t = [\star, \mathcal{L}, n]$ where $n > 0$ and \mathcal{L} is $\{c, r_i\}$ or $\{c, r_i, e\}$, then

$$\begin{aligned} P(t, [\star, \{a, S(r_i)\}, n-1]) &= \sigma^n(\kappa)_1, \\ P(t, [\star, \{h, b, S^2(r_i)\}, \star]) &= \sigma^n(\kappa)_2, \\ P(t, [\star, \{h, c, S^2(r_i), d\}, \star]) &= p_n, \\ P(t, [\star, \{h, c, S^2(r_i)\}, \star]) &= 1 - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2 - p_n \end{aligned}$$

where

$$p_n = \frac{\gamma - (1-q)\sigma^{n-1}(\kappa)_2}{1 - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2}$$

Note that $0 < p_n < 1 - \sigma^n(\kappa)_1 - \sigma^n(\kappa)_2$ due to the constraints imposed on γ and κ at the beginning of Section 5. More precisely, the constraint $\gamma > (1-q)\kappa_2$ ensures that $p_n > 0$, and the constraint $\gamma < \frac{3}{4}q - \frac{5}{4}q + \frac{1}{2}q^2$ ensures that $2p_n < 1 - (2(q-\frac{1}{2}) + (1-q))$

(recall that $\kappa_1 + \kappa_2 < q - \frac{1}{2}$ and $\sigma^m(\kappa)$ is strictly less than κ in both components for every $m \geq 0$). This is more than we need here; the constraints are chosen so that they also satisfy stronger requirements needed in the proof of Theorem 6.2.

Rule IV. If $t = [\star, \mathcal{L}, \star]$, then $P(t, t) = 1$.

Note that if ω is periodic, then M has finitely many states. It is easy to check that $s \models \psi$ by verifying that every state of M satisfies either **Fin**, **Transient**, or **Free**. In particular, every state of the form $[t, \{b, r_i, \ell_j\}, n]$ where Ins_j is a Type I instruction satisfies $\mathbf{F}_{=y}^2((b \wedge S^4(r_i)) \vee d)$ and $\mathbf{F}_{=y}^2((c \wedge S^4(r_i) \wedge e) \vee d)$. This is where we need the constant p_n of Rule III. Clearly, s covers the computation ω (and no other computation).

6 SIMULATING MINSKY MACHINES WITH TWO COUNTERS

In this section, we show how to simulate non-deterministic two-counter Minsky machines by PCTL formulae. Technically, we construct a PCTL formula simulating a synchronized products of two non-deterministic one-counter Minsky machines defined in the next paragraph.

Let

$$\begin{aligned} \mathcal{M}_1 &\equiv 1 : Ins_1^1; \dots m : Ins_m^1; \\ \mathcal{M}_2 &\equiv 1 : Ins_1^2; \dots m : Ins_m^2; \end{aligned}$$

be nondeterministic one-counter Minsky machines with m instructions and $I = (I_1, I_2)$ a partition of $\{1, \dots, m\}$, i.e., $I_1 \cup I_2 = \{1, \dots, m\}$ and $I_1 \cap I_2 = \emptyset$. An I -synchronized product of $\mathcal{M}_1, \mathcal{M}_2$, denoted by $\mathcal{M}_1 \times_I \mathcal{M}_2$, is an automaton⁶ operating over two counters in the following way.

A configuration of $\mathcal{M}_1 \times_I \mathcal{M}_2$ is a triple (j, n_1, n_2) where $j \in \{1, \dots, m\}$ and $n_1, n_2 \in \mathbb{N}$ are counter values. For every configuration (j, n_1, n_2) of $\mathcal{M}_1 \times_I \mathcal{M}_2$, the successor configurations (j', n'_1, n'_2) are determined as follows (recall that \mapsto denotes the “standard” computational step of a Minsky machine, see Section 2.3):

- If $j \in I_1$, then j', n'_1, n'_2 are integers satisfying $(j, n_1) \mapsto (j', n'_1)$ in \mathcal{M}_1 and $(j, n_2) \mapsto (j'', n'_2)$ in \mathcal{M}_2 (for some $j'' \in \{1, \dots, m\}$).
- If $j \in I_2$, then j', n'_1, n'_2 are integers satisfying $(j, n_1) \mapsto (j'', n'_1)$ in \mathcal{M}_1 and $(j, n_2) \mapsto (j', n'_2)$ in \mathcal{M}_2 (for some $j'' \in \{1, \dots, m\}$).

In other words, $\mathcal{M}_1 \times_I \mathcal{M}_2$ simultaneously executes the instructions Ins_j^1 and Ins_j^2 operating on the first and the second counter, and the next j' is determined by either Ins_j^1 or Ins_j^2 , depending on whether $j \in I_1$ or $j \in I_2$, respectively. Note that the counter values n'_1 and n'_2 are the same in every successor configuration of (j, n_1, n_2) .

We write $(j, n_1, n_2) \rightsquigarrow (j', n'_1, n'_2)$ when (j', n'_1, n'_2) is a successor of (j, n_1, n_2) . A computation of $\mathcal{M}_1 \times_I \mathcal{M}_2$ is an infinite sequence of configurations $\omega \equiv D_0, D_1, \dots$ such that $D_i \rightsquigarrow D_{i+1}$ for all $i \in \mathbb{N}$. The boundedness and the recurrent reachability

⁶ $\mathcal{M}_1 \times_I \mathcal{M}_2$ is a non-standard computational model introduced specifically for purposes of this paper. Encoding the computation of $\mathcal{M}_1 \times_I \mathcal{M}_2$ by a PCTL formula is substantially easier than encoding the computation of a two-counter Minsky machine.

problems for $\mathcal{M}_1 \times_I \mathcal{M}_2$ are defined in the same way as for non-deterministic Minsky machines (see Section 2.3).

A synchronized product of two one-counter Minsky machines can faithfully simulate a two-counter Minsky machine. Thus, we obtain the following:

PROPOSITION 6.1. *The boundedness problem for a synchronized product of two deterministic one-counter Minsky machines is Σ_1^0 -hard. The recurrent reachability problem for a synchronized product of two non-deterministic one-counter Minsky machines is Σ_1^1 -hard.*

Let $\mathcal{M}_1 \times_I \mathcal{M}_2$ be a synchronized product of two non-deterministic one-counter Minsky machines with m instructions. Recall the set \mathcal{A} defined in Section 5.1. For $k \in \{1, 2\}$, let $\mathcal{A}^k = \{p^k \mid p \in \mathcal{A}\}$ be a set of atomic propositions such that $\mathcal{A}^1 \cap \mathcal{A}^2 = \emptyset$.

