ON RETRACT VARIETIES OF ALGEBRAS

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Abstract. A retract variety is defined as a class of algebras closed under isomorphisms, retracts and products. Let a principal retract variety be generated by one algebra and a set-principal retract variety be generated by some set of algebras. It is shown that (a) not each set-principal retract variety is principal, and (b) not each retract variety is set-principal. A class of connected monounary algebras $\mathcal{S}$ such that every retract variety of monounary algebras is generated by algebras that have all connected components from $\mathcal{S}$ and at most two connected components are isomorphic is defined, this generating class is constructively described. All set-principal retract varieties of monounary algebras are characterized via degree function of monounary algebras.

1. Introduction

In many branches of mathematics, classes of structures were investigated, the classes being determined by whether they are closed with respect to certain operators. Among the first there were the operators **H** (homomorphic images), **S** (subalgebras), **P** (direct products), creating the class named variety, for recent papers see, e.g., [1, 6, 16, 22]. For years, several modifications and extensions of operators have been studied and we recall a only two: quasivariety (closed under isomorphisms, **P**, **S** and ultraproducts), pseudovariety (closed under **H**, **S** and finite direct products).

In 1981, Duffus and Rival [5] introduced the notion of order variety as a class of posets closed with respect to isomorphism, direct products and retracts. From papers in this direction let us mention [4, 9, 20, 21]. The notion of retract connects homomorphisms and subalgebras in some sense (see, e.g., [3, 15, 19, 14]).

For algebras,the notion of retract variety was introduced in [8] analogously as for posets: it is a class of algebras closed under operators **P** and **R** (= forming retracts and their isomorphic copies). A retract variety generated by a class of algebras $\mathcal{K}$ is equal to $\mathbf{RP}(\mathcal{K})$. Retract varieties were studied for lattice-ordered groups [8] and monounary algebras [10, 12].

We will concentrate to retract varieties of monounary algebras. Monounary algebras are very well represented by oriented graphs. In [10] it was shown that the system $\mathcal{L}$ of all retract varieties of monounary algebras forms a proper class (i.e., it is not a set). Moreover, ordering $\mathcal{L}$ by inclusion, there exist a chain and an antichain in $\mathcal{L}$ which are proper classes.

Notice that each variety of algebras is principal (= generated by one algebra). For quasivarieties this fails to hold [17]. We will show that this is not valid for retract

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varieties as well. Dealing with set-principal (= generated by *a set* of algebras) retract varieties, there appear questions as

- Is each retract variety set-principal?
- Is each set-principal retract variety principal?
- Is a system of all set-principal retract varieties a set?

The aim of this paper is a description of set-principal retract varieties of monounary algebras. It is done in Theorem 5.7 by using a function that is assigned to a monounary algebra as degree function, cf. [13] or as grade function, cf. [2]. As a particular results, not directly involved in the main ones, we have defined a class $\mathcal{S}$ of some reduced connected monounary algebras such that

(1) each retract variety generated by a class of connected monounary algebras is generated by a subclass of $\mathcal{S}$, see Corollary 4.8.
(2) each retract variety is generated by algebras that have all connected components from $\mathcal{S}$ and at most two connected components are isomorphic, see Corollary 5.4.

2. Preliminaries

The set of all positive integers will be denoted by $\mathbb{N}$; the set of all integers will be denoted by $\mathbb{Z}$. The cardinality of a set A will be denoted by $\operatorname{card} A$ and the class of all ordinals by Ord. We use braces $\{\}$ for collections of elements, not exclusively for sets.

We will apply notations and definitions concerning (partial) monounary algebras from [7, 13, 18]; let us recall some of them.

A partial monounary algebra is a pair (A, f), where A is a nonempty set and f is a partial unary operation on A. If domain of f is equal to A, then (A, f) is called a monounary algebra.

A partial monounary algebra $A = (A, f)$ is called *connected* if

$$\text{for every } x, y \in A \text{ there exist } m, n \in \mathbb{N} \cup \{0\} \text{ such that}$$
$$f^m(x), f^n(y) \text{ are defined and } f^m(x) = f^n(y).$$

Notation 2.1. Let us denote the class of all monounary algebras by $\mathcal{U}$ and the class of all connected monounary algebras by $\mathcal{U}_c$.

Further, let $\mathcal{T}$ be a class of all partial monounary algebras (A, f) such that

(1) (A, f) is connected partial monounary algebra and
(2) $\operatorname{card}(A \setminus \operatorname{dom} f) = 1$.

If $(A, f) \in \mathcal{T}$, then we denote by c_A the unique element of $A \setminus \operatorname{dom} f$.

Let (A, f) be a partial monounary algebra. We will also use A for this algebra, without distinguishing between the set and the algebra, if no misunderstanding can occur.

If $x \in A$, put

$$f^{-1}(x) = \{y \in A \colon y \in \operatorname{dom} f \text{ and } f(y) = x\}$$

and by induction for $n \in \mathbb{N}, n > 1$

$$f^{-n}(x) = \bigcup_{z \in f^{-(n-1)}} f^{-1}(z).$$

Further, put

$$P(x) = \{x\} \cup \bigcup_{n \in \mathbb{N}} f^{-n}(x).$$

It is obvious that $(P(x), f\restriction P(x))$ is a connected partial monounary algebra and the relationship $\operatorname{card}(P(x) \setminus \operatorname{dom} f) \leq 1$ is valid. Moreover, $(P(x), f\restriction P(x))) \in \mathcal{T}$ if and only if x is not cyclic element of (A, f).

Lemma 2.1. *Let (A, f) be a partial monounary algebra, $a, b \in A$, $a \neq b$ and $f(a) = a = f(b)$. Then the algebra $(P(a), f\restriction P(a))$ is not isomorphic to $(P(b), f \restriction P(b))$.*

Proof. The class $\mathcal{T}$ is closed under isomorphisms. We have $(P(a), f\restriction P(a)) \notin \mathcal{T}$ and $(P(b), f\restriction P(b)) \in \mathcal{T}$. $\square$

Notation 2.2. We define the following condition ($\bigstar$):
if $f(x_1) = f(x_2) = f(x_3)$ and algebras $(P(x_1), f\restriction P(x_1)), (P(x_2), f\restriction P(x_2))$ and $(P(x_3), f\restriction P(x_3))$ are isomorphic, then $\operatorname{card}\{x_1, x_2, x_3\} \leqq 2$.
We denote
$\mathcal{T}^\bigstar = \{(A, f) \in \mathcal{T}\colon$ the condition ($\bigstar$) is satisfied for all $x_1, x_2, x_3 \in A\}$.

Let us remark that if $(A, f) \in \mathcal{U}$ and $x_1, x_2, x_3 \in A$ are such that $f(x_1) = f(x_2) = f(x_3)$ and partial algebras $(P(x_1), f\restriction P(x_1)), (P(x_2), f\restriction P(x_2))$ $(P(x_3), f\restriction P(x_3))$ are isomorphic, then none of these three elements is a fixed element of (A, f) according to Lemma 2.1.

