RINGS WHOSE NON-INVERTIBLE ELEMENTS ARE STRONGLY NIL-CLEAN

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Abstract. We consider in-depth and characterize in certain aspects those rings whose non-units are strongly nil-clean in the sense that they are a sum of commuting nilpotent and idempotent. In addition, we examine those rings in which the non-units are uniquely nil-clean in the sense that they are a sum of a nilpotent and an unique idempotent. In fact, we succeeded to prove that these two classes of rings can completely be characterized in terms of already well-studied and fully described sorts of rings.

1. Introduction and Basic Concepts

Everywhere in the current paper, let R be an associative but *not* necessarily commutative ring having identity element, usually stated as 1. Standardly, for such a ring R, the letters $U(R)$, Nil(R) and $\mathrm{Id}(R)$ are designed for the set of invertible elements (also termed as the unit group of R), the set of nilpotent elements and the set of idempotent elements in R, respectively. Likewise, $J(R)$ denotes the Jacobson radical of R, and $\mathrm{Z}(R)$ denotes the center of R. The ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are denoted by $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$, respectively. Standardly, a ring is said to be *abelian* if each of its idempotents is central, that is, $\mathrm{Id}(R) \subseteq \mathrm{Z}(R)$.

In order to present our achievements here, we now need the necessary background material as follows: Mimicking [14], an element a from a ring R is called *clean* if there exists $e \in \mathrm{Id}(R)$ such that $a - e \in U(R)$. Then, R is said to be *clean* if each element of R is clean. In addition, a is called *strongly clean* provided $ae = ea$ and, if each element of R are strongly clean, then R is said to *strongly clean* too. On the other hand, imitating [24], $a \in R$ is called *uniquely clean* if there exists a unique $e \in \mathrm{Id}(R)$ such that $a - e \in U(R)$. In particular, a ring R is said to be *uniquely clean* (or just *UC* for short) if every element in R is uniquely clean. Generalizing these notions, in [5] was defined an element $a \in R$ to be *uniquely strongly clean* if there exists a unique $e \in \mathrm{Id}(R)$ such that $a - e \in U(R)$ and $ae = ea$. In particular, a ring R is *uniquely strongly clean* (or just *USC* for short) if each element in R is uniquely strongly clean. A ring R is generalized uniquely clean (or just GUC for short) if every non-invertible element of R is uniquely clean, which was introduced in [13]. A ring is called a *generalized uniquely strongly clean* ring (or just *GUSC* for short) if every non-invertible element is uniquely strongly clean. These rings are generalization of USC rings, which was introduced in [10].

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Let R be a ring. An element $r \in R$ is said to be *nil-clean* if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e + b$. Such an element r is further called *strongly nil-clean* if the existing idempotent and nilpotent can be chosen such that $be = eb$. A ring is called *nil-clean* (respectively, *strongly nil-clean*) if each of its elements is nil-clean (respectively, strongly nil-clean). Nil-clean and strongly nil-clean rings was introduced by Diesl in [12]. An element a in a ring R is called uniquely nil-clean (or just UNC for short) if there is a unique idempotent e such that $a - e$ is nilpotent. We will say that a ring is uniquely nil-clean (or just UNC for short) if each of its elements is uniquely nil-clean. These rings were also introduced by Diesl in [12]. A ring R is called a UU ring if $U(R) = 1 + Nil(R)$, which was introduced by Calugareanu [1] and studied in more details by Danchev and Lam in [**?**]. Diesl in [12] proved that a unit u of R is strangly nil-clean if and only if $u \in 1 + Nil(R)$. In particular, R is a UU ring if and only if every unit of R is strongly nil-clean. It is clear that the UU rings are generalization of strongly nil-clean rings. Also, Karimi-Mansoub et al in [16] proved that a ring R is a UU if and only if every unit of R is uniquely nil-clean. It is also clear that UU rings are generalization of uniquely nil-clean rings. So, this idea comes to mind that what can be said about rings whose non-invertible elements are strongly nil-clean and rings whose non-invertible elements are uniquely nil-clean. Also we know that UU ring need not be strongly clean. Thus, a natural problem is what generalizations of strongly nil-clean and uniquely nil-clean rings can be found that are strongly clean. In this paper, we introduce two families of rings. The first one is a generalization of uniquely nil-clean rings which is a subclass of strongly clean rings and, the second one is a generalization of strongly nil-clean rings which are strongly π-regular and strongly clean. These families include rings in which each non-invertible element is uniquely nil-clean (or just GUNC for short) and rings in which every non-invertible element is strongly nil-clean (or just GSNC for short). Various extensions of these rings will be studied.

We are now planning to give a brief program of our results established in the sequel: In the next second section, we establish some fundamental characterization properties of GUNC rings – for instance, we succeeded to establish a valuable necessary and sufficient condition, which totally classifies any ring to be GUNC (see Theorem **??**). In the subsequent third section, we explore GUNC group rings and obtain a good criterion for a group ring of a locally finite p-group, with p a prime, over an arbitrary ring to be GUNC. In the next fourth section, we give a comprehensive investigation of GSNC rings and characterize them in several ways (see, e.g., Theorems 4.9, 4.12, 4.34 and 4.36, respectively). Our fifth section is devoted to the examination in-depth of GSNC group rings and we receive some satisfactory characterization of their structure. We finish our study in the sixth section with two intriguing left-open questions that are of some interest and importance.

2. GUNC RINGS

We start here with the following key notion.

Definition 2.1. A ring R is called *generalized uniquely nil-clean* (or just *GUNC* for short) if every non-invertible element in R is uniquely nil-clean.

We now have the following diagram:

$$\begin{tikzcd}
\text{UNC} \arrow[r] \arrow[d] & \text{GUNC} \arrow[r] \arrow[d] & \text{Strongly clean} \\
\text{UC} \arrow[r] & \text{GUC} \arrow[ur] &
\end{tikzcd}$$

The next example gives us the opportunity to discover the complicate structure of these rings.

Example 2.2. (i) Any UNC ring is GUNC, but the converse is *not* true in general. In fact, a simple check shows that the ring $\mathbb{Z}_3$ is GUNC that is *not* UNC.

(ii) Any UNC ring is UC, but the converse is *not* true in general. In fact, a plain verification shows that the ring $\mathbb{Z}_4[[x]]$ is UC that is *not* UNC.

(iii) Any UC ring is GUC, but the converse is *not* true in general. Indeed, an easy inspection shows that the ring $\mathbb{Z}_5$ is GUC that is *not* UC.

(iv) Any GUC ring is strongly clean, but the converse is *not* true in general. Indeed, a quick trick shows that the ring $\mathrm{M}_2(R)$, where $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{Z}_2,\ b \in \mathbb{Z}_{(2)}[x] \right\}$, is strongly clean that is *not* GUC.

This substantiates our argumentation.

WE now need a series of preliminary technicalities.

Lemma 2.3. *Let R be a ring and $a \in R$. Then, a is UNC if, and only if, $1-a$ is UNC.*

Proof. Given $a \in R$ is UNC. Then, there exists $e^2 = e \in R$ and $q \in \mathrm{Nil}(R)$ such that $a = e + q$. Hence, $1 - a = (1-e) + (-q)$. Suppose $1 - a = f + q'$, where $f^2 = f \in R$ and $q' \in \mathrm{Nil}(R)$. Thus, $a = (1-f) + (-q')$. Since a is UNC, one sees that $1 - f = e$ whence $1 - e = f$, so $1 - a$ is too UNC. The converse is similar, so we omit the details. $\square$

Lemma 2.4. *Let R be a GUNC ring. Then, R is abelian.*

Proof. Given $e \in Id(R)$. If $e \in U(R)$, it must be that $e = 1$ and hence e is central. If, however, $e \notin U(R)$, it follows that e is UNC and, therefore, we find that e is central by [8, Corollary 2.4], as required. $\square$

Lemma 2.5. *If the direct product $\prod_{i=1}^{n} R_i$ is a GUNC ring, then each direct component R_i is a GUNC ring.*

Proof. Given $a \in R_i$, where $a \notin U(R_i)$, so one sees that the vector $(1, \dots, a, 1, \dots, 1)$ is not a unit in $\prod_{i=1}^{n} R_i$. However, as $\prod_{i=1}^{n} R_i$ is GUNC, the element $(1, \dots, a, 1, \dots, 1)$ is UNC, so that a is UNC as well; for otherwise, if a has two different nil-clean decompositions, then $(1, \dots, a, 1, \dots, 1)$ will also have two different nil-clean decompositions, which is a contradiction, as pursued. $\square$

It is worthy of noticing that the converse claim of Lemma 2.5 is *not* true. For instance, the ring $\mathbb{Z}_3$ is obviously GUNC, but however the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is *not* GUNC.

Nevertheless, we can offer the following.