Let $M = (S, P, v)$ be a Markov chain. For all $t \in S$ and $k \in \{1, 2\}$, we use $\mathbf{v}^k[t] \in [0, 1]^2$ to denote the k -th characteristic vector of t where

- $\mathbf{v}^k[t]_1$ is the probability of satisfying the path formula $\mathbf{X} a^k$ in the state t ;
- $\mathbf{v}^k[t]_2$ is the probability of satisfying the path formula $\mathbf{X} b^k$ in the state t .

A state $t \in S$ represents a configuration (i, n_1, n_2) of $\mathcal{M}_1 \times_I \mathcal{M}_2$ iff $t \models \ell_i^1 \wedge \ell_i^2$, $t \not\models \ell_j^1 \wedge \ell_j^2$ for all $j \neq i$, $\mathbf{v}^1[t] = \sigma^{n_1}(\boldsymbol{\kappa})$, and $\mathbf{v}^2[t] = \sigma^{n_2}(\boldsymbol{\kappa})$. Furthermore, we say that a state $s \in S$ simulates $\mathcal{M}_1 \times_I \mathcal{M}_2$ if s represents $(1, 0, 0)$ and every state t reachable from s satisfies the following condition: If t represents a configuration D of $\mathcal{M}_1 \times_I \mathcal{M}_2$, then at least one immediate successor of t represents a successor configuration of D . Furthermore, for every immediate successor t' of t that does not represent a successor of D we have that $t' \models \mathbf{G}_{=1} \bigwedge_{i=1}^m \neg(\ell_i^1 \wedge \ell_i^2)$.

Let $s \in S$ be a state simulating $\mathcal{M}_1 \times_I \mathcal{M}_2$, and let s_0, s_1, \dots be a run of M such that $s_0 = s$ and every s_i represents a configuration D_i of $\mathcal{M}_1 \times_I \mathcal{M}_2$. Then D_0, D_1, \dots is a computation of $\mathcal{M}_1 \times_I \mathcal{M}_2$ covered by s .

Now, we formulate the main technical result of this paper.

THEOREM 6.2. *Let $\mathcal{M}_1 \times_I \mathcal{M}_2$ be a synchronized product of two non-deterministic one-counter Minsky machines. Then there is an effectively constructible PCTL formula Ψ satisfying the following conditions:*

- For every Markov chain M and every state s of M , we have that if $s \models \Psi$, then s simulates $\mathcal{M}_1 \times_I \mathcal{M}_2$.
- For every computation ω of $\mathcal{M}_1 \times_I \mathcal{M}_2$, there exists a Markov chain M and a state s of M such that $s \models \Psi$ and s covers ω . Furthermore, if ω is periodic, then M has finitely many states.

Observe the following:

- Let $\mathcal{M}_1 \times_I \mathcal{M}_2$ be a synchronized product of two deterministic one-counter Minsky machines. Then, $\mathcal{M}_1 \times_I \mathcal{M}_2$ is bounded iff the only computation ω of $\mathcal{M}_1 \times_I \mathcal{M}_2$ is periodic. By Theorem 6.2, we obtain that $\mathcal{M}_1 \times_I \mathcal{M}_2$ is bounded iff Ψ is finite-satisfiable.
- Let $\mathcal{M}_1 \times_I \mathcal{M}_2$ be a synchronized product of two non-deterministic one-counter Minsky machines. Then $\mathcal{M}_1 \times_I \mathcal{M}_2$ has a recurrent computation iff the formula

$$\Psi \wedge \mathbf{G}_{=1} \left((\ell_1^1 \wedge \ell_1^2) \Rightarrow \mathbf{F}_{>0}(\ell_1^1 \wedge \ell_1^2) \right)$$

is (generally) satisfiable.

Thus, we obtain the following corollary to Theorem 6.2:

COROLLARY 6.3. *The finite satisfiability problem for PCTL is Σ_1^0 -hard, and the general satisfiability problem for PCTL is Σ_1^1 -hard.*

6.1 Constructing Ψ

For the rest of this section, we fix the following non-deterministic one-counter Minsky machines:

$$\begin{aligned} \mathcal{M}_1 &\equiv 1 : \text{Ins}_1^1; \dots m : \text{Ins}_m^1; \\ \mathcal{M}_2 &\equiv 1 : \text{Ins}_1^2; \dots m : \text{Ins}_m^2; \end{aligned}$$

Furthermore, we fix a partition $I = (I_1, I_2)$ of $\{1, \dots, m\}$.

Roughly speaking, the formula Ψ is obtained by “merging” the formulae ψ_1 and ψ_2 of Section 5 constructed for \mathcal{M}_1 and \mathcal{M}_2 using the disjoint sets of atomic propositions \mathcal{A}^1 and \mathcal{A}^2 . The main modification is in the subformulae $LTrans$ of ψ_1 and ψ_2 , where we need to adjust the way of selecting the propositions of *Labels* passed to the successors so that the operational semantics of $\mathcal{M}_1 \times_I \mathcal{M}_2$ is reflected properly. When constructing ψ_1 and ψ_2 , we assume the constants $q, \boldsymbol{\kappa}$, and γ satisfying the same constraints as in Section 5 (these constants are used in both ψ_1 and ψ_2).

The set of atomic propositions used in Ψ is $\mathcal{A}^1 \cup \mathcal{A}^2$. For every formula *Form* constructed in Section 5 and $k \in \{1, 2\}$, we use \mathbf{Form}^k to denote the formula obtained from *Form* by replacing all propositions of \mathcal{A} with \mathcal{A}^k . For example, $t \models \langle \mathbf{a}, \mathbf{r}_i \rangle^1$ if t satisfies both a^1 and r_i^1 , and no other proposition of \mathcal{A}^1 . The formula does not say anything about the validity of the propositions of \mathcal{A}^2 in t .

When we need to define new formulae over \mathcal{A}^1 and \mathcal{A}^2 with the same structure up to the upper indexes of atomic propositions, we write a definition of \mathbf{Form}^k parameterized by the k . For example, by stipulating $\mathbf{Form}^k \equiv h^k \vee \langle \mathbf{a}, \mathbf{r}_i \rangle^k$, we simultaneously define $\mathbf{Form}^1 \equiv h^1 \vee \langle \mathbf{a}, \mathbf{r}_i \rangle^1$ and $\mathbf{Form}^2 \equiv h^2 \vee \langle \mathbf{a}, \mathbf{r}_i \rangle^2$. Furthermore, for $k \in \{1, 2\}$, the other element of $\{1, 2\}$ is denoted by k' , i.e., $\{k, k'\} = \{1, 2\}$.

