AN EQUIVARIANT BGG CORRESPONDENCE AND PERFECT COMPLEXES FOR EXTENSIONS BY $\mathbb{Z}/2 \times \mathbb{Z}/2$

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Abstract. We provide an equivariant extension of Carlsson's BGG correspondence in characteristic two. As an application we classify perfect cochain complexes of $(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes Q$-representations with four-dimensional total homology for finite groups Q of odd order. We deduce that cochain complexes of finite, free A_4-CW complexes with four-dimensional total homology are rigid: They are determined by the degrees of the nonzero homology groups.

1. Introduction

A classical problem in the theory of transformation groups asks which finite groups G can act freely on a finite CW complex X homotopy equivalent to a sphere S^n. Smith [Smi44] showed that such a group G can not contain an elementary abelian subgroup of rank 2, i.e., of the form $\mathbb{Z}/p \times \mathbb{Z}/p$ for any prime p. Swan [Swa59] proved that this condition is also sufficient for the existence of $n \geq 1$ such that G acts freely on some $X \simeq S^n$. The problem which dimensions n can occur is still open and related to number theoretic questions; see the survey [Ham15].

The rank conjecture of Benson and Carlson [BC87] states more generally that a finite group G can act freely on a finite CW complex X homotopy equivalent to a product of r spheres $S^{n_1} \times \ldots \times S^{n_r}$ if and only if G does not contain an elementary abelian subgroup of rank $r+1$. For $r = 2$, the condition that G can not contain a subgroup of the form $(\mathbb{Z}/p)^3$ was already known by work of Heller [Hel54]. For $r = 2$ and finite p-groups, Adem and Smith [AS01] showed that it is also sufficient. Nevertheless, it is open whether the simple groups $\mathrm{PSL}_3(\mathbb{F}_p)$ of rank 2 for odd primes p can act freely on some $X \simeq S^m \times S^n$.

In collaboration with Yalçın [RSY22], we investigated which dimensions m and n can occur for free actions of the alternating group $G = A_4$ on $X \simeq S^m \times S^n$ following a question of Blaszczyk [Bla13] and extending Oliver's result [Oli79, Theorem 2] that A_4 can not act freely on a product of two equidimensional spheres. We proved that any such action yields a parameter ideal with parameters of degrees $m+1$, $n+1$ in the group cohomology ring $H^*(BA_4; \mathbb{F}_2)$ that is closed under Steenrod operations, and classified these ideals explicitly. These topological obstructions apply to free actions of any finite simple group of rank 2, as they all contain A_4 as a subgroup. In this paper, we approach the study of free A_4-actions from a more algebraic side.

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Algebraically, we are interested in perfect cochain complexes over the group ring $k[G]$ for a finite group G and field k in modular characteristic with total homology of dimension 2^r, as the cellular cochain complex of any finite G-CW complex homotopy equivalent to a product of r spheres is an example of such a perfect complex. Again, if $r = 1$ or $r = 2$, then these perfect complexes can only exist if G does not contain a subgroup of the form $(\mathbb{Z}/p)^{r+1}$. If the homology is concentrated in one degree, then we are just studying projective modules of dimension 2^r. For $r = 1$ and nonzero homology in two different degrees $m < n$, any bounded above cochain complex C^* is classified by the $k[G]$-representations $H^m(C^*)$, $H^n(C^*)$ and one k-invariant element $k(C^*) \in \mathrm{Ext}^{n-m+1}_{k[G]}(H^n(C^*), H^m(C^*))$; see [Dol60, 7.6 Satz]. If $H^*(C^*)$ is finite-dimensional, then the cochain complex C^* is perfect if and only if it has trivial support; see [BIK11, Theorem 11.4]. For $r \geq 2$, we can have nonzero homology in more than two degrees. In general, such cochain complexes are not classified anymore by their k-invariants. For the alternating group $A_4 = (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes C_3$ and, more generally, for any extension of a finite group Q of odd order by $\mathbb{Z}/2\times\mathbb{Z}/2$, we classify the perfect cochain complexes with four-dimensional total homology.

Theorem 1.1 (see Corollary 6.6)**.** *Let $\mathbb{F}$ be a field of characteristic two and let Q be a group of odd order acting on $P = \mathbb{Z}/2 \times \mathbb{Z}/2$. There is a bijection between quasi-isomorphism classes of perfect cochain complexes over $\mathbb{F}[P \rtimes Q]$ with four-dimensional homology and triples (l, L, J), where l is an integer, L a one-dimensional Q-representation, and $J \subset H^*(BP; \mathbb{F})$ is a Q-invariant parameter ideal.*

The bijection assigns to a perfect cochain complex C^* with $\dim_{\mathbb{F}} H^*(C^*) = 4$, the degree l of the lowest nonzero homology group, the Q-representation $\mathrm{Ext}^l_{\mathbb{F}[P]}(\mathbb{F}, C^*)$, and the annihilator ideal $J \subset \mathrm{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, \mathbb{F})$ of $\mathrm{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, C^*)$. Up to shifting, we may assume that $l = 0$ and that the four basis elements of $H^*(C^*)$ have degrees $0 \leq m \leq n \leq t$. We will show that $t = m+n$ and that the corresponding parameter ideal has parameters of degrees $m+1$ and $n+1$; see Proposition 4.3. In fact, $H^*(C^*)$ is isomorphic to L tensored with the exterior algebra $\Lambda(\Sigma^{-1}J/(H^{>0}(BP)J))$ as graded $\mathbb{F}[Q]$-modules.

The two main methods to establish Theorem 1.1 hold in much greater generality. In Section 3, we extend the spectral sequence of [RS22] from finite p-groups P to finite extensions by P. We show in Section 4 that the spectral sequence collapses for C^* as in Theorem 1.1 and establish that the annihilator ideal of $\mathrm{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, C^*)$ is a Q-equivariant parameter ideal.

The idea of constructing perfect complexes from parameter ideals in group cohomology is due to Benson and Carlson [BC94, Theorem 4.1]. Their method constructs perfect cochain complexes with trivial action on homology. In our situation, we also need perfect cochain complexes with nontrivial action on homology.

In Theorem 5.8 we extend Carlsson's BGG correspondence for perfect cochain complexes over an exterior algebra Λ over $\mathbb{F}$ from [Car86] Q-equivariantly. Our formulation holds for any finite group Q, not just groups of odd order.

We use the equivariant BGG correspondence to construct perfect cochain complexes and to establish the classification of Theorem 1.1 in Section 6. This uses an identification of the Q-algebra $\mathbb{F}[P]$ for a group of odd order Q acting on $P = (\mathbb{Z}/2)^n$ with an exterior algebra $\Lambda(V)$ for a Q-representation V.

Section 2 on augmented crossed product algebras $A *_\gamma Q$ over a field k provides a uniform treatment for augmented skew group algebras and algebras of group extensions to establish a Q-action on $\mathrm{Ext}^*_A(k, C^*)$ for cochain complexes C^* over $A *_\gamma Q$.

The cellular cochain complex of a finite, free G-CW complex is not just a perfect complex, but a finite, free cochain complex. In Section 7, we use Wall's finiteness obstruction to determine which perfect complexes from Theorem 1.1 are homotopy equivalent to a finite, free one.

Theorem 1.2 (see Theorem 7.8)**.** *Let $\mathbb{F}$ be a field of characteristic two and C^* a perfect cochain complex over $\mathbb{F}[A_4]$ with four-dimensional homology. Then C^* is homotopy equivalent to a finite, free $\mathbb{F}[A_4]$-cochain complex if and only if the corresponding parameter ideal J has a C_3-invariant parameter of even degree.*

The modular representation theory of A_4 depends on whether the polynomial $X^2 + X + 1$ is irreducible over $\mathbb{F}$, see [DR89, BTCB22], and so does the proof of Theorem 1.2 even though its statement does not. For instance, if $X^2 + X + 1$ is irreducible, then the only one-dimensional representation is the trivial representation.

For $\mathbb{F} = \mathbb{F}_2$ we can read off from the homology of C^* whether it is homotopy equivalent to a finite, free one: A perfect $\mathbb{F}_2[A_4]$-cochain complex C^* such that its homology is four-dimensional with basis elements in degrees $0 \leq m \leq n \leq t$ is homotopy equivalent to a finite, free cochain complex if and only if m or n is odd and $H^*(C^*)$ is a trivial C_3-representation, see Corollary 7.9.

Theorem 1.1 for $\mathbb{F} = \mathbb{F}_2$ allows us to count the perfect cochain complexes with four-dimensional homology in fixed degrees by computing Q-invariant parameter ideals with parameters in corresponding degrees. For instance, we will compute in future work that up to quasi-isomorphism, there exist 9831 perfect cochain complexes over $\mathbb{F}_2[A_4]$ with four-dimensional homology in degrees $0 \leq 21 \leq 35 \leq 56$. Each of these cochain complexes is homotopy equivalent to a finite, free one. A priori, it might seem hard to find a finite, free cochain complex that can not be realized topologically as the cochains on a finite, free A_4-CW complex. However, it turns out that it is actually harder to find the ones that can be realized topologically. None of the 9831 perfect cochain complexes above can be realized topologically, since there does not exist a C_3-invariant parameter ideal with parameters of degrees 22 and 36 that is closed under Steenrod operations by [RSY22, Corollary 6.13].

Moreover, we know by [RSY22, Corollary 6.13] that for any degrees $m + 1$ and $n + 1$, there is at most one Steenrod closed parameter ideal with parameters of degrees $m + 1$, $n + 1$ in $H^*(BA_4; \mathbb{F}_2) = H^*(BP; \mathbb{F}_2)^{C_3}$ for $P = \mathbb{Z}/2 \times \mathbb{Z}/2$. In combination with the current work, this fact has the unexpected consequence that given only the degrees of the nonzero cohomology groups of a finite, free A_4-CW complex with four-dimensional cohomology, we can produce its cellular cochain complex up to homotopy equivalence.

Theorem 1.3 (see Theorem 8.2)**.** *If there exists a finite, free A_4-CW complex X with four-dimensional total cohomology $H^*(X; \mathbb{F}_2)$ with a basis in degrees $0 \leq m \leq n \leq t$, then its cellular cochain complex is determined by m and n up to homotopy.*

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2. Augmented crossed product algebras

Crossed product algebras are a common generalization of skew group algebras and group algebras of group extensions. A fitting reference for their representation theory is [RR85]. We extend the notion of crossed product algebras to augmented crossed product algebras $A *_{\Psi,\gamma} Q$ for augmented algebras A over a field k and finite groups Q. For any cochain complex C^* of right $A *_{\Psi,\gamma} Q$-modules, we establish a Q-action on $\mathrm{Ext}^*_A(k, C^*)$ and show that the action of $\mathrm{Ext}^*_A(k,k)$ on $\mathrm{Ext}^*_A(k, C^*)$ is Q-equivariant. Similar results for skew group algebras are in [MV01] and generalizations thereof to smash product algebras for Hopf algebra actions are in [HVOZ11].

Definition 2.1. Let k be a field and $(A, \mathrm{aug}\colon A \to k)$ an augmented k-algebra. Let Q be a finite group together with a function $\Psi\colon Q \to \mathrm{Aut}(A)$ that assigns to each $q \in Q$ an automorphism $\Psi(q)\colon A \to A$ of augmented algebras. Let $\gamma\colon Q \times Q \to U(A)$ be a map to the units of A such that

(1) $\gamma(q,q')\gamma(qq',q'') = \Psi(q)(\gamma(q',q''))\gamma(q,q'q'')$ for any $q,q',q'' \in Q$,
(2) $\gamma(e,q) = 1 = \gamma(q,e)$ for any $q \in Q$ and e the neutral element of Q,
(3) $\gamma(q,q')\Psi(qq')(a) = \Psi(q)(\Psi(q')(a))\gamma(q,q')$ for any $q,q' \in Q$ and $a \in A$,
(4) $\mathrm{aug}(\gamma(q,q')) = 1$ for all $q,q' \in Q$.

The crossed product algebra $A *_{\Psi,\gamma} Q$ (or short $A *_\gamma Q$) is the k-algebra given as a vector space by the free A-module on the basis $(\overline{q})_{q\in Q}$ with multiplication

$$a\overline{q}a'\overline{q'} = a\Psi(q)(a')\gamma(q,q')\overline{qq'}.$$

It is augmented by $a\overline{q} \mapsto \mathrm{aug}(a)$.

We do not require Ψ to be a group homomorphism.

Example 2.2. If Ψ is a group homomorphism and γ constant with value 1, then $A *_{\Psi,\gamma} Q$ is the ordinary skew group algebra $A * Q$ of the Q-action on A.

Example 2.3. Let $N \to G \to Q$ be a short exact sequence of groups and let $s\colon Q \to G$ be a section of sets that preserves the neutral element, i.e., such that $s(e) = e$. For the group ring $A = k[N]$ and the choice of s, define

$$\gamma(q,q') = s(q)s(q')s(qq')^{-1} \quad \text{and} \quad \Psi(q)(a) = s(q)as(q)^{-1}\,.$$

It is elementary to show that γ and Ψ satisfy the conditions of Definition 2.1. Moreover, we obtain an isomorphism $k[N] *_{\Psi,\gamma} Q \to k[G]$ by extending the identity on $k[N]$ via $\overline{q} \mapsto s(q)$.

If $N \to G \to Q$ splits and s is a section of groups, then this isomorphism specializes to $k[N]*Q \cong k[N \rtimes Q]$ writing the group algebra of a semidirect product as a skew-group algebra.

We recall basic identities and properties.

Lemma 2.4. *Let $A *_{\Psi,\gamma} Q$ be a crossed product algebra. Then*

(1) $\Psi(e) = \mathrm{id}$*,*
(2) $\overline{e}$ *is the neutral element in* $A *_\gamma Q$*,*
(3) and each $\overline{q}$ *is a unit in* $A *_\gamma Q$ *with inverse* $\gamma(q^{-1},q)^{-1}\overline{q^{-1}} = \overline{q^{-1}}\gamma(q,q^{-1})^{-1}$.