We will substantially use the notion of the degree of an element $x \in A$, where (A, f) is a (partial) monounary algebra; cf. e.g. [13]. The degree of x is an ordinal or the symbol ∞ and it is denoted by $\mathrm{s}_f(x)$. It is - roughly speaking - an expression of how far we can go back from a point x in the graph of algebra A. We remind the definition.

Notation 2.3. Let us denote by $A^{(\infty)}$ the system of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to A with the property $x_0 = x$ and $x_n \in \operatorname{dom} f, f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put

$$A^{(0)} = \{x \in A\colon f^{-1}(x) = \emptyset\}.$$

Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal $\lambda > 0$ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \{x \in A - \bigcup_{\alpha < \lambda} A^{(\alpha)}\colon f^{-1}(x) \subseteq \bigcup_{\alpha < \lambda} A^{(\alpha)}\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. We set

$$\mathrm{s}_f(x) = \begin{cases} \lambda & \text{if there is } \lambda \in \mathrm{Ord} \text{ such that } x \in A^{(\lambda)}, \\ \infty & \text{otherwise.} \end{cases}$$

We put $\lambda < \infty$ for every ordinal λ.

Let us remark that for every $\lambda \in \mathrm{Ord}$ there exists an algebra $(A, f) \in \mathcal{U}$ such that $s_f(x) = \lambda$ for some $x \in A$.

We will use the following notation:

Notation 2.4. Let $(A, f) \in \mathcal{U}$. We denote

$$A^{'} = \{x \in A \setminus A^{(\infty)} \colon f(x) \in A^{(\infty)}\}.$$

Let $\beta \in$ Ord. Denote

$$\begin{aligned} \mathcal{M}_\beta &= \{(A, f) \in \mathcal{U} \colon \mathrm{s}_f(x) \leqq \beta \text{ for each } x \in A \setminus A^{(\infty)}\}, \\ \mathcal{T}_\beta &= \{(A, f) \in \mathcal{T}^\bigstar \colon 0 < \mathrm{s}_f(c_A) \leqq \beta\}. \end{aligned}$$

2.1. **Retracts and special algebras.** A nonempty subset M of A is said to be *retract* of $(A, f) \in \mathcal{U}$ if there is a mapping $h \colon A \to M$ such that h is an endomorphism of (A, f) and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M.

The following theorem proved in [11] is essentially applied in several proofs.

Theorem 2.2. *Let $(A, f) \in \mathcal{U}$ and (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following conditions are satisfied:*

(a) *If $y \in f^{-1}(M)$, then there is $z \in M$ such that*

$$f(y) = f(z) \text{ and } \mathrm{s}_f(y) \leq \mathrm{s}_f(z).$$

(b) *For any connected component K of (A, f) with $K \cap M = \emptyset$, the following conditions are satisfied.*

(b1) *If K contains a cycle with d elements, then there is a connected component K^* of (A, f) with $K^* \cap M \neq \emptyset$ and there is $n \in \mathbb{N}$ such that $n|d$ and K^* has a cycle with n elements.*

(b2) *If K contains no cycle and x_0 is a fixed element of K, then there is $y_0 \in M$ such that $\mathrm{s}_f(f^k(x_0)) \leq \mathrm{s}_f(f^k(y_0))$ for each $k \in \mathbb{N} \cup 0$.*

We name several meaningful monounary algebras for this paper in the next notation.

Notation 2.5. Denote by $Z = (\mathbb{Z}, f)$ a monounary algebra such that $f(k) = k+1$ for all $k \in \mathbb{Z}$. For $n \in \mathbb{N}$ let $\mathbb{Z}_n$ be the set of all integers modulo n and $\underline{n} = (\mathbb{Z}_n, f)$ be a monounary algebra such that if $k \in \mathbb{Z}_n$ then $f(k) = k+1 \pmod{n}$.

Let us define the monounary algebra (E, f) as follows:

$$\begin{aligned} &E = \mathbb{Z} \cup \{(k, 1) \colon k \in \mathbb{N}\}, \\ &f(k) = k+1 \quad \text{for each} \quad k \in \mathbb{Z}, \\ &f((k, 1)) = \begin{cases} (k-1, 1) & \text{if} \;\; k \in \mathbb{N} \;\; k > 1, \\ 0 & \text{if} \;\; k = 1. \end{cases} \end{aligned}$$

For $n \in \mathbb{N}$ we define a monounary algebra $\widehat{n} = (\widehat{n}, f)$ by putting

$$\begin{aligned} &\widehat{n} = Z_n \cup \{(n, 1) : n \in \mathbb{N}\}, \\ &f(k) = k+1 (\text{mod } n) \quad \text{for each} \quad k \in \mathbb{Z}_n, \\ &f((k, 1)) = \begin{cases} (k-1, 1) & \text{if} \;\; k \in \mathbb{N} \;\; k > 1, \\ 0 & \text{if} \;\; k = 1. \end{cases} \end{aligned}$$

2.2. **Several facts about retract varieties.**

Notation 2.6. Let $\mathcal{K}$ be a class of algebras of the same type. The retract variety generated by $\mathcal{K}$ is denoted by $\mathbf{V}(\mathcal{K})$. The class of algebras whose elements are all retracts (all products) of members of $\mathcal{K}$ and their isomorphic images is denoted by $\mathbf{R}(\mathcal{K})$ $(\mathbf{P}(\mathcal{K}))$.

If $n \in \mathbb{N}$, $A_1, \dots, A_n$ be algebras of the same signature, then we write $\mathbf{V}(A_1 \dots, A_n)$ instead of $\mathbf{V}(\{A_1 \dots, A_n\})$.

For a (partial) algebra A we denote by $[A]$ the class all B isomorphic to A. If $B \in [A]$, then we write $B \cong A$.

Proposition 1.3 in [12] says that $\mathbf{V}(\mathcal{K}) = \mathbf{RP}(\mathcal{K})$ for each $\mathcal{K} \subseteq \mathcal{U}$. The proof of this Proposition works generally, therefore we obtain

Lemma 2.3. *Let $\mathcal{K}$ be a class of algebras of the same type. Then $\mathbf{V}(\mathcal{K}) = \mathbf{RP}(\mathcal{K})$.*

Lemma 2.4. *Let $\mathcal{K}, \mathcal{L}$ be a classes of algebras of the same type. If $A \in \mathbf{V}(\mathcal{L})$ for each $A \in \mathcal{K}$, then $\mathbf{V}(\mathcal{K}) \subseteq \mathbf{V}(\mathcal{L})$.*

Proof. Let $B \in \mathbf{V}(\mathcal{K}) = \mathbf{RP}(\mathcal{K})$. Then there exist a set I and $A_i \in \mathcal{K}, i \in I$ such that B is a retract of $\prod_{i\in I} A_i$. The assumption of Lemma says that $A_i \in \mathbf{V}(\mathcal{L})$ for each $i \in I$. It yields

$$B \in \mathbf{RPV}(\mathcal{L}) = \mathbf{V}(\mathcal{L}).$$

$\square$

The following two properties of retract varieties follow from definitions.