We put

$$\Psi \equiv \mathbf{Init}^1 \wedge \mathbf{Init}^2 \wedge \mathbf{G}_{=1}(\mathbf{Invariant} \wedge \mathbf{LPass})$$

where

$$\mathbf{LPass} \equiv \left(\bigvee_{j=1}^m (\ell_j^1 \wedge \ell_j^2) \right) \Rightarrow \mathbf{X}_{>0} \left(\bigvee_{j=1}^m (\ell_j^1 \wedge \ell_j^2) \right)$$

$$\begin{aligned} \mathbf{Invariant} &\equiv (\mathbf{Fin}^1 \vee \mathbf{Transient}^1 \vee \mathbf{Free}^1) \\ &\wedge (\mathbf{Fin}^2 \vee \mathbf{Transient}^2 \vee \mathbf{Free}^2) \end{aligned}$$

Intuitively, if $t \models \ell_j^1 \wedge \ell_j^2$, then the instructions Ins_j^1 and Ins_j^2 are “simulated in parallel” in t . The formula \mathbf{LPass} enforces the existence of at least one immediate successor t' of t where the successor instructions $\text{Ins}_{j'}^1$ and $\text{Ins}_{j'}^2$ are again simulated jointly. For all $k \in \{1, 2\}$, we put

$$\mathbf{Transient}^k \equiv \mathbf{Trans}^k \vee \mathbf{CTrans}^k \vee \mathbf{LTrans}^k$$

where

$$\begin{aligned} LTrans^k &\equiv \left(\bigvee_{j=1}^m (\ell_j^k \wedge \ell_j^{k'}) \right) \Rightarrow Sim^k \\ &\wedge \left(\bigvee_{j=1}^m (\ell_j^k \wedge \neg \ell_j^{k'}) \right) \Rightarrow Abandon^k \end{aligned}$$

As we shall see, for every state t reachable from a state satisfying Ψ we have that $t \models \ell_j^k$ for at most one $\ell_j \in Labels$. Intuitively, the formula $LTrans^k$ says that when $t \models \ell_j^k \wedge \ell_j^{k'}$, we continue with simulating the instruction Ins_j^k . If t satisfies a proposition $\ell_j^k \in Labels^k$ but not the matching proposition $\ell_j^{k'} \in Labels^{k'}$, we “abandon” the simulation. In the latter case, the immediate successors of t must still have an appropriate structure so that no formula is “spoiled” in the immediate predecessor of t . This is enforced by the formula $Abandon^k$. More precisely, we put

$$Abandon^k \equiv (Zero^k \Rightarrow OZer^k) \wedge (\neg Zero^k \Rightarrow OPos^k).$$

The formulae $OZer^k$ and $OPos^k$ are defined as follows:

$$\begin{aligned} OZer^k &\equiv \bigvee_{i \in \{0, \dots, 4\}} (r_i^k \wedge OZSuc_i^k) \\ OPos^k &\equiv \bigvee_{i \in \{0, \dots, 4\}} (r_i^k \wedge OPSuc_i^k \wedge Interval^k \wedge Eq_i^k) \end{aligned}$$

where

$$\begin{aligned} OZSuc_i^k &\equiv \mathbf{x}_{=1} (\langle \mathbf{h}, \mathbf{a}, \mathbf{S}(\mathbf{r}_i) \rangle^k \vee \langle \mathbf{h}, \mathbf{b}, \mathbf{S}^2(\mathbf{r}_i) \rangle^k \vee \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i) \rangle^k \\ &\quad \vee \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i), \mathbf{e} \rangle^k) \wedge \mathbf{x}_{=1-q} \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i), \mathbf{e} \rangle^k \\ OPSuc_i^k &\equiv \mathbf{x}_{=1} (\langle \mathbf{a}, \mathbf{S}(\mathbf{r}_i) \rangle^k \vee \langle \mathbf{h}, \mathbf{b}, \mathbf{S}^2(\mathbf{r}_i) \rangle^k \vee \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i) \rangle^k \\ &\quad \vee \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i), \mathbf{e} \rangle^k) \wedge \mathbf{x}_{=1-q} \langle \mathbf{h}, \mathbf{c}, \mathbf{S}^2(\mathbf{r}_i), \mathbf{e} \rangle^k \end{aligned}$$

The formulae Sim^k , where $k \in \{1, 2\}$, enforce the simulation of one computational step of $\mathcal{M}_1 \times_I \mathcal{M}_2$. We put

$$Sim^k \equiv \bigvee_{i \in \{0, \dots, 4\}} \bigvee_{\ell \in Labels} \bigvee_{x \in \{a, b\}} (\langle \mathbf{x}, \mathbf{r}_i, \ell \rangle^k \wedge STEP_{i,\ell}^k)$$

where the formula $STEP_{i,\ell}^k$ is constructed as follows. Let Ins_j^k be the instruction associated with ℓ , i.e., $\ell = \ell_j$. If $j \in I_k$, then

$$STEP_{i,\ell}^k \equiv \mathbf{Step}_{i,\ell}^k$$

Now let $j \notin I_k$. For every $L \subseteq Labels$ where L has one or two elements, let $\mathbf{Step}_{i,\ell}^k[L]$ be the formula obtained from $\mathbf{Step}_{i,\ell}^k$ by substituting every occurrence of every set of target labels with $L^k = \{p^k \mid p \in L\}$. Note that this substitution affects only the “big disjunction” in the subformulae $\mathbf{ZSuc}_{i,\ell}^k$, $\mathbf{PSuc}_{i,\ell}^k$, $\mathbf{ISuc}_{i,\ell}^k$, and $\mathbf{IPSuc}_{i,\ell}^k$. Now, we distinguish two cases (recall that $k' \neq k$ is “the other index” of $\{1, 2\}$).

(A) $Ins_j^{k'} \equiv \text{if } c=0 \text{ then goto } L \text{ else dec } c; \text{ goto } L'$. Then,

$$STEP_{i,\ell}^k \equiv (\mathbf{Zero}^{k'} \Rightarrow \mathbf{Step}_{i,\ell}^k[L]) \wedge (\neg \mathbf{Zero}^{k'} \Rightarrow \mathbf{Step}_{i,\ell}^k[L'])$$

(B) $Ins_j^{k'} \equiv \text{inc } c; \text{ goto } L$. Then,

$$STEP_{i,\ell}^k \equiv \mathbf{Step}_{i,\ell}^k[L]$$

This completes the construction of Ψ .

