

# STRINGS IN METRIC SPACES

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ABSTRACT. We introduce strings in metric spaces and define string complexes of metric spaces. We describe the class of 2-dimensional topological spaces which arise in this way from finite metric spaces.

## 1. INTRODUCTION

Classical constructions due to Čech and Vietoris-Rips associate certain simplicial complexes with metric spaces. The study of these complexes yields various invariants of metric spaces including their Čech and Vietoris-Rips homology. This area of research has been active recently in the context of topological data analysis, see [JJ], [RB].

We discuss a different method deriving a simplicial complex from a metric space. This method is based on the study of points for which the triangle inequality is an equality. Our construction is formulated in terms of so-called strings which we view as metric analogues of straight segments. We call the resulting simplicial complexes the string complexes. Homology of the string complex may be a useful invariant of the original metric space.

Our main result yields necessary and sufficient conditions on a 2-dimensional topological space to be homeomorphic to the string complex of a finite metric space. In particular, we show that all compact surfaces arise in this way.

For the sake of generality, we define and study strings in so-called gap spaces which generalize metric spaces by dropping all conditions on the metric except the triangle inequality.

## 2. STRINGS IN GAP SPACES

**2.1. Gap spaces.** Given a set  $X$ , we call a mapping  $d : X \times X \rightarrow \mathbb{R}$  a *gap function* if it obeys the triangle inequality:  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ . Such a pair  $(X, d)$  is a *gap space*, and the number  $d(x, y)$  is the *gap* between  $x, y \in X$ . A repeated application of the triangle inequality shows that for any  $n \geq 3$  points  $x_1, x_2, \dots, x_n \in X$  we have the *n-gon inequality*

$$(2.1.1) \quad d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n) \geq d(x_1, x_n).$$

A standard source of gap functions are road maps with some of the roads one-way and other roads two-way. Here  $X$  is the set of all crossings of the roads and the gap between any two crossings is the length of the shortest path leading from the first crossing to the second. We have to require that the map allows to travel from any crossing to any other crossing along the roads.

**2.2. Strings.** A *string of length*  $n \geq 3$  in a gap space  $(X, d)$  is an  $n$ -element set  $S \subset X$  whose elements can be ordered  $x_1, x_2, \dots, x_n$  so that (2.1.1) is an equality:

$$(2.2.1) \quad d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n) = d(x_1, x_n).$$

Any order in  $S$  satisfying (2.2.1) is said to be *direct*. A string endowed with a direct order is *ordered*.

It is convenient to extend these notions to 1-element and 2-element subsets of the gap space  $X$ . For  $x \in X$ , a 1-element set  $\{x\}$  is a *string of length 1* if  $x$  belongs to a string of length 3 in  $X$ . The only order in the set  $\{x\}$  is direct. Similarly, a 2-element subset of  $X$  is a *string of length 2* if it is contained (as a subset) in a string of length 3 in  $X$ . An order in a string of length 2 is *direct* if it extends to a direct order in a string of length 3. (We agree that the order  $x, y, z$  in the 3-element set  $\{x, y, z\}$  is an extension of the order  $x, y$  in the set  $\{x, y\}$ , the order  $x, z$  in the set  $\{x, z\}$ , and the order  $y, z$  in the set  $\{y, z\}$ .)

**Lemma 2.1.** *Each non-void subset of a string is itself a string (called a substring). A direct order in a string restricts to a direct order in each substring.*

*Proof.* Let  $(X, d)$  be a gap space. Let  $S \subset X$  be a string of length  $n$  with direct order  $S = \{x_1, \dots, x_n\}$ . For  $n = 1, 2, 3$ , both claims of the lemma concerning the subsets of  $S$  follow directly from the definitions. Assume that  $n \geq 4$ . We have

$$(2.2.2) \quad d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \\ \geq d(x_1, x_{n-1}) + d(x_{n-1}, x_n) \geq d(x_1, x_n)$$

where we use the  $(n-1)$ -gon inequality and the triangle inequality. Since  $x_1, \dots, x_n$  is a direct order in  $S$ , Formula (2.2.1) implies that both inequalities in (2.2.2) are equalities. The first of these equalities implies that the set  $\{x_1, \dots, x_{n-1}\}$  is a string with direct order  $x_1, \dots, x_{n-1}$ . Similarly, we have

$$d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n) \geq d(x_1, x_2) + d(x_2, x_n) \geq d(x_1, x_n).$$