Proof. We have $\Psi(e)(a) = \Psi(e)(\Psi(e)(a))$ for any $a \in A$ by Definition 2.1(3) and Definition 2.1(2). Since $\Psi(e)$ is an isomorphism, it follows that $a = \Psi(e)(a)$, showing (1).

We have $(a\overline{q})\overline{e} = a\gamma(q,e)\overline{qe} = a\overline{q}$ and $\overline{e}a\overline{q} = \Psi(e)(a)\gamma(e,q)\overline{eq} = a\overline{q}$ showing (2).

The left inverse of $\overline{q}$ is $\gamma(q^{-1},q)^{-1}\overline{q^{-1}}$ since

$$\gamma(q^{-1},q)^{-1}\overline{q^{-1}}\overline{q} = \gamma(q^{-1},q)^{-1}\gamma(q^{-1},q)\overline{e} = \overline{e}.$$

The right inverse is $\overline{q^{-1}}\gamma(q,q^{-1})^{-1}$ as

$$\overline{q}\overline{q^{-1}}\gamma(q,q^{-1})^{-1} = \gamma(q,q^{-1})\overline{e}\gamma(q,q^{-1})^{-1} = \overline{e}.$$

Hence the two are equal and the inverse of $\overline{q}$. This shows (3). $\square$

The following two lemmas are well-known for skew group algebras. We consider elements in degree n of a hom complex as homomorphisms that are homogeneous of degree n.

Lemma 2.5. *Let C^*, D^*, E^* be right $A *_\gamma Q$-cochain complexes. The hom complex $\mathrm{Hom}_A(C^*, D^*)$ is a cochain complex of right $k[Q]$-modules via*

$$(fq)(x) := f(x\overline{q}^{-1})\overline{q}\,.$$

Moreover, composition $\mathrm{Hom}_A(D^*, E^*) \otimes_k \mathrm{Hom}_A(C^*, D^*) \to \mathrm{Hom}_A(C^*, E^*)$ is Q-linear with respect to the diagonal action on the tensor product.

Proof. The map fq is right A-linear since

$$\begin{aligned}(fq)(xa) =& f(x\overline{q}^{-1}\overline{q}a\overline{q}^{-1})\overline{q} = f(x\overline{q}^{-1}\Psi(q)(a))\overline{q}\\ =& f(x\overline{q}^{-1})\Psi(q)(a)\overline{q} = f(x\overline{q}^{-1})\overline{q}a = (fq)(x)a\,.\end{aligned}$$

The formula defines a Q-action as

$$\begin{aligned}(f(qq'))(x) =& f(x\overline{qq'}^{-1})\overline{qq'} = f(x(\gamma(q,q')^{-1}\overline{q}\overline{q'})^{-1})\gamma(q,q')^{-1}\overline{q}\overline{q'}\\ =& f(x\overline{q'}^{-1}\overline{q}^{-1}\gamma(q,q'))\gamma(q,q')^{-1}\overline{q}\overline{q'} = f(x\overline{q'}^{-1}\overline{q}^{-1})\overline{q}\overline{q'} = ((fq)q')(x)\,.\end{aligned}$$

We verify that the differential d_{Hom} on $\mathrm{Hom}_A(C^*, D^*)$ is Q-linear:

$$\begin{aligned}&(d_{\mathrm{Hom}}(f)q)(x)\\ =&((d_D \circ f) - (-1)^{|f|} f \circ d_C)q)(x) = d_D(f(x\overline{q}^{-1}))\overline{q} - (-1)^{|f|} f(d_C(x\overline{q}^{-1}))\overline{q}\\ =&d_D(f(x\overline{q}^{-1})\overline{q}) - (-1)^{|f|} f(d_C(x)\overline{q}^{-1})\overline{q} = d_D((fq)(x)) - (-1)^{|f|}(fq)(d_C(x))\\ =&d_{\mathrm{Hom}}(fq)(x)\,.\end{aligned}$$

Finally, we get for the composition of two composable morphisms f, f':

$$(fq \circ f'q)(x) = f(f'(x\overline{q}^{-1})\overline{q}\,\overline{q}^{-1})\overline{q} = f(f'(x\overline{q}^{-1}))\overline{q} = ((f' \circ f)q)(x) \quad \square$$

Lemma 2.6. *Let C^* be a right $A *_\gamma Q$ cochain complex and D^* be a left $A *_\gamma Q$ cochain complex. Then $C^* \otimes_A D^*$ is a cochain complex of right $k[Q]$-modules with Q-action given by $(x \otimes y)q = x\overline{q} \otimes \overline{q}^{-1}y$.*

Proof. The formula for the action of Q is well-defined as

$$\begin{aligned}xa\overline{q} \otimes_A \overline{q}^{-1}y =& x\overline{q}\Psi(q)^{-1}(a) \otimes_A \overline{q}^{-1}y = x\overline{q} \otimes_A \Psi(q)^{-1}(a)\overline{q}^{-1}y\\ =& x\overline{q} \otimes_A \overline{q}^{-1}\,\overline{q}\Psi(q)^{-1}(a)\overline{q}^{-1}y = x\overline{q} \otimes_A \overline{q}^{-1}ay\,.\end{aligned}$$

It is indeed a group action since

$$\begin{aligned}((x \otimes_A y)q)q' =& x\overline{q} \cdot \overline{q'} \otimes_A \overline{q'}^{-1} \cdot \overline{q}^{-1} y = x\gamma(q,q')\overline{qq'} \otimes_A \overline{qq'}^{-1}\gamma(q,q')^{-1}y \\ =& x\overline{qq'}\Psi(qq')^{-1}(\gamma(q,q')) \otimes_A \Psi(qq')^{-1}(\gamma(q,q')^{-1})\overline{qq'}^{-1}y \\ =& x\overline{qq'} \otimes_A \overline{qq'}^{-1}y = (x \otimes_A y)(qq')\,.\end{aligned}$$

We verify that the differential $d_\otimes$ on $C^* \otimes_A D^*$ is Q-linear:

$$\begin{aligned}(d_\otimes(x \otimes_A y))q =& (d_C(x) \otimes_A y + (-1)^{|x|} x \otimes_A d_D(y))q \\ =& d_C(x)\overline{q} \otimes_A \overline{q}^{-1}y + (-1)^{|x|} x\overline{q} \otimes_A \overline{q}^{-1} d_D(y) \\ =& d_C(x\overline{q}) \otimes_A \overline{q}^{-1}y + (-1)^{|x|} x\overline{q} \otimes_A d_D(\overline{q}^{-1}y) \\ =& d_\otimes(x\overline{q} \otimes_A \overline{q}^{-1}y) = d_\otimes((x \otimes_A y)q)\end{aligned}$$

□

The augmentation on $A *_\gamma Q$ equips k with an $A *_\gamma Q$-module structure.

Proposition 2.7. *For any cochain complex C^* of right $A *_\gamma Q$-modules, the graded vector space $\mathrm{Ext}^*_A(k, C^*)$ has an induced right Q-action. With this action, the map*

$$\mathrm{Ext}^*_A(k, C^*) \otimes_k \mathrm{Ext}^*_A(k, k) \to \mathrm{Ext}^*_A(k, C^*)$$

is a map of graded right Q-modules.

Proof. We take a projective resolution of k as a right $A *_\gamma Q$-module. Then the claims follow from Lemma 2.5 and taking homology. □

Example 2.8. For a short exact sequence of groups

$$N \to G \xrightarrow{pr} Q$$

as in Example 2.3 and C^*, D^* two cochain complexes of right $k[G]$-complexes the Q-action on $\mathrm{Hom}_{k[N]}(C^*, D^*)$ can be calculated by

$$(fq)(x) = f(xg^{-1})g\,,$$

where $g \in G$ is any element such that $pr(g) = q$. In particular, this is independent of the choice of representative for q in G.

3. A spectral sequence for extensions by p-groups

Let $\mathbb{F}$ be a field of characteristic $p > 0$. We generalize the spectral sequence from [RS22] to finite extensions by p-groups. Fix a short exact sequence

$$1 \to P \to G \xrightarrow{pr} Q \to 1$$

of finite groups such that P is a finite p-group. The spectral sequence in [RS22] is obtained from the coradical filtration of a cochain complex C^* of free $\mathbb{F}[P]$-modules. Here we will show that if C^* is the restriction of a cochain complex of $\mathbb{F}[G]$-modules, then the spectral sequence becomes a spectral sequence of $\mathbb{F}[Q]$-modules.

We view $\mathbb{F}[P]$ as a subring of $\mathbb{F}[G]$. Let C^* be a cochain complex of right $\mathbb{F}[G]$-modules such that their restrictions to $\mathbb{F}[P]$ are free.

Since group algebras of finite groups over a field are Frobenius algebras, their classes of injective and projective modules agree; see [Lam99, (15.9) Theorem]. Moreover, for a finite p-group P, the algebra $\mathbb{F}[P]$ is local. Hence projective modules over $\mathbb{F}[P]$ are free by Kaplansky's theorem on projective modules.

From the topological side, we are interested in cochain complexes of G-spaces whose isotropy groups intersect P trivially.

Example 3.1. Let $C_*(X;\mathbb{F})$ be the singular chain complex with coefficients in $\mathbb{F}$ of a G-space X such that the restricted P-action is free. Then the singular cochain complex $C^*(X;\mathbb{F})$ consists of right $\mathbb{F}[G]$-modules whose restrictions over $\mathbb{F}[P]$ are free.

From the algebraic side we can dualize perfect chain complexes over $\mathbb{F}[G]$ or consider perfect cochain complexes directly.

Example 3.2. Let C_* be a perfect chain complex over $\mathbb{F}[G]$. Dualizing yields a bounded cochain complex C^* of finitely generated, injective (and thus projective) right $\mathbb{F}[G]$-modules. Hence C^* is a perfect cochain complex over $\mathbb{F}[G]$ and since $\mathbb{F}[P]$ is local, it is free over $\mathbb{F}[P]$.

Let $I \subset \mathbb{F}[P]$ be the augmentation ideal. Since P is a p-group, the augmentation ideal is nilpotent. We write L for its nilpotency degree minus 1, i.e., L is the maximal number for which $I^L \neq 0$. We equip C^* with the increasing filtration of $\mathbb{F}[P]$-modules

$$0 = F^{-1}C^* \subset \ldots \subset F^L C^* = C^*,$$

given by

$$F^i C^* := \{x \mid x\lambda = 0 \quad \text{for all } \lambda \in I^{i+1}\}.$$

Lemma 3.3. *The filtration on C^* is a filtration of right $\mathbb{F}[G]$-modules.*

Proof. We show that right multiplication with any element in $\mathbb{F}[G]$ preserves the filtration degree. The group G acts on $\mathbb{F}[P]$ by conjugation. The unique maximal ideal I is invariant by the conjugation action. Thus all powers of I are invariant as well. If $x \in F^i C^*$, then for any $g \in G$ and $\lambda \in I^{i+1}$, we obtain

$$xg\lambda = (xg\lambda g^{-1})g = 0g = 0$$

by the definition of the filtration and since $g\lambda g^{-1} \in I^{i+1}$. Hence xg lies in $F^i C^*$. $\square$

We examine the associated spectral sequence. Since we do not follow standard grading conventions, we briefly recall the pages. Let $Z_r^{k,t}$ denote the module of r-almost cocycles in homological degree t and in filtration degree k, i.e.,

$$Z_r^{k,t} := \{x \in F^k C^t \mid dx \in F^{k-r} C^{t+1}\}.$$

With this notation, the pages of the spectral sequence are given by

$$E_r^{k,t} := \frac{Z_r^{k,t}}{Z_{r-1}^{k-1,t} + d(Z_{r-1}^{k+r-1,t-1})}.$$

The differential of the cochain complex induces differentials

$$d_r : E_r^{k,t} \to E_r^{k-r,t+1}$$

on each page. Finally, we have for the induced filtration on $H^*(C^*)$ that

$$E_\infty^{k,t} = F^k(H^t(C^*))/F^{k-1}(H^t(C^*)).$$

Lemma 3.4. *The G-action on $E_r^{k,t}$ descends to a Q-action with which the spectral sequence becomes a spectral sequence of $\mathbb{F}[Q]$-modules.*

Proof. If x represents a class $[x] \in E_r^{k,t}$, then $g \in G$ acts on $[x]$ by $[xg]$. We show that this action is independent of the representative of the coset Pg in $G/P \cong Q$. Indeed, for $p \in P$, we obtain

$$[xpg] = [x(p-1)g] + [xg].$$

Since $(p-1) \in I$, the element $x(p-1)g$ is in $Z_{r-1}^{k-1,t}$ and thus represents zero in $E_r^{k,t}$. Hence the G-action descends to a Q-action.

Since the differential of C^* is $\mathbb{F}[G]$-linear, all differentials in the spectral sequence are $\mathbb{F}[Q]$-linear. □

In [RS22, Proposition 3.6, Corollary 3.7], we calculated the E_0-page and E_1-page over $\mathbb{F}[P]$ involving the associated graded $\mathrm{gr}(\mathbb{F}[P])$ of $\mathbb{F}[P]$. We will show that these isomorphisms are Q-equivariant.

Lemma 3.5. *The graded ring*

$$\mathrm{gr}(\mathbb{F}[P]) \coloneqq \bigoplus_s I^s/I^{s+1}$$

has a Q-action induced by conjugation.