6.2 A Proof of Theorem 6.2

Theorem 6.2 is proven by reusing the arguments used in the proof of Theorem 5.1 with some modifications and extensions. Let $\mathcal{M}_1 \times_I \mathcal{M}_2$ be a synchronized product of two non-deterministic one-counter Minsky machines with m instructions, and let Ψ be the formula constructed for $\mathcal{M}_1 \times_I \mathcal{M}_2$ in Section 6.1. The two claims of Theorem 6.2 are proven separately in the following subsections.

6.2.1 A Proof of Theorem 6.2 (A). Let $M = (S, P, v)$ be a Markov chain such that $s \models \Psi$ for some $s \in S$. For every $k \in \{1, 2\}$, let T^k be the set of all $t \in S$ reachable from s such that $t \not\models h^k$. It is easy to verify that the conditions of Theorem 3.8 are satisfied for both T^1 and T^2 , where $\mathbf{v}^1[t]$ and $\mathbf{v}^2[t]$ play the role of $\mathbf{v}[t]$, respectively.

We show that s simulates $\mathcal{M}_1 \times_I \mathcal{M}_2$. Clearly, s represents the configuration $(1, 0, 0)$ because $s \models \mathbf{Init}^1 \wedge \mathbf{Init}^2$. Let t be a state reachable from s such that t represents a configuration (j, n_1, n_2) . Let $n'_1, n'_2 \in \mathbb{N}$ be the (unique) counter values in a successor configuration of (j, n_1, n_2) , and let L be the set of all j' such that $(j, n_1, n_2) \rightsquigarrow (j', n'_1, n'_2)$.

Observe that $t \models Sim^1 \wedge Sim^2$ and hence $t \models STEP_{i,\ell_j}^1 \wedge STEP_{i',\ell_j}^2$ for some $i, i' \in \{0, \dots, 4\}$. Let t' be an immediate successor of t such that $t' \models \ell_j^1 \wedge \ell_j^2$ for some $j' \leq m$. Then $j' \in L$ by the definition of $STEP_{i,\ell}^k$. Furthermore, at least one such t' must exist because $t \models LPass$. By using the arguments of Section 5.2.1, we obtain that $\mathbf{v}^1[t'] = \sigma^{n'_1}(\boldsymbol{\kappa})$ and $\mathbf{v}^2[t'] = \sigma^{n'_2}(\boldsymbol{\kappa})$. Hence, t' represents a successor configuration of (j, n_1, n_2) . Also observe that if t'' is an immediate successor of t satisfying the formula $\bigwedge_{j=1}^m \neg(\ell_j^1 \wedge \ell_j^2)$, then all states reachable from t'' also satisfy this formula.

6.2.2 A Proof of Theorem 6.2 (B). Let $\omega = D_0, D_1, \dots$ be a computation of $\mathcal{M}_1 \times_I \mathcal{M}_2$. If ω is periodic, we fix α, β such that $\alpha < \beta$ and the computations $D_{\alpha-1}, D_\alpha, D_{\alpha+1}, \dots$ and $D_{\beta-1}, D_\beta, D_{\beta+1}, \dots$ are the same. If ω is not periodic, then $\beta = \infty$ and α is undefined.

For every configuration $D_i = (j, n_1, n_2)$ of ω , let $C_i^1 = (j, n_1)$ and $C_i^2 = (j, n_2)$ be the corresponding configurations of \mathcal{M}_1 and \mathcal{M}_2 . Furthermore, we define the infinite sequences $\omega^1 = C_0^1, C_1^1, C_2^1, \dots$ and $\omega^2 = C_0^2, C_1^2, C_2^2, \dots$. Note that ω_1 and ω_2 are *not* necessarily computations of \mathcal{M}_1 and \mathcal{M}_2 . However, for all $k \in \{1, 2\}$ and $i \in \mathbb{N}$, we have that if $C_i^k = (j, n)$ and $C_{i+1}^k = (j', n')$, then there is $j'' \leq m$ such that $(j, n) \mapsto (j'', n')$ is a computational step of \mathcal{M}_k . In other words, n' is obtained from n by executing $Ins_{j''}$ in \mathcal{M}_k .

We construct a Markov chain $M = (S, P, v)$ and a state $s \in S$ such that $s \models \Psi$ and s covers ω . If $\beta < \infty$, then S is finite.

The states of S are tuples of the form $t = [i, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$ where $i \in \{i \in \mathbb{N} \mid 0 \leq i < \beta\} \cup \{\star\}$, $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{A}$, and $n_1, n_2 \in \mathbb{N} \cup \{\star\}$. The set $v(t)$ of atomic propositions satisfied in t is $\{p^1 \mid p \in \mathcal{L}_1\} \cup \{p^2 \mid p \in \mathcal{L}_2\}$.

Each state $t = [i, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$ of S determines two projections $t_1 = [i, \mathcal{L}_1, n_1]$ and $t_2 = [i, \mathcal{L}_2, n_2]$. Conversely, for all $u_1 = [i, \mathcal{L}_1, n_1]$ and $u_2 = [i, \mathcal{L}_2, n_2]$ where $i, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2$ satisfy the conditions of the previous paragraph we define a tuple $u_1 \uplus u_2 = [i, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$.

The Markov chain M is the least Markov chain M' such that $s = [0, \{a, r_0, \ell_1\}, \{a, r_0, \ell_1\}, 0, 0]$ is a state of M' , and if t is a state of M' , then M' contains all immediate successors of t defined by the rules

given below. The rules are designed so that for every $t \in S$, the immediate successors of the projections t_1 and t_2 are defined by the rules of Section 5.2.2 where \mathcal{M}_1 and \mathcal{M}_2 are used as the underlying one-counter Minsky machines, the infinite sequences ω_1 and ω_2 play the role of the fixed computation ω , and the constants α and β refer to the constants fixed above. Recall that ω is used only in Rule I of Section 5.2.2, and this rule makes a clear sense also for ω^1 and ω^2 . Also recall that the set $\text{succ}(t_k)$ of all immediate successors of t_k (where $k \in \{1, 2\}$) satisfies precisely one of the following conditions:

- $\text{succ}(t_k) = \{t_k\}$. This happens iff $t_k \models h$.
- $\text{succ}(t_k)$ contains precisely three states satisfying the propositions a , b , and c , respectively, such that the last state does not satisfy $d \vee e$. These states are denoted by ta_k , tb_k , and tc_k , respectively.
- $\text{succ}(t_k)$ has precisely four states; apart of ta_k , tb_k , and tc_k , there is also the fourth state satisfying d or e . This fourth state is denoted by tx_k .

Now we can define the closure rules.