Proposition 2.6. *Let R_i be rings for all $1 \leqslant i \leqslant n$. Then, the product $\prod_{i=1}^n R_i$ is a GUNC ring if, and only if, each direct factor R_i is a UNC ring.*

Proof. Assuming every R_i is UNC, so $\prod_{i=1}^n R_i$ is UNC using [12, Proposition 5.2] and hence $\prod_{i=1}^n R_i$ is necessarily GUNC.

Conversely, assume that $\prod_{i=1}^n R_i$ is GUNC and, in a way of contradiction, that R_j is *not* UNC for some index j. Then, there exists $a \in R_j$ which is not UNC, so the vector $(0, \dots, 0, a, 0, \dots, 0)$ is not UNC in $\prod_{i=1}^n R_i$ too. But, one verifies that

$$(0, \dots, 0, a, 0, \dots, 0) \notin U(\prod_{i=1}^n R_i)$$

and, by hypothesis, $(0, \dots, 0, a, 0, \dots, 0)$ is UNC contradicting our assumption. Therefore, every R_i is UNC, as claimed. $\square$

A ring R is called *division*, if every non-zero element of R is invertible. Also, a ring R is called *local* if $R/J(R)$ is a division ring. thereby, as a consequence, we yield:

Corollary 2.7. *Let R be a ring, and $e^2 = e \in \mathrm{Z}(R)$. If R is GUNC, then eRe is GUNC. In particular, if e is non-trivial, then eRe is UNC.*

Proof. If, for a moment, a GUNC ring R is *not* a local ring, then there exists an idempotent e which is *not* trivial such that $R = eRe \oplus (1-e)R(1-e)$. As R is GUNC, the corner eRe is GUNC in accordance with Lemma 2.5. In addition, if e is non-trivial, then the subring eRe is UNC in view of Proposition 2.6. $\square$

Recall that a ring R is *directly finite*, provided that $ab = 1$ implies $ba = 1$ for all $a, b \in R$ (or, equivalently, $aR = R$ implies $Ra = R$). We, thus, arrive at the following interesting property of GUNC rings.

Proposition 2.8. *Every GUNC ring is directly finite.*

Proof. Letting $ab = 1$, it must be that $(ba)^2 = baba = ba$. So, ba is an idempotent in R, and hence it is central by Lemma 2.4. Therefore,

$$ba = ba(ab) = a(ba)b = (ab)(ab) = 1,$$

as required. $\square$

Our next machinery, necessary to establish the global results, is the following.

Proposition 2.9. *Let R be a ring. If R is local with $J(R)$ nil, then R is GUNC.*

Proof. Supposing $a \in R$, where $a \notin U(R)$, so $a \in J(R) \subseteq \mathrm{Nil}(R)$. Then, a has the only nil-clean expression like this $a = 0 + a$. Hence, a is UNC, as needed. $\square$

Lemma 2.10. *Let R be a ring. Then, the following are equivalent:*
(i) R is either local with $J(R)$ nil, or R is UNC.
(ii) R is a GUNC ring.

Proof. (i) $\Rightarrow$ (ii). It is straightforward bearing in mind Proposition 2.9.
(ii) $\Rightarrow$ (i). Assuming that R is a GUNC ring which is *not* local, then there exists an idempotent e that is *not* trivial such that $R = eRe \oplus (1-e)R(1-e)$. Hence, both eRe and $(1-e)R(1-e)$ are UNC taking into account Corollary 2.7. Consequently, R is UNC in virtue of [12, proposition 5.2]. $\square$

Corollary 2.11. *A ring R is GUNC if, and only if, R is GUC and $J(R)$ is nil.*

Proof. It follows combining Lemma 2.10, [13, Theorem 2.10] and [2, Lemma 5.3.7]. $\square$

Corollary 2.12. *If R is GUNC, then $J(R)$ is nil.*

Proof. It is immediate by combination of Lemma 2.10 and [12, Theorem 5.9]. $\square$

A ring R is called *boolean* if every element of R is an idempotent.

Corollary 2.13. *If R is GUNC, then $J(R) = \mathrm{Nil}(R)$.*

Proof. Consulting with Corollary 2.12, we know that $J(R) \subseteq \mathrm{Nil}(R)$. Now, assume that $x \in \mathrm{Nil}(R)$. Then, $\bar{x} \in \bar{R} = \dfrac{R}{J(R)}$ is nilpotent. Exploiting [13, Corollary 2.11], the quotient-ring $\bar{R}$ is either boolean or division. If, foremost, $\bar{R}$ is boolean, thus $\bar{x}$ has to be an idempotent, whence $\bar{x} = \bar{0}$. So, $x \in J(R)$.

If, however, $\bar{R}$ is a division factor-ring, we have again $\bar{x} = \bar{0}$ and $x \in J(R)$. Thus, in both cases, $x \in J(R)$ and hence $J(R) = \mathrm{Nil}(R)$, as stated. $\square$

The following property sounds somewhat curiously.

Corollary 2.14. *If R is GUNC, then R is strongly clean.*

Proof. It follows at once applying Corollary 2.11 and [13, Lemma 2.3]. $\square$

The next statement is pivotal for our further presentation.

Proposition 2.15. *Let R be a ring and I is a nil-ideal of R. Then, the following two equivalencies hold:*

(i) R is GUNC if, and only if, $\dfrac{R}{J(R)}$ is GUNC, $J(R)$ is nil, and R is abelian.

(ii) R is GUNC if, and only if, $\dfrac{R}{I}$ is GUNC and R is abelian.

Proof. (i). Given R is GUNC and $\bar{a} \in \bar{R} = \dfrac{R}{J(R)}$, where $\bar{a} \notin U(R)$. So, one sees that $a \notin U(R)$, as for otherwise, if $a \in U(R)$, then it must be that $\bar{a} \in U(\bar{R})$ and this is a contradiction. Henceforth, we must show that $\bar{a}$ is UNC. To this goal, write $\bar{a} = \bar{e} + \bar{q}_1 = \bar{f} + \bar{q}_2$, where $\bar{e}, \bar{f} \in Id(\bar{R})$ and $\bar{q}_1, \bar{q}_2 \in Nil(\bar{R})$. Thus, we have $a - (e + q_1), a - (f + q_2) \in J(R)$ and hence $a = e + (q_1 + j_1) = f + (q_2 + j_2)$ for some $j_1, j_2 \in J(R)$, where $e, f \in Id(R)$, because idempotents lift modulo $J(R)$, and $(q_1 + j_1), (q_2 + j_2) \in \mathrm{Nil}(R)$. But we know that a is UNC, and so $e = f$ whence $\bar{e} = \bar{f}$. Therefore, $\bar{a}$ is UNC. Moreover, $J(R)$ is nil owing to Corollary 2.12, and R is abelian according to Lemma 2.4, as formulated.

Conversely, let $a \in R$, where $a \notin U(R)$. So, one observes that $\bar{a} \notin U(\bar{R})$, as for otherwise, if $\bar{a} \in U(\bar{R})$, then it must be that $a \in U(R)$, because units lift modulo $J(R)$ and this is a contradiction. Now, writing $a = e + q_1 = f + q_2$, where $e, f \in \mathrm{Id}(R)$ and $q_1, q_2 \in \mathrm{Nil}(R)$, so we have $\bar{a} = \bar{e} + \bar{q}_1 = \bar{f} + \bar{q}_2$, where $\bar{e}, \bar{f} \in \mathrm{Id}(\bar{R})$ and $\bar{q}_1, \bar{q}_2 \in \mathrm{Nil}(\bar{R})$. But, the element $\bar{a}$ is UNC, so that $\bar{e} = \bar{f}$ and, consequently, $e - f \in J(R)$. As R is abelian, $(e - f)^3 = (e - f)$, and thus $e - f = 0$ implying $e = f$. Finally, a is UNC, as expected.

(ii). The proof is quite similar to the preceding point (i), so we omit the details. $\square$

Proposition 2.16.
(i) For any ring R, the power series ring $R[[x]]$ is not GUNC.
(ii) If R is any commutative ring, then $R[x]$ is not GUNC.
(iii) The matrix rings $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$ are never GUNC for any $n \geqslant 2$.

Proof. (i). Note the principal fact that the Jacobson radical of $R[[x]]$ is *not* nil. Thus, invoking Corollary 2.12, $R[[x]]$ is really *not* a GUNC ring.
(ii). If we assume the contrary that $R[x]$ is GUNC, then Corollary 2.14 gives that $R[x]$ is clean. This, however, is impossible in conjunction with [14, Example 2].
(iii). It is pretty obvious referring to Lemma 2.4. □

The next constructions are worthwhile.

Example 2.17.
(i). Any field and even any division ring is GUNC.
(ii). Any GUNC ring is strongly clean but, the converse is manifestly *not* true in general. Indeed, the ring $\mathrm{M}_2(\hat{\mathbb{Z}}_p)$ is strongly clean, but is definitely *not* GUNC.
(iii). Any GUNC ring is GUC but, the converse is *not* true in general. In fact, it is *not* too hard to see that the ring $\mathbb{Z}_2[[x]]$ is GUC but is *not* GUNC.