Proof. Each power I^s is invariant under the conjugation action by G by the proof of Lemma 3.3. Thus it suffices to show that the action descends to a Q-action on I^s/I^{s+1}. Let $g \in G$ and $p \in P$. We show that g and pg act the same way. For $\lambda \in I^s$ we obtain

$$[g^{-1}p^{-1}\lambda pg] = [g^{-1}\lambda g] + [g^{-1}(p^{-1}-1)\lambda g] + [g^{-1}\lambda(p-1)g] + [g^{-1}(p^{-1}-1)\lambda(p-1)g]$$

and the last three summands lie in I^{s+1} since $(p-1), (p^{-1}-1) \in I$. □

Lemma 3.6. *The natural maps*

$$E_0^{L,t} \otimes_{\mathbb{F}} I^s/I^{s+1} \to E_0^{L-s,t}, \qquad [c] \otimes [\lambda] \mapsto [c\lambda],$$
$$E_1^{L,t} \otimes_{\mathbb{F}} I^s/I^{s+1} \to E_1^{L-s,t}, \qquad [c] \otimes [\lambda] \mapsto [c\lambda],$$

are Q-equivariant isomorphisms.

Proof. These are natural isomorphisms by [RS22, Proposition 3.6, Corollary 3.7]. Since the Q action is descended from G, it suffices to establish G-equivariance. For $g \in G$, we have

$$[c]g \otimes [\lambda]g = [cg] \otimes [g^{-1}\lambda g] \mapsto [(c\lambda)g] = [c\lambda]g. \qquad \square$$

To complete the calculation of the E_0- and E_1-page, we provide Q-equivariant identifications for $E_0^{L,*}$ and $E_1^{L,*}$. By [RS22, Proposition 3.6], we have

$$E_0^{L,t} = C^t/C^tI \cong C^t \otimes_{\mathbb{F}[P]} \mathbb{F}$$

with Q-action descended from right multiplication by G on C^t.

Lemma 3.7. *We have a natural Q-equivariant isomorphism $E_0^{L,t} \cong C^t \otimes_{\mathbb{F}[G]} \mathbb{F}Q$ and thus*

$$E_0^{L-s,t} \cong (C^t \otimes_{\mathbb{F}[G]} \mathbb{F}[Q]) \otimes_{\mathbb{F}} I^s/I^{s+1},$$
$$E_1^{L-s,t} \cong H^t(C^* \otimes_{\mathbb{F}[G]} \mathbb{F}[Q]) \otimes_{\mathbb{F}} I^s/I^{s+1}$$

as right $\mathbb{F}[Q]$-modules.

Proof. We let Q act on $\mathbb{F}[G] \otimes_{\mathbb{F}[P]} \mathbb{F}$ as when considering $\mathbb{F}[G]$ as a cochain complex concentrated in degree zero. Thus $Pg = gP$ acts on $c \otimes 1$ by $cg \otimes 1$. The isomorphism $\mathbb{F}[G/P] \cong \mathbb{F}[G] \otimes_{\mathbb{F}[P]} \mathbb{F}$ is Q-equivariant and yields a natural isomorphism

$$C^t \otimes_{\mathbb{F}[P]} \mathbb{F} \cong C^t \otimes_{\mathbb{F}[G]} \mathbb{F}[G] \otimes_{\mathbb{F}[P]} \mathbb{F} \cong C^t \otimes_{\mathbb{F}[G]} \mathbb{F}[Q]$$

of right $\mathbb{F}[Q]$-modules.

The identification of the E_0- and E_1-page follows now from Lemma 3.6. □

Let $\varepsilon^*(G)$ be a projective resolution of $\mathbb{F}$ by finitely generated projective right $\mathbb{F}[G]$-modules. We will use the following result for perfect cochain complexes C^*.

Lemma 3.8. *Suppose that C^* is bounded below. Then the natural map*

$$C^* \cong \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, C^*) \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), C^*)$$

induces an isomorphism of spectral sequences for the pages E_r with $r \geq 1$.

Proof. Since the restriction of C^* to $\mathbb{F}[P]$ is free, the cochain complex C^* is a bounded below complex of injective $\mathbb{F}[P]$-modules. The complex $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), C^*)$ is bounded below as well and consists of injective (and thus free) $\mathbb{F}[P]$-modules. Thus the natural map is a quasi-isomorphism between bounded below complexes of injectives and hence a homotopy equivalence. It follows from [RS22, Corollary 3.7] that the induced map on spectral sequences is an isomorphism. □

We say that a spectral sequence E is a *right module* over a multiplicative spectral sequence R if each page E_r is a bigraded right module over R_r and the Leibniz rule holds for the action $E_r \otimes R_r \to E_r$. Furthermore, we require that the induced multiplication $H^*(E_r) \otimes H^*(R_r) \to H^*(E_r)$ agrees with the multiplication on the $(r+1)$-page.

Lemma 3.9. *The spectral sequence $E^{*,*}_{r\geq 0}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G)))$ is multiplicative and the spectral sequence $E^{*,*}_{r\geq 0}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*))$ is a module over it for any cochain complex D^* of right $\mathbb{F}[G]$-modules.*

Proof. The composition

$$\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G)) \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*) \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*)$$

is compatible with the filtrations as in the proof of [RS22, Proposition 4.4]. For $D^* = \varepsilon^*(G)$, the composition is the multiplication of a filtered differential graded algebra that is compatible with the filtration and thus induces a multiplicative spectral sequence; see [McC01][Theorem 2.14]. For arbitrary C^*, we obtain analogously an induced module structure of spectral sequences. □

We will use the following description of the E_1-page.

Remark 3.10. By definition of the E_0-page we have

$$E_0^{0,*}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*)) = \mathrm{Hom}_{\mathbb{F}[P]}(\varepsilon^*(G), D^*)$$

and thus

$$E_1^{0,*}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*)) \cong \mathrm{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, D^*).$$

For $D^* = \varepsilon^*(G)$, we obtain the group cohomology ring

$$E_1^{0,*}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G))) \cong \mathrm{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, \mathbb{F}) = H^*(BP).$$

All higher pages $E_{r\geq 1}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*))$ inherit an $H^*(BP)$-module structure

$$E^{k,t}_{r\geq 1}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*)) \otimes_{\mathbb{F}} H^r(BP) \to E^{k,r+t}_{r\geq 1}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), D^*))$$

using the quotient map

$$H^*(BP) = E_1^{0,*}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G))) \to E_r^{0,*}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G))),$$

and the differentials are $H^*(BP)$-linear.

We will use this structure for $D^* = C^*$ bounded below such that the restriction to $\mathbb{F}[P]$ is free as in Lemma 3.8. Then we can replace $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), C^*)$ by C^* in the spectral sequence for $E_{r\geq 1}$. The map

$$\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G)) \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \mathbb{F})$$

induces an isomorphism on the E_1-page and thus all higher pages as well by [RS22, Corollary 3.7]. Hence $E_{r\geq 1}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G)))$ can be replaced by the spectral sequence $E_{r\geq 1}(\mathrm{Hom}_{\mathbb{F}}(\varepsilon^*(G), \mathbb{F}))$.

4. The spectral sequence for extensions by $\mathbb{Z}/2 \times \mathbb{Z}/2$

In this section, we consider the spectral sequence from Section 3 for an elementary abelian 2-group $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of rank 2 and $\mathbb{F}$ a field of characteristic 2. If f_1, f_2 are generators for P, then $\mathbb{F}[P]$ is an exterior algebra generated by $\lambda_1 \coloneqq f_1 - 1$ and $\lambda_2 \coloneqq f_2 - 1$. The augmentation ideal $I \subset \mathbb{F}[P]$ is (λ_1, λ_2) and

$$I^2/I^3 \cong \mathbb{F}\lambda_1\lambda_2, \quad I/I^2 \cong \mathbb{F}\lambda_1 \oplus \mathbb{F}\lambda_2, \quad \mathbb{F}[P]/I \cong \mathbb{F}.$$

Thus only three columns in the spectral sequence can be nontrivial.

Lemma 4.1. *Let C^* be cochain complex over $\mathbb{F}[G]$ such that its restriction to $\mathbb{F}[P]$ is a perfect complex. If $H^*(C^*) \neq 0$, then $\dim_{\mathbb{F}} H^*(C^*) \geq 4$ with*

$$\dim_{\mathbb{F}} E_\infty^{0,*} \geq 1, \quad \dim_{\mathbb{F}} E_\infty^{1,*} \geq 2, \quad \dim_{\mathbb{F}} E_\infty^{2,*} \geq 1.$$

If $\dim_{\mathbb{F}} H^*(C^*) = 4$, then the spectral sequence collapses on the E_2-page.

Proof. The assumption $H^*(C^*) \neq 0$ implies $H^*(C^*/C^*I) \neq 0$. Indeed, as $\mathbb{F}[P]$ is a commutative, noetherian, local ring, this can be proved for instance by considering the minimal free resolution of the perfect cochain complex C^* over $\mathbb{F}[P]$; see e.g. [Rob80, Chapter 2, 2.4 Theorem].

As graded vector spaces, we have

$$E_1^{0,*} \cong H^*(C^*/C^*I), \quad E_1^{1,*} \cong H^*(C^*/C^*I)^2, \quad E_1^{2,*} \cong H^*(C^*/C^*I).$$

Since the differentials have bidegree $d_r = (-r, 1)$, the lowest nonzero entry of $E_1^{0,*}$ and the highest nonzero entry of $E_1^{2,*}$ survive to E_∞. Setting $d \coloneqq \dim_{\mathbb{F}} H^*(C^*/C^*I)$, the total rank of $d_1 \colon E_1^{1,*} \to E_1^{0,*-1}$ is at most $d-1$ and the total rank of $d_1 \colon E_1^{2,*} \to E_1^{1,*-1}$ is at most $d-1$ as well. It follows that $\dim_{\mathbb{F}}(E_2^{1,*}) \geq 2d-(d-1)-(d-1) = 2$. For degree reasons, the classes in the middle column $E_2^{1,*}$ cannot support any further differentials and thus survive to the E_∞-page. Together with the two surviving corners from the E_1-page, we deduce $\dim_{\mathbb{F}} H^*(C) = \dim_{\mathbb{F}} E_\infty \geq 4$.

If $\dim_{\mathbb{F}} H^*(C) = 4$, then $\dim_{\mathbb{F}} E_2^{1,*} = 2$. It follows that the total ranks of d_1 from $E_1^{1,*}$ to $E_1^{0,*}$ and of d_1 from $E_1^{2,*}$ to $E_1^{1,*}$ are both $d-1$. Hence $E_2^{0,*}$ consists only of the surviving bottom left corner, $E_2^{2,*}$ consists only of the surviving top right corner, and $d_r = 0$ for $r \geq 2$. □

Remark 4.2. If $\dim_{\mathbb{F}}(H^*(C^*)) = 4$ and the degrees of the elements of a homogeneous basis are $0, m, n, t$ with $0 \leq m \leq n \leq t$ counted with multiplicities, then the surviving classes on E_∞ sit in bidegrees $(0,0), (1,m), (1,n), (2,t)$. Moreover, $H^0(C^*/C^*I)$ is one-dimensional since $H^0(C^*/C^*I) \cong E_1^{0,0} = E_\infty^{0,0}$.

Recall that $H^*(BP) \cong \mathbb{F}[x_1, x_2]$ is a polynomial ring with generators x_1, x_2 of degree one.

Proposition 4.3. *Let C^* be a perfect $\mathbb{F}[G]$-cochain complex such that its total homology is four-dimensional with basis elements in degrees $0 \leq m \leq n \leq t$. Let J be the annihilator ideal of the graded $H^*(BP)$-module $\operatorname{Ext}^*_{\mathbb{F}P}(\mathbb{F}, C^*)$. Then*

(1) there is a Q-equivariant isomorphism of graded $H^*(BP)$-modules

$$\operatorname{Ext}^*_{\mathbb{F}P}(\mathbb{F}, C^*) \cong \operatorname{Ext}^0_{\mathbb{F}P}(\mathbb{F}, C^*) \otimes_{\mathbb{F}} (H^*(BP)/J);$$

(2) J is generated by a regular sequence of two parameters;
(3) $H^*(BP)/J$ is a complete intersection;
(4) there is an isomorphism of graded $\mathbb{F}[Q]$-modules

$$\operatorname{Ext}^0_{\mathbb{F}[P]}(\mathbb{F}, C^*) \otimes_{\mathbb{F}} \Lambda(\Sigma^{-1} J/(x_1, x_2)J) \cong \operatorname{gr} H^*(C^*);$$

(5) $t = m + n$.

Proof. We show that the map

$$E_1^{0,0}(C^*) \otimes_{\mathbb{F}} E_1^{0,*}(\operatorname{Hom}_{\mathbb{F}}(\varepsilon^*(G), \mathbb{F})) \to E_1^{0,*}(C^*)$$

from Lemma 3.9 using Remark 3.10 is surjective by induction on the degree. In degree zero, the map is surjective by construction. Suppose that the map is surjective in degree i for some $i \geq 0$ and let $z \in E_1^{0,i+1}(C^*)$. Since the spectral sequence collapses on the E_2-page by Lemma 4.1 and $E_2^{0,>0}(C^*)$ is zero, the d_1-differential surjects onto $E_1^{0,>0}(C^*)$. Thus there exists a class $z' \in E_1^{1,i}(C^*)$ with $d_1(z') = z$. It follows from the induction hypothesis and Lemma 3.6 that there exists $z'' \in E_1^{0,0}(C^*) \otimes E_1^{1,i}(\varepsilon^*(G))$ that is mapped to z'. Then $d_1(z'')$ is mapped to $d_1(z') = z$ which concludes the induction step.

Since $E_1^{0,0}(C^*)$ is one-dimensional, it follows that $E_1^{0,0}(C^*) \otimes_{\mathbb{F}} (H^*(BP)/J) \cong \operatorname{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, C^*)$, showing (1). We will show that the map

$$E_1^{0,0}(C^*) \otimes_{\mathbb{F}} E_1^{*,*}(\operatorname{Hom}_{\mathbb{F}}(\varepsilon^*(G), \mathbb{F})) \to E_1^{*,*}(C^*) \tag{4.4}$$

on the whole E_1-page is surjective. Let $f\colon \varepsilon^*(G) \to C^*$ be a representative of a generator of $E_1^{0,0}(C^*) \cong \operatorname{Ext}^0_{\mathbb{F}[P]}(\mathbb{F}, C^*)$. This is a map of right $\mathbb{F}[P]$-cochain complexes. Postcomposing with f induces a map on spectral sequences

$$E_1^{*,*}(\operatorname{Hom}_{\mathbb{F}}(\varepsilon^*(G), \varepsilon^*(G))) \to E_1^{*,*}(\operatorname{Hom}_{\mathbb{F}}(\varepsilon^*(G), C^*))$$

over $\mathbb{F}$ that is isomorphic to the map from (4.4). By Lemma 3.7, its surjectivity on $E_1^{0,*}$ shows that it is surjective on the whole E_1-page.