Again, Formula (2.2.1) implies that both these inequalities are equalities. The first of them means that the set  $\{x_2, \dots, x_n\}$  is a string with direct order  $x_2, \dots, x_n$ . Now, pick any  $j \in \{2, \dots, n-1\}$ . We have

$$d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n) \\ = \sum_{k=1}^{j-2} d(x_k, x_{k+1}) + d(x_{j-1}, x_j) + d(x_j, x_{j+1}) + \sum_{k=j+1}^{n-1} d(x_k, x_{k+1}) \geq \\ \geq \sum_{k=1}^{j-2} d(x_k, x_{k+1}) + d(x_{j-1}, x_{j+1}) + \sum_{k=j+1}^{n-1} d(x_k, x_{k+1}) \geq d(x_1, x_n)$$

where we use the triangle inequality and the  $(n-1)$ -gon inequality. Since  $x_1, \dots, x_n$  is a direct order in  $S$ , both inequalities here have to be equalities. The second of them shows that the set  $\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$  is a string with direct order  $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ . This proves both claims of the lemma for all subsets of  $S$  having  $n-1$  elements. Proceeding by induction, we obtain both claims for all non-void subsets of  $S$ .  $\square$

The gaps between the points of an ordered string  $x_1, \dots, x_n$  of length  $n \geq 3$  can be computed by the formula

$$(2.2.3) \quad d(x_i, x_j) = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \dots + d(x_{j-1}, x_j)$$

for any  $1 \leq i < j \leq n$ . Indeed, by Lemma 2.1, the sequence  $x_i, x_{i+1}, \dots, x_j$  is an ordered string. Hence, we have (2.2.3).

**2.3. The case of metric spaces.** More can be said about strings in metric spaces. A gap space  $(X, d : X \times X \rightarrow \mathbb{R})$  is a metric space (and  $d$  is a metric) iff  $d(x, y) = d(y, x) \geq 0$  for all  $x, y \in X$  and  $d(x, y) = 0 \iff x = y$ . If  $x_1, \dots, x_n$  is a direct order in a string in a metric space  $X$  with  $n \geq 2$  then the opposite order  $x_n, x_{n-1}, \dots, x_1$  in the same string also is direct. Formula (2.2.3) implies that  $x_1, x_n$  are the points of this string with the greatest distance between them. Therefore the unordered pair of points  $x_1, x_n$  depends only on the string and not on the direct order in it. The points  $x_1, x_n$  are called the *endpoints* of the string. Using these observations and the induction on  $n$ , one easily concludes that in a metric space, every string of length  $\geq 2$  has exactly two direct orders, and they are opposite to each other.

Strings in metric spaces are unstable in the sense that a slight deformation of the metric can destroy strings. For example, increasing all distances between different points by the same positive number we obtain a new metric having no strings. On the other hand, every finite metric space can be *trimmed* (see [Tu1], [Tu2]) to produce a metric space  $(X, d)$  with the following property: for any  $y \in X$  there are  $x, z \in X \setminus \{y\}$  such that  $x \neq z$  and  $d(x, y) + d(y, z) = d(x, z)$ . Consequently, all points of  $X$  lie on strings.

### 3. THE STRING SPACE

**3.1. Basics.** Recall that an abstract simplicial complex  $C$  is a collection of non-empty finite subsets of a certain set such that for every element of the collection  $C$ , all its non-empty subsets also belong to  $C$ . By Lemma 2.1, the collection of strings in any gap space  $(X, d)$  is an abstract simplicial complex. We denote its geometric realization by  $|X, d|$ . This is a topological space glued from simplexes numerated by the strings in  $X$ . The simplex numerated by a string of length  $n \geq 1$  has  $n$  vertexes and dimension  $n - 1$ . The simplexes corresponding to smaller strings are faces of the simplexes corresponding to bigger strings. We call  $|X, d|$  the *string space* of  $(X, d)$ . By the very definition, the space  $|X, d|$  admits a triangulation whose all vertexes (0-simplexes) and edges (1-simplexes) are faces of certain 2-simplexes. The usual techniques of algebraic topology - homology, cohomology, homotopy groups etc. - apply to the space  $|X, d|$  and produce interesting algebraic objects associated with the gap space  $(X, d)$ . In particular, all these constructions apply to metric spaces.