Proof. (i). It is evident by the definition of a GUNC ring.
(ii). Note that, adapting [4, Theorem 2.4], the ring $\mathrm{M}_2(\hat{\mathbb{Z}}_p)$ is strongly clean. However, it is *not GUNC*, because all GUNC rings are always abelian.
(iii). Note that the ring $\mathbb{Z}_2[[x]]$ is uniquely clean, and hence is GUC, but it is *not* GUNC as Proposition 2.16 tell us. □

We now come to the following necessary and sufficient condition.

Proposition 2.18. *A ring R is UNC if, and only if, R is simultaneously GUNC and UU.*

Proof. Given R is UNC and $u \in U(R)$, so one may write that $u = e + q$, where $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$. Thus, $e = u - q$. But [2, Theorem 5.3.3] informs us that every UNC ring is abelian. Therefore, $e \in U(R)$ and hence $e = 1$. Then, $u \in 1 + \mathrm{Nil}(R)$ and R is a UU ring.

For the converse implication, it suffices to show that R is UU if, and only if, every unit of R is UNC. However, this has been proven in [16, Theorem 2.23]. □

Recall that a ring is called *reduced* if it has no non-zero nilpotent elements.

Lemma 2.19. *Let R be a GUNC ring and $J(R) = \{0\}$. Then, R is reduced.*

Proof. Assume that $x^2 = 0$, where $0 \neq x \in R$. Since R is GUNC, we know that R is clean, and hence R is semi-potent thanks to [13, Proposition 2.16]. Thus, consulting with [20, Theorem 2.1], there exists $0 \neq e^2 = e \in R$ such that $eRe \cong \mathrm{M}_2(S)$ for some non-trivial ring S. But, in regard to Proposition 2.16, $\mathrm{M}_2(S)$ is *not* a GUNC ring. That is why, eRe is *not* GUNC, and this contradicts Corollary 2.7. Finally, R is reduced, as promised. □

Our final assertion for this section, which states an interesting criterion for a ring to be GUNC, is the following one.

Proposition 2.20. *A ring R is GUNC if, and only if, all next three conditions are fulfilled:*
(i) $\mathrm{Nil}(R)$ *is an ideal of* R*;*
(ii) $\frac{R}{\mathrm{Nil}(R)}$ *is either a boolean ring, or a division ring;*
(iii) R *is an abelian ring.*

Proof. "$\Rightarrow$". With Corollary 2.13, Corollary 2.11, [13, Corollary 2.11] and Lemma 2.4 at hand, all things are rather easy.
"$\Leftarrow$". Firstly, letting $\frac{R}{\mathrm{Nil}(R)}$ is a boolean ring, we show that R is UNC. To this purpose, chosen $a \in R$, so $a - a^2 \in \mathrm{Nil}(R)$. By hypothesis, there exists a unique idempotent, $e \in R$ say, such that $q := a - e \in \mathrm{Nil}(R)$. Hence, $a = e + q$. Also, write $a = f + q$, where $f \in \mathrm{Id}(R)$ and $q' \in \mathrm{Nil}(R)$. However, treating the uniqueness, we get $e = f$. Therefore, R is UNC, and so GUNC.

Now, let $\frac{R}{\mathrm{Nil}(R)}$ is a division ring, it is readily to see that $\frac{R}{\mathrm{Nil}(R)}$ is GUNC. Then, R is GUNC using Proposition 2.15, as wanted. □

3. GUNC group rings

We know with the help of Lemma 2.10 that GUNC rings include both UNC rings and local rings, and vice versa. In addition, there are some results on UC group rings and local group rings proved in [6] and [25], respectively. Correspondingly, we can also obtain some achievements about GUNC group rings that could be of some interest and importance. To this aim, we recall that a group G is a *p-group* if every element of G is a power of p, where p is a prime. Likewise, a group G is called *locally finite* if every finitely generated subgroup is finite.

Suppose now that G is an arbitrary group and R is an arbitrary ring. As usual, RG stands for the group ring of G over R. The homomorphism $\varepsilon : RG \to R$, defined by $\varepsilon(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$, is called the *augmentation map* of RG and its kernel, denoted by $\Delta(RG)$, is called the *augmentation ideal* of RG.

Our two affirmations, motivated us in writing this section, are these:

Proposition 3.1. *Let R be a ring and let G be a group. If RG is a GUNC ring, then R is GUNC, G is a p-group and $p \in \mathrm{Nil}(R)$.*

Proof. Assume RG is GUNC. Then, exploiting Lemma 2.10, either RG is local with $J(RG)$ nil, or RG is an UNC ring. We consider these two possibilities in the sequel:

(1) If RG is a local ring with $J(RG)$ nil, then [25, Corollary] guarantees that $\Delta(RG) \subseteq J(RG)$. Hence, $\Delta(RG)$ is a nil-ideal. Therefore, with [7, Proposition 16] at hand, one deduces that G is a p-group, where $p \in \mathrm{Nil}(R)$. Moreover, since $RG/\Delta(RG) \cong R$, the application of Proposition 2.15 ensures that R is a GUNC ring, as desired.

(2) If RG is a UNC ring, then RG is obviously uniquely clean. Applying [6, Theorem 5], one infers that G is a 2-group, and employing [2, Corollary 5.3.5], one finds that R is an UNC ring. Furthermore, [12, Proposition 3.14] assures that $2 \in \mathrm{Nil}(R)$, as asked for. □

Proposition 3.2. *Let R be a ring and let G be a locally finite p-group. Then, RG is GUNC if, and only if, R is GUNC and $p \in \mathrm{Nil}(R)$.*

Proof. Assume R is a GUNC ring and $p \in \mathrm{Nil}(R)$. Then, utilizing [7, Proposition 16], one derives that $\Delta(RG)$ is a nil-ideal. Furthermore, Lemma 2.10 insures that either R is local with nil Jacobson radical, or R is an UNC ring.

If, firstly, R is local with nil Jacobson radical, then [25, Corollary] is a guarantor that RG is local. Thus, $\varepsilon(J(RG)) \subseteq J(R)$, because ε is an onto ring homomorphism. Since $J(R)$ is nil, for every $f \in J(RG)$, there exists $k \in \mathbb{N}$ such that $f^k \in \Delta(RG)$. Likewise, since $\Delta(RG)$ is nil, we have f is nil. Hence, RG is a local ring with nil Jacobson radical, and forcing Lemma 2.10, we conclude that RG is a GUNC ring.

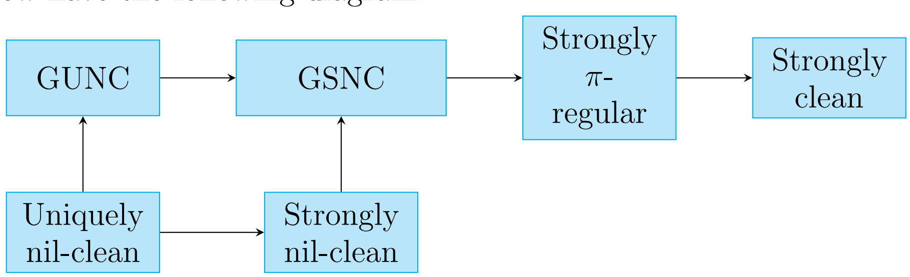
If now R is an UNC ring, then [12, Proposition 3.14] yields that $2 \in \mathrm{Nil}(R)$, and so $p = 2$. Therefore, [6, Theorem 12] reflects to get that RG is an UNC ring, as pursued. $\square$

4. GSNC rings

We begin here with the following key concept.

Definition 4.1. A ring R is called *generalized strongly nil-clean* (or just *GSNC* for short) if every non-invertible element in R is strongly nil-clean.

We now have the following diagram:



We continue here with a series of technicalities as follows.

Lemma 4.2. *Every GSNC ring is strongly clean.*

Proof. Let $a \in R$. Then, either $a \in U(R)$ or $a \notin U(R)$. If, firstly, $a \in U(R)$, then a is a strongly clean element. Now, if $a \notin U(R)$, so a is strongly nil-clean element, and hence a is strongly clean by [12, Corollary 3.6]. $\square$

Lemma 4.3. *Let R_i be a ring for all $i \in I$. If $\prod_{i=1}^n R_i$ is GSNC, then each R_i is GSNC.*

Proof. Let $a_i \in R_i$, where $a_i \notin U(R_i)$, whence $(1, 1, \dots, a_i, 1, \dots, 1) \notin U(\prod_{i=1}^n R_i)$. So, $(1, 1, \dots, a_i, 1, \dots, 1)$ is strongly nil-clean, and hence a_i is strongly nil-clean. If, however, a_i is not strongly nil clean, we clearly conclude that $(1, 1, \dots, a_i, 1, \dots, 1)$ is not strongly nil-clean and this is a contradiction. $\square$

We note that the converse of Lemma 4.3 is manifestly false. For example, $\mathbb{Z}_3$ is GSNC. But the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is *not* GSNC, because the element $(2, 0)$ is *not* invertible in $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $(0, 2)$ is really not strongly nil- clean element.