After shifting the i-th column of the E_1-page for $\operatorname{Hom}_{\mathbb{F}}(\varepsilon^*(G), \mathbb{F})$ up by i, we obtain the Koszul complex of $H^*(BP)$ by the description of the differential from [RS22, Corollary 6.3].

After the same shift, the E_1-page for C^* is the tensor product of the module $E_1^{0,0}(C^*)$ with the Koszul complex of $H^*(BP)/J$.

The first Koszul homology of $H^*(BP)/J = \mathbb{F}[x_1, x_2]/J$ for the sequence x_1, x_2 is $J/(x_1, x_2)J$ (see e.g. [GL69, Proof of Lemma 1.4.15]). Since $E_2^{1,*}(C^*)$ is two-dimensional by Lemma 4.1, so is $J/(x_1, x_2)J$. Thus J is generated by two parameters by the graded Nakayama lemma. Since $H^*(BP)/J$ is finite-dimensional and the Krull dimension of $H^*(BP)$ is 2, the ideal J is generated by a system of parameters. Since $H^*(BP)$ is Cohen-Macaulay, the two parameters form a regular sequence, showing (2). The ring $H^*(BP)/J$ is a complete intersection of embedding dimension 2; see [BH93, Theorem 2.3.3]. Thus (3) holds.

The Koszul homology of the complete intersection $H^*(BP)/J$ is the exterior algebra $\Lambda(J/(x_1,x_2)J)$; see [BH93, Theorem 2.3.11]. Since the spectral sequence collapses on the E_2-page, we obtain that $\mathrm{gr}(H^*(C^*)) \cong \bigoplus E_2^{k,*}(C^*)$ is isomorphic to $E_1^{0,0}(C^*) \otimes \Lambda(\Sigma^{-1}J/(x_1,x_2)J)$ showing (4). The generators of $\Sigma^{-1}J/(x_1,x_2)J$ have degrees m, n. Thus their exterior product has degree $t = m+n$, showing (5). $\square$

Remark 4.5. If C^* as in Proposition 4.3 is given by the cochains of a finite, free G-CW complex X, then the spectral sequence is multiplicative by [RS22, Theorem 6.9]. The one-dimensional Q-representation $\mathrm{Ext}^0_{\mathbb{F}[P]}(\mathbb{F}, C^*) \cong H^0(X/P)$ is trivial since the Q-action on the space X/P fixes $1 \in H^0(X/P)$. In this case Proposition 4.3 (4) is an isomorphism of graded rings

$$\Lambda(\Sigma^{-1}J/(x_1,x_2)J) \cong \mathrm{gr}\, H^*(C^*).$$

It is not known for which dimensions m, n the group A_4 can act freely on a finite, CW complex X homotopy equivalent to $S^m \times S^n$.

Remark 4.6. The obstruction result [RSY22, Theorem 7.5] for $G = A_4$ has the assumption that X is a finite, free G-CW complex with cohomology ring $H^*(X;\mathbb{F}_2) \cong H^*(S^m \times S^n;\mathbb{F}_2)$ for some $0 < m < n$. As explained on [RSY22, page 31], it suffices that the total cohomology of X is four-dimensional and such that the product of the two middle classes is the top class. This last assumption always holds by Remark 4.5.

5. An equivariant BGG correspondence

We will provide an explicit equivariant BGG correspondence in Theorem 5.8. There are different versions of the BGG correspondence. A related equivariant BGG correspondence is in [Fy01, Theorem 9.1.2]. We are interested in Carlsson's from [Car86]. In particular, we consider the exterior algebra as an ungraded algebra and work in characteristic 2. Similarly, there are many results on Koszul duality in the literature (see e.g. [Avr13]). We have not found a general statement that directly provides our equivariant BGG correspondence.

We begin with two general results for skew group algebras $A * Q$. We will use them for A an exterior algebra.

5.1. **Augmented skew group algebras.** Let k be a field and A an augmented k-algebra. Let Q be a finite group acting on the augmented k-algebra A. We write $\Psi(q)(a)$ for the left action of $q \in Q$ on $a \in A$. Recall that the skew group algebra $A * Q$ is the k-algebra given by the free A-module $\oplus_{q\in Q} Aq$ with basis Q and multiplication $(aq)(bp) = (a\Psi(q)(b))(qp)$ for $a, b \in A$ and $q, p \in Q$. In contrast to the notation for crossed product algebras, we just write $q \in A * Q$ instead of $\overline{q} \in A * Q$ for $q \in Q$.

Note that A is a right $A * Q$-module via $b \cdot (aq) = \Psi(q^{-1})(ba)$. Since this does not hold for crossed product algebras, we restrict to skew group algebras. The right $A * Q$-action on A does not commute with the left A-action, i.e., A is not an A-$A * Q$-bimodule and the following result is not just a formal consequence of bimodule structures. We transform the right Q-action from Lemma 2.5 to a left Q-action.

Lemma 5.1. *For any cochain complex C^* of right $A*Q$-modules, the hom complex $\mathrm{Hom}_A(C^*, A)$ is a cochain complex of left $A*Q$-modules with left A-action coming from the A-A-bimodule structure on A and Q-action $qf := fq^{-1}$, i.e., $(qf)(x) = f(xq)q^{-1}$.*

Proof. A left A-module structure and a Q-action yield an $A*Q$-module structure if $a(qf) = q(\Psi(q)^{-1}(a)f)$. This follows from the computation

$$\begin{aligned}(a(qf))(x) &= a(f(xq)q^{-1}) = a(\Psi(q)(f(xq))) \\ =&\Psi(q)(\Psi(q)^{-1}(a)f(xq)) = (q(\Psi(q)^{-1}(a)f))(x).\end{aligned}$$

□

Lemma 5.2. *Let C^* and P^* be cochain complexes of right $A*Q$-modules. If P^* is bounded above and consists of finitely generated A-projective modules and C^* is bounded below (i.e. $C^n = 0$ for all small enough n), then we have an isomorphism*

$$\Phi\colon C^* \otimes_A \mathrm{Hom}_A(P^*, A) \to \mathrm{Hom}_A(P^*, C^*), \qquad c \otimes f \mapsto c \cdot f(\_),$$

of right $k[Q]$-modules.

Proof. The left-hand side inherits a $k[Q]$-module structure by Lemma 2.6 and the right-hand side inherits a $k[Q]$-module structure by Lemma 2.5.

We check that Φ commutes with the Q-action:

$$\begin{aligned}\Phi((c \otimes_A f)q)(x) =&\Phi((cq \otimes q^{-1}f))(x) = (cq)(\Psi(q^{-1})(f(xq^{-1})) \\ =&cf(xq^{-1})q = (\Phi(c \otimes f)q)(x)\end{aligned}$$

Note that Φ is a chain map, thus it suffices to check that Φ is an isomorphism of graded modules.

For any right $A*Q$-module M and any finitely generated, A-projective right $A*Q$-module P, the map

$$M \otimes_A \mathrm{Hom}_A(P, A) \to \mathrm{Hom}_A(P, M), \qquad m \otimes f \mapsto mf(\_)$$

is an isomorphism.

For a fixed degree n, consider the map

$$\Phi^n\colon \bigoplus_m C^{n+m} \otimes \mathrm{Hom}_A(P^m, A) \to \prod_m \mathrm{Hom}_A(P^m, C^{m+n}).$$

The finiteness assumptions ensure that the right-hand side is a direct sum and thus the map is a direct sum of isomorphisms as above and hence an isomorphism. □

Remark 5.3. We write A^{op} for the opposite algebra. The Q-action on A induces a left action on A^{op} and $A^{\mathrm{op}} * Q \cong (A*Q)^{\mathrm{op}}$ via $aq \mapsto (\Psi(q^{-1})(a))q^{-1}$. Thus results for right $A*Q$-modules can be translated to results for left $A^{\mathrm{op}} * Q$-modules. If A is commutative, then $A^{\mathrm{op}} * Q = A*Q$.

For instance, A is a left $A*Q$-module via $(aq) \cdot b = a\Psi(q)(b)$. We will also need that for a left $A*Q$-module M and a left kQ-vector space V, the tensor product $M \otimes_k V$ is a left $A*Q$-module via

$$(aq)(m \otimes v) = (aqm) \otimes qv$$

for $a \in A$, $m \in M$, $v \in V$ and $q \in Q$.

5.2. **Equivariant BGG correspondence.** We specialize to an exterior algebra and work over a field $\mathbb{F}$ of characteristic two. Let V be a finite-dimensional vector space over $\mathbb{F}$ with a left Q-action denoted by qv for $q \in Q$ and $v \in V$. The group Q acts on the dual vector space $V^* = \mathrm{Hom}_k(V,k)$ by $(qf)(v) = f(q^{-1}v)$. Choose a basis $y_1, \ldots, y_n$ of V and a dual basis $x_1, \ldots x_n$ of V^*. Let Λ be the exterior algebra on V and $S = \mathbb{F}[x_1, \ldots, x_n]$ the symmetric algebra on V^*. We consider Λ as a graded algebra concentrated in degree 0 and grade S by $\deg(x_i) = 1$ for $1 \leq i \leq n$. Since Q acts on V and V^*, we obtain induced actions on the algebras Λ and S. In this subsection, we omit the notation Ψ for these actions.

Carlsson [Car86] established an equivalence of derived categories

$$\beta \colon D_{\Lambda\text{- perf}}(\Lambda) \to D^{hf}_{S\text{- perf}}(S)$$

from perfect chain complexes over the exterior algebra Λ to finitely generated, free S-dg modules M with finite-dimensional total homology. We provide an equivariant extension in Theorem 5.8 replacing Λ by $\Lambda * Q$ and S by $S * Q$.

Lemma 5.4. *The graded tensor product $\Lambda \otimes_{\mathbb{F}} S$ with differential*

$$d(c \otimes f) = \sum_i c y_i \otimes x_i f$$

is a Λ-injective resolution of $\mathrm{Hom}_\Lambda(\mathbb{F}, \Lambda)$ as left $\Lambda * Q$-module. If Q is of odd order, then $\Lambda \otimes_{\mathbb{F}} S$ is a $\Lambda * Q$-injective resolution.

Proof. Nonequivariantly, it is well-known that d is a differential on $\Lambda \otimes_{\mathbb{F}} S$ with homology concentrated in degree zero; see [Car83, (II) Proposition 2]. The differential commutes with the action of Λ by definition. It commutes with the Q-action as well as we show now. The element $\sum_{i=1}^n y_i \otimes x_i$ in $V \otimes V^*$ is fixed by Q since under the equivariant isomorphism of vector spaces $V \otimes V^* \cong \mathrm{End}(V)$, $v \otimes f \mapsto f(-)v$, the sum corresponds to id_V. The map

$$V \otimes V^* \otimes \Lambda \otimes S \to \Lambda \otimes S, \quad y \otimes x \otimes c \otimes f \mapsto cy \otimes xf,$$

is a homomorphism of $\mathbb{F}[Q]$-modules. It follows that

$$\sum_i q(c)q(y_i) \otimes q(x_i)q(f) = \sum_i q(c) y_i \otimes x_i q(f)$$

since the left-hand side is the image of $q(\sum_{i=1}^n y_i \otimes x_i) \otimes qc \otimes qf$ and the right-hand side is the image of $(\sum_{i=1}^n y_i \otimes x_i) \otimes qc \otimes qf$. We conclude that the differential is Q-equivariant:

$$\begin{aligned} q(d(c \otimes f)) &= \sum_{i=1}^n q(c)q(y_i) \otimes q(x_i)q(f) = \sum_{i=1}^n q(c) y_i \otimes x_i q(f) \\ &= d(q(c) \otimes q(f)) = d(q(c \otimes f)). \end{aligned}$$

Thus $\Lambda \otimes_{\mathbb{F}} S$ is a cochain complex of $\Lambda * Q$-modules. The modules are finitely generated and free over Λ. Since Λ is self-injective, the modules are injective over Λ so that $\Lambda \otimes_{\mathbb{F}} S$ is a Λ-injective resolution. We have

$$H^0(\Lambda \otimes_{\mathbb{F}} S) \cong \Lambda^n(V) \cong \mathrm{Hom}_\Lambda(\mathbb{F}, \Lambda)$$

as left $\Lambda * Q$-modules.

Suppose the order of Q does not divide the characteristic of $\mathbb{F}$. Then the skew group algebra $\Lambda * Q$ is self-injective as well; see [RR85, Theorem 1.1 and 1.3]. Moreover, a $\Lambda * Q$-module is projective over $\Lambda * Q$ if and only if it is projective over

Λ. Thus $\Lambda \otimes_{\mathbb{F}} S$ consists of finitely generated projective modules and since $\Lambda * Q$ is self-injective, it is indeed an injective resolution. $\square$

To emphasize the twisted differential, we write $\Lambda \tilde{\otimes}_{\mathbb{F}} S$ for the cochain complex from Lemma 5.4.

Lemma 5.5. *The hom complex*

$$\varepsilon^* = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda \tilde{\otimes}_{\mathbb{F}} S, \Lambda)$$

of left Λ-module homomorphisms is a Λ-projective resolution of $\mathbb{F}$ as right $\Lambda * Q$-module. For this ε^* and any bounded below cochain complex C^* of right $\Lambda * Q$-modules, there is a natural isomorphism

$$\mathrm{Hom}_\Lambda(\varepsilon^*, C^*) \cong C^* \tilde{\otimes}_{\mathbb{F}} S,$$

where $C^* \tilde{\otimes}_{\mathbb{F}} S$ is $C^* \otimes_{\mathbb{F}} S$ with differential given by $d(c \otimes f) = (dc) \otimes f + \sum_i cy_i \otimes x_i f$.