**3.2. Examples.** 1. Let a metric  $d$  in a set  $X$  be defined by  $d(x, y) = 1$  for all  $x \neq y$  and  $d(x, x) = 0$  for all  $x \in X$ . Then  $(X, d)$  has no strings and  $|X, d| = \emptyset$ .

2. Let  $X \subset \mathbb{R}$  be a finite set of real numbers with metric induced by the standard metric in  $\mathbb{R}$ . Suppose that  $\text{card}(X) \geq 3$ . Then all non-void subsets of  $X$  are strings. The string space of  $X$  is a simplex of dimension  $\text{card}(X) - 1$ .

3. Let  $X \subset \mathbb{R}^2$  be a set consisting of  $m \geq 3$  points lying on a straight line in  $\mathbb{R}^2$  and  $n \geq 3$  points lying on a parallel line. The metric in  $X$  is induced by the Euclidean metric in  $\mathbb{R}^2$ . Then the string space of  $X$  is the disjoint union of two simplexes, one of dimension  $m - 1$  and one of dimension  $n - 1$ .

4. Consider a convex polygon  $P \subset \mathbb{R}^2$  with  $k \geq 3$  vertexes and all interior angles  $< 180^\circ$ . Consider a finite set  $X \subset P$  including all vertexes of  $P$  and at least one point inside each edge of  $P$ . The metric in  $X$  is the usual Euclidean metric. Then the string space  $|X|$  is formed by  $k$  simplexes  $T_1, \dots, T_k$ . If  $\{e_i \mid i \in \mathbb{Z}/k\mathbb{Z}\}$  are the cyclically numerated edges of  $P$  then  $\dim(T_i) = \text{card}(X \cap e_i) - 1 \geq 2$ . For all  $i$ , the simplexes  $T_i, T_{i+1}$  meet in one common vertex; otherwise, the simplexes  $T_1, \dots, T_k$  do not meet. Note that the string space  $|X|$  is homotopy equivalent to the circle.

5. Let  $X \subset \mathbb{R}^2$  be a 4-element set formed by the vertexes of a planar rectangle. To define a metric in  $X$ , we inscribe  $X$  in a circle in  $\mathbb{R}^2$  and assign to any two points of  $X$  the length of the shortest arc which they bound in this circle. It is easy to see that all non-void subsets of  $X$  are strings except  $X$  itself. The string space of  $X$  is the boundary of a 3-simplex, i.e., a topological (and piecewise linear) 2-sphere.

**3.3. Remark.** For any pair of distinct points  $x, y$  of a metric space  $(X, d)$ , we can consider a subcomplex of the string complex  $|X, d|$  formed by simplexes corresponding to strings with endpoints  $x, y$ . Homology of this subcomplex yields interesting algebraic data associated with the metric space  $(X, d)$  and its points  $x, y$ .

#### 4. TWO-DIMENSIONAL STRING SPACES

What topological spaces arise as the string spaces of finite metric spaces? By definition, such a topological space has a finite triangulation whose all vertexes and edges are vertexes and edges of 2-simplexes. The author does not know whether this condition is sufficient (conjecturally, not). The next theorem - our main result - establishes sufficiency of this condition for 2-dimensional spaces.

**Theorem 4.1.** *Let  $T$  be a topological space which has a finite 2-dimensional triangulation whose every vertex is a vertex of a 2-simplex and every edge is an edge of a 2-simplex. Then there exists a finite metric space whose string space is homeomorphic to  $T$ .*

*Proof.* We first pick any real numbers  $k > 0$  and  $u, v \in (k/2, k)$ . Pick a triangulation  $\tau$  of  $T$  as in the assumptions of the theorem. Let  $X$  be the set of simplexes of  $\tau$ . Clearly, there is a unique function  $d : X \times X \rightarrow \mathbb{R}$  such that:

- $d$  is symmetric, i.e.,  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- $d(x, x) = 0$  for all  $x \in X$ ;
- $d(x, y) = k$  for all  $x, y \in X$  with  $x \not\subset y$  (i.e.,  $x$  not a face of  $y$ ) and  $y \not\subset x$ ;
- $d(x, y) = u$  for any 1-simplex  $y \in X$  and any vertex  $x$  of  $y$ ;
- $d(x, y) = v$  for any 2-simplex  $y \in X$  and any edge  $x$  of  $y$ ;
- $d(x, y) = u + v$  for any 2-simplex  $y \in X$  and any vertex  $x$  of  $y$ .