Lemma 4.4. *If R is GSNC, then $J(R)$ is nil.*

Proof. If we have $a \in J(R)$, then $a \notin U(R)$, so $a = e + q$, where $e = e^2 \in R$, $q \in \mathrm{Nil}(R)$ and $eq = qe$. Therefore,

$$1 - e = (1 + q) - a \in U(R) + J(R) \subseteq U(R),$$

implying $e = 0$. This, in turn, implies that $a = q \in \mathrm{Nil}(R)$. $\square$

Proposition 4.5. *Let R be a ring, and let $a \in R$. Then, the following are equivalent:*
(i) R is a GSNC ring.
(ii) For any $a \in R$, where $a \notin U(R)$, $a - a^2 \in \mathrm{Nil}(R)$.
(iii) For any $a \in R$, where $a \notin U(R)$, there exists $x \in R$ such that $a - ax \in \mathrm{Nil}(R)$, $ax = xa$ and $x = x^2 a$.

Proof. The proof is similar to [2, Theorem 5.1.1]. $\square$

Corollary 4.6. *Every GSNC ring is strongly π-regular.*

Proof. Let R be a GSNC ring. Choose $a \in R$. If $a \in U(R)$, then a is a strongly π-regular element. If, however, $a \notin U(R)$, then, by Proposition 4.5, there exists $k \in \mathbb{N}$ such that $(a - a^2)^k = 0$, so we have $a^k = a^{k+1} r$ for some $r \in R$. Thus, R is a strongly π-regular ring, as claimed. $\square$

Proposition 4.7. *(i) For any nil-ideal $I \subseteq R$, R is GSNC if, and only if, R/I is GSNC.*

(ii) A ring R is GSNC if, and only if, $J(R)$ is nil and $R/J(R)$ is GSNC.

(iii) The direct product $\prod_{i=1}^n R_i$ is GSNC if, and only if, each R_i is strongly nil-clean.

Proof. (i) Assume R is a GSNC ring and $\overline{R} := R/I$, where $\bar{a} \notin U(\overline{R})$. Then, $a \notin U(R)$, which insures, in view of Proposition 4.5, that $a - a^2 \in \mathrm{Nil}(R)$, so $\bar{a} - \bar{a}^2 \in \mathrm{Nil}(\overline{R})$.

Conversely, suppose $\overline{R}$ is a GSNC ring. If $a \notin U(R)$, then $\bar{a} \notin U(\overline{R})$, and Proposition 4.5 yields that $\bar{a} - \bar{a}^2 \in \mathrm{Nil}(\overline{R})$. Therefore, there exists $k \in \mathbb{N}$ such that $(a - a^2)^k \subseteq I \subseteq \mathrm{Nil}(R)$.

(ii) Using Lemma 4.4 and part (i) of the proof, the proof is clear.

(iii) Letting each R_i be strongly nil-clean, then $\prod_{i=1}^n R_i$ is strongly nil-clean employing [12, Proposition 3.13]. Hence, $\prod_{i=1}^n R_i$ is GSNC.

Conversely, assume that $\prod_{i=1}^n R_i$ is GSNC and that R_j is not strongly nil-clean for some index $1 \leqslant j \leqslant n$. Then, there exists $a \in R_j$ which is not strongly nil-clean. Consequently, $(0, \ldots, 0, a, 0, \ldots, 0)$ is not strongly nil-clean in $\prod_{i=1}^n R_i$. But, it is readily seen that $(0, \ldots, 0, a, 0, \ldots, 0)$ is not invertible in $\prod_{i=1}^n R_i$ and thus, by hypothesis, $(0, \ldots, 0, a, 0, \ldots, 0)$ is not strongly nil-clean, a contradiction. Therefore, each R_i is strongly nil-clean, as needed. $\square$

As a consequence, we derive:

Corollary 4.8. *Let R be a ring and $0 \neq e \in \mathrm{Id}(R)$. If R is GSNC, then so is eRe.*

Proof. Assuming $a \in eRe \setminus U(eRe)$, we have $a = ea = ae = eae$. If $a \in U(R)$, then there exists $b \in R$ such that $ab = ba = 1$, which implies $a(ebe) = (ebe)a = e$, leading to a contradiction. Therefore, $a \notin U(R)$. Hence, by Proposition 4.5, we have $a - a^2 \in \mathrm{Nil}(R) \cap eRe \subseteq \mathrm{Nil}(eRe)$, as required. $\square$

We now possess all the machinery necessary to establish the following assertion.

Theorem 4.9. *For any ring $S \neq 0$ and any integer $n \geq 3$, the ring $\mathrm{M}_n(S)$ is not GSNC.*

Proof. Since it is well known that $\mathrm{M}_3(S)$ is isomorphic to a corner ring of $\mathrm{M}_n(S)$ (for $n \geq 3$), it suffices to show that $\mathrm{M}_3(S)$ is *not* a GSNC ring by virtue of Corollary 4.8. To this target, consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin U(\mathrm{M}_3(S)).$$

But a plain inspection gives us that

$$A - A^2 \notin \mathrm{Nil}(\mathrm{M}_3(S)),$$

as expected. Therefore, Proposition 4.5 guarantees that R cannot be a GSNC ring, as asserted. □

The following is worthwhile to be recorded.

Example 4.10. The ring $\mathrm{M}_2(\mathbb{Z}_2)$ is a GSNC ring but is *not* strongly nil-clean. In fact, according to [9, Theorem 2.7], for any arbitrary ring R, the ring $\mathrm{M}_2(R)$ cannot be strongly nil-clean. However, from the obvious equality

$$\mathrm{M}_2(\mathbb{Z}_2) = U(\mathrm{M}_2(\mathbb{Z}_2)) \cup rmId(\mathrm{M}_2(\mathbb{Z}_2)) \cup \mathrm{Nil}(\mathrm{M}_2(\mathbb{Z}_2)),$$

, it is evident that $M_2(\mathbb{Z}_2)$ is a GSNC ring in conjunction with Proposition 4.5.

Corollary 4.11. *The ring* $\mathrm{M}_2(\mathbb{Z}_{2^k})$ *is a GSNC ring for each* $k \in \mathbb{N}$

Our next chief statement is:

Theorem 4.12. *Let R be a 2-primal, local and strongly nil-clean ring. Then,* $\mathrm{M}_2(R)$ *is a GSNC ring.*

Proof. Since R is simultaneously local and strongly nil-clean, we can write that $R/J(R) \cong \mathbb{Z}_2$, so Example 4.10 informs us that $\mathrm{M}_2(R/J(R))$ is a GSNC ring. On the other hand, since R is both 2-primal and strongly nil-clean, we may write $J(R) = \mathrm{Nil}(R) = \mathrm{Nil}_*(R)$, so that from this we extract that

$$\mathrm{M}_2(R/J(R)) = \mathrm{M}_2(R)/\mathrm{M}_2(J(R)) = \mathrm{M}_2(R)/\mathrm{M}_2(\mathrm{Nil}_*(R)) = \mathrm{M}_2(R)/\mathrm{Nil}_*(\mathrm{M}_2(R)),$$

and, moreover, as $\mathrm{Nil}_*(\mathrm{M}_2(R))$ is a nil-ideal, Proposition 4.7(i) forces that $\mathrm{M}_2(R)$ is a GSNC ring, as inferred. □

In the above theorem, the condition of being local is *not* redundant. In order to demonstrate that, supposing $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then R is 2-primal and strongly nil-clean, but definitely $\mathrm{M}_2(R)$ is *not* a GSNC ring as a plain check shows.

Besides, it is impossible to interchange the condition of being strongly nil-clean with GSNC in the above theorem. Indeed, to illustrate that such a replacement is really *not* possible, assuming $R = \mathbb{Z}_3$, then R is a 2-primal, GSNC, and local ring, but an easy verification shows that $\mathrm{M}_2(R)$ is *not* a GSNC ring.

Our next series of technical claims is like this.

Lemma 4.13. *Let* $\mathrm{M}_2(R)$ *be a GSNC ring. Then, R is a strongly nil-clean ring.*

Proof. Let us assume that $a \in R$. Then, one sees that

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \notin U(\mathrm{M}_2(R)).$$

Thus, referring to Proposition 4.5, we have $A - A^2 \in \mathrm{Nil}(\mathrm{M}_2(R))$ and, as a result, $a - a^2 \in \mathrm{Nil}(R)$. Therefore, consulting with [2, Theorem 5.1.1], we can conclude that R is a strongly nil-clean ring. □

Example 4.14. If R is a local ring with nil $J(R)$, then R is GSNC.