Rule A. If $t = [k, \{x, r_i, \ell_j\}, \{x', r_{i'}, \ell_{j'}\}, n_1, n_2]$ where $x, x' \in \{a, b\}$, $i, i' \in \{0, \dots, 4\}$, then the immediate successors of t are defined as follows. Let $t'_1 \in \text{succ}(t_1)$ and $t'_2 \in \text{succ}(t_2)$ be the unique states satisfying some proposition of *Labels*. Observe that $t'_1 = [k', \mathcal{L}_1, n_1]$ and $t'_2 = [k', \mathcal{L}_2, n_2]$ where \mathcal{L}_1 and \mathcal{L}_2 contain the *same* proposition of *Labels* (this follows immediately by inspecting Rule I of Section 5.2.2 and the definitions of ω^1 and ω^2). We put

$$P(t, t'_1 \uplus t'_2) = \min\{P(t_1, t'_1), P(t_2, t'_2)\}.$$

Furthermore,

- if $P(t_1, t'_1) > P(t_2, t'_2)$, then

$$P(t, t'_1 \uplus t'_2) = P(t_1, t'_1) - P(t_2, t'_2);$$
- if $P(t_2, t'_2) > P(t_1, t'_1)$, then

$$P(t, tc_1 \uplus t'_2) = P(t_2, t'_2) - P(t_1, t'_1);$$
- for all $t''_1 \in \text{succ}(t_1)$ where $t''_1 \neq t'_1$ and $t''_1 \in \{ta_1, tb_1, tx_1\}$,

$$P(t, t''_1 \uplus t'_2) = P(t_1, t''_1);$$
- for all $t''_2 \in \text{succ}(t_2)$ where $t''_2 \neq t'_2$ and $t''_2 \in \{ta_2, tb_2, tx_2\}$,

$$P(t, tc_1 \uplus t''_2) = P(t_2, t''_2);$$
- finally, we put $P(t, tc_1 \uplus t'_2) = 1 - s$ where s is the sum of all $P(t, \cdot)$ defined above.

Note that $s < 1$ due to the constraints on q and κ adopted in Section 5.

Rule B. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$ where $\mathcal{L}_1 \cap \text{Labels} \neq \emptyset$, $\mathcal{L}_2 \cap \text{Labels} = \emptyset$, and $h \notin \mathcal{L}_2$, then \mathcal{L}_1 contains precisely one $r_i \in \{r_0, \dots, r_4\}$, and we distinguish two subcases.

$n_1 = 0$. Then

$$\begin{aligned} P(t, [\star, \{h, a, S(r_i)\}, \star] \uplus tc_2) &= \kappa_1, \\ P(t, [\star, \{h, b, S^2(r_i)\}, \star] \uplus tc_2) &= \kappa_2, \\ P(t, [\star, \{h, c, S^2(r_i), e\}, \star] \uplus tc_2) &= 1 - q. \end{aligned}$$

Furthermore, for every $t''_2 \in \{ta_2, tb_2, tx_2\}$, we put

$$P(t, [\star, \{h, c, S^2(r_i)\}, \star] \uplus t''_2) = P(t_2, t''_2).$$

Finally, we put $P(t, [\star, \{h, c, S^2(r_i)\}, \star] \uplus tc_2) = 1 - s$ where s is the sum of all $P(t, \cdot)$ defined above.

$n > 0$. Then

$$\begin{aligned} P(t, [\star, \{a, S(r_i)\}, n-1] \uplus tc_2) &= \sigma^n(\kappa)_1, \\ P(t, [\star, \{h, b, S^2(r_i)\}, \star] \uplus tc_2) &= \sigma^n(\kappa)_2, \\ P(t, [\star, \{h, c, S^2(r_i), e\}, \star] \uplus tc_2) &= 1 - q. \end{aligned}$$

Furthermore, for every $t''_2 \in \{ta_2, tb_2, tx_2\}$,

$$P(t, [\star, \{h, c, S^2(r_i)\}, \star] \uplus t''_2) = P(t_2, t''_2).$$

Finally, we put $P(t, [\star, \{h, c, S^2(r_i)\}, \star] \uplus tc_2) = 1 - s$ where s is the sum of all $P(t, \cdot)$ defined above.

Rule C. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$ where $\mathcal{L}_2 \cap \text{Labels} \neq \emptyset$, $\mathcal{L}_1 \cap \text{Labels} = \emptyset$, and $h \notin \mathcal{L}_1$, then the immediate successors of t are defined similarly as in Rule B (Rule C is fully symmetric to Rule B).

Rule D. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, n_2]$ where $\mathcal{L}_1 \cap \text{Labels} = \emptyset$, $\mathcal{L}_2 \cap \text{Labels} = \emptyset$, $h \notin \mathcal{L}_1$, and $h \notin \mathcal{L}_2$, then the immediate successors of t are defined as follows. For every $t''_1 \in \{ta_1, tb_1, tx_1\}$,

$$P(t, t''_1 \uplus tc_2) = P(t_1, t''_1).$$

Similarly, for all $t''_2 \in \{ta_2, tb_2, tx_2\}$,

$$P(t, tc_1 \uplus t''_2) = P(t_2, t''_2).$$

Finally, $P(t, tc_1 \uplus tc_2) = 1 - s$, where s is the sum of all $P(t, \cdot)$ above.

Rule E. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, \star]$ where $h \notin \mathcal{L}_1$ and $h \in \mathcal{L}_2$, then $P(t, t'_1 \uplus t_2) = P(t_1, t'_1)$ for all $t'_1 \in \text{succ}(t_1)$.

Rule F. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, \star]$ where $h \in \mathcal{L}_1$ and $h \notin \mathcal{L}_2$, then $P(t, t_1 \uplus t'_2) = P(t_2, t'_2)$ for all $t'_2 \in \text{succ}(t_2)$.

Rule G. If $t = [k, \mathcal{L}_1, \mathcal{L}_2, n_1, \star]$ where $h \in \mathcal{L}_1$ and $h \in \mathcal{L}_2$, then $P(t, t) = 1$.

If the computation ω is periodic, then the constructed Markov chain M has finitely many states. A routine check (similar to the one performed in the proof of Theorem 5.1 (B)) reveals that the state $s = [0, \{a, r_0, \ell_1\}, \{a, r_0, \ell_1\}, 0, 0]$ satisfies Ψ and covers the computation ω .