Lemma 2.5. *Let A be an algebra. If $\mathcal{V}$ is a retract variety and $A \in \mathcal{V}$, then $\mathbf{V}(A) \subseteq \mathcal{V}$.*

Lemma 2.6. $\mathbf{V}(\underline{1}) = [\underline{1}]$ *and if $n \in \mathbb{N}, n > 1$, then $\mathbf{V}(\underline{n})$ is the class of all monounary algebras which have every component isomorphic to $\underline{n}$.*

Proposition 2.7. *The retract variety $\mathbf{V}(\underline{2}, \underline{3})$ is not principal.*

Proof. We will prove two statements:

(1) $\mathbf{V}(\underline{2}, \underline{3}) = \mathbf{V}(\underline{2}) \cup \mathbf{V}(\underline{3}) \cup \mathbf{V}(\underline{6})$
(2) If $A \in \mathcal{U}$ is such that $\mathbf{V}(A) \subseteq \mathbf{V}(\underline{2}, \underline{3})$, then $\mathbf{V}(A) \in \{\mathbf{V}(\underline{2}), \mathbf{V}(\underline{3}), \mathbf{V}(\underline{6})\}$.

Let $B \in \mathbf{V}(\underline{2}, \underline{3})$. We have $\mathbf{V}(\underline{2}, \underline{3}) = \mathbf{RP}(\underline{2}, \underline{3})$, thus there are sets I, J and monounary algebras A_i for each $i \in I$, B_j for each $j \in J$ such that $A_i \cong \underline{2}$ for each $i \in I$, $B_j \cong \underline{3}$ for each $j \in J$ and B is isomorphic to some retract M of $\prod\limits_{i\in I} A_i \times \prod\limits_{j \in J} B_j$.

If $I = \emptyset$, then M is a retract of $\prod\limits_{j\in J} B_j$, hence $B \in \mathbf{V}(\underline{3})$. Analogously, if $J = \emptyset$, then $B \in \mathbf{V}(\underline{2})$.

Let $I \neq \emptyset \neq J$. Then $\prod\limits_{i\in I} A_i \times \prod\limits_{j\in J} B_j$ is a disjoint union of cycles with cardinality 6, hence M is a disjoint union of 6-element cycles, therefore $B \in \mathbf{V}(\underline{6})$.

If $n \in \{2, 3\}$, then $\mathbf{V}(\underline{n}) \subset \mathbf{V}(\underline{2}, \underline{3})$. Suppose that $A \in \mathbf{V}(\underline{6})$. For each $a \in A$ let $D_a = \underline{2}$ and $E_a = \underline{3}$. Put $D = \prod\limits_{a\in A} D_a \times \prod\limits_{a\in A} E_a$. Then D is a disjoint union of at least card A cycles with cardinality 6. We have $A \in \mathbf{R}(D)$ according to Theorem 2.2 and therefore $A \in \mathbf{V}(\underline{2}, \underline{3})$.

It yields the statement (1).

Suppose that $\mathbf{V}(A) \subseteq \mathbf{V}(\underline{2}, \underline{3})$. Then $A \in \mathbf{V}(\underline{2}, \underline{3})$ and thus $A \in \mathbf{V}(\underline{n})$ for some $n \in \{2, 3, 6\}$. Therefore $\mathbf{V}(A) \subseteq \mathbf{V}(\underline{n})$ according to Lemma 2.5. Further, $\underline{n} \in \mathbf{R}(A)$. This implies $\mathbf{V}(\underline{n}) \subseteq \mathbf{V}(A)$ according to Lemma 2.4. $\square$

3. Auxiliary results

The definition of degree of an element yields

Lemma 3.1. *Let $(A, f) \in \mathcal{U}$ and $B \in \mathbf{V}(A)$. If $A^{(\infty)} = A$, then $B^{(\infty)} = B$.*

Let us remark that direct products preserve injectivity of operations. Therefore

Lemma 3.2. *Let $(A, f) \in \mathcal{U}$ and $B \in \mathbf{V}(A)$. If the operation f is injective on A, then the operation f is injective on B.*

Lemma 3.3. *Let $(A, f) \in \mathcal{U}$ and $B \in \mathbf{V}(A)$. If the operation f is injective on $A \setminus \bigcup_{a\in A'} P(a)$, then the operation f is injective on $B \setminus \bigcup_{b\in B'} P(b)$.*

Proof. Suppose that $B \in \mathbf{V}(A)$ and

$$B \in \mathbf{R}(D), \text{ where } D = \textstyle\prod_{i\in I} A_i, A_i \cong A \text{ for all } i \in I.$$

An element $d = (a_i, i \in I) \in D^{(\infty)}$ if and only if $a_i \in A^{(\infty)}$ for each $i \in I$. Therefore the operation of the algebra $D \setminus \bigcup_{d\in D'} P(d)$ is injective and the operation of $B \setminus \bigcup_{b\in B'} P(b)$ is injective, too, since it is a subalgebra of $D \setminus \bigcup_{d\in D'} P(d)$. □

Proposition 3.4. *Let $\mathcal{V}$ be a set-principal retract variety of monounary algebras. Then there is $\beta \in \mathrm{Ord}$ such that $\mathcal{V} \subseteq \mathcal{M}_\beta$.*

Proof. Suppose that $\mathcal{V} = \mathbf{V}(\{A_i : i \in I\})$, where I is a set, $A_i \in \mathcal{U}$ for each $i \in I$. Let

$$S = \{\mathrm{s}_f(x) : x \in A_i, i \in I\} \setminus \{\infty\}.$$

There exists $\beta \in \mathrm{Ord}$ such that

(1) $\beta \geqq \alpha$ for each $\alpha \in S$.

Assume that $A \in \mathcal{V}$ and $D \in \mathbf{P}(\{A_i : i \in I\})$ is such that $A \in \mathbf{R}(D)$. Let

$$D = \prod_{j\in J} B_j, \text{ where } B_j \in \{A_i : i \in I\} \text{ for each } j \in J.$$

Take $x \in D \setminus D^{(\infty)}$. If $j \in J$, then $\mathrm{s}_f(x(j)) \in S \cup \{\infty\}$ and there is $j_0 \in J$ with $\mathrm{s}_f(x(j_0)) \neq \infty$, therefore (1) yields $\mathrm{s}_f(x(j_0)) \leqq \beta$. From this we obtain $\mathrm{s}_f(x) \leqq \beta$. Therefore $A \in \mathcal{M}_\beta$ since $A \subseteq D$. □

3.1. **On classes $\mathcal{T}_\beta$.** Now we will deal with the class $\mathcal{T}$. If $(A, f) \in \mathcal{T}$, then (A, f) is connected and contains the unique element c_A such that the operation f is not defined in it, see Notation 2.1. Remind that classes $\mathcal{T}_\beta$ for $\beta \in \mathrm{Ord}$ were introduced in Notation 2.4.

Lemma 3.5. *Let $n \in \mathbb{N}$. Then $\mathcal{T}_n/\cong$ is a finite set.*

Proof. Let us prove the assertion by induction.