Proof. Given $a \in R$ and $a \notin U(R)$. Since R is local, $a \in J(R)$ and hence $a \in \mathrm{Nil}(R)$. So, a is a nilpotent element, and thus it is strongly nil-clean, as stated. □

Lemma 4.15. *Let R be a ring. Then, the following points are equivalent for a semi-local ring R:*
(i) R is a GSNC ring.
(ii) R is either a local ring with nil $J(R)$, or a strongly nil-clean ring, or $R/J(R) = \mathrm{M}_2(\mathbb{Z}_2)$ with $J(R)$ nil.

Proof. (i) $\Rightarrow$ (ii). Applying Proposition 4.7(ii), we have that $R/J(R)$ is a GSNC ring. Since R is a semi-local ring, we write $R/J(R) = \prod_{i=1}^{m} \mathrm{M}_{n_i}(D_i)$, where each component $\mathrm{M}_{n_i}(D_i)$ is a matrix ring over a division ring D_i. If, for a moment, $m = 1$, then combining Example 4.10 and Theorem 4.9, we infer that either $R/J(R) = D_1$ or $R/J(R) = \mathrm{M}_2(\mathbb{Z}_2)$.

If, however, $m > 1$, then employing Proposition 4.7(iii), each of the rings $\mathrm{M}_n(D)$ must be strongly nil-clean, which means that, for any i, the equality $\mathrm{M}_{n_i}(D_i) = \mathbb{Z}_2$ holds.
(ii) $\Rightarrow$ (i). If R is a strongly nil-clean ring, it is apparent that R is GSNC. If now R is local with nil $J(R)$, then R has to be GSNC invoking Example 4.14. □

Corollary 4.16. *Let R be a ring. Then, the following items are equivalent for a semi-simple ring R:*
(i) R is a GSNC ring.
(ii) R is either a division ring, or a strongly nil-clean ring, or $R = \mathrm{M}_2(\mathbb{Z}_2)$.

Proof. It is immediate by using Lemma 4.15. □

We now proceed by proving some structural results.

Proposition 4.17. *Let R be a ring. Then, the following two issues are equivalent for an abelian ring R:*
(i) R is a GSNC ring.
(ii) R is either a local with nil $J(R)$, or a strongly nil-clean ring.

Proof. It is similar to that of Lemma 2.10. □

The next claim also appeared in [9], but we will give it a bit more conceptual proof.

Proposition 4.18. *A ring R is strongly nil-clean if, and only if,*
(i) R is UU;
(ii) R is strongly clean.

Proof. "$\Rightarrow$". Let $u \in U(R)$. So, $a = e + q$, where $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$ with $eq = qe$. Thus, $e = u - q$, where $uq = qu$, and hence $e \in U(R)$. Therefore, $e = 1$ whence $u \in 1 + \mathrm{Nil}(R)$. That is why, R is a UU ring.
"$\Leftarrow$". Let $a \in R$. Write $a + 1 = e + u$, where $eu = ue$, $e \in \mathrm{Id}(R)$ and $u \in U(R)$. So, $a = e + (-1 + u)$, where $-1 + u \in \mathrm{Nil}(R)$ and $e(-1 + u) = (-1 + u)e$, guaranteeing that a is strongly nil-clean, as required. □

Lemma 4.19. [12] *A unit u of a ring R is strongly nil-clean if, and only if, $u \in 1 + \mathrm{Nil}(R)$. In particular, R is a UU ring if, and only if, every unit of R is strongly nil-clean.*

In our terminology alluded to above, we extract the following two assertions:

Corollary 4.20. *A ring R is strongly nil-clean if, and only if,*
(i) R is UU;
(ii) R is GSNC.

Proof. It follows directly from a combination of Proposition 4.18 and Lemma 4.19. □

Corollary 4.21. *Let R be a UU ring. Then, the following are equivalent:*
(i) R is a strongly clean ring.
(ii) R is a strongly nil-clean ring.
(iii) R is a GSNC ring.
(iv) R is a strongly π-regular ring.

Proposition 4.22. *A ring R is GUNC if, and only if,*
(i) R is abelian;
(ii) R is GSNC.

Proof. It follows immediately combining Lemma 2.10, Proposition 4.17 and [2, Theorem 5.3.3]. □

We call a ring an *NR* ring if its set of nilpotent elements forms a subring. Recall also that a ring R is called an *exchange* ring, provided that, for any a in R, there is an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$.

Lemma 4.23. *Let R be a NR ring. Then, the following two conditions are equivalent:*
(i) R is either local with nil $J(R)$, or R is strongly nil-clean.
(ii) R is a GSNC ring.

Proof. (i) $\Rightarrow$ (ii). It is clear by following Example 4.14.
(ii) $\Rightarrow$ (i). If R is a GSNC ring, then Lemma 4.2 applies to get that R is an exchange ring. Therefore, in virtue of [3, Corollary 2], $R/J(R)$ is an abelian ring. Consequently, Proposition 4.22 enables us that $R/J(R)$ is GUNC ring, which means that either $R/J(R)$ is local, or $R/J(R)$ is an UNC ring. Hence, the application of [24, Theorem 19] leads us to $R/J(R)$ is either a local ring, or a Boolean ring. Finally, inspired by [2, Theorem 5.1.5], we conclude that R is either a local ring, or is a strongly nil-clean ring. □

The next constructions are worthy of mentioning.

Example 4.24. (i) Any strongly nil-clean ring is GSNC, but the converse is *not* true in general. For instance, consider the ring $\mathbb{Z}_3$ which is GSNC, but is *not* strongly nil-clean.
(ii) Any GSNC ring is strongly clean, but the converse is *not* generally valid. For instance, the ring $\mathbb{Z}_2[[x]]$ is strongly clean, but is *not* GSNC.
(iii) Any GUNC ring is GSNC, but the converse is *not* fulfilled in generality. For instance, the ring $\mathrm{M}_2(\mathbb{Z}_2)$ is GSNC, but is *not* GUNC.

A ring R is said to be *semi-potent* if every one-sided ideal not contained in $J(R)$ contains a non-zero idempotent. Additionally, a semi-potent ring R is called *potent*, provided all of its idempotents lift modulo $J(R)$. Notice that semi-potent rings and potent rings were also named in [23] as I_0*-rings* and I*-rings*, respectively.

In the terminology we have introduced, we remember that the definitions of GUSC, GUC, GUNC and GSNC rings are given above.

Proposition 4.25. *Let R be a ring, $\mathrm{Id}(R) = \{0,1\}$ and $J(R)$ is nil. Then, the following are equivalent:*
(i) R is a local ring.
(ii) R is a GUSC ring.
(iii) R is a strongly clean ring.
(iv) R is a clean ring.
(v) R is an exchange ring.
(vi) R is a potent ring.
(vii) R is a semi-potent ring.
(viii) R is a GUC ring.
(ix) R is a GUNC ring.
(x) R is a GSNC ring.

Proof. (i) $\Rightarrow$ (ii). It follows from [10, Example 2.7].

(ii) $\Rightarrow$ (iii). It follows from [10, Lemma 3.2].

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii). These implications are pretty obvious, so leave all details to the interested reader.

(vii) $\Rightarrow$ (i). It is obvious since $\mathrm{Id}(R) = \{0,1\}$.

(i) $\Leftrightarrow$ (viii). It follows at once from [13, Proposition 2.9].

(viii) $\Leftrightarrow$ (ix). It follows directly from Example 2.11.

(ix) $\Leftrightarrow$ (x). It follows immediately from Proposition 4.22. □

Example 4.26. (i) For any ring R, the power series ring $R[[x]]$ is *not* GSNC.
(ii) If R is a ring, then the polynomial ring $R[x]$ is *not* GSNC.

Proof. (i) Note the principal fact that the Jacobson radical of $R[[x]]$ is *not* nil. Thus, in view of Lemma 4.4, $R[[x]]$ need *not* be a GSNC ring.
(ii) If we assume the contrary that $R[x]$ is GSNC, then $R[x]$ is clean in accordance with Lemma 4.2. This, however, contradicts [14, Example 2]. □

Example 4.27. Let R be a ring, and let

$$\mathrm{S}_n(R) = \{(a_{ij}) \in \mathrm{T}_n(R) \,|\, a_{11} = a_{22} = \cdots = a_{nn}\}\,.$$

Then, R is GSNC if, and only if, $\mathrm{S}_n(R)$ is GSNC for all $n \in \mathbb{N}$.

Proof. Given $\mathrm{S}_n(R)$ is GSNC. It is easy to see that R is a GSNC ring, so we omit the details.

