# SAMPLE COMPLEXITY OF THE LINEAR QUADRATIC REGULATOR: A REINFORCEMENT LEARNING LENS

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ABSTRACT. We provide the first known algorithm that provably achieves  $\varepsilon$ -optimality within  $\tilde{\mathcal{O}}(1/\varepsilon)$  function evaluations for the discounted discrete-time LQR problem with unknown parameters, without relying on two-point gradient estimates. These estimates are known to be unrealistic in many settings, as they depend on using the exact same initialization, which is to be selected randomly, for two different policies. Our results substantially improve upon the existing literature outside the realm of two-point gradient estimates, which either leads to  $\tilde{\mathcal{O}}(1/\varepsilon^2)$  rates or heavily relies on stability assumptions.

#### 1. Introduction

The Linear-Quadratic Regulator (LQR) has been used as a benchmark in optimal control theory since the sixties, see [15]. The key distinguishing property of LQR problems is that the optimal controller is linear and can be fully characterized by the celebrated Riccati equation [3]. Naturally, with the recent increase in interest in model-free and data-driven methods, the study of LQR problems has resurfaced in the literature in scenarios where the model parameters are unknown and either need to be estimated, or model-free strategies need to be used. Even though such settings fall within the realm of adaptive control, the majority of classical studies addressing this issue have concentrated on system identification or examining asymptotic outcomes [13, 6, 7, 5, 4].

Recently, the problem has been examined from a machine learning standpoint in both online and offline contexts. In online settings, least-square estimators have been demonstrated to achieve sublinear regret. This area has seen extensive research focusing on the details of these estimations [1, 8, 18, 2, 20]. This paper focuses on the offline setting and builds on a sequence of breakthrough results through a reinforcement learning lens, starting with [10]. By establishing a gradient domination/Polyak-Lojasiewicz property, the results of [10] first demonstrate that exact gradient descent, in the model-based case, converges to the global optimal solution, despite the non-convex landscape of the LQR problem under study. Using this and in the model-free settings, gradient estimations are derived from samples of the cost function value, leading to policy gradient methods. For the undiscounted discrete-time LQR under the random initialization setting, global convergence guarantees are provided using so-called one-point gradient estimates. As also explicitly pointed out in later work [17], the convergence rate for obtaining an  $\varepsilon$ -optimal policy established in [10] is only of the order  $\widetilde{\mathcal{O}}(1/\varepsilon^4)$  in zero-order evaluations. Note that by zero-order methods, we mean a setup where gradients are not available and can only be approximated using samples of the function value. The two most common such methods in the LQR problem are the one-point and two-point estimates where the former is obtained from a single function evaluation and the latter from two different such evaluations 1.

$$g_r^1(K) := C_{\text{init}}(K + rU; x_0) \frac{m \cdot n}{r} U,$$

for some smoothing radius  $r \in \mathbb{R}$  and a random matrix  $U \in \mathbb{R}^{m \times n}$  drawn uniformly over matrices with unit Frobenius norm. Likewise, the two-point estimate at a policy  $K \in \mathbb{R}^{m \times n}$  is given by

$$g_r^2(K) := \left[\mathcal{C}_{\text{init}}(K + rU; x_0) - \mathcal{C}_{\text{init}}(K - rU; x_0)\right] \frac{m.n}{2r}U,$$

<sup>&</sup>lt;sup>1</sup>To elaborate further, we provide the formulation of these two methods here. The one-point estimate at a policy  $K \in \mathbb{R}^{m \times n}$  is computed as

The next significant development related to our work is presented in [17], which considers the discounted discrete-time LQR and employs zero-order methods for gradient estimation. For the essential case of one-point gradient estimation, an enhanced analysis is proposed. This analysis does not rely on stability assumptions yet improves the convergence rate reported in [10] from  $\tilde{\mathcal{O}}(1/\varepsilon^4)$  to  $\tilde{\mathcal{O}}(1/\varepsilon^2)$ . Remarkably, with a two-point gradient estimate,  $\varepsilon$ -optimality can be achieved using only  $\tilde{\mathcal{O}}(1/\varepsilon)$  function evaluations. Similar findings are reported in [19], which are somewhat restrictive in terms of scaling of probability bounds with respect to dimensions. The substantial improvement in [17] stems from the application of sharp probabilistic estimates on stability regions using martingale techniques, a method we also heavily rely on. It should be noted that in both mentioned works, a constant learning rate is employed for the policy update. Interestingly, it is not difficult to observe that there is no advantage in using time-varying learning rates when the technique developed in [17] is applied directly.

It is worth pointing out the literature related to the discrete-time LQR problem with time-average cost. For instance, [25] employs an actor-critic approach to achieve a sample complexity of  $\mathcal{\tilde{O}}(1/\varepsilon^5)$ . Similarly, using actor-critic methods, [26] demonstrates that a sample complexity of  $\mathcal{\tilde{O}}(1/\varepsilon)$  is achievable, assuming almost sure stability and boundedness of the policy size throughout the algorithm. However, the assumption of boundedness may not always be realistic, and more so is the assumption on stability, considering the inherently noisy dynamics. For example, this issue is echoed in the recent work [11], which presupposes the boundedness of policies at every iteration.

As part of our contributions, and somewhat inspired by REINFORCE [24, 23], we propose a different gradient estimate scheme, which relies on a new take on using policy gradient for gradient estimation by appropriately sampling the deterministic policies. We are able to achieve high-probability upper bounds on our gradient estimations using moment concentration inequalities. Coupled with the adoption of time-varying learning rates, our methodology enables us to reach a  $\tilde{\mathcal{O}}(1/\varepsilon)$  convergence rate, circumventing the need for two-point gradient estimations.

### 2. Problem statement

We start with a few mathematical notations that will be used throughout. For arbitrary matrix  $M \in \mathbb{R}^{m \times n}$ , we use ||M||,  $||M||_F$ , and  $\sigma_{\min}(M)$  to denote the 2-norm, Frobenius norm, and the minimum singular value of M respectively. In addition, for a square matrix  $\tilde{M} \in \mathbb{R}^{n \times n}$ ,  $\operatorname{tr}(\tilde{M})$  represents the trace of  $\tilde{M}$ , and  $\mathcal{K}(\tilde{M})$  denotes the Kreiss constant of  $\tilde{M}$ :

(1) 
$$\mathcal{K}(\tilde{M}) := \sup_{|z| > 1, z \in \mathbb{C}} (|z| - 1) \|(zI - \tilde{M})^{-1}\|.$$

We also use  $\langle M_1, M_2 \rangle := \operatorname{tr}(M_1^{\top} M_2)$  to denote the inner product of the matrices  $M_1, M_2 \in \mathbb{R}^{m \times n}$ .

Let us now define the problem under study. We consider the discrete-time infinite-horizon discounted LQR problem

(2) 
$$\min \mathbb{E} \left[ \sum_{t \ge 0} \gamma^t (x_t^\top Q x_t + u_t^\top R u_t) \right] \quad \text{s.t. } x_{t+1} = A x_t + B u_t + z_t,$$

where  $x_t \in \mathbb{R}^n$  is the system state at time t, initialized, deterministically or randomly at  $x_0, u_t \in \mathbb{R}^m$  is the control input at time  $t, z_t \in \mathbb{R}^n$  is the additive noise of the system at time  $t, A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are the system matrices, and  $Q \in \mathbb{R}^{n \times n}$  and  $R \in \mathbb{R}^{m \times m}$  are positive-definite matrices that parameterize the quadratic costs. In most of what follows, we assume that the pair (A, B) is controllable.

which requires cost evaluations of implementing two different policies K + rU and K - rU with respect to the same initial state  $x_0$ .

As noted above, randomness is introduced in two different ways in the above problem formulation: through the initialization or as an added disturbance to the dynamics. This has led to two separate scenarios considered in the literature:

• Random initialization: where it is assumed that the additive noise  $z_t$  is zero for all  $t \ge 0$ , and that the initial state  $x_0$  is randomly chosen from an initial distribution  $\mathcal{D}_0$ . Given the initial state  $x_0$ , we let  $\mathcal{C}_{\text{init},\gamma}(K;x_0)$  be the random variable representing the cost of implementing the linear policy  $K \in \mathbb{R}^{m \times n}$ , i.e., choosing  $u_t = -Kx_t$  for  $t \ge 0$ , from the initial state  $x_0$ :

(3) 
$$\mathcal{C}_{\text{init}}(K; x_0) := \sum_{t=0}^{\infty} \gamma^t (x_t^{\top} Q x_t + u_t^{\top} R u_t),$$

where  $0 < \gamma \le 1$  is the discount factor, and the dynamics is given by (2) with  $z_t = 0$ . That is, in this case the trajectories satisfy the dynamics

$$x_{t+1} = Ax_t + Bu_t,$$

$$u_t = -Kx_t.$$

Naturally, the objective is to minimize the population cost defined as

(5) 
$$\mathcal{C}_{\text{init}}(K) := \mathbb{E}_{x_0 \sim \mathcal{D}_0}[\mathcal{C}_{\text{init}}(K; x_0)]$$

over choices of the policy K.

• Noisy dynamics: where it is assumed  $z_t$  is drawn i.i.d. for each t from a distribution  $\mathcal{D}_{\text{add}}$ , and that the initial state  $x_0$  is set deterministically to zero. Given a sequence of random variables  $\mathcal{Z} = \{z_t\}_{t \geq 0}$ , we let  $\mathcal{C}_{\text{dyn}}(K; \mathcal{Z})$  be the random variable representing the cost of implementing the linear policy K on a system where the additive noise is drawn from  $\mathcal{Z}$ , i.e.,

(6) 
$$\mathcal{C}_{\text{dyn}}(K; \mathcal{Z}) := \sum_{t=0}^{\infty} \gamma^t (x_t^{\top} Q x_t + u_t^{\top} R u_t),$$

where we have set  $x_0 = 0$ , the dynamics is given by (2) with  $u_t = -Kx_t$  for each  $t \ge 0$ , and  $0 < \gamma < 1$  is the discount factor. In contrast to the random initialization setting, the discount factor in this setting obeys  $\gamma < 1$  to ensure that the costs sum to something finite. Once again, the objective is to minimize the population cost

(7) 
$$\mathcal{C}_{\text{dyn}}(K) := \mathbb{E}_{\mathcal{Z} \sim \mathcal{D}_{\text{add}}^{\mathbb{N}}} [\mathcal{C}_{\text{dyn}}(K; \mathcal{Z})].$$

By classical results in optimal control theory, see e.g., [12, 15], the optimal controller in both cases is linear and can be expressed as  $u_t = -K^*x_t$  where  $t \ge 0$  and  $K^* \in \mathbb{R}^{m \times n}$  is the controller gain, and can be explicitly computed. When the system matrices are known, which is not the case in this paper, the policy  $K^*$  can be derived as follows

(8) 
$$K^* = \gamma (R + \gamma B^{\mathsf{T}} P B)^{-1} B^{\mathsf{T}} P A,$$

where P denotes the unique positive definite solution to the discounted discrete-time algebraic Riccati equation [3]:

$$(9) P = \gamma A^{\mathsf{T}} P A - \gamma^2 A^{\mathsf{T}} P B (R + \gamma B^{\mathsf{T}} P B)^{-1} B^{\mathsf{T}} P A + Q.$$

Throughout this paper, we closely follow the notation and terminology that is introduced in the seminal work [17]. To start, for a random variable  $v \sim \mathcal{D}$  where  $\mathcal{D} \in \{\mathcal{D}_0, \mathcal{D}_{add}\}$ , we assume that

(10) 
$$\mathbb{E}[v] = 0, \quad \mathbb{E}[vv^{\top}] = I, \text{ and } ||v||^2 \leqslant C_m \text{ a.s.}$$

where as per usual, "a.s." refers to almost surely. The assumption on the covariance being identity is without loss of generality, see [17]. Moreover, it is noteworthy to mention that using the definition (3) with the trajectories following (4), the cost for the random initialization setting can be rewritten as

(11) 
$$\mathcal{C}_{\text{init}}(K; x_0) = x_0^{\top} P_K x_0,$$

where  $P_K$  is the symmetric positive semi-definite solution to the fixed point equation:

(12) 
$$P_K = Q + K^{\top} R K + \gamma (A - B K)^{\top} P_K (A - B K).$$

Consequently, it also holds that

(13)
$$\mathcal{C}_{\text{init}}(K) = \mathbb{E}_{x_0 \sim \mathcal{D}_0} [\mathcal{C}_{\text{init}}(K; x_0)] \\
= \mathbb{E}_{x_0 \sim \mathcal{D}_0} [x_0^\top P_K x_0] \\
= \mathbb{E}_{x_0 \sim \mathcal{D}_0} [\text{tr}(P_K x_0 x_0^\top)] \\
= \text{tr}(P_K \mathbb{E}_{x_0 \sim \mathcal{D}_0} [x_0 x_0^\top]) \\
\stackrel{\text{(i)}}{=} \text{tr}(P_K),$$

where (i) follows from assumption (10) on the randomness. Although this formulation is stated for the cost under the random initialization setting, it turns out that the two costs are essentially equivalent when the respective systems are driven by noise with the same first two moments, in the sense that is shown in Lemma 2.4 to follow. For this reason, we focus on the random initialization scenario henceforth.

Let us now state the problem that we consider throughout this paper. We recall here that we assume that the pair (A, B) is controllable, however, unknown. A policy K is said to stabilize the system (A, B) if we have  $\rho(A - BK) < 1$ . Note that by the controllability assumption, there exists some policy K satisfying the condition  $\rho(A - BK) < 1$ . Furthermore, we assume access to some stable policy  $K_0$ ; this is a mild assumption that can be satisfied in a variety of ways; we refer the reader to [10, 9]. We use  $K_0$  to initialize our algorithms, which we shortly introduce.

With this in mind, the main objective of this paper is to find an  $\varepsilon$ -optimal policy  $\hat{K}$ , i.e., one satisfying

$$C_{\text{init}}(\hat{K}) - C_{\text{init}}(K^*) \leq \varepsilon,$$

where  $K^*$  is an optimal policy. The proposed scheme in the literature crucially involves forming form an estimation of the gradient of the cost function (3), which then is used for a gradient update with an appropriate learning rate. In accordance with this, we also propose an algorithm here, displayed as Algorithm 1, where we use an estimate inspired by the REINFORCE method [24, 23] with a time-varying learning rate to achieve  $\varepsilon$ -optimality.

2.1. **Regularity properties.** We introduce some notations related to the regularity properties of the cost functions; these will play a crucial role in some of our bounds; the next few results are borrowed from [17].