(I) Let $n = 1$. Suppose that $(A, f) \in \mathcal{T}_1$. Then Notation 2.4 implies that $A = \{c_A\} \cup f^{-1}(c_A)$. According to (★) we obtain that $\mathrm{card}\, f^{-1}(c_A) \leqq 2$. Thus $\mathrm{card}(\mathcal{T}_1/\cong) = 2$.

(II) Let $n \in \mathbb{N}$, $n > 1$ and suppose that $\mathcal{T}_m/\cong$ is a finite set for each $m \in \mathbb{N}$, $m < n$. Let $(A, f) \in \mathcal{T}_n$. By the definition of s_f we get

(1) if $y \in f^{-1}(c_A)$, then $\mathrm{s}_f(y) \in \{m \in \mathbb{N} : m < n\}$.

Hence,

(2) if $y \in f^{-1}(c_A)$, then $(P(y), f)$ belongs to $\mathcal{T}_m$ for some $m \in \mathbb{N}$, $m < n$.

Denote, for $m \in \mathbb{N}$, $m < n$,

$$A_m = \{(P(y), f) \colon y \in f^{-1}(c_A), \mathrm{s}_f(y) = m\}.$$

By ($\bigstar$) and by the induction assumption, there are finitely many possibilities for A_m ($m < n$), thus there are only finitely many non-isomorphic (A, f) in $\mathcal{T}_n$. □

Notation 3.1. *Let $\beta \in \mathrm{Ord}$. We define $\tau(\beta)$ by induction as follows:*

(i) *if β is finite, then $\tau(\beta) = \aleph_0$;*

(ii) *if β is infinite, then $\tau(\beta) = \sup\{2^{\tau(\alpha)} \colon \alpha \in \mathrm{Ord}, \alpha < \beta\}$.*

Lemma 3.6. *Let $\beta \in \mathrm{Ord}$. Then $\mathcal{T}_\beta/\cong$ is a set and $\mathrm{card}(\mathcal{T}_\beta/\cong) \leqq \tau(\beta)$.*

Proof. By induction:

(I) If $n \in \mathbb{N}$, then $\mathrm{card}(\mathcal{T}_n/\cong) < \aleph_0 = \tau(n)$ according to Lemma 3.5.

(II) Let β be an infinite ordinal and suppose that for each $\alpha \in \mathrm{Ord}$, $\alpha < \beta$ the assertion is valid. Consider $(A, f) \in \mathcal{T}_\beta$. The definition of s_f implies

(1) if $y \in f^{-1}(c_A)$, then $\mathrm{s}_f(y) \in \{\alpha \in \mathrm{Ord} \colon \alpha < \beta\}$.

Hence,

(2) if $y \in f^{-1}(c_A)$, then $(P(y), f)$ belongs to $\mathcal{T}_\alpha$ for some $\alpha < \beta$.

The condition ($\bigstar$) is valid in (A, f), thus there are $\prod\limits_{\alpha<\beta} 2 \cdot (2^{\mathrm{card}(\mathcal{T}_\alpha/\cong)})$ possibilities how (A, f) can look like. It yields that $\mathcal{T}_\beta/\cong$ is a set and

$$\mathrm{card}(\mathcal{T}_\beta/\cong) = \prod_{\alpha<\beta} 2 \cdot (2^{\mathrm{card}(\mathcal{T}_\alpha/\cong)}) \leqq \prod_{\alpha<\beta} 2^{\tau(\alpha)} = \tau(\beta).$$

□

4. Connected algebras

Notation 4.1. Consider the following sets of connected monounary algebras:

$\mathcal{U}_c^\bigstar = \{(A, f) \in \mathcal{U}_c \colon$ *the condition ($\bigstar$) is satisfied for all* $x_1, x_2, x_3 \in A\}$,

$\mathcal{S}^{(0)} = [Z, E] \cup \bigcup_{n\in\mathbb{N}} [\underline{n}, \widehat{n}]$,

$\mathcal{S}^{(1)} = \{A \in \mathcal{U}_c^\bigstar \colon$ *there is* $a \in A$ *such that* $A' = \{a\}, A \setminus P(a) \in \mathcal{S}^{(0)}\}$,

$\mathcal{S}^{(2)} = \{A \in \mathcal{U}_c^\bigstar \colon A^{(\infty)} = \emptyset\}$.

Put

$$\mathcal{S} = \mathcal{S}^{(0)} \cup \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}.$$

We remark that $\mathcal{S}^{(0)} \subset \mathcal{U}_c^\bigstar$.

Lemma 4.1. *Let $A \in \mathcal{U}_c$ possess no cycle.*
If $A \ncong Z$ and $A^{(\infty)} = A$, then $\mathbf{V}(A) = \mathbf{V}(E)$.

Proof. By Theorem 2.2, $E \in \mathbf{R}(A)$. Thus $\mathbf{V}(E) \subseteq \mathbf{V}(A)$ according to Lemma 2.4.

To see the opposite inclusion we need to prove $A \in \mathbf{V}(E)$. It holds trivially in the case $A \cong E$. Suppose that $A \ncong E$. Put $\alpha = \mathrm{card}\, A$. Let I_j for $j \in \mathbb{Z}$ be disjoint sets of indices such that $\mathrm{card}\, I_j = \alpha$ for each $j \in Z$ and $I = \bigcup\limits_{j\in\mathbb{Z}} I_j$. Put $B_i = E$ for each $i \in I$ and $B = \prod\limits_{i\in I} B_i$. Denote by K the connected component of B such that K contains the element $b \in B$ with $b(i) = j$ for each $i \in I_j$, $j \in \mathbb{Z}$.

Let $x \in K$. Then there are $k, l \in \mathbb{N}$ such that $f^k(x) = f^l(b)$ since K is connected. Denote $d = f^l(b)$. We have $d(i) = j$ for each $j \in \mathbb{N}$ and $i \in I_{-l+j}$. Therefore $d(i) = k$ for every $i \in I_{-l+k}$. Consequently $(f(x))(i) = 1$ and $x(i) = 0$ since $f^{-1}(1) = \{0\}$ in the algebra E. We get $\operatorname{card} f^{-1}(x) \geq \alpha$.

Thus K is a connected algebra such that $\mathrm{s}_f(x) = \infty$ and $\operatorname{card} f^{-1}(x) \geq \alpha$ for each $x \in K$. Such an algebra contains obviously a subalgebra T such that $T \cong A$. Further, T is a retract of B according to Theorem 2.2 and the fact that no connected component of B contains a cycle. Hence $A \in \mathbf{RP}(E) = \mathbf{V}(E)$. □

Lemma 4.2. *Let $n \in \mathbb{N}$ and $A \in \mathcal{U}_c$ possess n-element cycle. If $A \not\cong \underline{n}$ and $A^{(\infty)} = A$, then $\mathbf{V}(A) = \mathbf{V}(\widehat{n})$.*

Proof. Obviously $\widehat{n} \in \mathbf{R}(A)$ and therefore $\mathbf{V}(\widehat{n}) \subseteq \mathbf{V}(A)$. To see the opposite inclusion, we need to prove that $A \in \mathbf{V}(\widehat{n})$.