The last three conditions imply that the restriction of  $d$  to any subset of  $X$  consisting of three simplexes  $x \subset y \subset z$  of dimensions 0, 1, 2 respectively is isometric to the restriction of the standard metric in  $\mathbb{R}$  to the set  $\{0, u, u + v\} \subset \mathbb{R}$ .

The function  $d$  obviously satisfies all axioms of a metric except possibly the triangle inequality which needs to be verified. We check now that  $d(x, y) + d(y, z) \geq d(x, z)$  for any  $x, y, z \in X$ . If  $x = y$  or  $x = z$  or  $y = z$ , then this inequality holds because  $d \geq 0$  and  $d(x, x) = 0$  for all  $x$ . Assume from now on that  $x, y, z$  are three different simplexes of the triangulation  $\tau$ . If  $x \subset y$  (i.e., if  $x$  is a face of  $y$ ) and  $y \subset z$  then necessarily  $\dim(x) = 0, \dim(y) = 1, \dim(z) = 2$  and

$$d(x, y) = u, \quad d(y, z) = v, \quad d(x, z) = u + v.$$

In this case the triangle inequality is an equality. The same is true if  $x \supset y \supset z$ .

We claim that in all other cases  $d(x, y) + d(y, z) > d(x, z)$ . The symmetry of  $d$  ensures that exchanging  $x$  and  $z$  we get an equivalent inequality. Therefore it is enough to treat the case  $\dim(x) \leq \dim(z)$  and  $x \neq y \neq z \neq x$ . By the definition of  $d$ , the numbers  $d(x, y), d(y, z), d(x, z)$  belong to the set  $\{u, v, u + v, k\}$ . If  $x$  is not a face of  $z$ , then  $d(x, z) = k$ . We have  $d(x, y) + d(y, z) > k$  because in the set  $\{u, v, u + v, k\}$  the sum of any two terms is strictly bigger than  $k$  (as  $u, v > k/2$ ). For the rest of the argument we assume that  $x$  is a face of  $z$ . We have several cases to consider.

If  $x$  is a vertex of a 1-simplex  $z$ , then the sum of any two terms in the list  $\{u, v, u + v, k\}$  being strictly bigger than  $k$  is also strictly bigger than  $d(x, z) = u$ . If  $x$  is an edge of a 2-simplex  $z$ , then, similarly, the sum of any two terms in the set  $\{u, v, u + v, k\}$  is strictly bigger than  $d(x, z) = v$ . Suppose now that  $x$  is a vertex of a 2-simplex  $z$ . The case  $x \subset y \subset z$  being already considered, we suppose that  $x \not\subset y$  or  $y \not\subset z$  or both. If  $x \not\subset y \subset z$ , then either  $y$  is a vertex of  $z$  different from  $x$  or  $y$  is an edge of  $z$  opposite to  $x$ . In the first case