**Lemma 2.1** (LQR Cost is locally Lipschitz). [17, Lemma 4] Given any linear policy K with finite cost, there exist positive scalars  $(\lambda_K, \widetilde{\lambda_K}, \zeta_K)$ , depending on the function value  $C_{\text{init}}(K)$ , such that for all policies K' satisfying  $||K' - K||_F \leq \zeta_K$ , and for all initial states  $x_0$ , we have

$$|\mathcal{C}_{\text{init}}(K') - \mathcal{C}_{\text{init}}(K)| \leq \lambda_K ||K' - K||_F, \text{ and}$$

(14b) 
$$|\mathcal{C}_{\text{init}}(K'; x_0) - \mathcal{C}_{\text{init}}(K; x_0)| \leq \widetilde{\lambda_K} ||K' - K||_F.$$

**Lemma 2.2** (LQR Cost has locally Lipschitz Gradients). [17, Lemma 5] Given any linear policy K with finite cost, there exist positive scalars  $(\beta_K, \phi_K)$ , depending on the function value  $C_{\text{init}}(K)$ , such that for all policies K' satisfying  $||K' - K||_F \leq \beta_K$ , we have

(15) 
$$\|\nabla \mathcal{C}_{\text{init}}(K') - \nabla \mathcal{C}_{\text{init}}(K)\|_F \leqslant \phi_K \|K' - K\|_F.$$

**Lemma 2.3** (LQR satisfies PL). [17, Lemma 6] There exists a universal constant  $\mu_{lqr} > 0$  such that for all stable policies K, we have

(16) 
$$\|\nabla \mathcal{C}_{\text{init}}(K)\|_F^2 \geqslant \mu_{lgr}(\mathcal{C}_{\text{init}}(K) - \mathcal{C}_{\text{init}}(K^*)),$$

where  $K^*$  is a global minimizer of the cost function  $C_{\text{init}}$ .

For the sake of exposition, these properties are stated here without specifying the various smoothness and PL constants. The explicit expressions for  $\{\lambda_K, \widetilde{\lambda_K}, \phi_K, \beta_K, \zeta_K, \mu_{lqr}\}$  in terms of the parameters of the LQR problem are provided in [17, Appendix A]. Remark 2.1 to follow will provide further elaboration on these parameters as well.

**Lemma 2.4** (Equivalence of population costs up to scaling). [17, Lemma 7] For all policies K, we have

$$C_{\mathrm{dyn}}(K) = \frac{\gamma}{1-\gamma}C_{\mathrm{init}}(K).$$

This result shows that the noisy dynamics and random initialization population costs behave identically when their respective sources of randomness have the same first two moments. Therefore, we focus on the random initialization cost from now on and remind the reader that  $C(K) := C_{\text{init}}(K)$  for ease of notation.

We define the set

(17) 
$$\mathcal{G}^{lqr} := \{ K \mid \mathcal{C}(K) - \mathcal{C}(K^*) \leqslant 10\mathcal{C}(K_0) \}.$$

Since  $\mathcal{C}$  is  $(\zeta_K, \lambda_K)$  locally Lipschitz and  $(\beta_K, \phi_K)$  locally smooth, both properties hold simultaneously within a Frobenius norm radius  $\omega_K := \min\{\beta_K, \zeta_K\}$  of a point  $K \in \mathcal{G}^{lqr}$ . We define the quantities

$$\phi_{\operatorname{lqr}} := \sup_{x \in \mathcal{G}^{\operatorname{lqr}}} \phi_K, \qquad \lambda_{\operatorname{lqr}} := \sup_{K \in \mathcal{G}^{\operatorname{lqr}}} \lambda_K, \quad \text{and} \quad \omega_{\operatorname{lqr}} := \inf_{K \in \mathcal{G}^{\operatorname{lqr}}} \omega_K.$$

It is noteworthy to mention that these values are non-zero and finite, and their explicit formulation is provided in [17, Appendix A], see Remark 2.1 to follow for further clarification.

Observe that by the definition of these quantities, one can immediately show that for any  $K \in \mathcal{G}^{lqr}$  and  $K' \in \mathbb{R}^{m \times n}$  such that  $\|K' - K\|_F \leq \omega_{lqr}$ , we have that

$$\begin{split} |\mathcal{C}(K') - \mathcal{C}(K)| \leqslant & \lambda_{\mathrm{lqr}} \|K' - K\|_F, \text{ and } \\ \|\nabla \mathcal{C}(K') - \nabla \mathcal{C}(K)\|_F \leqslant & \phi_{\mathrm{lqr}} \|K' - K\|_F. \end{split}$$

**Remark 2.1.** We now describe how to specify the set of parameters  $\{\lambda_K, \widetilde{\lambda_K}, \phi_K, \beta_K, \zeta_K, \mu_{lqr}\}$  in our setting. We start by recalling that a set of parameters  $\{c_{K_0}, c_{K_1}, \ldots, c_{K_9}\}$  is defined in [17, Appendix A], which notably depend on  $\mathcal{C}(K)$ . Subsequently, by replacing said  $\mathcal{C}(K)$  with  $\sup_{K \in \mathcal{G}^{lqr}} \mathcal{C}(K)$ , they obtain a set of constants  $\{\widetilde{c_{K_0}}, \widetilde{c_{K_1}}, \ldots, \widetilde{c_{K_9}}\}$  which are independent of K. For ease of access for the reader, we point out that

(18) 
$$\omega_{lqr} = \widetilde{c_{K_9}}, \quad \phi_{lqr} = \widetilde{c_{K_7}}, \quad and \quad \lambda_{lqr} = \widetilde{c_{K_8}}.$$

Moreover, it holds that  $\max\{\|K\|, \|\nabla \mathcal{C}(K)\|_F\} \leqslant \widetilde{c_{K_1}}$  for any  $K \in \mathcal{G}^{lqr}$ , see [17, Appendix A] and [10, Lemma 22]. Note that the only required modification in the values of  $\widetilde{c_{K_0}}, \widetilde{c_{K_1}}, \ldots, \widetilde{c_{K_9}}$  for our case is having  $10\mathcal{C}(K_0) + \mathcal{C}(K^*)$  as  $\sup_{K \in \mathcal{G}^{lqr}} \mathcal{C}(K)$  instead of [17]'s  $10\mathcal{C}(K_0) - 9\mathcal{C}(K^*)$ , due to the difference in our definition of  $\mathcal{G}^{lqr}$  in (17).

We now provide an informal statement of our main result, which shows that our proposed algorithm obtains an  $\varepsilon$ -optimal policy after  $\widetilde{\mathcal{O}}(1/\varepsilon)$  iterations. As we outline precisely later, this algorithm forms an estimate  $\widehat{\nabla C}(K_t)$  of the gradient at a given time t and updates the policy  $K_t$  with a time-varying learning rate  $\alpha_t$ .

Theorem 2.1. (Informal Statement of Our Main Result): If the step-size is chosen as  $\alpha_t = C \frac{1}{t+N}$  with N "large enough", i.e.,  $N \sim \mathcal{O}\left((\log \frac{1}{\delta})^{3/2}\right)$  for any chosen  $\delta$ , and C being a known constant, then after  $T = \mathcal{O}\left(N/\varepsilon\right)$  iterations, provided the discount factor is close enough to 1, we have that

$$\mathcal{C}(K_T) - \mathcal{C}(K^*) \leqslant \varepsilon$$

with a probability of at least  $4/5 - \delta T$ . In particular, choosing  $\delta$  proportional to 1/T, we attain  $C(K_T) - C(K^*)$  with a constant probability with a sample complexity of  $\widetilde{\mathcal{O}}(1/\varepsilon)$ .

Let us first point out that this result substantially improves the ones in the literature by achieving a  $\tilde{\mathcal{O}}(1/\varepsilon)$  rate without any additional assumptions. The best previous result achieves a convergence rate of  $\tilde{\mathcal{O}}(1/\varepsilon^2)$  [17] in this setting. Indeed,  $\tilde{\mathcal{O}}(1/\varepsilon)$  rates were only available using so-called two-point estimates which re-use randomness (e.g., require being able to initialize the system at a given  $x_0$ ). As a result, one must have perfect control over a simulator to use such estimates; one cannot implement them for systems which need to learn in the real world, for example. In contrast, our result only uses gradient estimates with a single zero-order evaluation at each step.

We now begin the process of collecting the essentials needed to articulate our theorem precisely and to prove this result, beginning with a fresh examination of the policy gradient that we employ for gradient estimation.

#### 3. Policy Gradient

Most formulations of the policy gradient require probabilistic policies; in contrast, as can be seen in (4), we have used a deterministic policy given by  $u_t = -Kx_t$ . To remedy, we utilize the control input  $u_{\hat{t}}$ , to be defined shortly, where  $\hat{t}$  is sampled at random from the distribution  $\mu_{\gamma}(t) := (1-\gamma)\gamma^t$ , where  $t \in \{0, 1, 2, \dots\}$ . Keeping this in mind, we now compute

(20) 
$$\widehat{\nabla \mathcal{C}}(K) := \frac{1}{1 - \gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \nabla_K \log \pi_K(u_{\hat{t}} | x_{\hat{t}}),$$

where the control input  $u_{\hat{t}}$  is randomly chosen from the Gaussian distribution  $\mathcal{N}(-Kx_{\hat{t}}, \sigma^2 I_m)$  for some  $\sigma > 0$  only for the selected iteration  $\hat{t}$ , and  $x_{\hat{t}} = (A - BK)^{\hat{t}} x_0$  with  $x_0 \sim \mathcal{D}$  as before. Note that

(21) 
$$\mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \widehat{\nabla C}(K) \right] = \sum_{t=0}^{\infty} \gamma^{t} Q^{K}(x_{t}, u_{t}) \nabla_{K} \log \pi_{K}(u_{t}|x_{t}),$$

where

(22) 
$$\pi_K(u_t|x_t) = \frac{1}{\sqrt{(2\pi)^m (\sigma^2)^m}} e^{-\frac{(u_t + Kx_t)^\top (u_t + Kx_t)}{2\sigma^2}},$$

and

(23) 
$$Q^{K}(x_{t}, u_{t}) := x_{t}^{\mathsf{T}} Q x_{t} + u_{t}^{\mathsf{T}} R u_{t} + \gamma \mathcal{C}_{\mathrm{init}}(K; A x_{t} + B u_{t})$$
$$\stackrel{(i)}{=} x_{t}^{\mathsf{T}} Q x_{t} + u_{t}^{\mathsf{T}} R u_{t} + \gamma (A x_{t} + B u_{t})^{\mathsf{T}} P_{k} (A x_{t} + B u_{t}),$$

where (i) is on account of (11). Note that we can also rewrite  $u_{\hat{t}} \sim \mathcal{N}(-Kx_{\hat{t}}, \sigma^2 I_m)$  as

$$(24) u_{\hat{t}} = -Kx_{\hat{t}} + \sigma\eta_{\hat{t}},$$

where  $\eta_{\hat{t}} \sim \mathcal{N}(0, I_m)$ . Moreover, we have the following lemma to provide an alternative way of representing (20).

**Lemma 3.1.** The gradient estimate in (20) can be modified to get

(25) 
$$\widehat{\nabla \mathcal{C}}(K) = -\frac{1}{\sigma(1-\gamma)} Q^K(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}) \eta_{\hat{t}} x_{\hat{t}}^{\top}.$$

*Proof.* Following (20), we have that

$$\widehat{\nabla \mathcal{C}}(K) = \frac{1}{1 - \gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \nabla_K \log \pi_K(u_{\hat{t}} | x_{\hat{t}})$$

$$\stackrel{\text{(i)}}{=} \frac{1}{1 - \gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \nabla_K \left( -\frac{(u_{\hat{t}} + Kx_{\hat{t}})^\top (u_{\hat{t}} + Kx_{\hat{t}})}{2\sigma^2} \right)$$

$$= \frac{1}{1 - \gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \nabla_K \left( -\frac{u_{\hat{t}}^\top u_{\hat{t}} + 2u_{\hat{t}}^\top Kx_{\hat{t}} + x_{\hat{t}}^\top K^\top Kx_{\hat{t}}}{2\sigma^2} \right)$$

$$= \frac{1}{1 - \gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \nabla_K \left( -\frac{\text{tr}\left(2x_{\hat{t}}u_{\hat{t}}^\top K\right) + \text{tr}\left(x_{\hat{t}}x_{\hat{t}}^\top K^\top K\right)}{2\sigma^2} \right),$$
(26)

where (i) follows from (22). Now note that

(27) 
$$\nabla_K \operatorname{tr} \left( 2x_{\hat{t}} u_{\hat{t}}^\top K \right) = \nabla_K \operatorname{tr} \left( \left( 2u_{\hat{t}} x_{\hat{t}}^\top \right)^\top K \right) = \nabla_K \left\langle 2u_{\hat{t}} x_{\hat{t}}^\top, K \right\rangle = 2u_{\hat{t}} x_{\hat{t}}^\top,$$

and

$$\nabla_{K} \operatorname{tr} \left( x_{\hat{t}} x_{\hat{t}}^{\top} K^{\top} K \right) = \nabla_{K_{1}} \operatorname{tr} \left( x_{\hat{t}} x_{\hat{t}}^{\top} K^{\top} K_{1} \right) + \nabla_{K_{2}} \operatorname{tr} \left( x_{\hat{t}} x_{\hat{t}}^{\top} K_{2}^{\top} K \right)$$

$$= \nabla_{K_{1}} \operatorname{tr} \left( \left( K x_{\hat{t}} x_{\hat{t}}^{\top} \right)^{\top} K_{1} \right) + \nabla_{K_{2}} \operatorname{tr} \left( K_{2}^{\top} \left( K x_{\hat{t}} x_{\hat{t}}^{\top} \right) \right)$$

$$= \nabla_{K_{1}} \left\langle K x_{\hat{t}} x_{\hat{t}}^{\top}, K_{1} \right\rangle + \nabla_{K_{2}} \left\langle K x_{\hat{t}} x_{\hat{t}}^{\top}, K_{2} \right\rangle$$

$$= 2K x_{\hat{t}} x_{\hat{t}}^{\top}.$$

$$(28)$$

As a result, combining (27) and (28) with (26) yields

$$\begin{split} \widehat{\nabla \mathcal{C}}(K) = & \frac{1}{1-\gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \left( -\frac{1}{2\sigma^2} \left( 2(Kx_{\hat{t}}x_{\hat{t}}^\top + u_{\hat{t}}x_{\hat{t}}^\top) \right) \right) \\ = & \frac{1}{1-\gamma} Q^K(x_{\hat{t}}, u_{\hat{t}}) \left( -\frac{(u_{\hat{t}} + Kx_{\hat{t}})}{\sigma^2} x_{\hat{t}}^\top \right) \\ \stackrel{\text{(i)}}{=} & -\frac{1}{\sigma(1-\gamma)} Q^K(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}) \eta_{\hat{t}} x_{\hat{t}}^\top, \end{split}$$

where (i) follows from (24). This finishes the proof.