Suppose that A is not isomorphic to $\widehat{n}$. Denote $\alpha = \operatorname{card} A$. Put $B_a = \widehat{n}$ for each $a \in A$ and $B = \prod\limits_{a \in A} B_a$. We have $B^{(\infty)} = B$ according to Lemma 3.1. Next, each connected component of B contains a cycle with cardinality n and coordinates of cyclic elements of B create subsets of $\mathbb{Z}_n$.

Consider a component K of B such that there is $b \in K, b$ cyclic and every element of $\mathbb{Z}_n$ occurs α times in b. Analogously as in the proof of the previous lemma we can see that $\operatorname{card} f^{-1}(x) \geq \alpha$ for each $x \in K$. Therefore B contains a subalgebra isomorphic to A. Hence, according to Theorem 2.2, $A \in \mathbf{R}(B)$. This implies that $A \in \mathbf{V}(\widehat{n})$. □

Lemma 4.3. *Let $A \in \mathcal{U}_c$. Then there are a set I and algebras $B_i \in \mathbf{R}(A)$ with $\operatorname{card} B_i^{'} = 1$ for each $i \in I$ such that*

$$\mathbf{V}(A) = \mathbf{V}(\{B_i : i \in I\}).$$

Proof. Let $A^{'} = \{b_i : i \in I\}$ and $0 \notin I$. We have $\operatorname{card} I > 1$.

Put

$$A_0 = A - \bigcup_{i \in I} P(b_i)$$

and for each $i \in I$ put

$$B_i = A_0 \cup P(b_i).$$

We obtain $B_i \in \mathbf{R}(A)$ and $\operatorname{card} B_i^{'} = 1$, since $f(b_i) \in A^{(\infty)}$ for each $i \in I$. Further, $\mathbf{V}(\{B_i : i \in I\}) \subseteq \mathbf{V}(A)$ according to Lemma 2.4.

Obviously A_0 is a retract of A, thus there is a retraction endomorphism φ of A onto A_0. Denote $B = \prod\limits_{i \in I} B_i$. Put

$$T_0 = \{t \in B : (\exists a \in A_0)(t(i) = a \ \text{ for each } \ i \in I)\}$$

$$T_i = \{t \in B : t(i) \in P(b_i), t(j) = \varphi(t(i)) \ \text{ for } \ j \in I - \{i\}\} \text{ for } i \in I,$$

$$T = \bigcup_{i \in I \cup \{0\}} T_i.$$

In view of Theorem 2.2, T is a retract of B.

Let us define a mapping $\Phi : A \to T$ as follows. If $a \in A^{(\infty)}$, then $\Phi(a) = t$ where $t(j) = a$ for all $j \in I$. If $i \in I$ and $a \in P(b_i)$, then $\Phi(a) = t$ where $t(i) = a$ and $t(j) = \varphi(a)$ for each $j \in I \setminus \{i\}$. It can be verified that Φ is an isomorphism.

Therefore we get $A \in \mathbf{RP}(\{B_i : i \in I\}) = \mathbf{V}(\{B_i : i \in I\})$. □

Lemma 4.4. *Let $A \in \mathcal{U}_c$.*
Then there exists $B \in \mathbf{R}(A) \cap \mathcal{U}_c^\bigstar$ such that $\mathbf{V}(A) = \mathbf{V}(B)$.

Proof. If $A \in \mathcal{U}_c^\bigstar$ then the assertion holds trivially. Now assume that $x_1, x_2, x_3 \in A$ are such that the condition ($\bigstar$) is not valid. For each $x \in A$ let

$$U(x) = \{y \in A\colon f(x) = f(y), P(x) \cong P(y)\}.$$

We have $U(x_1) = U(x_2) = U(x_3)$ and $\operatorname{card} U(x_1) > 2$. Denote

$$\{U_i\colon i \in I\} = \{U(x)\colon x \in A, \operatorname{card} U(x) > 2\}.$$

For each $i \in I$ take two distinct fixed elements of U_i and denote them by u_1^i, u_2^i. Now put

$$B = (A \setminus (\bigcup_{i\in I} \bigcup_{u\in U_i} P(u))) \cup \bigcup_{i\in I}(P(u_1^i) \cup P(u_2^i)).$$

Then $B \in \mathbf{R}(A) \cap \mathcal{U}_c^\bigstar$ and $\mathbf{V}(B) \subseteq \mathbf{V}(A)$ according to Lemma 2.4.

Let $\kappa = \sup\{\operatorname{card} U_i\colon i \in I\}$ and let J be a set with $\operatorname{card} J = \kappa$. Put $B_j = B$ for each $j \in J$ and let $C = \prod_{j\in J} B_j$.

We have that for $i \in I$ the set

$$D_i = \{c \in C\colon c(j) \in \{u_1^i, u_2^i\} \ \text{ for each } \ j \in J\}$$

has 2^κ elements, thus there is an injection $\xi_i\colon U_i \to D_i$. Further, if $i \in I$ and $x \in U_i$, then there exists isomorphisms

$$\psi_1^x\colon P(x) \to P(u_1^i),$$
$$\psi_2^x\colon P(x) \to P(u_2^i).$$

Now let us define an injective homomorphism $\nu\colon A \to C$. If $a \in B$, then denote

(1) $\nu(a) = \overline{a} \in C$, where $\overline{a}(j) = a$ for each $j \in J$.

If $i \in I$, $a \in U_i \setminus \{u_1^i, u_2^i\}$, then we define

(2) $\nu(a) = \xi_i(a)$.

If $i \in I$, $x \in U_i \setminus \{u_1^i, u_2^i\}$ and $a \in P(x) \setminus \{x\}$ then we define

(3) $\nu(a) = z$, where

$$z(j) = \begin{cases} \psi_1^x(a) & \text{if } (\xi_i(x))(j) = u_1^i, \\ \psi_2^x(a) & \text{if } (\xi_i(x))(j) = u_2^i, \end{cases}$$

for each $j \in J$. Denote $T = \nu(A)$. It is a technical matter to verify that T is a retract of C; hence

$$A \in \mathbf{R}(C) \subseteq \mathbf{RP}(B) = \mathbf{V}(B).$$

We get $\mathbf{V}(A) \subseteq \mathbf{V}(B)$. □

Lemma 4.5. *Let $A \in \mathcal{U}_c$ and $a \in A$ be such that $A' = \{a\}$. If A possesses no cycle and and $A \setminus P(a) \not\cong Z$, then there is $B \in \mathbf{R}(A) \cap \mathcal{S}^{(1)}$ such that $\mathbf{V}(A) = \mathbf{V}(B, E)$.*

Proof. Let A^* be an algebra from Lemma 4.4. Then $(A^*)' = \{a\}$ according to Lemma 3.1 and $A^* \setminus P(a)$ is not isomorphic to Z according to Lemma 3.3. Therefore the same assumptions as for A are valid for A^*. Further, if B is such that the statement is true for A^*, then $B \in \mathbf{R}(A^*) \subseteq \mathbf{R}(A)$ and $\mathbf{V}(A) = \mathbf{V}(A^*) = \mathbf{V}(B, E)$.