Conversely, let R be GSNC. Thus, for any $(a_{ij}) \in \mathrm{S}_n(R)$, where $(a_{ij}) \notin U(\mathrm{S}_n(R))$, we see that $a_{11} = \cdots = a_{nn} \notin U(R)$. So, Proposition 4.5 allows us to infer that

$$a_{ii} - a_{ii}^2 \in \mathrm{Nil}(R)$$

for each i. Furthermore, write $(a_{ii} - a_{ii}^2)^m = 0$ for some $m \in \mathbb{N}$. Consequently, $\big((a_{ij}) - (a_{ij})^2\big)^{nm} = 0$. Now, according to Proposition 4.5, we infer that $\mathrm{S}_n(R)$ is GSNC, as stated. $\square$

Let R be a ring, and define the following rings thus:

$$\begin{aligned}
&\mathrm{A}_{n,m}(R) = R[x,y|x^n = xy = y^m = 0],\\
&\mathrm{B}_{n,m}(R) = R\left\langle x,y|x^n = xy = y^m = 0\right\rangle,\\
&\mathrm{C}_n(R) = R\langle x,y|x^2 = \underbrace{xyxyx...}_{n-1 \text{ words}} = y^2 = 0\rangle.
\end{aligned}$$

On the other vein, Wang introduced in [28] the matrix ring $\mathrm{S}_{n,m}(R)$ as follows: suppose R is a ring, then the matrix ring $\mathrm{S}_{n,m}(R)$ is representable like this

$$\left\{ \begin{pmatrix}
a & b_1 & \cdots & b_{n-1} & c_{1n} & \cdots & c_{1n+m-1}\\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & a & b_1 & c_{n-1,n} & \cdots & c_{n-1,n+m-1}\\
0 & \cdots & 0 & a & d_1 & \cdots & d_{m-1}\\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & \cdots & a & d_1\\
0 & \cdots & 0 & 0 & \cdots & 0 & a
\end{pmatrix} \in \mathrm{T}_{n+m-1}(R) : a, b_i, d_j, c_{i,j} \in R \right\}.$$

Also, let $\mathrm{T}_{n,m}(R)$ be

$$\left\{ \left(\begin{array}{ccccc|ccccc}
a & b_1 & b_2 & \cdots & b_{n-1} & & & & & \\
0 & a & b_1 & \cdots & b_{n-2} & & & & & \\
0 & 0 & a & \cdots & b_{n-3} & & & \mathbf{0} & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & & & & & \\
0 & 0 & 0 & \cdots & a & & & & & \\
\hline
 & & & & & a & c_1 & c_2 & \cdots & c_{m-1}\\
 & & & & & 0 & a & c_1 & \cdots & c_{m-2}\\
 & & \mathbf{0} & & & 0 & 0 & a & \cdots & c_{m-3}\\
 & & & & & \vdots & \vdots & \vdots & \ddots & \vdots\\
 & & & & & 0 & 0 & 0 & \cdots & a
\end{array}\right) \in \mathrm{T}_{n+m}(R) : a, b_i, c_j \in R \right\},$$

as well as we state

$$\mathrm{U}_n(R) = \left\{ \begin{pmatrix}
a & b_1 & b_2 & b_3 & b_4 & \cdots & b_{n-1}\\
0 & a & c_1 & c_2 & c_3 & \cdots & c_{n-2}\\
0 & 0 & a & b_1 & b_2 & \cdots & b_{n-3}\\
0 & 0 & 0 & a & c_1 & \cdots & c_{n-4}\\
\vdots & \vdots & \vdots & \vdots & & & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & a
\end{pmatrix} \in \mathrm{T}_n(R) : a, b_i, c_j \in R \right\}.$$

We now show in the following statement some of the existing relations between these rings.

Lemma 4.28. *Let R be a ring and $m, n \in \mathbb{N}$. Then, the following three isomorphisms hold:*
(i) $\mathrm{A}_{n,m}(R) \cong \mathrm{T}_{n,m}(R)$.
(ii) $\mathrm{B}_{n,m}(R) \cong \mathrm{S}_{n,m}(R)$.
(iii) $\mathrm{C}_n(R) \cong \mathrm{U}_n(R)$.

Proof. (i) We assume $f = a + \sum_{i=1}^{n-1} b_i x^i + \sum_{j=1}^{m-1} c_j y^j \in \mathrm{A}_{n,m}(R)$. We define $\varphi : \mathrm{A}_{n,m}(R) \to \mathrm{T}_{n,m}(R)$ as

$$\varphi(f) = \left(\begin{array}{ccccc|ccccc} a & b_1 & b_2 & \cdots & b_{n-1} & & & & & \\ 0 & a & b_1 & \cdots & b_{n-2} & & & & & \\ 0 & 0 & a & \cdots & b_{n-3} & & & \mathbf{0} & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & a & & & & & \\ \hline & & & & & a & c_1 & c_2 & \cdots & c_{m-1} \\ & & & & & 0 & a & c_1 & \cdots & c_{m-2} \\ & & \mathbf{0} & & & 0 & 0 & a & \cdots & c_{m-3} \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & 0 & \cdots & a \end{array}\right).$$

It can easily be shown that φ is a ring isomorphism, as required.

(ii) We assume $f \in \mathrm{B}_{n,m}(R)$ such that

$$\begin{aligned} f = {} & a_{00}y^0x^0 + a_{01}y^0x^1 + \cdots + a_{0,n-1}y^0x^{n-1} \\ & + a_{10}y^1x^0 + a_{11}y^1x^1 + \cdots + a_{1,n-1}y^1x^{n-1} \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & + a_{m-1,0}y^{m-1}x^0 + a_{m-1,1}y^{m-1}x^1 + \cdots + a_{m-1,n-1}y^{m-1}x^{n-1} \end{aligned}$$

We define $\psi : \mathrm{B}_{n,m}(R) \to \mathrm{S}_{n,m}(R)$ as

$$\psi(f) = \begin{pmatrix} a_{00} & a_{10} & \cdots & a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{00} & a_{10} & a_{11} & \cdots & a_{1,n-1} \\ 0 & \cdots & 0 & a_{00} & a_{01} & \cdots & a_{0,n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a_{00} & a_{0,1} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a_{00} \end{pmatrix}.$$

It can plainly be proved that ψ is an isomorphism, as needed.

(iii) We introduce the coefficients as follows:

$$f = \sum_{\substack{0 \le i_j \le 1 \\ 1 \le j \le n-1}} d_{(i_1,\ldots,i_{n-1})} \underbrace{y^{i_1}x^{i_2}y^{i_3}x^{i_4}\ldots}_{n-1 \text{ words}} \in \mathrm{C}_n(R)$$

We define $\phi : \mathrm{C}_n(R) \to \mathrm{S}_{n,m}(R)$ as

$$\phi(f) = \begin{pmatrix} d_{(0,0,0,\ldots,0)} & d_{(1,0,0,\ldots,0)} & d_{(1,1,0,\ldots,0)} & d_{(1,1,1,\ldots,0)} & \cdots & d_{(1,1,1,\ldots,1)} \\ 0 & d_{(0,0,0,\ldots,0)} & d_{(0,1,0,\ldots,0)} & d_{(0,1,1,\ldots,0)} & \cdots & d_{(0,1,1,\ldots,1)} \\ 0 & 0 & d_{(0,0,0,\ldots,0)} & d_{(1,0,0,\ldots,0)} & \cdots & d_{(1,\ldots,1,0,0)} \\ 0 & 0 & 0 & d_{(0,0,0,\ldots,0)} & \cdots & d_{(0,1,\ldots,1,0,0)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_{(0,0,0,\ldots,0)} \end{pmatrix}.$$

It can readily be checked that ϕ is an isomorphism, as asked for. $\square$

Our three concrete examples are the following.

Example 4.29. Letting R be a ring, then we have:

(i) $R\left[x, y | x^2 = xy = y^2 = 0\right] \cong \left\{ \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\}$.

(ii) $R\left\langle x, y | x^2 = xy = y^2 = 0 \right\rangle \cong \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\}$.

(iii) $R\left\langle x, y | x^2 = xyx = y^2 = 0 \right\rangle \cong \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_5 & a_6 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\} \cong \mathrm{T}(\mathrm{T}(R, R), \mathrm{M}_2(R))$.

Example 4.30. Let R be a ring. Then, the following statements are equivalent:
(i) R is a GSNC ring.
(ii) $\mathrm{S}_{n,m}(R)$ is a GSNC ring.
(iii) $\mathrm{T}_{n,m}(R)$ is a GSNC ring.
(iv) $\mathrm{U}_n(R)$ is a GSNC ring.

Proof. The proof is similar to that of Example 4.27, so remove the details. $\square$

Example 4.31. Let R be a ring. If $\mathrm{T}_n(R)$ is a GSNC ring, then R is GSNC. However, the converse is *not* true in general.

Proof. Choose $e = \mathrm{diag}(1, 0, \ldots, 0) \in \mathrm{T}_n(R)$. Then, one sees that $R \cong e\mathrm{T}_n(R)e$. Furthermore, with Corollary 4.8 at hand, we are done.

As for the converse, take $R = \mathbb{Z}_3$ and $S = \mathrm{T}_2(\mathbb{Z}_3)$. Clearly, R is a GSNC ring. But, an easy inspection leads to $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right) \notin U(S)$, and thereby A is definitely *not* a strongly nil-clean element in S, as required, whence S need *not* be GSNC. $\square$

Before stating and proving our next major result, the following two propositions are pretty welcome.