Taking the alternative formulation of our gradient estimate provided in Lemma 3.1 into consideration, we introduce the algorithm

## Algorithm 1 LQR with Policy Gradient

- 1: Given iteration number  $T \ge 1$ , initial policy  $K_0 \in \mathbb{R}^{m \times n}$ , noise parameter  $\sigma$ , and step size  $\alpha_t > 0$
- 2: **for**  $t \in \{0, 1, \dots, T-1\}$  **do**
- Sample  $x_0 \sim \mathcal{D}$ ,  $\hat{t} \sim \mu_{\gamma}$ , and  $\eta_{\hat{t}} \sim \mathcal{N}(0, I_m)$
- Simulate  $K_t$  for  $\hat{t}$  steps starting from  $x_0$  and observe  $x_{\hat{t}}$ . 4:
- $\widehat{\nabla \mathcal{C}}(K_t) \leftarrow \frac{1}{\sigma(1-\gamma)} \eta_{\hat{t}} x_{\hat{t}}^{\top} Q^{K_t}(x_{\hat{t}}, u_{\hat{t}})$   $K_{t+1} \leftarrow K_t \alpha_t \widehat{\nabla \mathcal{C}}(K_t)$  **return**  $K_T$

Before we state the next result, note that one can compute

(29) 
$$\nabla \mathcal{C}(K) = 2((R + \gamma B^{\top} P_K B)K - \gamma B^{\top} P_K A) \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \gamma^t x_t x_t^{\top} \right];$$

a proof can be found in [10] for the undiscounted case, where  $\gamma = 1$ , and in [17] for the discounted case. The following proposition plays a key role in our constructions.

**Proposition 3.1.** Suppose  $u_{\hat{t}} \sim \mathcal{N}(-Kx_{\hat{t}}, \sigma^2 I_m)$  as before. Then for any given K,

(30) 
$$\mathbb{E}[\widehat{\nabla C}(K)] = \nabla C(K).$$

*Proof.* Following (25),

$$\mathbb{E}[\widehat{\nabla C}(K)] \\
= \mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ \mathbb{E}_{\eta_{\hat{t}} \sim \mathcal{N}(0, I_{m})} \left[ \widehat{\nabla C}(K) | \hat{t}, x_{0} \right] | \hat{t} \right] \right] \\
\stackrel{(i)}{=} \mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ -\frac{1}{\sigma^{2}(1 - \gamma)} \mathbb{E}_{\eta_{\hat{t}} \sim \mathcal{N}(0, I_{m})} \left[ Q(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}) (\sigma \eta_{\hat{t}}) | \hat{t}, x_{0} \right] x_{\hat{t}}^{\top} | \hat{t} \right] \right] \\
(31) \qquad \stackrel{(ii)}{=} \frac{1}{1 - \gamma} \mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ \mathbb{E}_{\eta_{\hat{t}} \sim \mathcal{N}(0, I_{m})} \left[ -\nabla_{u} Q^{K}(x_{\hat{t}}, u) \Big|_{u = -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}} | \hat{t}, x_{0} \right] x_{\hat{t}}^{\top} | \hat{t} \right] \right],$$

where (i) follows from  $x_{\hat{t}}$  being determined when given  $x_0$  and  $\hat{t}$ , and (ii) from Stein's lemma [22]. Using (23), we compute

$$\nabla_u Q^K(x_{\hat{t}}, u) = \nabla_u \left( x_{\hat{t}}^\top Q x_{\hat{t}} + u^\top R u + \gamma (A x_{\hat{t}} + B u)^\top P_K (A x_{\hat{t}} + B u) \right)$$
$$= 2R u + 2\gamma B^\top P_K B u + 2\gamma B^\top P_K A x_{\hat{t}},$$

which evaluated at  $u = -Kx_{\hat{t}} + \sigma\eta_{\hat{t}}$  yields

$$\nabla_u Q^K(x_{\hat{t}}, u) \bigg|_{u = -Kx_{\hat{t}} + \sigma\eta_{\hat{t}}} = 2\left( (R + \gamma B^\top P_K B)(-Kx_{\hat{t}} + \sigma\eta_{\hat{t}}) + \gamma B^\top P_K Ax_{\hat{t}} \right).$$

Substituting in (31), we obtain

$$\mathbb{E}[\widehat{\nabla C}(K)]$$

$$= \frac{1}{1 - \gamma} \mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ 2 \left( (R + \gamma B^{\mathsf{T}} P_{K} B) K - \gamma B^{\mathsf{T}} P_{K} A \right) x_{\hat{t}} x_{\hat{t}}^{\mathsf{T}} \Big| \hat{t} \right] \right]$$

$$= \frac{2}{1 - \gamma} \mathbb{E}_{\hat{t} \sim \mu_{\gamma}} \left[ \left( (R + \gamma B^{\mathsf{T}} P_{K} B) K - \gamma B^{\mathsf{T}} P_{K} A \right) (A - BK)^{\hat{t}} \mathbb{E}_{x_{0} \sim \mathcal{D}} [x_{0} x_{0}^{\mathsf{T}}] \left( (A - BK)^{\hat{t}} \right)^{\mathsf{T}} \right]$$

$$= 2 \left( (R + \gamma B^{\mathsf{T}} P_{K} B) K - \gamma B^{\mathsf{T}} P_{K} A \right) \sum_{t=0}^{\infty} \gamma^{t} (A - BK)^{t} \mathbb{E}_{x_{0} \sim \mathcal{D}} [x_{0} x_{0}^{\mathsf{T}}] \left( (A - BK)^{t} \right)^{\mathsf{T}}$$

$$\stackrel{\text{(i)}}{=} 2 \left( (R + \gamma B^{\mathsf{T}} P_{K} B) K - \gamma B^{\mathsf{T}} P_{K} A \right) \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \gamma^{t} (A - BK)^{t} x_{0} x_{0}^{\mathsf{T}} \left( (A - BK)^{t} \right)^{\mathsf{T}} \right]$$

$$\stackrel{\text{(ii)}}{=} 2 \left( (R + \gamma B^{\mathsf{T}} P_{K} B) K - \gamma B^{\mathsf{T}} P_{K} A \right) \mathbb{E}_{x_{0} \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \gamma^{t} x_{t} x_{t}^{\mathsf{T}} \right]$$

$$\stackrel{\text{(iii)}}{=} \nabla \mathcal{C}(K),$$

where (i) is done by utilizing the linearity of expectation along with replacing  $\hat{t}$  by t as it is just a sum variable from that equation forward, (ii) is due to  $x_t = (A - BK)^t x_0$ , and (iii) follows from (29).

Before moving on to the next result, we define the undiscounted cost

(32) 
$$\mathcal{C}_{\text{und}}(K) = \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} (x_t^{\top} Q x_t + u_t^{\top} R u_t) \right],$$

subject to (4).

**Lemma 3.2.** Suppose  $K_0$  is stable and suppose that

$$\gamma \in \left(1 - \frac{\sigma_{\min}(Q)}{11\mathcal{C}_{\mathrm{und}}(K_0)}, 1\right).$$

Then

(33) 
$$\sup_{K \in \mathcal{G}^{lqr}} \rho(A - BK) \leqslant \frac{1}{\sqrt{\gamma}} \sqrt{1 - \frac{\sigma_{\min}(Q)}{10\mathcal{C}(K_0) + \mathcal{C}(K^*)}};$$

in particular, the set  $\mathcal{G}^{lqr}$  in (17) only contains stable policies.

Before we provide the proof, we point out that the condition of stability of  $K_0$  readily implies that  $\mathcal{C}_{\text{und}}(K_0)$  is finite.

*Proof.* Suppose  $\tilde{K}$  satisfies  $\rho(A - B\tilde{K}) \ge 1$ . Then we have

$$C(\tilde{K}) = \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \gamma^t (x_t^{\top} Q x_t + u_t^{\top} R u_t) \right]$$

$$\geqslant \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q) \mathbb{E} \| (A - B\tilde{K})^t x_0 \|^2$$

$$= \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q) \mathbb{E} [\text{tr}(((A - B\tilde{K})^t)^{\top} (A - B\tilde{K})^t x_0 x_0^{\top})]$$

$$\stackrel{(i)}{=} \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q) \| (A - B\tilde{K})^t \|_F^2$$

$$\geqslant \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q) \rho((A - B\tilde{K})^t)^2$$

$$\stackrel{(ii)}{\geq} \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q)$$

$$\stackrel{(iii)}{\geq} \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q)$$

$$= \frac{\sigma_{\min}(Q)}{1 - \gamma},$$
(34)

where (i) comes from the linearity of expectation along with the assumption on the noise from (10), and (ii) follows from the instability of  $\tilde{K}$  and that  $\rho(A^t) = (\rho(A))^t$  which holds for any square matrix A. Now as a result of this, if we also show  $\sup_{K \in \mathcal{G}^{lqr}} \mathcal{C}(K) < \frac{\sigma_{\min}(Q)}{1-\gamma}$ , we have proved stability of every K in the set  $\mathcal{G}^{lqr}$ . We do so as follows:

$$\frac{\sigma_{\min}(Q)}{1-\gamma} \stackrel{\text{(i)}}{>} 11\mathcal{C}_{\text{und}}(K_0) \stackrel{\text{(ii)}}{\geqslant} 11\mathcal{C}(K_0) \geqslant 10\mathcal{C}(K_0) + \mathcal{C}(K^*) \stackrel{\text{(iii)}}{\geqslant} \sup_{K \in \mathcal{G}^{\text{lqr}}} \mathcal{C}(K),$$

where (i) comes from the assumption on  $\gamma$ , (ii) from the fact that for a given policy, the undiscounted cost is not less than the discounted cost, and (iii) from the definition of the set  $\mathcal{G}^{lqr}$  from (17). This proves the second claim.

For the first part, since for any  $K \in \mathcal{G}^{lqr}$  we have that  $\rho(A - BK) < 1$ , we conclude that

$$C(K) = \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \gamma^t (x_t^\top Q x_t + u_t^\top R u_t) \right]$$

$$\stackrel{(i)}{\geqslant} \sum_{t=0}^{\infty} \gamma^t \sigma_{\min}(Q) \rho((A - BK)^t)^2$$

$$= \sigma_{\min}(Q) \sum_{t=0}^{\infty} (\gamma(\rho(A - BK))^2)^t$$

$$\stackrel{(ii)}{=} \frac{\sigma_{\min}(Q)}{1 - \gamma(\rho(A - BK))^2},$$

where (i) is done the same way as (34) and (ii) follows from  $\gamma(\rho(A-BK))^2 < 1$  for  $K \in \mathcal{G}^{lqr}$ . As a result, for  $K \in \mathcal{G}^{lqr}$ , we have that

$$1 - \gamma(\rho(A - BK))^{2} \geqslant \frac{\sigma_{\min}(Q)}{\mathcal{C}(K)} \Rightarrow$$
$$\gamma(\rho(A - BK))^{2} \leqslant 1 - \frac{\sigma_{\min}(Q)}{\mathcal{C}(K)} \Rightarrow$$
$$\rho(A - BK) \leqslant \frac{1}{\sqrt{\gamma}} \sqrt{1 - \frac{\sigma_{\min}(Q)}{\mathcal{C}(K)}},$$

which after taking a supremum gives

$$\sup_{K \in \mathcal{G}^{\text{lqr}}} \rho(A - BK) \leqslant \frac{1}{\sqrt{\gamma}} \sup_{K \in \mathcal{G}^{\text{lqr}}} \sqrt{1 - \frac{\sigma_{\min}(Q)}{\mathcal{C}(K)}} = \frac{1}{\sqrt{\gamma}} \sqrt{1 - \frac{\sigma_{\min}(Q)}{10\mathcal{C}(K_0) + \mathcal{C}(K^*)}},$$

concluding the proof.

We next introduce a high probability upper bound on our gradient estimate on any  $K \in \mathcal{G}^{lqr}$ .

**Lemma 3.3.** Suppose  $\delta \in (0, \frac{1}{e}]$ , and  $\gamma$  is chosen as in Lemma 3.2. Then for any  $K \in \mathcal{G}^{lqr}$ , we have that

(35) 
$$\|\widehat{\nabla C}(K)\|_F \leqslant \frac{\xi_3}{1-\gamma} \left(\log \frac{1}{\delta}\right)^{3/2}$$

with probability at least  $1 - \delta$ , where  $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$  are given by

(36) 
$$\xi_1 := \left( \|Q\| + 2\|R\| \widetilde{c_{K_1}}^2 + 2\gamma (10\mathcal{C}(K_0) + \mathcal{C}(K^*)) \right) e^3 n^3 \bar{\mathcal{K}}^3 C_m^{3/2}$$

(37) 
$$\xi_2 := (2\|R\| + 2\gamma \|B\|^2 (10\mathcal{C}(K_0) + \mathcal{C}(K^*))) en\bar{\mathcal{K}}C_m^{1/2}$$

(38) 
$$\xi_3 := \frac{1}{\sigma} \left( \xi_1 5^{1/2} m^{1/2} \right) + \sigma \left( \xi_2 5^{3/2} m^{3/2} \right),$$

where  $\bar{\mathcal{K}}$  is a positive constant. Moreover,

(39) 
$$\mathbb{E}\|\widehat{\nabla}\mathcal{C}(K)\|_F^2 \leqslant \frac{\xi_4}{(1-\gamma)^2},$$

where

(40) 
$$\xi_4 := \frac{1}{\sigma^2} \xi_1^2 m + 2\xi_1 \xi_2 m(m+2) + \sigma^2 \xi_2^2 m(m+2)(m+4).$$

*Proof.* Using the formulation of  $\widehat{\nabla \mathcal{C}}(K)$  derived in (25), we have

$$\|\widehat{\nabla \mathcal{C}}(K)\|_{F} = \left\| \frac{1}{\sigma(1-\gamma)} \eta_{\hat{t}} x_{\hat{t}}^{\top} Q^{K}(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}) \right\|_{F}$$

$$\leq \frac{1}{\sigma(1-\gamma)} \|\eta_{\hat{t}}\| \|x_{\hat{t}}\| Q^{K}(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma \eta_{\hat{t}}).$$
(41)

First, note that

(42) 
$$||x_{\hat{t}}|| = ||(A - BK)^{\hat{t}}x_0|| \le ||(A - BK)^{\hat{t}}|| ||x_0|| \le \sup_{t \ge 0} ||(A - BK)^t||C_m^{1/2},$$

where (i) follows from the assumption on the initial state noise mentioned in (10).