Proof. The cochain complex ε^* consists of finitely generated Λ-projective right $\Lambda * Q$-modules by Lemma 5.1 and Remark 5.3. It is a projective resolution of

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(\mathbb{F}, \Lambda), \Lambda) \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda^n(V), \Lambda) \cong \mathbb{F}$$

by Lemma 5.4 as Λ is self-injective.

Finally, using Lemma 5.2, we have a natural isomorphism

$$\mathrm{Hom}_\Lambda(\varepsilon^*, C^*) \cong C^* \otimes_\Lambda \mathrm{Hom}_\Lambda(\varepsilon^*, \Lambda) \cong C^* \otimes_\Lambda \Lambda \tilde{\otimes}_{\mathbb{F}} S \cong C^* \tilde{\otimes}_{\mathbb{F}} S$$

under which the differential of $C^* \tilde{\otimes}_{\mathbb{F}} S$ is as given in the statement. $\square$

Note that $\varepsilon^* = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda \tilde{\otimes}_{\mathbb{F}} S, \Lambda)$ is a right dg module over $(S^{\mathrm{op}} \otimes_{\mathbb{F}} \Lambda) * Q$ equipped with the trivial differential.

Remark 5.6. Nonequivariantly, $C^* \tilde{\otimes}_{\mathbb{F}} S$ is $\beta(C^*)$ for Carlsson's functor β from [Car83, Section (II)]. To compare to [ABIM10b, ABIM10a], there is an isomorphism

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda \tilde{\otimes}_{\mathbb{F}} S, \Lambda) \cong \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(\Lambda, \mathbb{F}) \tilde{\otimes}_{\mathbb{F}} S, \mathbb{F}) \cong \Lambda \tilde{\otimes}_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(S, \mathbb{F})$$

of dg modules over $(S^{\mathrm{op}} \otimes_{\mathbb{F}} \Lambda) * Q$, where the differential on the right-hand side is $d(\lambda \otimes h) = \sum_i \lambda y_i \otimes h x_i$.

The functor $\mathrm{Hom}_\Lambda(\varepsilon^*, -)$ from cochain complexes of right $\Lambda * Q$-modules to right dg modules over $S * Q$ is right adjoint to $M \mapsto M \otimes_S \varepsilon^*$. Here Q acts on $M \otimes_S \varepsilon^*$ diagonally.

The counit $\mathrm{Hom}_\Lambda(\varepsilon^*, C^*) \otimes_S \varepsilon^* \to C^*$ is given by evaluation $(f_i)_i \otimes \varphi \mapsto f_m(\varphi)$ for $\varphi \in \varepsilon^m$.

The unit $M \to \mathrm{Hom}_\Lambda(\varepsilon^*, M \otimes_S \varepsilon^*)$ sends $m \in M_n$ to $m \otimes_S - \colon \varepsilon^* \to M_n \otimes_S \varepsilon^*$.

Lemma 5.7. *For $M = S$, the unit $S \to \mathrm{Hom}_\Lambda(\varepsilon^*, \varepsilon^*)$ is a quasi-isomorphism. For $C^* = \mathbb{F}$ concentrated in degree zero, the counit $Hom_\Lambda(\varepsilon^*, \mathbb{F}) \otimes_S \varepsilon^* \to \mathbb{F}$ is a quasi-isomorphism.*

Proof. We show that the unit is a quasi-isomorphism in $M = S$. Using Remark 5.6, consider the composite

$$S \to \mathrm{Hom}_\Lambda(\varepsilon^*, \varepsilon^*) \simeq \mathrm{Hom}_\Lambda(\varepsilon^*, \mathbb{F}) \cong \mathrm{Hom}_\Lambda(\Lambda \tilde{\otimes}_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(S, \mathbb{F}), \mathbb{F}).$$

The target has trivial differential and is isomorphic to $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(S, \mathbb{F}), \mathbb{F}) \cong S$. It follows that the unit is a quasi-isomorphism in $M = S$ since the whole composite $S \to S$ is the identity.

For $C^* = \mathbb{F}$, the counit factors as an isomorphism followed by the resolution $\varepsilon^* \to \mathbb{F}$:

$$\mathrm{Hom}_\Lambda(\varepsilon^*, \mathbb{F}) \otimes_S \varepsilon^* \cong S \otimes_S \varepsilon^* \to \mathbb{F}$$

Hence, the counit is a quasi-isomorphism in $\mathbb{F}$. □

For a differential graded algebra R, we write $K(R)$ for the homotopy category of differential graded right modules over R and $D(R)$ for the corresponding derived category obtained by localization with respect to the quasi-isomorphisms. We equip $K(R)$ and $D(R)$ with the usual triangulated structures. For an object X of $D(R)$, we write $\mathrm{thick}(X)$ (or $\mathrm{thick}_R(X)$) for the thick subcategory of $D(R)$ generated by X, i.e., the intersection of all, full triangulated subcategories of $D(R)$ that contain X and are closed under taking summands. We use cohomological grading. In particular, the derived category of cochain complexes over $\Lambda * Q$ is $D(\Lambda * Q)$.

The adjoint functors $- \otimes_S \varepsilon^*$ and $\mathrm{Hom}_\Lambda(\varepsilon^*, -)$ induce exact, adjoint functors on the triangulated homotopy categories

$$K(S * Q) \rightleftarrows K(\Lambda * Q).$$

The functor $\mathrm{Hom}_\Lambda(\varepsilon^*, -)$ preserves quasi-isomorphisms and thus has a right derived functor $R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -)$ with $R\,\mathrm{Hom}_\Lambda(\varepsilon^*, C^*) = \mathrm{Hom}_\Lambda(\varepsilon^*, C^*)$ for $C^* \in D(\Lambda * Q)$. The left derived functor of $- \otimes_S \varepsilon^*$ exists as well and can be computed in a dg module M over $S * Q$ by applying $- \otimes_S \varepsilon^*$ to a semifree resolution of M. If the underlying graded S-module of M is finitely generated and free, then M is semifree over S, and $M \otimes_S^L \varepsilon^* = M \otimes_S \varepsilon^*$.

The functor $- \otimes_S^L \varepsilon^* \colon D(S * Q) \to D(\Lambda * Q)$ is exact and has exact right adjoint $R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -)$.

We write $D_{S\text{-}\mathrm{perf}}(S * Q)$ for the full triangulated subcategory of $D(S * Q)$ of objects isomorphic to dg $S * Q$-modules such that the underlying graded S-module is free and finitely generated. Moreover, we denote the full triangulated subcategory of $M \in D_{S\text{-}\mathrm{perf}}(S * Q)$ with $\dim_{\mathbb{F}} H^*(M) < \infty$ by $D^{hf}_{S\text{-}\mathrm{perf}}(S * Q)$. We identify the bounded derived category $D^b(\mathrm{mod}_{\Lambda * Q})$ of finitely generated right $\Lambda * Q$-modules with the full subcategory of $D(\Lambda * Q)$ consisting of the objects isomorphic to bounded cochain complexes of finitely generated modules. Equivalently, this is the full subcategory of cochain complexes in $D(\Lambda * Q)$ with finite-dimensional total homology; see e.g. [Kra22, Example 4.2.18].

We write $D_{\Lambda\text{-}\mathrm{perf}}(\Lambda * Q)$ for the full triangulated subcategory of $D(\Lambda * Q)$ of objects isomorphic to bounded cochain complexes whose underlying Λ-modules are finitely generated and projective. By [Lau23, Lemma 3.3], this category agrees with the full subcategory of objects in $D(\Lambda * Q)$ that are perfect in $D(\Lambda)$.

Theorem 5.8. *The adjunction* $(- \otimes_S^L \varepsilon^*, R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -))$ *restricts to equivalences of triangulated categories*

$$D_{S\text{-}\mathrm{perf}}(S * Q) \rightleftarrows D^b(\mathrm{mod}_{\Lambda * Q})$$

and

$$D^{hf}_{S\text{-}\mathrm{perf}}(S * Q) \rightleftarrows D_{\Lambda\text{-}\mathrm{perf}}(\Lambda * Q).$$

Proof. For any bounded complex C^* of finitely generated right modules over $\Lambda * Q$, the dg module $\mathrm{Hom}_\Lambda(\varepsilon^*, C^*)$ is S-free and finitely generated by Lemma 5.5. Thus $R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -)$ restricts to a functor $D^b(\mathrm{mod}_{\Lambda * Q}) \to D_{S\text{-}\mathrm{perf}}(S * Q)$. Moreover, the counit of the derived adjunction $\mathrm{Hom}_\Lambda(\varepsilon^*, C^*) \otimes_S^L \varepsilon^* \to C^*$ can be computed via the ordinary counit. We show that it is a quasi-isomorphism. Forgetting the Q-action, this counit agrees with the counit for the adjunction with Q the trivial group. This counit is a quasi-isomorphism since it is a quasi-isomorphism for $C^* = \mathbb{F}$ by Lemma 5.7 and since $\mathrm{thick}(\mathbb{F}) \subset D(\Lambda)$ is the bounded derived category of finitely generated Λ-modules. Hence, the counit between the derived adjunction is an isomorphism for $C^* \in D^b(\mathrm{mod}_{\Lambda * Q})$.

If a dg module $M \in D(S * Q)$ is S-free and finitely generated, then the unit $M \to \mathrm{Hom}_\Lambda(\varepsilon^*, M \otimes_S^L \varepsilon^*)$ can be computed by the ordinary unit. Forgetting the Q-action, this unit agrees with the unit for the adjunction with Q the trivial group. It is a quasi-isomorphism since it is a quasi-isomorphism for $M = S$ by Lemma 5.7 and $\mathrm{thick}(S) \subset D(S)$ contains all dg modules that are free and finitely generated. Hence, the unit between the derived adjunction is an isomorphism for $M \in D_{S\text{-}\mathrm{perf}}(S * Q)$.

To establish the first equivalence, we are left to show that $- \otimes_S^L \varepsilon^*$ restricts to a functor $D_{S\text{-}\mathrm{perf}}(S * Q) \to D^b(\mathrm{mod}_{\Lambda * Q})$. Forgetting the Q-action, this reduces to the case $M = S$ for which $S \otimes_S^L \varepsilon^* \cong \mathbb{F}$. Hence $M \otimes_S^L \varepsilon^*$ belongs to $D^b(\mathrm{mod}_{\Lambda * Q})$.

To establish the second equivalence, we check that the adjoint functors restrict further. If C^* is a bounded complex of finitely generated modules over $\Lambda * Q$ that are Λ-projective and thus injective over Λ, then $H^*(\mathrm{Hom}_\Lambda(\varepsilon^*, C^*)) \cong H^*(\mathrm{Hom}_\Lambda(\mathbb{F}, C^*))$ has finite total dimension.

On the other hand, if M over $S * Q$ is S-free and finitely generated with finite-dimensional total homology, then we will show that $M \otimes_S \varepsilon^*$ is perfect over Λ. This holds since for trivial Q the equivalence restricts to an equivalence $\mathrm{thick}_\Lambda(\Lambda) \simeq \mathrm{thick}_S(R\,\mathrm{Hom}_\Lambda(\varepsilon^*, \Lambda)) = \mathrm{thick}_S(\mathbb{F})$, and $\mathrm{thick}_S(\mathbb{F}) = D^{hf}_{S\text{-}\mathrm{perf}}(S)$ by [ABIM10b, Theorem 6.4]. We have shown that the first equivalence restricts as claimed, providing the second equivalence. $\square$

If Q is of odd order, then a right $\Lambda * Q$-module is projective if and only if it is projective over Λ. Thus we obtain the following consequence.

Corollary 5.9. *If Q is of odd order, then the adjunction $(- \otimes_S^L \varepsilon^*, R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -))$ restricts to equivalences of triangulated categories*

$$D^{hf}_{S\text{-}\mathrm{perf}}(S * Q) \rightleftarrows D_{\mathrm{perf}}(\Lambda * Q),$$

where the latter category is the perfect derived category of right $\Lambda * Q$-modules.

Remark 5.10. Carlsson used homological grading in [Car86]. Apart from that, our functor $R\,\mathrm{Hom}_\Lambda(\varepsilon^*, -)\colon D_{\Lambda-\mathrm{perf}}(\Lambda) \to D^{hf}_{S\text{-}\mathrm{perf}}(S)$ for Q the trivial group agrees with the functor H from [Car86, Theorem II.7]. This follows from Lemma 5.5. For a free, finitely generated dg S-module M with $\dim_{\mathbb{F}} H^*(M) < \infty$, we have $M \otimes_S^L \varepsilon^* \cong M \otimes_S (\Lambda \tilde{\otimes}_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(S, \mathbb{F}))$ instead of $G(M) = M \tilde{\otimes}_{\mathbb{F}} \Lambda$ from the proof of [Car86, Theorem II.7]. For a perfect Λ-cochain complex C^*, the composite

$$G(H(C^*)) = (C^* \tilde{\otimes}_{\mathbb{F}} S) \tilde{\otimes}_{\mathbb{F}} \Lambda \to C^* \otimes_{\mathbb{F}} \Lambda \to C^*$$

induced by the augmentation $S \to \mathbb{F}$ and the structure map for C^* over Λ is not a chain homotopy equivalence. Indeed, applying the functor $- \otimes_\Lambda \mathbb{F}$ to this map and precomposing with the quasi-isomorphism $\mathrm{Hom}_\Lambda(\mathbb{F}, C^*) \simeq C^* \tilde{\otimes}_{\mathbb{F}} S$ yields the zero map $\mathrm{Hom}_\Lambda(\mathbb{F}, C^*) \to C^* \tilde{\otimes}_\Lambda \mathbb{F}$.