Proposition 4.32. *Let R be a GSNC ring. Then, for any $n > 2$, there does not exist $0 \neq e \in \mathrm{Id}(R)$ such that $eRe \cong \mathrm{M}_n(S)$ for some ring S.*

Proof. Let us assume the opposite, namely that there exists $0 \neq e \in \mathrm{Id}(R)$ such that $eRe \cong \mathrm{M}_n(S)$ for some ring S. Since R is, by assumption, GSNC, it follows from Corollary 4.8 that the corner subring eRe is GSNC too, and hence $\mathrm{M}_n(S)$ is GSNC as well. This, however, is a contradiction with Theorem 4.9. $\square$

Recall that a set $\{e_{ij} : 1 \leq i, j \leq n\}$ of non-zero elements of R is said to be a *system of n^2 matrix units* if $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{jj} = 1$ and $\delta_{js} = 0$ for $j \neq s$. In this case, $e := \sum_{i=1}^{n} e_{ii}$ is an idempotent of R and $eRe \cong \mathrm{M}_n(S)$, where

$$S = \{r \in eRe : re_{ij} = e_{ij}r, \text{for all} i, j = 1, 2, ..., n\}.$$

Proposition 4.33. *Every GSNC ring is directly finite.*

Proof. Suppose R is a GSNC ring. If we assume the reverse, namely that R is *not* a directly finite ring, then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Putting $e_{ij} := a^i(1 - ba)b^j$ and $e := \sum_{i=1}^{n} e_{ii}$, a routine verification shows that there will exist a non-zero ring S such that $eRe \cong \mathrm{M}_n(S)$. However, according to Corollary 4.8, the corner eRe is a GSNC ring, so that $\mathrm{M}_n(S)$ must also be a GSNC ring, thus contradicting Theorem 4.9. □

We now have all the machinery necessary to establish the following.

Theorem 4.34. *Let R be a ring. Then, the following equivalencies hold:*
(i) R is GSNC.
(ii) $\dfrac{R[[x]]}{\langle x^n \rangle}$ is GSNC for all $n \in \mathbb{N}$.
(iii) $\dfrac{R[[x]]}{\langle x^n \rangle}$ is GSNC for some $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Set $S := \dfrac{R[[x]]}{\langle x^n \rangle}$. Thus,

$$S = \{\sum_{i=0}^{n-1} r_i x^i \,|\, r_0, \ldots, r_{n-1} \in R,\ x^n = 0\}.$$

Let $r(x) = \sum_{i=0}^{n-1} r_i x^i \in S$, where $r(x) \notin U(S)$. On the other hand, we know that

$$U(S) = \{\sum_{i=0}^{n-1} r_i x^i \in S \,|\, r_0 \in U(R)\}.$$

We also see that

$$r(x) - r^2(x) = (r_0 - r_0^2) + bx$$

for some $b \in R$. As $r_0 - r_0^2 \in \mathrm{Nil}(R)$, we can find some $m \in \mathbb{N}$ such that $(r_0 - r_0^2)^m = 0$, and so

$$(r(x) - r^2(x))^{2m+1} = ((r_0 - r_0^2) + bx)^{2m+1} = 0.$$

Therefore, one infers that $r(x) - r^2(x) \in \mathrm{Nil}(S)$. Furthermore, we apply Proposition 4.5 to get the assertion.
(ii) $\Rightarrow$ (iii). It is trivial.
(iii) $\Rightarrow$ (i). For any $r \in R$, where $r \notin U(R)$, we see that $r \in S$ with $r \notin U(S)$, and thus $r - r^2 \in \mathrm{Nil}(S)$. This implies that $r - r^2 \in \mathrm{Nil}(R)$. Therefore, R is GSNC, as asserted. □

An immediate consequence is the one:

Corollary 4.35. *Let R be a ring. Then, the following are equivalent:*
(i) R is GSNC.
(ii) $\dfrac{R[x]}{\langle x^n \rangle}$ is GSNC for all $n \in \mathbb{N}$.
(iii) $\dfrac{R[x]}{\langle x^n \rangle}$ is GSNC for some $n \in \mathbb{N}$.

Furthermore, let A, B be two rings, and let M, N be the (A, B)-bi-module and (B, A)-bi-module, respectively. Also, we consider the bilinear maps $\phi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ that apply to the following properties

$$\mathrm{Id}_M \otimes_B \psi = \phi \otimes_A \mathrm{Id}_M, \quad \mathrm{Id}_N \otimes_A \phi = \psi \otimes_B \mathrm{Id}_N.$$

For $m \in M$ and $n \in N$, we define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Thus, the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring equipped with the obvious matrix operations, which is called a *Morita context ring*. Denote the two-sided ideals $\mathrm{Im}\phi$ and $\mathrm{Im}\psi$ to MN and NM, respectively, that are called the *trace ideals* of the Morita context.

We now have all the ingredients needed to prove the following.

Theorem 4.36. *Let $R = \left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ be a Morita context such that MN and NM are nilpotent ideals of A and B, respectively. Then, R is GSNC if, and only if, both A and B are strongly nil-clean.*

Proof. Assume R is a GSNC ring. We show that A is a strongly nil-clean ring and, similarly, it can be shown that B is also a strongly nil-clean ring. To this goal, let us assume $a \in A$; then an elementary check gives that $C = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \notin U(R)$, so Proposition 4.5 yields $C - C^2 \in \mathrm{Nil}(R)$, that is, $a - a^2 \in \mathrm{Nil}(A)$. So, A is a strongly nil-clean ring. The converse implication can be obtained by [18, Theorem 3.4]. □

Now, let R, S be two rings, and let M be an (R, S)-bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-module M, we can set

$$\mathrm{T}(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it forms a ring with the usual matrix operations. The so-stated formal matrix $\mathrm{T}(R, S, M)$ is called a *formal triangular matrix ring*. In Theorem 4.36, if we set $N = \{0\}$, then we will obtain the following two corollaries.

Corollary 4.37. *Let R, S be rings and let M be an (R, S)-bi-module. Then, the formal triangular matrix ring $\mathrm{T}(R, S, M)$ is GSNC if, and only if, both A and B are strongly nil-clean.*

Corollary 4.38. *Let R be a ring and $n \geqslant 1$ is a natural number. Then, $\mathrm{T}_n(R)$ is GSNC if, and only if, R is strongly nil-clean.*

Given now a ring R and a central element s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition defined componentwise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This ring is denoted by $\mathrm{K}_s(R)$. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring* over R. It was observed in [19] that a ring S is a generalized matrix ring over R if, and only if, $S = \mathrm{K}_s(R)$ for some $s \in \mathrm{Z}(R)$, the center of R. Here $MN = NM = sR$, so that

$$MN \subseteq J(A) \Longleftrightarrow s \in J(R), NM \subseteq J(B) \Longleftrightarrow s \in J(R),$$

and MN, NM are nilpotent $\Longleftrightarrow$ s is a nilpotent. Thus, Theorem 4.36 has the following consequence, too.

Corollary 4.39. *Let R be a ring and $s \in \mathrm{Z}(R) \cap \mathrm{Nil}(R)$. Then, $\mathrm{K}_s(R)$ is GSNC if, and only if, R is strongly nil-clean.*

Following Tang and Zhou (cf. [27]), for $n \geq 2$ and for $s \in \mathrm{Z}(R)$, the $n \times n$ formal matrix ring over R defined with the help of s, and denoted by $\mathrm{M}_n(R; s)$, is the set of all $n \times n$ matrices over R with usual addition of matrices and with multiplication defined below:
For (a_{ij}) and (b_{ij}) in $\mathrm{M}_n(R; s)$, set

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where } (c_{ij}) = \sum s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, where δ_{jk}, δ_{ij}, δ_{ik} are the standard *Kroncker delta* symbols.

Thereby, we arrive at the following.

Corollary 4.40. *Let R be a ring and $s \in \mathrm{Z}(R)$. If $\mathrm{M}_n(R; s)$ is GSNC, then R is GSNC and $s \in J(R)$. The converse holds provided R is strongly nil-clean and $s \in \mathrm{Nil}(R)$.*

Let R be a ring and M a bi-module over R. The *trivial extension* of R and M is defined as

$$T(R, M) = \{(r, m) : r \in R \text{and} m \in M\},$$

with addition defined componentwise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

Notice that the trivial extension $\mathrm{T}(R, M)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M \right\}$$

consisting of the formal 2×2 matrix rings $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and, in particular, the isomorphism $\mathrm{T}(R, R) \cong R[x]/\langle x^2 \rangle$ is fulfilled. We also note that the set of units of the trivial extension $\mathrm{T}(R, M)$ is

$$U(\mathrm{T}(R, M)) = \mathrm{T}(U(R), M).$$

A Morita context is referred to as *trivial* if the context products are trivial, meaning that $MN = \{0\}$ and $NM = \{0\}$ (see, for instance, [21, p. 1993]). In this case, we have the isomorphism

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ represents a trivial Morita context, as stated in [17].