Sublemma 3.1. We have that

(43) 
$$\sup_{K \in \mathcal{G}^{lqr}} \sup_{t \ge 0} ||(A - BK)^t||$$

is finite.

Proof of Sublemma 3.1. We start by arguing that  $\mathcal{G}^{lqr}$  is a compact set. First, note that since  $||K|| \leqslant \widetilde{\mathcal{C}}_{K_1}$  (see Remark 2.1) for any  $K \in \mathcal{G}^{lqr}$ , the set  $\mathcal{G}^{lqr}$  is bounded. Secondly, since  $\mathcal{C}(K)$  is locally Lipschitz in  $\mathcal{G}^{lqr}$ , it is also continuous, and hence, by the definition of  $\mathcal{G}^{lqr}$  in (17), we have that  $\mathcal{G}^{lqr}$  is the pre-image of the closed interval  $[0, 10\mathcal{C}(K_0) + \mathcal{C}(K^*)]$  under a continuous map  $\mathcal{C}: \mathcal{G}^{lqr} \to \mathbb{R}$ , implying  $\mathcal{G}^{lqr}$  is closed as well. As a result of this, we have that  $\mathcal{G}^{lqr}$  is compact. Now we move on to show why (43) is finite.

First, let us define

$$S(x_0; K) := \sum_{t=0}^{\infty} ||x_t||^2,$$

where  $x_{t+1} = (A - BK)x_t$ . Moreover, we let

$$S(K) := \mathbb{E}_{x_0 \sim \mathcal{D}} S(x_0; K)$$

$$= \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \|x_t\|^2 \right]$$

$$= \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \sum_{t=0}^{\infty} \|(A - BK)^t x_0\|^2 \right]$$

$$= \sum_{t=0}^{\infty} \mathbb{E}_{x_0 \sim \mathcal{D}} \left[ \operatorname{tr} \left( \left( (A - BK)^t \right)^\top (A - BK)^t x_0 x_0^\top \right) \right]$$

$$= \sum_{t=0}^{\infty} \|(A - BK)^t\|_F^2$$

$$\geqslant \sum_{t=0}^{\infty} \|(A - BK)^t\|^2,$$

which after taking the square root of both sides gives

$$\sqrt{S(K)} \geqslant \sqrt{\sum_{t=0}^{\infty} \|(A - BK)^t\|^2}$$
$$\geqslant \sup_{t \geqslant 0} \|(A - BK)^t\|.$$

As a result, we have that

$$\sup_{t>0} \|(A - BK)^t\| \leqslant \sqrt{S(K)},$$

which after taking a supremum over  $\mathcal{G}^{lqr}$  yields

(44) 
$$\sup_{K \in \mathcal{G}^{\text{lqr}}} \sup_{t \geqslant 0} \|(A - BK)^t\| \leqslant \sup_{K \in \mathcal{G}^{\text{lqr}}} \sqrt{S(K)}.$$

Now it suffices to show  $\sup_{K \in \mathcal{G}^{\text{lqr}}} \sqrt{S(K)}$  is finite, which we prove by contradiction. Suppose that this is not the case. Therefore, there exists a sequence  $\{K_j\}_{j=1}^{\infty}$  such that  $\sqrt{S(K_j)} \xrightarrow{j \to \infty} \infty$ . By compactness, we can pick a convergent subsequence whose limit we denote by  $\bar{K}$ . We will abuse notation and henceforth use  $K_j$  to refer to the subsequence; observe that  $K_j$  should also satisfy  $\sqrt{S(K_j)} \xrightarrow{j \to \infty} \infty$ .

Now since  $\bar{K} \in \mathcal{G}^{lqr}$ , we have from Lemma 3.2 that  $A - B\bar{K}$  is strictly stable, and thus, there exists a Lyapunov function  $V(x) = x^{\top} \bar{P} x$  where  $\bar{P}$  is a positive definite matrix that satisfies

$$(A - B\bar{K})^{\top}\bar{P}(A - B\bar{K}) - \bar{P} = -I.$$

Therefore, for j large enough,

$$(A - BK_j)^{\top} \bar{P}(A - BK_j) - \bar{P} \le -(1/2)I.$$

Then

$$V((A - BK_j)x) - V(x) = x^T (A - BK_j) \bar{P}(A - BK_j)x - x^T \bar{P}x$$

$$\leq -(1/2) \|x\|^2$$

$$= -\frac{1}{2\lambda_{\max}(\bar{P})} \left(\lambda_{\max}(\bar{P}) \|x\|^2\right)$$

$$\stackrel{(i)}{\leq} -\frac{1}{2\lambda_{\max}(\bar{P})} V(x),$$

where (i) is due to the fact that  $V(x) \leq \lambda_{\max}(\bar{P}) ||x||^2$ . Thus,

$$(45) V((A - BK_j)x) \leqslant \left(1 - \frac{1}{2\lambda_{\max}(\bar{P})}\right)V(x).$$

As a result, we have that

$$\begin{split} S(x_0; K_j) &= \sum_{t=0}^{\infty} \|x_t\|^2 \\ &\stackrel{\text{(i)}}{\leqslant} \frac{1}{\lambda_{\min}(\bar{P})} \sum_{t=0}^{\infty} V((A - BK_j)^t x_0) \\ &\stackrel{\text{(ii)}}{\leqslant} \frac{1}{\lambda_{\min}(\bar{P})} \sum_{t=0}^{\infty} \left(1 - \frac{1}{2\lambda_{\max}(\bar{P})}\right)^t V(x_0) \\ &\stackrel{\text{(ii)}}{\leqslant} \frac{2\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})} V(x_0) \\ &\stackrel{\text{(ii)}}{\leqslant} \frac{2\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})} \|x_0\|^2, \end{split}$$

where (i) follows from  $V(x) \ge \lambda_{\min}(\bar{P}) \|x\|^2$  and (ii) from (45). Now taking an expectation over  $x_0 \sim \mathcal{D}$  yields

$$\begin{split} S(K_j) &\leqslant \frac{2\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})} \mathbb{E}_{x_0 \sim \mathcal{D}} \|x_0\|^2 \\ &= \frac{2\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})} \mathbb{E}_{x_0 \sim \mathcal{D}} \operatorname{tr}(x_0 x_0^\top) \\ &= \frac{2\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})} \operatorname{tr}\left(\mathbb{E}_{x_0 \sim \mathcal{D}}[x_0 x_0^\top]\right) \\ &= \frac{2\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})} \operatorname{tr}(I_n) \\ &= \frac{2n\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})}, \end{split}$$

and hence,

$$\sqrt{S(K_j)} \leqslant \sqrt{\frac{2n\lambda_{\max}^2(\bar{P})}{\lambda_{\min}(\bar{P})}},$$

which is finite, resulting in a contradiction, concluding the proof of (43) being finite.

We now continue with the proof of lemma. Let us first make a remark. By the Kreiss matrix theorem [16, 21], we have that

(46) 
$$\mathcal{K}(A - BK) \leqslant \sup_{t \geqslant 0} ||(A - BK)^t|| \leqslant e \ n \ \mathcal{K}(A - BK).$$

Consequently, we can define the following constant

(47) 
$$\bar{\mathcal{K}} := \sup_{K \in \mathcal{G}^{lqr}} \mathcal{K}(A - BK),$$

which is finite as a result of (46) along with the finiteness of  $\sup_{K \in \mathcal{G}^{lqr}} \sup_{t \ge 0} ||(A - BK)^t||$ . Combining (46) and (47) with (42) gives

(48) 
$$||x_{\hat{t}}|| \le e \ n \ C_m^{1/2} \ \mathcal{K}(A - BK) \le e \ n \ C_m^{1/2} \ \bar{\mathcal{K}},$$

for any  $\hat{t} \ge 0$ . Moreover,

$$Q^{K}(x_{\hat{t}}, -Kx_{\hat{t}} + \sigma\eta_{\hat{t}}) = x_{\hat{t}}^{\top}Qx_{\hat{t}} + (-Kx_{\hat{t}} + \sigma\eta_{\hat{t}})^{\top}R(-Kx_{\hat{t}} + \sigma\eta_{\hat{t}})$$

$$+ \gamma((A - BK)x_{\hat{t}} + \sigma B\eta_{\hat{t}})^{\top}P_{K}((A - BK)x_{\hat{t}} + \sigma B\eta_{\hat{t}})$$

$$\stackrel{(i)}{\leq} \|Q\|e^{2}n^{2}\bar{\mathcal{K}}^{2}C_{m} + \|R\|\|-Kx_{\hat{t}} + \sigma\eta_{\hat{t}}\|^{2} + \gamma\|P_{K}\|\|x_{\hat{t}+1} + \sigma B\eta_{\hat{t}}\|^{2}$$

$$\stackrel{(ii)}{\leq} \|Q\|e^{2}n^{2}\bar{\mathcal{K}}^{2}C_{m} + 2\|R\|\left(\|K\|^{2}\|x_{\hat{t}}\|^{2} + \sigma^{2}\|\eta_{\hat{t}}\|^{2}\right)$$

$$+ 2\gamma\mathcal{C}(K)\left(\|x_{\hat{t}+1}\|^{2} + \sigma^{2}\|B\|^{2}\|\eta_{\hat{t}}\|^{2}\right)$$

$$\stackrel{(iii)}{\leq} \left(\|Q\| + 2\|R\|\widetilde{c_{K_{1}}}^{2} + 2\gamma(10\mathcal{C}(K_{0}) + \mathcal{C}(K^{*}))\right)e^{2}n^{2}\bar{\mathcal{K}}^{2}C_{m}$$

$$+ \left(2\sigma^{2}\|R\| + 2\gamma\sigma^{2}\|B\|^{2}(10\mathcal{C}(K_{0}) + \mathcal{C}(K^{*}))\right)\|\eta_{\hat{t}}\|^{2},$$

$$(49)$$

where (i) follows from (48), (ii) from  $||P_K|| \leq \operatorname{tr}(P_K)$  along with  $\operatorname{tr}(P_K) = \mathcal{C}(K)$  as shown in (13), and (iii) from the fact that  $||K|| \leq \widetilde{c_{K_1}}$  for any  $K \in \mathcal{G}^{\operatorname{lqr}}$  (see Remark 2.1) along with reapplying (48) and utilizing the upper bound obtained on  $\mathcal{C}(K)$  by the definition of the set  $\mathcal{G}^{\operatorname{lqr}}$ . Now applying the derived bounds (48)

and (49) on (41), we conclude that

$$\|\widehat{\nabla C}(K)\|_{F} \leq \frac{\left(\|Q\| + 2\|R\|\widehat{c_{K_{1}}}^{2} + 2\gamma(10C(K_{0}) + C(K^{*}))\right)e^{3}n^{3}\bar{K}^{3}C_{m}^{3/2}}{\sigma(1 - \gamma)}\|\eta_{\hat{t}}\| + \frac{\sigma^{2}\left(2\|R\| + 2\gamma\|B\|^{2}(10C(K_{0}) + C(K^{*}))\right)e^{n\bar{K}}C_{m}^{1/2}}{\sigma(1 - \gamma)}\|\eta_{\hat{t}}\|^{3}$$

$$= \frac{1}{1 - \gamma}\left(\frac{1}{\sigma}\xi_{1}\|\eta_{\hat{t}}\| + \sigma\xi_{2}\|\eta_{\hat{t}}\|^{3}\right).$$
(50)

Furthermore, since  $\eta_{\hat{t}} \sim \mathcal{N}(0, I_m)$  for any  $\hat{t}$ ,  $\|\eta_{\hat{t}}\|^2$  is distributed according to the chi-squared distribution with m degrees of freedom ( $\|\eta_{\hat{t}}\|^2 \sim \chi^2(m)$  for any  $\hat{t}$ ). Therefore, the standard [14] bounds suggest that for arbitrary y > 0, we have that

(51) 
$$\mathbb{P}\{\|\eta_{\hat{t}}\|^2 \ge m + 2\sqrt{my} + 2y\} \le e^{-y}.$$

Now since by our assumption  $0 < \delta \le 1/e$ , it holds that  $y = m \log \frac{1}{\delta} \ge m$  and thus

$$\mathbb{P}\{\|\eta_{\hat{t}}\|^2 \geqslant 5y\} \leqslant \mathbb{P}\{\|\eta_{\hat{t}}\|^2 \geqslant m + 2\sqrt{my} + 2y\} \leqslant e^{-y},$$

which after substituting y with its value  $m \log \frac{1}{\delta}$  gives

$$\mathbb{P}\{\|\eta_{\hat{t}}\|^2 \geqslant 5m\log\frac{1}{\delta}\} \leqslant e^{-m\log\frac{1}{\delta}} = \delta^m \leqslant \delta.$$

As a result, we have  $\|\eta_{\hat{t}}\| \leqslant 5^{1/2} m^{1/2} (\log \frac{1}{\delta})^{1/2}$  and consequently

$$\|\eta_{\hat{t}}\|^3 \le 5^{3/2} m^{3/2} (\log \frac{1}{\delta})^{3/2}$$

with probability at least  $1 - \delta$ , which after applying on (50) yields

$$\|\widehat{\nabla C}(K)\|_{F} \leq \frac{1}{1-\gamma} \left( \frac{1}{\sigma} \xi_{1} 5^{1/2} m^{1/2} \left( \log \frac{1}{\delta} \right)^{1/2} + \sigma \xi_{2} 5^{3/2} m^{3/2} \left( \log \frac{1}{\delta} \right)^{3/2} \right)$$

$$\leq \frac{1}{1-\gamma} \left( \frac{1}{\sigma} \xi_{1} 5^{1/2} m^{1/2} + \sigma \xi_{2} 5^{3/2} m^{3/2} \right) \left( \log \frac{1}{\delta} \right)^{3/2}$$

$$= \frac{\xi_{3}}{1-\gamma} \left( \log \frac{1}{\delta} \right)^{3/2},$$

proving the first claim.