$$d(x, y) + d(y, z) = k + u + v > u + v = d(x, z).$$

In the second case

$$d(x, y) + d(y, z) = k + v > u + v = d(x, z).$$

If  $x \subset y \not\subset z$ , then either  $\dim(y) = 1$  and then

$$d(x, y) + d(y, z) = u + k > u + v = d(x, z)$$

or  $\dim(y) = 2$  and then

$$d(x, y) + d(y, z) = u + v + k > u + v = d(x, z).$$

If  $x \not\subset y \not\subset z$ , then

$$d(x, y) + d(y, z) = 2k > u + v = d(x, z).$$

We conclude that  $(X, d)$  is a metric space. Moreover, the arguments above show that the only strings of length 3 in  $X$  are the triples consisting of a 2-simplex of  $\tau$ , an edge of this 2-simplex, and a vertex of this edge. The direct order in such a string holds the edge as the middle term of the triple. This easily implies that there are no strings of length  $> 3$  in  $X$ . The triples as above bijectively correspond to the 2-simplexes of the barycentric subdivision of  $\tau$ . The string space  $|X, d|$  is obtained by gluing the 2-simplexes numerated by those triples. These gluings are the same as the gluings of the 2-simplexes in the barycentric subdivision of  $\tau$ . Consequently, the space  $|X, d|$  is homeomorphic to the space  $T$ . Under this homeomorphism the natural triangulation of  $|X, d|$  corresponds to the barycentric subdivision of  $\tau$ .  $\square$

**Corollary 4.2.** *Every compact surface (possibly, non-connected and/or with boundary) is homeomorphic to the string space of a finite metric space.*

It is interesting to compute for every compact surface  $T$  the minimal number  $n(T)$  of points in a metric space whose string space is homeomorphic to  $T$ . The proof of Theorem 4.1 shows that  $n(T)$  is smaller than or equal to the minimal total number of simplexes in a triangulation of  $T$ . This estimate seems to be rather weak. For example, the minimal triangulation of the 2-sphere  $S^2$  has 14 simplexes: 4 of dimension zero, 6 of dimension one, and 4 of dimension two. At the same time, Example 3.2.5 shows that  $n(S^2) \leq 4$ . It is straightforward to prove that  $n(S^2) = 4$ .

5.  $\varepsilon$ -STRINGS AND  $\varepsilon$ -STRING SPACES

In this section we let  $\varepsilon$  be any non-negative real number. In generalization of strings and string spaces we introduce  $\varepsilon$ -strings and  $\varepsilon$ -string spaces.

**5.1.  $\varepsilon$ -strings.** For  $n \geq 3$ , an  $\varepsilon$ -string of length  $n$  in a gap space  $(X, d)$  is an  $n$ -element set  $S \subset X$  whose elements can be ordered  $x_1, x_2, \dots, x_n$  so that

$$(5.1.1) \quad d(x_1, x_n) + \varepsilon \geq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n).$$

Any order in  $S$  satisfying (5.1.1) is said to be *direct*. An  $\varepsilon$ -string endowed with a direct order is *ordered*. These notions extend to 1-element and 2-element subsets of  $X$  as in Section 2. For  $x \in X$ , a 1-element set  $\{x\}$  is a  $\varepsilon$ -string of length 1 if  $x$  belongs to an  $\varepsilon$ -string of length 3. The only order in the set  $\{x\}$  is direct. A 2-element subset of  $X$  is an  $\varepsilon$ -string of length 2 if it is contained (as a subset) in an  $\varepsilon$ -string of length 3. An order in an  $\varepsilon$ -string of length 2 is *direct* if it extends to a direct order in an  $\varepsilon$ -string of length 3.

Note that the right-hand side of Formula (5.1.1) is always greater than or equal to  $d(x_1, x_n)$ . Therefore for  $\varepsilon = 0$ , this formula is equivalent to (2.2.1) and we recover the notion of a string from Section 2.

**Lemma 5.1.** *All non-void subsets of an  $\varepsilon$ -string are  $\varepsilon$ -strings (called  $\varepsilon$ -substrings). A direct order in an  $\varepsilon$ -string restricts to a direct order in each  $\varepsilon$ -substring.*

*Proof.* Let  $(X, d)$  be a gap space. Let  $S \subset X$  be an  $\varepsilon$ -string of length  $n$  with direct order  $S = \{x_1, \dots, x_n\}$ . For  $n = 1, 2, 3$ , both claims of the lemma concerning the subsets of  $S$  follow directly from the definitions. Assume that  $n \geq 4$ . We have

$$(5.1.2) \quad \begin{aligned} d(x_1, x_{n-1}) + d(x_{n-1}, x_n) + \varepsilon &\geq d(x_1, x_n) + \varepsilon \geq \\ &\geq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \end{aligned}$$

where we use the triangle inequality and the definition of an  $\varepsilon$ -string. Cancelling  $d(x_{n-1}, x_n)$ , we get

$$d(x_1, x_{n-1}) + \varepsilon \geq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-2}, x_{n-1}).$$