Remark 5.11. For $C^* = \mathbb{F}$ concentrated in degree zero, the isomorphism

$$\mathrm{Ext}^*_\Lambda(\mathbb{F}, \mathbb{F}) \cong H^*(\mathrm{Hom}_\Lambda(\varepsilon^*, \mathbb{F})) \cong S$$

is a graded ring isomorphism, which is compatible with the Q-actions. Moreover, for an arbitrary cochain complex C^* over $\Lambda * Q$, the $\mathrm{Ext}^*_\Lambda(\mathbb{F},\mathbb{F})$-action on $\mathrm{Ext}^*_\Lambda(\mathbb{F},C^*)$ from Proposition 2.7 agrees with the S-action on $H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C^*))$; see [AP93, Section 1.3].

6. Perfect complexes with small homology for extensions by $(\mathbb{Z}/2)^2$

In this section we restrict the equivalence $R\,\mathrm{Hom}_\Lambda(\varepsilon^*,-)$ from Corollary 5.9 to perfect cochain complexes over $\Lambda * Q$ with four-dimensional total homology for $\Lambda = \Lambda(y_1,y_2)$ and $S = \mathbb{F}[x_1,x_2]$. In particular, we assume that Q is of odd order. We will provide an explicit classification of these perfect complexes with small homology in Theorem 6.3.

Lemma 6.1. *The functor $\mathrm{Hom}_\Lambda(\varepsilon^*,-)$ induces a bijection between isomorphism classes of perfect dg modules C over $\Lambda*Q$ with four-dimensional homology in degrees $0 \leq m \leq n \leq l$ and isomorphism classes of objects M of $D^{hf}_{S\text{-}\mathrm{perf}}(S*Q)$ such that $M \otimes_S \mathbb{F}$ has four-dimensional homology in the same degrees. Moreover, if $J \subset S$ is the annihilator ideal of $H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$, then J is a Q-invariant parameter ideal, $H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C)) \cong H^0(\mathrm{Hom}_\Lambda(\varepsilon^*,C)) \otimes_\mathbb{F} S/J$ as right $S*Q$-modules, and $H^0(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$ is one-dimensional over $\mathbb{F}$.*

Proof. The bijection holds by Theorem 5.8 and since $\mathrm{Hom}_\Lambda(\varepsilon^*,C) \otimes_S \mathbb{F} \cong C$ as cochain complexes.

The ideal J is a parameter ideal by Proposition 4.3 considering Λ as $\mathbb{F}[P]$, and it is Q-invariant by Proposition 2.7. By Proposition 4.3(1), the S-action

$$H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C)) \otimes_\mathbb{F} S \to H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$$

induces an isomorphism

$$H^0(\mathrm{Hom}_\Lambda(\varepsilon^*,C)) \otimes_\mathbb{F} S/J \cong H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$$

of graded S-modules. Since the S-action is Q-equivariant, so is the induced isomorphism. Finally, $H^0(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$ is one-dimensional By Remark 4.2, . $\square$

There is a canonical ring homomorphism $\mathbb{F}[Q] \to S * Q$ as the inclusion of the degree zero part. Then for any graded right $\mathbb{F}[Q]$-module W the induced module $\mathrm{ind}^{S*Q}_{\mathbb{F}[Q]}(W)$ is just $W \otimes_\mathbb{F} S$ with the diagonal right Q-action. For a graded right $S*Q$-module M, we will use repeatedly the induction-restriction adjunction

$$\mathrm{Hom}_{S*Q}(W \otimes_\mathbb{F} S, M) \cong \mathrm{Hom}_{\mathbb{F}[Q]}(W,M)$$

to extend $\mathbb{F}[Q]$-linear maps $W \to M$ to $S*Q$-linear maps $W \otimes_\mathbb{F} S \to M$.

Lemma 6.2. *There is a bijection between isomorphism classes of objects M in $D^{hf}_{S\text{-}\mathrm{perf}}(S*Q)$ such that $M \otimes_S \mathbb{F}$ has four-dimensional homology in degrees $0 \leq m \leq n \leq t$, and pairs (L,J) of a one-dimensional Q-representation L and a Q-invariant parameter ideal J in S with parameters in degrees $m+1, n+1$. The map assigns to an object M in $D^{hf}_{S\text{-}\mathrm{perf}}(S*Q)$ its zeroth homology $H^0(M)$ and the annihilator ideal of $H^*(M)$. In particular, there is no such M with four-dimensional homology unless $l = m+n$.*

Proof. If M is an object of $D^{hf}_{S\text{-}\mathrm{perf}}(S*Q)$ such that $H^*(M\otimes_S\mathbb{F})$ is four-dimensional, then by Lemma 6.1, the annihilator ideal J in S of $H^*(M)$ is a Q-invariant parameter ideal and $H^*(M) \cong S/J \otimes H^0(M)$ with $H^0(M)$ one-dimensional.

Now suppose that L is a one-dimensional Q representation over $\mathbb{F}$ and that J is such a parameter ideal. Then $J/(x_1, x_2)J$ is a two-dimensional $\mathbb{F}[Q]$-module. The projection $J \to J/(x_1, x_2)J$ is an $\mathbb{F}[Q]$-linear map. Since Q is of odd order, it has a section $\sigma\colon J/(x_1, x_2)J \to J$.

We extend σ to the map $J/(x_1, x_2)J \otimes S \to J \otimes S \to S$ and let K be the Koszul complex of this map tensored with L, i.e.,

$$K = L \otimes \Lambda(J/(x_1, x_2)J) \otimes S$$

with differential $l \otimes j_1 \wedge \ldots \wedge j_r \otimes s \mapsto \sum_{0 \leq k \leq r} l \otimes j_1 \wedge \ldots \wedge \widehat{j_k} \wedge \ldots \wedge j_r \otimes \sigma(j_k)s$. Since σ is $\mathbb{F}[Q]$-linear, the differential in K is $S * Q$-linear. We consider K as an $S * Q$-dg module graded by $\deg(l \otimes j_1 \wedge \ldots \wedge j_r \otimes s) = (\sum_k \deg(j_k)) - r + \deg(s)$.

Note that $K \otimes_S \mathbb{F} = L \otimes \Lambda(J/(x_1, x_2)J)$ with zero differential and grading shift as above. Since $J/(x_1, x_2)J$ is a two-dimensional graded $\mathbb{F}$-module with generators in degrees $m+1, n+1$, the tensor product $L \otimes \Lambda(J/(x_1, x_2)J)$ is four-dimensional with generators in degrees $0, m, n, m+n$.

If r_1, r_2 is a basis of $J/(x_1, x_2)J$, then $\sigma(r_1), \sigma(r_2)$ is a minimal generating set of the parameter ideal J and thus a regular sequence. It follows that $H^*(K) \cong L \otimes S/J$ by [BH93, Corollary 1.6.14].

Thus for an arbitrary pair (L, J), we have constructed an object $K \in D^{hf}_{S\text{-}\mathrm{perf}}(S * Q)$ such that $H^*(K \otimes_S \mathbb{F})$ is four-dimensional, $H^0(K) \cong L$, and such that J is the annihilator ideal of $H^*(K)$.

Now consider $M \in D^{hf}_{S\text{-}\mathrm{perf}}(S * Q)$ such that $H^*(M \otimes_S \mathbb{F})$ is four-dimensional. Let $L = H^0(M)$ and let J be the annihilator ideal of $H^*(M)$. We construct a quasi-isomorphism $K \to M$, where K is constructed as above. Let W denote the graded Q-representation $J/(x_1, x_2)J$.

The S-module structure induces an isomorphism $H^0(M) \otimes S/J \cong H^*(M)$ by Lemma 6.1. Denoting the differential of M by d, consider a section $H^*(M) \to \ker d$ of graded $\mathbb{F}[Q]$-modules. We obtain an $\mathbb{F}[Q]$-linear map $f_0\colon L \otimes \Lambda^0(W) = H^0(M) \otimes \mathbb{F} \to M$. By adjunction we can extend it to an $S*Q$-linear map $f_0\colon L \otimes \Lambda^0(W) \otimes S \to M$.

Next we construct a map $L \otimes \Lambda^1(W) \otimes S = L \otimes W \otimes S \to M$. Consider the map $L \otimes J/(x_1, x_2)J \to M$ given by $l \otimes j \mapsto f_0(l) \cdot \sigma(j)$. Since $H^*(M) \cong L \otimes S/J$ by Lemma 6.1, it follows that this map hits only boundary elements in M. Thus it can be lifted to an $\mathbb{F}[Q]$-linear map $f_1\colon L \otimes W \to M$ satisfying $d \circ f_1(l \otimes w) = f_0(l) \cdot \sigma(w)$. Again by adjunction, we extend it to an $S * Q$-linear map $f_1\colon L \otimes \Lambda^1(W) \otimes S \to M$.

Consider the $\mathbb{F}[Q]$-linear map

$$g\colon L \otimes \Lambda^2(W) \to M, l \otimes (w_1 \wedge w_2) \mapsto f_1(l \otimes w_1) \cdot \sigma(w_2) + f_1(l \otimes w_2) \cdot \sigma(w_1).$$

The boundary of the right-hand side is

$$df_1(l \otimes w_1) \cdot \sigma(w_2) + df_1(l \otimes w_2) \cdot \sigma(w_1) = f_0(l)\sigma(w_1)\sigma(w_2) + f_0(l)\sigma(w_2)\sigma(w_1) = 0.$$

Since $\Lambda^2(W)$ is one-dimensional with a generator in degree $m+n+2$ and f_1 lowers degrees by one, the image of g is concentrated in degree $m+n+1$. By [NS02, Theorem 5.4.1], S/J is a Poincaré duality algebra with fundamental class in degree $m+n$, in particular $H^{m+n+1}(M) = 0$. Thus the image of g is contained in the boundaries of M and since $\mathbb{F}[Q]$ is semisimple we can lift g to a map $f_2\colon L \otimes \Lambda^2(W) \to M^{m+n}$ with $df_2 = g$ and extend it to an $S * Q$-linear map $f_2\colon L \otimes \Lambda^2(W) \otimes S \to M$.

Now define

$$F\colon L\otimes\Lambda(W)\otimes S = L\otimes(\Lambda^2(W)\oplus\Lambda^1(W)\oplus\Lambda^0(W))\otimes S\to M$$

by $F=(f_2,f_1,f_0)$. A straightforward computation shows that F is a chain map. By construction, F induces an isomorphism on H^0. It follows that $H^*(F)$ is an isomorphism since it is the composite

$$H^*(K)\cong S/J\otimes H^0(K)\cong S/J\otimes H^0(M)\cong H^*(M).$$

Note that $H^*(K\otimes_S\mathbb{F})=L\otimes\Lambda(W)$, and thus $t=m+n$. □

We deduce the main result of this section.

Theorem 6.3. *There is a bijection between isomorphism classes of perfect dg modules over $\Lambda * Q$ with four-dimensional homology and triples (l,L,J) where l is an integer, $J\subset\mathbb{F}[x_1,x_2]$ is a Q-invariant parameter ideal and L is a one-dimensional Q-representation.*

Proof. For a perfect dg module C^* over $\Lambda * Q$ with four-dimensional homology, let $l\in\mathbb{Z}$ be the lowest degree in which $H^*(C)$ is nonzero.

Let $L=H^l(\mathrm{Hom}_\Lambda(\varepsilon^*,C))=\mathrm{Ext}^l_\Lambda(\mathbb{F},C)$. Let $J\subset\mathbb{F}[x_1,x_2]\cong\mathrm{Ext}^*_\Lambda(\mathbb{F},\mathbb{F})$ be the annihilator ideal of $\mathrm{Ext}^*_\Lambda(\mathbb{F},C)\cong H^*(\mathrm{Hom}_\Lambda(\varepsilon^*,C))$. We show that the assignment $C\mapsto(l,L,J)$ is a bijection as desired.

After shifting C, we may assume that $l=0$. Then combining Lemma 6.1 with Lemma 6.2 shows that the assignment is indeed a bijection as claimed. □

For Q acting on $P=\mathbb{Z}/2\times\mathbb{Z}/2$, we will identify the Q-equivariant algebra $\mathbb{F}[P]$ with a suitable exterior algebra.

Example 6.4. For $Q=C_3=\langle q\rangle$ acting on $(\mathbb{Z}/2)^2$ nontrivially, the Q-action on $\mathbb{F}_2[(\mathbb{Z}/2)^2]\cong\mathbb{F}_2[\lambda_1,\lambda_2]/(\lambda_1^2,\lambda_2^2)$ is $q(\lambda_1)=\lambda_2$, $q(\lambda_2)=\lambda_1+\lambda_2+\lambda_1\lambda_2$. This Q-equivariant algebra is isomorphic to an exterior algebra $\Lambda(V)$ on an equivariant vector space $V=\mathbb{F}_2y_1\oplus\mathbb{F}_2y_2$ as follows. Let Q act on V by $q(y_1)=y_2$ and $q(y_2)=y_1+y_2$. Then

$$\Lambda(V)\to\mathbb{F}_2[(\mathbb{Z}/2)^2]\quad,y_1\mapsto\lambda_1+\lambda_1\lambda_2,\quad y_2\mapsto\lambda_2+\lambda_1\lambda_2,$$

is an equivariant isomorphism of algebras. Note that it does not preserve the internal gradings of the exterior algebras Λ and $\mathbb{F}_2[(\mathbb{Z}/2)^2]$. One can verify by hand that there is no equivariant isomorphism $\Lambda(V)\to\mathbb{F}_2[(\mathbb{Z}/2)^2]$ which preserves the grading.

The preceding example can be generalized.

Lemma 6.5. *Let $\mathbb{F}$ be a field of characteristic two, let $P=(\mathbb{Z}/2)^n$ and Q a finite group of odd order acting on P. Let I denote the maximal ideal in $\mathbb{F}[P]$. Then there is a Q-equivariant algebra isomorphism $\Lambda(I/I^2)\to\mathbb{F}[P]$ which respects the augmentations.*

Proof. Let $k: I/I^2\to I\subset\mathbb{F}[P]$ be a Q-equivariant section. Since the target algebra is also an exterior algebra, k extends to an equivariant algebra homomorphism $\Lambda(I/I^2)\to\mathbb{F}[P]$. It remains to show that it is surjective. This holds since k is a section and any lift of a basis of I/I^2 generates $\mathbb{F}[P]$.