As for the second claim, note that using (50), we have

(52) 
$$\|\widehat{\nabla C}(K)\|_F^2 \leq \frac{1}{(1-\gamma)^2} \left( \frac{1}{\sigma^2} \xi_1^2 \|\eta_{\hat{t}}\|^2 + 2\xi_1 \xi_2 \|\eta_{\hat{t}}\|^4 + \sigma^2 \xi_2^2 \|\eta_{\hat{t}}\|^6 \right).$$

Now since  $\|\eta_{\hat{t}}\| \sim \chi(m)$  whose moments are known, taking an expectation on both sides of (52) results in

$$\mathbb{E}\|\widehat{\nabla C}(K)\|_F^2 \leq \frac{1}{(1-\gamma)^2} \left( \frac{1}{\sigma^2} \xi_1^2 \mathbb{E} \|\eta_{\hat{t}}\|^2 + 2\xi_1 \xi_2 \mathbb{E} \|\eta_{\hat{t}}\|^4 + \sigma^2 \xi_2^2 \mathbb{E} \|\eta_{\hat{t}}\|^6 \right) \\
= \frac{1}{(1-\gamma)^2} \left( \frac{1}{\sigma^2} \xi_1^2 m + 2\xi_1 \xi_2 m(m+2) + \sigma^2 \xi_2^2 m(m+2)(m+4) \right) \\
= \frac{\xi_4}{(1-\gamma)^2},$$

concluding the proof.

Following Lemma 3.3, we now define the following event for each iteration t of Algorithm 1:

(53) 
$$\mathcal{A}_t = \left\{ \|\widehat{\nabla \mathcal{C}}(K_t)\|_F \leqslant \frac{\zeta_3}{1 - \gamma} \left( \log \frac{1}{\delta} \right)^{3/2} \right\}.$$

Having this, we introduce the following lemma:

**Lemma 3.4.** Suppose  $\delta \in (0, e^{-3/2}]$ , and  $\gamma$  is chosen as in Lemma 3.2. Then for any given  $K_t \in \mathcal{G}^{lqr}$ , we have that

(54) 
$$\|\mathbb{E}[\widehat{\nabla C}(K_t)1_{\mathcal{A}_t}] - \nabla C(K_t)\|_F \leqslant \frac{3\zeta_3}{1-\gamma}\delta \left(\log \frac{1}{\delta}\right)^{3/2}.$$

*Proof.* Following Proposition 3.1, we have that

$$\begin{split} \nabla \mathcal{C}(K_t) = & \mathbb{E}[\widehat{\nabla} \widehat{\mathcal{C}}(K_t)] \\ = & \mathbb{E}[\widehat{\nabla} \widehat{\mathcal{C}}(K_t) \mathbf{1}_{\mathcal{A}_t}] + \mathbb{E}[\widehat{\nabla} \widehat{\mathcal{C}}(K_t) \mathbf{1}_{\mathcal{A}_t^c}]. \end{split}$$

Therefore,

$$\|\mathbb{E}[\widehat{\nabla C}(K_{t})1_{\mathcal{A}_{t}}] - \nabla C(K_{t})\|_{F}$$

$$= \|\mathbb{E}[\widehat{\nabla C}(K_{t})1_{\mathcal{A}_{t}^{c}}]\|_{F}$$

$$\stackrel{(i)}{\leq} \mathbb{E}\left[\|\widehat{\nabla C}(K_{t})1_{\mathcal{A}_{t}^{c}}\|_{F}\right]$$

$$= \mathbb{E}\left[\|\widehat{\nabla C}(K_{t})\|_{F}1_{\mathcal{A}_{t}^{c}}\right]$$

$$\stackrel{(ii)}{\leq} \mathbb{E}\left[\|\widehat{\nabla C}(K_{t})\|_{F}1_{\mathcal{A}_{t}^{c}}\right]$$

$$\stackrel{(ii)}{\leq} \mathbb{E}\left[\|\widehat{\nabla C}(K_{t})\|_{F}1_{\mathcal{A}_{t}^{c}}\right]$$

$$= \mathbb{P}\left\{\|\widehat{\nabla C}(K_{t})\|_{F} \geqslant \frac{\zeta_{3}\left(\log\frac{1}{\delta}\right)^{3/2}}{1-\gamma}\right\}\mathbb{E}\left[\|\widehat{\nabla C}(K_{t})\|_{F} \geqslant \frac{\zeta_{3}\left(\log\frac{1}{\delta}\right)^{3/2}}{1-\gamma}\right],$$

$$(55)$$

where (i) follows from Jensen's inequality and (ii) from the fact that

$$\mathcal{A}_t^c = \left\{ \|\widehat{\nabla} \mathcal{C}(K_t)\|_F > \frac{\zeta_3}{1 - \gamma} \left( \log \frac{1}{\delta} \right)^{3/2} \right\} \subseteq \left\{ \|\widehat{\nabla} \mathcal{C}(K_t)\|_F \geqslant \frac{\zeta_3}{1 - \gamma} \left( \log \frac{1}{\delta} \right)^{3/2} \right\}.$$

Moreover, it holds that

(56) 
$$\mathbb{E}\left[\|\widehat{\nabla}\mathcal{C}(K_t)\|_F \left\| \|\widehat{\nabla}\mathcal{C}(K_t)\|_F \geqslant \frac{\zeta_3 \left(\log\frac{1}{\delta}\right)^{3/2}}{1-\gamma}\right] \right] = \frac{\zeta_3 \left(\log\frac{1}{\delta}\right)^{3/2}}{1-\gamma} + \frac{\int_{\frac{\zeta_3}{1-\gamma} \left(\log\frac{1}{\delta}\right)^{3/2}}^{\infty} \mathbb{P}\{\|\widehat{\nabla}\mathcal{C}(K_t)\|_F \geqslant z\} dz}{\mathbb{P}\left\{\|\widehat{\nabla}\mathcal{C}(K_t)\|_F \geqslant \frac{\zeta_3 \left(\log\frac{1}{\delta}\right)^{3/2}}{1-\gamma}\right\}}.$$

Now recall from Lemma 3.3 that

(57) 
$$\mathbb{P}\left\{\|\widehat{\nabla C}(K_t)\|_F \geqslant \frac{\zeta_3}{1-\gamma} \left(\log \frac{1}{\delta}\right)^{3/2}\right\} \leqslant \delta$$

for arbitrary  $\delta$ , which implies

(58) 
$$\mathbb{P}\left\{\|\widehat{\nabla \mathcal{C}}(K_t)\|_F \geqslant z\right\} \leqslant e^{-\left(\frac{z(1-\gamma)}{\zeta_3}\right)^{2/3}}.$$

Now combining (58), (56), and (55) yields

$$\|\mathbb{E}[\widehat{\nabla C}(K_{t})1_{\mathcal{A}_{t}}] - \nabla C(K_{t})\|_{F}$$

$$\leq \mathbb{P}\left\{\|\widehat{\nabla C}(K_{t})\|_{F} \geqslant \frac{\zeta_{3} \left(\log \frac{1}{\delta}\right)^{3/2}}{1 - \gamma}\right\} \frac{\zeta_{3} \left(\log \frac{1}{\delta}\right)^{3/2}}{1 - \gamma} + \int_{\frac{\zeta_{3}}{1 - \gamma} \left(\log \frac{1}{\delta}\right)^{3/2}}^{\infty} e^{-\left(\frac{z(1 - \gamma)}{\zeta_{3}}\right)^{2/3}} dz$$

$$\stackrel{(i)}{\leq} \frac{\zeta_{3}}{1 - \gamma} \delta \left(\log \frac{1}{\delta}\right)^{3/2} + \frac{\zeta_{3}}{1 - \gamma} \int_{\left(\log \frac{1}{\delta}\right)^{3/2}}^{\infty} e^{-u^{2/3}} du$$

$$= \frac{\zeta_{3}}{1 - \gamma} \delta \left(\log \frac{1}{\delta}\right)^{3/2} + \frac{\zeta_{3}}{1 - \gamma} \left(\frac{3}{2} \delta \left(\log \frac{1}{\delta}\right)^{1/2} + \frac{3}{4} \sqrt{\pi} \operatorname{erfc}\left(\sqrt{\log \frac{1}{\delta}}\right)\right)$$

$$\stackrel{(ii)}{\leq} \frac{\zeta_{3}}{1 - \gamma} \delta \left(\log \frac{1}{\delta}\right)^{3/2} + \frac{3}{2} \delta \left(\log \frac{1}{\delta}\right)^{1/2} + \frac{3}{4} \sqrt{\pi} \delta\right)$$

$$\stackrel{(iii)}{\leq} \frac{3\zeta_{3}}{1 - \gamma} \delta \left(\log \frac{1}{\delta}\right)^{3/2},$$

$$(59)$$

where (i) follows from (57) along with a change of variables  $u = \left(\frac{1-\gamma}{\zeta_3}\right)z$  in the integral, (ii) from the fact that  $\operatorname{erfc}\left(\sqrt{\log\frac{1}{\delta}}\right) \leqslant \delta$ , and (iii) from  $\delta \leqslant e^{-3/2}$ . This concludes the proof.

Before introducing the next lemma, let us denote the optimality gap of iterate t of the algorithm by

$$\Delta_t := \mathcal{C}(K_t) - \mathcal{C}(K^*).$$

Moreover, let  $\mathcal{F}_t$  denote the  $\sigma$ -algebra containing the randomness up to iteration t of Algorithm 1 (including  $K_t$  but not  $\widehat{\nabla \mathcal{C}}(K_t)$ ). We then define

(61) 
$$\tau_1 := \min\{t \mid \Delta_t > 10\mathcal{C}(K_0)\},\,$$

which is a stopping time with respect to  $\mathcal{F}_t$ .

**Lemma 3.5.** Suppose  $\delta \in (0, e^{-3/2}]$ ,  $\gamma$  is as suggested in Lemma 3.2, and the update rule follows

(62) 
$$K_{t+1} = K_t - \alpha_t \widehat{\nabla \mathcal{C}}(K_t)$$

with a step-size  $\alpha_t$  such that for all  $t \in \{0, 1, 2, ...\}$ ,

$$\alpha_t \leqslant \frac{\omega_{lqr}}{\frac{\zeta_3}{1-\gamma} \left(\log \frac{1}{\delta}\right)^{3/2}}.$$

Then for any  $t \in \{0, 1, 2, \dots\}$ , we have

$$(63) \qquad \mathbb{E}\left[\Delta_{t+1} 1_{\mathcal{A}_t} | \mathcal{F}_t\right] 1_{\tau_1 > t} \leqslant \left(\left(1 - \mu_{lqr}\alpha_t\right) \Delta_t + \frac{3\zeta_3 \widetilde{c_{K_1}}}{1 - \gamma} \delta\left(\log \frac{1}{\delta}\right)^{3/2} \alpha_t + \frac{\phi_{lqr}\alpha_t^2}{2} \frac{\xi_4}{(1 - \gamma)^2}\right) 1_{\tau_1 > t},$$

where  $\Delta_t$  and  $A_t$  are defined in (60) and (53) respectively.

*Proof.* First, note that by the definition of  $\tau_1$  in (61),  $\tau_1 > t$  implies  $K_t \in \mathcal{G}^{lqr}$ . In addition, since  $\alpha_t \leq \frac{\omega_{lqr}}{\frac{\zeta_3}{1-\gamma}(\log \frac{1}{\delta})^{3/2}}$ , the event  $\mathcal{A}_t$  implies that

$$||K_{t+1} - K_t||_F = ||\alpha_t \widehat{\nabla C}(K_t)||_F \le \omega_{\text{lqr}}.$$

Thus, by local smoothness of  $\mathcal{C}(K_t)$ , see Lemma 2.2, it holds that

$$\begin{split} (\Delta_{t+1} - \Delta_t) \mathbf{1}_{\tau_1 > t} \mathbf{1}_{\mathcal{A}_t} = & (\mathcal{C}(K_{t+1} - \mathcal{C}(K_t)) \mathbf{1}_{\tau_1 > t} \mathbf{1}_{\mathcal{A}_t} \\ \leqslant & \left( \langle \nabla \mathcal{C}(K_t), K_{t+1} - K_t \rangle + \frac{\phi_{\text{lqr}}}{2} \|K_{t+1} - K_t\|_F^2 \right) \mathbf{1}_{\tau_1 > t} \mathbf{1}_{\mathcal{A}_t} \\ = & \left( -\alpha_t \left\langle \nabla \mathcal{C}(K_t), \widehat{\nabla \mathcal{C}}(K_t) \right\rangle + \frac{\phi_{\text{lqr}} \alpha_t^2}{2} \|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 \right) \mathbf{1}_{\tau_1 > t} \mathbf{1}_{\mathcal{A}_t}, \end{split}$$

which after taking an expectation conditioned on  $\mathcal{F}_t$  gives

$$\mathbb{E}[\Delta_{t+1} 1_{\tau_1 > t} 1_{\mathcal{A}_t} | \mathcal{F}_t] - \mathbb{E}[\Delta_t 1_{\tau_1 > t} 1_{\mathcal{A}_t} | \mathcal{F}_t]$$

$$\leq -\alpha_t \left\langle \nabla \mathcal{C}(K_t), \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\tau_1 > t} 1_{\mathcal{A}_t} | \mathcal{F}_t] \right\rangle + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 1_{\tau_1 > t} 1_{\mathcal{A}_t} | \mathcal{F}_t].$$