Thus, the set  $\{x_1, \dots, x_{n-1}\}$  is an  $\varepsilon$ -string with direct order  $x_1, \dots, x_{n-1}$ . Similarly,

$$\begin{aligned} d(x_1, x_2) + d(x_2, x_n) + \varepsilon &\geq d(x_1, x_n) + \varepsilon \geq \\ &\geq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n). \end{aligned}$$

Cancelling  $d(x_1, x_2)$  we deduce that the set  $\{x_2, \dots, x_n\}$  is an  $\varepsilon$ -string with direct order  $x_2, \dots, x_n$ . Now, pick any  $j \in \{2, \dots, n-1\}$ . We have

$$\begin{aligned} d(x_1, x_n) + \varepsilon &\geq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n) \geq \\ &= \sum_{k=1}^{j-2} d(x_k, x_{k+1}) + d(x_{j-1}, x_j) + d(x_j, x_{j+1}) + \sum_{k=j+1}^{n-1} d(x_k, x_{k+1}) \geq \\ &\geq \sum_{k=1}^{j-2} d(x_k, x_{k+1}) + d(x_{j-1}, x_{j+1}) + \sum_{k=j+1}^{n-1} d(x_k, x_{k+1}) \end{aligned}$$

where we use the definition of an  $\varepsilon$ -string and the triangle inequality. Therefore the set  $\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$  is an  $\varepsilon$ -string with direct order  $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ . This proves both claims of the lemma for any subset of  $S$  having  $n-1$  elements. By induction, we obtain both claims for all non-void subsets of  $S$ .  $\square$

**5.2. The  $\varepsilon$ -string space.** By Lemma 5.1, the collection of  $\varepsilon$ -strings in a gap space  $(X, d)$  is an abstract simplicial complex. We denote its geometric realization by  $|X, d|_\varepsilon$ . This is a topological space glued from simplexes numerated by  $\varepsilon$ -strings in  $(X, d)$ . The simplexes corresponding to smaller  $\varepsilon$ -strings are faces of the simplexes corresponding to bigger  $\varepsilon$ -strings. We call  $|X, d|_\varepsilon$  the  $\varepsilon$ -string space of  $(X, d)$ . By the very definition, the space  $|X, d|_\varepsilon$  admits a triangulation whose all vertexes and edges are faces of certain 2-simplexes. For  $\varepsilon = 0$ , we obtain the same space as in Section 3.1, i.e.,  $|X, d|_0 = |X, d|$ .

For any real numbers  $\varepsilon' \geq \varepsilon \geq 0$ , each  $\varepsilon$ -string in  $(X, d)$  is also an  $\varepsilon'$ -string in  $(X, d)$ . This induces a simplicial embedding  $i_{\varepsilon, \varepsilon'} : |X, d|_\varepsilon \hookrightarrow |X, d|_{\varepsilon'}$ . Clearly, this construction is functorial, that is  $i_{\varepsilon, \varepsilon} = \text{id}$  and  $i_{\varepsilon', \varepsilon''} \circ i_{\varepsilon, \varepsilon'} = i_{\varepsilon, \varepsilon''}$  for all  $\varepsilon'' \geq \varepsilon' \geq \varepsilon \geq 0$ . In terminology of topological data analysis (TDA), we have got a filtered system of simplicial complexes  $\{|X, d|_\varepsilon\}_\varepsilon$ . The standard methods of TDA produce persistent homology and the barcode of this filtered system.

Given  $\varepsilon \geq 0$ , for any  $\varepsilon' \geq \varepsilon$  sufficiently close to  $\varepsilon$ , all  $\varepsilon'$ -strings in  $(X, d)$  are  $\varepsilon$ -strings and the mapping  $i_{\varepsilon, \varepsilon'}$  is a homeomorphism. In particular, for  $\varepsilon = 0$  and sufficiently small  $\varepsilon'$ , we have a homeomorphism  $i_{0, \varepsilon'} : |X, d| \approx |X, d|_{\varepsilon'}$ . For sufficiently big  $\varepsilon$ , all non-void subsets of  $X$  are  $\varepsilon$ -strings, so  $|X, d|_\varepsilon$  is a simplex with  $\text{card}(X)$  vertexes.

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