We, thus, come to the following symmetric relationship.

Lemma 4.41. *Let R be a ring and M a bi-module over R. Then,*

$$\mathrm{Nil}(\mathrm{T}(R, M)) = \mathrm{T}(\mathrm{Nil}(R), M).$$

Proof. It is technically straightforward, so we drop off the full details leaving them to the interested reader for an inspection. $\square$

A good information gives also the following necessary and sufficient condition.

Proposition 4.42. *Let R be a ring and M a bi-module over R. Then,* $\mathrm{T}(R, M)$ *is GSNC if, and only if, R is GSNC.*

Proof. **Method 1:** Assuming $I = (0, M)$, then clearly I is a nil-ideal of the ring $\mathrm{T}(R, M)$. Moreover, since the isomorphism $R \cong \mathrm{T}(R, M)/I$ is true, Proposition 4.7(i) employs to get the claim.

Method 2: Let $\mathrm{T}(R, M)$ be a GSNC ring and $a \notin U(R)$. Then, one verifies that $(a, 0) \notin U(\mathrm{T}(R, M))$, so Proposition 4.5 applies to detect that $(a, 0) - (a, 0)^2 \in \mathrm{Nil}(\mathrm{T}(R, M))$, hence $a - a^2 \in \mathrm{Nil}(R)$, as required.

Conversely, assuming R is a GSNC ring and $(a, m) \notin U(\mathrm{T}(R, M))$, we derive $a \notin U(R)$. Consequently, $a - a^2 \in \mathrm{Nil}(R)$, implying

$$(a, m) - (a, m)^2 \in \mathrm{Nil}(\mathrm{T}(R, M)),$$

as needed. $\square$

The next criterion is also worthy of documentation.

Corollary 4.43. *Let $R = \left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ be a trivial Morita context. Then, R is GSNC if, and only if, both A and B are strongly nil-clean.*

Proof. It is straightforward bearing in mind Propositions 4.42 and 4.7(iii). $\square$

Likewise, we can derive the following:

Corollary 4.44. *Let R be a ring and M a bi-module over R. Then, the following four statements are equivalent:*

(i) *R is a GSNC ring.*
(ii) $\mathrm{T}(R, M)$ *is a GSNC ring.*
(iii) $\mathrm{T}(R, R)$ *is a GSNC ring.*
(iv) $\dfrac{R[x]}{\langle x^2 \rangle}$ *is a GSNC ring.*

Now, consider R to be a ring and M to be a bi-module over R. Let

$$\mathrm{DT}(R, M) := \{(a, m, b, n) | a, b \in R, m, n \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1a_2, a_1m_2+m_1a_2, a_1b_2+b_1a_2, a_1n_2+m_1b_2+b_1m_2+n_1a_2).$$

Then, $\mathrm{DT}(R, M)$ is a ring which is isomorphic to the ring $\mathrm{T}(\mathrm{T}(R, M), \mathrm{T}(R, M))$. Also, we have

$$\mathrm{DT}(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} | a, b \in R, m, n \in M \right\}.$$

Besides, we also have the following isomorphism as rings: $\dfrac{R[x, y]}{\langle x^2, y^2 \rangle} \to \mathrm{DT}(R, R)$ defined by

$$a + bx + cy + dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, detect the following.

Corollary 4.45. *Let R be a ring and M a bi-module over R. Then, the following statements are equivalent:*

(i) *R is a GSNC ring.*
(ii) *$\mathrm{DT}(R, M)$ is a GSNC ring.*
(iii) *$\mathrm{DT}(R, R)$ is a GSNC ring.*
(iv) *$\dfrac{R[x, y]}{\langle x^2, y^2 \rangle}$ is a GSNC ring.*

Let α be an endomorphism of R and n a positive integer. It was introduced by Nasr-Isfahani in [22] the *skew triangular matrix ring* like this:

$$\mathrm{T}_n(R, \alpha) = \left\{ \left. \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \right| a_i \in R \right\}$$

with addition defined point-wise and multiplication given by:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} =$$

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ 0 & c_0 & c_1 & \cdots & c_{n-2} \\ 0 & 0 & c_0 & \cdots & c_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix},$$

where

$$c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_i),\ 1 \le i \le n-1.$$

We denote the elements of $\mathrm{T}_n(R,\alpha)$ by $(a_0, a_1, \ldots, a_{n-1})$. If α is the identity endomorphism, then obviously $\mathrm{T}_n(R,\alpha)$ is a subring of upper triangular matrix ring $\mathrm{T}_n(R)$.

We now establish the validity of the following.

Corollary 4.46. *Let R be a ring. Then, the following are equivalent:*

(i) *R is a GSNC ring.*

(ii) *$\mathrm{T}_n(R,\alpha)$ is a GSNC ring.*

Proof. Choose

$$I := \left\{ \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \middle| \, a_{ij} \in R \quad (i \le j) \right\}.$$

Therefore, one easily finds that $I^n = (0)$ and $\dfrac{\mathrm{T}_n(R,\alpha)}{I} \cong R$. Consequently, one verifies that Proposition 4.7 is applicable to get the pursued result. $\square$

5. GSNC group rings

As usual, for an arbitrary ring R and an arbitrary group G, the symbol RG stands for the *group ring* of G over R. Standardly, $\Delta(RG)$ designates the kernel of the classical augmentation map $RG \to R$.

We begin here with the following technicality.

Lemma 5.1. *Let $\phi : R \to S$ be a non-zero epimorphism of rings with* $\mathrm{Ker}(\phi) \cap \mathrm{Id}(R) = \{0\}$*. Then, R is a GSNC ring if, and only if, S is a GSNC ring and* $\mathrm{Ker}(\phi)$ *is a nil-ideal of S.*

Proof. Suppose R is a GSNC ring and $a \in \mathrm{Ker}(\phi)$. Thus, $a \notin U(R)$, so that there exist $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$ with $a = e + q$ and $eq = qe$. That is why, $0 = \phi(a) = \phi(e) + \phi(q)$, yielding $\phi(e) \in \mathrm{Id}(S) \cap \mathrm{Nil}(S) = \{0\}$. This unambiguously shows that $e \in \mathrm{Id}(R) \cap \mathrm{Ker}(\phi) = \{0\}$, hence $a = q \in \mathrm{Nil}(R)$.

Next, since ϕ is an epimorphism, we have $S \cong R/\mathrm{Ker}(\phi)$ and, in conjunction with Proposition 4.7(i), we conclude that R is a GSNC ring.

The converse relation can easily be extracted from Proposition 4.7(i). □

The following three lemmas are also useful for further applications.

Lemma 5.2. *Let R be a ring and let G be a group, where $\Delta(RG) \cap \mathrm{Id}(RG) = \{0\}$. Then, RG is a GSNC ring if, and only if, R is a GSNC ring and $\Delta(RG)$ is a nil-ideal of RG.*

Proof. There is an epimorphism $\varepsilon : RG \to R$ with $\mathrm{Ker}(\varepsilon) = \Delta(RG)$. □

Lemma 5.3. [29, Lemma 2]. *Let p be a prime with $p \in J(R)$. If G is a locally finite p-group, then $\Delta(RG) \subseteq J(RG)$.*

Lemma 5.4. *Let R be a ring and let G be a locally finite p-group, where p is a prime and $p \in J(R)$. Then, RG is a GSNC ring if, and only if, R is a GSNC ring and $\Delta(RG)$ is a nil-ideal of RG.*

In regard to the last lemma, an important question which could be raised is to find, as in the case of GUNC rings, a suitable criterion when a group ring RG of a locally finite p-group G over an arbitrary ring R to be a GSNC ring. In other words, is the restriction $p \in J(R)$ necessary in this claim and whether it could be deduced from the condition RG is GSNC?

6. Open Questions

We close the work with the following challenging problems.

A ring R is said to be *weakly nil-clean* provided that, for any $a \in R$, there exists an idempotent $e \in R$ such that $a - e$ or $a + e$ is nilpotent. Additionally, a ring R is said to be *strongly weakly nil-clean* provided $ae = ea$ or, equivalently, provided that, for any $a \in R$, at least one of the elements a or $-a$ is strongly nil-clean (see, e.g., [2, 11]).

We now can formulate the following.

Problem 6.1. Examine those rings whose non-invertible elements are (strongly) weakly nil-clean.

A ring R is called *strongly 2-nil-clean* if every element in R is the sum of two idempotents and a nilpotent that commute each other (see, for example, [2]). These rings are a common generalization of the aforementioned strongly weakly nil-clean rings.

Now, we may raise the following.

Problem 6.2. Examine those rings whose non-invertible elements are strongly 2-nil-clean.

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