Since  $\Delta_t$  and  $1_{\tau_1>t}$  are determined by  $\mathcal{F}_t$ ,

$$\begin{split} &\mathbb{E}[\Delta_{t+1} 1_{\mathcal{A}_t} | \mathcal{F}_t] 1_{\tau_1 > t} \\ &\leqslant \left( \Delta_t \mathbb{E}[1_{\mathcal{A}_t} | \mathcal{F}_t] - \alpha_t \left\langle \nabla \mathcal{C}(K_t), \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] \right\rangle + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 1_{\mathcal{A}_t} | \mathcal{F}_t] \right) 1_{\tau_1 > t} \\ &\leqslant \left( \Delta_t - \alpha_t \left\langle \nabla \mathcal{C}(K_t), \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] \right\rangle + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 | \mathcal{F}_t] \right) 1_{\tau_1 > t} \\ &= \Delta_t 1_{\tau_1 > t} - \alpha_t \left\langle \nabla \mathcal{C}(K_t), \nabla \mathcal{C}(K_t) + \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] - \nabla \mathcal{C}(K_t) \right\rangle 1_{\tau_1 > t} \\ &+ \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 | \mathcal{F}_t] 1_{\tau_1 > t} \\ &= \Delta_t 1_{\tau_1 > t} - \alpha_t \left\langle \nabla \mathcal{C}(K_t), \nabla \mathcal{C}(K_t) \right\rangle 1_{\tau_1 > t} \\ &= \Delta_t 1_{\tau_1 > t} - \alpha_t \left\langle \nabla \mathcal{C}(K_t), \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] - \nabla \mathcal{C}(K_t) \right\rangle 1_{\tau_1 > t} + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 | \mathcal{F}_t] 1_{\tau_1 > t} \\ &- \alpha_t \left\langle \nabla \mathcal{C}(K_t), \mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] - \nabla \mathcal{C}(K_t) \right\rangle 1_{\tau_1 > t} + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \mathbb{E}[\|\widehat{\nabla \mathcal{C}}(K_t)\|_F^2 | \mathcal{F}_t] 1_{\tau_1 > t} \\ &+ \alpha_t \|\nabla \mathcal{C}(K_t)\|_F \|\mathbb{E}[\widehat{\nabla \mathcal{C}}(K_t) 1_{\mathcal{A}_t} | \mathcal{F}_t] - \nabla \mathcal{C}(K_t)\|_F 1_{\tau_1 > t} + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \frac{\zeta_4}{(1 - \gamma)^2} 1_{\tau_1 > t} \\ &\leqslant \Delta_t 1_{\tau_1 > t} - \alpha_t \mu_{\text{lqr}} \Delta_t 1_{\tau_1 > t} + \frac{3\zeta_3 \widehat{c_{K_1}}}{1 - \gamma} \delta \left( \log \frac{1}{\delta} \right)^{3/2} \alpha_t 1_{\tau_1 > t} + \frac{\phi_{\text{lqr}}}{2} \alpha_t^2 \frac{\zeta_4}{(1 - \gamma)^2} 1_{\tau_1 > t} \\ &= \left( (1 - \mu_{\text{lqr}} \alpha_t) \Delta_t + \frac{3\zeta_3 \widehat{c_{K_1}}}{1 - \gamma} \delta \left( \log \frac{1}{\delta} \right)^{3/2} \alpha_t + \frac{\phi_{\text{lqr}} \alpha_t^2}{2} \frac{\xi_4}{(1 - \gamma)^2} \right) 1_{\tau_1 > t}, \end{split}$$

where (i) follows from  $1_{\mathcal{A}_t} \leq 1$ , (ii) from Lemma 3.3, and (iii) from applying the PL inequality (16), the fact that  $\|\nabla \mathcal{C}(K_t)\|_F \leq \widetilde{c_{K_1}}$  for any  $K_t \in \mathcal{G}^{\text{lqr}}$  (see Remark 2.1), and Lemma 3.4. This finishes the proof of Lemma 3.5.

We are now in a position to state a precise version of our main result.

**Theorem 3.1.** Suppose  $K_0$  is stable and  $\gamma$  is as suggested in Lemma 3.2. If the step-size  $\alpha_t$  is chosen as

(64) 
$$\alpha_t = \frac{2}{\mu_{lqr}} \frac{1}{t+N} \quad \text{for} \quad N = \max \left\{ N_1, \frac{2}{\mu_{lqr}} \frac{\xi_3 \left(\log \frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\omega_{lqr}} \right\},$$

where

(65) 
$$N_1 = \max \left\{ 2, \frac{4\phi_{lqr}\xi_4}{\mu_{lqr}^2(1-\gamma)^2} \frac{2}{\mathcal{C}(K_0)} \right\},\,$$

then for a given error tolerance  $\varepsilon$  such that  $\mathcal{C}(K_0) \geqslant \frac{\varepsilon}{20}$ , and  $\delta$  chosen arbitrarily to satisfy

$$\delta \leqslant \min \left\{ 2 \times 10^{-5}, \left( \frac{\phi_{lqr} \zeta_4 \omega_{lqr}}{960 \zeta_3^2 \widetilde{c_{K_1}} \mathcal{C}(K_0)} \right)^3 \varepsilon^3, \left( \frac{\phi_{lqr} \zeta_4}{480 (1 - \gamma) \mu_{lqr} \zeta_3 \widetilde{c_{K_1}} N_1 \mathcal{C}(K_0)} \right)^3 \varepsilon^3, \left( \frac{\mu_{lqr} (1 - \gamma)}{240 \zeta_3 \widetilde{c_{K_1}}} \right)^3 \varepsilon^3 \right\},$$
(66)

the iterate  $K_T$  of the Algorithm 1 after

(67) 
$$T = \frac{40}{\varepsilon} N \mathcal{C}(K_0)$$

steps satisfies

(68) 
$$C(K_T) - C(K^*) \leqslant \varepsilon$$

with a probability of at least  $4/5 - \delta T$ .

It is essential to re-emphasize that, as also evident from the statement of Theorem 3.1, there is no reliance on a stability assumption; rather, the result is proven to hold with a certain probability. In particular, the instances of the algorithm that lead to instability at any iteration before T are factored into the failure probability  $1/5 + \delta T$ .

The proof of Theorem 3.1 relies on an intermediate result, namely Proposition 3.2, which we establish next. Before doing so, we provide some observations regarding the statement of the theorem. First, we have the following remark for  $\delta$ :

Remark 3.1 (Selection of  $\delta$  for the probability of failure). The  $\delta T$  term in the probability of failure stated in Theorem 3.1 can be adjusted arbitrarily; however, since T depends on N which depends on  $\delta$  itself, we add some further discussion here. If we want the  $\delta T$  term to be less than some arbitrary small  $\delta'$ , it needs to hold that

$$\delta T = \delta \frac{40}{\varepsilon} \max \left\{ N_1 \mathcal{C}(K_0), \frac{2\zeta_3 \mathcal{C}(K_0)}{\mu_{lqr} \omega_{lqr} (1 - \gamma)} \left( \log \frac{1}{\delta} \right)^{3/2} \right\} \leqslant \delta'.$$

Therefore,  $\delta$  first needs to satisfy

(69) 
$$\frac{40}{\varepsilon} N_1 \mathcal{C}(K_0) \delta \leqslant \delta' \Rightarrow \delta \leqslant \frac{\delta' \varepsilon}{40 N_1 \mathcal{C}(K_0)},$$

and secondly.

(70) 
$$\frac{80\zeta_3 \mathcal{C}(K_0)}{\mu_{lqr}\omega_{lqr}(1-\gamma)} \frac{1}{\varepsilon} \delta \left(\log \frac{1}{\delta}\right)^{3/2} \leqslant \delta' \Rightarrow \delta \left(\log \frac{1}{\delta}\right)^{3/2} \leqslant \frac{\mu_{lqr}\omega_{lqr}(1-\gamma)}{80\zeta_3 \mathcal{C}(K_0)} \delta' \varepsilon.$$

Now since  $a^3 \left(\log \frac{1}{a^3}\right)^{3/2} \leq a$  for any  $a \in (0,1)$ , for (70) to hold, it would suffice to have

(71) 
$$\delta \leqslant \left(\frac{\mu_{lqr}\omega_{lqr}(1-\gamma)}{80\zeta_3 \mathcal{C}(K_0)}\right)^3 (\delta'\varepsilon)^3.$$

Note that (71) is only a loose sufficient bound on  $\delta$  that can be improved (for instance, the exponents in (71) can be reduced from 3 to 2 considering the other requirements on  $\delta$  in (66)); however, since the dependence of T on  $\delta$  is logarithmic, the looser requirement only adds a constant and does not change the order.

As a result, adding (69) and (71) to the existing requirements on  $\delta$  in (66), we will have

$$\delta \leqslant \min \left\{ 2 \times 10^{-5}, \left( \frac{\phi_{lqr} \zeta_4 \omega_{lqr}}{960 \zeta_3^2 \widetilde{c_{K_1}} \mathcal{C}(K_0)} \right)^3 \varepsilon^3, \left( \frac{\phi_{lqr} \zeta_4}{480 (1 - \gamma) \mu_{lqr} \zeta_3 \widetilde{c_{K_1}} N_1 \mathcal{C}(K_0)} \right)^3 \varepsilon^3, \left( \frac{\mu_{lqr} (1 - \gamma)}{240 \zeta_3 \widetilde{c_{K_1}}} \right)^3 \varepsilon^3, \frac{\delta' \varepsilon}{40 N_1 \mathcal{C}(K_0)}, \left( \frac{\mu_{lqr} \omega_{lqr} (1 - \gamma)}{80 \zeta_3 \mathcal{C}(K_0)} \right)^3 (\delta' \varepsilon)^3 \right\},$$
(72)

 $\Diamond$ 

which will lead to the result of Theorem 3.1 holding with probability  $4/5 - \delta'$  after

$$T \sim \frac{N}{\varepsilon} \sim \mathcal{O}\left(\frac{1}{\varepsilon} \left(\log \frac{1}{(\delta'\varepsilon)^3}\right)^{3/2}\right) = \mathcal{O}\left(\frac{1}{\varepsilon} \left(\log \frac{1}{\delta'} + \log \frac{1}{\varepsilon}\right)^{3/2}\right) = \widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon}\right)$$

iterations of Algorithm 1.

Secondly, we find it worthwile to provide the following observation on the choice of  $\sigma$ :

Remark 3.2 (Selection of  $\sigma$  and its impact on T). Note that the value of  $\sigma$  in (22) is at our discretion, so one natural question would be regarding the asymptotic analysis of  $\sigma$  and its impact on our rate T. Observe that the only effect of  $\sigma$  on T is through  $\xi_3$  and  $\xi_4$  defined in (38) and (40) respectively. Taking everything else as constants, following the choice of T and N suggested in Theorem 3.1, we have that  $T \geq \mathcal{O}(\max\{\xi_3, \xi_4\})$ . Now since both  $\xi_3$  and  $\xi_4$  will grow unbounded as  $\sigma$  approaches either zero or infinity, so does T. Therefore, we choose a non-zero value for  $\sigma$  instead. An optimal value can be derived, but given that this only affects the constants in the rate, we opt for  $\sigma = 1$ .

Thirdly, note that for any  $K_t \in \mathcal{G}^{lqr}$ , by our choice of  $\alpha_t$  and N in Theorem 3.1, we have

(73) 
$$||K_{t+1} - K_t||_F = ||\alpha_t \widehat{\nabla C}(K_t)||_F$$

$$= \frac{2}{\mu_{\text{lqr}}} \frac{1}{t+N} ||\widehat{\nabla C}(K_t)||_F$$

$$\leq \frac{2}{\mu_{\text{lqr}}} \frac{1}{N} ||\widehat{\nabla C}(K_t)||_F$$

$$\stackrel{\text{(i)}}{\leq} \omega_{\text{lqr}} \frac{||\widehat{\nabla C}(K_t)||_F}{\frac{\xi_3}{1-\gamma} \left(\log \frac{1}{\delta}\right)^{3/2}},$$

where (i) follows from (64). Now applying Lemma 3.3 on (73) yields

(74) 
$$||K_{t+1} - K_t||_F \leqslant \omega_{\text{lqr}} = \inf_{K \in \mathcal{C}^{\text{lqr}}} \omega_K$$

with probability at least  $1 - \delta$ , where  $\omega_K = \min\{\beta_K, \zeta_K\}$ . This implies that the local Lipschitzness and local smoothness properties of the cost hold for the update at iteration t with probability at least  $1 - \delta$ .

Fourthly, to help unravel the logical reasoning elucidated in the proof, we introduce the following stopping times:

(75) 
$$\tau_2 := \min \left\{ t \geqslant 1 \mid \|\widehat{\nabla C}(K_{t-1})\|_F > \frac{\xi_3}{1 - \gamma} \left( \log \frac{1}{\delta} \right)^{3/2} \right\}$$

with  $\tau_1$  previously defined in (61). Essentially, one can observe that as long as  $t < \tau_1$  and  $t + 1 < \tau_2$ , it holds that  $K_t \in \mathcal{G}^{\text{lqr}}$  and  $\|K_{t+1} - K_t\|_F \leq \omega_{\text{lqr}}$ , implying that local Lipschitzness and local smoothness properties of the cost hold until that iteration. By the definition of  $\tau$  in (75), we have that

$$1_{\tau>t} = 1_{\tau_1>t} 1_{\tau_2>t}.$$

Moreover, following the definition of  $A_t$  in (53), it also holds that

$$1_{\tau_2 > t+1} = 1_{\tau_2 > t} 1_{\mathcal{A}_t}.$$

Finally, we note that the idea of introducing a stopping time (75), which helps identify the failure of the algorithm and is also used to define a stopped process later on, is inspired by [17]. However, despite the similarity of our forthcoming statements to those in the proof of [17, Theorem 8], the paths we take to prove said statements are considerably different due to the differences in how we defined our stopping time (and

subsequently the stopped process to be defined later on), our gradient estimation method, the time-varying learning rate, etc.

Having covered all of the above, we are now ready to present the following proposition:

**Proposition 3.2.** Under the parameter settings of Theorem 3.1, we have

$$\mathbb{E}[\Delta_T 1_{\tau > T}] \leqslant \frac{\varepsilon}{20}.$$

Furthermore, the event  $\{\tau > T\}$  happens with a probability of at least  $\frac{17}{20} - \delta T$ .

*Proof.* The following provides us with a stepping stone for proving the first claim:

Sublemma 3.2. Under the parameter settings of Theorem 3.1, we have that

(78) 
$$\mathbb{E}[\Delta_t 1_{\tau > t}] \leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{t + N},$$

for all  $t \in [T]$ .

*Proof of Sublemma 3.2.* We prove this result by induction on t as follows:

Base case (t=0):

$$\Delta_0 1_{\tau > 0} \leqslant \Delta_0 \leqslant \mathcal{C}(K_0) = \frac{N\mathcal{C}(K_0)}{0 + N} \leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{0 + N}$$

which after taking expectation proves the claim for t = 0.