So, we can suppose that $A = (A, f) \in \mathcal{U}_c^\bigstar$. Obviously $E \in \mathbf{R}(A)$. Denote by A_0 a subalgebra of A such that $A_0 \cong Z$ and $f(a) \in A_0$. Put $B = A_0 \cup P(a)$. Then $B \in \mathbf{R}(A) \cap \mathcal{S}^{(1)}$. It yields $\mathbf{V}(B, E) \subseteq \mathbf{V}(A)$ according to Lemma 2.4.

There is a retraction homomorphism ψ of A onto A_0. Denote

$$T = \{t \in B \times A^{\infty} \colon (\exists x \in P(a))(t = (x, \psi(x))) \text{ or } (\exists x \in A - P(a))(t = (\psi(x), x))\}.$$

It is easy to see that $T \cong A$ and T is a retract of $B \times A^{(\infty)}$. Therefore $A \in \mathbf{V}(B, A^{(\infty)})$. Further, the algebra $A^{(\infty)}$ fulfills the assumptions of Lemma 4.1, hence we obtain $A^{(\infty)} \in \mathbf{V}(E)$. Thus $\mathbf{V}(B, A^{(\infty)}) \subseteq \mathbf{V}(B, E)$ according to Lemma 2.4. □

Analogously, we can prove

Lemma 4.6. *Let $A \in \mathcal{U}_c$ and $a \in A$ be such that $A^{'} = \{a\}$. If A possesses a cycle with n elements, $n \in \mathbb{N}$ and $A \setminus P(a) \notin \mathcal{S}^{(0)}$, then there is $B \in \mathbf{R}(A) \cap \mathcal{S}^{(1)}$ such that $\mathbf{V}(A) = \mathbf{V}(B, \widehat{n})$.*

Lemma 4.7. *If $A \in \mathcal{U}_c$, then there is a set I and algebras $B_i \in \mathbf{R}(A) \cap \mathcal{S}$ for each $i \in I$ such that $\mathbf{V}(A) = \mathbf{V}(\{B_i \colon i \in I\})$.*

Proof. If $A \in \mathcal{S}^{(0)}$, then the assertion holds. Let $A \notin \mathcal{S}^{(0)}$. By Lemma 4.4 there is an algebra $B \in \mathbf{R}(A) \cap \mathcal{U}_c^{\bigstar}$ such that

$$\mathbf{V}(A) = \mathbf{V}(B).$$

If $B \in \mathcal{S}$, then the proof is finished; therefore let $B \notin \mathcal{S}$. Then $B^{(\infty)} \neq \emptyset$ since $B \notin \mathcal{S}^{(2)}$.

Assume that A has a cycle of length n, $n \in \mathbb{N}$.

Let $B = B^{(\infty)}$. Then Lemma 4.2 implies that $\mathbf{V}(A) = \mathbf{V}(B) = \mathbf{V}(\widehat{n})$.

Suppose that $B \neq B^{(\infty)}$. Then $B^{'} \neq \emptyset$ and Lemma 4.3 yields that there is a set J and algebras $B_j \in \mathbf{R}(B)$ with $\mathrm{card}(B_j^{'}) = 1$ for each $j \in J$ such that

$$\mathbf{V}(B) = \mathbf{V}(\{B_j \colon j \in J\}).$$

Assume that $j \in J$. Then $B_j \in \mathcal{U}_c^{\bigstar}$ according to $B_j \in \mathbf{R}(B)$ and $B \in \mathcal{U}_c^{\bigstar}$. Further, $B_j' = \{b_j\}$ for some $b_j \in B_j$ since $\mathrm{card}(B_j^{'}) = 1$. If $B_j \setminus P(b_j) \in \mathcal{S}^{(0)}$, then $B_j \in \mathcal{S}^{(1)}$ by the definition of $\mathcal{S}^{(1)}$. If $B_j \setminus P(b_j) \notin \mathcal{S}^{(0)}$, then in view of 4.6 there is $C_j \in \mathbf{R}(B_j) \cap \mathcal{S}^{(1)}$ such that

$$\mathbf{V}(B_j) = \mathbf{V}(C_j, \widehat{n}).$$

Put

$$J_2 = \{j \in J \colon B_j \in \mathcal{S}^{(1)}\},\ J_1 = J \setminus J_2.$$

If $J_1 = \emptyset$, then the proof is finished.

If $J_1 \neq \emptyset$, then the set

$$M = \{B_j \colon j \in J_2\} \cup \{C_j \colon j \in J_1\} \cup \{\widehat{n}\} \subset \mathcal{S}^{(0)} \cup \mathcal{S}^{(1)} \subset \mathcal{S}$$

and

$$\mathbf{V}(\{B_j \colon j \in J\}) = \mathbf{V}(M)$$

according to Lemma 2.4. This completes the proof for A with a cycle.

If A has no cycle, then we can proceed analogously, use algebras Z, E and Lemmas 4.1, 4.3 and 4.5. □

Corollary 4.8. *Let $\mathcal{K} \subseteq \mathcal{U}_c$. Then there exists $\mathcal{L} \subseteq \mathcal{S}$ such that $\mathbf{V}(\mathcal{L}) = \mathbf{V}(\mathcal{K})$.*

4.1. **Retract varieties generated by a set of connected algebras.** Let $\beta \in \mathrm{Ord}$. The class $\mathcal{M}_\beta$ is defined in Notation 2.4.

Notation 4.2. Denote
$\mathcal{S}_\beta = \mathcal{M}_\beta \cap \mathcal{S}$, $\mathcal{S}_\beta^{(1)} = \mathcal{M}_\beta \cap \mathcal{S}^{(1)}$, $\mathcal{S}_\beta^{(2)} = \mathcal{M}_\beta \cap \mathcal{S}^{(2)}$.

We remark that $\mathcal{M}_\beta \cap \mathcal{S}^{(0)} = \mathcal{S}^{(0)}$ for every $\beta \in \mathrm{Ord}$.

Lemma 4.9. *Let* $\beta \in \mathrm{Ord}$. *Then* $\mathcal{S}_\beta^{(1)}/\cong$ *is a set and* $\mathrm{card}(\mathcal{S}_\beta^{(1)}/\cong) \leqq \tau(\beta)$.

Proof. We define a mapping

$$\varphi\colon (\mathcal{S}_\beta^{(1)}/\cong) \to (\mathbb{N} \cup \{0\}) \times \mathbb{Z} \times (\mathcal{T}_\beta/\cong)$$

as follows. Let $A \in \mathcal{S}^{(1)}$, $A' = \{a\}$. If $A \setminus P(a) \cong Z$, then put

$$\varphi([A]) = (0, 0, [P(a)]).$$

If $A \setminus P(a) \cong \underline{n}$ for some $n \in \mathbb{N}$, then put

$$\varphi([A]) = (n, 0, [P(a)]).$$

If $A \setminus P(a) \cong \widehat{n}$ for some $n \in \mathbb{N}$, then there is a uniquely determined $k \in \mathbb{N} \cup \{0\}$ such that $f^k(a)$ does not belong to a cycle, $f^{k+1}(a)$ belongs to a cycle; put

$$\varphi([A]) = (n, k+1, [P(a)]).$$

The mapping φ is injective, therefore $\mathcal{S}_\beta^{(1)}/\cong$ is a set and Lemma 3.6 implies

$$\mathrm{card}(\mathcal{S}_\beta^{(1)}/\cong) \leqq \aleph_0 \times \aleph_0 \times \tau(\beta) = \tau(\beta).$$

□

Lemma 4.10. *Let* $\beta \in \mathrm{Ord}$. *Then* $\mathcal{S}_\beta^2/\cong$ *is a set and* $\mathrm{card}(\mathcal{S}_\beta^{(2)}/\cong) \leqq (\tau(\beta))^{\aleph_0}$.