7 CONCLUSIONS

We have shown that the general/finite PCTL satisfiability problems are undecidable. Note that the formula Ψ constructed in the proof of Theorem 6.2 is *always* generally satisfiable, which implies that the finite satisfiability problem for PCTL is undecidable even for the subset of generally satisfiable PCTL formulae. Furthermore, the undecidability result remains valid even if the set of eligible finite models is restricted to *tree-like* models, where the underlying graph of a Markov chain is a tree with self-loops on all leaves (this requires a slight modification of our construction). Finally, let us note that the construction of Theorem 6.2 applies also to a universal Minsky machine. Hence, fixed parameterized PCTL formulae exist such that the general/finite satisfiability of their instances is undecidable.

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Appendix

Here, we provide the proofs omitted in the main body of the paper due to space constraints.

LEMMA 3.3. *For every $\mathbf{v} \in W$, we have the following:*

- (a) $\tau(\mathbf{v}), \sigma(\mathbf{v}) \in W$;
- (b) $\tau(\mathbf{v})_1 > \mathbf{v}_1$ and $\tau(\mathbf{v})_2 \geq \mathbf{v}_2$; if $\mathbf{v}_2 > 0$, then $\tau(\mathbf{v})_2 > \mathbf{v}_2$;
- (c) let $\mathbf{u} = (\mathbf{v}_1, 0)$; then $\text{slope}(\mathbf{u}, \tau(\mathbf{v})) = \text{slope}(\tau(\mathbf{v}), \tau^2(\mathbf{v}))$;
- (d) let $\mathbf{u} = (\mathbf{v}_1, y)$ where $0 \leq y < \mathbf{v}_2$. Then $\text{slope}(\mathbf{u}, \tau(\mathbf{v})) < \text{slope}(\mathbf{v}, \tau(\mathbf{v}))$;
- (e) $\sigma(\tau(\mathbf{v})) = \tau(\sigma(\mathbf{v})) = \mathbf{v}$.

PROOF. Recall that

$$\begin{aligned} I_q &= \left(\frac{1 - \sqrt{4q-3}}{2}, \frac{1 + \sqrt{4q-3}}{2} \right) \\ W &= I_q \times [0, \infty) \\ \tau(\mathbf{v}) &= \left(\frac{q-1+\mathbf{v}_1}{\mathbf{v}_1}, \frac{\mathbf{v}_2}{\mathbf{v}_1} \right) \\ \sigma(\mathbf{v}) &= \left(\frac{1-q}{1-\mathbf{v}_1}, \frac{\mathbf{v}_2(1-q)}{1-\mathbf{v}_1} \right) \end{aligned}$$

Item (a). Let $\mathbf{v} \in W$. We show that $\tau(\mathbf{v}) \in W$. Observe

$$\begin{aligned} \tau(\mathbf{v})_1 &= 1 - \frac{1-q}{\mathbf{v}_1} \\ &< 1 - \frac{2(1-q)}{1 - \sqrt{4q-3}} \\ &= 1 - \frac{(2-2q)}{1 - \sqrt{4q-3}} \cdot \frac{1 + \sqrt{4q-3}}{1 + \sqrt{4q-3}} \\ &= 1 - \frac{(2-2q)(1 + \sqrt{4q-3})}{4 - 4q} \\ &= \frac{1 + \sqrt{4q-3}}{2} \end{aligned}$$

Similarly, we obtain $\tau(\mathbf{v})_1 > \frac{1 - \sqrt{4q-3}}{2}$, and hence $\tau(\mathbf{v})_1 \in I_q$. Since $\tau(\mathbf{v})_2 = \mathbf{v}_2/\mathbf{v}_1 > 0$, we have that $\tau(\mathbf{v}) \in W$ as required.

Now we show that $\sigma(\mathbf{v}) \in W$. Observe

$$\begin{aligned} \sigma(\mathbf{v})_1 &= \frac{1-q}{1-\mathbf{v}_1} \\ &< \frac{1-q}{1 - \frac{1+\sqrt{4q-3}}{2}} \\ &= \frac{2(1-q)}{1 - \sqrt{4q-3}} \cdot \frac{1 + \sqrt{4q-3}}{1 + \sqrt{4q-3}} \\ &= \frac{1 + \sqrt{4q-3}}{2} \end{aligned}$$

Similarly, we obtain $\sigma(\mathbf{v})_1 > \frac{1 - \sqrt{4q-3}}{2}$. Since $\sigma(\mathbf{v})_2 = \frac{\mathbf{v}_2(1-q)}{1-\mathbf{v}_1} > 0$, we have that $\sigma(\mathbf{v}) \in W$.

Item (b). Let $\mathbf{v} \in W$. Observe that $\tau(\mathbf{v})_1 > \mathbf{v}_1$ iff $(q-1+\mathbf{v}_1)/\mathbf{v}_1 > \mathbf{v}_1$ iff $\mathbf{v}_1^2 - \mathbf{v}_1 - q + 1 < 0$ iff $\mathbf{v}_1 \in I_q$. This explains our choice of I_q . Furthermore, $\tau(\mathbf{v})_2 = \mathbf{v}_2/\mathbf{v}_1 \geq \mathbf{v}_2$; if $\mathbf{v}_2 > 0$, then $\tau(\mathbf{v})_2 > \mathbf{v}_2$.

Item (c). Let $\mathbf{v} \in W$. Observe

$$\text{slope}(\mathbf{u}, \tau(\mathbf{v})) = \frac{\tau(\mathbf{v})_2}{\tau(\mathbf{v})_1 - \mathbf{v}_1} = \frac{\mathbf{v}_2}{q-1+\mathbf{v}_1(1-\mathbf{v}_1)}$$

Similarly,

$$\begin{aligned} \text{slope}(\tau(\mathbf{v}), \tau^2(\mathbf{v})) &= \frac{\tau(\mathbf{v})_2(1-\tau(\mathbf{v})_1)}{q-1+\tau(\mathbf{v})_1(1-\tau(\mathbf{v})_1)} \\ &= \frac{\frac{\mathbf{v}_2}{\mathbf{v}_1} \left(1 - \frac{q-1+\mathbf{v}_1}{\mathbf{v}_1}\right)}{q-1 + \frac{q-1+\mathbf{v}_1}{\mathbf{v}_1} \left(1 - \frac{q-1+\mathbf{v}_1}{\mathbf{v}_1}\right)} \\ &= \frac{\frac{\mathbf{v}_2}{\mathbf{v}_1} \left(\frac{1-q}{\mathbf{v}_1}\right)}{q-1 + \frac{q-1+\mathbf{v}_1}{\mathbf{v}_1} \left(\frac{1-q}{\mathbf{v}_1}\right)} \\ &= \frac{\mathbf{v}_2}{q-1+\mathbf{v}_1(1-\mathbf{v}_1)} \end{aligned}$$

Hence, $\text{slope}(\mathbf{u}, \tau(\mathbf{v})) = \text{slope}(\tau(\mathbf{v}), \tau^2(\mathbf{v}))$.