**Inductive step:** Let  $k \in [T-1]$  be fixed and assume that

(79) 
$$\mathbb{E}[\Delta_k 1_{\tau > k}] \leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{k+N}$$

holds (the inductive hypothesis). Observe that

$$\mathbb{E}[\Delta_{k+1} 1_{\tau > k+1}] \stackrel{\text{(i)}}{=} \mathbb{E}[\Delta_{k+1} 1_{\tau_1 > k+1} 1_{\tau_2 > k+1}]$$

$$\stackrel{\text{(ii)}}{\leqslant} \mathbb{E}[\Delta_{k+1} 1_{\tau_1 > k} 1_{\tau_2 > k} 1_{\mathcal{A}_k}]$$

$$= \mathbb{E}[\mathbb{E}[\Delta_{k+1} 1_{\tau_1 > k} 1_{\tau_2 > k} 1_{\mathcal{A}_k} | \mathcal{F}_k]]$$

$$\stackrel{\text{(iii)}}{=} \mathbb{E}[\mathbb{E}[\Delta_{k+1} 1_{\mathcal{A}_k} | \mathcal{F}_k] 1_{\tau_1 > k} 1_{\tau_2 > k}],$$

$$(80)$$

where (i) follows from (76), (ii) from equation (77) along with the fact that  $1_{\tau_1>k+1} \leq 1_{\tau_1>k}$ , and (iii) is due to  $1_{\tau_2>k}$  and  $1_{\tau_1>k}$  being determined by  $\mathcal{F}_k$ . By Lemma 3.5, we have that

$$\begin{aligned}
&(\mathbb{E}[\Delta_{k+1} 1_{\mathcal{A}_{k}} | \mathcal{F}_{k}] 1_{\tau_{1} > k}) 1_{\tau_{2} > k} \\
&\leq \left( \left( \left( 1 - \mu_{\operatorname{lqr}} \alpha_{k} \right) \Delta_{k} + \frac{3\zeta_{3} \widetilde{c_{K_{1}}}}{1 - \gamma} \delta \left( \log \frac{1}{\delta} \right)^{3/2} \alpha_{k} + \frac{\phi_{\operatorname{lqr}} \alpha_{k}^{2}}{2} \frac{\xi_{4}}{(1 - \gamma)^{2}} \right) 1_{\tau_{1} > k} \right) 1_{\tau_{2} > k} \\
&\stackrel{\text{(i)}}{=} \left( \left( 1 - \frac{2}{k + N} \right) \Delta_{k} + \frac{6\zeta_{3} \widetilde{c_{K_{1}}} \delta \left( \log \frac{1}{\delta} \right)^{3/2}}{\mu_{\operatorname{lqr}} (1 - \gamma)} \frac{1}{k + N} + \frac{2\phi_{\operatorname{lqr}} \xi_{4}}{(1 - \gamma)^{2} \mu_{\operatorname{lqr}}^{2}} \frac{1}{(k + N)^{2}} \right) 1_{\tau > k}, \end{aligned}$$

where (i) follows from (76) along with replacing  $\alpha_k$  with its value in (64). Now due to the choice of  $\delta$  in (66), we have that

$$\delta \leqslant \left(\frac{\mu_{\text{lqr}}(1-\gamma)}{240\zeta_3\widetilde{c_{K_1}}}\right)^3 \varepsilon^3,$$

 $\Diamond$ 

which after noting that  $a^3 \left(\log \frac{1}{a^3}\right)^{3/2} \leqslant a$  for any  $a \in (0,1)$  implies

(82) 
$$\delta \left( \log \frac{1}{\delta} \right)^{3/2} \leqslant \frac{\mu_{\text{lqr}}(1-\gamma)}{240\zeta_3 \widetilde{c_{K_1}}} \varepsilon \Rightarrow \frac{6\zeta_3 \widetilde{c_{K_1}} \delta \left( \log \frac{1}{\delta} \right)^{3/2}}{\mu_{\text{lqr}}(1-\gamma)} \leqslant \frac{\varepsilon}{40}.$$

Applying (82) on (81) yields

$$\mathbb{E}[\Delta_{k+1} 1_{\mathcal{A}_k} | \mathcal{F}_k] 1_{\tau_1 > k} 1_{\tau_2 > k} \\
\leq \left( \left( 1 - \frac{2}{k+N} \right) \Delta_k + \frac{\varepsilon}{40} \frac{1}{k+N} + \frac{2\phi_{\text{lqr}} \xi_4}{(1-\gamma)^2 \mu_{\text{lqr}}^2} \frac{1}{(k+N)^2} \right) 1_{\tau > k} \\
\leq \left( 1 - \frac{2}{k+N} \right) \Delta_k 1_{\tau > k} + \frac{\varepsilon}{40} \frac{1}{k+N} + \frac{2\phi_{\text{lqr}} \xi_4}{(1-\gamma)^2 \mu_{\text{lqr}}^2} \frac{1}{(k+N)^2},$$

which after taking expectation results in

(83) 
$$\mathbb{E}\left[\mathbb{E}\left[\Delta_{k+1} 1_{\mathcal{A}_{k}} \middle| \mathcal{F}_{k}\right] 1_{\tau_{1} > k} 1_{\tau_{2} > k}\right] \\ \leq \left(1 - \frac{2}{k+N}\right) \mathbb{E}\left[\Delta_{k} 1_{\tau > k}\right] + \frac{\varepsilon}{40} \frac{1}{k+N} + \frac{2\phi_{\text{lqr}} \xi_{4}}{(1-\gamma)^{2} \mu_{\text{lqr}}^{2}} \frac{1}{(k+N)^{2}}.$$

Combining the hypothesis (inequality (79)) and inequality (80) with (83), we obtain

$$\begin{split} &\mathbb{E}[\Delta_{k+1} \mathbf{1}_{\tau > k+1}] \\ &\leqslant \left(1 - \frac{2}{k+N}\right) \left(\frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{k+N}\right) + \frac{\varepsilon}{40} \frac{1}{k+N} + \frac{2\phi_{\text{lqr}} \xi_4}{(1-\gamma)^2 \mu_{\text{lqr}}^2} \frac{1}{(k+N)^2} \right) \\ &\leqslant \frac{\varepsilon}{40} + \left(1 - \frac{1}{k+N}\right) \frac{N\mathcal{C}(K_0)}{k+N} - \frac{1}{(k+N)^2} \left(N\mathcal{C}(K_0) - \frac{2\phi_{\text{lqr}} \xi_4}{(1-\gamma)^2 \mu_{\text{lqr}}^2}\right) \\ &\leqslant \frac{\varepsilon}{40} + \left(\frac{k+N-1}{(k+N)^2}\right) N\mathcal{C}(K_0) \\ &\leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{k+N+1}, \end{split}$$

where (i) follows from the fact that

$$NC(K_0) \ge N_1C(K_0) \ge \left(\frac{4\phi_{\text{lqr}}\xi_4}{\mu_{\text{lqr}}^2(1-\gamma)^2}\frac{2}{C(K_0)}\right)C(K_0) = \frac{8\phi_{\text{lqr}}\xi_4}{(1-\gamma)^2\mu_{\text{lqr}}^2} \ge \frac{2\phi_{\text{lqr}}\xi_4}{(1-\gamma)^2\mu_{\text{lqr}}^2}$$

This proves the claim for k + 1, completing the inductive step.

Now utilizing Sublemma 3.2 and the choice of T from (67) in Theorem 3.1,

$$\mathbb{E}[\Delta_T 1_{\tau > T}] \leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{T + N} \leqslant \frac{\varepsilon}{40} + \frac{N\mathcal{C}(K_0)}{T} = \frac{\varepsilon}{20},$$

which finishes the proof of the first claim of Proposition 3.2. Now before moving on to the second claim, we introduce the following sublemma:

**Sublemma 3.3.** Under the parameter setup of Theorem 3.1, we have that for all  $t \in [T]$ ,

$$(84) \qquad \frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t} + \frac{\phi_{lqr}\zeta_{4}}{2(1-\gamma)^{2}}\alpha_{t}^{2} + \frac{4\phi_{lqr}\zeta_{4}}{(1-\gamma)^{2}\mu_{lqr}^{2}}\frac{1}{t+N+1} \leqslant \frac{4\phi_{lqr}\zeta_{4}}{(1-\gamma)^{2}\mu_{lqr}^{2}}\frac{1}{t+N}$$

Proof of Sublemma 3.3. First, substituting  $\alpha_t$  with its value in (64), inequality (84) becomes

$$\frac{6\zeta_{3}\widetilde{c_{K_{1}}}\delta\left(\log\frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\mu_{\text{lqr}}}\frac{1}{t+N} + \frac{2\phi_{\text{lqr}}\zeta_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}}\left(\frac{1}{(t+N)^{2}} + \frac{2}{t+N+1}\right) \leqslant \frac{2\phi_{\text{lqr}}\zeta_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}}\left(\frac{2}{t+N}\right)$$

$$\iff \frac{6\zeta_{3}\widetilde{c_{K_{1}}}\delta\left(\log\frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\mu_{\text{lqr}}}\frac{1}{t+N} \leqslant \frac{2\phi_{\text{lqr}}\zeta_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}}\left(\frac{2}{t+N} - \frac{2}{t+N+1} - \frac{1}{(t+N)^{2}}\right)$$

$$\iff \frac{6\zeta_{3}\widetilde{c_{K_{1}}}\delta\left(\log\frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\mu_{\text{lqr}}}\frac{1}{t+N} \leqslant \frac{2\phi_{\text{lqr}}\zeta_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}}\left(\frac{2}{(t+N)(t+N+1)} - \frac{1}{(t+N)^{2}}\right)$$

$$\iff \frac{6\zeta_{3}\widetilde{c_{K_{1}}}\delta\left(\log\frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\mu_{\text{lqr}}}\frac{1}{t+N} \leqslant \frac{2\phi_{\text{lqr}}\zeta_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}}\left(\frac{t+N-1}{(t+N)^{2}(t+N+1)}\right)$$

$$\iff \delta\left(\log\frac{1}{\delta}\right)^{3/2} \leqslant \frac{\phi_{\text{lqr}}\zeta_{4}}{3\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\text{lqr}}}\left(\frac{t+N-1}{(t+N)(t+N+1)}\right).$$
(85)

Note that for the right-hand side of (85), we have for all  $t \in [T]$  that

$$\frac{\phi_{\text{lqr}}\zeta_{4}}{3\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\text{lqr}}}\left(\frac{t+N-1}{t+N}\frac{1}{t+N+1}\right) \stackrel{\text{(i)}}{\geqslant} \frac{\phi_{\text{lqr}}\zeta_{4}}{6\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\text{lqr}}}\left(\frac{1}{t+N+1}\right) 
\geqslant \frac{\phi_{\text{lqr}}\zeta_{4}}{6\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\text{lqr}}}\left(\frac{1}{T+N+1}\right) 
\stackrel{\text{(ii)}}{\geqslant} \frac{\phi_{\text{lqr}}\zeta_{4}}{12\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\text{lqr}}}\left(\frac{1}{T}\right),$$
(86)

where (i) follows from the fact that  $\frac{t+N-1}{t+N} \geqslant \frac{1}{2}$  which is due to  $N \geqslant 2$  (see (64) and (65)), and (ii) from  $\mathcal{C}(K_0) \geqslant \frac{\varepsilon}{20}$  under the settings of Theorem 3.1, which results in

$$T = \frac{40}{\varepsilon} N \mathcal{C}(K_0) \geqslant 2N \geqslant N + 1 \Rightarrow \frac{1}{T + N + 1} \geqslant \frac{1}{2T}.$$

As a result of (85) and (86), in order to conclude the proof Sublemma 3.3, it would suffice to show that

$$\delta\left(\log\frac{1}{\delta}\right)^{3/2} \leq \frac{\phi_{\operatorname{lqr}}\zeta_{4}}{12\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\operatorname{lqr}}}\left(\frac{1}{T}\right)$$

$$= \frac{\phi_{\operatorname{lqr}}\zeta_{4}}{12\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\operatorname{lqr}}}\frac{\varepsilon}{40}\frac{1}{N\mathcal{C}(K_{0})}$$

$$= \frac{\phi_{\operatorname{lqr}}\zeta_{4}}{12\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\operatorname{lqr}}}\frac{\varepsilon}{40}\frac{1}{\max\left\{N_{1}\mathcal{C}(K_{0}), \frac{2\mathcal{C}(K_{0})}{\mu_{\operatorname{lqr}}}\frac{\xi_{3}\left(\log\frac{1}{\delta}\right)^{3/2}}{(1-\gamma)\omega_{\operatorname{lqr}}}\right\}}$$

$$= \frac{\phi_{\operatorname{lqr}}\zeta_{4}}{12\zeta_{3}\widetilde{c_{K_{1}}}(1-\gamma)\mu_{\operatorname{lqr}}}\frac{\varepsilon}{40}\min\left\{\frac{1}{N_{1}\mathcal{C}(K_{0})}, \frac{\mu_{\operatorname{lqr}}\omega_{\operatorname{lqr}}(1-\gamma)}{2\mathcal{C}(K_{0})\zeta_{3}\left(\log\frac{1}{\delta}\right)^{3/2}}\right\}.$$
(87)

For (87) to hold, we need two inequalities to hold as a result of the min{.,.} operator. First, we require

(88) 
$$\delta \left( \log \frac{1}{\delta} \right)^{3/2} \leqslant \frac{\phi_{\text{lqr}} \zeta_4}{480 \zeta_3 \widetilde{c_{K_1}} (1 - \gamma) \mu_{\text{lqr}} N_1 \mathcal{C}(K_0)} \varepsilon.$$

Now since  $a^3 \left(\log \frac{1}{a^3}\right)^{3/2} \leqslant a$  for all  $a \in (0,1)$  and the choice of  $\delta$  in (66), i.e.,

$$\delta \leqslant \left(\frac{\phi_{\text{lqr}}\zeta_4}{480\zeta_3\widetilde{c_{K_1}}(1-\gamma)\mu_{\text{lqr}}N_1\mathcal{C}(K_0)}\right)^3\varepsilon^3,$$

we conclude that (88) holds for the parameter setup of Theorem 3.1.

Secondly, it needs to hold that

(89) 
$$\delta \left(\log \frac{1}{\delta}\right)^{3/2} \leqslant \frac{\phi_{\text{lqr}}\zeta_4\omega_{\text{lqr}}}{960\zeta_3^2\widetilde{c_{K_1}}\mathcal{C}(K_0)\left(\log \frac{1}{\delta}\right)^{3/2}}\varepsilon$$
$$\iff \delta \left(\log \frac{1}{\delta}\right)^3 \leqslant \frac{\phi_{\text{lqr}}\zeta_4\omega_{\text{lqr}}}{960\zeta_3^2\widetilde{c_{K_1}}\mathcal{C}(K_0)}\varepsilon.$$

Now if

$$\frac{\phi_{\mathrm{lqr}}\zeta_4\omega_{\mathrm{lqr}}}{960\zeta_3^2\widetilde{c_{K_1}}\mathcal{C}(K_0)}\varepsilon\leqslant 0.028,$$

for any  $\delta \leqslant \left(\frac{\phi_{\text{lqr}}\zeta_4\omega_{\text{lqr}}}{960\zeta_2^2\overline{c_{K_1}}\mathcal{C}(K_0)}\right)^3\varepsilon^3$ , we have that

$$\delta \left( \log \frac{1}{\delta} \right)^3 \leqslant \frac{\phi_{\text{lqr}} \zeta_4 \omega_{\text{lqr}}}{960 \zeta_3^2 \widetilde{c_{K_1}} \mathcal{C}(K_0)} \varepsilon,$$

and if

$$\frac{\phi_{\mathrm{lqr}}\zeta_4\omega_{\mathrm{lqr}}}{960\zeta_3^2\widetilde{c_{K_1}}\mathcal{C}(K_0)}\varepsilon > 0.028,$$

it would suffice to have that

$$\delta \left(\log \frac{1}{\delta}\right)^3 \leqslant 0.028,$$

which would hold for any  $\delta \leq 2 \times 10^{-5}$ . As a result, due to the choice of  $\delta$  in (66), i.e.,

$$\delta \leqslant \min \left\{ 2 \times 10^{-5}, \left( \frac{\phi_{\text{lqr}} \zeta_4 \omega_{\text{lqr}}}{960 \zeta_3^2 \widetilde{c_{K_1}} \mathcal{C}(K_0)} \right)^3 \varepsilon^3 \right\},$$

we have that (89) will also hold under the parameter setup of Theorem 3.1. Finally, since both (88) and (89) hold for  $\delta$  as chosen in (66), inequality (87) is satisfied, finishing the proof.