Proof. For a class $[A] \in \mathcal{S}_\beta^{(2)}/\cong$ take a fixed representant $\overline{A}$ of this class and let a be an arbitrary (fixed) element of $\overline{A}$. We define a mapping

$$\varphi\colon (\mathcal{S}_\beta^{(2)}/\cong) \to (\mathcal{T}_\beta/\cong)^{\mathbb{N}}$$

as follows. If $[A] \in \mathcal{S}_\beta^{(2)}/\cong$, then put

$$\varphi([A]) = (P(a), P(f(a)), P(f^2(a)), \dots).$$

The mapping φ is injective, therefore $\mathcal{S}_\beta^{7}/\cong$ is a set and we obtain in view of Lemma 3.6,

$$\mathrm{card}(\mathcal{S}_\beta^{(2)}/\cong) \leqq (\tau(\beta))^{\aleph_0}.$$

□

Corollary 4.11. *Let* $\beta \in \mathrm{Ord}$. *Then* $\mathcal{S}_\beta/\cong$ *is a set and* $\mathrm{card}(\mathcal{S}_\beta/\cong) \leqq (\tau(\beta))^{\aleph_0}$.

Proposition 4.12. *Let* $\mathcal{K} \subseteq \mathcal{U}_c$. *The following conditions are equivalent:*

(i) $\mathbf{V}(\mathcal{K})$ *is set-principal,*
(ii) *there is* $\beta \in \mathrm{Ord}$ *such that* $\mathcal{K} \subseteq \mathcal{M}_\beta$.

Proof. Let (ii) hold. If $A \in \mathcal{K}$, then Lemma 4.7 implies that there are a set I_A and algebras $B_i^A \in \mathbf{R}(A) \cap \mathcal{S}$ for $i \in I_A$ such that $A \in \mathbf{V}(\{B_i^A : i \in I_A\})$.

We get

$$\mathbf{V}(\mathcal{K}) = \mathbf{V}(\{A\colon A \in \mathcal{K}\}) = \mathbf{V}(\bigcup_{A \in \mathcal{K}} \{B_i^A : i \in I_A\}).$$

Further, $B_i^A \in \mathcal{S}_\beta$ since $B_i^A \in \mathbf{R}(A)$ and $A \in \mathcal{M}_\beta$. That means that $\mathbf{V}(\mathcal{K})$ is set-principal with respect to Corollary 4.11.

The opposite implication is proved in Proposition 3.4. □

5. General case

In this section we finish a description of set principal retract varieties of monounary algebras.

For a monounary algebra A consider the following condition:

$\binom{\bigstar}{\bigstar}$ if C_1, C_2, C_3 are connected components of A such that $C_1 \cong C_2 \cong C_3$, then $\operatorname{card}\{C_1, C_2, C_3\} \leqq 2$.

We denote

$$\mathcal{U}_\bigstar^\bigstar = \{(A \in \mathcal{U}\colon \text{ the condition } \tbinom{\bigstar}{\bigstar} \text{ is satisfied for all connected components } C_1, C_2, C_3 \text{ of } A\}.$$

Lemma 5.1. *Let $A \in \mathcal{U}$. Then there is $B \in \mathbf{R}(A) \cap \mathcal{U}_\bigstar^\bigstar$ such that $\mathbf{V}(A) = \mathbf{V}(B)$.*

Proof. If $A \in \mathcal{U}_\bigstar^\bigstar$ then $B = A$. Assume that A does not fulfil $\binom{\bigstar}{\bigstar}$. For each connected component C of A let $U(C)$ be the set of all connected components of A isomorphic to C. Denote

$$\{U_i\colon i \in I\} = \{U(C)\colon C \text{ is a connected component of } A, \ \operatorname{card} U(C) > 2\}.$$

Since $\binom{\bigstar}{\bigstar}$ is not valid in A, the set I is nonempty. For each $i \in I$ take two distinct fixed elements of U_i and denote them by C_i and C_i'. Now put

$$B = (A \setminus \bigcup_{i \in I} \bigcup_{K \in U_i} K) \cup \bigcup_{i \in I} (C_i \cup C_i').$$

Then $B \in \mathcal{U}_\bigstar^\bigstar$. If $i \in I$ and $C \in U_i$ then there exist isomorphisms

$$\psi_C\colon C \to C_i,$$
$$\psi_C'\colon C \to C_i'.$$

Define a mapping $\varphi\colon A \to B$ as follows:

if $x \in B$, then $\varphi(x) = x$, or

if $x \in C \in U_i \setminus \{C_i, C_i'\}$ for some $i \in I$, then $\varphi(x) = \psi_C(x)$.

This φ is a retraction endomorphism on A, therefore $B \in \mathbf{R}(A)$ and $\mathbf{V}(B) \subset \mathbf{V}(A)$ according to Lemma 2.4.

We need to see $A \in \mathbf{V}(B)$. Let

$$\kappa = \sup\{\operatorname{card} U_i\colon i \in I\}$$

and let J be a set with $\operatorname{card} J = \kappa$. Put $B_j = B$ for each $j \in J$ and let $D = \prod_{j \in J} B_j$. We are going to show that $A \in \mathbf{R}(D)$. If $a \in B$, then denote $\nu(a) = \overline{a} \in D$, where $\overline{a}(j) = a$ for each $j \in J$. If $i \in I$, then the set

$$Q_i = \{Y : Y \text{ is a connected component of } D, \ Y(j) \in \{C_i, C_i'\} \text{ for each } j \in J\}$$

has at least 2^κ elements, thus there is an injection $\xi_i\colon U_i \to Q_i$. If $a \in C \in U_i - \{C_i, C_i'\}$, then define $\nu(a) = z$, where

$$z(j) = \begin{cases} \psi_C(a) & \text{if } j \in J \text{ and } (\xi_i(C))(j) = C_i, \\ \psi_C'(a) & \text{if } j \in J \text{ and } (\xi_i(C))(j) = C_i'. \end{cases}$$

Denote $T = \nu(A)$. It can be verified that

(1) $\nu\colon A \to T$ is an isomorphism,
(2) T is a retract of D.

Thus

$$A \in \mathbf{R}(D) \subseteq \mathbf{RP}(B) = \mathbf{V}(B).$$

□

Lemma 5.2. *Let $A \in \mathcal{U}$. Then there exist a set K and algebras $B_\xi \in \mathcal{U}$ for each $\xi \in K$ such that*

(1) *if $\xi \in K$, then every connected component of B_ξ belongs to $\mathcal{S}$,*
(2) $\mathbf{V}(A) = \mathbf{V}(\{B_\xi : \xi \in K\})$.