Item (d). Realize that

$$\text{slope}(\mathbf{u}, \tau(\mathbf{u})) = \frac{y(1-\mathbf{v}_1)}{q-1+\mathbf{v}_1(1-\mathbf{v}_1)}$$

$$\text{slope}(\mathbf{v}, \tau(\mathbf{v})) = \frac{\mathbf{v}_2(1-\mathbf{v}_1)}{q-1+\mathbf{v}_1(1-\mathbf{v}_1)}$$

Since $0 \leq y < \mathbf{v}_2$, we have that $\text{slope}(\mathbf{u}, \tau(\mathbf{u})) < \text{slope}(\mathbf{v}, \tau(\mathbf{v}))$.

Item (e). It is trivial to verify that $\sigma(\tau(\mathbf{v})) = \tau(\sigma(\mathbf{v})) = \mathbf{v}$ for every $\mathbf{v} \in W$. \square

LEMMA 3.5. *For every $\mathbf{v} \in W$ where $\mathbf{v}_2 > 0$ and every $\mathbf{u} \in \text{Points}(\mathbf{v})$, we have the following:*

- (a) If \mathbf{u} is a positive convex combination of $\mathbf{u}^1, \mathbf{u}^2, \dots$ where $\mathbf{u}^i \in \text{Area}(\mathbf{v})$ for all $i \in \mathbb{N}$, then $\mathbf{u}^i = \mathbf{u}$ for all $i \in \mathbb{N}$.
- (b) If $\mathbf{w} \in L(\mathbf{u})$ is a positive convex combination of $\mathbf{u}^1, \mathbf{u}^2, \dots$ where $\mathbf{u}^i \in \text{Area}(\mathbf{v})$ for all $i \in \mathbb{N}$, then $\mathbf{u}^i \in L(\mathbf{u}) \cup \{\tau(\mathbf{u})\}$ for all $i \in \mathbb{N}$.

PROOF. Let $\mathbf{v} \in W$. All claims follow directly from Lemma 3.3. More concretely, for every $\mathbf{u} \in \text{Points}(\mathbf{v})$, we have that both \mathbf{u} and $L(\mathbf{u})$ are faces of the convex set $\text{Area}(\mathbf{v})$ (see, e.g., Section 2.6 in [23]), and the claims (A) and (B) are just instances of the defining property of a face. \square

LEMMA 3.6. *Let $\mathbf{w} \in W$ and $\mathbf{u} \in L(\mathbf{w})$. Then $\tau(\mathbf{u}) \in L(\tau(\mathbf{w}))$.*

PROOF. It is easy to verify that for all $\mathbf{x}, \mathbf{y} \in W$ and all $\lambda \in (0, 1]$ we have that

$$\tau(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda'\tau(\mathbf{x}) + (1-\lambda')\tau(\mathbf{y})$$

where

$$\lambda' = \frac{\lambda\mathbf{x}_1}{\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{y}_1}$$

Observe that $\lambda' \in (0, 1]$. The lemma follows by putting $\mathbf{x} = \mathbf{w}$, $\mathbf{y} = \tau(\mathbf{w})$ and choosing λ so that $\mathbf{u} = \lambda\mathbf{w} + (1-\lambda)\tau(\mathbf{w})$. \square

LEMMA 3.7. *For all $\mathbf{v} \in W \setminus \text{Area}(\kappa)$ where $\mathbf{v}_1 \leq \kappa_1$, there exists $\mathbf{u} \in W \setminus \text{Area}(\kappa)$ such that $\mathbf{u}_1 = \sigma^k(\kappa)_1$ for some $k \geq 0$ and $\mathbf{v} \in L(\mathbf{u})$.*

PROOF. First, we show that

$$\lim_{k \rightarrow \infty} \sigma^k(\boldsymbol{\kappa})_1 = \frac{1 - \sqrt{4q-3}}{2} \quad (12)$$

Let

$$J_q = \left[\frac{1 - \sqrt{4q-3}}{2}, \boldsymbol{\kappa}_1 \right]$$

By Lemma 3.3, the infinite sequence $\boldsymbol{\kappa}_1, \sigma(\boldsymbol{\kappa})_1, \sigma^2(\boldsymbol{\kappa})_1, \dots$ is decreasing and bounded from below by $(1 - \sqrt{4q-3})/2$. Consequently, the sequence has a limit $\alpha \in J_q$, and hence it is also a Cauchy sequence, i.e.,

$$\lim_{k \rightarrow \infty} \sigma^{k+1}(\boldsymbol{\kappa})_1 - \sigma^k(\boldsymbol{\kappa})_1 = 0.$$

Consider the function $f : J_q \rightarrow \mathbb{R}$ where

$$f(x) = \frac{1 - q - x(1-x)}{1-x}$$

Observe that f is non-negative and continuous. Furthermore, $f(x) = 0$ iff $x = (1 - \sqrt{4q-3})/2$. Observe that for every $k \in \mathbb{N}$, we have that

$$\sigma^{k+1}(\boldsymbol{\kappa})_1 - \sigma^k(\boldsymbol{\kappa})_1 = f(\sigma^k(\boldsymbol{\kappa})_1).$$

Hence,

$$0 = \lim_{k \rightarrow \infty} \sigma^{k+1}(\boldsymbol{\kappa})_1 - \sigma^k(\boldsymbol{\kappa})_1 = \lim_{k \rightarrow \infty} f(\sigma^k(\boldsymbol{\kappa})_1) = f(\alpha)$$

which implies $\alpha = (1 - \sqrt{4q-3})/2$.