We now prove the second claim. Even though our proof strategy mimics the one in [17], the structure of the stopping times in (61) and (75) makes the arguments more involved. Note that this difference in the definition of the stopping time (and subsequently the stopped process) can be attributed to the fact that in contrast to [17]'s one scenario (leaving the stable region) which may lead their algorithm to fail, there are two possible scenarios that may cause the failure of our algorithm. We start by introducing the stopped process

(90) 
$$Y_t := \Delta_{\tau_1 \wedge t} 1_{\tau_2 > t} + \frac{4\phi_{\text{lqr}} \xi_4}{(1 - \gamma)^2 \mu_{\text{lqr}}^2} \frac{1}{t + N} \quad \text{for each } t \in [T].$$

We next show that this process is a supermartingale. First, we have that

$$\mathbb{E}[Y_{t+1}|\mathcal{F}_{t}]$$

$$=\mathbb{E}[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{2}>t+1}|\mathcal{F}_{t}] + \frac{4\phi_{\text{lqr}}\xi_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}} \frac{1}{t+N+1}$$

$$=\mathbb{E}[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{2}>t+1} (1_{\tau_{1}\leqslant t}+1_{\tau_{1}>t})|\mathcal{F}_{t}] + \frac{4\phi_{\text{lqr}}\xi_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}} \frac{1}{t+N+1}$$

$$=\mathbb{E}[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{2}>t+1}1_{\tau_{1}\leqslant t}|\mathcal{F}_{t}] + \mathbb{E}[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{2}>t+1}1_{\tau_{1}>t}|\mathcal{F}_{t}] + \frac{4\phi_{\text{lqr}}\xi_{4}}{(1-\gamma)^{2}\mu_{\text{lqr}}^{2}} \frac{1}{t+N+1}.$$
(91)

Then for the first term on the right-hand side of (91), it holds that

$$\mathbb{E}\left[\Delta_{\tau_{1} \wedge t+1} 1_{\tau_{2} > t+1} 1_{\tau_{1} \leqslant t} \middle| \mathcal{F}_{t}\right] \leqslant \mathbb{E}\left[\Delta_{\tau_{1} \wedge t+1} 1_{\tau_{2} > t} 1_{\tau_{1} \leqslant t} \middle| \mathcal{F}_{t}\right]$$

$$= 1_{\tau_{2} > t} \mathbb{E}\left[\Delta_{\tau_{1} \wedge t+1} 1_{\tau_{1} \leqslant t} \middle| \mathcal{F}_{t}\right]$$

$$= 1_{\tau_{2} > t} \mathbb{E}\left[\Delta_{\tau_{1} \wedge t} 1_{\tau_{1} \leqslant t} \middle| \mathcal{F}_{t}\right]$$

$$= \Delta_{\tau_{1} \wedge t} 1_{\tau_{2} > t} 1_{\tau_{1} \leqslant t}.$$

$$(92)$$

As for the second term, we have

$$\mathbb{E}\left[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{2}>t+1}1_{\tau_{1}>t}|\mathcal{F}_{t}\right]$$

$$\stackrel{(i)}{=}\mathbb{E}\left[\Delta_{\tau_{1}\wedge t+1}1_{\tau_{1}>t}1_{\tau_{2}>t}1_{\mathcal{A}_{t}}|\mathcal{F}_{t}\right]$$

$$=\mathbb{E}\left[\Delta_{t+1}1_{\tau_{1}>t}1_{\tau_{2}>t}1_{\mathcal{A}_{t}}|\mathcal{F}_{t}\right]$$

$$=\mathbb{E}\left[\Delta_{t+1}1_{\mathcal{A}_{t}}|\mathcal{F}_{t}\right]1_{\tau_{1}>t}1_{\tau_{2}>t}$$

$$\stackrel{(ii)}{\leq}\left(\left(1-\mu_{\operatorname{lqr}}\alpha_{t}\right)\Delta_{t}+\frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t}+\frac{\phi_{\operatorname{lqr}}\alpha_{t}^{2}}{2}\frac{\xi_{4}}{(1-\gamma)^{2}}\right)1_{\tau_{1}>t}1_{\tau_{2}>t}$$

$$=\left(\left(1-\frac{2}{t+N}\right)\Delta_{t}+\frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t}+\frac{\phi_{\operatorname{lqr}}\alpha_{t}^{2}}{2}\frac{\xi_{4}}{(1-\gamma)^{2}}\right)1_{\tau_{1}>t}1_{\tau_{2}>t}$$

$$\stackrel{(iii)}{\leq}\Delta_{t}1_{\tau_{1}>t}1_{\tau_{2}>t}+\frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t}+\frac{\phi_{\operatorname{lqr}}\alpha_{t}^{2}}{2}\frac{\xi_{4}}{(1-\gamma)^{2}}$$

$$\stackrel{(iv)}{=}\Delta_{\tau_{1}\wedge t}1_{\tau_{1}>t}1_{\tau_{2}>t}+\frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t}+\frac{\phi_{\operatorname{lqr}}\alpha_{t}^{2}}{2}\frac{\xi_{4}}{(1-\gamma)^{2}},$$

$$\stackrel{(iv)}{=}\Delta_{\tau_{1}\wedge t}1_{\tau_{1}>t}1_{\tau_{2}>t}+\frac{3\zeta_{3}\widetilde{c_{K_{1}}}}{1-\gamma}\delta\left(\log\frac{1}{\delta}\right)^{3/2}\alpha_{t}+\frac{\phi_{\operatorname{lqr}}\alpha_{t}^{2}}{2}\frac{\xi_{4}}{(1-\gamma)^{2}},$$

where (i) follows from (77), (ii) from Lemma 3.5, (iii) from  $1_{\tau_1>t}1_{\tau_2>t}\leqslant 1$  along with the fact that  $\frac{2}{t+N}\leqslant 1$  for all  $t\in [T]$ , and (iv) from  $\Delta_t1_{\tau_1>t}=\Delta_{\tau_1\wedge t}1_{\tau_1>t}$ .

Combining (91), (92), and (93), we obtain that for all  $t \in [T]$ ,

$$\mathbb{E}[Y_{t+1}|\mathcal{F}_{t}] \leq \Delta_{\tau_{1} \wedge t} 1_{\tau_{2} > t} 1_{\tau_{1} \leq t} + \Delta_{\tau_{1} \wedge t} 1_{\tau_{1} > t} 1_{\tau_{2} > t}$$

$$+ \frac{3\zeta_{3} c_{K_{1}}}{1 - \gamma} \delta \left(\log \frac{1}{\delta}\right)^{3/2} \alpha_{t} + \frac{\phi_{\text{lqr}} \alpha_{t}^{2}}{2} \frac{\xi_{4}}{(1 - \gamma)^{2}} + \frac{4\phi_{\text{lqr}} \xi_{4}}{(1 - \gamma)^{2} \mu_{\text{lqr}}^{2}} \frac{1}{t + N + 1}$$

$$\stackrel{(i)}{\leq} \Delta_{\tau_{1} \wedge t} 1_{\tau_{2} > t} (1_{\tau_{1} \leq t} + 1_{\tau_{1} > t}) + \frac{4\phi_{\text{lqr}} \zeta_{4}}{(1 - \gamma)^{2} \mu_{\text{lqr}}^{2}} \frac{1}{t + N}$$

$$= \Delta_{\tau_{1} \wedge t} 1_{\tau_{2} > t} + \frac{4\phi_{\text{lqr}} \zeta_{4}}{(1 - \gamma)^{2} \mu_{\text{lqr}}^{2}} \frac{1}{t + N}$$

$$= Y_{t}.$$

where (i) follows from Sublemma 3.3. This proves the claim that  $Y_t$  is a supermartingale. Moreover, define the following events:

(94) 
$$\mathcal{E}_1 := \{ \tau_2 \leqslant \tau_1 \text{ and } \tau_2 \in [T] \}$$

(95) 
$$\mathcal{E}_2 := \{ \tau_1 < \tau_2 \text{ and } \tau_1 \in [T] \}$$

(96) 
$$\mathcal{E}_3 := \left\{ \max_{t \in [T]} \Delta_{\tau_1 \wedge t} 1_{\tau_2 > t} \geqslant 10 \mathcal{C}(K_0) \right\},\,$$

and hence, we have  $\mathbb{P}\{\tau \leq T\} = \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2)$ . Now since  $\tau_2 \leq \tau_1$  in  $\mathcal{E}_1$  suggests that  $\|\widehat{\nabla \mathcal{C}}(K_{\tau_2-1})\|_F > \frac{\xi_3}{1-\gamma} \left(\log \frac{1}{\delta}\right)^{3/2}$  despite  $\Delta_{\tau_2-1} \leq 10\mathcal{C}(K_0)$  (which implies  $\mathcal{K}_{\tau_2-1} \in \mathcal{G}^{\text{lqr}}$ ), after applying union bound on the result of Lemma 3.3, we have

$$(97) \mathbb{P}(\mathcal{E}_1) \leqslant \delta T.$$

Furthermore, note that  $\tau_1 < \tau_2$  in  $\mathcal{E}_2$  implies that  $\Delta_{\tau_1 \wedge \tau_1} 1_{\tau_2 > \tau_1} = \Delta_{\tau_1}$  and since  $\tau_1 \in [T]$ , it holds that

$$\max_{t \in [T]} \Delta_{\tau_1 \wedge t} 1_{\tau_2 > t} \geqslant \Delta_{\tau_1 \wedge \tau_1} 1_{\tau_2 > \tau_1} = \Delta_{\tau_1} \stackrel{\text{(i)}}{>} 10\mathcal{C}(K_0),$$

where (i) follows the definition of  $\tau_1$ . As a result of this, we have that  $\mathcal{E}_2$  implies  $\mathcal{E}_3$ , and consequently,  $\mathbb{P}(\mathcal{E}_2) \leq \mathbb{P}(\mathcal{E}_3)$ . Finally, since  $Y_t \geq \Delta_{\tau_1 \wedge t} 1_{\tau_2 > t}$  for all  $t \in [T]$ , we have that

$$\mathbb{P}(\mathcal{E}_{2}) \leq \mathbb{P}(\mathcal{E}_{3})$$

$$= \mathbb{P}\left\{\max_{t \in [T]} \Delta_{\tau_{1} \wedge t} 1_{\tau_{2} > t} \geq 10\mathcal{C}(K_{0})\right\}$$

$$\leq \mathbb{P}\left\{\max_{t \in [T]} Y_{t} \geq 10\mathcal{C}(K_{0})\right\}$$

$$\stackrel{(i)}{\leq} \frac{\mathbb{E}[Y_{0}]}{10\mathcal{C}(K_{0})}$$

$$= \frac{\Delta_{\tau_{1} \wedge 0} 1_{\tau_{2} > 0} + \frac{4\phi_{1qr}\xi_{4}}{(1-\gamma)^{2}\mu_{1qr}^{2}} \frac{1}{N}}{10\mathcal{C}(K_{0})}$$

$$\stackrel{(ii)}{\leq} \frac{\Delta_{0} + \mathcal{C}(K_{0})/2}{10\mathcal{C}(K_{0})}$$

$$\leq \frac{\mathcal{C}(K_{0}) + \mathcal{C}(K_{0})/2}{10\mathcal{C}(K_{0})}$$

$$\leq \frac{\mathcal{C}(K_{0}) + \mathcal{C}(K_{0})/2}{10\mathcal{C}(K_{0})}$$

$$= \frac{3}{20},$$

$$(98)$$

where (i) follows from applying Doob/Ville's inequality for supermartingales, and (ii) from the condition on the choice of N in Theorem 3.1. Utilizing the acquired probability bounds (97) and (98), we observe that

$$\mathbb{P}\{\tau \leqslant T\} = \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2)$$
$$\leqslant \delta T + \frac{3}{20},$$

which verifies the second claim of Proposition 3.2, concluding the proof.

The proof of our main result is a straightforward corollary:

*Proof of Theorem 3.1.* We now show how Proposition 3.2 can be employed to validate the claims of Theorem 3.1. Note that

$$\begin{split} \mathbb{P}\left\{\Delta_{T} \geqslant \varepsilon\right\} \leqslant & \mathbb{P}\left\{\Delta_{T} \mathbf{1}_{\tau > T} \geqslant \varepsilon\right\} + \mathbb{P}\left\{\mathbf{1}_{\tau \leqslant T} = 1\right\} \\ \leqslant & \frac{(\mathrm{i})}{\varepsilon} \mathbb{E}\left[\Delta_{T} \mathbf{1}_{\tau > T}\right] + \mathbb{P}\left\{\tau \leqslant T\right\} \\ \leqslant & \frac{(\mathrm{ii})}{20} + \frac{3}{20} + \delta T = \frac{1}{5} + \delta T, \end{split}$$

where (i) follows from Markov's inequality and (ii) follows from Proposition 3.2.

## 4. Summary and Discussion

We have provided an algorithm with  $\varepsilon$ -optimality guarantees with a provable convergence rate of  $\widetilde{\mathcal{O}}(1/\varepsilon)$  for the discounted discrete-time LQR problem in the model-free setting. This was made possible by employing a gradient estimation technique inspired by REINFORCE, combined with a time-varying step-size. Our results contrast from the ones obtained by two-point methods – which make the stronger assumption of assuming access to cost for two different policies with the same realization of the initial conditions – as well as results that assume stability of the policies.

An interesting future direction would be to investigate an actor-critic approach that could maintain the rate without requiring further assumptions. Moreover, one could consider an extension of the presented results for

the undiscounted case; in particular, the current analysis of gradient estimation with one zero-order evaluation per iteration heavily relies on sampling from a distribution whose definition relies on the discount factor be strictly less that one.

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