Proof. Let $\{A_j, j \in J\}$ be a partition of A into connected components.

By using Lemma 4.7 we obtain that for each $j \in J$ there exist a set I_j and $D_{ji} \in \mathbf{R}(A_j) \cap \mathcal{S}, i \in I_j$ such that $A_j \in \mathbf{R}(\prod_{i\in I_j} D_{ji})$. Put K the set of all mappings ξ of J into $\bigcup_{j\in J} I_j$ such that $\xi(j) \in I_j$ for each $j \in J$. If $\xi \in K$, then put B_ξ this algebra that $\{D_{j\xi(j)}, j \in J\}$ is a partition of B_ξ into connected components.

If $\xi \in K$, then $D_{j\xi(j)} \in \mathbf{R}(A_j)$ for every $j \in J$. Thus $B_\xi \in \mathbf{R}(A)$ and

$$\mathbf{V}(\{B_\xi : \xi \in K\}) \subseteq \mathbf{V}(A)$$

according to Lemma 2.4.

To see the opposite inclusion suppose that ν_j is an isomorphism of A_j onto some retract T_j of $\prod_{i\in I_j} D_{ji}$. Let us define a mapping ν of A into $\prod_{\xi\in K} B_\xi$ as follows. If $a \in A$, then there is a uniquely determined $j \in J$ with $a \in A_j$. Then $\nu_j(a) \in T_j \subseteq \prod_{i\in I_j} D_{ji}$ and we put $\nu(a) = b$, where $b(\xi) = (\nu_j(a))(\xi(j))$ for each $\xi \in K$.

The mapping ν is injective, since if $a \in A_j$, $a' \in A_m$, $j, m \in J$, $\nu(a) = \nu(a')$, then, for each $\xi \in K$,

$$(\nu_j(a))(\xi(j)) = (\nu_m(a'))(\xi(m)),$$

thus $j = m$, $\nu_j(a) = \nu_j(a')$, hence $a = a'$. It can be shown that ν is a homomorphism, thus $\nu(A) \cong A$. Further, $\nu(A)$ is a retract of $\prod_{\xi\in K} B_\xi$ since $A_j \in \mathbf{R}(\prod_{i\in I_j} D_{ji})$. Therefore we have $A \in \mathbf{RP}(B_\xi : \xi \in K) = \mathbf{V}(B_\xi : \xi \in K)$. □

Corollary 5.3. *Let A be a monounary algebra. Then there are a set I and monounary algebras $B_i \in \mathcal{U}^\bigstar_\bigstar$ for each $i \in I$ such that*

(i) $\mathbf{V}(A) = \mathbf{V}(B_i : i \in I)$,
(ii) *if $i \in I$ and C is a connected component of B_i, then $C \in \mathcal{S}$.*

Proof. The assertion is a consequence of Lemmas 5.2, 5.1 and 2.4. □

Corollary 5.4. *Let $\mathcal{K} \subseteq \mathcal{U}$. Then there exists $\mathcal{L} \subseteq \mathcal{U}^\bigstar_\bigstar$ such that*

(i) $\mathbf{V}(\mathcal{L}) = \mathbf{V}(\mathcal{K})$,

(ii) *if* $A \in \mathcal{L}$ *and* C *is a connected component of* A*, then* $C \in \mathcal{S}$*.*

Notation 5.1. For $\beta \in \mathrm{Ord}$ let
$\mathcal{R}_\beta = \{B \in \mathcal{M}_\beta \cap \mathcal{U}^\bigstar_\bigstar \colon$ if C is a connected component of B, then $C \in \mathcal{S}\}$.

Lemma 5.5. *Let* $\beta \in \mathrm{Ord}$*. Then* $\mathcal{R}_\beta/\cong$ *is a set such that*

$$\mathrm{card}(\mathcal{R}_\beta/\cong) \leqq 2^{(\tau(\beta))^{\aleph_0}}$$

Proof. If $A \in \mathcal{R}_\beta$, then every connected component of A is from $\mathcal{S}_\beta$. In view of $A \in \mathcal{U}^\bigstar_\bigstar$ and Corollary 4.11 the algebra A consists of at most

$$2 \cdot \mathrm{card}(\mathcal{S}^{(1)}_\beta \cup \mathcal{S}^{(2)}_\beta) = (\tau(\beta))^{\aleph_0}$$

connected components. For each connected component there are at most $(\tau(\beta))^{\aleph_0}$ possibilities, therefore

$$\mathrm{card}(\mathcal{R}_\beta/\cong) \leqq [(\tau(\beta))^{\aleph_0}]^{(\tau(\beta))^{\aleph_0}} = 2^{(\tau(\beta))^{\aleph_0}}.$$

□

Proposition 5.6. *Let* $\beta \in \mathrm{Ord}$ *and* $\mathcal{V} \subseteq \mathcal{M}_\beta$ *be a retract variety. Then* $\mathcal{V}$ *is set-principal.*

Proof. If $A \in \mathcal{V}$, then Corollary 5.3 implies that there are a set I_A and monounary algebras $B^A_i \in \mathcal{U}^\bigstar_\bigstar$ for each $i \in I_A$ such that

(i) $\mathbf{V}(A) = \mathbf{V}(B^A_i \colon i \in I_A)$,
(ii) if $i \in I_A$ and C is a connected component of B^A_i, then $C \in \mathcal{S}$.

Then the assumption $\mathcal{V} \subseteq \mathcal{M}_\beta$ implies $B^A_i \in \mathcal{R}_\beta$ for each $i \in I_A$. We get

$$\mathcal{V} = \mathbf{V}(\{A \colon A \in \mathcal{V}\}) = \mathbf{V}(\{B^A_i \colon A \in \mathcal{V}, i \in I_A\}) \subseteq \mathbf{V}(\mathcal{R}_\beta).$$

and Lemma 5.5 implies that the retract variety $\mathcal{V}$ is set-principal. □

Theorem 5.7. *Let* $\mathcal{V}$ *be a retract variety of monounary algebras. The following conditions are equivalent:*

(1) $\mathcal{V}$ *is set-principal,*
(2) *there is* $\beta \in \mathrm{Ord}$ *such that if* $A \in \mathcal{V}$*,* $x \in A \setminus A^{(\infty)}$*, then* $\mathrm{s}_f(x) \leqq \beta$*.*

Proof. It follows from Propositions 5.6 and 3.4. □

Proposition 5.8. *There exists a retract variety* $\mathcal{T}$ *of monounary algebras such that* $\mathcal{T}$ *is not set-principal.*

Proof. For $\alpha \in \mathrm{Ord}$ there exists a (connected) monounary algebra A_α and an element $a_\alpha \in A_\alpha$ such that $\mathrm{s}_f(a_\alpha) = \alpha$. Then Theorem 5.7 implies that the retract variety $\mathbf{V}(\{A_\alpha \colon \alpha \in \mathrm{Ord}\})$ is not set-principal. □

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