By construction this map sends the augmentation ideal of $\Lambda(I/I^2)$ to the augmentation ideal of $\mathbb{F}[P]$ and thus it is compatible with the augmentations. □

The classification from Theorem 6.3 now yields a classification for perfect cochain complexes up to quasi-isomorphism (or equivalently up to isomorphism in the derived category) with small homology for extensions by $(\mathbb{Z}/2)^2$.

Corollary 6.6. *Let Q be a group of odd order acting on $P = \mathbb{Z}/2 \times \mathbb{Z}/2$. There is a bijection between isomorphism classes of perfect cochain complexes over $\mathbb{F}[P \rtimes Q]$ with four-dimensional homology and triples (l, L, J) where l is an integer, $J \subset \mathbb{F}[x_1, x_2]$ is a Q-invariant parameter ideal and L a one-dimensional Q-representation.*

Proof. The augmented Q-algebra $\mathbb{F}[P]$ is isomorphic to an augmented Q-algebra Λ on a two-dimensional Q-representation V by Lemma 6.5. Thus $\mathbb{F}[P \rtimes Q] \cong \mathbb{F}[P] * Q \cong \Lambda * Q$ and the bijection follows from Theorem 6.3. ☐

Methods of Benson and Carlson construct perfect cochain complexes with trivial action on homology; see [BC94, Theorem 4.1].

Remark 6.7. For $P = \mathbb{Z}/2 \times \mathbb{Z}/2$, any perfect cochain complex C^* over $\mathbb{F}[P]$ is either isomorphic to the zero complex in the perfect derived category or has at least four-dimensional homology. Thus if $\dim_{\mathbb{F}} H^*(C^*) = 4$, then C^* is an indecomposable object in the perfect derived category. Not all indecomposable objects have four-dimensional total homology, e.g., the cochain complex from [RS22, Example 3.8] has six-dimensional homology and in particular is indecomposable.

7. Computing finiteness obstructions

As in Section 3, consider a short exact sequence

$$1 \to P \to G \xrightarrow{pr} Q \to 1 \tag{7.1}$$

of finite groups such that P is a p-group and $\mathbb{F}$ a field of characteristic p. In topology, we are interested in cochain complexes of finite, free G-CW complexes. These are not just be perfect cochain complexes, but in fact bounded complexes of finitely generated free $\mathbb{F}[G]$-modules. In this section we consider the finiteness obstruction to determine whether a perfect cochain complex is homotopy equivalent to a finite, free one.

The finiteness obstruction is an element of the reduced Grothendieck group $\tilde{K}_0(\mathbb{F}[G])$. This group is isomorphic to $\tilde{K}_0(\mathbb{F}[Q])$ as we will explain below.

Lemma 7.2. *Let $J \subset \mathbb{F}[G]$ be the two-sided ideal generated by the augmentation ideal $I \subset \mathbb{F}[P]$. Then J is the kernel of $pr_*\colon \mathbb{F}[G] \to \mathbb{F}[Q]$ and J is nilpotent.*

Proof. For $p \in P$, the element $p - 1$ is in the kernel of $\mathbb{F}[G] \to \mathbb{F}[Q]$ and so is the ideal I generated by these elements. We verify that an arbitrary element $\sum_{g \in G} \lambda_g g$ of this kernel belongs to J. Since

$$\sum_{g \in G} \lambda_g g = \sum_{q \in Q} \sum_{g \in pr^{-1}(q)} \lambda_g g$$

it suffices to show that $\sum_{g \in pr^{-1}(q)} \lambda_g g \in J$ for any fixed $q \in Q$. For each $q \in Q$ we have $\sum_{g \in pr^{-1}(q)} \lambda_g = 0$. So for a fixed $g' \in pr^{-1}(q)$, we have $\sum_{p \in P} \lambda_{pg'} = 0$. Thus

$$\sum_{p \in P} \lambda_{pg'} pg' = \sum_{p \in P \setminus \{e\}} \lambda_{pg'} (p-1) g'$$

and this is an element of J. To prove the second statement, let S denote the set $\{p - 1 \mid p \in P\}$. Since P is normal, we have $\mathbb{F}[G] \cdot S \cdot \mathbb{F}[G] = \mathbb{F}[G] \cdot S$ as sets and inductively $(\mathbb{F}[G] \cdot S \cdot \mathbb{F}[G])^n = \mathbb{F}[G] \cdot (S^n)$. Since the augmentation ideal of a finite p-group in modular characteristic is nilpotent, the set S^n is zero for n large enough, and so is J. □

If p does not divide the order of Q then $I \cdot \mathbb{F}[G]$ is the Jacobson radical of $\mathbb{F}[G]$ by [Pot77, Corollary to Theorem 1], but we will not use this statement.

We denote the Grothendieck group of isomorphism classes of finitely generated projective modules over a ring R by $K_0(R)$.

Lemma 7.3. *The induced map*

$$K_0(pr_*)\colon K_0(\mathbb{F}[G]) \to K_0(\mathbb{F}[Q])$$

is an isomorphism.

Proof. Since $\ker(pr_*)$ is nilpotent by Lemma 7.2, the induced map on K_0 is an isomorphism; see [Wei13, Lemma II.2.2]. □

Definition 7.4. For a perfect cochain complex C^* over $\mathbb{F}[G]$, the *Euler characteristic* of C^* is

$$\chi(C^*) = \sum_i (-1)^i [C^i] \in K_0(\mathbb{F}[G]).$$

The image $\tilde{\chi}(C^*)$ of $\chi(C^*)$ in the reduced projective class group $\tilde{K}_0(\mathbb{F}[G])$ is called the *finiteness obstruction* of C^*.

The element $\tilde{\chi}(C^*)$ vanishes if and only if the perfect cochain complex C^* is homotopy equivalent to a finite, free complex; see [Ran85].

If p does not divide the order of Q, then any $\mathbb{F}[Q]$-module is projective. In particular $\operatorname{gr}(\mathbb{F}[P])$ from Lemma 3.5 represents a class in $K_0(\mathbb{F}[Q])$. Moreover, $P \subset G$ is a normal p-Sylow subgroup and the short exact sequence (7.1) splits by the Schur-Zassenhaus Theorem; see [CR81, (8.35) Theorem]. In the following result we choose a splitting so that G is a semidirect product $G = P \rtimes Q$ and $[\operatorname{gr}(\mathbb{F}[P])] = [\operatorname{res}^G_Q \mathbb{F}[P]]$ in $K_0(\mathbb{F}[Q])$. If Q is trivial, then the statement reduces to the fact that the ordinary Euler characteristic of a finite, free $\mathbb{F}[P]$-cochain complex is divisible by the order of P.

Proposition 7.5. *Let C^* be a perfect $\mathbb{F}[G]$-cochain complex and assume that p does not divide the order of Q. We have*

$$pr_*(\chi(C^*)) \cdot [\operatorname{gr}(\mathbb{F}[P])] = \chi(\operatorname{res}^G_Q H^*(C^*)) \in K_0(\mathbb{F}[Q]).$$

Hence if $[\operatorname{gr}(\mathbb{F}[P])]$ is not a zero-divisor in the ring $K_0(\mathbb{F}[Q])$, then the finiteness obstruction of C^* vanishes if and only if $\chi(\operatorname{res}^G_Q H^*(C^*))$ lies in the subgroup generated by the product $[\operatorname{gr}(\mathbb{F}[P])] \cdot [\mathbb{F}[Q]]$ in $K_0(\mathbb{F}[Q])$.

Proof. Let C^* be perfect $\mathbb{F}[G]$-cochain complex and choose a splitting $G = P \rtimes Q$. Since $\mathbb{F}[Q]$ is semisimple, the Euler characteristic commutes with taking homology and in a short exact sequence the Euler characteristic of the middle term is the sum of the Euler characteristics of the other two terms. It follows that

$$\chi(\operatorname{res}^G_Q H^*(C^*)) = \chi(\operatorname{res}^G_Q C^*) = \sum_i \chi(\operatorname{res}^G_Q (F^i C^*/F^{i-1} C^*))$$

for the filtration from Section 3. These filtration quotients are the columns $E_0^{i,*}$. By Lemma 3.7 and the definition of the functor K_0, we obtain

$$\sum_i \chi(\operatorname{res}_Q^G(F^iC^*/F^{i-1}C^*)) = \chi(C^* \otimes_{\mathbb{F}[G]} \mathbb{F}[Q]) \cdot [\operatorname{gr}(\mathbb{F}[P])] = pr_*(\chi(C^*)) \cdot [\operatorname{gr}(\mathbb{F}[P])].$$

It follows from Lemma 7.3 that the finiteness obstruction $\tilde{\chi}(C^*)$ vanishes if and only if $pr_*\tilde{\chi}(C^*) = 0$ in $\tilde{K}_0(\mathbb{F}[Q])$, i.e., if $pr_*(\chi(C^*))$ lies in the subgroup generated by $\mathbb{F}[Q]$ of $K_0(\mathbb{F}[Q])$. If $[\operatorname{gr}(\mathbb{F}[P])]$ is not a zero-divisor, this holds if and only if $pr_*(\chi(C^*)) \cdot [\operatorname{gr}(\mathbb{F}[P])] = \chi(\operatorname{res}_Q^G H^*(C^*))$ lies in the additive subgroup generated by the product $[\operatorname{gr}(\mathbb{F}[P])] \cdot [\mathbb{F}[Q]]$. □

We will use the following consequence.

Lemma 7.6. *Let C^* be a perfect $\mathbb{F}[G]$-cochain complex and assume that p does not divide the order of Q. If $[\operatorname{gr}(\mathbb{F}[P])]$ is not a zero-divisor in $K_0(\mathbb{F}[Q])$ and $\dim_{\mathbb{F}}(H^*(C^*)) < |G|$, then the following assertions are equivalent:*

(1) The cochain complex C^* is homotopy equivalent to a finite, free $\mathbb{F}[G]$-cochain complex;
(2) $\chi(\operatorname{res}_Q^G H^*(C^*)) = 0$ in $K_0(\mathbb{F}[Q])$.

Proof. By Proposition 7.5 and the assumption on $[\operatorname{gr}(\mathbb{F}[P])]$, the finiteness obstruction vanishes if and only if $\chi(\operatorname{res}_Q^G H^*(C^*))$ lies in the additive subgroup generated by $[\operatorname{gr}(\mathbb{F}[P])] \cdot \mathbb{F}[Q]$.

The ring homomorphism $\dim : K_0(\mathbb{F}[Q]) \to K_0(\mathbb{F}) \cong \mathbb{Z}$ maps this cyclic subgroup bijectively to $|G|\mathbb{Z}$. We have $-|G| < \dim(\chi(\operatorname{res}_{C_3}^G H^*(C^*))) < |G|$ by assumption. Thus $\chi(\operatorname{res}_Q^G H^*(C^*))$ lies in that subgroup if and only if it is zero. □

If the characteristic p of $\mathbb{F}$ does not divide the order of Q, then $K_0(\mathbb{F}[Q])$ coincides with the representation ring and is additively isomorphic to $\mathbb{Z}^s$, where s is the number of isomorphism classes of irreducible representations and the canonical generators are given by the classes of the irreducible representations.

We specialize to the case of $G = A_4 = (\mathbb{Z}/2)^2 \rtimes C_3$ with $Q = C_3$ acting on $P = \mathbb{Z}/2 \times \mathbb{Z}/2$ nontrivially.

If X^2+X+1 does not have a zero in $\mathbb{F}$, then the only two irreducible representations of C_3 are the trivial representation $\mathbb{F}$ and the two-dimensional representation $V = \mathbb{F}e_1 \oplus \mathbb{F}e_2$, where a generator of C_3 acts by $e_1 \mapsto e_2$ and $e_2 \mapsto e_1 + e_2$.

The tensor product $V \otimes V$ decomposes as $V \otimes V \cong V \oplus \mathbb{F}^2$ so that

$$K_0(\mathbb{F}[C_3]) \cong \mathbb{Z}[V]/(V^2 - V - 2)$$

as rings.

If $X^2 + X + 1$ has a root α, then $X^3 + 1$ has the three distinct roots 1, α and $\alpha^2 = \alpha + 1$ in $\mathbb{F}$. For each of the roots, we get a one-dimensional representation, where a fixed generator of C_3 acts by multiplication with that root. We denote these three one-dimensional representations again by $1, \alpha, \alpha^2$. Since $\alpha \otimes_{\mathbb{F}} \alpha \cong \alpha^2$ and $\alpha \otimes_{\mathbb{F}} \alpha^2 \cong 1$, we obtain

$$K_0(\mathbb{F}[C_3]) \cong \mathbb{Z}[\alpha]/(\alpha^3 - 1).$$

Lemma 7.7. *The element $[\operatorname{gr}\mathbb{F}[P]]$ is not a zero divisor in $K_0(\mathbb{F}[C_3])$. More precisely:*

(1) If X^2+X+1 does not have a zero in $\mathbb{F}$, then

$$[\mathrm{gr}(\mathbb{F}[P])] = V+2 \text{ in } K_0(\mathbb{F}[C_3]) \cong \mathbb{Z}[V]/(V^2-V-2).$$

(2) If X^2+X+1 has a zero in $\mathbb{F}$, then

$$[\mathrm{gr}(\mathbb{F}[P])] = \alpha^2+\alpha+2 \text{ in } K_0(\mathbb{F}[C_3]) \cong \mathbb{Z}[\alpha]/(\alpha^3-1).$$

Proof. Since C_3 acts nontrivially on $(\mathbb{Z}/2)^2$, there are generators f_1, f_2 of $(\mathbb{Z}/2)^2$ on which a generator $\tau \in C_3$ acts by $f_1 \mapsto f_2$ and $f_2 \mapsto f_1 f_2$. A basis for I/I^2 is given by $[f_1-1], [f_2-1]$. Conjugation by τ sends this basis to $[f_2-1]$ and

$$[f_1f_2-1] = [f_1f_2-1] - [(f_1-1)(f_2-1)] = [f_1-1]+[f_2-1].$$

Moreover, conjugation by τ is the identity on $\mathbb{F}[P]/I$ and fixes the generator $(f_1-1)(f_2-1)$ of I^2.