Now let $\boldsymbol{v} \in W \setminus \text{Area}(\boldsymbol{\kappa})$ where $\boldsymbol{v}_1 \leq \boldsymbol{\kappa}_1$. By (12), there exist $k \in \mathbb{N}$ such that $\sigma^k(\boldsymbol{\kappa})_1 \leq \boldsymbol{v}_1 < \sigma^{k-1}(\boldsymbol{\kappa})_1$. We put $\boldsymbol{u}_1 = \sigma^k(\boldsymbol{\kappa})_1$ and choose \boldsymbol{u}_2 so that

$$\text{slope}(\boldsymbol{u}, \tau(\boldsymbol{u})) = \text{slope}(\boldsymbol{u}, \boldsymbol{v}).$$

Hence, we require that

$$\frac{\boldsymbol{u}_2(1 - \boldsymbol{u}_1)}{q - 1 + \boldsymbol{u}_1(1 - \boldsymbol{u}_1)} = \frac{\boldsymbol{v}_2 - \boldsymbol{u}_2}{\boldsymbol{v}_1 - \boldsymbol{u}_1}$$

From this, we obtain

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2(q - 1 + \boldsymbol{u}_1(1 - \boldsymbol{u}_1))}{\boldsymbol{v}_1 - \boldsymbol{u}_1 + q - 1 + \boldsymbol{u}_1(1 - \boldsymbol{u}_1)}$$

and the proof is finished. \square

PROPOSITION 6.1. *The boundedness problem for a synchronized product of two deterministic one-counter Minsky machines is Σ_1^0 -hard. The recurrent reachability problem for a synchronized product of two non-deterministic one-counter Minsky machines is Σ_1^1 -hard.*

PROOF. Let $\mathcal{M} \equiv 1 : \text{Ins}_1; \dots; m : \text{Ins}_m$; be a non-deterministic two-counter Minsky machine. We start by transforming \mathcal{M} into another two-counter Minsky machine $\widehat{\mathcal{M}}$ with $3m$ instructions constructed as follows:

- The first m instructions of $\widehat{\mathcal{M}}$ are the same as the instructions of \mathcal{M} .
- For every $j \in \{1, \dots, m\}$, the machine $\widehat{\mathcal{M}}$ contains the following labeled instructions:
 - $m+j$: *if* $c_1=0$ *then goto* $\{1\}$ *else dec* c_1 ; *goto* $\{j\}$
 - $2m+j$: *if* $c_2=0$ *then goto* $\{1\}$ *else dec* c_2 ; *goto* $\{j\}$

As we shall see, the target labels in the *then* branches are insignificant and can be chosen arbitrarily.

Observe that $\widehat{\mathcal{M}}$ has the same set of computations as \mathcal{M} , because the newly added instructions are not reachable from the initial configuration. Furthermore, if \mathcal{M} is deterministic, then $\widehat{\mathcal{M}}$ is also deterministic.

Let $L_1, L_2 \subseteq \{1, \dots, 3m\}$ be the sets of labels of all instructions of $\widehat{\mathcal{M}}$ operating on c_1 and c_2 , respectively. Now, we construct two one-counter Minsky machines $\mathcal{M}_1, \mathcal{M}_2$ whose synchronized product simulates $\widehat{\mathcal{M}}$.

Both \mathcal{M}_1 and \mathcal{M}_2 have $6m$ instructions. For notation convenience, the labels of $\mathcal{M}_1, \mathcal{M}_2$ are written as pairs $(\ell, 0)$, $(\ell, +)$, where $\ell \in \{1, \dots, 3m\}$.

For every $\ell \in \{1, \dots, 3m\}$, the instructions labeled by $(\ell, 0)$ and $(\ell, +)$ are constructed as follows: Let Ins_ℓ be the instruction of $\widehat{\mathcal{M}}$ with label ℓ . If $\ell \in L_1$, then

- \mathcal{M}_1 contains the instruction $(\ell, 0) : \overline{\text{Ins}}$, where $\overline{\text{Ins}}$ is obtained from Ins_ℓ as follows:
 - c_1 is replaced with c ;
 - each set of target labels L occurring in Ins_ℓ is replaced with \overline{L} obtained from L by replacing every $u \in L$ with either $(u, +)$ or $(2m+u, 0)$, depending on whether $u \in L_1$ or $u \in L_2$, respectively.
- \mathcal{M}_2 contains the instruction $(\ell, 0) : \text{inc } c; \text{goto } \{1\}$
- \mathcal{M}_1 contains the instruction $(\ell, +) : \overline{\text{Ins}}$, where $\overline{\text{Ins}}$ is obtained from Ins_ℓ as follows:
 - c_1 is replaced with c ;
 - each set of target labels L occurring in Ins_ℓ is replaced with \overline{L} obtained from L by replacing every $u \in L$ with $(u, 0)$.
- \mathcal{M}_2 contains the instruction $(\ell, +) : \text{if } c=0 \text{ then goto } \{1\} \text{ else dec } c; \text{goto } \{1\}$

If $\ell \in L_2$, then

- \mathcal{M}_2 contains the instruction $(\ell, 0) : \overline{\text{Ins}}$, where $\overline{\text{Ins}}$ is obtained from Ins_ℓ as follows:
 - c_2 is replaced with c ;
 - each set of target labels L occurring in Ins_ℓ is replaced with \overline{L} obtained from L by replacing every $u \in L$ with either $(u, +)$ or $(m+u, 0)$, depending on whether $u \in L_2$ or $u \in L_1$, respectively.
- \mathcal{M}_1 contains the instruction $(\ell, 0) : \text{inc } c; \text{goto } \{1\}$
- \mathcal{M}_2 contains the instruction $(\ell, +) : \overline{\text{Ins}}$, where $\overline{\text{Ins}}$ is obtained from Ins_ℓ as follows:
 - c_2 is replaced with c ;
 - each set of target labels L occurring in Ins_ℓ is replaced with \overline{L} obtained from L by replacing every $u \in L$ with $(u, 0)$.
- \mathcal{M}_1 contains the instruction $(\ell, +) : \text{if } c=0 \text{ then goto } \{1\} \text{ else dec } c; \text{goto } \{1\}$

Furthermore, we put $I = (I_1, I_2)$, where I_1 is the set of all $(\ell, 0), (\ell, +)$ such that $\ell \in L_1$, and I_2 contains the other labels. The computation of $\mathcal{M}_1 \times_I \mathcal{M}_2$ starts by executing the instructions with label $(1, 0)$.

Intuitively, $\mathcal{M}_1 \times_I \mathcal{M}_2$ simulates $\widehat{\mathcal{M}}$ where the instructions on c_1 and c_2 are performed by \mathcal{M}_1 and \mathcal{M}_2 , respectively. As

long as \mathcal{M}_1 performs instructions on c_1 , \mathcal{M}_2 keeps incrementing/decrementing c_2 alternately (the flags + and 0 in the label indicate whether the “inactive” counter should be decremented or incremented; note that we only decrement the inactive counter when it was incremented before, and hence its value is certainly positive). When an instruction operating on c_2 is reached, the control is passed to \mathcal{M}_2 . Before executing the instruction on c_2 , \mathcal{M}_2

possibly decrements c_2 to restore its value. It is easy to check that \mathcal{M} is deterministic and bounded iff $\mathcal{M}_1 \times_I \mathcal{M}_2$ is deterministic and bounded, and \mathcal{M} has a recurrent computation iff $\mathcal{M}_1 \times_I \mathcal{M}_2$ has a recurrent computation. \square

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