The first formula for $[\mathrm{gr}(\mathbb{F}[P])]$ follows immediately. It is not a zero divisor since $V^2-V-2=(V-2)(V+1)$ in $\mathbb{Z}[V]$.

If X^2+X+1 has a zero in $\mathbb{F}$, then the ring homomorphism $K_0(\mathbb{F}_2[C_3]) \to K_0(\mathbb{F}[C_3])$ sends V to $\alpha+\alpha^2$ from which we deduce the second formula for $[\mathrm{gr}(\mathbb{F}[P])]$. Since $\alpha^3-1=(\alpha-1)(\alpha^2+\alpha+1)$ in the polynomial ring $\mathbb{Z}[\alpha]$, the element $\alpha^2+\alpha+2$ is not a zero divisor in $K_0(\mathbb{F}[C_3])$. □

Theorem 7.8. *Let C^* be a perfect cochain complex over $\mathbb{F}[A_4]$ with four-dimensional homology corresponding to the triple (l, L, J) as in Corollary 6.6.*

Then the following assertions are equivalent:

(1) The cochain complex C^* is homotopy equivalent to a finite, free $\mathbb{F}[A_4]$-cochain complex;
(2) $\chi(\mathrm{res}^{A_4}_{C_3} H^*(C^*)) = 0$ in $K_0(\mathbb{F}[C_3])$;
(3) the graded representation $J/(x_1,x_2)J$ has a C_3-invariant basis element of even degree;
(4) J has a C_3-invariant parameter in even degree.

Proof. The first two assertions are equivalent by Lemma 7.6 and Lemma 7.7.

After shifting, we may assume that the homology of C^* has a basis with elements in degrees $0 \le m_1 \le m_2 \le m_1+m_2$. Since $\mathbb{F}[C_3]$ is semisimple, we obtain $\chi(\mathrm{res}^{A_4}_{C_3} H^*(C^*)) = \chi(\mathrm{res}^{A_4}_{C_3} \mathrm{gr}\, H^*(C^*))$. Since one-dimensional representations are invertible in K_0, it follows from Proposition 4.3(4) that $\chi(\mathrm{res}^{A_4}_{C_3} \mathrm{gr}\, H^*(C^*)) = 0$ if and only if $\chi(\Lambda(\Sigma^{-1}J/(x_1,x_2)J)) = 0$.

We show that $\chi(\Lambda(\Sigma^{-1}J/(x_1,x_2)J)) = 0$ if and only if $J/(x_1,x_2)J$ has a trivial one-dimensional subrepresentation of even degree. We distinguish two cases.

First, assume that $J/(x_1,x_2)J$ is the sum of two one-dimensional graded, representations L_1 in degree m_1+1 and L_2 in degree m_2+1. We have

$$\chi(\Lambda(\Sigma^{-1}J/(x_1,x_2)J)) = (1+(-1)^{m_1}[L_1])(1+(-1)^{m_2}[L_2].$$

Obviously this vanishes if one of the factors is zero, i.e., one m_i is odd and L_i is the trivial representation. We show that it is nonzero otherwise. Neither 2 nor (if available) the elements $1+\alpha$, $1+\alpha^2$ are zero-divisors in $K_0(\mathbb{F}[C_3])$, since $(1+\alpha)(1+\alpha^2) = 2+\alpha+\alpha^2$ which is not a zero-divisor by Lemma 7.7. Thus we may assume that both m_i are odd and both L_i are nontrivial representations.

Then $X^2 + X + 1$ has a zero in $\mathbb{F}$, since otherwise all one-dimensional representations are trivial. Thus $L_i \in \{\alpha, \alpha^2\}$. A case distinction yields:

$$(1-\alpha)(1-\alpha) = 1 - 2\alpha + \alpha^2 \neq 0$$
$$(1-\alpha)(1-\alpha^2) = 2 - \alpha - \alpha^2 \neq 0$$
$$(1-\alpha^2)(1-\alpha^2) = 1 + \alpha - 2\alpha^2 \neq 0$$

Secondly, assume that $J/(x_1, x_2)J$ is not the sum of two one-dimensional representations, i.e., a two-dimensional irreducible representation in one single degree $m+1$. In that case (3) does not hold. Note that $X^2 + X + 1$ does not have a zero in $\mathbb{F}$, and we have

$$\chi(\Lambda(\Sigma^{-1} J/(x_1, x_2)J)) = 2 + (-1)^m V \neq 0 \in K_0(\mathbb{F}[C_3]).$$

Finally, we show that (3) is equivalent to (4). A C_3-invariant parameter in J generates a one-dimensional trivial subrepresentation of $J/(x_1, x_2)J$. Conversely, applying the Reynolds operator $\mathcal{R}(z) = 1/3 \sum_{q \in C_3} qz$ to a representative of a generator of a C_3-invariant subspace of $J/(x_1, x_2)J$ yields a C_3-invariant parameter. □

For $\mathbb{F} = \mathbb{F}_2$ we can read off from the homology of C^* whether it is homotopy equivalent to a finite, free one:

Corollary 7.9. *A perfect $\mathbb{F}_2[A_4]$-cochain complex C^* such that its homology is four-dimensional with basis elements in degrees $0 \leq m \leq n \leq t$ is homotopy equivalent to a finite, free cochain complex if and only if m or n is odd and $H^*(C^*)$ is a trivial C_3-representation.*

Proof. Over $\mathbb{F}_2$ all one-dimensional $\mathbb{F}_2[C_3]$-representations are trivial. Moreover, recall that $t = m + n$ by Proposition 4.3(5). It follows immediately from Theorem 7.8(2) that C^* is homotopy equivalent to a finite, free cochain complex if m or n is odd and C_3 acts trivially on $H^*(C^*)$.

Conversely, if C^* is homotopy equivalent to a finite, free cochain complex, then m or n is odd by Theorem 7.8(4) and Proposition 4.3(4). □

Corollary 7.9 is an algebraic result with a topological application. The following argument is our original proof of [RSY22, Theorem 7.1].

Remark 7.10. In [Oli79, Theorem 2], Oliver proved that A_4 can not act freely on a finite CW complex X with cohomology ring $H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^n; \mathbb{Z})$. The statement of [RSY22, Theorem 7.1] is that the assumption on the cohomology ring can be weakened to $\mathbb{F}_2$-coefficients. Indeed, by [Oli79, Theorem 1], it suffices to show that A_4 can not act freely on a finite CW-complex X with cohomology ring $H^*(X; \mathbb{F}_2) \cong H^*(S^n \times S^n; \mathbb{F}_2)$ on which A_4 acts nontrivially. This holds by Theorem 7.8 applied to the cellular cochain complex of X.

8. A topological application to free A_4-actions

We will show in Theorem 8.2 that cochain complexes of finite, free A_4-CW complexes with four-dimensional cohomology are rigid. They are determined by the degrees of the nonzero cohomology groups.

We begin with a comparison of the topological and algebraic Borel construction. For any short exact sequence of finite groups

$$1 \to N \to G \to Q \to 1$$

and field k, we have equipped the group cohomology $H^*(BN;k)$ with a Q-action. More generally, $\operatorname{Ext}^*_{k[N]}(k,D^*)$ has a Q-action for any cochain complex D^* of right $k[G]$-modules by Proposition 2.7. For the cochains $C^* = C^*(X;k)$ on a left G-CW complex X, this can be modeled topologically. The quotient of the universal left G-space EG by N is a model for BN on which Q acts. More generally, for the left G-space X, the Borel construction $(X)_{hN} = EG \times_N X$ inherits a left Q-action defined by $q[e,x] \coloneqq [ge,gx]$ for $q = Ng = gN$.

Let $\varepsilon^*(G)$ be a projective resolution of k over $k[G]$. Then $\operatorname{Hom}_{k[N]}(\varepsilon^*(G), D^*)$ is the algebraic Borel construction from [AP93, Section 1.2] extended to extensions. To connect to the topological Borel construction, note that for $C^* = \operatorname{Hom}_k(C_*,k)$, i.e., the cochain complex of the singular chain complex $C_* = C_*(X;k)$, we have:

$$\operatorname{Hom}_{k[N]}(\varepsilon^*(G), \operatorname{Hom}_{\mathbb{F}}(C_*,k)) \cong \operatorname{Hom}_k(\varepsilon^*(G) \otimes_{k[N]} C_*, k)$$

In Proposition 2.7, we have equipped $\operatorname{Ext}^*_{k[N]}(k,D^*)$ with a Q-equivariant action by the group cohomology $\operatorname{Ext}^*_{k[N]}(k,k) = H^*(BN;k)$ using Yoneda composition. Alternatively, the action can be defined with cup products using a diagonal approximation as in [AP93]; see [Ben98, Lemma 3.2.3]. We used Yoneda composition since this description makes clear that the action only depends on the structure of $k[N]$ as an augmented k-algebra and thus does not depend on the comultiplication of the Hopf algebra structure.

If the N-action on X is free, then $H^*(X/N;k) \cong H^*(EG \times_N X;k)$ and the $H^*(BN;k)$-action agrees with the action induced by $H^*(-;k)$ applied to the classifying map $f\colon X/N \to BN$.

Lemma 8.1. *Let X be a finite A_4-CW complex with four-dimensional cohomology $H^*(X;\mathbb{F}_2)$ such that the restriction of the A_4-action to $P = \mathbb{Z}/2 \times \mathbb{Z}/2$ is free. Then the kernel of $H^*(BP;\mathbb{F}_2) \to H^*(X/P;\mathbb{F}_2)$ is a Steenrod closed, C_3-invariant parameter ideal.*

Proof. Let $C^* = C^*(X;\mathbb{F}_2)$ be the cellular cochain complex of X and J the annihilator ideal of $\operatorname{Ext}^*_{\mathbb{F}_2[P]}(\mathbb{F}_2, C^*)$. Then J is C_3-invariant by Proposition 2.7 and a parameter ideal by Proposition 4.3. Since the $H^*(BP)$-module structure of $H^*(X/P) \cong \operatorname{Ext}^*_{\mathbb{F}[P]}(\mathbb{F}, C^*)$ is induced by a ring homomorphism $H^*(BP) \to H^*(X/P)$, the annihilator ideal J is the kernel of this homomorphism. The ring homomorphism is induced by a map of spaces and thus commutes with Steenrod operations. Hence J is closed under Steenrod operations. □

Theorem 8.2. *If there exists a finite, free A_4-CW complex X with four-dimensional total cohomology $H^*(X;\mathbb{F}_2)$ with a basis in degrees $0 \le m \le n \le t$, then its cellular cochain complex is determined by m and n up to homotopy.*

Proof. By Corollary 6.6 for $\mathbb{F} = \mathbb{F}_2$, the cellular cochain complex $C^* \coloneqq C^*(X;\mathbb{F}_2)$ is determined by just its corresponding parameter ideal $J \subset H^*(BP)$ for $P = \mathbb{Z}/2 \times \mathbb{Z}/2$ since for spaces the lowest degree l with nonzero homology is $l = 0$ and every one-dimensional C_3-representation over $\mathbb{F}_2$ is trivial.

Since $C^*(X;\mathbb{F}_2)$ is a finite, free $\mathbb{F}_2[A_4]$-cochain complex, the parameter ideal J has a C_3-invariant parameter x by Theorem 7.8. Over $\mathbb{F}_2$, the graded two-dimensional representation $J/(x_1,x_2)J$ has to be trivial. Thus we can also find a second C_3-invariant parameter y using the Reynolds operator (as in the proof of Theorem 7.8). Let $J' \subset H^*(BP)^{C_3}$ be the ideal generated by $x, y \in H^*(BP)^{C_3} = H^*(BA_4)$. In particular, J is the extension of J' to $H^*(BP)$. Then J' is a parameter ideal; see [RSY22, Lemma 2.3]. The ideal J is Steenrod closed by Lemma 8.1 and so is J' by [RSY22, Lemma 2.8].

Finally, [RSY22, Corollary 6.13] states that there is at most one Steenrod closed parameter ideal in $H^*(BA_4)$ with parameters in degrees $m+1, n+1$. □

Theorem 8.2 does not hold for $P = \mathbb{Z}/2 \times \mathbb{Z}/2$ instead of A_4.

Example 8.3. Let $P = \mathbb{Z}/2 \times \mathbb{Z}/2$. Consider the product action of the antipodal actions on $X = S^2 \times S^3$. Then the cohomology ring of the quotient is

$$H^*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]/(x_1^3, x_2^4).$$

Thus the corresponding parameter ideal of the cellular cochain complex of X is (x_1^3, x_2^4). On the other hand, Oliver [Oli79] constructed a free A_4-action on $S^2 \times S^3$. The only Steenrod closed parameter ideal in $H^*(BA_4; \mathbb{F}_2)$ with parameters of degrees 3 and 4 is $(x_1x_2(x_1+x_2), x_1^4 + (x_1x_2)^2 + x_2^4)$. Its extension to $\mathbb{F}_2[x_1, x_2]$ classifies the cochain complex of the restriction from A_4 to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Since the two ideals are different, the cellular cochain complexes of the two P-actions are not homotopy equivalent. Note that the product action can not be extended to an A_4-action since the corresponding parameter ideal is not C_3-invariant.

We do not know if rigidity holds topologically.

Question 8.4. Given two finite, free A_4-CW complexes homotopy equivalent to $S^m \times S^n$. Are they A_4-equivariantly homotopy equivalent?

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